

# ASYMPTOTIC PROPERTIES OF SUBORDINATORS AND APPLICATIONS IN POTENTIAL THEORY

ANTE MIMICA

ABSTRACT. We consider a particular class of subordinators whose Laplace exponent varies regularly with index  $\alpha \in [0, 2]$  and obtain asymptotic behavior of the Lévy density and potential density. This results can be used to obtain asymptotic behavior of the Green function and the Lévy density of the corresponding subordinate Brownian motion. We prove a-priori regularity estimates of harmonic functions. These results are applied to the case of variance gamma and iterated variance gamma processes as well as to the case of geometric stable processes.

## CONTENTS

1. Introduction	2
2. Preliminaries	6
3. Asymptotic properties of subordinators	9
3.1. Slow subordinators.	10
3.2. Fast subordinators.	12
3.3. Stable-like subordinators.	13
4. Subordinate Brownian motion	15
5. Difference estimates	19
6. Harmonic functions	24
7. Examples	25
7.1. Variance gamma process	25
7.2. Iterated variance gamma process	26
7.3. Geometric stable process	28
7.4. Conjugate of the iterated geometric stable process	29
References	30

---

2000 *Mathematics Subject Classification.* Primary 60J45, 60J75, Secondary 60J25.

*Key words and phrases.* Geometric stable process, Green function, harmonic function, modulus of continuity, subordinator, iterated variance gamma process.

Research supported in part by German Science Foundation DFG via IGK "Stochastics and real world models" and SFB 701.

## 1. INTRODUCTION

Let  $S = (S_t : t \geq 0)$  be a subordinator defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with the Laplace exponent

$$\phi(\lambda) := -\log \mathbb{E} [e^{-\lambda S_1}], \quad \lambda > 0$$

and such that  $S_0 = 0$ . Then

$$\phi(\lambda) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) M(dt),$$

where  $b \geq 0$  and  $M$  is a measure on  $(0, \infty)$  satisfying

$$\int_{(0, \infty)} (1 \wedge t) M(dt) < \infty, \quad (1.1)$$

usually called the Lévy measure of  $S$ .

It is shown in [SSV10] that the Lévy measure of the geometric stable subordinator, i.e. a subordinator with  $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$  ( $0 < \beta \leq 2$ ), has a density  $\mu$  satisfying

$$\mu(t) \sim \frac{\beta}{2t}, \quad t \rightarrow 0+. \quad (1.2)$$

This result was proved by using the explicit formula

$$\mu(t) = \frac{\beta}{2t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{n\beta/2}}{\Gamma(1 + n\beta/2)}.$$

When  $\beta = 2$  this subordinator is also known as the gamma subordinator.

Usually, behavior of the Lévy density can be obtained by using one of the classical Tauberian theorems. These types of theorems are not applicable to the above example or, more generally, to the case when  $\phi$  varies slowly at infinity, i.e.

$$\lim_{\lambda \rightarrow \infty} \frac{\phi(\lambda x)}{\phi(\lambda)} = 1 \quad \text{for all } x > 0.$$

Our first aim is to obtain the behavior of the Lévy density within this class of subordinators. In particular, we can treat the class of iterated gamma subordinators. For example, the subordinator with the Laplace exponent

$$\phi(\lambda) = \log(1 + \log(1 + \lambda)) \quad (1.3)$$

has the Lévy density  $\mu$  with the following behavior

$$\mu(t) \sim \frac{1}{t \log(1/t)}, \quad t \rightarrow 0+ \quad (1.4)$$

and such that  $t \mapsto te^t \mu(t)$  is a non-decreasing function.

From (1.2) and (1.4) we see that Lévy measure of subordinators in this class is almost integrable, meaning that these subordinators jump very slowly.

Another way to see this is by considering the potential measure  $U$ , which measures the mean time spent by the subordinator in some set, i.e.

$$U(A) = \mathbb{E} \left[ \int_0^\infty \mathbb{1}_{\{S_t \in A\}} dt \right].$$

For the gamma subordinator we get

$$U([0, t]) \sim \frac{1}{\log(1/t)}, \quad t \rightarrow 0+$$

and for the iterated gamma subordinator with the Laplace exponent given in (1.3)

$$U([0, t]) \sim \frac{1}{\log \log(1/t)}, \quad t \rightarrow 0+.$$

In other words, these subordinators spend, on average, more time in small intervals  $[0, t]$  than any stable subordinator, i.e. a subordinator with the Laplace exponent  $\phi(\lambda) = \lambda^{\alpha/2}$  ( $0 < \alpha < 2$ ).

Knowing asymptotic properties of potential density allows us to deduce asymptotic properties of the Lévy densities of the corresponding 'conjugate' subordinators. For example, the conjugate of geometric subordinator has the Laplace exponent

$$\phi(\lambda) = \frac{\lambda}{\log(1 + \lambda)}.$$

The Lévy density have the following behavior

$$\mu(t) \sim \frac{1}{t^2(\log t)^2}, \quad t \rightarrow 0+.$$

As a consequence, we see that this Lévy measure barely satisfies the integrability condition (1.1), meaning that the intensity of small jumps is very high. In this sense this subordinator is very fast.

We can use asymptotic results concerning these subordinators to say something about Green function, jumping function and regularity of harmonic functions for corresponding subordinate Brownian motions.

Now let us be more precise. Throughout the paper we will have the following assumption.

**Assumptions 1.1.** Let  $\alpha \in [0, 2]$  and let  $S$  be a subordinator with the Laplace exponent  $\phi$  satisfying:

(A-1) derivative  $\phi'$  varies regularly at infinity with index  $\alpha/2 - 1$ , i.e.

$$\lim_{\lambda \rightarrow \infty} \frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\alpha/2 - 1} \quad \text{for any } x > 0;$$

(A-2) the Lévy measure  $M$  is infinite and has a non-increasing density  $\mu$ ;

(A-3) the potential measure  $U$  has a non-increasing density  $u$ .

In the case  $\alpha = 0$  we have an additional assumption:

(B)  $t \mapsto te^{at}\mu(t)$  is monotone on  $(0, T)$  for some  $a \geq 0$  and  $T > 0$  ;  
while in the case  $\alpha = 2$  we assume

(C)  $(\lambda/\phi(\lambda))'$  varies regularly at infinity with index  $-1$  .

*Remark 1.2.* The most important assumption is (A-1). Assumptions (A-2) and (A-3) are satisfied for a large class of subordinators (e.g. when the Laplace exponent is a complete Bernstein function, cf. Section 2).

Our results are new for  $\alpha \in \{0, 2\}$ . More precisely, for  $\alpha = 0$  we have the following results.

**Theorem 1.3.** *Assume (A-1), (A-2) and (B) with  $\alpha = 0$ . The Lévy density  $\mu$  has the following behavior near zero*

$$\mu(t) \sim \frac{\phi'(1/t)}{t^2}, \quad t \rightarrow 0+ .$$

**Theorem 1.4.** *Assume (A-1) and (A-3) with  $\alpha = 0$ . The potential density  $u$  satisfies*

$$u(t) \sim \frac{\phi'(1/t)}{t^2\phi(1/t)^2}, \quad t \rightarrow 0+ .$$

In the case  $\alpha = 2$  we get the following results.

**Theorem 1.5.** *Assume (A-1), (A-2) and (C) with  $\alpha = 2$ . Then*

$$\mu(t) \sim \frac{1}{t} \frac{d}{dt} (t\phi(1/t)), \quad t \rightarrow 0+ .$$

**Theorem 1.6.** *Assume (A-1) and (A-3) with  $\alpha = 2$ . Then the following is true*

$$u(t) \sim \frac{1}{\phi'(1/t)} \sim \frac{1}{t\phi(1/t)}, \quad t \rightarrow 0+ .$$

Let  $B = (B_t : t \geq 0)$  be a Brownian motion in  $\mathbb{R}^d$  with transition density

$$p_0(t, x, y) = (4\pi t)^{-d/2} \exp \left\{ -\frac{|x-y|^2}{4t} \right\}$$

(note that  $B$  is twice faster than the standard Brownian motion in  $\mathbb{R}^d$ ). We define a new stochastic process  $X = (X_t : t \geq 0)$  by setting  $X_t = B_{S_t}$ ,  $t \geq 0$ . The process  $X$  is then a Lévy process such that

$$\mathbb{E} e^{i\langle \xi, X_t \rangle} = e^{-t\phi(|\xi|^2)}, \quad \xi \in \mathbb{R}^d, \quad t \geq 0 .$$

For example, if  $S$  is a geometric stable subordinator, then the corresponding subordinate Brownian motion  $X$  is called geometric stable process.

We say that a bounded function  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  is *harmonic* in a nonempty bounded open set  $D \subset \mathbb{R}^d$  if for any open set  $B \subset \mathbb{R}^d$  such that  $\bar{B} \subset D$  we have

$$h(x) = \mathbb{E}_x [h(X_{\tau_B}); \tau_B < \infty], \quad \text{for all } x \in B .$$

The following regularity result covers known results for  $\alpha \in (0, 2)$ , but is new for  $\alpha \in \{0, 2\}$ .

**Theorem 1.7.** *Assume  $d \geq 3$ . There is a constant  $c > 0$  such that for any  $0 < r < 1/4$  and any bounded function  $h: \mathbb{R}^d \rightarrow \mathbb{R}$  which is harmonic in  $B_{4r}(0)$  we have*

$$|h(x) - h(y)| \leq c \|h\|_\infty \frac{\phi(r^{-2})}{\phi(|x-y|^{-2})} \varphi_\alpha \left( \frac{r}{|x-y|} \right) \quad \text{for all } x, y \in B_{r/4}(0),$$

where  $\varphi_\alpha: (0, 1) \rightarrow \mathbb{R}$  is defined by

$$\varphi_\alpha(s) = \begin{cases} (1 - 2^{\alpha-1})^{-1} & 0 \leq \alpha < 1 \\ \log s & \alpha = 1 \\ (2^{\alpha-1} - 1)^{-1} s^{\alpha-1} & 1 < \alpha \leq 2. \end{cases}$$

It is interesting that for  $\alpha = 0$  we do not get Hölder continuity of harmonic functions. For example, in the case of geometric stable subordinator we get logarithmic modulus of continuity of harmonic functions

$$|h(x) - h(y)| \leq c \log(r^{-1}) \|h\|_\infty \frac{1}{\log(|x-y|^{-1})}.$$

On the other hand, for the subordinator conjugate to the geometric stable subordinator, the corresponding subordinate Brownian motion will be closer to Brownian motion than any isotropic stable process. In this case harmonic functions are more than locally Hölder continuous:

$$|h(x) - h(y)| \leq \frac{c}{r \log(r^{-1})} \|h\|_\infty |x-y| \log(|x-y|^{-1}).$$

In the case of the isotropic symmetric  $\alpha$ -stable process (i.e.  $\phi(\lambda) = \lambda^{\alpha/2}$ ) with  $\alpha \in (1, 2)$  Theorem 1.7 gives local Lipschitz continuity of harmonic functions.

