Exit problem of McKean-Vlasov diffusions in convex landscape*

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Abstract

The exit time and the exit location of a non-markovian diffusion is analyzed. More particularly, we focus on the so-called self-stabilizing process. The question has been studied by Herrmann, Imkeller and Peithmann in [HIP08]. Some results similar to the ones of Freidlin and Wentzell for classical diffusions have been proved. We aim to provide the same results by a method more intuitive. Our arguments are the uniform propagation of chaos and the application of the Freidlin-Wentzell theory to a mean-field system. Moreover, we provide a new kind of uniform propagation of chaos in the small noise.

Key words and phrases: Self-stabilizing diffusion ; Exit time ; Exit location ; Large deviations ; Interacting particle systems ; Propagation of chaos ; Granular media equation

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Introduction

We investigate the exit problem (exit time and exit location) of the following so-called self-stabilizing process on $\mathbb{R}^d$ from a domain $\mathcal{D}$:

$$\begin{cases} X_t = X_0 + \sqrt{\epsilon} B_t - \int_0^t \nabla W_s\left(X_s\right) ds \\ W_s := V + F * u_s := V + F * \mathcal{L}\left(X_s\right) \end{cases} . \tag{I}$$

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Here, $*$ denotes the convolution and $X_0 \in \mathcal{D}$ is *deterministic*. Since the own law of the process intervenes in the drift, this equation is nonlinear - in the sense of McKean. ϵ is omitted for the comfort of the reading.

The motion of the process is generated by three essential elements. The first one is the gradient of a potential V - the confining potential. The second force is a Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ in $\mathbb{R}^d$ with intensity $\frac{\epsilon}{2}$. It allows the particle to move upwards the potential V. The third term describes the attraction between one trajectory $t \mapsto X_t(\omega_0)$ and the whole ensemble of trajectories. Indeed, we remark: $\nabla F * u_s\left(X_s(\omega_0)\right) = \int_{\omega\in\Omega} \nabla F\left(X_s(\omega_0) - X_s(\omega)\right) d\mathbb{P}\left(\omega\right)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying measurable space.

This kind of processes were introduced by McKean, see [McK67] or [McK66].

The key of the article is the following: the diffusion X_t which verifies (I) can be seen as a particle in a continuous mean-field system of an infinite number of particles. The mean-field system that we will consider is a random dynamical system like

$$\left\{\begin{array}{l} X_t^1 = X_0 + \sqrt{\epsilon}B_t^1 - \int_0^t \nabla W_s^{(N)}\left(X_s^1\right) ds \\ \vdots \\ X_t^i = X_0 + \sqrt{\epsilon}B_t^i - \int_0^t \nabla W_s^{(N)}\left(X_s^i\right) ds \\ \vdots \\ X_t^{\epsilon,N} = X_0 + \sqrt{\epsilon}B_t^N - \int_0^t \nabla W_s^{(N)}\left(X_s^N\right) ds \end{array}\right. \tag{II}$$

where the N brownian motions $\left(B_t^i\right)_{t\in\mathbb{R}_+}$ are independent and the potential $W_s^{(N)}$ is defined as $W_s^{(N)} := V + F * \left(\frac{1}{N}\sum_{i=1}^N \delta_{X_s^i}\right)$. For simpliying the reading, we write X_t^i instead of $X_t^{\epsilon,i,N}$.

As written previously, the diffusion X_t which satisfies (I) can be seen as a particle X_t^1 among the whole ensemble of particles in (II) when N goes to ∞. This particular phenomenon - which is called the propagation of chaos - has been investigated in [Szn91] (under Lipschitz properties) and on [BRTV98] (when V is equal to 0). Ben Arous and Zeitouni went further by proving that for any $f(N)$ such that $f(N) \to \infty$ and $\frac{f(N)}{N} \to 0$, $f(N)$ particles among N become independent. Malrieu established a uniform propagation of chaos if both V and F are convex, see [Mal01, Mal03]. Concentration inequalities and large deviations have also been studied in [BGV07, DPdH96, DG87]. Finally, Cattiaux, Guillin and Malrieu established a propagation of chaos uniform with respect to the time in the non-uniformly strictly convex case, see [CGM08]:

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}\left\{\left|\left|X_t - X_t^1\right|\right|^2\right\} \leq K(N) \to 0\,.$$

However, the supremum is not under the expectation. One of the two keystones of this paper is the uniform propagation of chaos.

Equation (II) can be rewritten:

$$d\mathcal{X}_t = \sqrt{\epsilon}\mathcal{B}_t - N\nabla\Upsilon^N\left(\mathcal{X}_t\right) dt \tag{II}$$

where $\mathcal{X}_t := \left(X_t^1, \cdots, X_t^N\right)$, $\mathcal{B}_t := \left(B_t^1, \cdots, B_t^N\right)$ and for all $\mathcal{X} \in \left(\mathbb{R}^d\right)^N$:

$$\Upsilon^N(\mathcal{X}) := \frac{1}{N}\sum_{j=1}^{N} V(X_j) + \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} F(X_i - X_j) \,. \tag{III}$$

In [HIP08], Herrmann, Imkeller and Peithmann studied the exit problem of self-stabilizing processes in convex landscape and provided some Kramers type law closed to the one of standard diffusions like in [DZ10] or [FW98]. They follow and extend the same method. The aim of this paper is to provide a much simpler and intuitive proof of the result.

Let us recall briefly some of the previous results on diffusions like (I). The existence problem has been stated by two different methods. The first one consists in the application of a fixed point theorem, see [McK67, BRTV98, HIP08]. The other consists in a propagation of chaos, see for example [Mél96].
In [McK67], the author proved - by using Weyl lemma - that the law $\mathcal{L}(X_t)$ of the unique strong solution X_t admits a $\mathcal{C}^\infty$-continuous density u_t with respect to the Lebesgue measure for all $t > 0$. Furthermore, this density satisfies a nonlinear partial differential equation of the following type:

$$\frac{\partial}{\partial t}u_t = \operatorname{div}\left\{\frac{\epsilon}{2}\nabla u_t + u_t\Big(\nabla V + \nabla F * u_t\Big)\right\} \,.$$

This equation is a useful tool for characterizing the stationary measure(s) and the long-time behavior, see [BRV98, Tam84, Tam87, Ver06]. It has been proved in [HT10a, HT10b, HT09, Tug11a] that diffusion (I) admits several stationary measures if V is not convex in dimension one. Convergence towards these equilibria have been investigated in [Tug10]. The general dimension case has been made in [Tug11b, Tug11c].
Large deviations is the natural framework but a potential theory approach is possible, see for example [Bov06].
The most common one is the one of the large deviations. In the case of a classical diffusion

$$dx_t = \sqrt{\epsilon}dB_t - \nabla U(x_t)dt \,,$$

the exit time τ_ϵ from a domain $\mathcal{D}$ satisfies the following estimate under conditions easy to verify:

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{H-\delta}{\epsilon}\right] \le \tau_\epsilon \le \exp\left[\frac{H+\delta}{\epsilon}\right]\right\} = 0$$

where $H := 2\inf_{z\in\partial\mathcal{D}} U(z) - 2\inf_{z\in\mathcal{D}} U(z)$; for all $\delta > 0$. If we consider Lévy noise (but not brownian motion), similar result can be obtain with a different rate of ϵ, see [IPW09, IPS10].
By using exactly the same method than in [DZ10], Herrmann, Imkeller and Peithmann proved the same result with $V + F * \delta_{a_0}$ instead of V ; if a_0 is the

unique wells of V. Let us note that δ_{a_0} corresponds here to the small-noise limit of the unique stationary measure. This means that exit time and convergence towards steady state are closed. Consequently, we will look at the methods for getting the convergence in order to deduce the exit time. One of this method consists in using the propagation of chaos in order to derive the convergence of the self-stabilizing process from the one of the mean-field system. However, we shall use it independently of the time and the classical result which is on a finite interval of time is not sufficiently strong. Cattiaux, Guillin and Malrieu proceeded a uniform propagation of chaos in [CGM08] and obtained the convergence in the convex case, including the non-uniformly convex case. See also [Mal03].

Since (II) is a classical diffusion in $\mathbb{R}^{dN}$, the exit problem can be obtained by the classical Freidlin-Wentzell theory. Then, the uniform propagation of chaos permits to tract the exit time of (I) from the one of the first particle X_t^1 of (II).

