Binary jumps in continuum. II. Non-equilibrium process and a Vlasov-type scaling limit

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July 5, 2011

Abstract

Let Γ denote the space of all locally finite subsets (configurations) in $\mathbb{R}^d$. A stochastic dynamics of binary jumps in continuum is a Markov process on Γ in which pairs of particles simultaneously hop over $\mathbb{R}^d$. We discuss a non-equilibrium dynamics of binary jumps. We prove the existence of an evolution of correlation functions on a finite time interval. We also show that a Vlasov-type mesoscopic scaling for such a dynamics leads to a generalized Boltzmann non-linear equation for the particle density.

Key words: continuous system, binary jumps, non-equilibrium dynamics, correlation functions, scaling limit, Vlasov scaling, Poisson measure

2010 Mathematics Subject Classification: 60K35, 82C21, 60J75, 35Q83

1 Introduction

Let $\Gamma = \Gamma(\mathbb{R}^d)$ denote the space of all locally finite subsets (configurations) in $\mathbb{R}^d$, $d \in \mathbb{N}$. A stochastic dynamics of jumps in continuum is a Markov process on Γ in which groups of particles simultaneously hop over $\mathbb{R}^d$, i.e., at each jump time several points of the configuration change their positions.

The simplest case corresponds to the so-called Kawasaki-type dynamics in continuum. This dynamics is a Markov process on Γ in which particles hop over $\mathbb{R}^d$ so that, at each jump time, only one particle changes its position. For a study of an equilibrium Kawasaki dynamics in continuum, we refer the reader

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to the papers [11–13, 17, 18, 20, 21] and the references therein. Under the so-called balance condition on the jump rate, a proper Gibbs distribution is an invariant, and even symmetrizing measure for such a dynamics, see [13]. To obtain a simpler measure, e.g. Poissonian, as a symmetrizing measure for a Kawasaki-type dynamics, we should either suppose quite unnatural conditions on the jump rate, or consider a free dynamics. In the free Kawasaki dynamics, in the course of a random evolution, each particle of the configuration randomly hops over $\mathbb{R}^d$ without any interaction with other particles (see [19] for details).

In the dynamics of binary jumps, at each jump time two points of the (infinite) configuration change their positions in $\mathbb{R}^d$. A randomness for choosing a pair of points provides a random interaction between particles of the system, even in the case where the jump rate only depends on the hopping points (and does not depend on the other points of the configuration). A Poisson measure may be invariant, or even symmetrizing for such a dynamics. In the first part of the present paper [10], we considered such a process with generator

$$(LF)(\gamma) = \sum_{\{x_1, x_2\} \subset \gamma} \int_{(\mathbb{R}^d)^2} Q(x_1, x_2, dh_1 \times dh_2)$$
$$\times \big(F(\gamma \setminus \{x_1, x_2\} \cup \{x_1 + h_1, x_2 + h_2\}) - F(\gamma)\big). \quad (1.1)$$

Here, the measure $Q(x_1, x_2, dh_1 \times dh_2)$ describes the rate at which two particles, x_1 and x_2, of configuration γ simultaneously hop to $x_1 + h_1$ and $x_2 + h_2$, respectively. Under some additional conditions on the measure Q, we studied a corresponding equilibrium dynamics for which a Poisson measure is a symmetrizing measure. We also considered two different scalings of the rate measure Q, which led us to a diffusive dynamics and to a birth-and-death dynamics, respectively.

In the present paper, we restrict our attention to the special case of the generator (1.1). We denote the arrival points by $y_i = x_i + h_i$, $i = 1, 2$. An (informal) generator of a binary jump process of our interest is given by

$$(LF)(\gamma) = \sum_{\{x_1, x_2\} \subset \gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(x_1, x_2, y_1, y_2)$$
$$\times \big(F(\gamma \setminus \{x_1, x_2\} \cup \{y_1, y_2\}) - F(\gamma)\big) dy_1 dy_2. \quad (1.2)$$

Here

$$c(x_1, x_2, y_1, y_2) = c(\{x_1, x_2\}, \{y_1, y_2\}) \qquad (1.3)$$

is a proper non-negative measurable function. Our aim is to study a non-equilibrium dynamics corresponding to (1.2). Note that the Poisson measure with any positive constant intensity will be invariant for this dynamics. If, additionally,

$$c(\{x_1, x_2\}, \{y_1, y_2\}) = c(\{y_1, y_2\}, \{x_1, x_2\}), \qquad (1.4)$$

then any such measure will even be symmetrizing.

It should be noted that similar dynamics of finite particle systems were studied in [2, 3, 14]. In particular, in [3], the authors studied a non-equilibrium dynamics of velocities of particles, such that the law of conservation of momentum is satisfied for this system. However, the methods applied to finite particle systems seem not to be applicable to infinite systems of our interest.

Let us also note an essential difference between lattice and continuous systems. An important example of a Markov dynamics on lattice configurations is the so-called exclusion process. In this process, particles randomly hop over the lattice under the only restriction to have no more than one particle at each site of the lattice. This process may have a Bernoulli measure as an invariant (and even symmetrizing) measure, but a corresponding stochastic dynamics has non-trivial properties and possesses an interesting and reach scaling limit behavior. A straightforward generalization of the exclusion process to the continuum gives a free Kawasaki dynamics, because the exclusion restriction (yielding an interaction between particles) obviously disappears for configurations in continuum. To introduce the simplest (in certain sense) interaction, we consider the generator above.

The generator (1.2) informally provides a functional evolution via the (backward) Kolmogorov equation

$$\frac{\partial F_t}{\partial t} = LF_t, \qquad F_t \big|_{t=0} = F_0. \tag{1.5}$$

However, the problem of existence of a solution to (1.5) in some functional space seems to be a very difficult problem. Fortunately, in applications, we usually need only an information about a mean value of a function on Γ with respect to some probability measure on Γ, rather than a full point-wise information about this function. Therefore, we turn to a weak evolution of probability measures (states) on Γ. This evolution of states is informally given as a solution to the initial value problem:

$$\frac{d}{dt}\langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \qquad \mu_t \big|_{t=0} = \mu_0, \tag{1.6}$$

provided, of course, that a solution exists. Here $\langle F, \mu \rangle = \int_\Gamma F(\gamma) d\mu(\gamma)$.

The problem (1.6) may be rewritten in terms of correlation functionals k_t of states μ_t. The corresponding dynamics of an infinite vector of correlation functions has a chain structure which is similar to the BBGKY-hierarchy for Hamiltonian dynamics. The corresponding generator of this dynamics has a two-diagonal (upper-diagonal) structure in a Fock-type space. Note that the most important case from the point of view of applications, being also the most interesting from the mathematical point of view, is the case of bounded (non-integrable) correlation functions. Because of (1.6), the dynamics of correlation functions should be treated in a weak form. Therefore, we consider a pre-dual evolution of the so-called quasi-observables, whose generator has a low-diagonal structure in a Fock-type space of integrable functions.

The paper is organized as follows. In Section 2, we describe the model and give some necessary preliminary information. Section 3 is devoted to the func-

tional evolution of both quasi-observables and correlation functions. We derive some information about the structure and properties of the two-diagonal generator of the dynamics of quasi-observables (Propositions 3.1 and 3.3). We further prove a general result for a low-diagonal generator, which shows that the evolution may be obtained, for all times, recursively in a scale of Banach spaces whose norms depend on time (Theorem 3.5). Next, we construct a dual dynamics on a finite time interval (Theorem 3.6). We show that this dynamics is indeed an evolution of correlation functions, which in turn leads to an evolution of probability measures on Γ (Theorem 3.9). In Section 4 we consider a Vlasov-type scaling for our dynamics. The limiting evolution (which exists by Proposition 4.4) has a chaos preservation property. This means that the corresponding dynamics of states transfers a Poisson measure with a non-homogeneous intensity into a Poisson measure whose non-homogeneous intensity satisfies a non-linear evolution (kinetic) equation. We finally present sufficient conditions for the existence and uniqueness of a solution to this equation (Proposition 4.6).

It is worth noting that we rigorously prove the convergence of the scaled evolution of the infinite particle system to the limiting evolution (which in turn leads to the kinetic equation). This seems to be a new step even for finite particle systems.

2 Preliminaries

Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-algebra on $\mathbb{R}^d$, $d \in \mathbb{N}$, and let $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denote the system of all bounded sets from $\mathcal{B}(\mathbb{R}^d)$. The configuration space over $\mathbb{R}^d$ is defined as the set of all locally finite subsets of $\mathbb{R}^d$:

$$\Gamma := \big\{ \gamma \subset \mathbb{R}^d \,\big|\, |\gamma_\Lambda| < \infty \text{ for any } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d) \big\}.$$

Here $| \cdot |$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x \in \gamma} \delta_x \in \mathcal{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on $\mathcal{B}(\mathbb{R}^d)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps $\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x)\, \gamma(dx) = \sum_{x \in \gamma} f(x)$, $f \in C_0(\mathbb{R}^d)$, are continuous. Here, $C_0(\mathbb{R}^d)$ is the space of all continuous functions on $\mathbb{R}^d$ with compact support. The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ coincides with the smallest σ-algebra on Γ for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, see e.g. [1]. It is worth noting that Γ is a Polish space (see e.g. [16]).

Let $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ denote the set of all measurable *cylinder functions* on Γ. Each $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ is characterized by the following property: $F(\gamma) = F(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and for any $\gamma \in \Gamma$.

A stochastic dynamics of binary jumps in continuum is a Markov process on Γ in which pairs of particles simultaneously hop over $\mathbb{R}^d$, i.e., at each jump time two points of the configuration change their positions. Thus, an (informal)

generator of such a process has the form (1.2), where $c(x_1, x_2, y_1, y_2)$ is a nonnegative measurable function which satisfies (1.3) and

$$c(x_1, x_2, \cdot, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d) \quad \text{for a.a. } x_1, x_2 \in \mathbb{R}^d.$$

The function c describes the rate at which pairs of particles hop over $\mathbb{R}^d$.

Remark 2.1. Note that, in general, the expression on the right hand side of (1.2) is not necessarily well defined for *all* $\gamma \in \Gamma$, even if $F \in \mathcal{F}_{\text{cyl}}(\Gamma)$. Nevertheless, for such F, $(LF)(\gamma)$ has sense at least for all $\gamma \in \Gamma$ with $|\gamma| < \infty$.

In various applications, the evolution of *states of the system* (i.e., *measures on the configuration space* Γ) helps one to understand the behavior of the process and gives possible candidates for invariant states. In fact, various properties of such an evolution form the main information needed in applications. Using the duality between functions and measures, this evolution may be considered in a weak form, given, as usual, by the expression

$$\langle F, \mu \rangle = \int_\Gamma F(\gamma) d\mu(\gamma). \tag{2.1}$$

Therefore, the evolution of states is informally given as a solution to the initial value problem:

$$\frac{d}{dt}\langle F, \mu_t \rangle = \langle LF, \mu_t \rangle, \qquad \mu_t \big|_{t=0} = \mu_0, \tag{2.2}$$

provided, of course, that a solution exists. For a wide class of probability measures on Γ, one can consider a corresponding evolution of their correlation functionals, see below.

The space of n-point configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma^{(n)}(Y) := \left\{ \eta \subset Y \mid |\eta| = n \right\}, \quad n \in \mathbb{N}.$$

We set $\Gamma^{(0)}(Y) := \{\emptyset\}$. As a set, $\Gamma^{(n)}(Y)$ may be identified with the quotient of $\widetilde{Y^n} := \left\{ (x_1, \ldots, x_n) \in Y^n \mid x_k \neq x_l \text{ if } k \neq l \right\}$ with respect to the natural action of the permutation group S_n on $\widetilde{Y^n}$. Hence, one can introduce the corresponding Borel σ-algebra, which will be denoted by $\mathcal{B}(\Gamma^{(n)}(Y))$. The space of finite configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_0(Y) := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}(Y).$$

This space is equipped with the topology of the disjoint union. Therefore, we consider the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0(Y))$. In the case where $Y = \mathbb{R}^d$, we will omit the index Y in the notation, namely, $\Gamma_0 = \Gamma_0(\mathbb{R}^d)$, $\Gamma^{(n)} = \Gamma^{(n)}(\mathbb{R}^d)$.

The image of the Lebesgue product measure $(dx)^n$ on $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$ will be denoted by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. Let $z > 0$ be fixed. The Lebesgue–Poisson measure λ_z on Γ_0 is defined by

$$\lambda_z := \sum_{n=0}^\infty \frac{z^n}{n!} m^{(n)}. \tag{2.3}$$

For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, the restriction of λ_z to $\Gamma(\Lambda) := \Gamma_0(\Lambda)$ will also be denoted by λ_z. The space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the family of spaces $\{(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda)))\}_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$. The Poisson measure π_z on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_z^\Lambda\}_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, where $\pi_z^\Lambda := e^{-zm(\Lambda)}\lambda_z$ is a probability measure on $(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda)))$ and $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$; for details see e.g. [1]. We will mostly use the measure $\lambda := \lambda_1$.

