On the existence of Evans potentials

WOLFHARD HANSEN and IVAN NETUKA *

Abstract

It is shown that, for every noncompact parabolic Riemannian manifold X and every nonpolar compact K in X, there exists a positive harmonic function on $X \setminus K$ which tends to ∞ at infinity. (This is trivial for $\mathbb{R}$, easy for $\mathbb{R}^2$, and known for parabolic Riemann surfaces.) In fact, the statement is proven, more generally, for any noncompact connected Brelot harmonic space X, where constants are the only positive superharmonic functions and, for every nonpolar compact set K, there is a *symmetric* (positive) Green function for $X \setminus K$. This includes the case of parabolic Riemannian manifolds. Without symmetry, however, the statement may fail. This is shown by an example, where the underlying space is a graph (the union of the parallel half-lines $[0,\infty) \times \{0\}$, $[0,\infty) \times \{1\}$, and the line segments $\{n\} \times [0,1]$, $n = 0, 1, 2, \dots$).

1 Introduction

In classical potential theory on $\mathbb{R}^d$, $d \geq 1$, constant functions are harmonic. If $d = 1$ or $d = 2$, sometimes restrictions on open subsets are necessary, since – in contrast to $d \geq 3$ – constants are the *only* positive superharmonic functions on $\mathbb{R}^d$, and hence $\mathbb{R}$ and $\mathbb{R}^2$ are not $\mathcal{P}$-harmonic ($\mathcal{P}$-harmonicity requires the existence of a strictly positive potential, that is, a superharmonic function $p > 0$ for which the constant function 0 is the only positive harmonic minorant). On the other hand, for every $x_0 \in \mathbb{R}$, the function $h\colon x \mapsto |x - x_0|$ is harmonic on $\mathbb{R} \setminus \{x_0\}$ and, for all $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$, the function $h\colon x \mapsto \log(|x - x_0|/\varepsilon)$ is harmonic on $\{x \in \mathbb{R}^2 \colon |x - x_0| > \varepsilon\}$; and in both cases the functions are strictly positive and satisfy $\lim_{x\to\infty} h(x) = \infty$. Such harmonic functions outside a compact set which tend to ∞ at infinity and which, for historical reasons, are called *Evans potentials*, cannot exist for $\mathbb{R}^d$, $d \geq 3$, because of transience.

The existence of Evans potentials for Riemann surfaces which are not $\mathcal{P}$-harmonic (here we avoid the term *parabolic*, since it has a different meaning in potential theory) has been shown in [21, 22]. Surprisingly, to the best of our knowledge, the case of higher dimensional manifolds has not been treated explicitly until now. It is natural to ask, if, more generally, Evans potentials exist for arbitrary Brelot harmonic spaces[1], where constants are harmonic, but no strictly positive potential exists.

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[1]The definition of Brelot harmonic spaces, a concept unifying the theory of solutions to large classes of second order elliptic equations (see, for example, [3, 4, 14, 15]), is recalled in Section 2.

In this paper we shall see that, even in the case of Brelot spaces, Evans potentials exist provided there is a corresponding *symmetric* Green function outside some compact set.

THEOREM 1.1. *Let $(X, \mathcal{H})$ be a noncompact connected Brelot space such that constants are harmonic and X is not $\mathcal{P}$-harmonic. Assume that K is a compact in X such that $X \setminus K$ admits a symmetric Green function.*

Then there exists a harmonic function $h \geq 0$ on $X \setminus K$ tending to ∞ at infinity.

Rightly assuming that the proof given in [22] for Riemann surfaces can be carried over to higher dimensional manifolds, the following corollary has been stated as Theorem 5.5 in [13]. Let us note that parabolic Riemannian manifolds exist in profusion and can be characterized in various different ways (see, for instance, the survey [13], the recent publication [10], and the many references in both papers).

COROLLARY 1.2. *Let X be a noncompact parabolic Riemannian manifold. Then, for every nonpolar compact K in X, there exists a positive harmonic function on $X \setminus K$ which tends to ∞ at infinity.*

However, the statement of Theorem 1.1 may fail without symmetry of the Green function, as the following result shows (see also Remark 4.9 for an example given by an elliptic operator on $\mathbb{R}^d$, $d \geq 2$):

THEOREM 1.3. *Let X be the graph obtained from the set $\{0, 1, 2, 3, \dots\} \times \{0, 1\}$ of vertices adding edges between any two points having Euclidean distance* 1. *Then there exists a sheaf $\mathcal{H}$ on X having the following properties:*

- *$(X, \mathcal{H})$ is a Brelot space which is not $\mathcal{P}$-harmonic.*
- *Constant functions are harmonic.*
- *For every $x \in X$, there exists a Green function on $X \setminus \{x\}$.*
- *There are no compact K in X and $h \in \mathcal{H}^+(X \setminus K)$ such that $\lim_{x \to \infty} h(x) = \infty$.*

In Section 2 we shall discuss some properties of Brelot spaces which are not $\mathcal{P}$-harmonic. These properties will be used in Section 3, where we prove Theorem 1.1, and in Section 4 which is devoted to the counterexample. Finally, we include an Appendix on symmetric Green functions for open subsets and on topological properties of locally connected spaces.

2 Harmonic spaces without positive potentials

Let $(X, \mathcal{H})$ be a *Brelot harmonic space*, that is, X is a locally connected, locally compact space with countable base and $\mathcal{H}$ is a sheaf of vector spaces of continuous real functions such that the following holds. The sets which are regular for the Dirichlet problem (the definition recalled below) form a base of X and $\mathcal{H}$ has the Brelot convergence property: If $U \subset X$ is a connected open set in X and $h_n \in \mathcal{H}(U)$, $h_n \uparrow h$, $h(x) < \infty$ for some $x \in U$, then $h \in \mathcal{H}(U)$.

In this section, we assume that X is connected and noncompact, that constants are harmonic and the constant 0 is the only potential on X.

Let us collect a few simple facts on such harmonic spaces (which partly can be found elsewhere; see, for example, [6, 2, 9, 1]). For the convenience of the reader we include some proofs. For every open set U in X, let $\mathcal{S}(U)$ denote the set of all superharmonic functions on U.

LEMMA 2.1. $\mathcal{S}^+(X) = \mathcal{H}^+(X) = \mathbb{R}^+$.

Proof. Since every positive superharmonic function is the sum of a potential and a positive harmonic function, it is clear that $\mathcal{S}^+(X) = \mathcal{H}^+(X)$. Let $h \in \mathcal{H}^+(X)$ and $x, y \in X$ such that $h(x) \le h(y) =: c$. Then $s := h \wedge c \in \mathcal{S}^+(X) = \mathcal{H}^+(X)$ and hence $g := c - s \in \mathcal{H}^+(X)$. Since $g(y) = 0$, we obtain that $g = 0$. In particular, $h(x) = s(x) = c = h(y)$. □

By [8, Theorem 7], every open set U in X such that $X \setminus U$ is nonpolar is $\mathcal{P}$-harmonic. In particular, every relatively compact open set in X is $\mathcal{P}$-harmonic.