The paper is organized as follows. After presenting the assumptions, we state the first results, in particular the precise majoration of the moments and the boundedness with respect to the time and to the noise of all the moments. Then, the exit time and the exit location of the particle X_t^1 from a domain $\mathcal{D}$ is provided by using the classical Freidlin-Wentzell theory ; when N is fixed. Also, we show that the associated cost H_N converges towards $H > 0$. Subsequently, a result of propagation of chaos uniform with respect to time and with respect to noise is proved. Finally, the two previous results are combined and the main results are stated:

Theorem: *Let an open $\mathcal{D}$ which contains the unique wells a_0 of the uniformly strictly convex potential V. We assume that for all $x \in \mathcal{D}$, we have $\{x_t\,;\, t \in \mathbb{R}_+\} \subset \mathcal{D}$ and $x_t \to a_0$ with $\dot{x_t} = -\left(\nabla V + \nabla F * \delta_{a_0}\right)(x_t)$. Also, let us assume that $\{y_t\,;\, t \in \mathbb{R}_+\} \subset \mathcal{D}$ with $\dot{y_t} = -\nabla V\left(y_t\right)$ and $y_0 = X_0 \in \mathcal{D}$. We introduce*

$$H := 2\left(\inf_{z \in \partial\mathcal{D}} W(z) - W(a_0)\right)$$

with $W(z) := V(z) + F(z - a_0)$. Let us note τ_ϵ the first exit time of the diffusion X_t. Then, for all $\delta > 0$, we have the following result:

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{\exp\left[\frac{H-\delta}{\epsilon}\right] < \tau_\epsilon < \exp\left[\frac{H+\delta}{\epsilon}\right]\right\} = 1\,.$$

Furthermore, if $\mathcal{N} \subset \partial\mathcal{D}$ verifies $\inf_{z \in \mathcal{N}} V(z) + F(z - a_0) - V(a_0) > H$, then:

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{\overline{X_{\tau_\epsilon}} \in \mathcal{N}\right\} = 0\,.$$

Theorem: *Let $\delta, \kappa > 0$. We assume that V is uniformly strictly convex on $\mathbb{R}^d$. Then, for all $K > 0$, there exists $N_0 \in \mathbb{N}^*$ and $\epsilon_0 > 0$ such that*

$$\sup_{N \geq N_0} \sup_{\epsilon < \epsilon_0} \mathbb{P}\left[\sup_{0 \leq t \leq \exp\left[\frac{K}{\epsilon}\right]} \left|\left|X_t - X_t^1\right|\right| \geq \kappa\right] \leq \delta\,.$$

Finally, we postpone classical Freidlin-Wentzell theory in annex.

Assumptions

Firstly, we note $||\,.\,||$ the norm that we consider on $\mathbb{R}^d$. We assume the following properties of the confining potential V:

(V-1) V is a polynomial function on $\mathbb{R}^d$ with $\deg(V) =: 2m$.

(V-2) $\mathrm{Hess}\, V \geq \vartheta > 0$.

(V-3) V admits exactly one critical point: a_0, which is a wells.

(V-4) For all $x \in \mathbb{R}^d$: $\langle x\,;\, \nabla V(x)\rangle \geq C_2\,||x||^2 - C_0$ with $C_2, C_0 > 0$.

We would like to stress that weaker assumptions could be considered but all the mathematical difficulties are present in the polynomial case and it permits to avoid some technical and tedious computations. Let us present now the assumptions on the interaction potential F:

(F-1) There exists an even polynomial function G on $\mathbb{R}$ with $\deg(G) =: 2n \geq 2$ such that $F(x) = G(||x||)$.

(F-2) G is convex.

(F-3) Initialization: $G(0) = 0$.

By using method similar to the ones of [CGM08, Tug11b], we know that there is a unique stationary measure which converges towards δ_{a_0} in the small-noise limit.

Let us note that with assumptions (F-1)–(F-3), we have:

$$\nabla F(x) = G'(||x||)\frac{x}{||x||}$$

Since all the moments of the initial law $u_0 = \delta_x$ are finite, we know by Theorem 2.12 in [HIP08] that (I) admits a unique strong solution. Moreover:

$$\max_{1\leq j\leq Q}\sup_{t\in\mathbb{R}_+}\mathbb{E}\left[||X_t||^j\right] \leq M_0(Q)$$

for all $Q \in \mathbb{N}^*$. We deduce immediately the tightness of the family $(u_t)_{t\in\mathbb{R}_+}$.

1 Preliminaries

We begin by presenting some notations concerning the space $\left(\mathbb{R}^d\right)^N$.

Definition 1.1. 1. *For all* $x \in \mathbb{R}^d$, *we write* $\overline{x} := (x, \cdots, x) \in \left(\mathbb{R}^d\right)^N$.

2. *For all* $\mathcal{X} = (X_1, \cdots, X_N) \in \left(\mathbb{R}^d\right)^N$ *and for all* $k \geq 1$, *we consider the following norm:*

$$||\mathcal{X}||_k := \left\{ \frac{1}{N} \sum_{i=1}^{N} ||X_i||^k \right\}^{\frac{1}{k}} .$$

If we do not specify k, *that means that* $k = 2$.

3. *Let a set* $\mathcal{D} \subset \left(\mathbb{R}^d\right)^N$ *and a potential* V *from* $\left(\mathbb{R}^d\right)^N$ *to* $\left(\mathbb{R}^d\right)^N$. *We say that* $\mathcal{D}$ *is stable by* $-\nabla V$ *if for all* $x \in \mathcal{D}$, *the orbit* $\{x_t\,;\, t \in \mathbb{R}_+\}$ *is included in* $\mathcal{D}$ *with* $x_0 = x$ *and* $\dot{x_t} = -\nabla V(x_t)$.

4. *Finally, for all* $\rho > 0$, *for all* $\mathcal{X} \in \left(\mathbb{R}^d\right)^N$ *and for all* $p \in \mathbb{N}^*$, *we introduce the ball:*

$$\mathbb{B}_N^{2p}(\mathcal{X}\,;\,\rho) := \left\{ \mathcal{Y} \in \left(\mathbb{R}^d\right)^N \;\middle|\; ||\mathcal{X} - \mathcal{Y}||_{2p} \leq \rho \right\} .$$

When there is no confusion possible, we write $\mathbb{B}_\rho$.

We state now two important results about the moments. Indeed, the uniform propagation of chaos with respect to the time needs a uniform majoration of the moments. Such a result has been proved in [HIP08]. However, it has been used for obtaining the existence of the solution for fixed ϵ. Since, the work of this paper is in the small-noise limit, we also aim to obtain a uniform boundedness with respect to the noise.

Proposition 1.2. *Let* $(X_t)_{t\in\mathbb{R}_+}$ *the unique strong solution of* (I). *For all* $p \in \mathbb{N}^*$, *there exists* $C_p < \infty$ *which satisfies:*

$$\sup_{0<\epsilon<1} \sup_{t\geq 0} \mathbb{E}\left[||X_t||^{2p} \right] \leq C_p \,. \tag{1.1}$$

Proof. Since the initial value X_0 is deterministic, the quantity $\mathbb{E}\left[||X_0||^{2p} \right]$ is finite. We apply Itô formula to the function $x \mapsto ||x||^{2p}$:

$$\begin{aligned} d\,||X_t||^{2p} =& 2p\sqrt{\epsilon}\,||X_t||^{2p-2}\,\langle X_t\,;\,dB_t\rangle - 2p\,||X_t||^{2p-2}\,\langle X_t\,;\,\nabla V\,(X_t)\rangle\,dt \\ & - 2p\,||X_t||^{2p-2}\,\langle X_t\,;\,\nabla F * u_t\,(X_t)\rangle\,dt \\ & + p(2p-1)\epsilon\,||X_t||^{2p-2}\,dt\,. \end{aligned}$$

After integration, it yields

$$
\begin{aligned}
||X_t||^{2p} = &||X_0||^{2p} + 2p\sqrt{\epsilon}\int_0^t ||X_s||^{2p-2} \langle X_t\,;\, dB_s\rangle \\
&- 2p\int_0^t ||X_s||^{2p-2}\Big\{ \langle X_t\,;\, \nabla V\left(X_s\right)\rangle + \langle X_s\,;\, \nabla F * u_s\left(X_s\right)\rangle \Big\} ds \\
&+ p(2p-1)\epsilon\int_0^t ||X_s||^{2p-2}\, ds\,.
\end{aligned}
$$