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exist $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^{N} \Gamma^{(n)}(\Lambda)$. The set of bounded measurable functions with bounded support will be denoted by $B_{\mathrm{bs}}(\Gamma_0)$, i.e., $G \in B_{\mathrm{bs}}(\Gamma_0)$ if $G \restriction_{\Gamma_0 \setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. We also consider the larger set $L_{\mathrm{ls}}^0(\Gamma_0)$ of all measurable functions on Γ_0 with local support, which means: $G \in L_{\mathrm{ls}}^0(\Gamma_0)$ if $G \restriction_{\Gamma_0 \setminus \Gamma(\Lambda)} = 0$ for some $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0 is, in fact, defined by a sequence of functions $\{G^{(n)}\}_{n \in \mathbb{N}_0}$, where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. Functions on Γ and Γ_0 will be called *observables* and *quasi-observables*, respectively.

We consider the following mapping from $L_{\mathrm{ls}}^0(\Gamma_0)$ into $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$:

$$(KG)(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.4}$$

where $G \in L_{\mathrm{ls}}^0(\Gamma_0)$, see, e.g., [15,23,24]. The summation in (2.4) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible with

$$(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.5}$$

Remark 2.2. Note that, using formula (2.5), we can extend the mapping K^{-1} to functions F which are well-defined, at least, on Γ_0.

Let $\mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ denote the set of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ which have finite local moments of all orders, i.e., $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$. A measure ρ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is called locally finite if $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$. The set of all such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

One can define a transform $K^* : \mathcal{M}_{\mathrm{fm}}^1(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e., for every $\mu \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ and $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$, we have

$$\int_\Gamma (KG)(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu := K^*\mu$ is called the *correlation measure* of μ.

As shown in [15], for any $\mu \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ and $G \in L^1(\Gamma_0, \rho_\mu)$, the series (2.4) is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma_0} G(\eta)\,\rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\,\mu(d\gamma). \tag{2.6}$$

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to $\pi := \pi_1$ if $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi^\Lambda := \pi_1^\Lambda$ for all $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$. Here $\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) := \gamma \cap \Lambda \in \Gamma(\Lambda)$. In this case, the correlation measure ρ_μ is absolutely continuous with respect to $\lambda = \lambda_1$. A *correlation functional of* μ is then defined by

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda}(\eta), \quad \eta \in \Gamma_0.$$

The functions $k_\mu^{(0)} := 1$ and

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+, \quad n \in \mathbb{N}, \tag{2.7}$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are called *correlation functions of* μ, and they are well known in statistical physics, see e.g [26], [27].

In view of Remark 2.1 and (2.5), the mapping

$$(\widehat{L}G)(\eta) := (K^{-1}LKG)(\eta), \qquad \eta \in \Gamma_0,$$

is well defined for any $G \in B_{\mathrm{bs}}(\Gamma_0)$, where K^{-1} is understood in the sense of Remark 2.2.

Let k be a measurable function on Γ_0 such that $\int_B k \, d\lambda < \infty$ for any bounded $B \in \mathcal{B}(\Gamma_0)$. Then, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$, we can consider an analog of pairing (2.1),

$$\langle\!\langle G, k \rangle\!\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda. \tag{2.8}$$

Using this duality, we may consider a dual mapping $\widehat{L}^*$ of $\widehat{L}$. As a result, we obtain two initial value problems:

$$\frac{\partial G_t}{\partial t} = \widehat{L}G_t, \qquad G_t \big|_{t=0} = G_0, \tag{2.9}$$

$$\frac{d}{dt} \langle\!\langle G, k_t \rangle\!\rangle = \langle\!\langle G, \widehat{L}^* k_t \rangle\!\rangle = \langle\!\langle \widehat{L}G, k_t \rangle\!\rangle, \qquad k_t \big|_{t=0} = k_0. \tag{2.10}$$

We are going to solve the first problem in a proper functional space over Γ_0. The second problem, corresponding to (2.2), can be realized by means of (2.8), or solved independently.

The next section is devoted to a (rigorous) solution of these two problems.

3 Functional evolution

We denote, for any $n \in \mathbb{N}$,

$$X_n := L^1\big((\mathbb{R}^d)^n, dx^{(n)}\big), \tag{3.1}$$

where we set $dx^{(n)} = dx_1 \cdots dx_n$ and $X_0 := \mathbb{R}$. The symbol $\| \cdot \|_{X_n}$ stands for the norm of the space (3.1).

For an arbitrary $C > 0$, we consider the functional Banach space

$$\mathcal{L}_C := L^1\big(\Gamma_0, C^{|\eta|} d\lambda(\eta)\big). \tag{3.2}$$

Throughout this paper, the symbol $\|\cdot\|_{\mathcal{L}_C}$ denotes for the norm of the space (3.2). Then, for any $G \in \mathcal{L}_C$, we may identify G with the sequence $(G^{(n)})_{n \geq 0}$, where $G^{(n)}$ is a symmetric function on $(\mathbb{R}^d)^n$ (defined almost everywhere) such that

$$\|G\|_{\mathcal{L}_C} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} |G^{(n)}(x^{(n)})| C^n dx^{(n)} = \sum_{n=0}^{\infty} \frac{C^n}{n!} \|G^{(n)}\|_{X_n} < \infty. \tag{3.3}$$

In particular, $G^{(n)} \in X_n$, $n \in \mathbb{N}_0$. Here and below we set $x^{(n)} = (x_1, \ldots, x_n)$.

We consider the dual space $(\mathcal{L}_C)'$ of $\mathcal{L}_C$. It is evident that this space can be realized as the Banach space

$$\mathcal{K}_C := \big\{ k : \Gamma_0 \to \mathbb{R} \ \big| \ k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, d\lambda) \big\}$$

with the norm $\|k\|_{\mathcal{K}_C} := \|k(\cdot) C^{-|\cdot|}\|_{L^\infty(\Gamma_0, \lambda)}$, where the pairing between any $G \in \mathcal{L}_C$ and $k \in \mathcal{K}_C$ is given by (2.8). In particular,

$$|\langle\!\langle G, k \rangle\!\rangle| \leq \|G\|_C \cdot \|k\|_{\mathcal{K}_C}.$$

Clearly, $k \in \mathcal{K}_C$ implies

$$|k(\eta)| \leq \|k\|_{\mathcal{K}_C} C^{|\eta|} \qquad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0. \tag{3.4}$$

Proposition 3.1. *Let for a.a. $x_1, x_2, y_1 \in \mathbb{R}^d$*

$$\tilde{c}(x_1, x_2, y_1) = \tilde{c}(\{x_1, x_2\}, y_1) := \int_{\mathbb{R}^d} c(x_1, x_2, y_1, y_2) \, dy_2 < \infty. \tag{3.5}$$

Then, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$, the following formula holds

$$(\widehat{L}G)(\eta) = (L_0 G)(\eta) + (WG)(\eta), \tag{3.6}$$

where

$$(L_0 G)(\eta) = \sum_{\{x_1, x_2\} \subset \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(x_1, x_2, y_1, y_2)$$
$$\times \big(G(\eta \setminus \{x_1, x_2\} \cup \{y_1, y_2\}) - G(\eta)\big) dy_1 dy_2, \tag{3.7}$$

$$(WG)(\eta) = \sum_{x_2 \in \eta} \sum_{x_1 \in \eta \setminus x_2} \int_{\mathbb{R}^d} \tilde{c}(x_1, x_2, y_1)$$
$$\times \big(G((\eta \setminus x_2) \setminus x_1 \cup y_1) - G(\eta \setminus x_2)\big) dy_1. \tag{3.8}$$

Proof. We have, for $F = KG$,

$$
(KG)(\gamma \setminus \{x_1, x_2\} \cup \{y_1, y_2\}) - (KG)(\gamma)
$$

$$
= \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\} \cup \{y_1,y_2\}} G(\eta) - \sum_{\eta \Subset \gamma} G(\eta)
$$

$$
= \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\}} G(\eta \cup y_1) + \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\}} G(\eta \cup y_2)
$$

$$
+ \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\}} G(\eta \cup y_1 \cup y_2) - \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\}} G(\eta \cup x_1)
$$

$$
- \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\}} G(\eta \cup x_2) - \sum_{\eta \Subset \gamma \setminus \{x_1,x_2\}} G(\eta \cup x_1 \cup x_2)
$$

$$
= (K \left(G(\cdot \cup y_1) + G(\cdot \cup y_2) + G(\cdot \cup \{y_1, y_2\}) \right.
$$

$$
\left. - G(\cdot \cup x_1) - G(\cdot \cup x_2) - G(\cdot \cup \{x_1, x_2\}) \right)) (\gamma \setminus \{x_1, x_2\}).
$$

Hence, for any measurable h on $\Gamma \times \mathbb{R}^d \times \mathbb{R}^d$,

$$
K^{-1} \left(\sum_{\{x_1,x_2\} \subset \cdot} h(\cdot \setminus \{x_1, x_2\}, x_1, x_2) \right)(\eta)
$$

$$
= \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} \sum_{\{x_1,x_2\} \subset \xi} h(\xi \setminus \{x_1, x_2\}, x_1, x_2)
$$

$$
= \sum_{\{x_1,x_2\} \subset \eta} \sum_{\xi \subset \eta \setminus \{x_1,x_2\}} (-1)^{|\eta \setminus \{x_1,x_2\} \setminus \xi|} h(\xi, x_1, x_2)
$$

$$
= \sum_{\{x_1,x_2\} \subset \eta} \left(K^{-1} h(\cdot, x_1, x_2) \right)(\eta \setminus \{x_1, x_2\}).
$$

Therefore,

$$
(\widehat{L}G)(\eta) = \left(K^{-1} L K G \right)(\eta)
$$

$$
= \sum_{\{x_1,x_2\} \subset \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(\{x_1, x_2\}, \{y_1, y_2\}) \left(G(\eta \setminus \{x_1, x_2\} \cup y_1) \right.
$$

$$
\left. + G(\eta \setminus \{x_1, x_2\} \cup y_2) + G(\eta \setminus \{x_1, x_2\} \cup \{y_1, y_2\}) \right) dy_1 dy_2
$$

$$
- \sum_{\{x_1,x_2\} \subset \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(\{x_1, x_2\}, \{y_1, y_2\}) \left(G(\eta \setminus \{x_1, x_2\} \cup x_1) \right.
$$

$$
\left. + G(\eta \setminus \{x_1, x_2\} \cup x_2) + G(\eta \setminus \{x_1, x_2\} \cup \{x_1, x_2\}) \right) dy_1 dy_2,
$$

from where the statement follows. $\qquad \square$

As we noted before, any function G on Γ_0 may be identified with the infinite vector $\left(G^{(n)} \right)_{n \geq 0}$ of symmetric functions. Due to this identification, any operator on functions on Γ_0 may be considered as an infinite operator matrix. By Proposition 3.1, the operator matrix $\widehat{L}$ has a two-diagonal structure.

More precisely, on the main diagonal, we have operators $L_0^{(n)}$, $n \in \mathbb{N}_0$, where $L_0^{(0)} = L_0^{(1)} = 0$ and for $n \geq 2$

$$
\left(L_0^{(n)} G^{(n)}\right)(x^{(n)}) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_i, x_j, y_1, y_2,\right)
$$
$$
\times \left(G^{(n)}\left(x_1, \ldots, \underset{\hat{i}}{y_1}, \ldots, \underset{\hat{j}}{y_2}, \ldots, x_n\right) - G^{(n)}\left(x^{(n)}\right)\right) dy_1 dy_2. \quad (3.9)
$$

On the low diagonal, we have operators $W^{(n)}$, $n \in \mathbb{N}$, where $W^{(1)} = 0$ and for $n \geq 2$

$$
\left(W^{(n)} G^{(n-1)}\right)(x^{(n)}) = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_{\mathbb{R}^d} \tilde{c}\left(x_i, x_j, y_1\right)
$$
$$
\times \left(G^{(n-1)}\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, \underset{\hat{j}}{y_1}, \ldots, x_n\right)\right.
$$
$$
\left. - G^{(n-1)}\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right)\right) dy_1. \quad (3.10)
$$

Let us formulate our main conditions on the rate c:

$$
c_1 := \operatorname*{ess\,sup}_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_1, x_2, y_1, y_2\right) dy_1 dy_2 < \infty, \quad (3.11)
$$

$$
c_2 := \operatorname*{ess\,sup}_{x_1, x_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(y_1, y_2, x_1, x_2\right) dy_1 dy_2 < \infty, \quad (3.12)
$$

$$
c_3 := \operatorname*{ess\,sup}_{x_1 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_1, x_2, y_1, y_2\right) dy_1 dy_2 dx_2 < \infty, \quad (3.13)
$$

$$
c_4 := \operatorname*{ess\,sup}_{x_1 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(y_1, y_2, x_1, x_2\right) dy_1 dy_2 dx_2 < \infty. \quad (3.14)
$$

Under conditions (3.11)–(3.12), we define the following functions

$$
a_1(x_1, x_2) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_1, x_2, y_1, y_2\right) dy_1 dy_2 \in [0, \infty), \quad (3.15)
$$

$$
a_2(x_1, x_2) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(y_1, y_2, x_1, x_2\right) dy_1 dy_2 \in [0, \infty). \quad (3.16)
$$

Remark 3.2. Note that, if the function c satisfies the symmetry condition (1.4), then conditions (3.12), (3.14) follow from (3.11), (3.13), respectively, and

$$
a_1(x_1, x_2) = a_2(x_1, x_2). \quad (3.17)
$$

In this case, the operator $L_0^{(n)}$ is symmetric in $L^2\left((\mathbb{R}^d)^n, dx^{(n)}\right)$. Moreover, the operator L given by (1.2) is (informally) symmetric in $L^2(\Gamma, \pi_z)$ for any $z > 0$.