Subordinate Brownian motions with  $\alpha \in (0, 2)$  belong to the class of stable-like jump processes. The potential theory of these processes is well investigated (cf. [BL02, CK03, SV04, BS05, BK05, RSV06, KS07, CK08, Mim10, Szt10]). It is known that for many stable-like jump processes harmonic functions are Hölder continuous, scale invariant Harnack inequality holds and two-sided heat kernel estimates are obtained.

On the other hand, not much is known about harmonic functions in the case when the corresponding subordinators are slow or fast (cf. [ŠSV06, Mim11a, Mim11b]). For the class of geometric stable processes only the non-scale invariant Harnack inequality was proved and on-diagonal heat kernel upper estimate is not finite (cf. [ŠSV06]).

The paper is organized as follows. In Section 2 we introduce all concepts we need throughout the paper. Section 3 is devoted to the study of subordinators. We obtain asymptotic properties of Lévy and potential densities with the help of Tauberian theorems. Section 4 is devoted to subordinate Brownian motions and obtaining asymptotics of the corresponding Green function and Lévy density. Difference estimates of the Green function are the main subject of Section 5. This type of estimates are the main ingredient in the proof of the regularity result in Section 6. In Section 7 we apply our results to a few examples including iterated variance gamma processes.

**Notation.** For two functions  $f$  and  $g$  we write  $f \sim g$  if  $f/g$  converges to 1. The  $n$ -th derivative of  $f$  (if exists) is denoted by  $f^{(n)}$ .

The logarithm with base  $e$  is denoted by  $\log$  and we introduce the following notation for iterated logarithms:  $\log_1 = \log$  and  $\log_{k+1} = \log \circ \log_k$  for  $k \in \mathbb{N}$ .

The standard Euclidian norm and the standard inner product in  $\mathbb{R}^d$  are denoted by  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$ , respectively. By  $B_r(x) = \{y \in \mathbb{R}^d: |y - x| < r\}$  we denote the open ball centered at  $x$  with radius  $r > 0$ . The Gamma function is defined by  $\Gamma(\rho) = \int_0^\infty t^{\rho-1} e^{-t} dt$  for  $\rho > 0$ .

## 2. PRELIMINARIES

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A stochastic process  $Y = (Y_t: t \geq 0)$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a *Lévy process* in  $\mathbb{R}^d$  ( $d \geq 1$ ) if it has stationary and independent increments with paths that are  $\mathbb{P}$ -a.s. right-continuous with left limits and starting at 0.

The characteristic function of  $Y_t$  is of the form

$$\mathbb{E} e^{i\langle \xi, Y_t \rangle} = e^{-t\Phi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

where  $\Phi$  is called the *characteristic exponent* of  $Y$  and it is given by

$$\Phi(\xi) = i\langle \xi, \gamma \rangle + \frac{1}{2}\langle Q\xi, \xi \rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, x \rangle} + i\langle \xi, x \rangle \mathbb{1}_{\{|x|<1\}}) \Pi(dx).$$

Here  $\gamma \in \mathbb{R}^d$ ,  $Q$  is a  $d \times d$  positive semidefinite matrix and  $\Pi$  is a measure on  $\mathbb{R}^d$  satisfying

$$\Pi(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge |x|^2) \Pi(dx) < \infty,$$

called the *Lévy measure*.

A stochastic process  $S = (S_t: t \geq 0)$  is called a *subordinator* if it is a Lévy process taking values in  $[0, \infty)$  and such that  $S_0 = 0$ . In this case it is convenient to use Laplace transform of  $S_t$ , which is always of the form

$$\mathbb{E} e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad \lambda > 0,$$

We call  $\phi$  the *Laplace exponent* of  $S$  and it has the following representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) M(dt), \quad (2.1)$$

where  $b \geq 0$  and the Lévy measure  $M$  is now a measure on  $(0, \infty)$  satisfying

$$\int_{(0,\infty)} (1 \wedge t) M(dt) < \infty. \quad (2.2)$$

The *potential measure*  $U$  of the subordinator  $S$  is a measure on  $(0, \infty)$  defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{\{S_t \in A\}} dt.$$

For a measure  $N$  on  $[0, \infty)$  we define its *Laplace transform* of  $N$  by

$$\mathcal{L}N(\lambda) = \int_0^\infty e^{-\lambda t} N(dt), \quad \lambda > 0.$$

When  $N$  has a density  $\nu$  (with respect to the Lebesgue measure) we have

$$\mathcal{L}N(\lambda) = \int_0^\infty e^{-\lambda t} \nu(t) dt.$$

We can calculate the Laplace transform of  $U$  as follows

$$\mathcal{L}U(\lambda) = \int_0^\infty \int_{[0,\infty)} e^{-\lambda u} \mathbb{P}(S_t \in du) dt = \int_0^\infty \mathbb{E} e^{-\lambda S_t} dt = \int_0^\infty e^{-t\phi(\lambda)} dt = \frac{1}{\phi(\lambda)}. \quad (2.3)$$

A function  $f: (0, \infty) \rightarrow (0, \infty)$  is a *Bernstein function* if

$$f \in C^\infty((0, \infty)) \quad \text{and} \quad (-1)^n f^{(n)} \leq 0 \quad \text{for all } n \in \mathbb{N}.$$

We say that  $g: (0, \infty) \rightarrow (0, \infty)$  is a *completely monotone function* if

$$g \in C^\infty((0, \infty)) \quad \text{and} \quad (-1)^n g^{(n)} \geq 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

If  $f$  is a Bernstein function, then  $f'$  and  $e^{-uf}$  are completely monotone functions for any  $u > 0$ .

Any Bernstein function  $f$  has the following representation

$$f(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) M(dt), \quad (2.4)$$

where  $a, b \geq 0$  and  $M$  is a measure on  $(0, \infty)$  satisfying (2.2). Therefore the Laplace exponent of subordinator is always a Bernstein function.

Conversely, every Bernstein function  $f$  with representation (2.4) is the Laplace exponent of a (possibly killed) subordinator. More precisely, let  $S$  be a subordinator with Laplace exponent  $\phi(\lambda) = f(\lambda) - a$  and let  $E$  be an independent exponential

random variable with mean  $1/a$ . The subordinator *killed with rate  $a$*  is a stochastic process  $\hat{S} = (\hat{S}_t : t \geq 0)$  defined by

$$\hat{S}_t = \begin{cases} S_t & t < E \\ \infty & t \geq E. \end{cases}$$

Then

$$\mathbb{E}e^{-\lambda\hat{S}_t} = \mathbb{E}[e^{\lambda S_t}] \mathbb{P}(E > t) = e^{t\phi(\lambda)} e^{-at} = e^{-f(\lambda)}.$$

A Bernstein function  $f$  is called a *complete Bernstein function* if the Lévy measure in representation (2.4) has a completely monotone density  $\mu$  (with respect to the Lebesgue measure).

For example, the following functions are complete Bernstein:

$$\lambda^\beta \ (0 \leq \beta \leq 1), \ \log(1 + \lambda) \quad \text{and} \quad \frac{\lambda}{a + \lambda} \ (a > 0). \quad (2.5)$$

If  $f_1, f_2$  and  $f$  are complete Bernstein functions the following functions are also complete Bernstein

$$f_1 + f_2, \ f_1 \circ f_2 \quad \text{and} \quad f^*, \quad (2.6)$$

where  $f^*$  is the *conjugate* of  $f$  defined by  $f^*(\lambda) = \lambda/f(\lambda)$ .

A Bernstein function  $f$  is called a *special Bernstein function* if  $f^*$  is also a Bernstein function. In particular, any complete Bernstein function is also a special Bernstein functions.

If the Laplace exponent  $\phi$  of a subordinator  $S$  is a Bernstein function such that its Lévy measure  $M$  is infinite, then the potential measure  $U$  has a non-increasing density  $u$  if and only if  $\phi$  is a special Bernstein function. Moreover, if  $T$  is the (possibly killed) subordinator corresponding to the Bernstein function  $\phi^*$  and if its Lévy measure is denoted by  $N$ , we have

$$u(t) = N(t, \infty), \ t > 0. \quad (2.7)$$

For all the details concerning Bernstein functions the reader is referred to [SSV10].

We say that  $f: (0, \infty) \rightarrow (0, \infty)$  *varies regularly* at infinity with index  $\rho \in \mathbb{R}$  if

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho \quad \text{for} \quad \lambda > 0.$$

If  $f$  varies regularly with index  $\rho = 0$  we say also that  $f$  *varies slowly* at infinity.

Any function  $f$  varying regularly with index  $\rho \in \mathbb{R}$  can be represented as

$$f(x) = x^\rho \ell(x)$$

for some slowly varying function  $\ell$ . Regular and slow variation at 0 is defined similarly. In this paper we will repeatedly use the following theorems from the theory of regular variation.

- (1) *Karamata's Theorem.* Assume that  $\ell$  is locally bounded function that varies slowly at infinity. If  $\rho > -1$ , then

$$\int_1^x t^\rho \ell(t) dt \sim \frac{x^{\rho+1} \ell(x)}{\rho+1}, \quad x \rightarrow \infty$$

and if  $\rho < -1$ , then

$$\int_x^\infty t^\rho \ell(t) dt \sim \frac{x^{\rho+1}}{-\rho-1}, \quad x \rightarrow \infty.$$

If  $\ell$  varies slowly at 0, then for  $\rho < -1$

$$\int_x^1 t^\rho \ell(t) dt \sim \frac{x^{\rho+1} \ell(x)}{-\rho-1}, \quad x \rightarrow 0+$$

and for  $\rho > -1$

$$\int_0^x t^\rho \ell(t) dt \sim \frac{x^{\rho+1}}{\rho+1}, \quad x \rightarrow 0+.$$

- (2) *Karamata's Tauberian theorem.* Let  $U$  be a measure on  $[0, \infty)$ ,  $c > 0$ ,  $\rho \geq 0$  and let  $\ell$  be a locally bounded function that varies slowly at infinity. Then the following is true

$$U([0, x]) \sim c \frac{x^\rho \ell(x)}{\Gamma(1+\rho)}, \quad x \rightarrow \infty \quad \iff \quad \mathcal{L}U(\lambda) \sim c \lambda^{-\rho} \ell(1/\lambda), \quad \lambda \rightarrow 0+.$$

The statement stays true if  $\ell$  varies slowly at 0,  $x \rightarrow 0+$  and  $\lambda \rightarrow \infty$ .