We introduce the function $\xi_\epsilon(t) := \mathbb{E}\left[||X_t||^{2p}\right]$ (we recall that ϵ intervenes on the diffusion X_t and on the probability measure u_t). The previous equality leads to:

$$
\begin{aligned}
\xi_\epsilon'(t) = &- 2p\mathbb{E}\left[||X_t||^{2p-2}\langle X_t\,;\, \nabla V\left(X_t\right)\rangle\right] \\
&- 2p\mathbb{E}\left[||X_t||^{2p-2}\langle X_t\,;\, \nabla F * u_t\left(X_t\right)\rangle\right] + p(2p-1)\epsilon\mathbb{E}\left[||X_t||^{2p-2}\right]\,.
\end{aligned}
$$

By definition, the second term can be written as in the following:

$$
\mathbb{E}\left[||X_t||^{2p-2}\langle X_t\,;\, \nabla F * u_t\left(X_t\right)\rangle\right] = \mathbb{E}\left[||X_t||^{2p-2}\langle X_t\,;\, \nabla F\left(X_t - Y_t\right)\rangle\right]
$$

where Y_t is a solution of (I) independent from X_t. We can exchange X_t and Y_t. Thereby, by using hypotheses (F-1) and (F-2), it yields:

$$
\begin{aligned}
&\mathbb{E}\left[||X_t||^{2p-2}\langle X_t\,;\, \nabla F\left(X_t - Y_t\right)\rangle\right] \\
&= \mathbb{E}\left[||X_t||^{2p-1}\,\frac{G'\left(||X_t - Y_t||\right)}{||X_t - Y_t||}\left\langle \frac{X_t}{||X_t||}\,;\, X_t - Y_t\right\rangle\right] \\
&= \frac{1}{2}\mathbb{E}\left\{G'\left(||X_t - Y_t||\right)\left\langle X_t\,||X_t||^{2p-2} - Y_t\,||Y_t||^{2p-2}\,;\, X_t - Y_t\right\rangle\right\}\,.
\end{aligned}
$$

This last term is nonnegative. Indeed, Cauchy-Schwarz inequality implies

$$
\begin{aligned}
&\left\langle X_t\,||X_t||^{2p-2} - Y_t\,||Y_t||^{2p-2}\,;\, X_t - Y_t\right\rangle \\
&\geq \left(||X_t||^{2p-2} - ||Y_t||^{2p-2}\right)\left(||X_t|| - ||Y_t||\right) \geq 0\,.
\end{aligned}
$$

Therefore, we obtain the inequality $\mathbb{E}\left[||X_t||^{2p-1}\,F' * u_t\left(X_t\right)\right] \geq 0$. Moreover, hypothesis (V-4) implies

$$
\mathbb{E}\left[||X_t||^{2p-2}\langle X_t\,;\, \nabla V\left(X_t\right)\rangle\right] \geq C_2\mathbb{E}\left[||X_t||^{2p}\right] - C_0\mathbb{E}\left[||X_t||^{2p-2}\right]\,.
$$

Hence, by using Jensen inequality, we deduce:

$$
\begin{aligned}
\xi_\epsilon'(t) &\leq -2pC_2\xi_\epsilon(t) + 2pC_0\xi_\epsilon(t)^{1-\frac{1}{p}} + p(2p-1)\epsilon\xi_\epsilon(t)^{1-\frac{1}{p}} \\
&\leq -2pC_2\xi_\epsilon(t)^{1-\frac{1}{p}}\left\{\xi_\epsilon(t)^{\frac{1}{p}} - \left(\frac{C_0}{C_2} + \frac{(2p-1)\epsilon}{2C_2}\right)\right\}\,. \qquad (1.2)
\end{aligned}
$$

As $\xi_\epsilon(t) \geq 0$, the inequality

$$\xi_\epsilon(t) \geq \left(\frac{C_0}{C_2} + \frac{(2p-1)\epsilon}{2C_2} \right)^p$$

would imply $\xi'_\epsilon(t) \leq 0$. It yields:

$$\xi_\epsilon(t) \leq \max \left\{ ||X_0||^{2p} \; ; \; \left(\frac{C_0}{C_2} + \frac{(2p-1)\epsilon}{2C_2} \right)^p \right\}$$

if $\epsilon > 0$. This achieves the proof. □

Inequality (1.1) means that the self-stabilizing process tends to be captiv in a ball. Indeed, we can go further than Proposition 1.2:

Proposition 1.3. *Let $(X_t)_{t\in\mathbb{R}_+}$ the unique strong solution of* (I). *Let $p \in \mathbb{N}^*$ and $\rho > 0$. We introduce the following deterministic time:*

$$T_\epsilon(\rho) := \min \left\{ t \geq 0 \;\middle|\; \mathbb{E}\left[||X_t - a_0||^{2p} \right] \leq \rho^p \right\} .$$

Then, for ϵ small enough, we have the inequality:

$$T_\epsilon(\rho) \leq \frac{1}{p\vartheta\rho^p} ||X_0 - a_0||^{2p} . \tag{1.3}$$

Moreover, for all $t \geq T_\epsilon(\rho)$, $\mathbb{E}\left\{ ||X_t - a_0||^{2p} \right\} \leq \rho^p$.

Proof. We introduce $\psi_\epsilon(t) := \mathbb{E}\left[||X_t - a_0||^{2p} \right]$. Under the assumption (V-2), we have the following inequality:

$$\langle x - a_0 \, ; \, \nabla V(x) \rangle \geq \vartheta \, ||x - a_0||^2 .$$

Then, by proceeding like in the proof of Proposition 1.2, we recover an inequality similar to (1.2):

$$\psi'_\epsilon(t) \leq -2p\vartheta\psi_\epsilon(t)^{1-\frac{1}{p}} \left\{ \psi_\epsilon(t)^{\frac{1}{p}} - \frac{(2p-1)\epsilon}{2\vartheta} \right\} . \tag{1.4}$$

From now, we take ϵ sufficiently small such that $\frac{\rho}{2} > \frac{(2p-1)\epsilon}{2\vartheta}$. Consequently, for all $t \leq T_\epsilon(\rho)$, it yields:

$$-\psi'_\epsilon(t) \geq 2p\vartheta\psi_\epsilon(t)^{1-\frac{1}{p}} \left(\psi_\epsilon(t)^{\frac{1}{p}} - \frac{(2p-1)\epsilon}{2\vartheta} \right) \geq p\vartheta\rho^p .$$

By definition, if $||X_0 - a_0||^{2p} \geq \rho^p$:

$$\int_0^{T_\epsilon(\rho)} -\psi'_\epsilon(t)dt = \psi_\epsilon(0) - \rho^p .$$

(1.3) holds immediatly. Finally, (1.4) implies that for all $t \geq T_\epsilon(\rho)$, we have $\mathbb{E}\left\{ ||X_t - a_0||^{2p} \right\} \leq \rho^p$. □

This result concerns the law u_t and not the trajectories $t \mapsto X_t(\omega)$. But, it points out the importance of δ_{a_0} in the study. Indeed, Propositions 1.2 and 1.3 imply:

$$\lim_{\epsilon \to 0} \limsup_{t \to \infty} \mathbb{E}\left\{ ||X_t - a_0||^2 \right\} = 0$$

in the uniformly strictly convex case. Consequently, the relevant sets for the exit problem of the McKean-Vlasov diffusions are the ones which contains the attractive point that is to say a_0. Moreover, this will permit to prove the main results.

We give now a classical result on the geometry of the potential Υ^N defined in (III):

Lemma 1.4. *Under the hypotheses (V-1)–(V-4) and (F-1)–(F-3), Υ^N admits exactly one critical point: $\overline{a_0}$. Moreover, it is a wells of Υ^N.*

The proof is similar - up to some details due to the dimension d and the number 1 of critical point - to the one of Proposition 2.1 in [Tug11d]. Thereby, it is left to the attention of the reader.

2 Exit problem for X_t^1

Propagation of chaos means that a self-stabilizing process has a behavior closed to the one of a particle of a mean-field system whose size N goes to infinity. Consequently, we provide results on the exit problem for the particle X_t^1. Then, the same will hold for $\overline{X_t}$. Main Freidlin-Wentzell theory results are described in the annex.