Proposition 3.3. (i) *Let* (3.11), (3.12) *hold. Then, for any* $G^{(n)} \in X_n$,

$$\|L_0^{(n)}G^{(n)}\|_{X_n} \leq \frac{n(n-1)}{2}(c_1 + c_2)\|G^{(n)}\|_{X_n}. \tag{3.18}$$

Moreover, if additionally

$$a_2(x_1, x_2) \leq a_1(x_1, x_2) \quad \text{for a.a. } x_1, x_2 \in \mathbb{R}^d, \tag{3.19}$$

then $L_0^{(n)}$ *is the generator of a contraction semigroup in* X_n.
(ii) *Let* (3.13), (3.14) *hold. Then, for any* $G^{(n-1)} \in X_{n-1}$,

$$\|W^{(n)}G^{(n-1)}\|_{X_n} \leq n(n-1)(c_3 + c_4)\|G^{(n-1)}\|_{X_{n-1}}. \tag{3.20}$$

Proof. The estimates (3.18), (3.20) follow directly from (3.9), (3.10) and (3.11)–(3.14). To prove that the bounded operator $L_0^{(n)}$ is the generator of a contraction semigroup in X_n, it is enough to show that $L_0^{(n)}$ is dissipative (see e.g. [25]). For any $\kappa > 0$ and $G^{(n)} \in X_n$,

$$\|L_0^{(n)}G^{(n)} - \kappa G^{(n)}\|_{X_n}$$

$$= \int_{(\mathbb{R}^d)^n} \left| \sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_i, x_j, y_1, y_2, \right) G^{(n)}\left(x_1, \ldots, \underset{\hat{i}}{y_1}, \ldots, \underset{\hat{j}}{y_2}, \ldots, x_n\right) dy_1 dy_2 \right.$$

$$\left. - \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_1(x_i, x_j) G^{(n)}\left(x^{(n)}\right) - \kappa G^{(n)}\left(x^{(n)}\right) \right| dx^{(n)},$$

and, using the obvious inequality $\|f - g\|_{X_n} \geq \left| \|f\|_{X_n} - \|g\|_{X_n} \right|$, we continue

$$\geq \left| \int_{(\mathbb{R}^d)^n} \left| \sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_i, x_j, y_1, y_2, \right) \right. \right.$$

$$\left. \times G^{(n)}\left(x_1, \ldots, \underset{\hat{i}}{y_1}, \ldots, \underset{\hat{j}}{y_2}, \ldots, x_n\right) dy_1 dy_2 \right| dx^{(n)}$$

$$\left. - \int_{(\mathbb{R}^d)^n} \left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_1(x_i, x_j) + \kappa \right) \left| G^{(n)}\left(x^{(n)}\right) \right| dx^{(n)} \right|.$$

Since $G^{(n)}$ is a symmetric function, we get

$$\int_{(\mathbb{R}^d)^n} \left| \sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(x_i, x_j, y_1, y_2, \right) \right.$$

$$\left. \times G^{(n)}\left(x_1, \ldots, \underset{\hat{i}}{y_1}, \ldots, \underset{\hat{j}}{y_2}, \ldots, x_n\right) dy_1 dy_2 \right| dx^{(n)}$$

$$\leq \frac{n(n-1)}{2} \int_{(\mathbb{R}^d)^n} a_2(y_1, y_2) \left| G^{(n)}\left(y_1, y_2, x_1, \ldots, x_{n-2}\right) \right| dy_1 dy_2 dx_1 \ldots dx_{n-2}.$$

Therefore, if (3.19) holds, then

$$\|L_0^{(n)}G^{(n)} - \kappa G^{(n)}\|_{X_n} \geq \kappa \|G^{(n)}\|_{X_n},$$

which proves the dissipativity of $L_0^{(n)}$, see e.g. [5, Proposition 3.23]. □

Remark 3.4. If the function c satisfies the symmetry condition (1.4), then (3.19) trivially follows from (3.17).

In Theorems 3.5 and 3.6 below, we formulate general results which are applicable to our dynamics under assumptions (3.11)–(3.14), (3.19).

Theorem 3.5. *Consider the initial value problem*

$$\frac{\partial}{\partial t}G_t(\eta) = (L_0 G_t)(\eta) + (W G_t)(\eta), \quad t > 0, \ \eta \in \Gamma_0,$$

$$G_t\big|_{t=0} = G_0. \tag{3.21}$$

Here, for any $G = \big(G^{(n)}\big)_{n \geq 0}$,

$$(L_0 G)^{(n)} = L_0^{(n)}G^{(n)}, \ n \geq 1;$$
$$(W G)^{(n)} = W^{(n)}G^{(n-1)}, \ n \geq 2;$$
$$(L_0 G)^{(0)} = (W G)^{(0)} = (W G)^{(1)} = 0.$$

Further suppose that $L_0^{(n)}$ *is a bounded generator of a strongly continuous contraction semigroup* $e^{tL_0^{(n)}}$ *in* X_n*, while* $W^{(n)}$ *is a bounded operator from* X_{n-1} *into* X_n *whose norm satisfies*

$$\big\|W^{(n)}\big\|_{X_{n-1} \to X_n} \leq Bn(n-1), \quad n \geq 1,$$

for some $B \geq 1$ *which is independent of* n.

Let $C > 0$ *and* $G_0 \in \mathcal{L}_C$. *Then the initial value problem (3.21) has a unique solution* $G_t \in \mathcal{L}_{\rho(t,C)}$, *where*

$$\rho(t, C) := \frac{C}{1 + BCt}. \tag{3.22}$$

Furthermore,

$$\|G_t\|_{\mathcal{L}_{\rho(t,C)}} \leq \|G_0\|_{\mathcal{L}_C}. \tag{3.23}$$

Proof. Let us rewrite the initial value problem (3.21) as an infinite system of differential equations. Namely, for any $n \geq 1$,

$$\frac{\partial}{\partial t}G_t^{(n)}\big(x^{(n)}\big) = \big(L_0^{(n)}G_t^{(n)}\big)\big(x^{(n)}\big) + \big(W^{(n)}G_t^{(n-1)}\big)\big(x^{(n)}\big) \tag{3.24}$$

with $G_t^{(0)} = G_0^{(0)}$. This system may be solved recurrently: for each $n \geq 1$

$$G_t^{(n)}\big(x^{(n)}\big) = \Big(e^{tL_0^{(n)}} G_0^{(n)}\Big)\big(x^{(n)}\big)$$
$$+ \int_0^t \Big(e^{(t-s)L_0^{(n)}} W^{(n)} G_s^{(n-1)}\Big)\big(x^{(n)}\big) ds. \qquad (3.25)$$

Iterating (3.25), we obtain

$$G_t^{(n)}(x^{(n)}) = \sum_{k=0}^n \big(V_{k,n}(t) G_0^{(n-k)}\big)(x^{(n)}),$$

where $V_{k,n}(t) : X_{n-k} \to X_n$ is given by

$$V_{k,n}(t) := \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} e^{(t-s_1)L_0^{(n)}} W^{(n)} e^{(s_1-s_2)L_0^{(n-1)}} W^{(n-1)} \cdots$$
$$\times e^{(s_{k-1}-s_k)L_0^{(n-k+1)}} W^{(n-k+1)} e^{s_k L_0^{(n-k)}} ds_k \cdots ds_1$$

for $2 \leq k \leq n-1$, and

$$V_{1,n}(t) G^{(n-1)} := \int_0^t e^{(t-s_1)L_0^{(n)}} W^{(n)} e^{s_1 L_0^{(n-1)}} G^{(n-1)} ds_1,$$
$$V_{0,n}(t) G^{(n)} := e^{tL_0^{(n)}} G^{(n)},$$
$$V_{n,n}(t) G^{(n)} := \chi_{\{n=0\}} G^{(0)}.$$

Then, using the conditions on L_0 and W, for $1 \leq k \leq n-1$, $n \geq 2$

$$\big\|V_{k,n}(t) G^{(n-k)}\big\|_{X_n} \leq \frac{t^k}{k!} B^k n(n-1)(n-1)(n-2) \cdots (n-k+1)(n-k)$$
$$= (tB)^k \frac{n!}{k!(n-k)!} \frac{(n-1)!}{(n-k-1)!} \big\|G^{(n-k)}\big\|_{X_{n-k}},$$

and $\big\|V_{0,n}(t) G^{(n)}\big\|_{X_n} \leq \|G^{(n)}\|_{X_n}$ for $n \geq 1$. Therefore, for $n \geq 1$,

$$\big\|G_t^{(n)}\big\|_{X_n} \leq \sum_{k=0}^{n-1} (tB)^k \frac{n!}{k!(n-k)!} \frac{(n-1)!}{(n-k-1)!} \big\|G^{(n-k)}\big\|_{X_{n-k}}$$
$$= \sum_{k=1}^n (tB)^{(n-k)} \frac{n!}{(n-k)!k!} \frac{(n-1)!}{(k-1)!} \big\|G_0^{(k)}\big\|_{X_k}. \qquad (3.26)$$

Then, for any $q(t) > 0$,

$$\|G_t\|_{\mathcal{L}_{Cq(t)}}$$

$$= |G_t^{(0)}| + \sum_{n=1}^{\infty} \frac{C^n q^n(t)}{n!} \|G_t^{(n)}\|_{X_n}$$

$$\leq |G_0^{(0)}| + \sum_{n=1}^{\infty} \frac{C^n q^n(t)}{n!} \sum_{k=1}^{n} \|G_0^{(k)}\|_{X_k} (tB)^{n-k} \frac{n!}{k!\,(n-k)!} \frac{(n-1)!}{(k-1)!}$$

$$= |G_0^{(0)}| + \sum_{k=1}^{\infty} \|G_0^{(k)}\|_{X_k} \frac{1}{k!} \sum_{n=k}^{\infty} C^n q^n(t) (tB)^{n-k} \frac{1}{(n-k)!} \frac{(n-1)!}{(k-1)!}$$

$$= |G_0^{(0)}| + \sum_{k=1}^{\infty} \|G_0^{(k)}\|_{X_k} \frac{C^k q^k(t)}{k!} \sum_{n=0}^{\infty} q^n(t)(tBC)^n \frac{(n+k-1)!}{n!\,(k-1)!} \,.$$

Now, let $q(t) = \dfrac{1}{1+BCt}$. For any $x \in [0,1)$ and $m \in \mathbb{N}$,

$$\left(\frac{1}{1-x}\right)^{m+1} = \sum_{n=0}^{\infty} x^n \frac{(n+m)!}{n!m!}.$$

Applying this equality to $x = q(t)BCt < 1$ and $m = k-1$, we obtain

$$q^k(t) \sum_{n=0}^{\infty} q^n(t)(tBC)^n \frac{(n+k-1)!}{n!\,(k-1)!} = \left(\frac{q(t)}{1-q(t)BCt}\right)^k = 1.$$

Therefore,

$$\|G_t\|_{\mathcal{L}_{Cq(t)}} \leq |G_0^{(0)}| + \sum_{k=1}^{\infty} \|G_0^{(k)}\|_{X_k} \frac{C^k}{k!} = \|G_0\|_{\mathcal{L}_C},$$

which proves the statement. $\qquad\square$

In fact, we have a linear evolution operator

$$V(t) : \mathcal{L}_C \to \mathcal{L}_{\rho(t,C)},$$

satisfying $G_t = V(t)G_0$ and

$$\|V(t)\|_{\mathcal{L}_C \to \mathcal{L}_{\rho(t,C)}} \leq 1 \tag{3.27}$$

Theorem 3.6. *Let the conditions of Theorem 3.5 be satisfied. Further suppose that there exists $A > 0$ such that $\|L_0^{(n)}\|_{X_n \to X_n} \leq An(n-1)$, $n \geq 1$. Let $C_0 > 0$, $k_0 \in \mathcal{K}_{C_0}$, $T = \frac{1}{BC_0}$. Then for any $t \in (0,T)$, there exists $k_t \in \mathcal{K}_{C_t}$ with*

$$C_t = \frac{C_0}{1 - BC_0 t}, \tag{3.28}$$

such that, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$,

$$\frac{d}{dt}\langle\!\langle G, k_t \rangle\!\rangle = \langle\!\langle (L_0 + W)G, k_t \rangle\!\rangle, \quad t \in (0, T).$$

Moreover, for any $t \in (0, T)$,

$$\|k_t\|_{\mathcal{K}_{C_t}} \le \|k_0\|_{\mathcal{K}_{C_0}}. \tag{3.29}$$

Proof. Let $t \in (0, T)$ be arbitrary. The function $f(x) = \rho(t, x) = \frac{x}{1+xBt}$, $x \ge 0$, increases to $\frac{1}{tB}$ as $x \to +\infty$. Since $C_0 < \frac{1}{tB}$, there exists a unique solution to $f(x) = C_0$, namely, $x = C_t$, given by (3.28). Take any $G_0 \in \mathcal{L}_{C_t}$. By Theorem 3.5, there exists an evolution $G_0 \mapsto G_\tau$ for any $\tau > 0$ such that $G_\tau \in \mathcal{L}_{\rho(\tau, C_t)}$. Consider this evolution at the moment $\tau = t$. Since

$$\rho(t, C_t) = \frac{C_t}{1 + BC_t t} = C_0, \tag{3.30}$$

we have $G_t \in \mathcal{L}_{C_0}$. Therefore, $\langle\!\langle G_t, k_0 \rangle\!\rangle$ is well-defined. Moreover, by (3.23),

$$\begin{aligned} \left|\langle\!\langle G_t, k_0 \rangle\!\rangle\right| &\le \|G_t\|_{\mathcal{L}_{C_0}} \|k_0\|_{\mathcal{K}_{C_0}} = \|G_t\|_{\mathcal{L}_{\rho(t, C_t)}} \|k_0\|_{\mathcal{K}_{C_0}} \\ &\le \|G_0\|_{\mathcal{L}_{C_t}} \|k_0\|_{\mathcal{K}_{C_0}}. \end{aligned} \tag{3.31}$$

Therefore, the mapping $G_0 \mapsto \langle\!\langle G_t, k_0 \rangle\!\rangle$ is a linear continuous functional on the space $\mathcal{L}_{C_t}$. Hence, there exists $k_t \in \mathcal{K}_{C_t}$ such that, for any $G_0 \in \mathcal{L}_{C_t}$,

$$\langle\!\langle G_0, k_t \rangle\!\rangle = \langle\!\langle G_t, k_0 \rangle\!\rangle = \langle\!\langle V(t)G_0, k_0 \rangle\!\rangle. \tag{3.32}$$

We note that k_t depends on k_0 and does not depend on G_0. Further, (3.31) implies (3.29).