- (3) *Karamata's monotone density theorem.* If  $U$  has density  $u$  which is ultimately monotone,  $c \in \mathbb{R}$  and  $\rho \in \mathbb{R}$ , then

$$U([0, x]) \sim c \frac{x^\rho \ell(x)}{\Gamma(1+\rho)}, \quad x \rightarrow \infty \quad \iff \quad u(x) \sim c \rho x^{\rho-1} \ell(x), \quad x \rightarrow \infty.$$

The above equivalence holds true when  $\ell$  varies slowly at 0,  $\rho > 0$  and  $x \rightarrow 0+$ .

Concerning all the details of this theory, the reader is referred to [\[BGT87\]](#).

### 3. ASYMPTOTIC PROPERTIES OF SUBORDINATORS

Let  $S = (S_t : t \geq 0)$  be a subordinator with the Laplace exponent satisfying **(A-1)**, **(A-2)** and **(A-3)**. The aim of this section is to obtain behavior of Lévy and potential density of  $S$  near zero.

**3.1. Slow subordinators.** In this subsection we treat the case  $\alpha = 0$  and thus we additionally assume **(B)**. This case requires finer analysis than the other cases.

Our first goal is to prove Theorem 1.3. First we need the following technical lemma.

**Lemma 3.1.** (i) *The Laplace exponent  $\phi$  varies slowly at infinity.*  
(ii) *For any  $x > 0$*

$$\frac{\phi(\lambda x) - \phi(\lambda)}{\lambda \phi'(\lambda)} \rightarrow \log x, \quad \lambda \rightarrow \infty.$$

*In particular, for any  $x > 1$ , the function  $\lambda \mapsto \phi(\lambda x) - \phi(\lambda)$  varies slowly at infinity.*

*Proof.* (i) Since  $\ell(\lambda) = \lambda \phi'(\lambda)$  varies slowly at infinity, by [BGT87, Proposition 1.5.9a] we conclude that  $\phi(\lambda) = \int_0^\lambda \ell(t) \frac{dt}{t}$  also varies slowly at infinity.

(ii) Let  $x > 0$ . Then

$$\frac{\phi(\lambda x) - \phi(\lambda)}{\ell(\lambda)} = \int_1^x \frac{\ell(\lambda t)}{\ell(\lambda)} \frac{dt}{t} \rightarrow \log x, \quad \lambda \rightarrow \infty, \quad (3.1)$$

since, by the uniform convergence theorem for slowly varying functions (cf. [BGT87,

Theorem 1.2.1])  $\lim_{\lambda \rightarrow \infty} \frac{\ell(\lambda t)}{\ell(\lambda)} = 1$  uniformly in  $t \in [1, x]$ .

Let  $x > 1$ . Then by (3.1) we have

$$\frac{\phi(\lambda xy) - \phi(\lambda y)}{\phi(\lambda x) - \phi(\lambda)} \sim \frac{\ell(\lambda y)}{\ell(\lambda)} \sim 1, \quad \lambda \rightarrow \infty.$$

□

*Remark 3.2.* Since  $\phi$  varies slowly at infinity, it can be easily showed that the drift coefficient  $b$  in representation (2.1) is necessary zero.

*Proof of Theorem 1.3.* We assume that  $t \mapsto e^{at} \mu(t)$  is non-decreasing. Let  $x > 1$  and let  $V_x$  be a measure on  $(0, \infty)$  with density

$$v_x(t) = \int_t^{xt} e^{as} \mu(s) ds = t \int_1^x e^{ast} \mu(st) ds, \quad t > 0.$$

For  $\lambda > ax$  we can calculate the Laplace transform of  $V_x$

$$\begin{aligned} \mathcal{L}V_x(\lambda) &= \int_0^\infty e^{-\lambda t} v_x(t) dt \\ &= \int_0^\infty \int_1^x e^{-(\lambda-as)t} t \mu(st) ds dt \\ &= \int_1^x \left( \int_0^\infty e^{-(\frac{\lambda}{s}-a)t} t \mu(st) dt \right) \frac{ds}{s^2}. \end{aligned}$$

Since  $\phi'(\lambda) = \int_0^\infty e^{-\lambda t} t \mu(t) dt$ , we obtain

$$\begin{aligned} \mathcal{L}V_x(\lambda) &= \int_1^x \phi' \left( \frac{\lambda}{s} - a \right) \frac{ds}{s^2} \\ &= \frac{\phi(\lambda - a) - \phi\left(\frac{\lambda}{x} - a\right)}{\lambda}. \end{aligned}$$

Then Lemma 3.1 (ii) yields

$$\mathcal{L}V_x(\lambda) \sim \phi'(\lambda) \log x, \quad \lambda \rightarrow \infty$$

and thus by the Karamata's Tauberian theorem we obtain

$$V_x([0, t]) \sim \phi'(1/t) \log x, \quad t \rightarrow 0+,$$

since  $t \mapsto \phi'(1/t)$  varies regularly at 0 with index 1.

Note that for  $0 < t_1 \leq t_2 < \frac{T}{x}$  we have

$$v_x(t_1) = \int_1^{xt_1} st_1 e^{ast_1} \mu(st_1) \frac{ds}{s} \leq \int_1^{xt_2} st_2 e^{ast_2} \mu(st_2) \frac{ds}{s} = v_x(t_2)$$

by (B) and thus we can apply Karamata's monotone density theorem to get

$$v_x(t) \sim \frac{\phi'(1/t)}{t} \log x, \quad t \rightarrow 0+. \quad (3.2)$$

Now we have

$$\frac{(x-1)t^2 e^{at} \mu(t)}{\phi'(1/t)} \leq \frac{\int_t^{xt} e^{as} s \mu(s) ds}{\phi'(1/t)} \leq \frac{xt \int_t^{xt} e^{as} \mu(s) ds}{\phi'(1/t)}$$

which by (3.2) implies

$$\limsup_{t \rightarrow 0+} \frac{t^2 e^{at} \mu(t)}{\phi'(1/t)} \leq \frac{x \log x}{x-1} \quad \text{for any } x > 1.$$

By letting  $x \rightarrow 1+$  we get

$$\limsup_{t \rightarrow 0+} \frac{t^2 \mu(t)}{\phi'(1/t)} \leq 1.$$

For  $0 < x < 1$  we define

$$v_x(t) = \int_x^1 e^{as} \mu(s) ds$$

and proceed similarly to obtain

$$v_x(t) \sim \frac{\phi'(1/t)}{t} \log(1/x), \quad t \downarrow 0. \quad (3.3)$$

Then

$$\frac{(1-x)t^2 e^{at} \mu(t)}{\phi'(1/t)} \geq \frac{\int_{xt}^t e^{as} s \mu(s) ds}{\phi'(1/t)} \geq \frac{xt \int_{xt}^t e^{as} \mu(s) ds}{\phi'(1/t)},$$

which with (3.3) gives

$$\liminf_{t \rightarrow 0+} \frac{t^2 e^{at} \mu(t)}{\phi'(1/t)} \geq \frac{x \log(1/x)}{1-x} \quad \text{for any } 0 < x < 1.$$

Letting  $x \rightarrow 1-$  yields

$$\liminf_{t \rightarrow 0+} \frac{t^2 \mu(t)}{\phi'(1/t)} \geq 1.$$

□

*Proof of Theorem 1.4.* By formula (2.3) and Lemma 3.1 we have

$$\frac{\mathcal{L}U\left(\frac{1}{\lambda x}\right) - \mathcal{L}U\left(\frac{1}{\lambda}\right)}{\frac{1}{\lambda} \phi'\left(\frac{1}{\lambda}\right) \phi\left(\frac{1}{\lambda}\right)^2} = \frac{\phi\left(\frac{1}{\lambda}\right) - \phi\left(\frac{1}{\lambda x}\right)}{\frac{1}{\lambda} \phi'\left(\frac{1}{\lambda}\right)} \frac{\phi\left(\frac{1}{\lambda}\right)}{\phi\left(\frac{1}{\lambda x}\right)} \rightarrow \log x, \quad \lambda \rightarrow 0+$$

for any  $x > 0$ . Now we can apply de Haan's Tauberian theorem (cf. 0-version of [BGT87, Theorem 3.9.1]) to deduce

$$\frac{U(\lambda x) - U(\lambda)}{\frac{1}{\lambda} \phi'\left(\frac{1}{\lambda}\right) \phi\left(\frac{1}{\lambda}\right)^2} \rightarrow \log x, \quad \lambda \rightarrow 0+.$$

If we apply de Haan's monotone density theorem (cf. [BGT87, Theorem 3.6.8]) we finally obtain

$$u(t) \sim \frac{\phi'(1/t)}{t^2 \phi(1/t)^2}, \quad t \rightarrow 0+.$$

□

**Corollary 3.3.** *We have the following asymptotic properties of the Lévy and potential densities*

$$\mu(t) \sim -\frac{d}{dt} \phi(1/t) \quad \text{and} \quad u(t) \sim \frac{d}{dt} \frac{1}{\phi(1/t)}, \quad t \rightarrow 0+.$$

**3.2. Fast subordinators.** We assume that  $S$  satisfies (A-1), (A-2), (A-3) and (C) with  $\alpha = 2$ . In this case  $\phi'$  varies slowly at infinity. Using Karamata's theorem we have

$$\phi(\lambda) = \phi(1) + \int_1^\lambda \phi'(t) dt \sim \lambda \phi'(\lambda), \quad \lambda \rightarrow \infty. \quad (3.4)$$

In particular,  $\phi$  varies regularly at infinity with index 1.

*Proof of Theorem 1.5.* Since potential density exists by (A-3) we see that  $\phi$  is a special Bernstein function and thus  $\phi^*(\lambda) = \lambda/\phi(\lambda)$  defines the Laplace exponent of a (possibly killed) subordinator, which we denote by  $T$ .