For the moment, let us fix a relatively compact open set U in X and recall a few well known notions and facts. If $s \in \mathcal{S}(U)$ such that $\liminf_{x \to z} s(x) \ge 0$, for all $x \in \partial U$, then $s \ge 0$ (*minimum principle*). Given a bounded Borel measurable function f on X, we may define a function $H_U f$ on X by $H_U f(x) := f(x)$, if $x \in X \setminus U$, and, for $x \in U$,

$$H_U f(x) := \inf\{u(x)\colon u \in \mathcal{S}(U),\ \liminf_{y \to z} u(y) \ge f(z) \text{ for every } z \in \partial U\}.$$

By the minimum principle, $H_U 1 = 1$ (consider $s := u - 1$) and $H_U f \ge 0$, if $f \ge 0$. Moreover, the functions $H_U f$ are harmonic on U and the mapping $f \mapsto H_U f$ is linear. A point $z \in \partial U$ is called *regular* (for U), if $\lim_{x \to z} H_U f(x) = f(z)$ provided f is continuous. This regularity is characterized by *thinness* of $X \setminus U$ at z. All that matters for us here is that thinness of a set at a point is a local property. The set U is called *regular* (for the Dirichlet problem), if every point in ∂U is a regular point. By [17, Proposition 7.1], for every neighborhood V of a compact set L in X, there exists a regular set U such that $L \subset U \subset V$.

We fix a compact set $K \ne \emptyset$ in X and a sequence (U_n) of regular open neighborhoods of K such that $\overline{U}_n \subset U_{n+1}$ for every $n \in \mathbb{N}$ and $X = \bigcup_{n \in \mathbb{N}} U_n$. Replacing K by a slightly larger set, we may and shall assume that K is not thin at any of its boundary points. (Indeed, given an open neighborhood V of K in U_1, there is a regular open set W such that $\overline{U}_1 \setminus V \subset W \subset X \setminus K$. Then $\tilde{K} := U_1 \setminus W$ is compact, $K \subset \tilde{K} \subset V$ and $U_1 \setminus \tilde{K} = U_1 \cap W$ is regular.) Then all sets $U_n \setminus K$, $n \in \mathbb{N}$, are regular. We define

$$S_n := \partial U_n, \qquad n \in \mathbb{N}.$$

LEMMA 2.2. *For every compact set L containing K and every $x \in X \setminus L$,*

$$\lim_{n \to \infty} H_{U_n \setminus L} 1_L(x) = 1 \quad \textit{and} \quad \lim_{n \to \infty} H_{U_n \setminus L} 1_{S_n}(x) = 0.$$

Proof. Let $n \in \mathbb{N}$ such that $L \subset U_n$. By the minimum principle, $s_n := H_{U_n \setminus K} 1_K \le H_{U_n \setminus L} 1_L$. Since s_n is continuous on X, $s_n \le 1$, and s_n is harmonic on $U_n \setminus K$, we obtain that s_n is superharmonic on U_n. Clearly, the sequence (s_n) is increasing, by the minimum principle. Therefore $s := \lim_{n\to\infty} s_n$ is superharmonic on X, and hence $s = 1$, by Lemma 2.1. The proof is finished, since $H_{U_n\setminus L}1_L + H_{U_n\setminus L}1_{S_n} = 1$ on $U_n \setminus L$. □

PROPOSITION 2.3. *Let L be a compact set in X such that $K \subset L$, $c > 0$, and $s \in \mathcal{S}^+(X\setminus L)$ such that $\liminf_{x\to z} s(x) \geq c$, for every $z \in \partial L$. Then $s \geq c$ on $X \setminus L$.*

Proof. Let $n \in \mathbb{N}$ such that $L \subset U_n$ and define

$$u_n := s - cH_{U_n\setminus L}1_L.$$

Then $u_n \in \mathcal{H}^+(U_n \setminus L)$ and $\liminf_{x\to z} u_n(x) \geq 0$ for every $z \in \partial(U_n \setminus L)$. Hence, by the minimum principle, $u_n \geq 0$ on $U_n \setminus L$. The proof is finished by Lemma 2.2. □

It is fairly easy to construct a superharmonic function $s \geq 0$ on $X \setminus K$ such that $\lim_{x\to\infty} s(x) = \infty$ or a harmonic function $h \geq 0$ on $X \setminus K$ which is unbounded. To that end we define

$$s_n := H_{U_n\setminus K}1_{X\setminus U_n}, \qquad n \in \mathbb{N}.$$

Then, for every $n \in \mathbb{N}$, $0 \leq s_n \leq 1$, s_n is continuous on X, $s_n = 0$ on K, and s_n is harmonic on $U_n \setminus K$. Moreover, $s_n = 1$ on $X \setminus U_n$, and hence s_n is superharmonic on $X \setminus K$. By the minimum principle, the sequence (s_n) is decreasing.

PROPOSITION 2.4. *There exists a continuous real function $s \geq 0$ on X such that $s = 0$ on K, s is superharmonic on $X \setminus K$, and $\lim_{x\to\infty} s(x) = \infty$.*

REMARKS 2.5. 1. Essentially, this result could be obtained combining [1, Corollaire 1.6] with Lemma 5.5.

2. Let us note that, conversely, for an arbitrary noncompact connected Brelot space Y, where constants are harmonic, the existence of a compact K in Y and a continuous superharmonic function $s \geq 0$ on $Y \setminus K$ which tends to ∞ at infinity implies that there is no strictly positive potential on Y. Indeed, suppose that there exists a potential $p > 0$ on Y. We may suppose $p \geq 1$ on K and we choose relatively compact open neighborhoods U_n of K, $n \in \mathbb{N}$, such that $s \geq n$ on $Y \setminus U_n$. Clearly, for every $n \in \mathbb{N}$, $f_n := H_{U_n\setminus K}1_{Y\setminus U_n} \leq s/n$ on $Y \setminus K$ and, by the minimum principle, $p+f_n \geq 1$ on $Y\setminus K$. Therefore $p \geq 1$ on $Y\setminus K$, that is, $p \geq 1$, which is a contradiction, since constants are harmonic.

For Riemannian manifolds, this is shown in [16] (cf. also [13, Corollary 5.4]).

Proof of Proposition 2.4. By Lemma 2.2, $s_n \downarrow 0$ locally uniformly as $n \to \infty$. Thus, for every $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that $s_{m_n} \leq 2^{-n}$ on $\overline{U}_n$. It now suffices to take

$$s := \sum_{n=1}^{\infty} s_{m_n}.$$

Indeed, by our choice of m_n, the series is locally uniformly convergent. Therefore s is a continuous real function on X, $s = 0$ on K, and s is superharmonic on $X\setminus K$. And, of course, for every $n \in \mathbb{N}$, $s \geq n$ on $X\setminus(U_{m_1}\cup\cdots\cup U_{m_n})$. So $\lim_{x\to\infty} s(x) = \infty$. □

PROPOSITION 2.6. *Let h be a positive continuous function on X such that $h = 0$ on K and h is harmonic on $X \setminus K$. Then $h = 0$ or h is unbounded.*

Proof. Let us suppose that h is bounded by some constant c. By the minimum principle, $h \leq cs_n$ on $U_n \setminus K$. Since $\lim_{n\to\infty} s_n = 0$ on X, we see that $h = 0$. □

The following result is a version of [1, Théorème 1.17].

COROLLARY 2.7. *Let V be a connected component of $X \setminus K$ which is not relatively compact. Then there exists an unbounded continuous function $h \geq 0$ on X such that $h = 0$ on $X \setminus V$ and h is harmonic on V (hence harmonic on $X \setminus K$).*

Proof. Let us fix $x_0 \in V$. Taking $Y := V$ we choose W_m, $m \in \mathbb{N}$, according to Lemma 5.4 such that $x_0 \in W_1$. Since $V \cap U_n \uparrow V$ as $n \to \infty$, we see that for every $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that the compact set $\overline{W}_m$ is contained in $V \cap U_{n_m}$. For every $n \in \mathbb{N}$, let V_n denote the connected component of $V \cap U_n$ containing x_0. Then obviously $W_m \subset V_{n_m}$, for every $m \in \mathbb{N}$. Therefore $V_n \uparrow V$ as $n \to \infty$.