Let now $G_0 \in B_{\mathrm{bs}}(\Gamma_0)$. Consider a function $g = g_{G_0, k_0} : [0, T) \to \mathbb{R}$, $g(t) := \langle\!\langle G_t, k_0 \rangle\!\rangle = \langle\!\langle G_0, k_t \rangle\!\rangle$. We have

$$g(t) = \langle\!\langle G_t, k_0 \rangle\!\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G_t^{(n)}\big(x^{(n)}\big) k_0^{(n)}\big(x^{(n)}\big) dx^{(n)}. \tag{3.33}$$

By (3.31), for any $[0, T'] \subset [0, T)$,

$$\left|\langle\!\langle G_t, k_0 \rangle\!\rangle\right| \le \|G_0\|_{\mathcal{L}_{C_{T'}}} \|k_0\|_{\mathcal{K}_{C_0}}, \quad t \in [0, T'].$$

Hence, the series (3.33) converges on $[0, T']$. Using the well-known representation

$$e^{tL_0^{(n)}} G_0 = G_0 + \int_0^t e^{sL_0^{(n)}} L_0^{(n)} G_0 ds \tag{3.34}$$

(see e.g. [5, Lemma 1.3 (iv)]), we derive from (3.25) and Fubini's theorem:

$$
\begin{aligned}
g_n(t) :&= \int_{(\mathbb{R}^d)^n} G_t^{(n)}\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}\\
&= \int_{(\mathbb{R}^d)^n} \Big(e^{tL_0^{(n)}}G_0^{(n)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}\\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^n} \Big(e^{(t-s)L_0^{(n)}}W^{(n)}G_s^{(n-1)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds\\
&= \int_{(\mathbb{R}^d)^n} G_0^{(n)}\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}\\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^n} \Big(e^{sL_0^{(n)}}L_0^{(n)}G_0^{(n)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds\\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^n} \Big(e^{(t-s)L_0^{(n)}}W^{(n)}G_s^{(n-1)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds. \quad (3.35)
\end{aligned}
$$

Since $L_0^{(n)}: X_n \to X_n$ and $W^{(n)}: X_{n-1} \to X_n$ are bounded, the functions inside the time integrals are continuous in s. Therefore, by (3.35) and (3.25), $g_n(t)$ is differentiable on $(0,T)$ and

$$
\begin{aligned}
g_n'(t) &= \int_{(\mathbb{R}^d)^n} \Big(L_0^{(n)}e^{tL_0^{(n)}}G_0^{(n)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}\\
&\quad + \int_0^t \int_{(\mathbb{R}^d)^n} \Big(L_0^{(n)}e^{(t-s)L_0^{(n)}}W^{(n)}G_s^{(n-1)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds\\
&\quad + \int_{(\mathbb{R}^d)^n} \Big(W^{(n)}G_t^{(n-1)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds\\
&= \int_{(\mathbb{R}^d)^n} \big(L_0^{(n)}G_t^{(n)}\big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds\\
&\quad + \int_{(\mathbb{R}^d)^n} \Big(W^{(n)}G_t^{(n-1)}\Big)\big(x^{(n)}\big)k_0^{(n)}\big(x^{(n)}\big)dx^{(n)}ds. \quad (3.36)
\end{aligned}
$$

Hence, for any $n \geq 2$,

$$
|g_n'(t)| \leq \|k_0\|_{\mathcal{K}_{C_0}}C_0^n n(n-1)\big(A\|G_t\|_{X_n} + B\|G_t\|_{X_{n-1}}\big).
$$

Analogously to the proof of Theorem 3.5, we obtain, for all $t \in [0,T'] \subset [0,T)$,

$$
\begin{aligned}
&\sum_{n=1}^\infty \frac{1}{n!}|g_n'(t)|\\
&\leq \text{const} \cdot \sum_{n=1}^\infty \frac{1}{n!}C_0^n n(n-1)\sum_{k=1}^n (tB)^{(n-k)}\frac{n!}{(n-k)!k!}\frac{(n-1)!}{(k-1)!}\big\|G_0^{(k)}\big\|_{X_k}\\
&\leq \text{const} \cdot \sum_{k=1}^\infty \frac{1}{k!}\big\|G_0^{(k)}\big\|_{X_k}\sum_{n=k}^\infty C_0^n(T'B)^{(n-k)}\frac{n(n-1)}{(n-k)!}\frac{(n-1)!}{(k-1)!}
\end{aligned}
$$

$$= \text{const} \cdot \sum_{k=1}^{\infty} \frac{C_0^k}{k!} \|G_0^{(k)}\|_{X_k} \sum_{n=0}^{\infty} C_0^n (T'B)^n \frac{(n+k-1)}{n!} \frac{(n+k)!}{(k-1)!} < \infty, \quad (3.37)$$

since $G_0 \in B_{\text{bs}}(\Gamma_0)$ (and so there exists $K \in \mathbb{N}$ such that $G_0^{(k)} = 0$ for all $k \geq K$) and the inner series converges as $C_0 T'B < 1$.

Hence, $g(t)$ is differentiable on any $[0, T'] \subset [0, T)$. Next, (3.33), (3.36), and (3.37) imply that

$$g'(t) = \frac{d}{dt} \langle\!\langle G_t, k_0 \rangle\!\rangle = \langle\!\langle (L_0 + W)G_t, k_0 \rangle\!\rangle \qquad (3.38)$$

and, moreover, $(L_0 + W)G_t \in \mathcal{L}_{C_0}$. Therefore, using (3.32), (3.38) and the obvious inclusion $(L_0 + W)G_0 \in \mathcal{L}_{C_t}$, we obtain

$$\frac{d}{dt} \langle\!\langle G_0, k_t \rangle\!\rangle = \frac{d}{dt} \langle\!\langle G_t, k_0 \rangle\!\rangle = \langle\!\langle (L_0 + W)G_t, k_0 \rangle\!\rangle = \langle\!\langle (L_0 + W)V(t)G_0, k_0 \rangle\!\rangle$$
$$= \langle\!\langle V(t)(L_0 + W)G_0, k_0 \rangle\!\rangle = \langle\!\langle (L_0 + W)G_0, k_t \rangle\!\rangle,$$

provided

$$(L_0 + W)V(t)G_0 = V(t)(L_0 + W)G_0. \qquad (3.39)$$

To prove (3.39), we consider, for each $N \in \mathbb{N}$, the space $\mathbb{X}_N := \bigoplus_{n=0}^{N} X_n$ with the norm $\|\cdot\|_{\mathbb{X}_N} := \sum_{n=0}^{N} \|\cdot\|_{X_n}$. For any $\mathbb{G} \in \mathbb{X}_N$, $\mathbb{G} = (\mathbb{G}^{(0)}, \ldots, \mathbb{G}^{(N)})$, we define the following function on Γ_0:

$$\mathbb{I}_N \mathbb{G} := (\mathbb{G}^{(0)}, \ldots, \mathbb{G}^{(N)}, 0, 0, \ldots).$$

For any function G on Γ_0, $G = (G^{(0)}, \ldots, G^{(n)}, \ldots)$, with $G^{(n)} \in X_n$, we define the following element of $\mathbb{X}_N$:

$$\mathbb{P}_N G := (G^{(0)}, \ldots, G^{(n)}).$$

The system of differential equations (3.24) for $1 \leq n \leq N$ can be considered as one equation $\frac{\partial}{\partial t} \mathbb{G}_t = \mathbb{L}_N \mathbb{G}_t$ in $\mathbb{X}_N$ with

$$\mathbb{L}_N := \mathbb{P}_N (L_0 + W) \mathbb{I}_N.$$

Clearly, $\mathbb{L}_N$ is a bounded operator in $\mathbb{X}_N$. Hence, there exists a unique vector-valued solution of this equation, $\mathbb{G}_t = e^{t\mathbb{L}_N} \mathbb{G}_0$. The n-th component of $\mathbb{G}_t$, i.e., $\mathbb{G}_t^{(n)}$, coincides with the $G_t^{(n)}$ obtained in Theorem 3.5, for each $0 \leq n \leq N$, where $G_0 = \mathbb{I}_N \mathbb{G}_0$. More precisely, for $0 \leq n \leq N$,

$$(V(t)G_0)^{(n)} = (\mathbb{I}_N e^{t\mathbb{L}_N} \mathbb{G}_0)^{(n)} = (e^{t\mathbb{L}_N} \mathbb{G}_0)^{(n)} = (e^{t\mathbb{L}_N} \mathbb{P}_N G_0)^{(n)}. \qquad (3.40)$$

It is well known that a bounded operator $\mathbb{L}_N$ commutes with its semigroup $e^{t\mathbb{L}_N}$. Note also that, for $0 \leq n \leq N$, $G^{(n)} = (\mathbb{P}_N G)^{(n)}$. Therefore, for all $N \geq 1$,

$0 \leq n \leq N$, and for $G_0 = \mathbb{I}_N \mathbb{G}_0$, we obtain

$$
\begin{aligned}
((L_0 + W)V(t)G_0)^{(n)} &= (\mathbb{P}_N(L_0 + W)V(t)G_0)^{(n)} = (\mathbb{L}_N e^{t\mathbb{L}_N}\mathbb{G}_0)^{(n)} \\
&= (e^{t\mathbb{L}_N}\mathbb{L}_N\mathbb{G}_0)^{(n)} = (e^{t\mathbb{L}_N}\mathbb{P}_N(L_0 + W)\mathbb{I}_N\mathbb{G}_0)^{(n)} \\
&= (V(t)(L_0 + W)G_0)^{(n)},
\end{aligned}
$$

where in the last equality we applied (3.40) for $(L_0 + W)G_0$ instead of G_0. Hence, (3.39) holds. $\qquad\square$

Remark 3.7. Note that the initial value problem $\dfrac{\partial}{\partial t}k_t = \widehat{L}^* k_t$, $k_t\big|_{t=t_1} = k_{t_1} \in \mathcal{K}_{C_{t_1}}$ for some $t_1 < T = \dfrac{1}{BC_0}$, has a solution only on the time interval $[t_1, t_1 + T_1) = [t_1, T)$, since $T_1 = \dfrac{1}{BC_{t_1}} = \dfrac{1 - BC_0 t_1}{BC_0} = T - t_1$.

Remark 3.8. Using an estimate analogous to (3.37), one can show that $\dfrac{\partial G_t}{\partial t} \in \mathcal{L}_{\rho(t,C)}$ if G_0 belongs to $B_{\mathrm{bs}}(\Gamma_0)$ (or even to a larger subset of $\mathcal{L}_C$).

Thus, by Theorems 3.5 and 3.6, under conditions (3.11)–(3.14), (3.19) for the binary jumps dynamics with generator (1.2) we have the evolution of quasi-observables and the corresponding dual one. We will now show that the latter evolution generates an evolution of probability measures on Γ.

Theorem 3.9. *Let* (3.11)–(3.14) *and* (3.19) *hold. Fix a measure* $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *which has a correlation functional* $k_\mu \in \mathcal{K}_{C_0}$, $C_0 > 0$. *Consider the evolution* $k_\mu \mapsto k_t \in \mathcal{K}_{C_t}$, $t \in (0, T)$, *where* $T = 1/(c_3 + c_4)C_0$. *Then, for any* $t \in (0, T)$, *there exists a unique measure* $\mu_t \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ *such that* k_t *is the correlation functional of* μ_t.

Proof. We first recall the following definition. Let a measurable, non-negative function k on Γ_0 be such that $\int_M k(\eta)\, d\lambda(\eta) < \infty$ for any bounded $M \in \mathcal{B}(\Gamma_0)$. The function k is said to be *Lenard positive definite* if $\langle\!\langle G, k \rangle\!\rangle \geq 0$ for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ such that $KG \geq 0$. It was shown in [24] that any such k is the correlation functional of some probability measure on Γ. If, additionally, $k \in \mathcal{K}_C$ for some $C > 0$, then this measure is uniquely defined (cf. [22]) and belongs to $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ (cf. [15]). Therefore, to prove the theorem, it is enough to show that k_t is Lenard positive definite for any $t \in (0, T)$.