Note that the subordinator  $T$  corresponds to the case of  $\alpha = 0$ . If we denote potential density of  $T$  by  $v$ , we have (cf. [SSV10, Corollary 10.8] or (2.7))

$$v(t) = M(t, \infty), \quad t > 0.$$

Theorem 1.4 yields

$$M(t, \infty) \sim \frac{(\phi^*)'(1/t)}{t^2 \phi^*(1/t)^2}, \quad t \rightarrow 0+.$$

By assumption (C) we know that  $t \mapsto (\phi^*)'(1/t)$  varies regularly at 0 with index 1 and thus  $t \mapsto \frac{(\phi^*)'(1/t)}{t^2 \phi^*(1/t)^2}$  varies regularly at 0 with index  $-1$ . Using the Karamata's monotone density theorem we deduce

$$\mu(t) \sim \frac{(\phi^*)'(1/t)}{t^3 \phi^*(1/t)^2} = \frac{1}{t} \frac{d}{dt} (t \phi(1/t)), \quad t \rightarrow 0+.$$

□

*Proof of Theorem 1.6.* By (3.4) we get

$$\mathcal{L}U(\lambda) = \frac{1}{\phi(\lambda)} \sim \frac{1}{\lambda \phi'(\lambda)}, \quad \lambda \rightarrow \infty$$

and thus by Karamata's Tauberian theorem it follows that

$$U([0, t]) \sim \frac{1}{\Gamma(2)} \frac{t}{\phi'(1/t)}, \quad t \rightarrow 0+$$

since  $\lambda \mapsto \lambda \phi'(\lambda)$  varies regularly at infinity with index 1. By applying Karamata's monotone density theorem we deduce

$$u(t) \sim \frac{1}{\phi'(1/t)}, \quad t \rightarrow 0+.$$

Now the result follows from (3.4). □

**3.3. Stable-like subordinators.** Here we assume that subordinator  $S$  satisfies **(A-1)**, **(A-2)** and **(A-3)** with  $\alpha \in (0, 2)$ . In this case  $\phi'$  varies regularly with index  $\alpha/2 - 1$  and so

$$\phi'(\lambda) = \lambda^{\alpha/2-1} \ell(\lambda), \quad (3.5)$$

where  $\ell$  varies slowly at infinity. Using the Karamata's theorem we obtain

$$\phi(\lambda) = \phi(1) + \int_1^\lambda \phi'(t) dt \sim \frac{2}{\alpha} \lambda^{\alpha/2} \ell(\lambda), \quad \lambda \rightarrow \infty. \quad (3.6)$$

**Theorem 3.4.** *We have the following asymptotic properties of the Lévy and potential density:*

$$\mu(t) \sim \frac{\alpha}{2\Gamma(1 - \frac{\alpha}{2})} \frac{\phi(1/t)}{t} \quad \text{and} \quad u(t) \sim \frac{\alpha}{2\Gamma(1 - \frac{\alpha}{2})} \frac{1}{t \phi(1/t)}, \quad t \rightarrow 0+.$$

*Proof.* We follow ideas from the proof of [Mim10, Lemma 4.1]. Since

$$\mathcal{L}U(\lambda) = \frac{1}{\phi(\lambda)}$$

varies regularly at infinity with index  $-\alpha/2$ , Karamata's Tauberian theorem implies

$$U([0, t]) \sim \frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{1}{\phi(1/t)}, \quad t \rightarrow 0+.$$

Then by Karamata's monotone density theorem we deduce

$$u(t) \sim \frac{\alpha}{2\Gamma(1 - \frac{\alpha}{2})} \frac{1}{t\phi(1/t)}, \quad t \rightarrow 0+.$$

Define measure  $V$  by

$$V(A) = \int_A M(s, \infty) ds, \quad A \in \mathcal{B}([0, \infty)),$$

where  $M$  is the Lévy measure of  $S$ . Integration by parts yields

$$\phi(\lambda) = \lambda \int_0^\infty e^{-\lambda t} M(t, \infty) dt.$$

Therefore

$$\mathcal{L}V(\lambda) = \frac{\phi(\lambda)}{\lambda}$$

and so by the Karamata Tauberian theorem we have

$$V([0, t]) \sim \frac{1}{\Gamma(2 - \frac{\alpha}{2})} t\phi(1/t), \quad t \rightarrow 0+.$$

Now Karamata's monotone density gives

$$M(t, \infty) \sim \frac{1 - \frac{\alpha}{2}}{\Gamma(2 - \frac{\alpha}{2})} \phi(1/t), \quad t \rightarrow 0+$$

and applying it one more time, we finally obtain

$$\mu(t) \sim \frac{\alpha(1 - \frac{\alpha}{2})}{2\Gamma(2 - \frac{\alpha}{2})} \frac{\phi(1/t)}{t}, \quad t \rightarrow 0+.$$

□

*Remark 3.5.* Similar results are obtained in [KSV, Theorems 2.9 and 2.10].

**Corollary 3.6.** *We have*

$$\mu(t) \sim \frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{\phi'(1/t)}{t^2} = -\frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{d}{dt} \phi(1/t), \quad t \rightarrow 0+$$

and

$$u(t) \sim \frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{\phi'(1/t)}{t^2\phi(1/t)^2} = \frac{1}{\Gamma(1 - \frac{\alpha}{2})} \frac{d}{dt} \frac{1}{\phi(1/t)}, \quad t \rightarrow 0+.$$

*Proof.* Using (3.5) and (3.6) we deduce

$$\frac{\phi'(\lambda)}{\phi(\lambda)} \sim \frac{\alpha}{2\lambda}, \quad \lambda \rightarrow \infty. \quad (3.7)$$

Combining this with Theorem 3.4 we finish the proof.  $\square$

#### 4. SUBORDINATE BROWNIAN MOTION

In this section we consider subordinate Brownian motions. Our main goal here is to obtain asymptotic properties of their Lévy density and Green function. Now let us be more precise.

Assume that  $S = (S_t : t \geq 0)$  is a subordinator satisfying properties **(A-1)**, **(A-2)** and **(A-3)** and additionally **(B)** when  $\alpha = 0$  and **(C)** when  $\alpha = 2$ . Let  $B = (B_t : t \geq 0)$  be an independent Brownian motion in  $\mathbb{R}^d$  with transition density

$$p_0(t, x, y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}.$$

Define new process  $X = (X_t : t \geq 0)$  by setting  $X_t = B_{S_t}$ ,  $t \geq 0$ . It follows from [Sat99, Theorem 30.1] that  $X$  is a purely discontinuous Lévy process with characteristic exponent  $\Phi(\xi) = \phi(|\xi|^2)$ . We can write it also as

$$\Phi(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, x \rangle} + i\langle \xi, x \rangle \mathbb{1}_{\{|x| \leq 1\}}) J(x) dx$$

where  $J(x) = j(|x|)$  with

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(t) dt. \quad (4.1)$$

In other words, the Lévy measure of  $X$  has a density  $J$  usually called the *Lévy density*.

The process  $X$  has also a transition density given by

$$p(t, x, y) = \int_0^\infty (4\pi s)^{-d/2} e^{-\frac{|x-y|^2}{4s}} \mathbb{P}(S_t \in ds). \quad (4.2)$$

The *Green function* of  $X$  is defined by

$$G(x, y) = \int_0^\infty p(t, x, y) dt. \quad (4.3)$$

If the integral in (4.3) is finite, we say that  $X$  is *transient*.

*Remark 4.1.* If the following integral is finite

$$\int_0^a \frac{\lambda^{d/2-1}}{\phi(\lambda)} d\lambda < \infty$$

for some  $a > 0$ , then the process  $X$  is transient (cf. [Sat99, Corollary 37.6]). In particular,  $X$  is transient if  $d \geq 3$ .

*Remark 4.2.* An interesting interpretation of the Green function can be seen from the following identity

$$\int_A G(x, y) dy = \mathbb{E}_x \left[ \int_0^\infty 1_{\{X_t \in A\}} dt \right].$$

In other words, if we integrate  $y \mapsto G(x, y)$  over a measurable set  $A \subset \mathbb{R}^d$ , we obtain the mean time spent in the set  $A$  by the process  $X$  started at  $x$ .

For the rest of the paper we assume that  $d \geq 3$ . In particular,  $X$  is transient.

*Remark 4.3.* We can prove similar results in the case when  $X$  is transient and  $d = 1$  or  $2$ . Then we need some additional assumptions on the behavior of the Laplace exponent  $\phi$  at infinity.

From (4.2) and (4.3) we can see that

$$G(x, y) = G(y - x) = g(|y - x|)$$

with

$$g(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} u(t) dt, \quad r > 0. \quad (4.4)$$

Note that functions  $j$  and  $g$  are non-increasing.

The following lemma will be useful in this and in the next section.

**Lemma 4.4.** *Let  $w: (0, \infty) \rightarrow (0, \infty)$  be a decreasing function satisfying*

$$w(t) \sim t^{-b} \ell(t), \quad t \rightarrow 0+,$$

*for a locally bounded function  $\ell: (0, \infty) \rightarrow (0, \infty)$  that varies slowly at 0 and  $b \geq 0$ .*

*If  $p > 1$  and  $a > 0$ , then*

$$I(r) = \int_0^\infty t^{-p} e^{-\frac{ar}{t}} w(t) dt, \quad r > 0,$$

*satisfies*

$$I(r) \sim a^{-p-b+1} \Gamma(p+b-1) r^{-p+1} w(r), \quad r \rightarrow 0+.$$

*Proof.* We follow ideas from [KSV, Lemma 3.1]. After change of variable we have

$$\begin{aligned} I(r) &= (ar)^{-p+1} \int_0^\infty e^{-t} t^{p-2} w\left(\frac{ar}{t}\right) dt \\ &= a^{-p-b+1} r^{-p+1} w(r) \int_0^\infty e^{-t} t^{p+b-2} \frac{w\left(\frac{ar}{t}\right)}{\left(\frac{ar}{t}\right)^{-b} \ell\left(\frac{ar}{t}\right)} \frac{r^{-b} \ell(r)}{w(r)} \frac{\ell\left(\frac{ar}{t}\right)}{\ell(r)} dt. \end{aligned}$$

We may apply dominated convergence theorem, since by theorem of Potter (cf. [BGT87, Theorem 1.5.6 (ii)]) for any  $0 < \delta < p + b - 1$  there exists a constant  $A(\delta) > 0$  such that

$$\frac{\ell\left(\frac{ar}{t}\right)}{\ell(r)} \leq A(\delta) \left[ \left(\frac{a}{t}\right)^\delta \vee \left(\frac{a}{t}\right)^{-\delta} \right] \quad \text{for any } t > 0.$$

Thus

$$\lim_{r \rightarrow 0+} \frac{I(r)}{r^{-p-b+1} \ell(r)} = a^{-p-b+1} \int_0^\infty e^{-t} t^{p+b-2} dt.$$

