Determinantal point processes with J-Hermitian correlation kernels

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Abstract

Let X be a locally compact Polish space and let m be a reference Radon measure on X. Let Γ_X denote the configuration space over X, i.e., the space of all locally finite subsets of X. A point process on X is a probability measure on Γ_X. A point process μ is called determinantal if its correlation functions have the form $k^{(n)}(x_1, \ldots, x_n) = \det[K(x_i, x_j)]_{i,j=1,\ldots,n}$. The function $K(x, y)$ is called the correlation kernel of the determinantal point process μ. Assume that the space X is split into two parts: $X = X_1 \sqcup X_2$. A kernel $K(x, y)$ is called J-Hermitian if it is Hermitian on $X_1 \times X_1$ and $X_2 \times X_2$, and $K(x, y) = -\overline{K(y, x)}$ for $x \in X_1$ and $y \in X_2$. We derive a necessary and sufficient condition of existence of a determinantal point process with a J-Hermitian correlation kernel $K(x, y)$.

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1 Introduction and preliminaries

1.1 Macchi–Soshnikov theorem

Let X be a locally compact Polish space, let $\mathcal{B}(X)$ be the Borel σ-algebra on X, and let $\mathcal{B}_0(X)$ denote the collection of all sets from $\mathcal{B}(X)$ which are pre-compact. The configuration space over X is defined as the set of all locally finite subsets of X:

$$\Gamma := \Gamma_X := \{\gamma \subset X \mid \text{for all } \Delta \in \mathcal{B}_0(X) \ |\gamma \cap \Delta| < \infty\}.$$

Here, for a set Λ, $|\Lambda|$ denotes its capacity. Elements $\gamma \in \Gamma$ are called configurations. The space Γ can be endowed with the vague topology, i.e., the weakest topology on Γ with respect to which all maps $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x)$, $f \in C_0(X)$, are continuous. Here $C_0(X)$ is the space of all continuous real-valued functions on X with compact support.

The configuration space Γ equipped with the vague topology is a Polish space. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ. A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a point process on X. For more detail, see e.g. [8, 10, 12, 15].

A point process μ can be described with the help of correlation functions, if they exist. Let m be a reference Radon measure on $(X, \mathcal{B}(X))$. The n-th correlation function of μ ($n \in \mathbb{N}$) is an $m^{\otimes n}$-a.e. non-negative measurable symmetric function $k_\mu^{(n)}(x_1, \ldots, x_n)$ on X^n such that, for any measurable symmetric function $f^{(n)} : X^n \to [0, \infty]$,

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\} \subset \gamma} f^{(n)}(x_1, \ldots, x_n) \, \mu(d\gamma)$$
$$= \frac{1}{n!} \int_{X^n} f^{(n)}(x_1, \ldots, x_n) k_\mu^{(n)}(x_1, \ldots, x_n) \, m(dx_1) \cdots m(dx_n). \quad (1)$$

Under a mild condition on the growth of correlation functions as $n \to \infty$, they determine a point process uniquely [12].

A point process μ is called determinantal if there exists a function $K(x, y)$ on X^2, called the *correlation kernel*, such that

$$k_\mu^{(n)}(x_1, \ldots, x_n) = \det \left[K(x_i, x_j) \right]_{i,j=1}^n, \quad n \in \mathbb{N}, \quad (2)$$

see e.g. [20]. The integral operator K in $L^2(X, m)$ which has integral kernel $K(x, y)$ is called the *correlation operator of* μ.

Assume that the correlation operator K is self-adjoint and bounded on the (real or complex) Hilbert space $L^2(X, m)$. In particular, the integral kernel $K(x, y)$ is Hermitian (symmetric in the real case). If the correlation functions $(k_\mu^{(n)})_{n \in \mathbb{N}}$ in (2) are *pointwisely* non-negative, then $K(x, y)$ is a positive definite kernel. Hence, if additionally the function $K(x, y)$ is continuous (it being possible to weaken the latter condition), then the operator K must be non-negative ($K \geq 0$).

A bounded linear operator K on $L^2(X, m)$ is called a locally trace-class operator if, for each $\Delta \in \mathcal{B}_0(X)$, the operator $K^\Delta := P^\Delta K P^\Delta$ is trace-class. Here P^Δ denotes the operator of multiplication by χ_Δ, the indicator function of the set Δ. (Thus, P^Δ is the orthogonal projection of $L^2(X, m)$ onto $L^2(\Delta, m)$.) If the operator K is self-adjoint and non-negative, then we can and will choose its integral kernel, $K(x, y)$, so that

$$\text{Tr } K^\Delta = \int_\Delta K(x, x) \, m(dx) \quad \text{for each } \Delta \in \mathcal{B}_0(X),$$

see [20] and [9]. By (1) and (2), for each $\Delta \in \mathcal{B}_0(X)$,

$$\int_\Gamma |\gamma \cap \Delta| \, \mu(d\gamma) = \int_\Delta K(x, x) \, m(dx).$$

Hence, in order that the correlation functions of μ be finite, we must indeed assume that the operator K is locally trace-class.

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The following theorem, which is due to Macchi [14] and Soshnikov [20], plays a fundamental role in the theory of point processes.

Theorem 1 (Macchi–Soshnikov). *Let K be a self-adjoint, non-negative, locally trace-class, bounded linear operator on $L^2(X, m)$. Then the integral kernel $K(x, y)$ of the operator K is the correlation kernel of a determinantal point process if and only if $0 \leq K \leq 1$.*

Note that, in the above theorem, the condition of boundedness of the operator K is not essential. One may instead initially assume that K is a Hermitian, non-negative, locally trace-class operator which is defined on a proper domain in $L^2(X, m)$.

Determinantal point processes with Hermitian correlation kernels occur in various fields of mathematics and physics, see e.g. the review paper [20] and Chapter 4 in [1].

1.2 Complementation principle (particle-hole duality)

Assume that the underlying space X is split into two disjoint parts: $X = X_1 \sqcup X_2$. Hence, we get $L^2(X, m) = L^2(X_1, m) \oplus L^2(X_2, m)$. For $i = 1, 2$, let P_i denote the orthogonal projection of $L^2(X, m)$ onto $L^2(X_i, m)$. Let us define a bounded linear operator J on $L^2(X, m)$ by $J := P_1 - P_2$. Following e.g. [2], we define an (indefinite) J-scalar product on $L^2(X, m)$ by

$$[f, g] := (Jf, g) = (P_1 f, P_1 g) - (P_2 f, P_2 g), \quad f, g \in L^2(X, m).$$

Here $(\cdot, \cdot)$ denotes the usual scalar product in $L^2(X, m)$. A bounded linear operator K on $L^2(X, m)$ is called J-self-adjoint if $[Kf, g] = [f, Kg]$ for all $f, g \in L^2(X, m)$. An integral kernel $K(x, y)$ of a J-self-adjoint, integral operator K is called J-Hermitian. More precisely, $K(x, y)$ is J-Hermitian if $K(x, y) = \overline{K(y, x)}$ if $x, y \in X_1$ or $x, y \in X_2$, and $K(x, y) = -\overline{K(y, x)}$ if $x \in X_1$, $y \in X_2$.

For a bounded linear operator K on $L^2(X, m)$, we denote

$$\widehat{K} := KP_1 + (1 - K)P_2. \tag{3}$$

As easily seen, K is J-self-adjoint if and only if $\widehat{K}$ is self-adjoint.

Assume now that the underlying space X is discrete, i.e., X is a countable set, and as a topological space X it totally disconnected. Thus, a configuration γ in X is an arbitrary subset of X. Let m be the counting measure on X: $m(\{x\}) = 1$ for each $x \in X$. Any linear operator K in $L^2(X, m)$ may be identified with its matrix $[K(x, y)]_{x,y \in X}$ ($K(x, y)$ being the integral kernel of K in this case).

Let μ be a point process on X. By (1),

$$k_\mu^{(n)}(x_1, \ldots, x_n) = \mu(\gamma \in \Gamma : \{x_1, \ldots, x_n\} \subset \gamma)$$

for distinct points $x_1, \ldots, x_n \in X$, otherwise $k_\mu^{(n)}(x_1, \ldots, x_n) = 0$. In particular, the correlation functions uniquely identify the corresponding point process.

Following [3], we will now present a complementation principle (a particle-hole duality) for determinantal point processes. (This observation is referred by the authors of [3] to a private communication by S. Kerov.) Assume, as above, that the underlying space X is divided into two disjoint parts: $X = X_1 \sqcup X_2$. Consider the mapping $I : \Gamma \to \Gamma$ defined by

$$I\gamma := \widehat{\gamma} := (\gamma \cap X_1) \cup (X_2 \setminus \gamma).$$

Thus, on the X_1 part of the space, the configuration $\widehat{\gamma}$ coincides with γ, while on the X_2 part the configuration $\widehat{\gamma}$ consists of all points from X_2 which do not belong to γ (holes). The mapping I is clearly an involution, i.e., I^2 is the identity mapping. For a point process μ on X, we denote by $\widehat{\mu}$ the push-forward of μ under I.

Proposition 1 ([3]). *Let μ be an arbitrary determinantal point process on a discrete space $X = X_1 \sqcup X_2$, with a correlation kernel $K(x,y)$. Then $\widehat{\mu}$ is the determinantal point process on X with the correlation kernel $\widehat{K}(x,y)$, the integral kernel of the operator $\widehat{K}$ defined by (3).*

Combining the Macchi–Soshnikov theorem with Proposition 1, we immediately get the following

Proposition 2. *Let $X = X_1 \sqcup X_2$ be a discrete space and let m be a counting measure on X. Let K be a bounded linear operator on $L^2(X, m)$ and let K be J-self-adjoint. Then $K(x,y)$ is the correlation kernel of a determinantal point process on X if and only if $0 \leq \widehat{K} \leq 1$.*

1.3 Formulation of the problem and the main result

In the case of a discrete underlying space X, determinantal point processes with J-Hermitian correlation kernels occurred in Borodin and Olshanski's studies on harmonic analysis of the infinite symmetric group, see e.g. [4, 5, 6, 7, 16], and the references therein. The paper [6, p. 1332] also contains references to some earlier works of mathematical physicists on solvable models of systems with positive and negative charged particles. In these papers, one finds further examples of determinantal point processes with J-Hermitian correlation kernels.

Furthermore, in their studies, Borodin and Olshanski derived three classes of determinantal point processes with J-Hermitian correlation kernels in the case where the underlying space X is *continuous*: the Whittaker kernel [5] ($X = \mathbb{R}_- \sqcup \mathbb{R}_+$), its scaling limit—the matrix tail kernel [16] ($X = \mathbb{R} \sqcup \mathbb{R}$), and the continuous hypergeometric kernel [6] ($X = (-\frac{1}{2}, \frac{1}{2}) \sqcup \{x \in \mathbb{R} : |x| > \frac{1}{2}\}$). It is important to note that, in all these examples, the self-adjoint operator $\widehat{K}$ appears to be an orthogonal projection. This follows from Proposition 5.1 in [7] and the respective results of [5], [16] (see also [7,

Proposition 6.6]), and [6]. (It should be, however, noted that, in the case of a continuous hypergeometric kernel, the corresponding projection property was proved only under an additional assumption, see the last two paragraphs of Section 10 in [6].)

The aim of the present paper is to *derive, in the case of a general underlying space X, a necessary and sufficient condition of existence of a determinantal point process with a J-Hermitian correlation kernel.* This problem was formulated to the author by Grigori Olshanski. I am extremely grateful to him for this and for many useful discussions and suggestions.

Our main result may be stated as follows. (We will omit a technical detail related to the choice of an integral kernel of the operator K.)

Main result. *Assume that K is a J-self-adjoint bounded linear operator on $L^2(X, m)$. Assume that the operators $P_1 K P_1$ and $P_2 K P_2$ are non-negative. Assume that, for any $\Delta_1, \Delta_2 \in \mathcal{B}_0(X)$ such that $\Delta_1 \subset X_1$ and $\Delta_2 \subset X_2$, the operators K^{Δ_i} $(i = 1, 2)$ are trace-class, while $P^{\Delta_2} K P^{\Delta_1}$ is a Hilbert–Schmidt operator. Then the integral kernel $K(x, y)$ of the operator K is the correlation kernel of a determinantal point process if and only if $0 \le \widehat{K} \le 1$.*

Let us make two remarks regarding the conditions of the main result. First, we note that, if the correlation operator K of μ is J-self-adjoint, then the restrictions of the point process μ to X_1 and X_2 are determinantal point processes on X_1 and X_2 with self-adjoint, correlation operators $P_1 K P_1$ and $P_2 K P_2$, respectively. Therefore, we assume that the latter operators are non-negative.