Let us recall the mean-field system (II):

$$\begin{cases} X_t^1 = X_0 + \sqrt{\epsilon} B_t^1 - \int_0^t \nabla V\left(X_s^1\right) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N \nabla F\left(X_s^1 - X_s^j\right) ds \\ \vdots \\ X_t^i = X_0 + \sqrt{\epsilon} B_t^i - \int_0^t \nabla V\left(X_s^i\right) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N \nabla F\left(X_s^i - X_s^j\right) ds \\ \vdots \\ X_t^N = X_0 + \sqrt{\epsilon} B_t^N - \int_0^t \nabla V\left(X_s^N\right) ds - \int_0^t \frac{1}{N} \sum_{j=1}^N \nabla F\left(X_s^N - X_s^j\right) ds \end{cases}$$

Let an open domain $\mathcal{D} \subset \mathbb{R}^d$ which contains the wells a_0 and the initial value X_0. For all $N \geq 2$, we introduce the open $\mathcal{G}_N := \mathcal{D} \times \left(\mathbb{R}^d\right)^{N-1}$. Let us call τ_ϵ^1 the first exit time of the diffusion X_t^1 from the domain $\mathcal{D} \subset \mathbb{R}^d$.

Remark 2.1. *This exit problem is equivalent to the one of the whole system $\mathcal{X}_t := \left(X_t^1, \cdots, X_t^N\right)$ from the domain $\mathcal{G}_N \subset \left(\mathbb{R}^d\right)^N$.*

We will study the exit problem from $\mathcal{G}_N$ starting by $\overline{X_0}$. However, the domain $\mathcal{G}_N$ does not satisfy *a priori* Assumption A.2. Indeed, the domain is

not necessary stable under the action of $-\nabla\Upsilon^N$. Consequently, we will restrict the domain that we will consider. Let us present the assumptions on $\mathcal{D}$:

Assumption 2.2. *We consider the equation:* $\dot{x_t} = -\nabla V(x_t)$ *with* $x_0 = X_0$. *Then, for all* $t > 0$, $x_t \in \mathcal{D}$ *and* $x_t \to a_0 \in \mathcal{D}$.

This hypothesis will be used in the following since it permits the stability by $-\nabla\Upsilon^N$ when the initial point is $\overline{X_0} \in \mathcal{G}_N$. Let us note that this assumption is weaker than Assumption 4.1.i) in [HIP08]. We present now the other hypothesis:

Assumption 2.3. *For all* $x \in \mathcal{D}$, *we consider the ordinary differential equation:* $\dot{y_t} = -\nabla V(y_t) - \nabla F(y_t - a_0)$ *and* $y_0 = x$. *Then, for all* $t > 0$, $y_t \in \mathcal{D}$. *Moreover,* $\lim_{t\to\infty} y_t = a_0 \in \mathcal{D}$ *for all* $x \in \overline{\mathcal{D}}$.

This hypothesis is natural according to Propositions 1.2 and 1.3. Indeed, after a deterministic time, the whole system (II) will be captiv in a small ball around $\overline{a_0}$. Consequently, the particles system evolution will be closed to the following:

$$X_t^i = X_0 + \sqrt{\epsilon}B_t^i - \int_0^t \nabla V\left(X_s^i\right) ds - \int_0^t \nabla F * \delta_{a_0}\left(X_s^i\right) ds\,.$$

Thereby, Assumption 2.3 means intuitively that $\mathcal{G}_N \bigcap \mathbb{B}_N^{2n}\left(\overline{0}\,;\,\rho\right)$ is stable by the dynamic of (II).

However, this statement is not necessary true. Consequently, we will consider two sequences of sets which frame the domain and which satisfy Assumption 2.3. Then we will be able to provide the result for $\mathcal{G}_N$

Definition 2.4. *Let* $\rho > 0$, *arbitrarily small. We recall* $2n = \deg(G)$. *We consider* $\mathcal{M}_\rho$ *the set of all the probability measures* μ *on* $\mathbb{R}^d$ *which satisfy* $\int_{\mathbb{R}^d} ||x||^{2n} \mu(dx) \leq \rho^n$. *For all* $\nu \in \mathcal{M}_\rho^{\mathbb{R}_t} =: \mathcal{N}_\rho$, *we also consider the dynamical system:*

$$\dot{x_t^\nu} = -\nabla W_{\nu_t} = -\nabla V(x_t^\nu) - \nabla F * \nu_t(x_t^\nu)$$

with $\nu_t \in \mathcal{M}_\rho$ *for all* $t \geq 0$. We do not assume any other hypotheses on the function $t \mapsto \nu_t$. *We introduce the two following domains:*

$$\mathcal{I}_\rho := \left\{ x \in \mathcal{D} \;\middle|\; \inf_{\nu\in\mathcal{N}_\rho} \inf_{t\in\mathbb{R}_+} d\left(x_t^\nu\,;\mathcal{D}^c\right) > 0 \right\} \tag{2.1}$$

$$\text{and} \quad \mathcal{E}_\rho := \left\{ x_t^\mu \;\middle|\; t \geq 0,\, \nu \in \mathcal{N}_\rho,\, d\left(x_0\,,\,\mathcal{D}\right) \leq \rho \right\}. \tag{2.2}$$

Obviously, for all $\rho > 0$, the two sets $\mathcal{I}_\rho$ and $\mathcal{E}_\rho$ satisfy Assumption 2.3. Moreover, we have the inclusions

$$\mathcal{I}_{\rho_1} \subset \mathcal{I}_{\rho_2} \subset \mathcal{D}^0 \subset \overline{\mathcal{D}} \subset \mathcal{E}_{\rho_2}^0 \subset \mathcal{E}_{\rho_1}^0\,,$$

for all $0 < \rho_2 < \rho_1$. Now we will justify why the two sequences of sets represent a good description of the domain $\mathcal{D}$.

Proposition 2.5. *The following limit holds:* $\lim_{\rho\to 0}\int_{\mathcal{E}_\rho} dx - \int_{\mathcal{I}_\rho} dx = 0$.

Proof. **Step 1.** First, we note that the drift $\nabla F * \mu$ is a polynomial function with a finite number of parameters of the form:

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} \langle x\,;\, e_i\rangle^{l_i}\, ||x||^l \, d\mu(x)$$

where $l + \sum_{i=1}^d l_i \leq 2n$. Consequently, the set of functions

$$\{x \mapsto -\nabla V(x) - \nabla F * \mu(x) \;\mid\; \mu \in \mathcal{M}_\rho\}$$

is compact. Thereby, for all compact K which contains $\mathcal{D}$, there exists $f(\rho)$ which tends towards 0 when ρ goes to 0 such that

$$\sup_{\mu\in\mathcal{M}_\rho}\sup_{x\in K} ||\nabla F * \mu(x) - \nabla F(x - a_0)|| \leq f(\rho)\,.$$

Moreover, (V-2) and (F-2) imply

$$\inf_{x\in K}\inf_{\mu\in\mathcal{M}_\rho} \operatorname{Hess} W_\mu(x) \geq \vartheta$$

for any compact K which contains $\mathcal{D}$.

Step 2. Let $x_0 \in \mathcal{D}$. We will prove that $x_0 \in \mathcal{I}_\rho$ when ρ is small enough. We introduce the dynamical system:

$$\dot{x_t} = -\nabla V(x_t) - \nabla F(x_t - a_0) =: -\nabla W(x_t)\,.$$

Assumption 2.3 implies that $\kappa := \inf_{t\geq 0} d\,(x_t\,;\,\mathcal{D}^c) > 0$. Then, for all $\nu \in \mathcal{N}_\rho$, if $x_t^\nu \in K$, it yields

$$\begin{aligned}
\frac{d}{dt}\,||x_t^\nu - x_t||^2 =& -2\,\langle \nabla W_{\nu_t}(x_t^\nu) - \nabla W(x_t)\,;\, x_t^\nu - x_t\rangle \\
\leq& -2\vartheta\,||x_t^\nu - x_t||^2 \\
&+ 2\,||x_t^\nu - x_t||\sup_{\mu\in\mathcal{M}_\rho}\sup_{x\in K}||\nabla F * \mu(x) - \nabla F(x - a_0)|| \\
\leq& 2\,||x_t^\nu - x_t||\,\{f(\rho) - \vartheta\,||x_t^\nu - x_t||\}\,.
\end{aligned}$$

By taking ρ sufficiently small, we have $\frac{f(\rho)}{\lambda} < \kappa$. Since $\inf_{t\geq 0} d\,(x_t\,;\,\mathcal{D}^c) = \kappa$, we deduce that $x_t^\nu \in \mathcal{D}$ for all $t \geq 0$ and for all $\nu \in \mathcal{N}_\rho$. This means that $x_0 \in \mathcal{I}_\rho$ for ρ small enough.

