Brauer spaces for commutative rings and structured ring spectra

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Abstract

Using an analogy between the Brauer groups in algebra and the Whitehead groups in topology, we first use algebraic K-theory to give a natural definition of Brauer spectra for commutative rings, such that their homotopy groups are given by the Brauer group, the Picard group and the group of units. Then, in the context of structured ring spectra, the same idea leads to two-fold non-connected deloopings of the spectra of units. Natural maps relate these in the case of extensions and in the case of Eilenberg-Mac Lane spectra.

Introduction

Let K be a field, and consider all finite-dimensional K-algebras A up to isomorphism. For many purposes, a much coarser equivalence relation than isomorphism is appropriate: Morita equivalence. Recall that two such algebras A and B are called *Morita equivalent* if their categories of modules are K-equivalent. While this description makes it clear that we get an equivalence relation, Morita theory actually shows that there is a convenient explicit description of the relation: A and B are Morita equivalent if and only if there are non-trivial, finite-dimensional K-vector spaces V and W such that $A\otimes_K \mathrm{End}_K(V)$ and $B\otimes_K \mathrm{End}_K(W)$ are isomorphic. In other words, Morita equivalence is generated by simple extensions. There is an abelian monoid structure on the set of Morita equivalence classes of algebras: The sum of the classes of two R-algebras A

and B is class of the tensor product $A \otimes_R B$, and the class of R is the neutral element, up to isomorphism. Not all elements have an inverse here, but those algebras A for which the natural map $A \otimes_K A^\circ \to \mathrm{End}_K(A)$ is an isomorphism do. These are precisely the central simple K-algebras. The abelian group of invertible classes of algebras is the Brauer group $\mathrm{Br}(K)$ of K. These notions have been generalized from the context of fields K to local rings by Azumaya [Azu51], and further to arbitrary commutative rings R by Auslander and Goldman [AG60]. A version for structured ring spectra in the context of commutative $\mathbb{S}$-algebras is defined in joint work with Baker and Richter [BRS].

Our first aim here is to use algebraic K-theory in order to define spaces such that their groups of components are naturally isomorphic to the Brauer groups $\mathrm{Br}(R)$. In fact, we will also offer the description of a(n infinite loop) space $\mathbf{Br}(R)$ such that there is a natural isomorphism

$$\pi_1 \mathbf{Br}(R) \cong \mathrm{Br}(R).$$

Then the first space may be replaced by $\Omega\mathbf{Br}(R)$. Of course, these properties will not characterize our results, so that we will have to provide motivation why the choice given here is appropriate. Therefore, in Section 1, we review Waldhausen's work [Wal85] on the Whitehead space of a space in sufficient detail so that it will become clear how this inspired the definition of the Brauer spaces to be given in Section 2 in the case of commutative rings. As will be discussed in Section 3, this gives an arguably more conceptual result than earlier efforts of Duskin [Dus88] and Street [Str04]. This becomes particularly evident when we discuss comparison maps later. Another advantage of the present approach is the fact that it naturally produces infinite loop space structures on these spaces.

The following Section 4 introduces Brauer spectra in the context of structured ring spectra. In both cases, we can also relate the Brauer spaces to the Picard groupoids, and prove that we have produced a delooping, see Theorem 2.6 and Theorem 4.9. This will imply that there are natural isomorphisms

$$\pi_2 \mathbf{Br}(R) \cong \mathrm{Pic}(R)$$

and

$$\pi_3 \mathbf{Br}(R) \cong \mathbb{G}_m(R),$$

in the case of ordinary rings, and similarly for structured ring spectra.

The final two sections indicate how to apply the theory. In Section 5, we discuss the functoriality of our construction and define relative invariants which dominate the relative invariants introduced in [BRS], see Proposition 5.3. In Section 6, we turn to the relation between the Brauer spectra of commutative rings and those of structured ring spectra. The Eilenberg-Mac Lane functor H produces structured ring spectra from ordinary rings, and it induces a homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(\mathrm{H}R)$ between the corresponding Brauer groups, see [BRS, Proposition 5.2]. However, it is difficult to say something about this homomorphism just from the definitions of both sides, and the present setting gives a new handle on this question, as we provide for a map of spectra which induces the displayed homomorphism after passing to homotopy groups, see Proposition 6.1.

1 A review of Whitehead groups and spectra

In this section we will review just enough of Waldhausen's work on the algebraic K-theory of spaces so that it will become clear how it inspired the definition of Brauer spaces to be given in the following Section 2.

A geometric definition of the Whitehead group of a space has been suggested by many people, see [Stö69], [EM70], [Sie70], [FW72], and [Coh73]. Take a space B. Ideally, it will be a nice topological space which has a universal covering, but it could also be a simplicial set if the reader prefers so. One considers finite cell extensions (cofibrations) $B \to X$ up to homeomorphism under B. An equivalence relation coarser than homeomorphism is generated by the elementary extensions $X \to Y$ (or their inverses, the elementary collapses). By [Coh67], this is the same as the equivalence relation generated by the simple maps. Recall that a map of simplicial sets is *simple* if its geometric realization has contractible point inverses. The standard reference for simple maps is now [WJR]. The sum of two extensions X and Y is obtained by glueing $X \cup_B Y$, and B itself is the neutral element, up to homeomorphism. Not all elements have an inverse here, but those X for which the structure map $B \to X$ is invertible (a homotopy equivalence) do. The abelian group of invertible extensions is the Whitehead group $\mathrm{Wh}(B)$ of B.

This description of the Whitehead group, which exactly parallels the description of the Brauer group given in the introduction, makes it clear that these are very similar constructions. We point out a slight difference:

Remark 1.1. The Whitehead group $\mathrm{Wh}(B)$ depends on the fundamental group(s) of B only. However, it is not quite true that the Brauer group $\mathrm{Br}(K)$ of a field K depends only on the absolute Galois group G_K of K. Instead, one has to take the Galois module $K_{\mathrm{sep}}^{\times}$ into account as well.