Second, choose any $\Delta \in \mathcal{B}_0(X)$ such that $m(\Delta_i) > 0$, where $\Delta_i := \Delta \cap X_i$, $i = 1, 2$. Then, since the operator K is not self-adjoint, the assumption in the main result is weaker than the requirement that the operator K^Δ be trace-class. In fact, K being locally trace-class seems to be a rather unnatural assumption for J-self-adjoint operators. This, of course, leads us to some additional difficulties in the proof.

Clearly, Proposition 2 is the special case of our main result in the case where the underlying space X is discrete. The drastic difference between the discrete and the continuous cases is that the mapping $\gamma \mapsto \widehat{\gamma}$ has no analog in the case of a continuous space X. Furthermore, if the space X is not discrete, the self-adjoint operator $\widehat{K}$ is not even an integral operator, so it can not be a correlation operator of a determinantal point process.

To prove the main result, we follow the strategy of dealing with determinantal point processes through the corresponding Fredholm determinants (compare with [14, 18, 20]), or rather the extension of Fredholm determinant as proposed in [3].

Combining the main result and Proposition 5.1 in [7], we also derive a method of constructing a big class of determinantal point processes with J-self-adjoint correlation operators K such that the corresponding operators $\widehat{K}$ are orthogonal projections. This class includes the above mentioned examples of determinantal point processes obtained by Borodin and Olshanski.

The paper is organized as follows. In Section 2, we prove a couple of results related to the mentioned extension of Fredholm determinant. In Section 3, we prove a series of auxiliary statements regarding J-self-adjoint operators and their extended Fredholm determinants. Finally, in Section 4, we formulate and prove the main results of the paper.

2   An extension of Fredholm determinant

We first recall the classical definition of a Fredholm determinant, see e.g. [19] for further detail. Let H be a complex, separable Hilbert space with scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$. We denote by $\mathscr{L}(H)$ the space of all bounded linear operators on H. An operator $A \in \mathscr{L}(H)$ is called a trace-class operator if $\|A\|_1 = \mathrm{Tr}(|A|) < \infty$, where $|A| = (A^*A)^{1/2}$. The set of all trace-class operators in H will be denoted by $\mathscr{L}_1(H)$. The trace of an operator $A \in \mathscr{L}_1(H)$ is given by $\mathrm{Tr}(A) = \sum_{n=1}^{\infty}(Ae_n, e_n)$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H. $\mathrm{Tr}(A)$ is independent of the choice of a basis. Note also that $|\mathrm{Tr}(A)| \leq \mathrm{Tr}(|A|)$. For any $A \in \mathscr{L}_1(H)$ and $B \in \mathscr{L}(H)$, we have $AB, BA \in \mathscr{L}_1(H)$ with

$$\max\{\|AB\|_1, \|BA\|_1\} \leq \|A\|_1 \|B\|,$$

where $\|B\|$ denotes the usual operator norm of B. In the latter case, we have

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA). \tag{4}$$

Denote by $\wedge^n(H)$ the n-th antisymmetric tensor power of the Hilbert space H, which is a closed subspace of $H^{\otimes n}$, the n-th tensor power of H. For any $A \in \mathscr{L}(H)$, the operator $A^{\otimes n}$ in $H^{\otimes n}$ acts invariantly on $\wedge^n(H)$ and we denote by $\wedge^n(A)$ the restriction of $A^{\otimes n}$ to $\wedge^n(H)$. If $A \in \mathscr{L}_1(H)$, then $\wedge^n(A) \in \mathscr{L}_1(\wedge^n(H))$ and

$$\| \wedge^n (A)\|_1 \leq \frac{1}{n!} \|A\|_1^n. \tag{5}$$

The Fredholm determinant is then defined by

$$\mathrm{Det}(1 + A) = 1 + \sum_{n=1}^{\infty} \mathrm{Tr}(\wedge^n(A)). \tag{6}$$

The Fredholm determinant can be characterized as the unique function which is continuous in A with respect to the trace norm $\|A\|_1$ and which coincides with the usual determinant when A is a finite-dimensional operator.

In [3], an extension of the Fredholm determinant was proposed. Assume that we are given a splitting of H into two subspaces:

$$H = H_1 \oplus H_2. \tag{7}$$

According to this splitting, we write an operator $A \in \mathscr{L}(H)$ in block form,

$$A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}, \tag{8}$$

where $A_{ij} : H_j \to H_i$, $i, j = 1, 2$. We define the even and odd parts of A as follows:

$$A_{\text{even}} := \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad A_{\text{odd}} := \begin{bmatrix} 0 & A_{21} \\ A_{12} & 0 \end{bmatrix}.$$

We denote by $\mathscr{L}_{1|2}(H)$ the set of all operators $A \in \mathscr{L}(H)$ such that $A_{\text{even}} \in \mathscr{L}_1(H)$ and $A_{\text{odd}} \in \mathscr{L}_2(H)$. Here $\mathscr{L}_2(H)$ denotes the space of all Hilbert–Schmidt operators on H, equipped with the norm

$$\|A\|_2 = \left(\sum_{n=1}^{\infty} \|Ae_n\|^2 \right)^{1/2},$$

where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H. Since $\mathscr{L}_1(H) \subset \mathscr{L}_2(H)$, one concludes that

$$\mathscr{L}_1(H) \subset \mathscr{L}_{1|2}(H) \subset \mathscr{L}_2(H).$$

We endow $\mathscr{L}_{1|2}(H)$ with the topology induced by the trace norm on the even part and by the Hilbert–Schmidt norm on the odd part.

Proposition 3 ([3]). *The function $A \mapsto \text{Det}(1 + A)$ admits a unique extension to $\mathscr{L}_{1|2}(H)$ which is continuous in the topology of $\mathscr{L}_{1|2}(H)$. This extension is given by the formula*

$$\text{Det}(1 + A) = \text{Det}((1 + A)e^{-A}) \cdot e^{\text{Tr}(A_{\text{even}})}. \tag{9}$$

Remark 1. Note that, for each $A \in \mathscr{L}_2(H)$, $(1 + A)e^{-A} - 1 \in \mathscr{L}_1(H)$. Therefore, $\text{Det}((1 + A)e^{-A})$ is a classical Fredholm determinant.

We will now derive an analog of formula (6) for $A \in \mathscr{L}_{1|2}(H)$. As follows from the proof of [18, Theorem 2.4], for each $A \in \mathscr{L}_1(H)$, we have:

$$\text{Tr}(\wedge^n(A)) = \frac{1}{n!} \sum_{\xi \in S_n} \text{sign}(\xi) \prod_{\eta \in \text{Cycle}(\xi)} \text{Tr}(A^{|\eta|}). \tag{10}$$

Here S_n denotes the set of all permutations of $1, \ldots, n$, the product in (10) is over all cycles η in permutation ξ, and $|\eta|$ denotes the length of cycle η. For $A \in \mathscr{L}_1(H)$, we clearly have $\text{Tr}(A) = \text{Tr}(A_{\text{even}})$. We further note that, for each $A \in \mathscr{L}_2(H)$, we have $A^k \in \mathscr{L}_1(H)$ for $k \geq 2$. Thus, for each $A \in \mathscr{L}_{1|2}(H)$, we set $C_n(A)$ to be equal to the right hand side of (10) in which we set

$$\text{Tr}(A) := \text{Tr}(A_{\text{even}}), \quad A \in \mathscr{L}_{1|2}(H).$$

Hence, $C_n(A)$ is well defined for each $A \in \mathscr{L}_{1|2}(H)$, and $C_n(A) = \text{Tr}(\wedge^n(A))$ for each $A \in \mathscr{L}_1(H)$.

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Proposition 4. *For each $A \in \mathscr{L}_{1|2}(H)$, we have*

$$\mathrm{Det}(1 + A) = 1 + \sum_{n=1}^{\infty} C_n(A). \tag{11}$$

Proof. We know that formula (11) hold for all $A \in \mathscr{L}_1(H)$. Next, for each $A \in \mathscr{L}_2(H)$,

$$\|A^k\|_1 \leq \|A\|^{k-2} \|A^2\|_1 \leq \|A\|^{k-2} \|A\|_2^2 \leq \|A\|_2^k, \quad k \geq 2.$$

We used that, for any $B, C \in \mathscr{L}_2(H)$,

$$\|BC\|_1 \leq \|B\|_2 \|C\|_2,$$

see e.g. [19, Theorem 2.8]. Hence, by the definition of $C_n(A)$,

$$|C_n(A)| \leq \|A\|_{1|2}^n, \tag{12}$$

where

$$\|A\|_{1|2} := \max\{\|A\|_2, \|A_{\mathrm{even}}\|_1\}.$$

(Note that $\| \cdot \|_{1|2}$ is a norm on $\mathscr{L}_{1|2}(H)$ which determines its topology.) Hence, if $\|A\|_{1|2} < 1$, the series on the right hand side of (11) converges absolutely.

Now, we fix any $A \in \mathscr{L}_{1|2}(H)$ with $\|A\|_{1|2} < 1$. For $i = 1, 2$, let $\{P_i^{(k)}\}_{k=1}^{\infty}$ be an ascending sequence of finite-dimensional orthogonal projections in H_i such that $P_i^{(k)}$ strongly converges to the identity operator in H_i as $k \to \infty$. Set $P^{(k)} := P_1^{(k)} + P_2^{(k)}$, $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, $A^{(k)} := P^{(k)} A P^{(k)}$ is a finite-dimensional operator in H, and

$$\|A^{(k)} - A\|_{1|2} \to 0 \quad \text{as } k \to \infty.$$

Hence, by (12) and the dominated convergence theorem,

$$\mathrm{Det}(1 + A^{(k)}) = 1 + \sum_{n=1}^{\infty} C_n(A^{(k)}) \to 1 + \sum_{n=1}^{\infty} C_n(A).$$

Therefore, by Proposition 3, formula (11) holds in this case.

Now we fix an arbitrary $A \in \mathscr{L}_{1|2}(H)$. Then, by (12), the function

$$z \mapsto 1 + \sum_{n=1}^{\infty} C_n(zA) = 1 + \sum_{n=1}^{\infty} z^n C_n(A)$$

is analytic on the set $\{z \in \mathbb{C} : |z| < \|A\|_{1|2}^{-1}\}$. Thus, by the uniqueness of analytic continuation, to prove, the proposition, it suffices to show that the function

$$\mathbb{C} \ni z \mapsto \mathrm{Det}(1 + zA) = \mathrm{Det}((1 + zA)e^{-zA}) \cdot e^{z\,\mathrm{Tr}(A_{\mathrm{even}})}$$

is entire. Equivalently, we have to show that the function

$$\mathbb{C} \ni z \mapsto \text{Det}((1 + zA)e^{-zA}).$$

is entire. Clearly,

$$(1 + zA)e^{-zA} - 1 = \sum_{k=2}^{\infty} \frac{(-z)^k(1-k)A^k}{k!}.$$

Hence,

$$\text{Det}((1 + zA)e^{-zA}) = 1 + \sum_{n=1}^{\infty} \text{Tr}\left(\wedge^n \left(\sum_{k=2}^{\infty} \frac{(-z)^k(1-k)A^k}{k!}\right)\right).$$

By (5), and we have

$$\left\|\wedge^n \left(\sum_{k=2}^{\infty} \frac{(-z)^k(1-k)A^k}{k!}\right)\right\|_1 \leq \frac{1}{n!}\left(\sum_{k=2}^{\infty} \frac{|z|^k|1-k|\,\|A^2\|_1\|A\|^{k-2}}{k!}\right)^n.$$

Hence, by the theorem on uniform convergence of analytic functions, it suffices to show that, for each $n \in \mathbb{N}$, $\text{Tr}\left(\wedge^n \left(\sum_{k=2}^{\infty} \frac{(-z)^k(1-k)A^k}{k!}\right)\right)$ is entire in z. By (10), this follows, since, for each $m \in \mathbb{N}$,

$$\text{Tr}\left(\left(\sum_{k=2}^{\infty} \frac{(-z)^k(1-k)A^k}{k!}\right)^m\right)$$

$$= \sum_{k_1,k_2,\ldots,k_m=2,3,\ldots} \frac{(-z)^{k_1+k_2+\cdots+k_m}(1-k_1)(1-k_2)\cdots(1-k_m)}{k_1!k_2!\cdots k_m!}\,\text{Tr}(A^{k_1+k_2+\cdots+k_m})$$

is entire in z. $\qquad\square$

Let us now assume that $H = L^2(X, m)$, where X is a locally compact Polish space and m is a Radon measure on $(X, \mathcal{B}(X))$. We fix any $X_1, X_2 \in \mathcal{B}(X)$ such that $X = X_1 \sqcup X_2$. By setting $H_i := L^2(X_i, m)$, $i = 1, 2$, we get a splitting (7) of H.