Since the functions s_n, $n \in \mathbb{N}$, are harmonic on V_n, there is a subsequence (h_n) of $(s_n/s_n(x_0))$ which converges locally uniformly on V to a function $h \in \mathcal{H}^+(V)$. Of course, $h(x_0) = 1$, and hence $h > 0$ on V. We extend h to a function on X defining $h(x) := 0$, if $x \in X \setminus V$.

Clearly, $\partial(V \cap U_1) \subset K \cup (S_1 \cap V)$, where $S_1 \cap V$ is a compact set in V, since S_1 is covered by the (pairwise disjoint) connected components of $X \setminus K$. For all $x \in V \cap U_1$,

$$s_n(x) = H_{U_1 \setminus K} s_n(x) = H_{V \cap U_1} s_n(x).$$

Since the functions s_n vanish on K, we conclude that

$$h(x) = H_{U_1 \setminus K} h(x), \qquad x \in V \cap U_1.$$

Therefore $\lim_{x \to z} h(x) = 0$ for all $z \in \partial K$. So h is continuous on X. By Proposition 2.6, h is unbounded. □

3 Symmetric Green functions and Evans potentials

Let $(X, \mathcal{H})$ be as in the previous section. In addition, we suppose the existence of a compact K in X such that there is a *symmetric* Green function G on $U := X \setminus K$ (see Appendix 5.1). By our previous considerations, we may assume that K is not thin at any of its points (a symmetric Green function on an open set induces a symmetric Green function on any open subset; see Proposition 5.3). We stress that we do not suppose that U is connected so that, for each $z \in U$, the set $\{G(\cdot, z) > 0\}$ is the connected component of U containing z (see Remark 5.1.1).[2]

We do know, however, that outside of any neighborhood of y the potential $G(\cdot, y)$ is bounded (by the minimum principle for arbitrary open sets in $\mathcal{P}$-harmonic spaces; see [2, Korollar 2.4.3]).

Finally, let (U_n) be again an exhaustion of X by regular neighborhoods of K and $S_n := \partial U_n$, $n \in \mathbb{N}$. Our aim is the following result.

THEOREM 3.1. *There exists $h \in \mathcal{H}^+(U)$ such that* $\lim_{x \to \infty} h(x) = \infty$.

[2] We note, however, that we might assume that the components of U are not relatively compact in X, since we may replace K by the union $\hat{K}$ of K and all relatively compact components of U; see Lemma 5.5.

The arguments in [22] yield a proof for Riemann surfaces (see also [21]). Some modifications will lead to a proof in our much more general situation.

We extend G to a function on $X \times X$ defining $G(x,y) = 0$, if x or y are not points in U. Then G is symmetric and lower semicontinuous on $X \times X$ and, for every $y \in U$, the function $G(\cdot, y)$ is continuous on $X \setminus \{y\}$. Let X^* denote the Stone-Čech compactification of X and

$$S := X^* \setminus X.$$

Given $x \in X$, we define, for every $z \in S$,

$$G(x,z) := \lim_{y \in X, y \to z} G(x,y) \quad \text{ and } \quad G(z,x) := \lim_{y \in X, y \to z} G(y,x)$$

so that $G(x, \cdot)$ and $G(\cdot, x)$ are continuous real functions on $X^* \setminus \{x\}$. Clearly, for every $z \in S$, the function $G(\cdot, z)$ is harmonic on U (locally being pointwise limit of positive harmonic functions), and we may continuously extend it to X^* defining

$$G(z', z) := \lim_{x \in X, x \to z'} G(x,z), \qquad z' \in S.[3]$$

For every compact $A \neq \emptyset$ in U, let $w(A)$ denote the *Wiener energy of* A, that is,

$$w(A) := \inf\{\int G^\nu \, d\nu \colon \nu \text{ probability measure on } A\}$$

(so that $1/w(A)$ is the capacity of A with respect to G). We recall that the symmetry of G implies that, for all measures μ, ν on X,

$$\left(\int G^\mu \, d\nu \right)^2 \leq \int G^\mu \, d\mu \cdot \int G^\nu \, d\nu. \tag{3.1}$$

Essential tools in proving Theorem 3.1 are the next two lemmas (cf. [22, pp. 482-483]).

LEMMA 3.2. $\lim_{n\to\infty} w(S_n) = \infty$.[4]

Proof. Let $n \in \mathbb{N}$. By Proposition 2.3, $p_n := H_{U_n \setminus K} 1_{X \setminus U_n}$ is the equilibrium potential for S_n with respect to U. Let μ_n be the the corresponding equilibrium measure, that is, μ_n is a measure on S_n such that $G^{\mu_n} = p_n$. Since $p_n = 1$ on S_n, we see, by (3.1), that, for every probability measure ν on S_n,

$$1 = \left(\int G^{\mu_n} \, d\nu \right)^2 \leq \mu_n(S_n) \int G^\nu \, d\nu. \tag{3.2}$$

Since $p_1 = 1$ on S_n, we know that

$$\mu_n(S_n) = \int G^{\mu_1} \, d\mu_n = \int G^{\mu_n} \, d\mu_1.$$

By Lemma 2.2, $p_n \downarrow 0$ on S_1 as $n \to \infty$. So $\lim_{n\to\infty} \mu_n(S_n) = 0$. A look at (3.2) and the definition of $w(S_n)$ finishes the proof. $\square$

[3] We do not claim the G is symmetric or lower semicontinuous on $X^* \times X^*$.

[4] Our proof has to be different from the proof in [22], since U need not be connected and hence G may have zeros.

For $k \in \mathbb{N}$, $k \geq 2$, and $A \subset X^*$, $A \neq \emptyset$, we introduce the *k-th* (*transfinite*) *diameter of* A, the *k-th Chebyshev constant of* A, respectively, by

$$D_k(A) := \inf_{z_1,\ldots,z_k \in A} \binom{k}{2}^{-1} \sum_{i<j} G(z_i, z_j), \quad M_k(A) := \sup_{z_1,\ldots,z_k \in A} \inf_{z \in A} k^{-1} \sum_i G(z, z_i).$$

Let us supply just the information about D_k and M_k we need for the proof of our theorem. For results on the convergence of $(D_k(A))$, $(M_k(A))$, and the relation of the limits to $w(A)$ for compact A in X (they coincide) we refer the reader to [11].

LEMMA 3.3. *For all* $k, n \in \mathbb{N}$, $k \geq 2$, $D_k(S_n) \leq D_k(S)$.