Since the measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ has the correlation functional k_μ, μ is locally absolutely continuous with respect to π, and for any $\Lambda \in B_{\mathrm{bs}}(\Gamma_0)$ and λ-a.a $\eta \in \Gamma(\Lambda)$

$$
\frac{d\mu^\Lambda}{d\lambda}(\eta) = \int_{\Gamma(\Lambda)} (-1)^{|\xi|} k_\mu(\eta \cup \xi) d\lambda(\xi), \tag{3.41}
$$

see [15, Proposition 4.3]. Since $k_\mu \in \mathcal{K}_{C_0}$, we have, by (3.41), (3.4), and (2.3),

$$
\frac{d\mu^\Lambda}{d\lambda}(\eta) \leq \|k_\mu\|_{\mathcal{K}_{C_0}} e^{C_0 m(\Lambda)} C_0^{|\eta|} \tag{3.42}
$$

for λ-a.a $\eta \in \Gamma(\Lambda)$.

We fix $\Lambda_0 \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and consider the projection $\mu_0 := \mu^{\Lambda_0}$ on $\Gamma(\Lambda_0)$. By (3.42), for λ-a.a. $\eta \in \Gamma(\Lambda_0)$

$$R_0(\eta) := \frac{d\mu_0}{d\lambda}(\eta) \le A_0 C_0^{|\eta|}, \tag{3.43}$$

where $A_0 := \|k_\mu\|_{\mathcal{K}_{C_0}} e^{C_0 m(\Lambda_0)}$. Clearly, μ_0 may be considered as a measure on the whole of Γ if we set R_0 equal to 0 outside of $\Gamma(\Lambda_0)$. In this case

$$\mu_0(A) = \int_{\Gamma(\Lambda_0) \cap A} R_0(\eta) d\lambda(\eta), \quad A \in \mathcal{B}(\Gamma).$$

On the other hand, R_0 extended by zero outside of $\Gamma(\Lambda_0)$ can also be regarded as a $\mathcal{B}(\Gamma_0)$ measurable function. Evidently, that in this case $0 \le R_0 \in L^1(\Gamma_0, d\lambda)$ with $\int_{\Gamma_0} R_0 d\lambda = 1$. Note that

$$k_0 := \mathbb{1}_{\Gamma(\Lambda_0)} k_\mu \in \mathcal{K}_{C_0} \tag{3.44}$$

is the correlation functional of μ_0. Here and below, $\mathbb{1}_\Delta$ stands for the indicator function of a set Δ. By [15, Proposition 4.2], for λ-a.a. $\eta \in \Gamma(\Lambda_0)$

$$k_0(\eta) = \int_{\Gamma(\Lambda_0)} R_0(\eta \cup \xi) d\lambda(\xi). \tag{3.45}$$

There exists an $N_0 \in \mathbb{N}$ such that $\int_{(\mathbb{R}^d)^{N_0}} R_0^{(N_0)} dx^{(N_0)} > 0$ (otherwise $R_0 = 0$ λ-a.e.). We set

$$r := \int_{\bigsqcup_{n=0}^{N_0} \Gamma^{(n)}} R_0(\eta) d\lambda(\eta) \in (0, 1].$$

For each $N \ge N_0$, we define

$$R_{0,N}(\eta) = \mathbb{1}_{\{|\eta| \le N\}} R_0(\eta) \left(\int_{\bigsqcup_{n=0}^{N} \Gamma^{(n)}} R_0(\eta) d\lambda(\eta) \right)^{-1}. \tag{3.46}$$

Then, clearly, $0 \le R_{0,N} \in L^1(\Gamma_0, d\lambda)$, with $\int_{\Gamma_0} R_{0,N} d\lambda = 1$. Moreover, $R_{0,N}$ has a bounded support on Γ_0. By (3.46) and (3.43) we have

$$R_{0,N}(\eta) \le r^{-1} R_0(\eta) \le r^{-1} A_0 C_0^{|\eta|} \tag{3.47}$$

for λ-a.a. $\eta \in \Gamma_0$.

We define a probability measure $\mu_{0,N} \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$, concentrated on Γ_0, by $d\mu_{0,N} = R_{0,N} d\lambda$. By [15, Proposition 4.2], the correlation functional $k_{0,N}$ of $\mu_{0,N}$ has the following representation

$$k_{0,N}(\eta) = \int_{\Gamma(\Lambda_0)} R_{0,N}(\eta \cup \xi) d\lambda(\xi) \tag{3.48}$$

for λ-a.a. $\eta \in \Gamma(\Lambda_0)$. It is evident now that $k_{0,N}$ has a bounded support on Γ_0. Moreover, by (3.45), (3.48), and the first inequality in (3.47), we get $k_{0,N} \leq \frac{1}{r}k_0 \in \mathcal{K}_{C_0}$ and

$$\|k_{0,N}\|_{\mathcal{K}_{C_0}} \leq \frac{1}{r}\|k_0\|_{\mathcal{K}_{C_0}}. \tag{3.49}$$

By the definition of a correlation functional, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$

$$\left|\langle\!\langle G, k_0\rangle\!\rangle - \langle\!\langle G, k_{0,N}\rangle\!\rangle\right| = \left|\int_{\Gamma_0}(KG)(\eta)\big(R_0(\eta) - R_{0,N}(\eta)\big)d\lambda(\eta)\right|$$

$$\leq D\int_{\Gamma(\Lambda_0)}(1 + |\eta|)^M\big|R_0(\eta) - R_{0,N}(\eta)\big|d\lambda(\eta) \tag{3.50}$$

for some $D = D(G) > 0$ and $M = M(G) \in \mathbb{N}$ (see [15, Proposition 3.1]). By (3.46), $R_{0,N}(\eta) \to R_0(\eta)$ for λ-a.a. $\eta \in \Gamma(\Lambda_0)$. Indeed, by (3.43) and (3.47),

$$\big|R_0(\eta) - R_{0,N}(\eta)\big| \leq A_0(1 + r^{-1})C_0^{|\eta|}.$$

By (2.3),

$$\int_{\Gamma(\Lambda_0)}(1 + |\eta|)^M C_0^{|\eta|}d\lambda(\eta) < \infty.$$

Therefore, by the dominated convergence theorem, (3.50) yields

$$\lim_{N\to\infty}\langle\!\langle G, k_{0,N}\rangle\!\rangle = \langle\!\langle G, k_0\rangle\!\rangle. \tag{3.51}$$

As before, we identify a function F on Γ_0 with a sequence of symmetric functions $F^{(n)}$ on $(\mathbb{R}^d)^n$, $n \in \mathbb{N}_0$. Fix any $G \in B_{\mathrm{bs}}(\Gamma_0)$ and let F be the restriction of KG to Γ_0. Then there exist $\Lambda = \Lambda_F \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, $M = M_F \in \mathbb{N}$, and $D = D_F > 0$ such that for all $\eta \in \Gamma_0$,

$$|F(\eta)| = |F(\eta \cap \Lambda)| \leq D\big(1 + |\eta \cap \Lambda|\big)^M$$

(see [15, Proposition 3.1]). In particular, $F^{(n)}$ is bounded on $(\mathbb{R}^d)^n$ for each n. We restrict the operator L given by (1.2) to functions on Γ_0. This restriction, L_0, is given by (3.7).

We define, for any $R \in L^1(\Gamma_0, \lambda)$, the function $L_0^* R$ on Γ_0 by $(L_0^* R)^{(n)} := (L_0^{(n)})^* R^{(n)}$, where $(L_0^{(n)})^*$ is given by the right hand side of (3.9) in which $c(x_i, x_j, y_1, y_2)$ is replaced by $c(y_1, y_2, x_i, x_j)$. Analogously to the proof of Proposition 3.3, we conclude that $(L_0^{(n)})^*$ is a bounded generator of a strongly continuous semigroup on $X_n = L^1\big((\mathbb{R}^d)^n, dx^{(n)}\big)$. In the dual space $X_n^* := L^\infty\big((\mathbb{R}^d)^n, dx^{(n)}\big)$, we consider the dual operator to $(L_0^{(n)})^*$, which is just the $L_0^{(n)}$ given by (3.9). It is easy to see that, under condition (3.12), $L_0^{(n)}$ is a bounded operator on X_n^*. Note that $L_0^{(n)}1 = 0$ implies

$$\int_{(\mathbb{R}^d)^n}e^{t(L_0^{(n)})^*}R^{(n)}dx^{(n)} = \int_{(\mathbb{R}^d)^n}\big(e^{tL_0^{(n)}}1\big)R^{(n)}dx^{(n)} = \int_{(\mathbb{R}^d)^n}R^{(n)}dx^{(n)}.$$

To show that $e^{t(L_0^{(n)})^*}$ preserves the cone X_n^+ of all positive functions in X_n, we write $(L_0^{(n)})^* = L_1 + L_2$, where

$$\left(L_1 R^{(n)}\right)(x^{(n)}) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c\left(y_1, y_2, x_i, x_j\right)$$
$$\times R^{(n)}\left(x_1, \ldots, \underset{\underset{i}{\wedge}}{y_1}, \ldots, \underset{\underset{j}{\wedge}}{y_2}, \ldots, x_n\right) dy_1 dy_2$$

and

$$\left(L_2 R^{(n)}\right)(x^{(n)}) = -\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_2(x_i, x_j)\right) R^{(n)}(x^{(n)}).$$

Clearly, L_1 and L_2 are bounded operators on X_n. Since $c \geq 0$, L_1 preserves the cone X_n^+, hence so does the semigroup e^{tL_1}. Since L_2 is a bounded multiplication operator, the semigroup e^{tL_2} is a positive multiplication operator in X_n. Therefore, $e^{t(L_0^{(n)})^*}$ preserves X_n^+ by the Lie–Trotter product formula.

Therefore, if we define functions $R_{t,N}$, $N \geq N_0$ (cf. (3.46)) on Γ_0 by

$$R_{t,N}^{(n)} := e^{t(L_0^{(n)})^*} R_{0,N}^{(n)},$$

then $0 \leq R_{t,N} \in L^1(\Gamma_0, d\lambda)$ with $\int_{\Gamma_0} R_{t,N} d\lambda = 1$. Note that $R_{t,N}^{(n)} \equiv 0$ for $n > N$. Therefore, we can define a measure $\tilde{\mu}_{t,N} \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$, concentrated on Γ_0, by $d\tilde{\mu}_{t,N} = R_{t,N} d\lambda$ (in fact, the measure $\tilde{\mu}_{t,N}$ is concentrated on $\bigsqcup_{n=0}^{N} \Gamma^{(n)}$). We denote by $\tilde{k}_{t,N}$ the correlation functional of $\tilde{\mu}_{t,N}$.

For each function G on Γ_0 we define $K_0 G := (KG)\!\restriction_{\Gamma_0}$. Take any $G_0 \in B_{\mathrm{bs}}(\Gamma_0)$ such that $KG_0 \geq 0$ on Γ. We denote $F_0 := K_0 G_0 \geq 0$ on Γ_0. We have, by the definition of a correlation functional,

$$\langle\!\langle G_0, k_{t,N} \rangle\!\rangle = \langle K_0 G_0, R_{t,N} \rangle \geq 0. \tag{3.52}$$

On the other hand, if we define a function $U(t)F_0$ on Γ_0 by $(U(t)F_0)^{(n)} = e^{tL_0^{(n)}} F_0^{(n)}$, we obtain

$$\langle F_0, R_{t,N} \rangle = \sum_{n=0}^{N} \frac{1}{n!} \langle F_0^{(n)}, R_{t,N}^{(n)} \rangle = \sum_{n=0}^{N} \frac{1}{n!} \langle F_0^{(n)}, e^{t(L_0^{(n)})^*} R_{0,N}^{(n)} \rangle$$

$$= \sum_{n=0}^{N} \frac{1}{n!} \langle e^{tL_0^{(n)}} F_0^{(n)}, R_{0,N}^{(n)} \rangle = \langle U(t)F_0, R_{0,N} \rangle$$

$$= \langle\!\langle K_0^{-1} U(t) K_0 G_0, k_{0,N} \rangle\!\rangle. \tag{3.53}$$

It is evident, by Proposition 3.1, that

$$(K_0^{-1} U(t) K_0 G_0)^{(n)} = e^{t(L_0^{(n)} + W^{(n)})} G_0^{(n)} = (V(t)G_0)^{(n)}, \tag{3.54}$$

where $V(t)$ is as in (3.27).