Proposition 5. *Let $K \in \mathscr{L}_{1|2}(L^2(X, m))$ be an integral operator with integral kernel $K(x, y)$ such that*

$$\text{Tr}(K_{\text{even}}) = \int_X K(x, x)\, m(dx). \tag{13}$$

Then

$$\text{Det}(1 + K) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} \det[K(x_i, x_j)]_{i,j=1,\ldots,n}\, m(dx_1)\cdots m(dx_n). \tag{14}$$

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Proof. For each $l = 2, 3, \ldots$, we have

$$\mathrm{Tr}(K^l) = \int_{X^l} K(x_1, x_2) K(x_2, x_3) \cdots K(x_l, x_1) \, m(dx_1) \cdots m(dx_l). \qquad (15)$$

Note that the integral in (15) is independent of the choice of a version of the integral kernel of K. Hence, by the definition of $C_n(K)$ and formulas (13), (15), we conclude that

$$C_n(K) = \frac{1}{n!} \int_{X^n} \det[K(x_i, x_j)]_{i,j=1,\ldots,n} \, m(dx_1) \cdots m(dx_n).$$

Now formula (14) follows from Proposition 4. $\qquad\qquad\square$

3 J-self-adjoint operators

We again assume that a Hilbert space H is split into two subspaces, see (7). According to this splitting, we write a vector $f \in H$ as $f = (f_2, f_2)$ and an operator $A \in \mathscr{L}(H)$ in the block form (8). Denote by P_1 and P_2 the orthogonal projections of the Hilbert space H onto H_1 and H_2, respectively. Setting $J := P_1 - P_2$, we introduce an (indefinite) J-scalar product on H by

$$[f, g] := (Jf, g) = (f_1, g_1) - (f_2, g_2), \quad f, g \in H.$$

An operator $A \in \mathscr{L}(H)$ is called self-adjoint in the indefinite scalar product $[\cdot, \cdot]$, or J-self-adjoint, if

$$[Af, g] = [f, Ag], \quad f, g \in H,$$

see e.g. [2]. In terms of the block form (8), an operator $A \in \mathscr{L}(H)$ is J-self-adjoint if and only if

$$A_{11}^* = A_{11}, \quad A_{22}^* = A_{22}, \quad A_{21}^* = -A_{12}. \qquad (16)$$

Remark 2. Assume A is a usual matrix which has a block form (8). If the blocks of A satisfy (16), then we will call A a J-Hermitian matrix.

For any $A \in \mathscr{L}(H)$, we denote by $\widehat{A}$ the operator from $\mathscr{L}(H)$ given by

$$\widehat{A} := AP_1 + (1 - A)P_2, \qquad (17)$$

or equivalently, in the block form,

$$\widehat{A} = \begin{bmatrix} A_{11} & A_{21} \\ -A_{12} & 1 - A_{22} \end{bmatrix}.$$

Clearly, if the operator A is self-adjoint, then $\widehat{A}$ is J-self-adjoint, while if A is J-self-adjoint, then $\widehat{A}$ is self-adjoint. Also $\widehat{\widehat{A}} = A$.

We will use below the following results.

Lemma 1. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint. Then $\|A\| = \|\widehat{A} - P_2\|$.*

Proof. We have $A = \widehat{A}P_1 + (1 - \widehat{A})P_2$, hence

$$A^* = P_1\widehat{A} + P_2(1 - \widehat{A}).$$

Denote by B_{A^*} the quadratic form on H with generator A^*. For any $f, g \in H$,

$$\begin{aligned}
B_{A^*}(f, g) &= (A^*f, g) \\
&= (\widehat{A}_{11}f_1, g_1) + (\widehat{A}_{12}f_2, g_1) + ((1 - \widehat{A}_{22})f_2, g_2) + (-\widehat{A}_{21}f_1, g_2).
\end{aligned}$$

Denote $\tilde{g} = (g_1, -g_2) = (\tilde{g}_1, \tilde{g}_2)$. Then

$$\begin{aligned}
B_{A^*}(f, g) &= (\widehat{A}_{11}f_1, \tilde{g}_1)_H + (\widehat{A}_{12}f_2, \tilde{g}_1)_H + ((\widehat{A}_{22} - 1)f_2, \tilde{g}_2)_H + (\widehat{A}_{21}f_1, \tilde{g}_2) \\
&= (\widehat{A}f, \tilde{g}) - (f_2, \tilde{g}) \\
&= ((\widehat{A} - P_2)f, \tilde{g}) \\
&= B_{\widehat{A} - P_2}(f, \tilde{g}).
\end{aligned}$$

From here

$$\|\widehat{A} - P_2\| = \|B_{\widehat{A} - P_2}\| = \|B_{A^*}\| = \|A^*\| = \|A\|.$$

$\square$

Proposition 6. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint and assume that $0 \le \widehat{A} \le 1$. Then $\|A\| \le 1$.*

Proof. By Lemma 1, it suffices to show that $\|\widehat{A} - P_2\| \le 1$. Note that $\widehat{A} - P_2$ is self-adjoint. For each $f \in H$,

$$((\widehat{A} - P_2)f, f) = (\widehat{A}f, f) - (f_2, f_2) \le (\widehat{A}f, f) \le (f, f).$$

Hence $\widehat{A} - P_2 \le 1$. Next,

$$((\widehat{A} - P_2)f, f) = (\widehat{A}f, f) - (f_2, f_2) \ge -(f_2, f_2) \ge -(f, f),$$

and so $\widehat{A} - P_2 \ge -1$. Thus, $-1 \le \widehat{A} - P_2 \le 1$, which implies the statement. $\square$

Proposition 7. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint and assume that $0 \le \widehat{A} \le 1$. Then $\|A\| = 1$ if and only if $\|A_{\text{even}}\| = 1$.*

Proof. By Lemma 1, it suffices to prove that $\|\widehat{A} - P_2\| = 1$ if and only if $\|A_{\text{even}}\| = 1$. Let us first assume that that $\|\widehat{A} - P_2\| = 1$.

Since $0 \le \widehat{A} \le 1$, we have $0 \le \widehat{A}_{11} \le 1$ and $0 \le \widehat{A}_{22} \le 1$. Hence, $0 \le A_{11} \le 1$ and $0 \le A_{22} \le 1$, and so $0 \le A_{\text{even}} \le 1$, which in turn implies that $\|A_{\text{even}}\| \le 1$. We have to consider two cases.

Case 1. $-1 \in \sigma(\widehat{A} - P_2)$. (Here, $\sigma(B)$ denotes the spectrum of an operator $B \in \mathscr{L}(H)$.) Then there exists a sequence $(f^{(n)})_{n=1}^{\infty}$ in H such that $\|f^{(n)}\| = 1$ and

$$((\widehat{A} - P_2)f^{(n)}, f^{(n)}) \to -1.$$

Since $(\widehat{A}f^{(n)}, f^{(n)}) \geq 0$ and $(P_2 f^{(n)}, f^{(n)}) \leq 1$, we get

$$(\widehat{A}f^{(n)}, f^{(n)}) \to 0, \quad \|f_2^{(n)}\| \to 1.$$

Hence $f_1^{(n)} \to 0$. From here

$$(\widehat{A}_{11}f_1^{(n)}, f_1^{(n)}) + (\widehat{A}_{21}f_1^{(n)}, f_2^{(n)}) + (\widehat{A}_{12}f_2^{(n)}, f_1^{(n)}) \to 0.$$

Thus

$$(\widehat{A}_{22}f_2^{(n)}, f_2^{(n)}) \to 0.$$

Hence

$$\left(\widehat{A}_{22}\frac{f_2^{(n)}}{\|f_2^{(n)}\|}, \frac{f_2^{(n)}}{\|f_2^{(n)}\|}\right) \to 0.$$

Hence, $0 \in \sigma(\widehat{A}_{22})$, and so $1 \in \sigma(1 - \widehat{A}_{22}) = \sigma(A_{22})$.

Case 2. $1 \in \sigma(\widehat{A} - P_2)$. Then there exists a sequence $(f^{(n)})_{n=1}^{\infty}$ in H such that $\|f^{(n)}\| = 1$ and

$$((\widehat{A} - P_2)f^{(n)}, f^{(n)}) \to 1.$$

Since $(\widehat{A}f^{(n)}, f^{(n)}) \leq 1$ and $(P_2 f^{(n)}, f^{(n)}) \geq 0$, we get

$$(\widehat{A}f^{(n)}, f^{(n)}) \to 1, \quad \|f_2^{(n)}\| \to 0.$$

From here, analogously to the above we conclude that $1 \in \sigma(\widehat{A}_{11}) = \sigma(A_{11})$.

Thus, in both cases, we get $\|A_{\text{even}}\| = 1$. By inverting the arguments, we conclude the inverse statement. $\qquad\square$

Proposition 8. *Let $A \in \mathscr{L}(H)$ be J-self-adjoint and let $A \in \mathscr{L}_{1|2}(H)$. Assume that $\|A\| < 1$ and $A_{11} \geq 0$. Then*

$$\text{Det}(1 - A) = \text{Det}(1 - A_{11}) \cdot \text{Det}(1 - A_{22} - A_{21}(1 - A_{11})^{-1}A_{12}) > 0. \qquad (18)$$

Proof. Analogously to the proof of Proposition 4, we define operators $A^{(n)}$, $n \in \mathbb{N}$. Then

$$A^{(n)} = \begin{bmatrix} A_{11}^{(n)} & A_{21}^{(n)} \\ A_{12}^{(n)} & A_{22}^{(n)} \end{bmatrix}, \qquad (19)$$

where $A_{ij}^{(n)} = P_i^{(n)} A_{ij} P_j^{(n)}$, $i, j = 1, 2$. Clearly, each $A^{(n)}$ is J-self-adjoint. By [3, Corollary A.2],

$$\text{Det}(1 - A) = \lim_{n \to \infty} \text{Det}(1 - A^{(n)}). \tag{20}$$

For each $n \in \mathbb{N}$, the operator $A^{(n)}$ is finite-dimensional, hence $\text{Det}(1 - A^{(n)})$ is a classical Fredholm determinant. Therefore,

$$\text{Det}(1 - A^{(n)}) = \text{Det} \begin{bmatrix} 1 - A_{11}^{(n)} & -A_{21}^{(n)} \\ -A_{12}^{(n)} & 1 - A_{22}^{(n)} \end{bmatrix}, \tag{21}$$

the latter (in fact, usual) determinant refers to the finite-dimensional Hilbert space $P^{(n)} H$. Since $\|A\| < 1$, we get $\|A_{11}\| < 1$, and hence $\|A_{11}^{(n)}\| < 1$ for all n. Thus, $1 - A_{11}^{(n)}$ is invertible in $P_{11}^{(n)} H$. Employing the well-known formula for the determinant of a block matrix, we get from (20) and (21):

$$\text{Det}(1 - A) = \lim_{n \to \infty} \text{Det}(1 - A_{11}^{(n)}) \cdot \text{Det}(1 - A_{22}^{(n)} - A_{21}^{(n)}(1 - A_{11}^{(n)})^{-1} A_{12}^{(n)}). \tag{22}$$

We state that

$$A_{11}^{(n)} \to A_{11}, \quad A_{22}^{(n)} \to A_{22}, \tag{23}$$

$$A_{21}^{(n)}(1 - A_{11}^{(n)})^{-1} A_{12}^{(n)} \to A_{21}(1 - A_{11})^{-1} A_{12} \tag{24}$$

in $\mathscr{L}_1(H)$ as $n \to \infty$. Formula (23) is evident, while (24) will require some additional consideration. Assume, for a moment, that (24) is proven. Then, by (22)

$$\begin{aligned} \text{Det}(1 - A) &= \text{Det}(1 - A_{11}) \cdot \text{Det}(1 - A_{22} - A_{21}(1 - A_{11})^{-1} A_{12}) \\ &= \text{Det}(1 - A_{11}) \cdot \text{Det}(1 - A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12}). \end{aligned} \tag{25}$$

Note that the determinants on the right hand side of (25) are classical Fredholm determinants, while the operators A_{11}, A_{22} and $A_{12}^*(1 - A_{11})^{-1} A_{12}$ are trace-class. Note also that the operators $-A_{11}$ and $-A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12}$ are self-adjoint. Since $\|A_{11}\| < 1$, we get $\text{Det}(1 - A_{11}) > 0$. Further $\|A_{22}\| < 1$, and hence there exists $\varepsilon > 0$ such that $1 - A_{22} \geq \varepsilon 1$. Clearly, since $A_{11} \geq 0$,

$$A_{12}^*(1 - A_{11})^{-1} A_{12} \geq 0,$$

which implies

$$1 - A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12} \geq \varepsilon 1.$$

From here

$$\text{Det}(1 - A_{22} + A_{12}^*(1 - A_{11})^{-1} A_{12}) > 0,$$

and formula (18) is proven.