Step 3. We will prove now that any point $x \in \overline{\mathcal{D}}^c$ is in $\mathcal{E}_\rho^c$ for ρ small enough. We assume $d\,(x,\mathcal{D}) =: \kappa > 0$. Let a point $x_0 \in \mathcal{D} + \mathbb{B}\,(0\,;\,\rho)$. There exists $y_0 \in \mathcal{D}$ such that $d(x_0, y_0) \leq 2\rho$. We consider the two dynamical systems: $\dot{x_t} = -\nabla W(x_t)$ and $\dot{y_t} = -\nabla W(y_t)$. Since W is uniformly strictly convex on the compact K (defined in Step 1), the distance $d(x_t, y_t)$ is nonincreasing if $\mathcal{D} + \mathbb{B}\,(0\,;\,\rho) \subset K$. This means $d(x_t, y_t) \leq 2\rho$ for all $t \geq 0$. By proceeding like in Step 2, the distance $d(y_t, y_y^\nu)$ is less than $f(\rho)$. By taking ρ small enough, we have $d(x_t, x_t^\nu) \leq \frac{\kappa}{3}$. Hence: $d(x_t^\nu, y_t) \leq \frac{2\kappa}{3}$. This result is uniform with respect to $x \in \mathcal{D} + \mathbb{B}\,(0\,;\,\rho)$. Consequently, $x \notin \mathcal{E}_\rho$. This achieves the proof. □

From now, we introduce the two exit times which will run the main proof of the article:

Definition 2.6. *We call τ_ρ (respectively τ_ρ^0) the first exit time of $\mathcal{X}_t$ from the domain $\mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$ (respectively $\mathcal{I}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$).*

It is important to stress that the domains $\mathcal{I}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$ and $\mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$ do not satisfy *a priori* Assumption A.2 for the potential Υ^N. Consequently, we will apply Freidlin-Wentzell theory to other domains. And, we will prove that the exit times and the exit locations of the new domains are exactly the same that the ones which are interesting. We begin with $\mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$.

Proposition 2.7. *Let $\rho > 0$. Let $N \geq 2$ and $\delta > 0$ arbitrarily small. We have the following limit for ρ small enough:*

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ \exp\left[\frac{H_N(\rho) - \delta}{\epsilon}\right] < \tau_\rho < \exp\left[\frac{H_N(\rho) + \delta}{\epsilon}\right] \right\} = 1$$
$$\text{with} \quad H_N(\rho) = H(\rho) + o_N(1)$$

where $H(\rho) := 2 \inf_{z \in \partial \mathcal{E}_\rho} W(z)$ and $W(z) := V(z) + F(z - a_0) - V(a_0)$. Furthermore, we have informations on the exit location. Indeed, for all $\mathcal{N} \subset \partial \mathcal{E}_\rho$ such that $\inf_{z \in \mathcal{N}} W(z) > \inf_{z \in \partial \mathcal{E}_\rho} W(z)$, we have:

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ X^1_{\tau_\rho} \in \mathcal{N} \right\} = 0 \tag{2.3}$$

for N large enough.

Proof. **Plan** The global idea is the following. First, we prove that the whole system $\mathcal{X}_t$ enters before a time T_ρ (finite, independent of N, independent of ϵ and deterministic) in the ball $\mathbb{B}_N^{2n}\left(\overline{a_0}\,;\, \rho\right) =: \mathbb{B}_\rho$. We go further by proving that it will enter into

$$\mathcal{H}_N(\rho) := \mathbb{B}_\rho \bigcap \left(\Upsilon^N\right)^{-1} \left(\left[0\,;\, \frac{\inf_{Z \in \partial \mathbb{B}_\rho} \Upsilon^N(Z) - \Upsilon^N(\overline{a_0})}{2} \right] \right)$$

which is stable by $-\nabla \Upsilon^N$. Moreover, we prove that the system does not leave $\mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$ before this time T_ρ.

The construction of $\mathcal{E}_\rho$ guarentees that the set $\mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1} \bigcap \mathcal{H}_N(\rho)$ is stable by the gradient of Υ^N. We apply Freidlin-Wentzell theory. Finally, we prove that $\mathcal{X}_t$ exits from $\mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1}$ before exiting from $\mathcal{H}_N(\rho)$.

Step 1 We note $\mathcal{Y}_t$ the following dynamical system:

$$\dot{\mathcal{Y}}_t = -\nabla \Upsilon^N\left(\mathcal{Y}_t\right) .$$

As $\mathcal{Y}_0 = \overline{X_0}$, we deduce that for all $t \geq 0$ and for all $N \geq 2$, $\mathcal{Y}_t = \overline{y}_t$ with $\dot{y_t} = -\nabla V(y_t)$. Assumption 2.2 means that $\{\mathcal{Y}_t \mid t \geq 0\} \subset \mathcal{D}^N \subset \mathcal{E}_\rho^N$.
The potential V is uniformly strictly convex according to (V-2). Consequently, $\mathcal{Y}_t$ converges towards $\overline{a_0}$ and there exists T_ρ deterministic and independent from N such that

$$\begin{aligned} &\mathcal{Y}_{T_\rho} \in \mathcal{H}_N\left(\frac{\rho}{2}\right) \\ &\text{with} \quad \mathcal{H}_N(\rho) := \mathbb{B}_\rho \bigcap \left(\Upsilon^N\right)^{-1}\left(\left[0\,;\, \frac{\inf_{Z\in\partial\mathbb{B}_\rho} \Upsilon^N(Z) - \Upsilon^N(\overline{a_0})}{2}\right]\right). \end{aligned}$$

Classical large deviations technics (see [DZ10]) permit to obtain the following limits:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau_\rho \leq T_\rho\right\} = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \mathbb{P}\left\{\mathcal{X}_{T_\rho} \in \mathcal{H}_N(\rho)\right\} = 1\,.$$

Step 2 From now, we consider the new exit time:

$$\begin{aligned} \eta_\rho &:= \inf\left\{t \geq T_\rho \mid \mathcal{X}_t \notin \mathcal{E}_{\rho,N}\right\} \\ \text{with} \quad \mathcal{E}_{\rho,N} &:= \mathcal{E}_\rho \times \left(\mathbb{R}^d\right)^{N-1} \bigcap \mathcal{H}_N(\rho)\,. \end{aligned}$$

Subsequently, it will be proved that - for N large enough and ϵ small enough - the two exit times η_ρ and τ_ρ are equal with probability closed to 1.
Step 2.1 By construction of $\mathcal{E}_\rho$, the domain $\mathcal{E}_{\rho,N}$ is stable by $-\nabla\Upsilon^N$.
Step 2.2 We apply Proposition A.1 to the domain $\mathcal{E}_{\rho,N}$ and we obtain

$$\begin{aligned} &\lim_{\epsilon\to 0} \mathbb{P}\left\{\exp\left[\frac{\widetilde{H_N}(\rho) - \delta}{\epsilon}\right] < \eta_\rho < \exp\left[\frac{\widetilde{H_N}(\rho) + \delta}{\epsilon}\right]\right\} = 1 \\ &\text{with} \quad \widetilde{H_N}(\rho) := 2N\left\{\inf_{Z\in\partial\mathcal{E}_{\rho,N}} \Upsilon^N(Z) - \Upsilon^N(\overline{a_0})\right\}. \end{aligned}$$

It remains now to prove that $\widetilde{H_N}(\rho) = H(\rho) + o_N(1)$.
Step 2.3 Since V is uniformly strictly convex and since $\operatorname{Hess} F \geq \alpha$, the potential Υ^N is convex on $\mathbb{B}_\rho$. Therefore, we have $\Upsilon^N(Z) - \Upsilon^N(\overline{a_0}) \geq \vartheta\,||Z||^2$. Elementary computation leads to $||Z|| \geq N^{\frac{1}{2}-\frac{1}{2n}}\,||Z||_{2n}$ for all $Z \in \left(\mathbb{R}^d\right)^N$. Consequently:

$$2N\left\{\inf_{Z\in\partial\mathcal{H}_N(\rho)} \Upsilon^N(Z) - \Upsilon^N(\overline{a_0})\right\} \geq \vartheta N^{\frac{1}{n}}\rho^2\,.$$

Step 2.4 Let us compute now

$$2N\left\{\inf_{Z\in\left(\partial\mathcal{E}_\rho\times\left(\mathbb{R}^d\right)^{N-1}\right)\bigcap\mathcal{H}_N(\rho)} \Upsilon^N(Z) - \Upsilon^N(\overline{a_0})\right\}.$$