As a result, from (3.52)–(3.54), we get

$$\langle\!\langle G_0, k_{t,N} \rangle\!\rangle = \langle\!\langle G_t, k_{0,N} \rangle\!\rangle, \tag{3.55}$$

where $G_t = V(t)G_0$. By (3.44), $k_0 \in \mathcal{K}_{C_0}$. Hence, by Theorem 3.6, for any $t \in (0,T)$, there exists $\tilde{k}_t \in \mathcal{K}_{C_t}$ such that

$$\langle\!\langle G_0, \tilde{k}_t \rangle\!\rangle = \langle\!\langle G_t, k_0 \rangle\!\rangle. \tag{3.56}$$

Note that here, for a given $t \in (0,T)$, we may consider $G_0 \in B_{\mathrm{bs}}(\Gamma_0) \subset \mathcal{L}_{C_t}$, where C_t is given by (3.28). Then, by the proof of Theorem 3.6, $G_t = V(t)G_0 \in \mathcal{L}_{C_0}$. By (3.55) and (3.56), to prove that

$$\langle\!\langle G_0, \tilde{k}_t \rangle\!\rangle = \lim_{N\to\infty} \langle\!\langle G_0, k_{t,N} \rangle\!\rangle, \tag{3.57}$$

we only need to show that

$$\lim_{N\to\infty} \langle\!\langle G_t, k_{0,N} \rangle\!\rangle = \langle\!\langle G_t, k_0 \rangle\!\rangle. \tag{3.58}$$

The latter fact is a direct consequence of (3.51) if we take into account that $G_t \in \mathcal{L}_{C_0}$ and that the set $B_{\mathrm{bs}}(\Gamma_0)$ is dense in $\mathcal{L}_{C_0}$. Indeed, let us consider, for a fixed $t \in (0,T)$ and for any $\varepsilon > 0$, a function $G \in B_{\mathrm{bs}}(\Gamma_0)$ such that $\|G - G_t\|_{\mathcal{L}_{C_0}} < \varepsilon$. Then, by (3.51), there exists an $N_1 \geq N_0$, such that, for any $N \geq N_1$

$$\left| \langle\!\langle G, k_{0,N} \rangle\!\rangle - \langle\!\langle G, k_0 \rangle\!\rangle \right| < \varepsilon.$$

Therefore, by (3.49), for any $N \geq N_1$

$$
\begin{aligned}
&\left| \langle\!\langle G_t, k_{0,N} \rangle\!\rangle - \langle\!\langle G_t, k_0 \rangle\!\rangle \right| \\
&\leq \|G - G_t\|_{\mathcal{L}_{C_0}} \|k_{0,N}\|_{\mathcal{K}_{C_0}} + \left| \langle\!\langle G, k_{0,N} \rangle\!\rangle - \langle\!\langle G, k_0 \rangle\!\rangle \right| + \|G - G_t\|_{\mathcal{L}_{C_0}} \|k_0\|_{\mathcal{K}_{C_0}} \\
&< \varepsilon(r^{-1} + 1)\|k_0\|_{\mathcal{K}_{C_0}} + \varepsilon,
\end{aligned}
$$

which proves (3.58). Therefore, (3.57) holds. Hence, by (3.52), $\tilde{k}_t =: \tilde{k}_t^{\Lambda_0}$ is Lenard positive definite for any $t \in (0,T)$.

As a result, for each $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$, the evolution $k_{\mu^\Lambda} \mapsto \tilde{k}_t^\Lambda$, $t \in (0,T)$, preserves positive-definiteness and $\langle\!\langle G_0, \tilde{k}_t^\Lambda \rangle\!\rangle = \langle\!\langle G_t, k_{\mu^\Lambda} \rangle\!\rangle$. On the other hand, by Theorem 3.6, we have the evolution $k_\mu \mapsto k_t$, $t \in (0,T)$ satisfying $\langle\!\langle G_0, k_t \rangle\!\rangle = \langle\!\langle G_t, k_\mu \rangle\!\rangle$.

Since $k_{\mu^\Lambda} = \mathbb{1}_{\Gamma(\Lambda)} k_\mu$, it is evident that, for any $t \in (0,T)$, $\langle\!\langle G_t, k_{\mu^\Lambda} \rangle\!\rangle \to \langle\!\langle G_t, k_\mu \rangle\!\rangle$ as $\Lambda \nearrow \mathbb{R}^d$. Therefore,

$$\langle\!\langle G_0, k_t \rangle\!\rangle = \lim_{\Lambda\nearrow\mathbb{R}^d} \langle\!\langle G_0, \tilde{k}_t^\Lambda \rangle\!\rangle \geq 0.$$

Hence, for each $t \in (0,T)$, there exists a unique measure $\mu_t \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ whose correlation functional is k_t. $\qquad\square$

4 Vlasov-type scaling

For the reader's convenience, we start with explaining the idea of the Vlasov-type scaling. A general scheme for both birth-and-death and conservative dynamics may be found in [7]. Certain realizations of this approach were studied in [4, 6, 8, 9].

We would like to construct a scaling of the generator L, say L_ε, $\varepsilon > 0$, such that the following requirements are satisfied. Assume we have an evolution $V_\varepsilon(t)$ corresponding to the equation $\frac{\partial}{\partial t} G_{t,\varepsilon} = \widehat{L}_\varepsilon G_{t,\varepsilon}$. Assume that, in a proper functional space, we have the dual evolution $V_\varepsilon^*(t)$ with respect to the duality (2.8). Let us choose an initial function for this dual evolution with a big singularity in ε, namely, $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$ as $\varepsilon \to 0$ ($\eta \in \Gamma_0$), with some function r_0, being independent of ε. Our first requirement on the scaling $L \mapsto L_\varepsilon$ is that the evolution $V_\varepsilon^*(t)$ preserve the order of the singularity:

$$\left(V_\varepsilon^*(t) k_0^{(\varepsilon)} \right)(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta) \quad \text{as } \varepsilon \to 0, \quad \eta \in \Gamma_0, \tag{4.1}$$

where r_t is such that the dynamics $r_0 \mapsto r_t$ preserves the so-called Lebesgue–Poisson exponents. Namely, if $r_0(\eta) = e_\lambda(p_0, \eta) := \prod_{x \in \eta} p_0(x)$, then $r_t(\eta) = e_\lambda(p_t, \eta) = \prod_{x \in \eta} p_t(x)$. (Here, $\prod_{x \in \emptyset} := 1$.) Furthermore, we require that the p_t's satisfy a (nonlinear, in general) differential equation

$$\frac{\partial}{\partial t} p_t(x) = \upsilon(p_t)(x), \tag{4.2}$$

which will be called a *Vlasov-type equation*.

For any $c > 0$, we set $\left(R_c G \right)(\eta) = c^{|\eta|} G(\eta)$. Roughly speaking, (4.1) means that

$$R_\varepsilon V_\varepsilon^*(t) R_{\varepsilon^{-1}} r_0 \sim r_t.$$

This gives us a hint to consider the map $R_\varepsilon \widehat{L}_\varepsilon^* R_{\varepsilon^{-1}}$, which is dual to

$$\widehat{L}_{\varepsilon,\text{ren}} = R_{\varepsilon^{-1}} \widehat{L}_\varepsilon R_\varepsilon.$$

Remark 4.1. We expect that the limiting dynamics for $R_\varepsilon V_\varepsilon^*(t) R_{\varepsilon^{-1}}$ will preserve the Lebesgue–Poisson exponents. Note that the Lebesgue–Poisson exponent $e_\lambda(p_t)$, $t \geq 0$, is the correlation functional of the Poisson measure π_{p_t} on Γ with the non-constant intensity p_t (for a rigorous definition of such a Poisson measure, see e.g. [1]). Therefore, at least heuristically, we expect to have the limiting dynamics: $\pi_{p_0} \mapsto \pi_{p_t}$, where p_t satisfies the equation (4.2).

Below we will realize this scheme in the case of the generator L given by (1.2). We will consider, for any $\varepsilon > 0$, the scaled operator $L_\varepsilon = \varepsilon L$. Then, obviously, $\widehat{L}_\varepsilon G = \varepsilon \widehat{L} G = \varepsilon L_0 G + \varepsilon W G$, where L_0 and W are given by Proposition 3.1.

Proposition 4.2. *For any $\varepsilon > 0$*

$$\widehat{L}_{\varepsilon,\text{ren}} = \varepsilon L_0 + W. \tag{4.3}$$

Moreover, let (3.11)–(3.14), (3.19) hold. Then, for any $C > 0$, the initial value problem

$$\frac{\partial}{\partial t} G_{t,\varepsilon}(\eta) = \left(\widehat{L}_{\varepsilon,\mathrm{ren}} G_{t,\varepsilon}\right)(\eta),$$

$$G_{t,\varepsilon}\big|_{t=0} = G_{0,\varepsilon} \in \mathcal{L}_C \tag{4.4}$$

has a unique solution $G_{t,\varepsilon} \in \mathcal{L}_{\rho(t,C)}$, where $\rho(t,C)$ is given by (3.22).

Proof. By Proposition 3.1, $L_0 R_\varepsilon = R_\varepsilon L_0$ and $W R_\varepsilon = \varepsilon^{-1} R_\varepsilon W$. Therefore,

$$\widehat{L}_{\varepsilon,\mathrm{ren}} = R_{\varepsilon^{-1}} \varepsilon L_0 R_\varepsilon + R_{\varepsilon^{-1}} \varepsilon W R_\varepsilon$$
$$= \varepsilon R_{\varepsilon^{-1}} R_\varepsilon L_0 + \varepsilon^{-1} \varepsilon R_{\varepsilon^{-1}} R_\varepsilon W = \varepsilon L_0 + W.$$

By Proposition 3.3, $(\varepsilon L_0)^{(n)}$ is a generator of a contraction semigroup in X_n for any $n \geq 1$. Hence, the statement is a direct consequence of Theorem 3.5. Note that the solution of (4.4) can be found recursively:

$$G_{t,\varepsilon}^{(n)}\left(x^{(n)}\right) = \left(e^{\varepsilon t L_0^{(n)}} G_{0,\varepsilon}^{(n)}\right)\left(x^{(n)}\right)$$
$$+ \int_0^t \left(e^{\varepsilon(t-s)L_0^{(n)}} W^{(n)} G_{s,\varepsilon}^{(n-1)}\right)\left(x^{(n)}\right) ds, \ n \geq 1, \tag{4.5}$$

and $G_{t,\varepsilon}^{(0)} = G_{0,\varepsilon}^{(0)}$. $\qquad\square$

By (4.3), $\widehat{L}_{\varepsilon,\mathrm{ren}} G(\eta) \to W G(\eta)$ as $\varepsilon \to 0$ ($\eta \in \Gamma_0$). Let (3.13)–(3.14) hold. By Theorem 3.5, for any $C > 0$, the initial value problem

$$\frac{\partial}{\partial t} G_{t,V}(\eta) = (W G_{t,V})(\eta),$$

$$G_{t,V}|_{t=0} = G_{0,V} \in \mathcal{L}_C \tag{4.6}$$

has a unique solution $G_{t,V} \in \mathcal{L}_{\rho(t,C)}$, with $\rho(t,C)$ given by (3.22). This solution can be constructed recursively, namely, $G_{t,V}^{(0)} = G_{0,V}^{(0)}$ and

$$G_{t,V}^{(n)}\left(x^{(n)}\right) = G_{0,V}^{(n)}\left(x^{(n)}\right) + \int_0^t \left(W^{(n)} G_{s,V}^{(n-1)}\right)\left(x^{(n)}\right) ds, \ n \geq 1. \tag{4.7}$$

Theorem 4.3. *Suppose that conditions (3.11)–(3.14) and (3.19) hold. Let $C > 0$ and let $\{G_{0,V}, G_{0,\varepsilon}, \varepsilon > 0\} \subset \mathcal{L}_C$ be such that*

$$\left\|G_{0,\varepsilon} - G_{0,V}\right\|_{\mathcal{L}_C} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.8}$$

Then, for any $T > 0$ and any $r < \rho(T,C)$,

$$\sup_{t \in [0,T]} \left\|G_{t,\varepsilon} - G_{t,V}\right\|_{\mathcal{L}_r} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.9}$$

Proof. By (4.5) and (4.7), we have

$$\left\| G_{t,\varepsilon}^{(n)} - G_{t,V}^{(n)} \right\|_{X_n}$$

$$\leq \left\| e^{\varepsilon t L_0^{(n)}} G_{0,\varepsilon}^{(n)} - G_{0,V}^{(n)} \right\|_{X_n}$$

$$+ \int_0^t \left\| e^{\varepsilon(t-s)L_0^{(n)}} W^{(n)} G_{s,\varepsilon}^{(n-1)} - W^{(n)} G_{s,V}^{(n-1)} \right\|_{X_n} ds$$

$$\leq \left\| e^{\varepsilon t L_0^{(n)}} \big(G_{0,\varepsilon}^{(n)} - G_{0,V}^{(n)} \big) \right\|_{X_n} + \left\| \big(e^{\varepsilon t L_0^{(n)}} - 1 \big) G_{0,V}^{(n)} \right\|_{X_n}$$

$$+ \int_0^t \left\| e^{\varepsilon(t-s)L_0^{(n)}} W^{(n)} \big(G_{s,\varepsilon}^{(n-1)} - G_{s,V}^{(n-1)} \big) \right\|_{X_n} ds$$

$$+ \int_0^t \left\| \big(e^{\varepsilon(t-s)L_0^{(n)}} - 1 \big) W^{(n)} G_{s,V}^{(n-1)} \right\|_{X_n} ds$$

By Proposition 3.3, $L_0^{(n)}$ is the generator of a contraction semigroup in X_n, hence we continue

$$\leq \left\| G_{0,\varepsilon}^{(n)} - G_{0,V}^{(n)} \right\|_{X_n} + \left\| \big(e^{\varepsilon t L_0^{(n)}} - 1 \big) G_{0,V}^{(n)} \right\|_{X_n}$$

$$+ \int_0^t \left\| W^{(n)} \big(G_{s,\varepsilon}^{(n-1)} - G_{s,V}^{(n-1)} \big) \right\|_{X_n} ds$$

$$+ \int_0^t \left\| \big(e^{\varepsilon(t-s)L_0^{(n)}} - 1 \big) W^{(n)} G_{s,V}^{(n-1)} \right\|_{X_n} ds. \qquad (4.10)$$