□

**Theorem 4.5.** (i) For  $0 \leq \alpha < 2$  we have

$$j(r) \sim \frac{2^\alpha}{\pi^{d/2}} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} r^{-d-2} \phi'(r^{-2}), \quad r \rightarrow 0+$$

and

$$g(r) \sim \frac{1}{2^\alpha \pi^{d/2}} \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})} r^{-d-2} \frac{\phi'(r^{-2})}{[\phi(r^{-2})]^2}, \quad r \rightarrow 0+.$$

(ii) For  $\alpha = 2$  we have

$$j(r) \sim \frac{4\Gamma(\frac{d+2}{2})}{\pi^{d/2}} r^{-d-2} [r^2 \phi(r^{-2}) - \phi'(r^{-2})], \quad r \rightarrow 0+$$

and

$$g(r) \sim \frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}} r^{-d+2} \frac{1}{\phi'(r^{-2})}, \quad r \rightarrow 0+.$$

*Proof.* Let  $0 \leq \alpha < 2$ . Then by (4.1) we have

$$j(r) = (4\pi)^{-d/2} \int_0^\infty t^{-d/2} e^{-\frac{r^2}{4t}} \mu(t) dt$$

with

$$\mu(t) \sim \frac{t^{-2} \phi'(1/t)}{\Gamma(1-\frac{\alpha}{2})}, \quad t \rightarrow 0+$$

by Theorem 1.3 and Corollary 3.6. Since  $\mu$  varies regularly at 0 with index  $-\alpha/2 - 1$  we can use Lemma 4.4 with  $a = 1/4$ ,  $b = 1 + \alpha/2$  and  $p = d/2$  to get

$$\begin{aligned} j(r) &= \frac{4^{\alpha/2} \Gamma(\frac{d}{2} + \frac{\alpha}{2})}{\pi^{d/2}} (r^2)^{-d/2+1} \mu(r^2) \\ &\sim \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2} \Gamma(1-\frac{\alpha}{2})} r^{-d-2} \phi'(r^{-2}), \quad r \rightarrow 0+. \end{aligned}$$

Similarly, by (4.4) we have

$$g(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} u(t) dt, \quad r > 0$$

and

$$u(t) \sim \frac{\phi'(1/t)}{\Gamma(1-\frac{\alpha}{2}) t^2 \phi(1/t)^2}, \quad t \rightarrow 0+$$

by Theorem 1.4 and Corollary 3.6. Since  $u$  varies regularly at 0 with index  $\alpha/2 - 1$ , from Lemma 4.4 with  $a = 1/4$ ,  $b = 1 - \alpha/2$  and  $p = d/2$  we get

$$\begin{aligned} g(r) &= \frac{4^{-\alpha/2} \Gamma\left(\frac{d}{2} - \frac{\alpha}{2}\right)}{\pi^{d/2}} (r^2)^{-d/2+1} u(r^2) \\ &\sim \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2} \Gamma\left(1 - \frac{\alpha}{2}\right)} r^{-d-2} \frac{\phi'(r^{-2})}{\phi(r^{-2})^2}, \quad r \rightarrow 0+. \end{aligned}$$

When  $\alpha = 2$  we use the same ideas with the help of Theorems 1.5 and 1.6, which give

$$\mu(t) \sim \frac{t^2 \phi(1/t) - \phi'(1/t)}{t^2}, \quad t \rightarrow 0+$$

and

$$u(t) \sim \frac{1}{\phi'(1/t)}, \quad t \rightarrow 0+.$$

□

**Corollary 4.6.** *There exist constants  $c, c' > 0$  depending on  $d$  and  $\alpha$  such that for all  $0 < r < 1$  we have*

$$\int_0^r s^{d-1} g(s) ds \leq \frac{c}{\phi(r^{-2})} \quad \text{and} \quad \int_r^1 s^{d-1} j(s) ds \leq c' \phi(r^{-2}).$$

*Proof.* Assume first that  $0 \leq \alpha < 2$ . Then by Theorem 4.5 (i) we get

$$\int_0^r s^{d-1} g(s) ds \leq c_1 \int_0^r s^{-3} \frac{\phi'(s^{-2})}{[\phi(s^{-2})]^2} ds = \frac{c_1}{2\phi(r^{-2})}$$

and

$$\int_r^1 s^{d-1} j(s) ds \leq c_2 \int_r^1 s^{-3} \phi'(s^{-2}) ds \leq \frac{c_2}{2} \phi(r^{-2}).$$

Let  $\alpha = 2$ . Then  $t \mapsto \phi'(t^{-2})$  varies slowly at 0 and  $t \mapsto \phi(r^{-2})$  varies regularly at 0 with index  $-2$ . Using Karamata's theorem and (3.4) we obtain from Theorem 4.5 (ii) the following

$$\int_0^r s^{d-1} g(s) ds \leq c_3 \int_0^r \frac{s}{\phi'(s^{-2})} ds \leq c_4 \frac{r^2}{\phi'(r^{-2})} \leq \frac{c_5}{\phi(r^{-2})}$$

and

$$\int_r^1 s^{d-1} j(s) ds \leq c_6 \left[ r^{-1} \int_r^1 \phi(s^{-2}) ds - \int_r^1 s^{-3} \phi'(s^{-2}) ds \right] \leq c_7 \phi(r^{-2}).$$

□

## 5. DIFFERENCE ESTIMATES

Let  $D \subset \mathbb{R}^d$  ( $d \geq 3$ ) be an open set and let  $X$  be a subordinate Brownian motion considered in Section 4. By  $X^D$  we denote the stochastic process obtained by killing the process  $X$  upon exiting set  $D$ , i.e.

$$X_t^D = \begin{cases} X_t & t < \tau_D \\ \Delta & t \geq \tau_D \end{cases}$$

where  $\Delta$  is an extra point adjoined to  $D$  and

$$\tau_D := \inf\{t > 0: X_t \notin D\}$$

is the *first exit time* from the set  $D$ . The process  $X$  has also a transition probability density  $p^D(t, x, y)$  given by

$$p^D(t, x, y) = p(t, x, y) - \mathbb{E}_x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t], \quad x, y \in D.$$

The Green function of the process  $X^D$  is denoted by  $G_D(x, y)$  and we can easily check that the following formula holds

$$G_D(x, y) = \int_0^\infty p^D(t, x, y) dt = G(x, y) - \mathbb{E}_x[G(X_{\tau_D}, y)], \quad x, y \in D. \quad (5.1)$$

We remark that  $G_D(x, y)$  is a symmetric function.

Concerning exit distribution of the process  $X$  it follows from the fact that  $X$  is in particular isotropic Lévy process that for any  $r > 0$  we have

$$\mathbb{P}_x(X_{\tau_{B_r(0)}} \in \partial B_r(0)) = 0, \quad x \in B_r(0).$$

(cf. [Szt00]). This allows us to use the Ikeda-Watanabe formula proved in [IW62, Theorem 1]:

$$\mathbb{P}_x(X_{\tau_{B_r(0)}} \in F) = \int_F \int_{B_r(0)} G_{B_r(0)}(x, y) j(|z - y|) dy dz, \quad (5.2)$$

for  $x \in B_r(0)$  and  $F \subset \overline{B_r(0)}^c$ .

If we define a function  $K_{B_r(0)}: B_r(0) \times \overline{B_r(0)} \rightarrow [0, \infty)$  by

$$K_{B_r(0)}(x, z) = \int_{B_r(0)} G_{B_r(0)}(x, y) j(|z - y|) dy, \quad (5.3)$$

then we can rewrite the Ikeda-Watanabe formula (5.2) as

$$\mathbb{P}_x(X_{\tau_{B_r(0)}} \in F) = \int_F K_{B_r(0)}(x, z) dz.$$

The function  $K_{B_r(0)}$  will be called the *Poisson kernel* for the ball  $B_r(0)$ .

**Proposition 5.1.** *There is a constant  $c > 0$  such that for all  $0 < r < 1$  and  $x, y \notin B_r(0)$*

$$|G(x) - G(y)| \leq cg(r) \left( 1 \wedge \frac{|x - y|}{r} \right).$$

*Proof.* First we assume that  $|x - y| < r/2$ . By the mean value theorem we know that for any  $t > 0$  there exists  $\vartheta = \vartheta(x, y, t) \in [0, 1]$  such that

$$\begin{aligned} \left| e^{-\frac{|x|^2}{4t}} - e^{-\frac{|y|^2}{4t}} \right| &\leq \frac{|x + \vartheta(y - x)|}{2t} e^{-\frac{|x + \vartheta(y - x)|^2}{4t}} |x - y| \\ &\leq \frac{|x - y|}{\sqrt{t}} e^{-\frac{|x + \vartheta(y - x)|^2}{8t}}, \end{aligned}$$

where in the last line we have used the following simple inequality

$$se^{-s^2} < 2e^{-\frac{s^2}{2}}, \quad s > 0.$$

Since

$$|x + \vartheta(y - x)| \geq |x| - \vartheta|y - x| \geq \frac{r}{2},$$

we have

$$\left| e^{-\frac{|x|^2}{4t}} - e^{-\frac{|y|^2}{4t}} \right| \leq 2 \frac{|x - y|}{\sqrt{t}} e^{-\frac{r^2}{32t}}. \quad (5.4)$$

Now we can use (5.4) to get

$$\begin{aligned} |G(x) - G(y)| &\leq (4\pi)^{-d/2} \int_0^\infty t^{-d/2} \left| e^{-\frac{|x|^2}{4t}} - e^{-\frac{|y|^2}{4t}} \right| u(t) dt \\ &\leq 2(4\pi)^{-d/2} |x - y| \int_0^\infty t^{-d/2-1/2} e^{-\frac{r^2}{32t}} u(t) dt. \end{aligned}$$

Since  $u$  is non-increasing and varies regularly at 0 with index  $\alpha/2 - 1$ , by Lemma 4.4 we get that for some constant  $c_1 > 0$

$$\int_0^\infty t^{-d/2-1/2} e^{-\frac{r^2}{32t}} u(t) dt \leq c_1 r^{-d+1} u(r^2) \quad \text{for } 0 < r < 1$$

and thus, by Theorem 4.5,

$$|G(x) - G(y)| \leq c_2 g(r) \frac{|x - y|}{r}.$$

In the other case we have a simpler estimate

$$|G(x) - G(y)| \leq G(x) + G(y) \leq 2g(r)$$

since  $|x|, |y| \geq r$ . □

**Proposition 5.2.** *There is a constant  $c > 0$  such that for all  $0 < R < 1$ ,  $0 < r \leq R/2$ ,  $y \in B_R(0)$  and  $x_1, x_2 \in B_{R/2}(0) \setminus B_r(y)$*

$$|G_{B_R(0)}(x_1, y) - G_{B_R(0)}(x_2, y)| \leq cg(r) \left( 1 \wedge \frac{|x_1 - x_2|}{r} \right).$$