Thus, it remains to show (24). We have

$$\|(A_{21}^{(n)} - A_{21})(1 - A_{11}^{(n)})^{-1}A_{12}^{(n)}\|_1 \leq \|A_{21}^{(n)} - A_{21}\|_2 \|(1 - A_{11}^{(n)})^{-1}A_{12}^{(n)}\|_2$$
$$\leq \|A_{21}^{(n)} - A_{21}\|_2 \|(1 - A_{11}^{(n)})^{-1}\| \|A_{12}^{(n)}\|_2$$
$$\leq \|A_{21}^{(n)} - A_{21}\|_2 (1 - \|A_{11}^{(n)}\|)^{-1} \|A_{12}^{(n)}\|_2$$
$$\leq \|A_{21}^{(n)} - A_{21}\|_2 (1 - \|A_{11}\|)^{-1} \|A_{12}\|_2 \to 0 \qquad (26)$$

as $n \to \infty$. Analogously,

$$\|A_{21}(1 - A_{11}^{(n)})^{-1}(A_{12}^{(n)} - A_{12})\|_1 \to 0 \qquad (27)$$

as $n \to \infty$. Therefore, by (26) and (27), it suffices to show that

$$\|A_{21}[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]A_{12}\|_1 \to 0,$$

which, in turn, would follow from

$$\|[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]A_{12}\|_2 \to 0. \qquad (28)$$

To this end, we fix an orthonormal basis $\{e_j\}_{j=1}^{\infty}$. Then

$$\|[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]A_{12}\|_2^2 = \sum_{j=1}^{\infty} \|[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]A_{12}e_j\|^2.$$

For each $j \in \mathbb{N}$,

$$\|[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]A_{12}e_j\|^2 \leq 4(1 - \|A_{11}\|)^{-1}\|^2 \|A_{12}e_j\|^2.$$

Hence, by the dominated convergence theorem, (28) will follow if we show that, for each $j \in \mathbb{N}$,

$$\|[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]A_{12}e_j\|^2 \to 0$$

as $n \to \infty$, or more generally $(1 - A_{11}^{(n)})^{-1} \to (1 - A_{11})^{-1}$ strongly in H. For each $f \in H$,

$$\|[(1 - A_{11}^{(n)})^{-1} - (1 - A_{11})^{-1}]f\| \leq \sum_{k=1}^{\infty} \|[(A_{11}^{(n)})^k - (A_{11})^k]f\|.$$

Hence, by the dominated convergence, it suffices to show that, for each $k \in \mathbb{N}$, $(A_{11}^{(n)})^k \to (A_{11})^k$ strongly as $n \to \infty$. This is clearly true for $k = 1$, and hence for all $k \in \mathbb{N}$. $\qquad \square$

Proposition 9. *Let $A \in \mathscr{L}_{1|2}(H)$ and let A be J-self-adjoint. Let $0 \leq \widehat{A} \leq 1$ and let $\|A\| < 1$. Let $L := A(1 - A)^{-1}$. Then L is J-selfdjoint, $L \in \mathscr{L}_{1|2}(H)$, and $L_{11} \geq 0$, $L_{22} \geq 0$.*

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Proof. We have $L = A + \sum_{n=2}^{\infty} A^n$, and

$$\sum_{n=2}^{\infty} \|A^n\|_1 \le \|A\|_2^2 \sum_{n=0}^{\infty} \|A\|^n < \infty.$$

Hence, $\sum_{n=2}^{\infty} A^n \in \mathscr{L}_1(H)$, so $L \in \mathscr{L}_{1|2}(H)$.

Let us show that the operator L is J-self-adjoint. For any $f, g \in H$, we have

$$(Lf, g) = \sum_{n=1}^{\infty} (A^n f, g) = \sum_{n=1}^{\infty} (f, (A^*)^n g) = \sum_{n=1}^{\infty} (f, (A_{11} - A_{21} - A_{12} + A_{22})^n g).$$

Denoting $A'_{11} := A_{11}$, $A'_{22} := A_{22}$, $A'_{12} := -A_{12}$, $A'_{21} := -A_{21}$, we get

$$(Lf, g) = \sum_{n=1}^{\infty} \sum_{\substack{i_k, j_k = 1,2 \\ k=1,\ldots,n}} (f, A'_{i_1 j_1} A'_{i_2 j_2} \cdots A'_{i_n j_n} g). \tag{29}$$

Assume that $f = f_1 \in H_1$, $g = g_1 \in H_1$. Then, in the latter sum, the terms in which the number of the A'_{12} operators is not equal to the number of the A'_{21} operators, are equal to zero. Hence,

$$(L_{11} f_1, g_1) = (Lf_1, g_1) = \sum_{n=1}^{\infty} \sum_{\substack{i_k, j_k = 1,2 \\ k=1,\ldots,n}} (f_1, A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} g_1)$$

$$= \sum_{n=1}^{\infty} (f_1, A^n g_1) = (f_1, L g_1) = (f_1, L_{11} g_1). \tag{30}$$

Thus, $L_{11}^* = L_{11}$. Analogously, $L_{22}^* = L_{22}$.

In the case where $f = f_1 \in H_1$ and $g = g_2 \in H_2$, those terms in the sum in (29) in which the number of the A'_{21} operators is not equal to the the number of the A'_{12} operators plus one, are equal to zero. Hence, similarly to (30), we get

$$(L_{21} f_1, g_2) = (f_1, -L_{12} g_2),$$

so $L_{21}^* = -L_{12}$. Thus, L is J-self-adjoint.

Next, we will show that $L_{11} \ge 0$. Analogously to the proofs of Proposition 4 and 8, we define operators $A^{(n)}$, $n \in \mathbb{N}$. Thus, each $A^{(n)}$ is J-self-adjoint and

$$\|A^{(n)}\| \le \|A\| < 1. \tag{31}$$

Let $\widehat{A^{(n)}}$ denote the corresponding transformation of the operator $A^{(n)}$ in the Hilbert space $P^{(n)} H$. Recalling representation (19) of $A^{(n)}$, we thus get

$$\widehat{A^{(n)}} = \begin{bmatrix} A_{11}^{(n)} & A_{21}^{(n)} \\ -A_{12}^{(n)} & P_2^{(n)} - A_{22}^{(n)} \end{bmatrix} = P^{(n)} \begin{bmatrix} A_{11} & A_{21} \\ -A_{12} & 1 - A_{22} \end{bmatrix} P^{(n)} = P^{(n)} \widehat{A} P^{(n)}.$$

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Since $0 \leq \widehat{A} \leq 1$, we therefore conclude that $0 \leq \widehat{A^{(n)}} \leq 1$. In particular, $A_{11}^{(n)} \geq 0$.

We may assume that the dimension of the Hilbert space $P^{(n)}H$ is n. Choose an orthonormal basis $(e^{(i)})_{i=1,\ldots,n}$ of $P^{(n)}H$ such that $e^{(i)} \in P_1^{(n)}H$, $i = 1, \ldots, k$, and $e^{(i)} \in P_2^{(n)}H$, $i = k+1, \ldots, n$. In terms of this orthonormal basis, we may treat the operator $A^{(n)}$ in $P^{(n)}H$ as an $n \times n$ J-Hermitian matrix $[A_{ij}^{(n)}]_{i,j=1,\ldots,n}$. Let

$$X^{(n)} := \{1, 2, \ldots, n\}, \quad X_1^{(n)} := \{1, 2, \ldots, k\}, \quad X_2^{(n)} := \{k+1, k+2, \ldots, n\},$$

so that $X^{(n)} = X_1^{(n)} \sqcup X_2^{(n)}$. In view of Proposition 2, there exists a determinantal point process $\mu^{(n)}$ on $\Gamma_{X^{(n)}}$ with correlation kernel

$$K^{(n)}(i, j) := A_{ij}^{(n)}, \quad i, j = 1, \ldots, n.$$

By Proposition 8, we have $\det(1 - A^{(n)}) > 0$. Let

$$L^{(n)} := A^{(n)}(1 - A^{(n)})^{-1}.$$

We define a possibly signed measure $\rho^{(n)}$ on the configuration space $\Gamma_{X^{(n)}}$ by setting,

$$\rho^{(n)}(\{\varnothing\}) := \mathrm{Det}(1 - A^{(n)})$$

and for each non-empty configuration $\{i_1, i_2, \ldots, i_m\} \in \Gamma_{X^{(n)}}$

$$\rho^{(n)}(\{i_1, i_2, \ldots, i_m\}) := \det(1 - A^{(n)}) \cdot \det(L^{(n)}(i_u, i_v))_{u,v=1,\ldots,m}.$$

Analogously to the proof of Theorem 2 below, we may show that $\rho^{(n)} = \mu^{(n)}$. Hence, for each non-empty configuration $\{i_1, i_2, \ldots, i_m\} \in \Gamma_{X^{(n)}}$,

$$\det(L^{(n)}(i_u, i_v))_{u,v=1,\ldots,m} \geq 0.$$

In particular, for any non-empty configuration $\{i_1, i_2, \ldots, i_m\} \in \Gamma_{X_1^{(n)}}$

$$\det(L_{11}^{(n)}(i_u, i_v))_{u,v=1,\ldots,m} \geq 0.$$

Hence, by the Sylvester criterion, $L_{11}^{(n)} \geq 0$, and so

$$(L_{11}^{(n)} f_1, f_1) \geq 0, \quad f_1 \in H_1. \tag{32}$$

By (31) and the dominated convergence theorem, we get

$$\lim_{n \to \infty} (L_{11}^{(n)} f_1, f_1) = \lim_{n \to \infty} (L^{(n)} f_1, f_1)$$

$$= \lim_{n \to \infty} \sum_{l=1}^{\infty} ((A^{(n)})^l f_1, f_1)$$

$$= \sum_{l=1}^{\infty} \lim_{n \to \infty} ((A^{(n)})^l f_1, f_1)$$

$$= \sum_{l=1}^{\infty} (A^l f_1, f_1) = (L f_1, f_1) = (L_{11} f_1, f_1). \tag{33}$$

Thus, by (32) and (33), $(L_{11} f_1, f_1) \geq 0$ for all $f_1 \in H_1$. Analogously, we get $L_{22} \geq 0$. $\square$

The following statement about J-Hermitian matrices was proven in [16].

Proposition 10 ([16]). *Assume that A is a J-Hermitian matrix and assume that its diagonal blocks, A_{11}, A_{22}, are nonnegative definite. Then $\det(A) \geq 0$.*

Remark 3. Note that the arguments we used in the (first part of the) proof of Proposition 8 are similar to the arguments Olshanski [16] used to prove Proposition 10.