Proof. For $z_1, \ldots, z_k \in X^*$, let

$$v(z_1, \ldots, z_k) := \sum_{i<j} G(z_i, z_j).$$

Let us fix $z_1^*, \ldots, z_k^* \in S$. The function $z \mapsto v(z, z_2^*, \ldots, z_k^*)$ is harmonic on U. It admits a minimum on S_n, say at $z_1 \in S_n$. Then, by Proposition 2.3,

$$v(z_1^*, z_2^*, \ldots, z_k^*) \geq v(z_1, z_2^*, \ldots, z_k^*). \tag{3.3}$$

Now assume that $1 \leq m < k$ and $z_1, \ldots, z_m \in S_n$ such that

$$v(z_1, \ldots, z_{m-1}, z_m^*, \ldots, z_k^*) \geq v(z_1, \ldots, z_m, z_{m+1}^*, \ldots, z_k^*)$$

(which is (3.3), if $m = 1$). Due to the symmetry of G, the function

$$z \mapsto v(z_1, \ldots, z_m, z, z_{m+2}^*, \ldots, z_k^*)$$

is harmonic on $X \setminus \overline{U}_n$, lower semicontinuous on U. It admits a minimum on S_n, say at $z_{m+1} \in S_n$. Then, by Proposition 2.3,

$$v(z_1, \ldots z_m, z_{m+1}^*, \ldots, z_k^*) \geq v(z_1, \ldots, z_{m+1}, z_{m+2}^*, \ldots, z_k^*).$$

Thus we can find points $z_1, \ldots, z_k \in S_n$ such that

$$v(z_1^*, \ldots, z_k^*) \geq v(z_1, \ldots, z_k).$$

By definition of D_k, the claim follows. □

The statements of the following two lemmas are well known (see, for example, Proposition 5 and the proof [5] of Theorem 3 in [11]). For the convenience of the reader we include the (short) proofs.

LEMMA 3.4. *For every compact* $A \neq \emptyset$ *in* X, $w(A) \leq \limsup_{k\to\infty} D_k(A)$.

[5] A citation of Theorem 3 itself is not possible, since G might not be symmetric on $X^* \times X^*$.

Proof. For every $k \in \mathbb{N}$, $k \geq 2$, there exist $z_{1k}, \ldots, z_{kk} \in A$ such that

$$\binom{k}{2} D_k(A) \geq \sum_{i<j} G(z_{ik}, z_{jk}) - 1. \tag{3.4}$$

Let $\mu_k := k^{-1} \sum_{i=1}^k \varepsilon_{z_{ik}}$. There exists a subsequence (ν_k) of (μ_k) which converges weakly to a probability measure μ on A.

There exist symmetric continuous real functions $G_n \colon U \times U \to [0, n]$, $n \in \mathbb{N}$, such that $G_n \uparrow G$ as $n \to \infty$. Let $n \in \mathbb{N}$. Since G_n is uniformly continuous on $A \times A$, the sequence $(G_n^{\nu_k})_{k \geq 1}$ converges uniformly to G_n^μ, and we see that

$$\lim_{k\to\infty} \int G_n^{\nu_k} \, d\nu_k = \int G_n^\mu \, d\mu. \tag{3.5}$$

Using the symmetry of G_n we obtain that, for every $k \in \mathbb{N}$,

$$\sum_{i<j} G(z_{ik}, z_{jk}) \geq \sum_{i<j} G_n(z_{ik}, z_{jk}) \geq \frac{k^2}{2} \int G_n^{\nu_k} \, d\nu_k - kn. \tag{3.6}$$

Combining (3.4), (3.6), and (3.5) we conclude that $\limsup_{k\to\infty} D_k(A) \geq \int G_n^\mu \, d\mu$, and hence, letting $n \to \infty$,

$$\limsup_{k\to\infty} D_k(A) \geq \int G^\mu \, d\mu \geq w(A).$$

$\square$

LEMMA 3.5. *For every set $A \neq \emptyset$ in X^*, $\sup_{k\geq 2} D_k(A) \leq \sup_{k \geq 2} M_k(A)$.*[6]

Proof. Let $a := \sup_k M_k(A)$. For the moment, let us fix $k \in \mathbb{N}$, $k \geq 2$, and $\eta > 0$. We can recursively choose points $z_k, \ldots, z_1 \in A$ such that, for every $1 \leq i \leq k-1$,

$$\sum_{j=k-i+1}^{k} G(z_{k-i}, z_j) \leq \inf_{z \in A} \sum_{j=k-i+1}^{k} G(z, z_j) + \eta. \tag{3.7}$$

Indeed, let z_k be any point in A. Assume that $i < k-1$ and $z_k, \ldots, z_{k-i+1}$ have already been chosen. Clearly, there exists $z_{k-i} \in A$ such that (3.7) holds.

By definition of $M_i(A)$ and a, $\inf_{z\in A} \sum_{j=k-i+1}^k G(z, z_j) \leq ia$, whence, by (3.7),

$$\sum_{j=k-i+1}^{k} G(z_{k-1}, z_j) \leq ia + \eta, \qquad 1 \leq i \leq k-1.$$

Summing up these $k-1$ inequalities we see that

$$D_k(A) \leq \binom{k}{2}^{-1} \sum_{i<j} G(z_i, z_j) \leq \binom{k}{2}^{-1} (\sum_{i=1}^{k-1} ia + (k-1)\eta) = a + \frac{2\eta}{k}.$$

Since $\eta > 0$ is arbitrary, the proof is finished. $\square$

[6] Note that this holds for any set X^* and any function $G \colon X^* \times X^* \to (-\infty, \infty] \times (-\infty, \infty]$.

Now we may finish in a similar way as in [22].

Proof of Theorem 3.1. By Lemmas 3.2, 3.3, 3.4, and 3.5, $\sup_{k\ge 2} M_k(S) = \infty$. So there exist $k_m \in \mathbb{N}$, $m \in \mathbb{N}$, such that $M_{k_m}(S) > 2^m$. For every $m \in \mathbb{N}$, there are points $z_{mi} \in S$, $1 \le i \le k_m$, such that

$$\inf_{z\in S} \sum_{i=1}^{k_m} G(z, z_{mi}) > 2^m k_m.$$

We define

$$\nu_m := (2^m k_m)^{-1} \sum_{i=1}^{k_m} \varepsilon_{z_{mi}} \quad \text{and} \quad \nu := \sum_{m=1}^{\infty} \nu_m.$$

Clearly, ν is a probability measure on S and $G^\nu := \int G(\cdot, z)\, d\nu(z) = \infty$ on S. Since G^ν is lower semicontinuous on X^*, we see that $h := G^\nu|_U$ has the desired properties. $\square$

4 A Brelot space without Evans potential

In this section, we shall present an example of a Brelot harmonic space X such that constants are harmonic, X is not $\mathcal{P}$-harmonic, every proper open set in X admits a Green function, but there is no compact K in X such that there exists a harmonic function $h > 0$ on $X \setminus K$ which tends to ∞ at infinity.

The basic idea is the construction of a (nonsymmetric) random walk on two copies of $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ which are connected in the natural way such that on one copy we have very strong recurrence[7]. The harmonic space X is then obtained in the usual way adding edges between adjacent vertices and defining corresponding harmonic functions which are affinely linear on the edges. This procedure is well known, but to keep our paper self-contained we shall include the few necessary details.

To begin with let

$$X_0 := \mathbb{N}_0 \times \{0, 1\} = \{y_0, y_1, \dots\} \cup \{z_0, z_1, z_2, \dots\},$$

where, for every $n \in \mathbb{N}_0$,

$$y_n := (n, 0), \quad z_n := (n, 1).$$

Let X be the union of $\mathbb{R}^+ \times \{0, 1\}$ and all line segments $[y_n, z_n]$, $n \in \mathbb{N}_0$.