*Proof.* By symmetry of the Green function we have

$$\begin{aligned} G_{B_R(0)}(x_i, y) &= G_{B_R(0)}(y, x_i) = G(x_i - y) - \mathbb{E}_y[G(X_{\tau_{B_R(0)}} - x_i)] \\ &= G(x_i - y) - \mathbb{E}_y[G(X_{\tau_{B_R(0)}} - x_i)], \end{aligned}$$

for  $i \in \{1, 2\}$ . Now the result follows from Proposition 5.1.  $\square$

The last part of this section is devoted to difference estimates of the Poisson kernel. Recall that  $\varphi_\alpha: (0, 1) \rightarrow \mathbb{R}$  is defined by

$$\varphi_\alpha(s) = \begin{cases} (1 - 2^{\alpha-1})^{-1} & 0 \leq \alpha < 1 \\ \log s & \alpha = 1 \\ (2^{\alpha-1} - 1)^{-1} s^{\alpha-1} & 1 < \alpha \leq 2. \end{cases}$$

**Proposition 5.3.** *There is a constant  $c > 0$  such that for any  $0 < r < 1$  and  $x, y \in B_{r/8}(0)$ ,  $x \neq y$  the following is true:*

(i) *if  $z \in B_{2r}(0) \setminus B_r(0)$ , then*

$$|K_{B_r(0)}(x, z) - K_{B_r(0)}(y, z)| \leq c|z|^{-d} \frac{\phi((|z| - r)^{-2})}{\phi(|x - y|^{-2})} \varphi_\alpha \left( \frac{r}{|x - y|} \right);$$

(ii) *if  $z \notin B_{2r}(0)$ , then*

$$|K_{B_r(0)}(x, z) - K_{B_r(0)}(y, z)| \leq c \frac{j\left(\frac{|z|}{2}\right)}{\phi(|x - y|^{-2})} \varphi_\alpha \left( \frac{r}{|x - y|} \right).$$

*Proof.* In the estimate

$$|K_{B_r(0)}(x, z) - K_{B_r(0)}(y, z)| \leq \int_{B_r(0)} |G_{B_r(0)}(x, v) - G_{B_r(0)}(y, v)| j(|z - v|) dv$$

we split the integral into three parts:

$$\begin{aligned} I_1 &= \int_{B_{2|x-y|}(x)} |G_{B_r(0)}(x, v) - G_{B_r(0)}(y, v)| j(|z - v|) dv \\ I_2 &= \int_{B_{r/4}(x) \setminus B_{2|x-y|}(x)} |G_{B_r(0)}(x, v) - G_{B_r(0)}(y, v)| j(|z - v|) dv \\ I_3 &= \int_{B_r(0) \setminus B_{r/4}(x)} |G_{B_r(0)}(x, v) - G_{B_r(0)}(y, v)| j(|z - v|) dv. \end{aligned}$$

For the first part we have

$$\begin{aligned}
I_1 &\leq \int_{B_{2|x-y|}(x)} G_{B_r(0)}(x, v) j(|z-v|) dv + \int_{B_{3|x-y|}(y)} G_{B_r(0)}(y, v) j(|z-v|) dv \\
&\leq 2j\left(\frac{|z|}{2}\right) \int_{B_{3|x-y|}(0)} G(v) dv \leq c_1 \frac{j\left(\frac{|z|}{2}\right)}{\phi(|x-y|^{-2})}, \tag{5.5}
\end{aligned}$$

for any  $z \notin B_r(0)$ . In the last inequality in (5.5) we have used Corollary 4.6.

In order to estimate  $I_2$  we split the integral in the following way. We let  $N = \left\lfloor \frac{\log \frac{r}{4|x-y|}}{\log 2} \right\rfloor$  and write

$$I_2 \leq \sum_{n=1}^N \int_{B_{2^{n+1}|x-y|}(x) \setminus B_{2^n|x-y|}(x)} |G_{B_r(0)}(x, v) - G_{B_r(0)}(y, v)| j(|z-v|) dv.$$

Now, for each  $n \in \{1, \dots, N\}$  we can apply Proposition 5.2 (with  $r = (2^n - 1)|x-y|$  and  $R = 2^{n+1}|x-y|$ ) to get

$$\begin{aligned}
&\int_{B_{2^{n+1}|x-y|}(x) \setminus B_{2^n|x-y|}(x)} |G_{B_r(0)}(x, v) - G_{B_r(0)}(y, v)| j(|z-v|) dv \\
&\leq c_3 \frac{g((2^n - 1)|x-y|)}{2^n - 1} \int_{B_{2^{n+1}|x-y|}(x)} j(|z-v|) dv.
\end{aligned}$$

From Theorem 4.5 it follows that  $g$  varies regularly at zero with index  $\alpha - d$ . By the uniform convergence theorem for regularly varying functions (cf. [BGT87, Theorem 1.5.2]) we conclude that for some constant  $c_4 > 0$  we have

$$\frac{g((2^n - 1)|x-y|)}{g(|x-y|)} \leq c_4 (2^n - 1)^{\alpha-d} \quad \text{for } n \in \mathbb{N}.$$

By Theorem 4.5 we obtain for  $0 \leq \alpha < 2$  that

$$g(|x-y|) \leq c_5 \frac{|x-y|^{-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \leq \frac{c_5}{\phi(|x-y|^{-2})}$$

since  $\lambda \phi'(\lambda) \leq \phi(\lambda)$  (recall that  $\phi^*$  is also a Bernstein function), while for  $\alpha = 2$  we have

$$g(|x-y|) \leq \frac{c_6}{|x-y|^{-2} \phi'(|x-y|^{-2})} \leq \frac{c_7}{\phi(|x-y|^{-2})},$$

since  $\lambda\phi'(\lambda) \sim \phi(\lambda)$ ,  $\lambda \rightarrow 0+$ . This gives

$$\begin{aligned}
I_2 &\leq c_8 \sum_{n=1}^N (2^n - 1)^{\alpha-d-1} g(|x-y|) (2^{n+1}|x-y|)^d j\left(\frac{|z|}{2}\right) \\
&\leq c_9 \frac{j\left(\frac{|z|}{2}\right)}{\phi(|x-y|^{-2})} \sum_{n=1}^N 2^{(\alpha-1)n} \\
&\leq c_{10} \frac{j\left(\frac{|z|}{2}\right)}{\phi(|x-y|^{-2})} \varphi_\alpha\left(\frac{r}{|x-y|}\right), \tag{5.6}
\end{aligned}$$

for any  $z \notin B_r(0)$ , since

$$\sum_{n=1}^N 2^{(\alpha-1)n} \leq \begin{cases} \sum_{n=1}^{\infty} 2^{-(1-\alpha)n} \leq (1-2^{\alpha-1})^{-1} & 0 \leq \alpha < 1 \\ N \leq \frac{\log \frac{r}{4|x-y|}}{\log 2} & \alpha = 1 \\ 2^{(\alpha-1)N} (2^{\alpha-1} - 1)^{-1} \leq \left(\frac{r}{4|x-y|}\right)^{\alpha-1} (2^{\alpha-1} - 1)^{-1} & 1 < \alpha \leq 2. \end{cases}$$

It remains to estimate  $I_3$ . Applying Proposition 5.2 we get

$$\begin{aligned}
I_3 &\leq c_{11} g(r) \frac{|x-y|}{r} \int_{B_r(z)} j(|v|) dv \\
&\leq c_{12} \frac{|x-y|\phi(|x-y|^{-2})}{r\phi(r^{-2})} \frac{r^{-d}}{\phi(|x-y|^{-2})} \int_{B_r(z)} j(|v|) dv \\
&\leq c_{13} \frac{r^{-d}}{\phi(|x-y|^{-2})} \int_{B_r(z)} j(|v|) dv \tag{5.7}
\end{aligned}$$

where in the last inequality we have used the theorem of Potter (cf. [BGT87, Theorem 1.5.6 (ii)]) to conclude that

$$\frac{|x-y|\phi(|x-y|^{-2})}{r\phi(r^{-2})} \leq c' \frac{|x-y|}{r} \leq c'.$$

For  $z \notin B_{2r}(0)$  we have

$$j(|v|) \geq j\left(\frac{|z|}{2}\right) \quad \text{for all } v \in B_r(z)$$

and so in this case

$$I_3 \leq c_{14} \frac{j\left(\frac{|z|}{2}\right)}{\phi(|x-y|^{-2})},$$

If  $z \in B_{2r}(0) \setminus B_r(0)$  we deduce from

$$B_r(z) \subset B_3(0) \setminus B_{|z|-r}(0)$$

and Corollary 4.6 the following

$$\begin{aligned} \int_{B_r(z)} j(|v|) dv &\leq c_{15} \int_{B_3(0) \setminus B_{|z|-r}(0)} j(|v|) dv \\ &\leq c_{16} \phi(|z| - r)^{-2}. \end{aligned}$$

Thus, in this case from (5.7) we see that

$$I_3 \leq c_{14} c_{16} |z|^{-d} \frac{\phi(|z| - r)^{-2}}{\phi(|x - y|^{-2})}.$$

□

## 6. HARMONIC FUNCTIONS

In this section we prove our regularity result. By the Ikeda-Watanabe formula (5.3), for a bounded function  $h: \mathbb{R}^d \rightarrow \mathbb{R}$  which is harmonic in  $B_{2r}(x_0)$  for some  $x_0 \in \mathbb{R}^d$  and  $r > 0$  we have

$$h(x) = \mathbb{E}_x \left[ h \left( X_{\tau_{B_r(x_0)}} \right) \right] = \int_{B_r(x_0)^c} K_{B_r(x_0)}(x, z) h(z) dz, \quad x \in B_r(x_0). \quad (6.1)$$

*Proof of Theorem 1.7.* From (6.1) we deduce

$$|h(x) - h(y)| \leq \|h\|_\infty \int_{B_{2r}(0)^c} |K_{B_{2r}(0)}(x, z) - K_{B_{2r}(0)}(y, z)| dz. \quad (6.2)$$

It remains to estimate the integral in (6.2), which we split in the following way

$$\begin{aligned} I_1 &= \int_{B_{4r}(0) \setminus B_{2r}(0)} |K_{B_{2r}(0)}(x, z) - K_{B_{2r}(0)}(y, z)| dz \\ I_2 &= \int_{B_1(0) \setminus B_{4r}(0)} |K_{B_{2r}(0)}(x, z) - K_{B_{2r}(0)}(y, z)| dz \\ I_3 &= \int_{B_1(0)^c} |K_{B_{2r}(0)}(x, z) - K_{B_{2r}(0)}(y, z)| dz \end{aligned}$$

In order to estimate  $I_1$  we use Proposition 5.3 (i). More precisely,

$$\begin{aligned} I_1 &\leq \frac{c_1}{\phi(|x - y|^{-2})} \int_{B_{4r}(0) \setminus B_{2r}(0)} |z|^{-d} \phi(|z| - 2r)^{-2} dz \\ &= \frac{c_2}{\phi(|x - y|^{-2})} \int_{2r}^{4r} t^{-1} \phi(t - 2r)^{-2} dt \\ &\leq \frac{c_2}{\phi(|x - y|^{-2})} (2r)^{-1} \int_0^{2r} \phi(t^{-2}) dt \\ &\leq \frac{c_3}{\phi(|x - y|^{-2})} \phi(r^{-2}), \end{aligned}$$

where in the last inequality we have used Karamata's theorem.