From now on we will again assume that $H = L^2(X, m)$, where X is a locally compact Polish space, m is a Radon measure on $(X, \mathcal{B}(X))$, and $X_1, X_2 \in \mathcal{B}(X)$ are such that $X = X_1 \sqcup X_2$. We also set $H_i := L^2(X_i, m)$, $i = 1, 2$. We further define

$$\mathcal{B}(X_i) := \{\Lambda \in \mathcal{B}(X) \mid \Lambda \subset X_i\},$$

and analogously $\mathcal{B}_0(X_i)$, for $i = 1, 2$.

For $\Delta \in \mathcal{B}_0(X)$, we denote by P^Δ the orthogonal projection of $L^2(X, m)$ onto $L^2(\Delta, m)$, i.e., the operator of multiplication by χ_Δ. For an operator $K \in \mathscr{L}(L^2(X, m))$, we denote $K^\Delta := P^\Delta K P^\Delta$. We will say that an operator $K \in \mathscr{L}(L^2(X, m))$ is locally trace-class on X_1 and X_2 if, for each $\Delta \in \mathcal{B}_0(X_i)$, $i = 1, 2$, we have $K^\Delta \in \mathscr{L}_1(L^2(X, m))$.

Proposition 11. *Let $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint and a locally trace-class operator on X_1 and X_2, and let $0 \leq \widehat{K} \leq 1$. Then, for each $\Delta \in \mathcal{B}_0(X)$, $K^\Delta \in \mathscr{L}_{1|2}(L^2(X, m))$.*

Proof. For each $\Delta_1 \in \mathcal{B}_0(X_1)$, we have

$$P^{\Delta_1} \widehat{K} P^{\Delta_1} = K^{\Delta_1} \in \mathscr{L}_1(L^2(X, m)). \tag{34}$$

Since $\widehat{K} \geq 0$, we get $P^{\Delta_1} \widehat{K} P^{\Delta_1} \geq 0$. Hence, by (34), $\sqrt{\widehat{K}} \, P^{\Delta_1} \in \mathscr{L}_2(L^2(X, m))$. Next, for each $\Delta_2 \in \mathcal{B}_0(X_2)$,

$$P^{\Delta_2} (1 - \widehat{K}) P^{\Delta_2} = K^{\Delta_2} \in \mathscr{L}_1(L^2(X, m)).$$

Hence, analogously to the above, $\sqrt{1 - \widehat{K}} \, P^{\Delta_2} \in \mathscr{L}_2(L^2(X, m))$. From here

$$K P^{\Delta_1} = \widehat{K} P^{\Delta_1} = \sqrt{\widehat{K}} \sqrt{\widehat{K}} \, P^{\Delta_1} \in \mathscr{L}_2(L^2(X, m)),$$

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$$KP^{\Delta_2} = (1 - \widehat{K})P^{\Delta_2} = \sqrt{1 - \widehat{K}}\,\sqrt{1 - \widehat{K}}\,P^{\Delta_2} \in \mathscr{L}_2(L^2(X, m)).$$

Therefore, $K(P^{\Delta_1} + P^{\Delta_2}) \in \mathscr{L}_2(L^2(X, m))$. Thus, for each $\Delta \in \mathcal{B}_0(X)$, $KP^{\Delta} \in \mathscr{L}_2(L^2(X, m))$, and so $K^{\Delta} \in \mathscr{L}_2(L^2(X, m))$.

By our assumption, for each $\Delta \in \mathcal{B}_0(X)$,

$$K^{\Delta}_{\text{even}} = P^{\Delta} K_{\text{even}} P^{\Delta} = P^{\Delta_1} K_{11} P^{\Delta_1} + P^{\Delta_2} K_{22} P^{\Delta_2} = K^{\Delta_1} + K^{\Delta_2} \in \mathscr{L}_1(L^2(X, m)).$$

(Here $\Delta_i := \Delta \cap X_i$, $i = 1, 2$.) Thus, $K^{\Delta} \in \mathscr{L}_{1|2}(L^2(X, m))$. $\qquad\square$

Proposition 12. *Let $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint, let $K^{\Delta} \in \mathscr{L}_{1|2}(L^2(X, m))$ for each $\Delta \in \mathcal{B}_0(X)$, and let $K_{11} \geq 0$, $K_{22} \geq 0$. Then K is an integral operator and its integral kernel $K(x, y)$ can be chosen so that:*

i) The kernel $K(x, y)$ is J-Hermitian.

ii) For $i = 1, 2$ and any $x_1, \dots, x_n \in X_i$ $(n \in \mathbb{N})$, the matrix $\big[K(x_i, x_j)\big]_{i,j=1,\dots,n}$ is nonnegative definite.

iii) For each $\Delta \in \mathcal{B}_0(X)$,

$$\operatorname{Tr}(K^{\Delta}_{\text{even}}) = \int_{\Delta} K(x, x)\, m(dx). \tag{35}$$

Proof. For any $\Delta_1 \in \mathcal{B}_0(X_1)$ and $\Delta_2 \in \mathcal{B}_0(X_2)$, $P^{\Delta_2} K P^{\Delta_1}$ is a Hilbert–Schmidt operator, hence an integral operator. Therefore, we can choose an integral kernel of K_{21}, which is a function $K_{21}(x, y)$ on $X_2 \times X_1$. We now define an integral kernel $K_{12}(x, y)$ of the operator K_{12} by setting $K_{12}(x, y) := -\overline{K_{21}(y, x)}$ for $(x, y) \in X_1 \times X_2$. Next, the operators K_{11} and K_{22} are non-negative, locally trace class operators. Hence, we can choose their integral kernels according to [9, Lemma A.3], see also [13, Section 3]. By combining the integral kernels $K_{ij}(x, y)$, $i, j = 1, 2$, we obtain an integral kernel $K(x, y)$ of K with needed properties.

$\qquad\square$

From now on, for an operator K as in Proposition 12, we will always assume that its integral kernel satisfies statements i)–iii) of this proposition.

We denote by $B_0(X)$ the space of all measurable bounded real-valued functions on X with compact support. For each $\varphi \in B_0(X)$, we preserve the notation φ for the bounded linear operator of multiplication by φ in $L^2(X, m)$.

Proposition 13. *Let $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint, let $K_{11} \geq 0$, $K_{22} \geq 0$, and let $K^{\Delta} \in \mathscr{L}_{1|2}(L^2(X, m))$ for each $\Delta \in \mathcal{B}_0(X)$. Fix any $\Delta \in \mathcal{B}_0(X)$ and any $\varphi \in B_0(X)$ which vanishes outside Δ. Then $K^{\Delta}\varphi$, $\operatorname{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|} \in \mathscr{L}_{1|2}(L^2(X, m))$ and*

$$\operatorname{Det}(1 + K^{\Delta}\varphi) = \operatorname{Det}(1 + \operatorname{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|}) \tag{36}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} \varphi(x_1) \cdots \varphi(x_n) \det[K(x_i, x_j)]_{i,j=1,\dots,n}\, m(dx_1) \cdots m(dx_n).$$

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Proof. Since $K^\Delta \in \mathscr{L}_2(L^2(X,m))$, $K^\Delta \varphi \in \mathscr{L}_2(L^2(X,m))$. Since $K^\Delta_{\text{even}} \in \mathscr{L}_1(L^2(X,m))$,

$$(K^\Delta \varphi)_{\text{even}} = K^\Delta_{\text{even}} \varphi \in \mathscr{L}_1(L^2(X,m)).$$

Thus, $K^\Delta \varphi \in \mathscr{L}_{1|2}(L^2(X,m))$.

Denote $\psi_1 := \text{sgn}(\varphi)\sqrt{|\varphi|}$ and $\psi_2 := \sqrt{|\varphi|}$, $\psi_1, \psi_2 \in B_0(X)$. Since ψ_1 and ψ_2 vanish outside Δ, we get

$$\psi_1 K \psi_2 = \psi_1 K^\Delta \psi_2,$$

and analogously to the above, we conclude that $\psi_1 K \psi_2 \in \mathscr{L}_{1|2}(L^2(X,m))$.

Since $K^\Delta_{\text{even}} \in \mathscr{L}_1(L^2(X,m))$ and since $\psi_1, \psi_2 \in \mathscr{L}(L^2(X,m))$, by (4),

$$\begin{aligned}
\text{Tr}((K^\Delta \varphi)_{\text{even}}) = \text{Tr}(K^\Delta_{\text{even}} \psi_2 \psi_1) &= \text{Tr}(\psi_1 K^\Delta_{\text{even}} \psi_2) \\
&= \text{Tr}(\psi_1 K_{\text{even}} \psi_2) = \text{Tr}((\psi_1 K \psi_2)_{\text{even}}).
\end{aligned} \tag{37}$$

Next, for $l = 2, 3, \dots$,

$$\begin{aligned}
\text{Tr}((\psi_1 K \psi_2)^l) = \text{Tr}(\psi_1 K^\Delta \varphi K^\Delta \varphi \cdots K^\Delta \varphi K^\Delta \psi_2) \\
= \text{Tr}(K^\Delta \varphi K^\Delta \varphi \cdots K^\Delta \varphi K^\Delta \psi_2 \psi_1) = \text{Tr}((K^\Delta \varphi)^l).
\end{aligned} \tag{38}$$

By (37) and (38), $C_n(K^\Delta \varphi) = C_n(\psi_1 K \psi_2)$ for each $n \in \mathbb{N}$, hence formula (36) holds.

Next, we note that the integral kernel $K^\Delta(x,y)$ of the operator K^Δ is the restriction of $K(x,y)$ to Δ^2. Clearly, the integral kernel of $K^\Delta \varphi$ is $K^\Delta(x,y)\varphi(y)$. Using, (35), it is not hard to show that

$$\text{Tr}((K^\Delta \varphi)_{\text{even}}) = \int_X K^\Delta(x,x)\varphi(x)\, m(dx).$$

Hence, by Proposition 5,

$$\begin{aligned}
\text{Det}(1 + K^\Delta \varphi) &= 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{X^n} \det[K^\Delta(x_i,x_j)\varphi(x_j)]_{i,j=1,\dots,n}\, m(dx_1)\cdots m(dx_n) \\
&= 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{X^n} \varphi(x_1)\cdots\varphi(x_n) \det[K^\Delta(x_i,x_j)]_{i,j=1,\dots,n}\, m(dx_1)\cdots m(dx_n) \\
&= 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{X^n} \varphi(x_1)\cdots\varphi(x_n) \det[K(x_i,x_j)]_{i,j=1,\dots,n}\, m(dx_1)\cdots m(dx_n).
\end{aligned}$$

$\square$

4   Main results

We again assume that X is a locally compact Polish space and m is a Radon measure on $(X, \mathcal{B}(X))$. We will also assume that m takes a positive value on each open non-empty set in X Let $\Gamma = \Gamma_X$ be the configuration space over X. Let μ be a point process on X, i.e., a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. Assume that μ satisfies

$$\int_\Gamma C^{|\gamma \cap \Delta|}\, \mu(d\gamma) \quad \text{for all } \Delta \in \mathcal{B}_0(X) \text{ and all } C > 0. \tag{39}$$

Then the Bogoliubov functional of μ is defined as

$$B_\mu(\varphi) := \int_\Gamma \prod_{x \in \gamma}(1 + \varphi(x))\, \mu(d\gamma), \quad \varphi \in B_0(X). \tag{40}$$

Note that, since the function φ has compact support, only a finite number of terms in the product $\prod_{x\in\gamma}(1 + \varphi(x))$ are not equal to one. Note also that the integrability of the function $\prod_{x\in\gamma}(1 + \varphi(x))$ for each $\varphi \in B_0(X)$ is equivalent to condition (39). If a point process μ has correlation functions $(k_\mu^{(n)})_{n=1}^\infty$ (see (1)), then condition (39) is also equivalent to

$$\sum_{n=1}^\infty \frac{C^n}{n!} \int_{\Delta^n} k_\mu^{(n)}(x_1, \ldots, x_n)\, m(dx_1) \cdots m(dx_n) < \infty \quad \text{for all } \Delta \in \mathcal{B}_0(X) \text{ and all } C > 0,$$

and the Bogoliubov functional of μ is given by

$$B_\mu(\varphi) = 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{X^n} \varphi(x_1) \cdots \varphi(x_n) k_\mu^{(n)}(x_1, \ldots, x_n)\, m(dx_1) \cdots m(dx_n) \tag{41}$$

for each $\varphi \in B_0(X)$. The Bogoliubov functional of μ uniquely determines this point process. For more detail about the Bogoliubov functional, see e.g. [11].