We look at the function ξ_z defined as

$$\xi_z(x_2,\cdots,x_N) := \frac{1}{N}V(z) + \frac{1}{N}\sum_{j=2}^{N} V(x_j) + \frac{1}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} F(x_i - x_j)$$

with $x_1 := z \in \partial\mathcal{E}_\rho$. Each partial derivative provides:

$$\frac{\partial}{\partial x_k}\xi_z(x_2,\cdots,x_N) = \frac{1}{N}\nabla V(x_k) + \frac{1}{N^2}\sum_{j=1}^{N}\nabla F(x_k - x_j) = \frac{1}{N}\nabla W_\mu$$

where $\mu \in \mathcal{M}_\rho$ because $\mathcal{M} \in \mathbb{B}_\rho$. W_μ is convex. We deduce that - for all $(x_1,\cdots,x_N) \in \left(\mathbb{R}^d\right)^N$ - the function $u \mapsto \frac{1}{N}\nabla V(u) + \frac{1}{N^2}\sum_{j=1}^{N}\nabla F(u - x_j)$ is an injection. Consequently, the critical points of ξ_z have all the forms $(x_2,\cdots,x_2)$. Moreover, the point x_2 - which depends on N and on z - is the solution of

$$\nabla V(x_2) + \frac{1}{N}\nabla F(x_2 - z) = 0\,.$$

This implies the uniqueness of x_2 which satisfies

$$x_2 = a_0 + \frac{1}{N}\left(\mathrm{Hess}\, V(a_0)\right)^{-1}\nabla F(z) + o_z(\frac{1}{N})\,.$$

It yields:

$$\begin{aligned}
&\Upsilon^N(z,x_2,\cdots,x_2)\\
&= \frac{V(z)}{N} + \left(1 - \frac{1}{N}\right)V(x_2) + \frac{1}{N^2}F(z - x_2)\\
&= \frac{1}{N}\left\{V(z) + (N-1)V(a_0) + F(z - a_0)\right\} + O\left(\frac{1}{N^2}\right)\,.
\end{aligned}$$

Then:

$$N\Upsilon^N(z,x_2,\cdots,x_2) - N\Upsilon^N(\overline{a_0}) = W(z) - W(a_0) + o(1)\,.$$

This last term does not depend on z since we take the maximum over the compact K. So: $2N\left\{\inf_{\left(\partial\mathcal{E}_\rho\times\left(\mathbb{R}^d\right)^{N-1}\right)\bigcap\mathcal{H}_N(\rho)}\Upsilon^N(\mathcal{X}) - \Upsilon^N(\overline{a_0})\right\} = H(\rho) + o(\frac{1}{N})$. For proving $\eta_\rho = \tau_\rho$ with probability closed to 1, we just remark that

$$\inf_{Z\in\left(\partial\mathcal{E}_\rho\times\left(\mathbb{R}^d\right)^{N-1}\right)\bigcap\mathcal{H}_N(\rho)}\Upsilon^N(Z) - \Upsilon^N(\overline{a_0}) < \inf_{Z\in\partial\mathcal{H}_N(\rho)}\Upsilon^N(Z) - \Upsilon^N(\overline{a_0})$$

for N large enough, and we apply (A.1) of Proposition A.1.

Step 3 We can also prove (2.3) by noting the following inequality for N large enough:

$$2N\left\{\inf_{Z\in\mathcal{N}\times\left(\mathbb{R}^d\right)^{N-1}}\Upsilon^N(Z) - \Upsilon^N(\overline{a_0})\right\} > 2N\left\{\inf_{Z\in\partial\mathcal{E}_{\rho,N}}\Upsilon^N(Z) - \Upsilon^N(\overline{a_0})\right\}$$

if $\mathcal{N} \subset \partial\mathcal{E}_\rho$ such that $\inf_{z\in\mathcal{N}} W(z) > \inf_{z\in\partial\mathcal{E}_\rho} W(z)$. □

The same holds with $\mathcal{I}_\rho$:

Proposition 2.8. *Let $\rho > 0$. Let $N \geq 2$ and $\delta > 0$ arbitrarily small. We have the following limit for ρ small enough:*

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ \exp\left[\frac{H_N^0(\rho) - \delta}{\epsilon}\right] < \tau_\rho^0 < \exp\left[\frac{H_N^0(\rho) + \delta}{\epsilon}\right] \right\} = 1$$

where $H_N^0(\rho)$ verifies:

$$H_N^0(\rho) = H^0(\rho) + o_N(1)$$

where $H^0(\rho) := 2 \inf_{z \in \partial \mathcal{I}_\rho} W(z)$ with $W(z) := V(z) + F(z - a_0) - V(a_0)$.
Furthermore, for all $\mathcal{N} \subset \partial \mathcal{I}_\rho$ such that $\inf_{z \in \mathcal{N}} W(z) > \inf_{z \in \partial \mathcal{I}_\rho} W(z)$, we have:

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ X^1_{\tau^0_\rho} \in \mathcal{N} \right\} = 0$$

if N is sufficiently large.

The proof is similar except the argument which explains the inclusion

$$\{\mathcal{Y}_t \mid t \geq 0\} \subset \mathcal{I}_\rho$$

for ρ small enough. It is a consequence of Proposition 2.5.
Proposition 2.7 and Proposition 2.8 permit to obtain the result on $\mathcal{D}$:

Corollary 2.9. *Let $\kappa > 0$ arbitrarily small. We recall that τ_ϵ^1 is the first exit time of X_t^1 from $\mathcal{D} \times \left(\mathbb{R}^d\right)^{N-1}$. Then, there exists $N_0 \geq 2$ such that for all $N \geq N_0$, we have the following limit:*

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ \exp\left[\frac{H - \kappa}{\epsilon}\right] < \tau_\epsilon^1 < \exp\left[\frac{H + \kappa}{\epsilon}\right] \right\} = 1 \tag{2.4}$$

with $H := 2 \inf_{z \in \partial \mathcal{D}} W(z)$.
Furthermore, for all $\mathcal{N} \subset \partial \mathcal{D}$ such that $\inf_{z \in \mathcal{N}} W(z) > \inf_{z \in \partial \mathcal{D}} W(z)$, there exists $N_1 \geq 2$ such that for all $N \geq N_1$, we have:

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ X^1_{\tau^1_\epsilon} \in \mathcal{N} \right\} = 0\,. \tag{2.5}$$

Proof. **Step 1.** X_t^1 needs to exit from $\mathcal{I}_\rho$ before exiting from $\mathcal{D}$. Consequently:

$$\mathbb{P}\left\{ \tau_\epsilon^1 \leq \exp\left[\frac{H - \kappa}{\epsilon}\right] \right\} \leq \mathbb{P}\left\{ \tau_\rho^0 \leq \exp\left[\frac{H - \kappa}{\epsilon}\right] \right\}.$$

By taking ρ sufficiently small, we have $H(\rho) \geq H - \frac{\delta}{3}$. Then, by taking N sufficiently large, we have $H_N(\rho) \geq H(\rho) - \frac{\delta}{3}$. We apply Proposition 2.8 with $\delta := \kappa$ and we obtain:

$$\lim_{\epsilon \to 0} \mathbb{P}\left\{ \tau_\epsilon^1 \leq \exp\left[\frac{H - \kappa}{\epsilon}\right] \right\} = 0$$

for N large enough.

Step 2. If X_t^1 does not exit from $\mathcal{D}$, it means that it does not exit from $\mathcal{E}_\rho$. By using the same method than in Step 1, the application of Proposition 2.7 implies

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\tau_\epsilon^1 \geq \exp\left[\frac{H+\kappa}{\epsilon}\right]\right\} = 0$$

for N large enough.