We estimate  $I_2$  and  $I_3$  with the help of Proposition 5.3 (ii). Since the Lévy measure is finite away from the origin, we have

$$I_3 \leq \frac{c_4}{\phi(|x-y|^{-2})} \int_{B_1(0)^c} j\left(\frac{|z|}{2}\right) dz \leq \frac{c_5}{\phi(|x-y|^{-2})}.$$

Finally,

$$I_2 \leq \frac{c_6}{\phi(|x-y|^{-2})} \int_{B_1(0) \setminus B_{4r}(0)} j\left(\frac{|z|}{2}\right) dz \leq \frac{c_7 \phi(r^{-2})}{\phi(|x-y|^{-2})},$$

where in the last inequality we have used Corollary 4.6.  $\square$

## 7. EXAMPLES

In this section we consider some examples. In all examples  $S$  is a subordinator with the Laplace exponent  $\phi$  and  $X$  is the corresponding subordinate Brownian motion. By  $\mu$  and  $u$  we denote the Lévy and potential density. Green function and jump function are denoted by  $G$  and  $J$ .

First we consider examples with  $\alpha = 0$ .

**7.1. Variance gamma process.** Let  $S$  be a subordinator with the Laplace exponent  $\phi(\lambda) = \log(1 + \lambda)$ . In this case the Lévy density is given by  $\mu(t) = \frac{e^{-t}}{t}$  since

$$\log(1 + \lambda) = \int_0^\infty (1 - e^{-\lambda t}) \frac{e^{-t}}{t} dt.$$

It is easy to check that  $\mu$  is completely monotone and thus  $\phi$  is a complete Bernstein function. Therefore conditions (A-1), (A-2) and (A-3) hold with  $\alpha = 0$ . Also the condition (B) is satisfied with  $a = 1$ . We have the following asymptotic of  $u$

$$u(t) \sim \frac{d}{dt} \frac{1}{\phi(1/t)} \sim \frac{1}{t(\log t)^2}, \quad t \rightarrow 0+.$$

The corresponding subordinate Brownian motion  $X$  is known as the *variance gamma process* and it is transient when  $d \geq 3$ . By Theorem 4.5 we obtain the following behavior of the jumping and Green function

$$J(x) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} \frac{1}{|x|^d}, \quad |x| \rightarrow 0+$$

and

$$G(x, y) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} \frac{1}{|x-y|^d (\log|x-y|)^2}, \quad |x-y| \rightarrow 0+$$

Using Theorem 1.7 we see that if  $h$  is harmonic and bounded in  $B_r(0)$ , then

$$|h(x) - h(y)| \leq c \log(r^{-1}) \|h\|_\infty \frac{1}{\log(|x-y|^{-1})}, \quad x, y \in B_{r/16}(0).$$

We do not know whether harmonic functions are Hölder continuous.

**7.2. Iterated variance gamma process.** Let  $n \in \mathbb{N}$  and let  $S^n$  be a subordinator whose Laplace exponent is  $\phi_n$ , where

$$\begin{aligned}\phi_1(\lambda) &= \log(1 + \lambda) \\ \phi_{n+1}(\lambda) &= \log(1 + \phi_n(\lambda)) \quad \text{for } n \in \mathbb{N}.\end{aligned}$$

For any  $n \in \mathbb{N}$   $\phi_n$  is complete Bernstein function (as a composition of such functions) and thus conditions **(A-1)**, **(A-2)** and **(A-3)** hold with  $\alpha = 0$ . Denote the Lévy and potential density of  $S^n$  by  $\mu_n$  and  $u_n$ . Processes  $S^n$  are called iterated gamma subordinators.

In the following lemma we prove that condition **(B)** holds with  $a = 1$ .

**Proposition 7.1.** *For every  $n \in \mathbb{N}$*

$$t \mapsto e^t t \mu_n(t), \quad t > 0$$

*is a non-decreasing function.*

*Proof.* If  $n = 1$ , then  $e^t t \mu(t) = 1$ . Let  $n \geq 2$ . Taking derivative in

$$\phi_n(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu_n(t) dt$$

we obtain

$$\phi'_n(\lambda) = \int_0^\infty e^{-\lambda t} t \mu_n(t) dt.$$

Therefore  $\phi'$  is the Laplace transform of a measure whose density is given by

$$t \mapsto t \mu_n(t), \quad t > 0.$$

We are going to use the Widder-Post inversion formula for the Laplace transform (cf. [Wid46]):

$$t \mu_n(t) = \lim_{m \rightarrow \infty} \frac{(-1)^{m-1}}{(m-1)!} \left(\frac{m}{t}\right)^m \phi_n^{(m)}\left(\frac{m}{t}\right), \quad t > 0. \quad (7.1)$$

Let us first prove that for any  $n \in \mathbb{N}$  the function

$$\lambda \mapsto (-1)^{m-1} (1 + \lambda)^m \phi_n^{(m)}(\lambda) \quad \text{is non-increasing.} \quad (7.2)$$

The following equality follows from the defining properties of  $\phi_n$ :

$$(1 + \lambda) \phi'_n(\lambda) = e^{-(\phi_2(\lambda) + \phi_3(\lambda) + \dots + \phi_n(\lambda))}. \quad (7.3)$$

Note that  $f_n(\lambda) := \phi_2(\lambda) + \phi_3(\lambda) + \dots + \phi_n(\lambda)$  is a Bernstein function and thus the right hand side of (7.3) is a completely monotone function.

Taking  $m$ -th derivative in (7.3) and using the Leibniz formula for the  $m$ -th derivative of the product of two functions we get

$$(1 + \lambda) \phi_n^{(m+1)}(\lambda) + m \phi_n^{(m)}(\lambda) = (e^{-f_n(\lambda)})^{(m)}$$

Using this we get

$$\begin{aligned} \frac{d}{d\lambda} \left( (-1)^{m-1} (1+\lambda)^m \phi_n^{(m)}(\lambda) \right) &= (-1)^{n-1} (1+\lambda)^{m-1} \left( m \phi_n^{(m)} + (1+\lambda) \phi_n^{(m+1)}(\lambda) \right) \\ &= (-1)^{m-1} (1+\lambda)^{m-1} \left( e^{-f_n(\lambda)} \right)^{(m)} \leq 0 \end{aligned}$$

since  $e^{-f_n}$  is a completely monotone function. This proves that the function given in (7.2) is non-increasing.

Now let  $0 < s < t$ . Then by (7.1) and (7.2) we get

$$\begin{aligned} s\mu_n(s) &= \lim_{m \rightarrow \infty} \frac{(-1)^{m-1}}{(m-1)!} \left( \frac{m}{s} \right)^m \phi_n^{(m)} \left( \frac{m}{s} \right) \\ &= \lim_{m \rightarrow \infty} \frac{(-1)^{m-1}}{(m-1)!} \left( \frac{m}{m+s} \right)^m \left( 1 + \frac{m}{s} \right)^m \phi_n^{(m)} \left( \frac{m}{s} \right) \\ &\leq \lim_{m \rightarrow \infty} \frac{(-1)^{m-1}}{(m-1)!} \left( \frac{m}{m+s} \right)^m \left( 1 + \frac{m}{t} \right)^m \phi_n^{(m)} \left( \frac{m}{t} \right) \\ &= \lim_{m \rightarrow \infty} \frac{(-1)^{m-1}}{(m-1)!} \left( \frac{m+t}{m+s} \right)^m \left( \frac{m}{t} \right)^m \phi_n^{(m)} \left( \frac{m}{t} \right) \\ &= e^{t-s} t \mu_n(t). \end{aligned}$$

□

Since

$$\begin{aligned} \phi_n'(\lambda) &\sim \frac{\phi_{n-1}'(\lambda)}{\phi_{n-1}(\lambda)} \sim \frac{\phi_1'(\lambda)}{\phi_1(\lambda) \cdots \phi_{n-1}(\lambda)} \\ &\sim \frac{1}{\lambda \log \lambda \cdot \log_2 \lambda \cdots \log_{n-1} \lambda}, \quad \lambda \rightarrow \infty, \end{aligned}$$

we have

$$\mu(t) \sim \frac{1}{t} \frac{1}{\log_{n-1}(1/t) \cdots \log_2(1/t) \log(1/t)}, \quad t \rightarrow 0+$$

and

$$u(t) \sim \frac{1}{t (\log_n t)^2} \frac{1}{\log_{n-1}(1/t) \cdots \log_2(1/t) \log(1/t)}, \quad t \rightarrow 0+.$$

Asymptotic properties of the jumping function and the Green function of the corresponding subordinate Brownian motion are given by (cf. Theorem 4.5)

$$J(x) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} \frac{1}{|x|^d \log_{n-1}(1/|x|) \cdots \log_2(1/|x|) \log(1/|x|)}, \quad |x| \rightarrow 0+$$

and

$$G(x, y) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} \frac{1}{|x - y|^d (\log_n |x - y|)^2} \times \\ \times \frac{1}{\log_{n-1}(1/|x - y|) \cdots \log_2(1/|x - y|) \log(1/|x - y|)}, \quad |x - y| \rightarrow 0+$$

Using Theorem 1.7 we see that if  $h$  is harmonic and bounded in  $B_r(0)$ , then

$$|h(x) - h(y)| \leq c \log_n(r^{-1}) \|h\|_\infty \frac{1}{\log_n(|x - y|^{-1})}, \quad x, y \in B_{r/16}(0).$$

**7.3. Geometric stable process.** We say that  $S$  is a *geometric  $\beta/2$ -stable subordinator* if

$$\phi(\lambda) = \log(1 + \lambda^{\beta/2})$$

for some  $0 < \beta \leq 2$ . The case  $\beta = 2$  has been already investigated in 7.1. The corresponding subordinator Brownian motion is called the *geometric  $\beta$ -stable process*. Since  $\phi$  is a complete Bernstein function, we only need to check the condition (B).