Let us now briefly recall some known facts about configuration spaces and point processes, see e.g. [8, 15] for further details. The σ-algebra $\mathcal{B}(\Gamma)$ coincides with the minimal σ-algebra on Γ with respect to which all mappings of the form $\Gamma \ni \gamma \mapsto |\gamma \cap \Lambda|$ with $\Lambda \in \mathcal{B}_0(X)$ are measurable. For a fixed set $\Delta \in \mathcal{B}(X)$, we denote by $\mathcal{B}_\Delta(\Gamma)$ the minimal σ-algebra on Γ with respect to which all mappings of the form $\Gamma \ni \gamma \mapsto |\gamma \cap \Lambda|$ with $\Lambda \in \mathcal{B}_0(X)$, $\Lambda \subset \Delta$, are measurable. In particular, $\mathcal{B}_\Delta(\Gamma)$ is a sub-σ-algebra of $\mathcal{B}(\Gamma)$. The σ-algebras $\mathcal{B}(\Gamma_\Delta)$ and $\mathcal{B}_\Delta(\Gamma)$ can be identified in the sense that, for each $A \in \mathcal{B}(\Gamma_\Delta)$, $\{\gamma \in \Gamma \mid \gamma \cap \Delta \in A\} \in \mathcal{B}_\Delta(\Gamma)$ and each set from $\mathcal{B}_\Delta(\Gamma)$ has a unique such representation. Hence, the restriction of a point process μ on X to the σ-algebra $\mathcal{B}_\Delta(\Gamma)$—denoted by μ_Δ—can be identified with a point process on Δ, i.e., with a probability measure on $(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta))$.

Let Δ be a compact subset of X. Then the configuration space Γ_Δ consists of all finite configurations in Δ, i.e., $\Gamma_\Delta = \bigsqcup_{n=0}^\infty \Gamma_\Delta^{(n)}$, where $\Gamma_\Delta^{(0)} := \{\varnothing\}$ and for $n \in \mathbb{N}$, $\Gamma_\Delta^{(n)}$ consists of all n-point configurations in Δ. Denote

$$\widetilde{\Delta^n} := \{(x_1, \ldots, x_n) \in \Delta^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Let $\mathcal{B}(\Gamma_\Delta^{(n)})$ denote the image of the σ-algebra $\mathcal{B}(\widetilde{\Delta^n})$ under the mapping

$$\widetilde{\Delta^n} \ni (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \in \Gamma_\Delta^{(n)}.$$

Then $\mathcal{B}(\Gamma_\Delta)$ is the minimal σ-algebra on Γ_Δ which contains all $\mathcal{B}(\Gamma_\Delta^{(n)})$, $n \in \mathbb{N}$. A point process μ on X has local densities in Δ if, for each $n \in \mathbb{N}$, there exists a nonnegative measurable symmetric function $d_\mu^{(n)}[\Delta](x_1, \ldots, x_n)$ on $\widetilde{\Delta^n}$ such that

$$\int_{\Gamma_\Delta^{(n)}} f^{(n)}(\gamma)\, \mu_\Delta(d\gamma) = \frac{1}{n!} \int_{\widetilde{\Delta^n}} f^{(n)}(\{x_1, \ldots, x_n\}) d_\mu^{(n)}[\Delta](x_1, \ldots, x_n)\, m(dx_1) \cdots m(dx_n)$$

for each measurable functions $f^{(n)} : \Gamma_\Delta^{(n)} \to [0, \infty)$. We also denote $d_\mu^{(0)}[\Delta] := \mu_\Delta(\{\varnothing\})$. In the case where $X = \Delta$ (so that X is a compact Polish space), we will write $d_\mu^{(n)}$ instead of $d_\mu^{(n)}[\Delta]$.

Theorem 2. *Let $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint. Let K be a locally trace-class operator on X_1 and X_2, and let $0 \leq \widehat{K} \leq 1$. Then there exists a unique point process μ on X which has correlation functions*

$$k_\mu^{(n)}(x_1, \ldots, x_n) = \det\left[K(x_i, x_j)\right]_{i,j=1,\ldots,n}. \tag{42}$$

The Bogoliubov functional of μ is given by

$$B_\mu(\varphi) = \mathrm{Det}\left(1 + \mathrm{sgn}(\varphi)\sqrt{|\varphi|}\, K \sqrt{|\varphi|}\right), \quad \varphi \in B_0(X). \tag{43}$$

If additionally $\|K\| < 1$, then for each $\Delta \in \mathcal{B}_0(X)$, the point process μ has local densities in Δ:

$$d_\mu^{(0)}[\Delta] = \mathrm{Det}(1 - K^\Delta),$$
$$d_\mu^{(n)}[\Delta](x_1, \ldots, x_n) = \mathrm{Det}(1 - K^\Delta) \det\left[L[\Delta](x_i, x_j)\right]_{i,j=1,\ldots,n}, \tag{44}$$

where $L[\Delta] := K^\Delta(1 - K^\Delta)^{-1}$.

Proof. By Proposition 6, $\|K\| \leq 1$. We first assume that $\|K\| < 1$. We fix any compact $\Delta \subset X$. By Proposition 11, $K^\Delta \in \mathscr{L}_{1|2}(L^2(X, m))$, hence $K^\Delta \in \mathscr{L}_{1|2}(L^2(\Delta, m))$. Clearly, K^Δ is J-self-adjoint. Setting $\Delta_i := \Delta \cap X_i$, $i = 1, 2$, we get

$$P^\Delta \widehat{K} P^\Delta = P^\Delta(KP_1 + (1 - K)P_2)P^\Delta$$

$$= K^\Delta P^{\Delta_1} + (1 - K^\Delta) P^{\Delta_2} = \widehat{K^\Delta}, \tag{45}$$

where the latter operator is understood as the transformation (17) of the operator K^Δ in the Hilbert space $L^2(\Delta, m) = L^2(\Delta_1, m) \oplus L^2(\Delta_2, m)$. As $0 \le \widehat{K} \le 1$, we conclude from (45) that $0 \le \widehat{K^\Delta} \le 1$. Since $\|K\| < 1$, we have $\|K^\Delta\| < 1$. Hence, by Proposition 8, $\mathrm{Det}(1 - K^\Delta) > 0$.

Furthermore, by Proposition 9, the operator $L[\Delta]$ is J-selfdjoint and

$$L[\Delta] \in \mathscr{L}_{1|2}(L^2(\Delta, m)), \quad L[\Delta]_{11} \ge 0, \ L[\Delta]_{22} \ge 0.$$

Hence, we can choose an integral kernel $L[\Delta](x, y)$ of the operator $L[\Delta]$ according to Proposition 12. Therefore, for any $x_1, \dots, x_n \in \Delta_1$, $x_{n+1}, \dots, x_{n+m} \in \Delta_2$, the matrix $\left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n+m}$ is J-Hermitian and the diagonal blocks

$$\left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n}, \quad \left[L[\Delta](x_i, x_j) \right]_{i,j=n+1,\dots,n+m}$$

are nonnegative definite. Hence, by Proposition 10,

$$\det \left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n+m} \ge 0.$$

Therefore, for each $n \in \mathbb{N}$, the function

$$\widetilde{\Delta^n} \ni (x_1, \dots, x_n) \mapsto \det \left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n}$$

is symmetric and takes nonnegative values.

Hence, we can define a positive measure μ_Δ on $(\Gamma_\Delta, \mathcal{B}(\Gamma_\Delta))$ for which

$$d_{\mu_\Delta}^{(0)} = \mathrm{Det}(1 - K^\Delta),$$
$$d_{\mu_\Delta}^{(n)}(x_1, \dots, x_n) = \mathrm{Det}(1 - K^\Delta) \det \left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n}, \quad n \in \mathbb{N}. \tag{46}$$

Note that

$$\det \left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n} = 0 \quad \text{for all } (x_1, \dots, x_n) \in \Delta^n \setminus \widetilde{\Delta_n}, \ n \in \mathbb{N}.$$

Hence, the Bogoliubov functional of μ_Δ is given by

$$B_{\mu_\Delta}(\varphi) = \mathrm{Det}(1 - K^\Delta) \left(1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Delta^n} (1 + \varphi(x_1)) \cdots (1 + \varphi(x_n)) \right.$$
$$\left. \times \det \left[L[\Delta](x_i, x_j) \right]_{i,j=1,\dots,n} m(dx_1) \cdots m(dx_n) \right), \quad \varphi \in B(\Delta). \tag{47}$$

Here $B(\Delta)$ denotes the set of all bounded measurable functions on Δ. It follows from Proposition 13 and (47) that

$$B_{\mu_\Delta}(\varphi) = \mathrm{Det}(1 - K^\Delta) \, \mathrm{Det}(1 + L[\Delta](1 + \varphi)), \quad \varphi \in B(\Delta). \tag{48}$$

Hence, by [3, Corollary A.3] and Proposition 13,

$$
\begin{aligned}
B_{\mu_\Delta}(\varphi) &= \mathrm{Det}\left((1 - K^\Delta)(1 + L[\Delta](1 + \varphi))\right) \\
&= \mathrm{Det}(1 + K^\Delta \varphi) \\
&= \mathrm{Det}(1 + \mathrm{sgn}(\varphi)\sqrt{|\varphi|}K\sqrt{|\varphi|}), \quad \varphi \in B(\Delta). \tag{49}
\end{aligned}
$$

Now we take any sequence of compact subsets of X, $\{\Delta_n\}_{n=1}^\infty$, such that

$$
\Delta_1 \subset \Delta_2 \subset \cdots, \quad \bigcup_{n=1}^\infty \Delta_n = X.
$$

By (49), the probability measures μ_{Δ_n} on $(\Gamma, \mathcal{B}_{\Delta_n}(\Gamma))$ form a consistent family of probability measures. Therefore, by the Kolmogorov theorem, there exists a unique probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ such that the restriction of μ to each $\mathcal{B}_{\Delta_n}(\Gamma)$ coincides with μ_{Δ_n}. By (49), the Bogoliubov functional of μ is given by (43), while the statement about the local densities of μ follows from (46). The determinantal form of the correlation functions of μ—formula (42)—follows from (41), (43) and Proposition 13.

Let us now consider the case where $\|K\| = 1$ For each $\varepsilon \in (0,1)$, set $K_\varepsilon := \varepsilon K$. Hence $\|K_\varepsilon\| < 1$. We have

$$
\widehat{K}_\varepsilon = \varepsilon K P_1 + (1 - \varepsilon K)P_2 = \varepsilon \widehat{K} + (1 - \varepsilon)P_2. \tag{50}
$$

Since $\widehat{K} \geq 0$ and $P_2 \geq 0$, we get $\widehat{K}_\varepsilon \geq 0$, and since $\widehat{K} \leq 1$ and $P_2 \leq 1$, we get $\widehat{K}_\varepsilon \leq 1$. Hence by the proved above, there exists a point process μ_ε which has correlation functions

$$
k_{\mu_\varepsilon}^{(n)}(x_1, \ldots, x_n) = \varepsilon^n \det\left[K(x_i, x_j)\right]_{i,j=1,\ldots,n}. \tag{51}
$$

Hence, the corresponding correlation measure is $\star$-positive definite in the sense of [10], see also [13]. By taking the limit as $\varepsilon \to 0$, we therefore conclude that the functions

$$
k_\mu^{(n)}(x_1, \ldots, x_n) := \det\left[K(x_i, x_j)\right]_{i,j=1,\ldots,n}, \quad n \in \mathbb{N}, \tag{52}
$$

determine a $\star$-positive definite correlation measure. By Proposition 13, for each $\Delta \in \mathcal{B}_0(X)$ and $C > 0$,

$$
1 + \sum_{n=1}^\infty \frac{C^n}{n!} \int_{\Delta^n} k_\mu^{(n)}(x_1, \ldots, x_n)\, m(dx_1) \cdots m(dx_n) = \mathrm{Det}(1 + CK^\Delta) < \infty.
$$

Hence, by [13, Corollary 1], we conclude that there exists a unique probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ which has correlation functions (52). By Proposition 13 and formula (41), the Bogoliubov functional of μ is given by (43). $\qquad\square$

The following corollary easily follows from Theorem 2 and Proposition 5.1 in [7] and its proof.