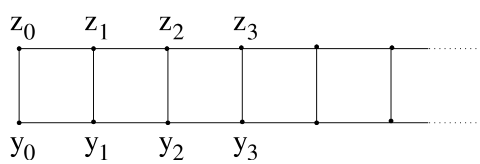

Figure 1. The space X

[7]For a counterexample on $\mathbb{R}^d$, $d \ge 2$, see Remark 4.9

For $x \in X_0$ and $0 < r \le 1$, let

$$B(x,r) := \{x' \in X \colon |x'-x| < r\}, \qquad N(x) := \{x' \in X_0 \colon |x'-x| = 1\}$$

so that $N(x)$ is the boundary of $B(x,1)$. Clearly, $N(y_0) = \{z_0, y_1\}$, $N(z_0) = \{y_0, z_1\}$, and, for $n \in \mathbb{N}$,

$$N(y_n) = \{y_{n-1}, z_n, y_{n+1}\}, \qquad N(z_n) = \{z_{n-1}, y_n, z_{n+1}\}.$$

For $x \in X_0$ and $x' \in N(x)$, we define $p(x,x') > 0$ (transition probability from x to x') by $p(x,x') := 1/2$, if $x \in \{y_0, z_0\}$, and for $n \in \mathbb{N}$ and $x \in \{y_n, z_n\}$ (see Figure 2),

$$p(y_n, x') := 1/3, \qquad p(z_n, x') := \begin{cases} 4^{-n}, & \text{if } x' = y_n \text{ or } x' = z_{n+1} \\ \\ 1 - 2 \cdot 4^{-n}, & \text{if } x' = z_{n-1}. \end{cases}$$

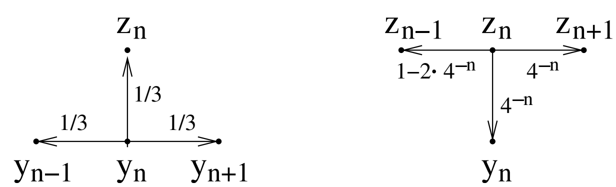

Figure 2. The transition probabilities

For every open set U in X, let $\mathcal{H}(U)$ be the set of all continuous real functions h on U which are locally affinely linear on $U \setminus X_0$ and have the following mean value property. For every $x \in U \cap X_0$, there exists $0 < r \le 1$ such that $\overline{B}(x,r) \subset U$ and

$$h(x) = \sum_{x' \in N(x)} p(x,x') h(x + r(x'-x)). \tag{4.1}$$

Of course, $1 \in \mathcal{H}(U)$, and it is easily verified that, for every $h \in \mathcal{H}(U)$, (4.1) holds for *every* $0 < r \le 1$ such that $\overline{B}(x,r) \subset U$. This implies that $\mathcal{H} \colon U \mapsto \mathcal{H}(U)$ is a sheaf of vector spaces of continuous real functions.

We intend to prove the following.

THEOREM 4.1. $(X, \mathcal{H})$ *is a connected Brelot harmonic space such that constants are harmonic, X is not $\mathcal{P}$-harmonic, every proper open set in X admits a Green function, but there are no Evans potentials on X.*

To prove that $(X, \mathcal{H})$ has the Brelot convergence property, let U be a connected open set in X and $h_n \in \mathcal{H}(U)$ such that $h_n \uparrow h$ as $n \to \infty$. Then h satisfies (4.1) for every $0 < r \le 1$ such that $\overline{B}(x,r) \subset U$ and hence (taking into account that the functions h_n are locally affinely linear on $U \setminus X_0$) the sets $\{h < \infty\}$ and $\{h = \infty\}$ are open. Therefore either $h = \infty$ on U or $h \in \mathcal{H}(U)$.

Of course, every open line segment I contained in $X \setminus X_0$ is regular. Moreover, for every $x \in X_0$ and $0 < r \le 1$, $B(x,r)$ is regular. Indeed, the solution h to the Dirichlet problem for f is obtained by first taking

$$h(x) := \sum_{x' \in N(x)} p(x,x') f(x + r(x'-x)) \tag{4.2}$$

(see (4.1)) and then extending h by a function on $B(x,r) \setminus \{x\}$ which is locally affinely linear on the (two or three) line segments forming $B(x,r) \setminus \{x\}$ and tends to $h(x)$ at x and to f at $\partial B(x,r)$. So we already know the following.

PROPOSITION 4.2. *$(X, \mathcal{H})$ is a connected Brelot harmonic space, where constants are harmonic.*

Given an open set U in X, a lower semicontinuous function $s \colon U \mapsto (-\infty, \infty]$ is superharmonic on U, if it is locally concave (and hence continuous) on $U \setminus X_0$ and, for all $x \in U \cap X_0$ and $0 < r \le 1$ such that $\overline{B}(x,r) \subset U$,

$$\sum_{x' \in N(x)} p(x,x') s(x + r(x'-x)) \le s(x). \tag{4.3}$$

As usual, let $\mathcal{S}(U)$ denote the set of all superharmonic functions on U. Let us note that all functions $s \in \mathcal{S}(U)$ are continuous also at each point $x \in U \cap X_0$. Indeed, by lower semicontinuity, $s(x) \le \liminf_{y \ne x, y \to x} s(y) < \infty$. Hence, by (4.3), it is impossible that $\limsup_{y \ne x, y \to x} s(y) > s(x)$. So $\lim_{y \to x} s(y) = s(x)$. Moreover, arguing as above for $h = \sup h_n$, we see that any superharmonic function $s \ge 0$ on a connected open set U vanishes identically, if it vanishes at some point of U.

For all open V in X, $A \subset V$ and $s \in \mathcal{S}^+(V)$, let

$${}^V R_s^A := \inf\{v \in \mathcal{S}^+(V) \colon v \ge s \text{ on } A\}.$$

Let $x \in X$ and $0 < r \le 1$ such that $B(x,r) \cap X_0 \subset \{x\}$. Then it is easily seen that

$${}^{B(x,r)} R_1^{\{x\}}(x') = 1 - |x - x'|/r \qquad (x' \in B(x,r)). \tag{4.4}$$

In particular, the singleton $\{x\}$ is not thin at x, hence $\{x\}$ is nonpolar.

PROPOSITION 4.3. *$\mathcal{S}^+(X) = \mathcal{H}^+(X) = \mathbb{R}^+$, and X is not $\mathcal{P}$-harmonic.*

Proof. Let $s \in \mathcal{S}^+(X)$ such that $s \ne 0$ and hence, $s > 0$. By (4.3), for every $n \in \mathbb{N}$, $s(z_n) \ge (1 - 2 \cdot 4^{-n}) s(z_{n-1})$. Clearly $a := \prod_{n=1}^\infty (1 - 2 \cdot 4^{-n}) > 0$, since $\sum_{n=1}^\infty 4^{-n} < \infty$. By induction, we see that, for every $n \in \mathbb{N}$,

$$s(z_n) \ge \prod_{j=1}^n (1 - 2 \cdot 4^{-j}) s(z_0) > a s(z_0),$$

and hence $s(y_n) \ge s(z_n)/3 > (a/3) s(z_0)$. So the constant function $(a/3)s(z_0)$ is a harmonic minorant of s. This shows that s is no potential. The proof is finished by Lemma 2.2. $\square$

For every $k \in \mathbb{N}$, let

$$U_k := X \cap ([0,k) \times \mathbb{R}).$$

Given $k \in \mathbb{N}$ and $g \in \mathcal{H}(X \setminus \overline{U}_k)$, we know that, for every $n \in \mathbb{N}$,

$$g(y_n) = \frac{1}{3}\big(g(y_{n+1}) + g(z_n) + g(y_{n-1})\big), \tag{4.5}$$

$$g(z_n) = \frac{1}{4^n}\big(g(z_{n+1}) + g(y_n)\big) + \big(1 - \frac{2}{4^n}\big) g(z_{n-1}). \tag{4.6}$$

In particular, if $g \geq 0$, then

$$g(y_{n+1}) \leq 3g(y_n), \tag{4.7}$$

and if $g(z_{n-1}) = 0$, then

$$g(z_{n+1}) = 4^n g(z_n) - g(y_n). \tag{4.8}$$

The next result will finish the proof of Theorem 4.1.