**Proposition 7.2.** *For any  $0 < \beta < 2$*

$$t \mapsto t\mu(t), \quad t > 0$$

*is a non-increasing function.*

*Proof.* We use the same idea as in Proposition 7.1. Note that

$$\lambda\phi'(\lambda) = \frac{\beta}{2} \frac{\lambda^{\beta/2}}{1 + \lambda^{\beta/2}}$$

is a (complete) Bernstein function. By the Leibniz formula we obtain

$$(-1)^{n-1} (\lambda\phi^{(n+1)}(\lambda) + n\phi^{(n)}(\lambda)) = (-1)^{n-1} (\lambda\phi'(\lambda))^{(n)} \geq 0$$

which implies

$$\frac{d}{d\lambda} ((-1)^{n-1} \lambda^n \phi^{(n)}(\lambda)) = (-1)^{n-1} \lambda^{n-1} (\lambda\phi^{(n+1)}(\lambda) + n\phi^{(n)}(\lambda)) \geq 0$$

for any  $n \in \mathbb{N}$  and  $\lambda > 0$ . Therefore

$$\lambda \mapsto (-1)^{n-1} \lambda^n \phi^{(n)}(\lambda), \quad \lambda > 0$$

is a non-decreasing function and thus by the Widder-Post inversion formula (7.1), for all  $0 < s < t$  we get

$$s\mu(s) = \lim_{n \rightarrow \infty} \frac{(-1)^{n-1}}{(n-1)!} \binom{n}{s}^n \phi^{(n)}\left(\frac{n}{s}\right) \\ \geq \lim_{n \rightarrow \infty} \frac{(-1)^{n-1}}{(n-1)!} \binom{n}{t}^n \phi^{(n)}\left(\frac{n}{t}\right) \\ = t\mu(t).$$

□

*Remark 7.3.* It can be proved that  $\phi$  is also a *Thorin-Bernstein function*, i.e. a Bernstein function such that  $t \mapsto t\mu(t)$  is a completely monotone function (cf. [SSV10]).

From Proposition 7.2 we see that condition **(B)** is satisfied with  $a = 0$ . We have

$$\mu(t) \sim \frac{\beta}{2t} \quad \text{and} \quad u(t) \sim \frac{2}{\beta t (\log t)^2}, \quad t \rightarrow 0+.$$

For the geometric stable process we have the following asymptotic (cf. Theorem 4.5)

$$J(x) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} \frac{\beta}{2|x|^d}, \quad |x| \rightarrow 0+$$

and

$$G(x, y) \sim \frac{\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}} \frac{1}{2\beta|x-y|^d (\log|x-y|)^2}, \quad |x-y| \rightarrow 0+.$$

*Remark 7.4.* These asymptotic properties were first obtained in [ŠSV06]. In particular, the behavior of Lévy density of the subordinator  $S$  near zero was obtained from the fact that  $S_1$  has the Mittag-Leffler distribution.

Concerning continuity properties of harmonic functions, we obtain the same behavior as in the case of the variance gamma process (i.e. when  $\beta = 2$ ).

Now we turn to the case  $\alpha = 2$ . It will turn out that (by considering the behavior of the Green function) these processes are 'close' to the Brownian motion.

**7.4. Conjugate of the iterated geometric stable process.** Define

$$\begin{aligned} \psi_1(\lambda) &= \log(1 + \lambda^{\beta/2}) \\ \psi_{n+1}(\lambda) &= \log(1 + \psi_n(\lambda)^{\beta/2}) = (\phi_1 \circ \phi_n)(\lambda), \quad n \in \mathbb{N} \end{aligned}$$

for some  $0 < \beta \leq 2$ . Furthermore, for  $n \in \mathbb{N}$  we set

$$\phi_n(\lambda) = \frac{\lambda}{\psi_n(\lambda)}.$$

Note that  $\phi_n$  is a complete Bernstein function and in the case  $\beta = 2$  the corresponding subordinator  $S$  is actually a killed subordinator, but this will not affect the behavior of Lévy and potential density near zero. Thus the conditions **(A-1)**, **(A-2)** and **(A-3)** hold with  $\alpha = 2$ . Direct computation shows that condition **(C)** also holds. Since

$$\begin{aligned} \psi'_n(\lambda) &\sim \frac{\beta}{2} \frac{\psi'_{n-1}(\lambda)}{\psi_{n-1}(\lambda)} \sim \left(\frac{\beta}{2}\right)^n \frac{\psi'_1(\lambda)}{\psi_1(\lambda) \cdots \psi_{n-1}(\lambda)} \\ &\sim \frac{\beta}{2\lambda \log \lambda \cdot \log_2 \lambda \cdots \log_{n-1} \lambda}, \quad \lambda \rightarrow \infty, \end{aligned}$$

we have

$$\begin{aligned} \mu(t) &\sim \frac{1}{t} \frac{d}{dt} (t\phi_n(1/t)) = \frac{1}{t^3} \frac{\psi'_n(1/t)}{\psi_n(1/t)^2} \\ &\sim \frac{2}{\beta t^2 (\log_n(1/t))^2} \frac{1}{\log_{n-1}(1/t) \cdots \log(1/t)}, \quad t \rightarrow 0+ . \end{aligned}$$

and

$$u(t) \sim \frac{1}{t\phi_n(1/t)} = \psi_n(1/t) \sim \frac{\beta}{2} \log_n(1/t), \quad t \rightarrow 0+ .$$

Asymptotic of jumping and Green function is then (cf. Theorem 4.5)

$$J(x) \sim \frac{4\Gamma\left(\frac{d+2}{2}\right)}{\pi^{d/2}} \frac{2}{\beta |x|^{d+2} (\log_n(1/|x|))^2} \frac{1}{\log_{n-1}(1/|x|) \cdots \log(1/|x|)}, \quad |x| \rightarrow 0+$$

and

$$G(x, y) \sim \frac{\Gamma\left(\frac{d+2}{2}\right)}{4\pi^{d/2}} \frac{\beta |x - y|^{2-d} \log_n(1/|x - y|)}{2}, \quad |x - y| \rightarrow 0+ .$$

*Remark 7.5.* Comparing this result to the Green functions of the Brownian motion and isotropic  $\alpha$ -stable process ( $0 < \alpha < 2$ ) given by

$$G^{(2)}(x, y) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} |x - y|^{2-d} \quad \text{and} \quad G^{(\alpha)}(x, y) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{d/2}} |x - y|^{\alpha-d},$$

we see that our process is in this sense closer to Brownian motion than any stable process.

Using Theorem 1.7 we see that if  $h$  is harmonic and bounded in  $B_r(0)$ , then

$$|h(x) - h(y)| \leq \frac{c}{r \log_n(r^{-1})} \|h\|_\infty |x - y| \log_n(|x - y|^{-1}), \quad x, y \in B_{r/16}(0) .$$

Thus harmonic functions locally are more than Hölder continuous.

## REFERENCES

- [BGT87] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Cambridge University Press, Cambridge, 1987.
- [BK05] R. F. Bass and M. Kassmann, *Harnack inequalities for non-local operators of variable order*, Trans. Amer. Math. Soc. **357** (2005), 837–850.
- [BL02] R. F. Bass and D. Levin, *Harnack inequalities for jump processes*, Potential Anal. **17** (2002), 375–388.
- [BS05] K. Bogdan and P. Sztonyk, *Harnack's inequality for stable Lévy processes*, Potential Anal. **22** (2005), 133–150.
- [CK03] Z.-Q. Chen and T. Kumagai, *Heat kernel estimates for stable-like processes on  $d$ -sets*, Stochastic Process. Appl. **108** (2003), no. 1, 27–62.
- [CK08] ———, *Heat kernel estimates for jump processes of mixed types on metric measure spaces*, Probab. Theory Related Fields **140** (2008), 277–317.

- [IW62] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. **2** (1962), 79–95.
- [KS07] P. Kim and R. Song, *Potential theory of truncated stable processes*, Math. Z. **256** (2007), 139–173.
- [KSV] P. Kim, R. Song, and Z. Vondraček, *Potential theory for subordinate Brownian motions revisited*, (preprint, 2011).
- [KSV09] ———, *Boundary Harnack principle for Subordinate Brownian Motions*, Stoch. Proc. Appl. **119** (2009), 1601–1631.
- [Mim10] A. Mimica, *Harnack inequalities for some Lévy processes*, Potential Anal. **32** (2010), 275–303.
- [Mim11a] ———, *Harnack inequality and Hölder regularity estimates for a Lévy process with small jumps of high intensity*, J. Theor. Probab. (2011), to appear.
- [Mim11b] ———, *Heat kernel estimates for symmetric jump processes with small jumps of high intensity*, Potential Anal. (2011), to appear.
- [RSV06] M. Rao, R. Song, and Z. Vondraček, *Green function estimates and Harnack inequality for subordinate Brownian motions*, Potential Anal. **25** (2006), 1–27.
- [Sat99] K.-I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.
- [ŠSV06] H. Šikić, R. Song, and Z. Vondraček, *Potential theory of geometric stable processes*, Prob. Theory Related Fields **135** (2006), 547–575.
- [SSV10] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein functions: theory and applications*, Walter de Gruyter, Berlin, 2010.
- [SV04] R. Song and Z. Vondraček, *Harnack inequalities for some classes of Markov processes*, Math. Z. **246** (2004), 177–202.
- [Szt00] P. Sztonyk, *On harmonic measure for Lévy processes*, Probab. Math. Statist. **20** (2000), 383–390.
- [Szt10] ———, *Regularity of harmonic functions for anisotropic fractional Laplacians*, Math. Nachr. **283** (2010), 289–311.
- [Wid46] D. V. Widder, *Laplace transform*, Princeton University Press, 1946.

DEPARTMENT OF MATHEMATICS, BIELEFELD UNIVERSITY, GERMANY  
*E-mail address:* amimica@math.hr