Corollary 1. *Let $G : L^2(X_1, m) \to L^2(X_2, m)$ be a bounded linear operator such that, for any $\Delta_1 \in \mathcal{B}_0(X_1)$ and $\Delta_2 \in \mathcal{B}_0(X_2)$, the operators GP^{Δ_1} and $P^{\Delta_2}G$ are Hilbert–Schmidt. Let an operator $L \in \mathscr{L}(L^2(X, m))$ be defined by*

$$L := \begin{bmatrix} 0 & G \\ -G^* & 0 \end{bmatrix}.$$

Then operator $1 + L$ is invertible, and we set $K := L(1 + L)^{-1}$. We further have:

   i) The operator K is J-self-adjoint.

   ii) The operator K is locally trace-class on X_1 and X_2.

   iii) The operator $\widehat{K}$ is the orthogonal projection of $L^2(X, m)$ onto the subspace

$$\left\{ h \oplus Gh \mid h \in L^2(X_1, m) \right\}.$$

   Thus, by Theorem 2, there exists a unique determiminantal point process with correlation kernel $K(x, y)$.

Remark 4. As we mentioned in Section 1, the Whittaker kernel [5], the matrix tail kernel [16], and the continuous hypergeometric kernel [6] have their L operators as in Corollary 1, and so their $\widehat{K}$ operators are orthogonal projections.

Proof of Corollary 1. That the operator $1 + L$ is invertible is shown in [7, Section 5]. Statement iii) is just [7, Proposition 5.1]. By the proof of [7, Proposition 5.1], the operator L has the following block form:

$$K_{11} = G^*G(1 + G^*G)^{-1},$$
$$K_{22} = GG^*(1 + GG^*)^{-1},$$
$$K_{21} = G(1 + G^*G)^{-1},$$
$$K_{21} = -G^*(1 + G^*G)^{-1}.$$

Hence, statement i) obviously follows. So we only need to prove statement ii). To this end, we fix any $\Delta_1 \in \mathcal{B}_0(X_1)$ and $\Delta_2 \in \mathcal{B}_0(X_2)$. By the assumption of the corollary, $P^{\Delta_2}G$ is a Hilbert-Schmidt operator. Therefore,

$$P^{\Delta_2}K_{21}P^{\Delta_1} = P^{\Delta_2}G(1 + G^*G)^{-1}P^{\Delta_1}$$

is a Hilbert–Schmidt operator, hence so is the operator $P^{\Delta_1}K_{12}P^{\Delta_2}$. Again by the assumption of the corollary, GP^{Δ_1} is a Hilbert–Schmidt operator, hence

$$(GP^{\Delta_1})^*(GP^{\Delta_1}) = P^{\Delta_1}G^*GP^{\Delta_1}$$

is a trace-class operator. Let $\{e^{(n)}\}_{n \geq 1}$ be an orthonormal basis in $L^2(\Delta_1, m)$. Then, by the spectral theorem

$$\sum_{n \geq 1}(K_{11}e^{(n)}, e^{(n)})_{L^2(\Delta_1, m)} = \sum_{n \geq 1}(G^*G(1 + G^*G)^{-1}e^{(n)}, e^{(n)})_{L^2(\Delta_1, m)}$$

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$$\leq \sum_{n \geq 1} (G^* G e^{(n)}, e^{(n)})_{L^2(\Delta_1, m)} < \infty.$$

Therefore, the operator $P^{\Delta_1} K_{11} P^{\Delta_1}$ is trace-class. Analogously, we may also show that the operator $P^{\Delta_2} K_{22} P^{\Delta_2}$ is trace-class. Thus, statement ii) is proven. $\qquad \square$

Corollary 2. *Let an operator $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint and locally trace-class on X_1 and X_2. Let $0 \leq \widehat{K} \leq 1$ and let $\|K\| = 1$. Let μ be the corresponding determinantal point process. Assume that $\Delta \in \mathcal{B}_0(X)$ is such that $\|K^\Delta\| = 1$. Then*

$$\mu_\Delta(\{\varnothing\}) = \mathrm{Det}(1 - K^\Delta) = 0,$$

i.e., the μ probability of the event that there are no particles in Δ is equal to zero.

Proof. By (43), for each $\Delta \in \mathcal{B}_0(X)$ and $z > 0$,

$$\int_\Gamma e^{-z|\gamma \cap \Delta|} \, \mu(d\gamma) = \int_\Gamma \prod_{x \in \gamma} (1 + (e^{-z} - 1)\chi_\Delta) \, \mu(d\gamma)$$
$$= \mathrm{Det}(1 - (1 - e^{-z})K^\Delta).$$

Letting $z \to \infty$ and using the dominated convergence theorem, we get

$$\mu_\Delta(\{\varnothing\}) = \mathrm{Det}(1 - K^\Delta).$$

Since $\|K^\Delta\| = 1$, by Proposition 7, at least one of the operators $K^{\Delta_1} = K_{11}^{\Delta_1}$, $K^{\Delta_2} = K_{22}^{\Delta_2}$ must have norm 1. (Here, as above, $\Delta_i = \Delta \cap X_i$, $i = 1, 2$.) Assume $\|K^{\Delta_1}\| = 1$ (the other case is analogous). As $\mathrm{Det}(1 - K^{\Delta_1})$ is a classical Fredholm determinant and the operator K^{Δ_1} is self-adjoint, we get $\mathrm{Det}(1 - K^{\Delta_1}) = 0$. Thus, we have $\mu_{\Delta_1}(\{\varnothing\}) = 0$, i.e., the μ probability of the event that there are no particles in the set Δ_1 is equal to 0. From here the statement follows. $\qquad \square$

Remark 5. Note that, for a determinantal point precess μ with a J-self-adjoint correlation operator K, the restriction of μ to the σ-algebra $\mathcal{B}_{X_i}(\Gamma)$ $(i = 1, 2)$ may be identified with the determinantal point process on X_i whose correlation operator is the self-adjoint operator K_{ii}.

We will now show that the conditions on a J-self-adjoint operator K in Theorem 2 are, in fact, necessary for a determinantal point process with correlation kernel $K(x, y)$ to exist.

Theorem 3. *Let $K \in \mathscr{L}(L^2(X, m))$ be J-self-adjoint, let $K_{11} \geq 0$, $K_{22} \geq 0$, and let $K^\Delta \in \mathscr{L}_{1|2}(L^2(X, m))$ for each $\Delta \in \mathcal{B}_0(X)$. Let an integral kernel $K(x, y)$ of the operator K be chosen so that statements i)–iii) of Proposition 12 are satisfied. Then there exists a unique point process μ on X which has correlation functions (42) if and only if $0 \leq \widehat{K} \leq 1$.*

Proof. We only have to prove that, if a point process μ exists, then $0 \le \widehat{K} \le 1$. We divide the proof into several steps.

1) Fix any compact $\Delta \subset X$. By Proposition 13 and (41), the Bogoliubov functional of μ is given by (43). Hence, analogously to the proof of Corollary 2, we get

$$\mu_\Delta(\{\varnothing\}) = \mathrm{Det}(1 - K^\Delta).$$

In particular,

$$\mathrm{Det}(1 - K^\Delta) \ge 0. \tag{53}$$

2) From now on we will additionally assume that $\|K\| < 1$. Then $\|K^\Delta\| < 1$ and we set $L[\Delta] := K^\Delta(1 - K^\Delta)^{-1}$. Just as in the proof of Proposition 9, we derive that $L[\Delta]$ is J-self-adjoint and $L[\Delta] \in \mathscr{L}_{1|2}(L^2(X, m))$. To choose an integral kernel of the operator $L[\Delta]$, we represent it in the form

$$L[\Delta] = K^\Delta + K^\Delta L[\Delta],$$

As $L[\Delta] \in \mathscr{L}_2(L^2(\Delta, m))$, we first choose an arbitrary J-Hermitian integral kernel of this operator, which we denote by $\widetilde{L}[\Delta](x, y)$. Now we set

$$L[\Delta](x, y) := K(x, y) + \int_\Delta K(x, z)\widetilde{L}[\Delta](z, y)\, m(dy), \quad x, y \in \Delta.$$

As easily seen, this integral kernel satisfies

$$\mathrm{Tr}(L[\Delta]_{\mathrm{even}}^\Lambda) = \int_\Lambda L[\Delta](x, x)\, m(dx) \quad \text{for each } \Lambda \in \mathcal{B}_0(X),\ \Lambda \subset \Delta. \tag{54}$$

3) By (54),

$$\mathrm{Tr}\big((L[\Delta](1 + \varphi))_{\mathrm{even}}\big) = \int_\Delta L[\Delta](x, x)(1 + \varphi(x)), \quad \varphi \in B_0(X). \tag{55}$$

Since formula (48) clearly holds for the Boliubov functional of μ_Δ, using (55) and Proposition 5, we get, for each $\varphi \in B(\Delta)$,

$$B_{\mu_\Delta}(\varphi) = \mathrm{Det}(1 - K^\Delta)\bigg(1 + \sum_{n=1}^\infty \int_{\Delta^n} (1 + \varphi(x_1)) \cdots (1 + \varphi(x_n))$$
$$\times \det\big[L[\Delta](x_i, x_j)\big]_{i,j=1,\ldots,n}\, m(dx_1) \cdots m(dx_n)\bigg).$$

This implies that the measure μ_Δ has densities (46). Hence, by (53), for $m^{\otimes n}$-a.a. $(x_1, \ldots, x_n) \in \Delta^n$,

$$\det\big[L[\Delta](x_i, x_j)\big]_{i,j=1,\ldots,n} \ge 0.$$

In particular, for $i = 1, 2$, for $m^{\otimes n}$-a.a. $(x_1, \ldots, x_n) \in \Delta_i^n$,

$$\det \left[L[\Delta]_{ii}(x_i, x_j) \right]_{i,j=1,\ldots,n} \geq 0. \tag{56}$$

Here $\Delta_i := \Delta \cap X_i$, $i = 1, 2$.

4) Following [16, Proposition 1.5], let us find a representation of $L[\Delta]_{11}$ in terms of the blocks of the operator K^Δ. Since $L[\Delta](1 - K^\Delta) = K^\Delta$, we have

$$L[\Delta]_{11}(1 - K_{11}^\Delta) - L[\Delta]_{12}K_{21}^\Delta = K_{11}^\Delta, \tag{57}$$

$$-L[\Delta]_{11}K_{12}^\Delta + L[\Delta]_{12}(1 - K_{22}^\Delta) = K_{12}^\Delta. \tag{58}$$

From (58),

$$-L[\Delta]_{11}K_{12}^\Delta(1 - K_{22}^\Delta)^{-1} + L[\Delta]_{12} = K_{12}^\Delta(1 - K_{22}^\Delta)^{-1},$$

hence

$$-L[\Delta]_{11}K_{12}^\Delta(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta + L[\Delta]_{12}K_{21}^\Delta = K_{12}^\Delta(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta.$$

Adding this to (57) yields

$$L[\Delta]_{11}(1 - Q[\Delta]_{11}) = Q[\Delta]_{11},$$

where

$$\begin{aligned} Q[\Delta]_{11} :&= K_{11}^\Delta + K_{12}^\Delta(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta \\ &= K_{11}^\Delta - (K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta. \end{aligned} \tag{59}$$

Since the operator $1 - K_{11}^\Delta$ is strictly positive and the operator $(K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta$ is nonnegative, the operator $1 - Q[\Delta]_{11}$ is strictly positive, hence invertible. Therefore,

$$L[\Delta]_{11} = Q[\Delta]_{11}(1 - Q[\Delta]_{11})^{-1}. \tag{60}$$

5) By (59), the operator $Q[\Delta]_{11}$ is self-adjoint and trace-class. Since the operator $1 - Q[\Delta]_{11}$ is strictly positive, we therefore get

$$\mathrm{Det}(1 - Q[\Delta]_{11}) > 0.$$

(Note that $\mathrm{Det}(1 - Q[\Delta]_{11})$ is a usual Fredholm determinant.) Therefore, by (56), we can define a non-negative, finite measure $\nu[\Delta_1]$ on $(\Gamma_{\Delta_1}, \mathcal{B}(\Gamma_{\Delta_1}))$ whose local densities are:

$$d_{\nu[\Delta_1]}^{(0)} = \mathrm{Det}(1 - Q[\Delta]_{11}),$$

$$d_{\nu[\Delta_1]}^{(n)}(x_1, \ldots, x_n) = \mathrm{Det}(1 - Q[\Delta]_{11}) \det \left[L[\Delta]_{11}(x_i, x_j) \right]_{i,j=1,\ldots,n}, \quad n \in \mathbb{N}. \tag{61}$$

Analogously to (46)–(49), we conclude from (61) that the Bogoliubov transform of the measure $\nu[\Delta_1]$ is given by

$$B_{\nu[\Delta_1]}(\varphi) = \int_{\Gamma_{\Delta_1}} \prod_{x \in \gamma}(1 + \varphi(x))\,\nu[\Delta_1](d\gamma)$$

$$= \mathrm{Det}\left(1 + \mathrm{sgn}(\varphi)\sqrt{|\varphi|}\,Q[\Delta]_{11}\sqrt{|\varphi|}\right), \quad \varphi \in B_0(\Delta_1). \tag{62}$$

Setting $\varphi \equiv 0$, we see that $\nu[\Delta_1](\Gamma_{\Delta_1}) = 1$, i.e., $\nu[\Delta_1]$ is a point process in Δ_1.