PROPOSITION 4.4. *Let K be a compact in X and $h \in \mathcal{H}^+(X \setminus K)$. Then*

$$\limsup_{n\to\infty} h(z_n) < \infty. \tag{4.9}$$

Proof. There exists $k \in \mathbb{N}$ such that $K \subset U_k$. We may assume without loss of generality that $h(y_k) = 1$. Let us suppose that (4.9) does not hold. Then

$$h(z_k) + \sup_n \sum_{m=k+1}^{n} (h(z_m) - h(z_{m-1})) = \sup_n h(z_n) = \infty.$$

Since $\sum_{m=1}^{\infty} 1/m^2 < \infty$, we see that there are infinitely many $m > k$ such that

$$h(z_m) - h(z_{m-1}) > 1/m^2. \tag{4.10}$$

So there exists $m > k$ such that (4.10) holds and

$$m^2 3^{m-k} < 4^{m-1}. \tag{4.11}$$

We assume next that $n \in \mathbb{N}$ such that $n \geq m$ and

$$g := h - h(z_{n-1})$$

satisfies $g(z_n) \geq 1/m^2$ (which is true, if $n = m$). Then $g(y_n) \leq h(y_n) \leq 3^{n-k}$, by (4.7), and hence

$$4^{n-1}g(z_n) - g(y_n) > 4^{n-1}m^{-2} - 3^{n-k} > 3^{n-m}(4^{m-1}m^{-2} - 3^{m-k}) > 0.$$

Therefore, by (4.8),

$$g(z_{n+1}) = 3 \cdot 4^{n-1}g(z_n) + 4^{n-1}g(z_n) - g(y_n) > 3 \cdot 4^{n-1}g(z_n).$$

So we see that

$$\begin{aligned} h(z_{n+1}) - h(z_n) &> 3 \cdot 4^{n-1}(h(z_n) - h(z_{n-1})) - (h(z_n) - h(z_{n-1})) \\ &> 4^{n-1}(h(z_n) - h(z_{n-1})). \end{aligned}$$

A trivial induction (using $4^{n-1} \geq 1$) now shows that, for *every* $n \geq m$,

$$h(z_{n+1}) - h(z_n) > 4^{n-1}(h(z_n) - h(z_{n-1})). \tag{4.12}$$

Thus, by (4.5) and (4.7),

$$h(y_{n+1}) - h(y_n) = 2h(y_n) - h(z_n) - h(y_{n-1}) \leq 2 \cdot 3^{n-k} - 4^{n-1}m^{-2},$$

which tends to $-\infty$ as $n \to \infty$. This is impossible, since $h \geq 0$. Therefore (4.9) holds. $\square$

Of course, the preceding results imply that there are no *symmetric* Green functions on complements of compact sets in X. It may be interesting to note that only the symmetry is missing. To that end let U be a proper open subset of X and

$$p_x := {}^U R_1^{\{x\}}, \qquad x \in U. \tag{4.13}$$

For the moment, let us fix $x \in U$. If $r > 0$ and $\overline{B}(x,r) \subset U$, then ${}^{B(x,r)}R_1^{\{x\}} \le p_x$, and hence, by (4.4), $\lim_{y\to x} p_x(y) = 1 = p_x(x)$. Therefore p_x is a continuous real potential on U which is harmonic on $U \setminus \{x\}$. If q is any potential on U which is harmonic on $U \setminus \{x\}$, then the continuity of q and the minimum principle imply that

$$q = q(x)p_x. \tag{4.14}$$

By [18, Theorem 3.4], we have the following consequence.

PROPOSITION 4.5. *For every proper open subset U of X, there exists a Green function on U.*

REMARKS 4.6. 1. In fact, using (4.4) it is easily verified directly that, given $x \in U$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in U \cap B(x,\delta)$,

$$\min\{p_x(x'), p_{x'}(x)\} \ge 1 - \varepsilon,$$

hence $p_x \ge (1-\varepsilon)p_{x'}$ and $p_{x'} \ge (1-\varepsilon)p_x$, which finally implies that $|p_x - p_{x'}| < \varepsilon$. Therefore $(x,y) \mapsto p_x(y)$ is continuous and bounded on $U \times U$.

Moreover, given a continuous real potential q on U, there exist potentials q_n on U such that $q = \sum_{n=1}^\infty q_n$ and, for every $n \in \mathbb{N}$, the carrier of q_n is a singleton $\{x_n\}$ in X_0 or a compact set K_n in some open line segment I_n contained in $U \setminus X_0$ (see [4, II.6.17]). In the first case, $q_n = q_n(x_n)p_{x_n}$, by (4.14). In the second case, there exists a measure ν_n on K_n such that $q_n - H_{I_n} q_n = \int G_{I_n}(\cdot,x)\,d\nu_n(x)$, where G_{I_n} denotes the classical Green function on I_n. For every $x \in I_n$, there exists $\varphi_n(x) > 0$ such that $p_x - H_{I_n} p_x = \varphi_n(x) G_{I_n}(\cdot,x)$, and then $q_n = \int p_x \varphi_n(x)\,d\nu_n(x)$. Taking $\mu := \sum_{n=1}^\infty \mu_n$ with $\mu_n := q_n(x)\varepsilon_{x_n}$ in the first case and $\mu_n := \varphi_n \nu_n$ in the second case, we obtain that $q = \int p_x\,d\mu(x)$. So $G\colon (x,y) \mapsto p_x(y)$ is a continuous bounded Green function for U.

2. As we know, there is no compact set K in X such that $X \setminus K$ admits a symmetric Green function. More generally, if V is an open set in X, $V \ne X$, such that $B(y_n,1) \cup B(y_{n+1},1) \cup B(z_n,1) \cup B(z_{n+1},1) \subset V$ for some large $n \in \mathbb{N}$, then there is no symmetric Green function for V. This follows from Proposition 5.3 and Lemma 4.7 below.

3. However, for every point $x_0 \in X_0$, there exists a symmetric Green function on $B(x_0,1)$ (see Lemma 4.8 below). Given any $x \in X$, the ball $B(x,1/2)$ is contained in some ball $B(x_0,1)$, $x_0 \in X_0$, and hence there exists a symmetric Green function on $B(x,1/2)$, by Proposition 5.3.

LEMMA 4.7. *Let $n \in \mathbb{N}$ and $U := B(y_n,1) \cup B(y_{n+1},1) \cup B(z_n,1) \cup B(z_{n+1},1)$. If n is large, then there is no symmetric Green function for U.*

Proof. Assuming that there exists a symmetric Green function G for U we have to get to a contradiction. By (4.14), for every $x_n \in P_n := \{y_n, y_{n+1}, z_n, z_{n+1}\}$,

$$G(\cdot, x_n) = G(x_n, x_n) p_{x_n}.$$

Representing a function v_n on P_n by the matrix with rows $(v_n(z_n),\ v_n(z_{n+1}))$, $(v_n(y_n), v_n(y_{n+1}))$ we have for p_{z_n}, $p_{z_{n+1}}$, p_{y_n}, respectively,

$$\begin{pmatrix} 1 & \approx 1 \\ \approx \frac{1}{2} & \approx \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} \approx \frac{9}{8} \cdot 4^{-n} & 1 \\ \approx \frac{1}{8} & \approx \frac{3}{8} \end{pmatrix}, \quad \begin{pmatrix} \approx 4^{-n} & \approx \frac{13}{12} \cdot 4^{-n} \\ 1 & \approx \frac{1}{3} \end{pmatrix}$$