Step 3. In order to prove (2.5), we introduce the following set:

$$\mathcal{G}_\kappa := \left\{x \in \mathcal{D} \;\mid\; W(x) \leq \inf_{z\in\partial\mathcal{D}} W(z) + \kappa\right\}$$

with $\kappa > 0$ such that $\inf_{z\in\partial\mathcal{G}_\kappa} W(z) < \inf_{z\in\mathcal{N}} W(z)$. $\mathcal{G}_\kappa$ satisfies Assumption 2.2 since V is convex on this domain. Assumption 2.3 holds immediatly by construction. Thereby, we can construct $\mathcal{E}_{\kappa,\rho}$ as an enlargement of $\mathcal{G}_\kappa$. Without loss of generality, we can assume that $x \in \mathcal{G}_\kappa$. If not, by the same argument than the one of Step 1 in the proof of Proposition 2.7, there exists a deterministic time independent from N and from ϵ such that - with probability arbitrarily closed to 1 - $\mathcal{X}_t$ enters in $\mathcal{G}_\kappa$ without leaving $\mathcal{D}$. By definition of the set $\mathcal{E}_{\kappa,\rho}$ associated to $\mathcal{G}_\kappa$, there exists $\gamma > 0$ such that

$$\inf_{z\in\partial\mathcal{D}} W(z) < H + 2\gamma < \inf_{z\in\partial\mathcal{E}_{\kappa,\rho}} W(z)\,.$$

Moreover, for κ and ρ small enough, we have the following inclusion: $\mathcal{N} \subset \mathcal{E}_{\kappa,\rho}^c$. We note $\tau_{\kappa,\rho}$ the first exit time of X_t^1 from $\mathcal{E}_{\kappa,\rho}$. We remark:

$$\begin{aligned}\mathbb{P}\left\{X^1_{\tau_\epsilon^1} \in \mathcal{N}\right\} \leq \mathbb{P}\left\{X^1_{\tau_\epsilon^1} \notin \mathcal{E}_{\kappa,\rho}\right\} \leq \mathbb{P}\left\{\tau_{\kappa,\rho} \leq \tau_\epsilon^1\right\}\\ \leq \mathbb{P}\left\{\tau_{\kappa,\rho} \leq \exp\left[\frac{H+\gamma}{\epsilon}\right]\right\} + \mathbb{P}\left\{\exp\left[\frac{H+\gamma}{\epsilon}\right] \leq \tau_\epsilon^1\right\}\,.\end{aligned}$$

For N large enough, the second term tends towards 0 after applying (2.4). The first one is a consequence of Proposition 2.7. □

3 Propagation of chaos

As well knwon, we know that propagation of chaos uniform with respect to the time holds. We provide the two main results about this propagation of chaos. On this purpose, we introduce a system of independent self-stabilizing processes:

$$\overline{X_t^i} = X_0 + \sqrt{\epsilon}B_t^i - \int_0^t \nabla V\left(\overline{X_s^i}\right) ds - \int_0^t \nabla F * u_s\left(\overline{X_s^i}\right) ds\,. \tag{3.1}$$

and we still consider the mean-field system (II). We omit the proof since the methods are classical. See [BRTV98, CGM08, Tug11d].

Proposition 3.1. *There exists $K > 0$ such that:*

$$\sup_{0<\epsilon<1} \sup_{t\geq 0} \mathbb{E}\left\{ \left|\left| X_t^1 - \overline{X_t^1} \right|\right|^2 \right\} \leq \frac{K}{N} .$$

Remark 3.2. *We can do the same if V is non-uniformly strictly convex. We only need the existence of $\kappa \geq 2$ and $\zeta > 0$ such that:*

$$\langle \nabla V(x) - \nabla V(y) \, ; \, x - y \rangle \geq \zeta \, ||x - y||^{\kappa} .$$

In this case, we would have:

$$\sup_{0<\epsilon<1} \sup_{t\geq 0} \mathbb{E}\left\{ \left|\left| X_t^1 - \overline{X_t^1} \right|\right|^2 \right\} \leq K N^{-\frac{1}{\kappa-1}} .$$

We provide now a result where the supremum (on a finite interval) is under the expectation:

Proposition 3.3. *Let $T > 0$. There exists $K > 0$ such that:*

$$\mathbb{E}\left\{ \sup_{0\leq t\leq T} \left|\left| X_t^1 - \overline{X_t^1} \right|\right|^2 \right\} \leq \frac{KT}{N} .$$

We would like to stress that this inequality is uniform with respect to ϵ. We provide now a more precise result. We write $\overline{X_t}$ instead of $\overline{X_t^1}$. From now, we consider a domain $\mathcal{D} \subset \mathbb{R}^d$ which satisfies Assumptions 2.2 and 2.3. We call τ_ϵ the first exit time of $\overline{X_t}$ from $\mathcal{D}$ and τ_ϵ^1 the one of X_t^1. By Step 1 in the proof of Proposition 2.7, we know that for all $\rho > 0$, there exists a time T_ρ deterministic, independent from ϵ and independent from N which verifies $\lim_{\epsilon\to 0} \mathbb{P}\left\{ \mathcal{X}_{T_\rho} \in \mathbb{B}_N^{2n}\left(\overline{a_0} \, ; \, \rho\right) \right\} = 1$. It permits to introduce the following exit time:

$$\tau_\epsilon^N := \inf\left\{ t \geq T_\rho \;\mid\; \mathcal{X}_t \notin \mathbb{B}_N^{2n}\left(\overline{a_0} \, ; \, \rho\right) \right\} .$$

Definition 3.4. *Also, we call $\mathcal{T}(\epsilon) := \min\left\{ \tau_\epsilon \, ; \, \tau_\epsilon^1 \, ; \, \tau_\epsilon^N \right\}$.*

We will now state that the propagation of chaos is uniform on the interval $[0 \, ; \, \mathcal{T}(\epsilon)]$.

Theorem 3.5. *Let $\delta > 0$. There exists $N_0 \in \mathbb{N}^*$ and $\epsilon_0 > 0$ such that for all $N \geq N_0$ and for all $\epsilon < \epsilon_0$, we have:*

$$\mathbb{E}\left[\sup_{0\leq t\leq \mathcal{T}(\epsilon)} \left|\left| \overline{X_t} - X_t^1 \right|\right|^2 \right] \leq \delta .$$

Proof. **Step 1.** By applying Proposition 3.3, we can prove the existence of a time deterministic, independent from ϵ and N - and we continue to write it T_ρ - such that:

$$\mathbb{E}\left[\left|\left| \overline{X_{T_\rho}} - a_0 \right|\right|^{2n} \right] < \rho^{2n} \quad \text{and} \quad \mathbb{E}\left[\sup_{0\leq t\leq T_\rho} \left|\left| \overline{X_t} - X_t^1 \right|\right|^2 \right] \leq \frac{\delta}{2}$$

for N large enough and ϵ sufficiently small. The first inequality holds in the small-noise case, independently of N. And, the second one is true for N large enough, uniformly with respect to ϵ.

Step 2. We apply Proposition 1.3 and we deduce $\mathbb{E}\left[\left|\left|\overline{X_t} - a_0\right|\right|^{2n}\right] < \rho^{2n}$ for all $t \geq T_\rho$.

Step 3. We note $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$. We recall the notation $W_\mu := V + F * \mu$. W_μ is convex and $\mathrm{Hess}\, W_\mu \geq \vartheta > 0$. By definition of $\overline{X_t}$ and X_t^1, if $\overline{X_{T_\rho}}, X_{T_\rho}^1 \in \mathcal{D}$ and if $\mathcal{X}_{T_\rho} \in \mathbb{B}_N^{2n}\left(\overline{a_0}\,;\,\rho\right)$, we have:

$$\begin{aligned}
\frac{d}{dt}\left|\left|\overline{X_t} - X_t^1\right|\right|^2 &= -2\left\langle \nabla W_{u_t}\left(\overline{X_t}\right) - \nabla W_{\mu_t^N}\left(X_t^1\right)\,;\,\overline{X_t} - X_t^1\right\rangle \\
&= -2\left\langle \nabla W_{u_t}\left(\overline{X_t}\right) - \nabla W_{u_t}\left(X_t^1\right)\,;\,\overline{X_t} - X_t^1\right\rangle \\
&\quad -2\left\langle \nabla F * u_t\left(\overline{X_t}\right) - \nabla F * \mu_t^N\left(X_t^1\right)\,;\,\overline{X_t} - X_t^1\right\rangle \\
&\leq -2\vartheta\left|\left|\overline{X_t} - X_t^1\right|\right|^2 \\
&\quad +2\left|\left|\overline{X_t} - X_t^1\right|\right| \sup_{x,y\in\mathcal{D}}\sup_{\mu_1,\mu_2\in\mathcal{M}_\rho} ||\nabla W_{\mu_1}(x) - \nabla W_{\mu_2}(y)||
\end{aligned}$$

for all $T_\rho \leq t \leq \mathcal{T}_\epsilon$. Thereby, by taking ρ small enough, we deduce:

$$\sup_{T_\rho\leq t\leq\mathcal{T}_\epsilon}\left|\left|\overline{X_t} - X_t^1\right|\right|^2 \leq \max\left\{\left|\left|\overline{X_{T_\rho}} - X_{T_\rho}^1\right|\right|^2\,;\,\frac{\delta}{2}\right\}.$$

Hence, it yields

$$\mathbb{E}\left\{\sup_{T_\rho\leq t\leq\mathcal{T}_\epsilon}\left|\left|\overline{X_t} - X_t^1\right|\right|^2\right\} \leq \frac{\delta}{2}.$$

Since $\max\{a,b\} \leq a+b$ for all $a,b\in\mathbb{R}_+$, it achieves the proof. □

4 Exit problem for the non-markovian process

We have now all the keys for proving the main result of the paper. Let us consider a domain $\mathcal{D}\subset\mathbb{R}^d$ which satisfies Assumptions 2.2 and 2.3. We recall that τ_ϵ is the first exit time of the diffusion (I) from the domain $\mathcal{D}$, τ_ϵ^1 is the one of the diffusion X_t^1 and τ_ϵ^N is the first exit time of (II) from $\mathbb{B}_N^{2n}\left(\overline{a_0}\,;\,\rho\right)$. Let us also recall that $\mathcal{T}_\epsilon := \min\left\{\tau_\epsilon\,;\,\tau_\epsilon^1\,;\,\tau_\epsilon^N\right\}$. The time $\mathcal{T}_\epsilon$ depends on ρ and on N but we do not write it for the comfort of the reading.