6) We can now choose an integral kernel of the operator $Q[\Delta]_{11}$ analogously to [9, Lemma A.3] and [13, Section 3]. Indeed, since K_{21}^Δ is a Hilbert–Schmidt operator, $(K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta$ is a nonnegative trace-class operator in $L^2(\Delta_1, m)$. The operator $\left((K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta\right)^{1/2}$ is Hilbert–Schmidt, hence an integral operator. We choose its integral kernel, denoted by $\theta(x, y)$, so that

$$\theta(x, y) = \overline{\theta(y, x)} \quad \text{for all } x, y \in \Delta_1, \qquad \theta(x, \cdot) \in L^2(\Delta_1, m) \quad \text{for all } x \in \Delta_1.$$

(Recall that $\int_{\Delta_1^2} |\theta(x, y)|^2\,m(dx)\,m(dy) = \left\|\left((K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta\right)^{1/2}\right\|_2^2 < \infty$.) Now, we set an integral kernel of the operator $(K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta$ to be

$$(K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1}K_{21}^\Delta(x, y) := \int_{\Delta_1} \theta(x, z)\theta(z, y)\,m(dz)$$

$$= (\theta(x, \cdot), \theta(y, \cdot))_{L^2(\Delta_1, m)}, \quad x, y \in \Delta_1.$$

We similarly construct an integral kernel of the operator K_{11}^Δ:

$$K_{11}^\Delta(x, y) = (\eta(x, \cdot), \eta(y, \cdot))_{L^2(\Delta_1, m)}, \quad x, y \in \Delta_1.$$

Hence, by virtue of (59), we may choose an integral kernel of the operator $Q[\Delta]_{11}$ as folllows:

$$Q[\Delta]_{11}(x, y) = \left(\eta(x, \cdot), \eta(y, \cdot)\right)_{L^2(\Delta_1, m)} - \left(\theta(x, \cdot), \theta(y, \cdot)\right)_{L^2(\Delta_1, m)}. \tag{63}$$

As easily seen, for each $\Delta \in \mathcal{B}_0(X)$, $\Lambda \subset \Delta_1$,

$$\mathrm{Tr}(Q[\Delta]_{11}^\Lambda) = \int_\Lambda Q[\Delta]_{11}(x, x)\,m(dx).$$

Now, analogously to Proposition 13, we get from (62):

$$B_{\nu[\Delta_1]}(\varphi) = 1 + \sum_{n=1}^\infty \frac{1}{n!} \int_{\Delta_1^n} \varphi(x_1)\cdots\varphi(x_n)\det\left[Q[\Delta]_{11}(x_i, x_j)\right]_{i,j=1,\ldots,n} m(dx_1)\cdots m(dx_n)$$

for each $\varphi \in B_0(\Delta_1)$. Hence, the correlation functions of the point process $\nu[\Delta_1]$ are

$$k_{\nu[\Delta_1]}^{(n)}(x_1, \ldots, x_n) = \det \left[Q[\Delta]_{11}(x_i, x_j) \right]_{i,j=1,\ldots,n}, \quad n \in \mathbb{N}.$$

Therefore, for each $n \in \mathbb{N}$,

$$\det \left[Q[\Delta]_{11}(x_i, x_j) \right]_{i,j=1,\ldots,n} \geq 0 \quad \text{for } m^{\otimes n}\text{-a.a. } (x_1, \ldots, x_n) \in \Delta_1^n. \tag{64}$$

7) Obviously, the following two mappings are measurable:

$$\Delta_1 \ni x \mapsto \eta(x, \cdot) \in L^2(\Delta_1, m), \quad \Delta_1 \ni x \mapsto \theta(x, \cdot) \in L^2(\Delta_1, m).$$

Therefore, by Lusin's theorem (see e.g. [17]), for each $\varepsilon > 0$, there exists a compact set $\Lambda_\varepsilon \subset \Delta_1$ such that $m(\Delta_1 \setminus \Lambda_\varepsilon) \leq \varepsilon$ and the mappings

$$\Lambda_\varepsilon \ni x \mapsto \eta(x, \cdot) \in L^2(\Delta_1, m), \quad \Lambda_\varepsilon \ni x \mapsto \theta(x, \cdot) \in L^2(\Delta_1, m).$$

are continuous. Therefore, by (63), the function

$$\Lambda_\varepsilon^2 \ni (x, y) \mapsto Q[\Delta]_{11}(x, y) \in \mathbb{C} \tag{65}$$

is continuous. Hence, by (64),

$$\det \left[Q[\Delta]_{11}(x_i, x_j) \right]_{i,j=1,\ldots,n} \geq 0 \quad \text{for all } (x_1, \ldots, x_n) \in \Lambda_\varepsilon^n.$$

Thus, the continuous kernel (65) is positive definite, and therefore the operator $Q[\Delta]_{11}$ is non-negative on $L^2(\Lambda_\varepsilon, m)$. By letting $\varepsilon \to 0$, we conclude that $Q[\Delta]_{11} \geq 0$ on $L^2(\Delta_1, m)$. Hence, by (59),

$$K_{11}^\Delta \geq (K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1} K_{21}^\Delta \quad \text{on } L^2(\Delta_1, m). \tag{66}$$

8) We denote by $\widehat{K^\Delta}$ the corresponding transformation of the operator K^Δ in the Hilbert space $L^2(\Delta, m) = L^2(\Delta_1, m) \oplus L^2(\Delta_2, m)$. Hence $\widehat{K^\Delta} = P^\Delta \widehat{K} P^\Delta$ and

$$\widehat{K^\Delta} = \begin{bmatrix} K_{11}^\Delta & K_{21}^\Delta \\ (K_{21}^\Delta)^* & 1 - K_{22}^\Delta \end{bmatrix}.$$

By (66), for each $f = (f_1, f_2) \in L^2(\Delta, m)$,

$$\begin{aligned}
(\widehat{K^\Delta} f, f) &= (K_{11}^\Delta f_1, f_1) + (K_{21}^\Delta f_1, f_2) + ((K_{21}^\Delta)^* f_2, f_1) + ((1 - K_{22}^\Delta) f_2, f_2) \\
&\geq ((K_{21}^\Delta)^*(1 - K_{22}^\Delta)^{-1} K_{21}^\Delta f_1, f_1) + ((1 - K_{22}^\Delta) f_2, f_2) - 2|(K_{21}^\Delta f_1, f_2)| \\
&= ((1 - K_{22}^\Delta)^{-1} K_{21}^\Delta f_1, K_{21}^\Delta f_1) + ((1 - K_{22}^\Delta) f_2, f_2) - 2|(K_{21}^\Delta f_1, f_2)|. \tag{67}
\end{aligned}$$

Since K_{22}^{Δ} is a compact self-adjoint operator in $L^2(\Delta_2, m)$, we can choose an orthnormal basis of $L^2(\Delta_2, m)$ which consists of eigenvectors of the operator K_{22}^{Δ}, and we denote by λ_n the eigenvalue belonging to eigenvector e_n, $n \geq 1$. Clearly $\lambda_n < 1$ for all n. Then, by (67),

$$
\begin{aligned}
(\widehat{K^{\Delta}}f, f) &\geq \sum_{n=1}^{\infty}(1 - \lambda_n)^{-1}|(K_{21}^{\Delta}f_1, e_n)|^2 + \sum_{n=1}^{\infty}(1 - \lambda_n)|(f_2, e_n)|^2 \\
&\quad - \sum_{n=1}^{\infty}2|(K_{21}^{\Delta}f_1, e_n)(f_2, e_n)| \\
&= \sum_{n=1}^{\infty}\left((1 - \lambda_n)^{-1/2}|(K_{21}^{\Delta}f_1, e_n)| - (1 - \lambda_n)^{1/2}|(f_2, e_n)|\right)^2 \geq 0.
\end{aligned}
$$

Thus, for each compact $\Delta \subset X$, the operator $\widehat{K^{\Delta}} = P_{\Delta}\widehat{K}P_{\Delta}$ is nonnegative. Hence, $\widehat{K} \geq 0$. Exchanging the role of the sets X_1 and X_2 and using instead of the operator K the operator $1 - K$, we therefore get $1 - \widehat{K} \geq 0$. Thus $0 \leq \widehat{K} \leq 1$.

9) We now assume that $\|K\| = 1$. Using the procedure of thinning of the point process μ, see e.g. [8, Example 8.2 (a)], we conclude that, for each $\varepsilon \in (0, 1)$, there exists a point process μ_{ε} which has correlation functions as in formula (51), i.e., a determinantal point process corresponding to the operator $K_{\varepsilon} := \varepsilon K$. By the proved above $0 \leq \widehat{K}_{\varepsilon} \leq 1$. Hence, by (50),

$$
0 \leq \varepsilon\widehat{K} + (1 - \varepsilon)P_2 \leq 1.
$$

Letting $\varepsilon \to 1$, we get $0 \leq \widehat{K} \leq 1$.

10) Finally, we assume that $\|K\| > 1$ and we have to show that a determinantal point process does not exist in this case. Assume the contrary, i.e., assume that there exists a determinantal point process with correlation kernel $K(x, y)$. Since $\|K\| > 1$, there exists a compact set $\Delta \subset X$ such that $\|K^{\Delta}\| > 1$. Analogously to part 9), using the procedure of thinning, we conclude that, for each $\varepsilon \in (0, 1)$, there exists a determinantal point process with correlation kernel $K_{\varepsilon}(x, y) := \varepsilon K(x, y)$. We choose $\varepsilon := \|K^{\Delta}\|^{-1}$, so that $\|K_{\varepsilon}^{\Delta}\| = 1$. We take the restriction of the corresponding probability measure to the σ-algebra $\mathcal{B}_{\Delta}(\Gamma)$, i.e., a point process on $(\Gamma_{\Delta}, \mathcal{B}(\Gamma_{\Delta}))$. We denote this point process by $\mu_{\varepsilon, \Delta}$. By part 9), we have $0 \leq \widehat{K_{\varepsilon}^{\Delta}} \leq 1$. Then, by Corollary 2,

$$
\mathrm{Det}(1 - K_{\varepsilon}^{\Delta}) = 0.
$$

Next, following the idea of [20, Remark 4], we consider

$$
\int_{\Gamma}(1 - \varepsilon)^{|\gamma \cap \Delta|}\,\mu(d\gamma) = \int_{\Gamma}\prod_{x \in \gamma}(1 - \varepsilon\chi_{\Delta}(x))\,\mu(d\gamma)
$$

$$= \mathrm{Det}(1 - \varepsilon K^\Delta) = \mathrm{Det}(1 - K^\Delta_\varepsilon) = 0.$$

On the other hand, $(1 - \varepsilon)^{|\gamma \cap \Delta|} > 0$ for all $\gamma \in \Gamma$. Hence,

$$\int_\Gamma (1 - \varepsilon)^{|\gamma \cap \Delta|} \, \mu(d\gamma) > 0,$$

which is a contradiction. $\qquad\square$

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