(where $v_n(x_n) \approx a_n$ means that $\lim_{n\to\infty} v_n(x_n)/a_n = 1$). Assuming without loss of generality that $G(z_n, z_n) = 1$, we conclude that

$$\begin{aligned} G(z_{n+1}, z_{n+1}) \tfrac{9}{8} \cdot 4^{-n} &\approx G(z_n, z_{n+1}) = G(z_{n+1}, z_n) \approx 1, \\ G(y_n, y_n) 4^{-n} &\approx G(z_n, y_n) = G(y_n, z_n) \approx \tfrac{1}{2}. \end{aligned}$$

So $G(z_{n+1}, z_{n+1}) \approx \frac{8}{9} \cdot 4^n$, $G(y_n, y_n) \approx \frac{1}{2} \cdot 4^n$, and therefore

$$\begin{aligned} G(y_n, z_{n+1}) &= G(z_{n+1}, z_{n+1}) p_{z_{n+1}}(y_n) \approx \tfrac{8}{9} \cdot 4^n \cdot \tfrac{1}{8} = \tfrac{1}{9} \cdot 4^n, \\ G(z_{n+1}, y_n) &= G(y_n, y_n) p_{y_n}(z_{n+1}) \approx \tfrac{1}{2} \cdot 4^n \cdot \tfrac{13}{12} \cdot 4^{-n} = \tfrac{13}{24}, \end{aligned}$$

contradicting the symmetry of G. $\square$

LEMMA 4.8. *For every $x \in X_0$, there exists a symmetric Green function on $B(x_0, 1)$.*

Proof. This can be seen by a very general argument. Indeed, let x_0 be one of the points y_n, z_n, $n \geq 1$, so that the boundary $N(x_0)$ of $U := B(x_0, 1)$ consists of three points $x_1, x_2, x_3 \in X_0$ (in the case $n = 0$ we would have only two points in $N(x_0)$). For every $x \in U$, we may choose $c(x) > 0$ such that

$$G(\cdot, x) := c(x) p_x \quad \text{and} \quad G(x_0, x) = 1 - |x - x_0|. \tag{4.15}$$

Then, of course, $G(x_0, x_0) = 1$ and $G(x, x_0) = G(x_0, x)$, since, for every $1 \leq j \leq 3$, $G(\cdot, x_0)$ is affinely linear on the open line segment I_j from x_0 to x_j and vanishes at x_j. To see that, for all $x, y \in U$, $G(x, y) = G(y, x)$, let $x \in I_j$ for some $1 \leq j \leq 3$. If $y \in U \setminus I_j$, then

$$G(y, x) = (1 - |y - x_0|) G(x_0, x) = (1 - |y - x_0|)(1 - |x - x_0|), \tag{4.16}$$

by affine linearity, and therefore $G(x, y) = G(y, x)$, since the right hand side of (4.16) is symmetric in x and y. So it suffices to consider $y \in I_j$. Without loss of generality $\eta_x := |x - x_0| \leq \eta_y := |y - x_0|$. Then

$$G(y, x) = \frac{1 - \eta_y}{1 - \eta_x} G(x, x) \quad \text{and} \quad G(x, y) = \frac{1}{\eta_y} \big[G(y, y) \eta_x + (1 - \eta_y)(\eta_y - \eta_x) \big].$$

Let $\alpha := p(x_0, x_j)$. Since $G(\cdot, x)$ is harmonic on $U \setminus I_j$, we know that $G(\cdot, x) = (1 - \eta_x) G(x_0, x) = (1 - \eta_x)^2$ on $\partial B(x_0, \eta_x) \setminus \{x\}$, and therefore

$$1 - \eta_x = G(x_0, x) = \alpha G(x, x) + (1 - \alpha)(1 - \eta_x)^2.$$

Hence

$$\frac{G(x,x)}{\eta_x(1-\eta_x)} = \frac{1}{\alpha} - 1 + \frac{1}{\eta_x}\,.$$

Having the analogous expression involving $G(y,y)$ we finally conclude that

$$\frac{G(y,x)-G(x,y)}{\eta_x(1-\eta_y)} = \frac{1}{\eta_x} - \frac{1}{\eta_y} - \frac{\eta_y-\eta_x}{\eta_x\eta_y} = 0.$$

So G is symmetric. $\square$

REMARK 4.9. An identification of X with the union of $\mathbb{R}\times\{0\}$ and the half-circles $\{-me^{it}\colon t\in(0,\pi)\}$, $m\in\mathbb{N}$, which sends z_n to $(n+1,0)$ and y_n to $(-(n+1),0)$, indicates how to define a second order elliptic operator $\mathcal{L}$ on $\mathbb{R}^d$, $d\ge 2$, such that the corresponding Brelot harmonic space $(\mathbb{R}^d,\mathcal{H}_{\mathcal{L}})$, exhibits the same behavior (constants harmonic, no strictly positive potential, no Evans function outside compact sets). In polar coordinates (r,θ), the operator $\mathcal{L}$ could be of the form

$$\mathcal{L} := \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \max\{\frac{1}{r^2},1\}\Delta_{S^{d-1}} - f(\theta)g(r)\frac{\partial}{\partial r},$$

where f is a continuous positive function on S^{d-1} such that $f(e_1)>0$ and $f=0$ outside $S^{d-1}\cap B(e_1,1/4)$, say, and $g\colon[0,\infty)\to[0,\infty)$ vanishes on $[0,1]$ and is rapidly increasing on $(1,\infty)$.

5 Appendix

5.1 Green functions on open subsets

In this part, we shall recall the definition of Green functions and discuss Green functions on open subsets. Let $(Y,\mathcal{H})$ be a Brelot harmonic space. A lower semi-continuous function $G\colon Y\times Y\to[0,\infty]$ is a *Green function* if the following holds:

(i) For every $y\in Y$, $G(\cdot,y)\ne 0$ is a potential on Y, which is harmonic on $Y\setminus\{y\}$.

(ii) *Representation of potentials*: For every continuous real potential p on Y, which is harmonic outside a compact L in Y, there exists a measure μ on Y such that

$$p = G^\mu := \int G(\cdot,y)\,d\mu(y). \tag{5.1}$$

G is called *symmetric*, if $G(x,y)=G(y,x)$, $x,y\in Y$.

REMARKS 5.1. 1. Clearly, $G(\cdot,y)=0$ on any connected component V of Y not containing y, since the function $h=1_V G(\cdot,y)$ is a harmonic minorant of $G(\cdot,y)$.

2. If G is symmetric, the lower semicontinuity follows from (i) (see [7, p. 342]).

3. If (i) holds and G is symmetric, then (ii) holds if and only if, for each $y\in Y$, every potential on Y which is harmonic on $Y\setminus\{y\}$ is a multiple of $G(\cdot,y)$ (see [17, Théorème 18.2] and [5, Satz 3.2]). Moreover, the measure μ is uniquely determined by p (and supported by L).

Let us assume that the Brelot harmonic space $(Y, \mathcal{H})$ admits a symmetric Green function G.

In particular, Y is $\mathcal{P}$-harmonic. We recall that, for all $A \subset Y$ and $s \in \mathcal{S}^+(Y)$, the reduced function R_s^A is defined by

$$R_s^A := \inf\{v \in \mathcal{S}^+(Y) \colon v \geq s \text{ on } A\}. \tag{5.2}$$

For every $y \in Y$, the mapping $s \mapsto R_s^A(y)$ is not only increasing and positively homogeneous, but also additive (see [4, VI.1.1]), and hence there exists a unique measure ν_y^A such that

$$R_s^A(y) = \int s \, d\nu_y^A, \qquad \text{for all } s \in \mathcal{S}^+(Y)$$

(see [4, VI.5.4]). Of course, $\nu_y^A = \varepsilon_y$, if $y \in A$; if $y \in Y \setminus A$, then ν_y^A is the swept measure ε_y^A.