By (3.3), for any $n \in \mathbb{N}_0$

$$\frac{C^n}{n!} \left\| G_{0,\varepsilon}^{(n)} - G_{0,V}^{(n)} \right\|_{X_n} \leq \left\| G_{0,\varepsilon} - G_{0,V} \right\|_{\mathcal{L}_C}. \qquad (4.11)$$

Hence, condition (4.8) implies that $\left\| G_{0,\varepsilon}^{(n)} - G_{0,V}^{(n)} \right\|_{X_n} \to 0$ as $\varepsilon \to 0$. Since $L_0^{(n)}$ is a bounded generator of a strongly continuous contraction semigroup on X_n and since $G_{0,V}^{(n)} \in X_n$, we get from (3.34):

$$\sup_{t \in [0,T]} \left\| \big(e^{\varepsilon t L_0^{(n)}} - 1 \big) G_{0,V}^{(n)} \right\|_{X_n} \leq \varepsilon T \left\| L_0^{(n)} G_{0,V}^{(n)} \right\|_{X_n} \to 0 \quad \text{as } \varepsilon \to 0.$$

Next, by the inclusion $W^{(n)} G_{s,V}^{(n-1)} \in X_n$, formula (3.34) also yields that, for any fixed $t > 0$ and any $s \in [0, t]$,

$$\left\| \big(e^{\varepsilon(t-s)L_0^{(n)}} - 1 \big) W^{(n)} G_{s,V}^{(n-1)} \right\|_{X_n} \leq \int_0^{\varepsilon(t-s)} \left\| L_0^{(n)} W^{(n)} G_{\tau,V}^{(n)} \right\|_{X_n} d\tau$$

By Proposition 3.3, we continue with $A := \dfrac{c_1 + c_2}{2}$ and $B := c_3 + c_4$:

$$\leq A B n^2 (n-1)^2 \int_0^{\varepsilon(t-s)} \left\| G_{\tau,V}^{(n-1)} \right\|_{X_{n-1}} d\tau$$

By (3.3), similarly to (4.11), we estimate:

$$\leq ABn^2 (n-1)^2 \int_0^{\varepsilon(t-s)} (\rho(\tau,C))^{-(n-1)} (n-1)! \|G_{\tau,V}\|_{\mathcal{L}_{\rho(\tau,C)}} d\tau$$

For $0 \leq s \leq t \leq T$ and $0 < \varepsilon < 1$, by (3.22) and (3.23), we continue:

$$\leq \varepsilon T ABn^2 (n-1)^2 (\rho(T,C))^{-(n-1)} (n-1)! \|G_{0,V}\|_{\mathcal{L}_C}.$$

Therefore,

$$\sup_{t \in [0,T]} \int_0^t \left\|\left(e^{\varepsilon(t-s)L_0^{(n)}} - 1\right)W^{(n)}G_{s,V}^{(n-1)}\right\|_{X_n} ds \to 0 \quad \text{as } \varepsilon \to 0.$$

Suppose now that

$$\sup_{t \in [0,T]} \left\|G_{t,\varepsilon}^{(n-1)} - G_{t,V}^{(n-1)}\right\|_{X_{n-1}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.12}$$

Then, by (3.20),

$$\sup_{t \in [0,T]} \int_0^t \left\|W^{(n)}\left(G_{s,\varepsilon}^{(n-1)} - G_{s,V}^{(n-1)}\right)\right\|_{X_n} ds$$

$$\leq Bn(n-1)T \sup_{t \in [0,T]} \left\|G_{t,\varepsilon}^{(n-1)} - G_{t,V}^{(n-1)}\right\|_{X_{n-1}} \to 0 \quad \text{as } \varepsilon \to 0.$$

Note that, for any $t \in [0,T]$, using (4.11), we obtain

$$\left|G_{t,\varepsilon}^{(0)} - G_{t,V}^{(0)}\right| = \left|G_{0,\varepsilon}^{(0)} - G_{0,V}^{(0)}\right| \to 0 \quad \text{as } \varepsilon \to 0.$$

As a result, by the induction principle, we conclude from (4.10) that, for any $n \in \mathbb{N}_0$,

$$\sup_{t \in [0,T]} \left\|G_{t,\varepsilon}^{(n)} - G_{t,V}^{(n)}\right\|_{X_n} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.13}$$

Let now $0 < r < \rho(T,C)$. Then

$$\sup_{t \in [0,T]} \left\|G_{t,\varepsilon} - G_{t,V}\right\|_{\mathcal{L}_r} \leq \sum_{n=0}^{\infty} \frac{r^n}{n!} \sup_{t \in [0,T]} \left\|G_{t,\varepsilon}^{(n)} - G_{t,V}^{(n)}\right\|_{X_n}. \tag{4.14}$$

Hence, by (4.13) and (4.14), to prove the theorem it suffices to show that the series (4.14) converges uniformly in ε. But the latter series may be estimated, similarly to the considerations above:

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} \sup_{t \in [0,T]} \left(\left\|G_{t,\varepsilon}^{(n)}\right\|_{X_n} + \left\|G_{t,V}^{(n)}\right\|_{X_n}\right)$$

$$\leq \sum_{n=0}^{\infty} \frac{r^n}{n!} \sup_{t \in [0,T]} \left((\rho(t,C))^{-n}n!\left\|G_{t,\varepsilon}\right\|_{\mathcal{L}_{\rho(t,C)}} + (\rho(t,C))^{-n}n!\left\|G_{t,V}\right\|_{\mathcal{L}_{\rho(t,C)}}\right)$$

$$\leq \left(\|G_{0,V}\|_{\mathcal{L}_C} + \sup_{\varepsilon>0}\|G_{0,\varepsilon}\|_{\mathcal{L}_C}\right) \sum_{n=0}^{\infty} \left(\frac{r}{\rho(T,C)}\right)^n < \infty,$$

since $G_{0,\varepsilon} \to G_{0,V}$ in $\mathcal{L}_C$ yields $\sup_{\varepsilon>0} \|G_{0,\varepsilon}\|_{\mathcal{L}_C} < \infty$. $\qquad\square$

Proposition 4.4. *Let the conditions of Theorem 3.6 hold. Let $C_0 > 0$, $T = \frac{1}{BC_0}$, and $\left\{k_{0,V}, k_{0,\varepsilon}, \varepsilon > 0\right\} \subset \mathcal{K}_{C_0}$. Then, for any $t \in (0, T)$, there exist functions $\left\{k_{t,V}, k_{t,\varepsilon}, \varepsilon > 0\right\} \subset \mathcal{K}_{C_t}$, with C_t given by (3.28), such that, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$,*

$$\frac{\partial}{\partial t}\langle\!\langle G, k_{t,\varepsilon}\rangle\!\rangle = \langle\!\langle (\varepsilon L_0 + W)G, k_{t,\varepsilon}\rangle\!\rangle, \quad \frac{\partial}{\partial t}\langle\!\langle G, k_{t,V}\rangle\!\rangle = \langle\!\langle WG, k_{t,V}\rangle\!\rangle.$$

Moreover, $\|k_{t,\varepsilon}\|_{\mathcal{K}_{C_t}} \le \|k_{0,\varepsilon}\|_{\mathcal{K}_{C_0}}$, $\|k_{t,V}\|_{\mathcal{K}_{C_t}} \le \|k_{0,V}\|_{\mathcal{K}_{C_0}}$. If, additionally,

$$\lim_{\varepsilon \to 0} \|k_{0,\varepsilon} - k_{0,V}\|_{\mathcal{K}_{C_0}} = 0, \tag{4.15}$$

then for any $T' \in (0, T)$, $r_0 > C_{T'}$, and $G_0 \in \mathcal{L}_{r_0}$,

$$\sup_{t \in [0,T']} \left|\langle\!\langle G_0, k_{t,\varepsilon} - k_{t,V}\rangle\!\rangle\right| \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.16}$$

Proof. The first part of the statement follows from Theorem 3.6. Since the function $[0, T'] \ni t \mapsto C_t$ is (strictly) increasing, we have $\left\{k_{t,V}, k_{t,\varepsilon}, \varepsilon > 0\right\} \subset \mathcal{K}_{C_{T'}} \subset \mathcal{K}_{r_0}$. Moreover, by the proof of Theorem 3.6, for any $G_0 \in \mathcal{L}_{r_0} \subset \mathcal{L}_{C_t}$,

$$\langle\!\langle G_0, k_{t,\varepsilon}\rangle\!\rangle = \langle\!\langle G_{t,\varepsilon}, k_{0,\varepsilon}\rangle\!\rangle, \quad \langle\!\langle G_0, k_{t,V}\rangle\!\rangle = \langle\!\langle G_{t,V}, k_{0,V}\rangle\!\rangle, \tag{4.17}$$

where $G_{t,\varepsilon}$ and $G_{t,V}$ are solutions to (4.4) and (4.6), respectively. Therefore, by Theorem 3.5, $\left\{G_{t,\varepsilon}, G_{t,V}\right\} \subset \mathcal{L}_{\rho(t,r_0)}$, where the function ρ is given by (3.22). Since ρ is (strictly) increasing in the second variable and since $r_0 > C_{T'}$, we have

$$\rho(T', r_0) > \rho(T', C_{T'}) = C_0.$$

Applying Theorem 4.3 with $C = r_0$, $r = C_0$, $T = T'$, and $G_{0,\varepsilon} = G_{0,V} = G_0$ we obtain

$$\sup_{t \in [0,T']} \left\|G_{t,\varepsilon} - G_{t,V}\right\|_{\mathcal{L}_{C_0}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.18}$$

Then, by (4.17),

$$\sup_{t \in [0,T']} \left|\langle\!\langle G_0, k_{t,\varepsilon} - k_{t,V}\rangle\!\rangle\right| = \sup_{t \in [0,T']} \left|\langle\!\langle G_{t,\varepsilon}, k_{0,\varepsilon}\rangle\!\rangle - \langle\!\langle G_{t,V}, k_{0,V}\rangle\!\rangle\right|$$

$$\le \sup_{t \in [0,T']} \left|\langle\!\langle G_{t,\varepsilon} - G_{t,V}, k_{0,\varepsilon}\rangle\!\rangle\right| + \sup_{t \in [0,T']} \left|\langle\!\langle G_{t,V}, k_{0,\varepsilon} - k_{0,V}\rangle\!\rangle\right|$$

$$\le \sup_{t \in [0,T']} \|G_{t,\varepsilon} - G_{t,V}\|_{\mathcal{L}_{C_0}} \|k_{0,\varepsilon}\|_{\mathcal{K}_{C_0}}$$

$$+ \sup_{t \in [0,T']} \|G_{t,V}\|_{\mathcal{L}_{C_0}} \|k_{0,\varepsilon} - k_{0,V}\|_{\mathcal{K}_{C_0}}. \tag{4.19}$$

Since $k_{0,\varepsilon} \to k_{0,V}$ in $\mathcal{K}_{C_0}$, we get $\sup_{\varepsilon > 0} \|k_{0,\varepsilon}\|_{\mathcal{K}_{C_0}} < \infty$. Then, by (4.18), the first summand in (4.19) converges to 0 as $\varepsilon \to 0$. Next, by (3.23),

$$\|G_{t,V}\|_{\mathcal{L}_{C_0}} = \|G_{t,V}\|_{\mathcal{L}_{\rho(t,C_t)}} \le \|G_0\|_{\mathcal{L}_{C_t}} \le \|G_0\|_{\mathcal{L}_{C_{T'}}} \le \|G_0\|_{\mathcal{L}_{r_0}}.$$

Hence, by (4.15), the second summand in (4.19) also converges to 0 as $\varepsilon \to 0$, which proves the second part of the statement. $\qquad \square$

We will now show that the evolution $k_{0,V} \mapsto k_{t,V}$ satisfies our second requirement on the initial scaling $L \mapsto L_\varepsilon$, namely, $e_\lambda(p_0, \eta) \mapsto e_\lambda(p_t, \eta)$.

Proposition 4.5. *Let the conditions of Theorem 3.6 hold. Then for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ and any $k \in \mathcal{K}_C$ with $C > 0$,*

$$\int_{\Gamma_0} (WG)(\eta) \, k(\eta) \, d\lambda(\eta) = \int_{\Gamma_0} G(\eta) (W^* k)(\eta) \, d\lambda(\eta), \qquad (4.20)$$

where

$$(W^* k)(\eta) = \sum_{y_1 \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{c}(x_1, x_2, y_1) \, k(\eta \cup x_2 \cup x_1 \setminus y_1) \, dx_2 dx_1$$

$$- \sum_{x_1 \in \eta} \int_{\mathbb{R}^d} a_1(x_1, x_2) k(\eta \cup x_2) \, dx_2. \qquad (4.21)$$

Here the functions $\tilde{c}$ and a_1 are defined by (3.5) and (3.15), respectively.