Theorem 4.1. *For all $\xi > 0$, we have the limit:*

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{H-\xi}{\epsilon}\right] < \tau_\epsilon < \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} = 1$$

with $H := \inf_{z\in\partial\mathcal{D}} W(z)$ with $W(z) := V(z) + F(z-a_0) - V(a_0)$.

Proof. The method is closed to the one used in Corollary 2.9. Let $\rho > 0$. We consider the two domains $\mathcal{I}_\rho$ and $\mathcal{E}_\rho$ previously introduced in (2.1) and (2.2). We take ρ sufficiently small such that $H(\rho) < H + \frac{\xi}{2}$. Let $\kappa := d\left(\mathcal{E}_\rho^c\,;\,\mathcal{D}\right) > 0$.
Step 1. In this step, $\tau_\epsilon^1(\rho)$ denotes the first exit time of X_t^1 from $\mathcal{E}_\rho$. Then, it yields:

$$\begin{aligned}\mathbb{P}\left(\tau_\epsilon \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\right) \leq & \mathbb{P}\left(\tau_\epsilon^1(\rho) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\right) + \mathbb{P}\left(\tau_\epsilon^N < \tau_\epsilon^1(\rho)\right) \\ & + \mathbb{P}\left(\left|\left|\overline{X}_{\tau_\epsilon} - X^1_{\tau_\epsilon}\right|\right| \geq \kappa\right)\end{aligned}$$

The application of Proposition 2.7 implies that the first term tends towards 0 for N large enough since $H(\rho) < H + \frac{\xi}{2}$. The convergence of the second one to 0 has been made in Step 2 of the proof of Proposition 2.7. The third term is as small as we want for N large enough and ϵ sufficiently small according to Theorem 3.5.
Step 2. Similar arguments with Proposition 2.8 instead of Proposition 2.7 provide the limit

$$\lim_{\epsilon\to 0} \mathbb{P}\left(\tau_\epsilon \leq \exp\left[\frac{H-\xi}{\epsilon}\right]\right) = 0\,.$$

□

This permits to obtain a good approximation of the self-stabilizing process on infinite interval:

Corollary 4.2. *Let $\delta, \xi, \kappa > 0$. There exists $N_0 \in \mathbb{N}^*$ and $\epsilon_0 > 0$ such that:*

$$\sup_{N\geq N_0}\sup_{\epsilon<\epsilon_0} \mathbb{P}\left\{\sup_{0\leq t\leq \exp\left[\frac{H-\xi}{\epsilon}\right]} \left|\left|\overline{X_t} - X_t^1\right|\right| \geq \kappa\right\} \leq \delta\,.$$

This is an immediate application of both Theorems 3.5 and 4.1, after recalling that the probability of the event $\left\{\tau_\epsilon^N > \tau_\epsilon^1(\rho)\right\}$ converges towards 1 for N large enough in the small-noise limit.
We provide now the result on the exit location.

Theorem 4.3. *Let $\mathcal{N} \subset \partial\mathcal{D}$ which verifies the inequality*

$$\inf_{z\in\mathcal{N}} V(z) + F(z-a_0) > \inf_{z\in\partial\mathcal{D}} V(z) + F(z-a_0)\,.$$

Then:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\overline{X_{\tau_\epsilon}} \in \mathcal{N}\right\} = 0\,.$$

The proof is similar to the one of the second part of Corollary 2.9 about the exit location of X_t^1. The only difference is the application of Theorem 3.5. The proof is left to the attention of the reader.

Before recalling the classical results on the Freidlin-Wentzell theory, we would like to stress the following point:

Remark 4.4. *We did not use the convexity of V in the whole space $\mathbb{R}^d$ but only its convexity on a compact which contains a wells a_0 and the captivity of the law u_t in a small ball which contains δ_{a_0}. Consequently, by using the result of convergence in [Tug11c], it is possible by using the ideas of this paper to characterize the exit time even if V is not convex.*

A Freidlin-Wentzell theory

We recall the main results of the Freidlin-Wentzell theory, see [DZ10] for proofs. We consider a classical diffusion on $\mathbb{R}^d$, $d \geq 1$:

$$x_t = x_0 + \sqrt{\epsilon} B_t - \int_0^t \nabla U\left(x_s\right) ds\,.$$

Let an open domain $\mathcal{G}$ which contains x_0 and a_0, a wells of the potential U. We introduce τ_ϵ the first exit time of the diffusion x_t from the domain $\mathcal{G}$. Then, under Assumptions A.2, A.3, A.4 and A.6 (described subsequently), we have:

Proposition A.1. *Let us introduce $H := 2\inf_{z\in\partial\mathcal{G}} U(z) - 2U(a_0)$. Then, for all $\delta > 0$, the following limit holds:*

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\exp\left[\frac{H-\delta}{\epsilon}\right] < \tau_\epsilon < \exp\left[\frac{H+\delta}{\epsilon}\right]\right\} = 1\,.$$

Moreover, for each subset $\mathcal{N} \subset \partial\mathcal{G}$ satisfying $\inf_{z\in\mathcal{N}} U(z) > \inf_{z\in\partial\mathcal{D}} U(z)$, we have:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{X_{\tau_\epsilon} \in \mathcal{N}\right\} = 0\,. \tag{A.1}$$

We describe now the four assumptions:

Assumption A.2. *The unique stable equilibrium point in the domain $\mathcal{G}$ of the d-dimensional ordinary differential equation $\dot{\phi}_t = -\nabla U\left(\phi_t\right)$ is at a_0. Moreover, for all $\phi_0 \in \mathcal{G}$, $\phi_t \in \mathcal{G}$ for all $t > 0$ and $\lim_{t\to\infty} \phi_t = a_0$.*

This assumption is automatic if $\mathcal{G}$ is the basin of attraction of a wells of U.

Assumption A.3. *All the trajectories of the deterministic system starting at $\phi_0 \in \partial\mathcal{G}$ converges to 0 as $t \to \infty$.*

The convexity of U on $\overline{\mathcal{G}}$ is sufficient.

Assumption A.4. *The following quantity is finite:*

$$\overline{V} := \inf_{z\in\mathcal{G}} \inf_{T>0} \inf_{\{u\in L_2^z([0;T])\,:\,\Phi_t = -\int_0^t \nabla U(\phi_s)ds + \sqrt{\epsilon}\int_0^t u_s ds\}} \frac{1}{2}\int_0^T ||u_t||^2 dt$$

where $L_2^z([0;T]) := \{u \in L_2([0;T])\ :\ u(T) = z\}$.

This hypothesis is satisfied. Indeed, as well known, we have the following proposition:

Proposition A.5. *For all $z \in \mathcal{G}$, it yields:*

$$\inf_{T>0} \inf_{\{u \in L_2^z([0;T])\,:\,\Phi_t = -\int_0^t \nabla U(\phi_s)ds + \sqrt{\epsilon}\int_0^t u_s ds\}} \frac{1}{2}\int_0^T ||u_t||^2 dt = 2U(z)\,.$$

See the proof of Theorem 3.1 in [FW98]. The last assumption is more technical:

Assumption A.6. *There exist an $M < \infty$ and $\rho_0 > 0$ such that for all $\rho > 0$, $\rho < \rho_0$, and for all x, y with $\inf_{z\in\mathcal{G}\cup\{0\}} ||x-z|| + ||y-z|| \leq \rho$, there is a function u satisfying $||u|| < M$ and $\phi_{T(\rho)} = y$ where*

$$\phi_t = x + \int_0^t -\nabla \Upsilon^N(\phi_s)\, ds + \sqrt{\epsilon}\int_0^t u_s ds$$

and $T(\rho) \to 0$ as $\rho \to 0$.

Since the diffusion coefficient is constant and strictly positive, this assumption is verified according to [DZ10].

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