The following lemma follows from [17, Théorème 31.1] provided there exists a base of sets which are "complètement déterminants".

LEMMA 5.2. *1. For all open sets V in Y and $x, y \in Y$, $R^V_{G(\cdot,y)}(x) = R^V_{G(\cdot,x)}(y)$.*

2. For all subsets A of Y and $x, y \in Y \setminus A$, $R^A_{G(\cdot,y)}(x) = R^A_{G(\cdot,x)}(y)$.

Proof. 1. By the minimum principle,

$$R^V_{G(\cdot,y)} = G(\cdot, y), \qquad \text{if } y \in V. \tag{5.3}$$

Let $x \in Y$ and $\nu := \nu_x^V$. By the symmetry of G, for every $y \in Y$,

$$R^V_{G(\cdot,y)}(x) = \int G(z, y) \, d\nu(z) = \int G(y, z) \, d\nu(z) = G^\nu(y). \tag{5.4}$$

By (5.3) and (5.4), $G^\nu = G(\cdot, x)$ on V. Since $G^\nu \in \mathcal{S}^+(Y)$, we see that $G^\nu \geq R^V_{G(\cdot,x)}$. So, for all $x, y \in Y$, $R^V_{G(\cdot,y)}(x) = G^\nu(y) \geq R^V_{G(\cdot,x)}(y)$.

2. By (1), it suffices to prove that, for $y \in Y \setminus A$,

$$v := \inf\{R^V_{G(\cdot,y)} \colon V \text{ open}, \ A \subset V\} = R^A_{G(\cdot,y)}.$$

Trivially, $R^A_{G(\cdot,y)} \leq v$. To prove the reverse inequality we fix $s_0 \in \mathcal{S}^+(Y)$, $0 < s_0 < \infty$. If $\varepsilon > 0$ and $s \in \mathcal{S}^+(Y)$ such that $s \geq G(\cdot, y)$ on A, then there is an open set V containing A such that $s + \varepsilon s_0 > G(\cdot, y)$ on V, and hence $R^V_{G(\cdot,y)} \leq s + \varepsilon s_0$. This implies that $v \leq R^A_{G(\cdot,y)}$. □

Let U be an open set in Y. We define

$$G_U(\cdot, y) := G(\cdot, y) - R^{U^c}_{G(\cdot,y)}, \qquad y \in U.$$

Taking into account Remark 5.1.3, the following result is Proposition 4.2 in [20], where this is shown (under an additional assumption) using [17] in a rather implicit way. For the convenience of the reader, we give a more direct proof.

PROPOSITION 5.3. *G_U is a symmetric Green function for the Brelot harmonic space $(U, \mathcal{H}|_U)$.*

Proof. By Lemma 5.2 and the symmetry of G, the function G_U is symmetric as well.

For every $y \in U$, the function $G_U(\cdot, y)$ is a potential on U which is harmonic on $U \setminus \{y\}$, since $R^{U^c}_{G(\cdot,y)}$ is the greatest harmonic minorant of $G(\cdot, y)$ on U.

Finally, let q be a continuous real potential on U which is harmonic outside a compact L in U. By [17, Théorème 13.2], there exists a (unique) continuous real potential p on Y such that p is harmonic outside L and $p - q$ is harmonic on U. Then $q = p - R^{U^c}_p$. By (ii), there exists a measure μ on L such that $p = G^\mu$. By (5.2) and Fubini's theorem (the mapping $(x, y) \mapsto R^{U^c}_{G(\cdot,y)}(x)$ is continuous on $U \times U$), it follows that $q = G^\mu_U$. $\square$

5.2 Two topological lemmas

Let $Y \neq \emptyset$ be a locally compact space with countable base which is connected and locally connected. Then, for every open set W in Y, $\emptyset \neq W \neq Y$, every connected component V of W is open (and hence closed in W) so that $\emptyset \neq \partial V \subset \partial W$.

LEMMA 5.4. *There exist relatively compact connected open sets W_m in Y, $m \in \mathbb{N}$, such that $W_m \uparrow Y$ as $m \to \infty$.*

Proof. Our assumptions on V imply that there exists a covering of Y by relatively compact connected open sets $V_m \neq \emptyset$, $m \in \mathbb{N}$. By [19, § 41,II,10], we may assume that, for every $m \in \mathbb{N}$,

$$(V_1 \cup \cdots \cup V_m) \cap V_{m+1} \neq \emptyset$$

(the idea is to rearrange the sequence step by step taking the first element which has not already been chosen and which leads to the non-empty intersection above). By induction, it follows that the relatively compact open sets $W_m := V_1 \cup \cdots \cup V_m$ are connected. Clearly, $W_m \uparrow Y$ as $m \to \infty$. $\square$

Let $K \neq \emptyset$ be a compact set in $\mathbb{R}^d$, $d \geq 1$, and $V := \mathbb{R} \setminus K$. If $d = 1$, then obviously V has exactly two connected components which are unbounded. If $d \geq 2$, there is only one unbounded connected component V_∞ in V (clearly, the connected set $\{x \in \mathbb{R}^d : |x| > R\}$ is contained in V_∞ if R is sufficiently large).

Finally, let us note the following (where the first part of the statement is Lemma 1 in [12]).

LEMMA 5.5. *Let K be a compact in Y. Then the union $\hat{K}$ of K and all relatively compact connected components of $Y \setminus K$ is compact and there are only finitely many connected components of $Y \setminus K$ which are not relatively compact.*

Proof. Of course, we may assume that $K \neq \emptyset$. We fix a relatively compact open neighborhood U of K. Let $\mathcal{V}$ denote the set of all connected components of $Y \setminus K$,

$$\mathcal{V}' := \{V \in \mathcal{V} \colon V \cap \partial U \neq \emptyset\}, \quad \mathcal{V}'' := \{V \in \mathcal{V} \colon V \text{ is not relatively compact}\}.$$

We know that, for every $V \in \mathcal{V}$, ∂V is a nonempty subset of $\partial(Y \setminus K) \subset K$, and hence $V \cap U \neq \emptyset$. If $V \in \mathcal{V} \setminus \mathcal{V}'$, then $V = (V \cap U) \cup (V \setminus \overline{U})$, and therefore $V = V \cap U$, $V \subset U$. In particular, $\mathcal{V}'' \subset \mathcal{V}'$.

Further, $\mathcal{V}'$ is a finite set, since ∂U is a compact subset of $Y \setminus K$ and $Y \setminus K$ is the disjoint union of all $V \in \mathcal{V}$. In particular, the second statement of our lemma holds. Moreover, $\hat{K}$ is contained in the (finite) union of U and all $V \in \mathcal{V}'$ which are relatively compact. Therefore $\hat{K}$ is relatively compact. Finally, $\hat{K}$ is closed, since $Y \setminus \hat{K}$ is the union of the open sets $V \in \mathcal{V}''$. □

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Wolfhard Hansen, Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany, e-mail: hansen@math.uni-bielefeld.de
Ivan Netuka, Charles University, Faculty of Mathematics and Physics, Mathematical Institute, Sokolovská 83, 186 75 Praha 8, Czech Republic, email: netuka@karlin.mff.cuni.cz