Proof. First, we note that, under conditions (3.11)–(3.14), for $G \in B_{\mathrm{bs}}(\Gamma_0)$ and $k \in \mathcal{K}_C$, both integrals in (4.20) are well defined. Then, using (3.8) and e.g. [7, Lemma 1], we have

$$\int_{\Gamma_0} (WG)(\eta) \, k(\eta) \, d\lambda(\eta)$$

$$= \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x_1 \in \eta} \int_{\mathbb{R}^d} \tilde{c}(x_1, x_2, y_1) \, G(\eta \setminus x_1 \cup y_1) \, dy_1 k(\eta \cup x_2) \, dx_2 d\lambda(\eta)$$

$$- \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x_1 \in \eta} \int_{\mathbb{R}^d} \tilde{c}(x_1, x_2, y_1) \, G(\eta) \, dy_1 k(\eta \cup x_2) \, dx_2 d\lambda(\eta)$$

$$= \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{y_1 \in \eta} \tilde{c}(x_1, x_2, y_1) \, G(\eta) \, k(\eta \cup x_2 \cup x_1 \setminus y_1) \, dx_2 dx_1 d\lambda(\eta)$$

$$- \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{x_1 \in \eta} \int_{\mathbb{R}^d} \tilde{c}(x_1, x_2, y_1) \, G(\eta) \, dy_1 k(\eta \cup x_2) \, dx_2 d\lambda(\eta),$$

which proves the statement. $\qquad \square$

Thus, for any $p_t \in L^\infty(\mathbb{R}^d)$,

$$(W^* e_\lambda(p_t))(\eta) = \sum_{y \in \eta} e_\lambda(p_t, \eta \setminus y) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{c}(x_1, x_2, y) \, p_t(x_1) p_t(x_2) dx_2 dx_1$$

$$- \sum_{y \in \eta} e_\lambda(p_t, \eta \setminus y) p_t(y) \int_{\mathbb{R}^d} a_1(y, x_2) p_t(x_2) dx_2.$$

On the other hand, if $\frac{d}{dt} p_t$ exists, then

$$\frac{\partial}{\partial t} e_\lambda(p_t, \eta) = \sum_{y \in \eta} e_\lambda(p_t, \eta \setminus y) \frac{\partial}{\partial t} p_t(y).$$

Therefore, there exists a (point-wise) solution $k_t = e_\lambda(p_t, \eta)$ of the initial value problem

$$\frac{\partial k_t}{\partial t} = W^* k_t, \qquad k_t \big|_{t=0} = e_\lambda(p_0, \eta), \tag{4.22}$$

provided p_t satisfies the non-linear Vlasov-type equation

$$\frac{\partial}{\partial t} p_t(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{c}(y_1, y_2, x) p_t(y_1) p_t(y_2) \, dy_1 dy_2$$
$$- p_t(x) \int_{\mathbb{R}^d} a_1(x, x_2) p_t(x_2) \, dx_2. \tag{4.23}$$

If the symmetry condition (1.4) holds, then we may rewrite (4.23) in the Boltzmann-type form

$$\frac{\partial}{\partial t} p_t(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(x, x_2, y_1, y_2)$$
$$\times [p_t(y_1) p_t(y_2) - p_t(x) p_t(x_2)] \, dy_1 dy_2 dx_2. \tag{4.24}$$

We are interested in positive bounded solutions of (4.23).

Proposition 4.6. *Let $C > 0$ and let $0 \le p_0 \in L^\infty(\mathbb{R}^d)$ with $\|p_0\|_{L^\infty(\mathbb{R}^d)} \le C$. Assume that (3.11) and (3.14) hold and, moreover,*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(y, u_1, x, u_2) du_1 du_2 \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(x, y, u_1, u_2) du_1 du_2. \tag{4.25}$$

Then, for any $T > 0$, there exists a function $0 \le p_t \in L^\infty(\mathbb{R}^d)$, $t \in [0, T]$, which solves (4.23) and, moreover,

$$\max_{t \in [0,T]} \|p_t\|_{L^\infty(\mathbb{R}^d)} \le C. \tag{4.26}$$

This functions is a unique non-negative solution to (4.23) which satisfies (4.26).

Proof. Let us fix an arbitrary $T > 0$ and define the Banach space $X_T := C([0, T], L^\infty(\mathbb{R}^d))$ of all continuous functions on $[0, T]$ with values in $L^\infty(\mathbb{R}^d)$; the norm on X_T is given by

$$\|u\|_T := \max_{t \in [0,T]} \|u_t\|_{L^\infty(\mathbb{R}^d)}.$$

We denote by X_T^+ the cone of all nonnegative functions from X_T. For a given $C > 0$, denote by $B_{T,C}^+$ the set of all functions u from X_T^+ with $\|u\|_T \le C$.

Let Φ be a mapping which assigns to any $v \in X_T$ the solution u_t of the linear Cauchy problem

$$\frac{\partial}{\partial t} u_t(x) = - u_t(x) \int_{\mathbb{R}^d} a_1(x, y) v_t(y) \, dy$$
$$+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{c}(y_1, y_2, x) v_t(y_1) v_t(y_2) \, dy_1 dy_2, \tag{4.27}$$
$$u_t(x) \Big|_{t=0} = p_0(x),$$

namely,

$$(\Phi v)_t(x) = \exp\left\{-\int_0^t \int_{\mathbb{R}^d} a_1(x,y)v_s(y)\,dyds\right\}p_0(x)$$

$$+ \int_0^t \exp\left\{-\int_s^t \int_{\mathbb{R}^d} a_1(x,y)v_\tau(y)\,dyd\tau\right\}$$

$$\times \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{c}(y_1,y_2,x)v_s(y_1)\,v_s(y_2)\,dy_1dy_2ds. \qquad (4.28)$$

Clearly, $v_t \geq 0$ implies $(\Phi v)_t \geq 0$. Moreover, $v \in X_T^+$ yields

$$\left|(\Phi v_t)(x)\right| \leq |p_0(x)| + c_4 T \|v\|_T^2.$$

Therefore, $\Phi : X_T^+ \to X_T^+$. Obviously, u_t solves (4.23) if and only if u is a fixed point of the map Φ.

Since $0 \leq p_0(x) \leq C$ for a.a. $x \in \mathbb{R}^d$, we get from (4.28) and (4.25), for any $v \in B_{T,C}^+$,

$$0 \leq (\Phi v)_t(x) \leq C\exp\left\{-\int_0^t \int_{\mathbb{R}^d} a_1(x,y)v_s(y)\,dyds\right\}$$

$$+ C\int_0^t \exp\left\{-\int_s^t \int_{\mathbb{R}^d} a_1(x,y)v_\tau(y)\,dyd\tau\right\}\int_{\mathbb{R}^d} a_1(x,y)v_s(y)\,dyds$$

$$\leq C\exp\left\{-\int_0^t \int_{\mathbb{R}^d} a_1(x,y)v_s(y)\,dyds\right\}$$

$$+ C\int_0^t \frac{\partial}{\partial s}\exp\left\{-\int_s^t \int_{\mathbb{R}^d} a_1(x,y)v_\tau(y)\,dyd\tau\right\}ds = C.$$

Therefore, $\Phi : B_{T,C}^+ \to B_{T,C}^+$.

Let us choose any $\Upsilon \in (0,T)$ such that $2C(c_3 + c_4)\Upsilon < 1$. Clearly, $v \in B_{T,C}^+$ implies $v \in B_{\Upsilon,C}^+$. By the proved above, $\Phi : B_{\Upsilon,C}^+ \to B_{\Upsilon,C}^+$. Note the elementary inequalities $|e^{-a} - e^{-b}| \leq |a - b|$ and

$$|pe^{-a} - qe^{-b}| \leq e^{-a}|p - q| + qe^{-b}|e^{-(a-b)} - 1| \leq e^{-a}|p - q| + qe^{-b}|a - b|,$$

which hold for any $a,b,p,q \geq 0$. Hence, for any $v,w \in B_{\Upsilon,C}^+$, $t \in [0,\Upsilon]$, we get

$$\left|(\Phi v)_t(x) - (\Phi w)_t(x)\right|$$

$$\leq \left|\int_0^t \int_{\mathbb{R}^d} a_1(x,y)v_s(y)\,dyds - \int_0^t \int_{\mathbb{R}^d} a_1(x,y)w_s(y)\,dyds\right|p_0(x)$$

$$+ \int_0^t \exp\left\{-\int_s^t \int_{\mathbb{R}^d} a_1(x,y)v_\tau(y)\,dyd\tau\right\}$$

$$\times \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{c}(y_1,y_2,x)v_s(y_1)\,v_s(y_2)\,dy_1dy_2\right.$$

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{c}(y_1, y_2, x) w_s(y_1) w_s(y_2)\, dy_1 dy_2 \bigg| ds$$

$$+ \int_0^t \exp\bigg\{ - \int_s^t \int_{\mathbb{R}^d} a_1(x, y) w_\tau(y)\, dy d\tau \bigg\}$$

$$\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{c}(y_1, y_2, x) w_s(y_1) w_s(y_2)\, dy_1 dy_2$$

$$\times \bigg| \int_s^t \int_{\mathbb{R}^d} a_1(x, y) v_\tau(y)\, dy d\tau - \int_s^t \int_{\mathbb{R}^d} a_1(x, y) w_\tau(y)\, dy d\tau \bigg| ds$$

$$=: I_1 + I_2 + I_3.$$

By (3.11), (3.14), and (4.25), we have

$$I_1 \le C c_3 \|v - w\|_\Upsilon \Upsilon,$$

$$I_3 \le C c_3 \|v - w\|_\Upsilon t \int_0^t \frac{\partial}{\partial s} \exp\bigg\{ - \int_s^t \int_{\mathbb{R}^d} a_1(x, y) w_\tau(y)\, dy d\tau \bigg\} ds$$

$$\le C c_3 \|v - w\|_\Upsilon \Upsilon,$$

$$I_2 \le 2 C c_4 \|v - w\|_\Upsilon \Upsilon.$$

Therefore,

$$\|\Phi v - \Phi w\|_\Upsilon \le 2C(c_3 + c_4) \Upsilon \|v - w\|_\Upsilon$$

for any $v, w \in B_{\Upsilon, C}^+$. Since $B_{\Upsilon, C}^+$ is a metric space (with the metric induced by the norm $\| \cdot \|_\Upsilon$) and since $2C(c_3 + c_4)\Upsilon < 1$, there exists a unique $u^* \in B_{\Upsilon, C}^+$ such that $u^* = \Phi(u^*)$. Hence, u_t^* solves (4.23) for $t \in [0, \Upsilon]$. Since $0 \le u_\Upsilon^*(x) \le C$ for a.a. $x \in \mathbb{R}^d$, we can consider the equation (4.23) with the initial value $p_t(x)\big|_{t=\Upsilon} = u_\Upsilon^*(x)$. Then we obtain a unique non-negative solution which satisfies $\max_{t \in [\Upsilon, 2\Upsilon]} \|u_t\|_{L^\infty(\mathbb{R}^d)} \le C$, and so on. As a result, we obtain a solution of (4.23) on $[0, T]$. The uniqueness among all solutions from $B_{T,C}^+$ is now obvious. $\qquad\square$

Remark 4.7. Note that, in the proof of Proposition 4.6, we did not use the property (1.3). On the other hand, all considerations remain true if, instead of (4.25), we assume that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(u_1, y, x, u_2)\, du_1 du_2 \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c(x, y, u_1, u_2)\, du_1 du_2. \qquad (4.29)$$

Suppose we have an expansion

$$c(\{x_1, x_2\}, \{y_1, y_2\}) = c'(x_1, x_2, y_1, y_2) + c''(x_1, x_2, y_1, y_2) \qquad (4.30)$$

where c', c'' are functions which satisfy conditions (4.25) and (4.29), respectively, as well as conditions (3.11) and (3.14) (but do not necessarily satisfy (1.3)). It is easy to see that all considerations in the proof of Proposition 4.6 remain true for this c. Let us give a quite natural example of functions c', c''.

Example 4.8 (cf. [10])**.** Let c be given by (4.30) with

$$c'(x_1, x_2, y_1, y_2) = \varkappa a(x_1 - y_1)a(x_2 - y_2)\big[b(x_1 - x_2) + b(y_1 - y_2)\big].$$

and

$$c''(x_1, x_2, y_1, y_2) = c'(x_2, x_1, y_1, y_2).$$

Here $\varkappa > 0$, $0 \le a, b \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\|a\|_{L^1(\mathbb{R}^d)} = \|b\|_{L^1(\mathbb{R}^d)} = 1$ and b is an even function. Then the condition (4.29) for c'' coincides with the condition (4.25) for c'. Note also that (3.11)–(3.14) are satisfied for c. For example, $c_4 = 4\varkappa$ and

$$c_1 \le 2\varkappa \|b\|_{L^\infty(\mathbb{R}^d)} + 2\varkappa \|a\|_{L^\infty(\mathbb{R}^d)}\|b\|_{L^\infty(\mathbb{R}^d)} < \infty.$$

Next, let us check whether (4.25) holds for c'. We have

$$\varkappa \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c'(y, u_1, x, u_2) du_1 du_2$$
$$= \varkappa \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(y - x)a(u_1 - u_2)\big(b(y - u_1) + b(x - u_2)\big) du_1 du_2 = 2\varkappa a(y - x)$$

and

$$\varkappa \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} c'(x, y, u_1, u_2) du_1 du_2$$
$$= \varkappa \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - u_1)a(y - u_2)\big(b(x - y) + b(u_1 - u_2)\big) du_1 du_2$$
$$= \varkappa b(x - y) + \varkappa \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - u_1)a(y - u_2)b(u_1 - u_2) du_1 du_2.$$

Since b is even, (4.25) holds if, for example, $2a(x) \le b(x)$, $x \in \mathbb{R}^d$.

Acknowledgements

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