The Whitehead group of a space B as described above is actually a homotopy group of the Whitehead space. Let us recall from [Wal85, Section 3.1] how this space can be constructed.

If B is a space we denote by $\mathcal{C}(B)$ the category of the cofibrations under B; the objects are the cofibrations $B \to X$ as above, and the morphisms from X to Y are the maps under B. The subscript f will denote the subcategory of finite objects, where X is generated by the image of B and finitely many cells. The superscript h will denote the subcategory of the invertible objects, where the structure map is an equivalence. The prefix s will denote the subcategory of simple maps.

Proposition 1.2. [WJR, Corollary 3.2.4] *There is a natural bijection*

$$\pi_0|s\mathcal{C}^h_f(B)| \cong \mathrm{Wh}(B).$$

This bijection is an isomorphism of groups if one takes into account the fact that the category $\mathcal{C}(B)$ has (finite) sums. Indeed, this leads to a delooping of the space $|s\mathcal{C}^h_f(B)|$ as follows.

Monoidal categories, such as $\mathcal{C}(B)$, can be used to define spaces whose loop spaces group complete their classifying spaces. One way to to do this has been described by Segal, compare [Seg74]. Waldhausen constructs natural deloopings of $|s\mathcal{C}^h_f(B)|$ using a variant of the theory which takes various notions of weak equivalences $w\mathcal{C} \subseteq \mathcal{C}$ (such as those given by the simple maps $s\mathcal{C}(B) \subseteq \mathcal{C}(B)$ in this case) into account. See [Wal85, Section 1.8]. In this case, this results in a simplicial category $N_\bullet\mathcal{C}(B)$ and its decorated variants. This is the $N_\bullet$-construction; the more general $S_\bullet$-construction will not be needed here.

Proposition 1.3. [Wal85, Proposition 3.1.1] *There is a natural homotopy equivalence*

$$|s\mathcal{C}^h_f(B)| \simeq \Omega|sN_\bullet\mathcal{C}^h_f(B)|.$$

This is true because the canonical map $|w\mathcal{C}| \to \Omega|wN_\bullet\mathcal{C}|$ is a group completion in general, and here the abelian monoid $\pi_0|s\mathcal{C}^h_f(B)|$ is already a group by Proposition 1.2.

Definition 1.4. [Wal85, Section 3.1] The space

$$\mathbf{Wh}(B) = |sN_\bullet\mathcal{C}^h_f(B)|$$

is the *Whitehead space* of B.

Thus, the Whitehead space $\mathbf{Wh}(B)$ of B is a path connected space, whose fundamental group is isomorphic to the Whitehead group $\mathrm{Wh}(B)$ of B.

Remark 1.5. The category $s\mathcal{C}^h_f(B)$ is symmetric monoidal. Thomason [Tho79a] and Shimada-Shimakawa [SS79], building on earlier work of Segal [Seg74] in the case where the monoidal structure is the categorical sum, have explained that this implies that the Whitehead space is an infinite loop space. See also [Tho79b] and the appendix to [Tho82] for nice expositions. Therefore, there is a *Whitehead spectrum* $\mathbf{wh}(B)$ such that there is an equivalence $\Omega^\infty\mathbf{wh}(B) \simeq \mathbf{Wh}(B)$ of infinite loop spaces, and a natural equivalence

$$\Omega^{\infty+1}\mathbf{wh}(B) \simeq |s\mathcal{C}^h_f(B)|.$$

As spectra have a richer structure than their underlying infinite loop spaces, it will be good to remember that this structure is around.

2 Brauer spectra for commutative rings

In this section, we will complete the analogy between Brauer groups and Whitehead groups by defining Brauer spaces and spectra in nearly the same way as we have described the Whitehead spaces and spectra in the previous section.

Throughout this section, let R denote an ordinary commutative ring, and [Bas68, Chapter II] will be our standard reference for the facts used from Morita theory.

The category $\mathcal{A}(R)$ has objects the associative R-algebras A, but this will just be their name, and will be better to think of A as a placeholder for the R-linear category (mod-A) of right A-modules. While categories of modules can be intrinsically characterized, it will not be necessary to know this here. The morphisms $A \to B$, or rather (mod-A) $\to$ (mod-B), will be the R-linear functors. Composition in $\mathcal{A}(R)$ is composition of functors and identities are the identity functors, so that it is evident that $\mathcal{A}(R)$ is a category. As it would be careless to forget about the natural transformations between functors, we will actually think of $\mathcal{A}(R)$ as a simplicial category, i.e., a category enriched in simplicial sets. The n-simplices in the space of morphisms $A \to B$ are the functors (mod-A) $\times [n] \to$ (mod-B). Here $[n]$ denotes the usual category with object set $\{0,\dots,n\}$ and standard order.

There are full subcategories $\mathcal{A}_f(R)$ and $\mathcal{A}^h(R)$ of $\mathcal{A}(R)$, defined as follows. Recall the following result from [Bas68, IX.4.6].

Proposition 2.1. *For finitely generated projective R-modules P, the following are equivalent.*
(1) *The R-module P is a generator of the category of R-modules.*
(2) *The rank function* $\mathrm{Spec(R)} \to \mathbb{Z}$ *of P is everywhere positive.*
(3) *There is a finitely generated projective R-module Q such that $P \otimes_R Q \cong R^{\oplus n}$ for some positive integer n.*

For $\mathcal{A}_f(R)$, we impose the equivalent finiteness conditions from Proposition 2.1 on the R-algebras A. For $\mathcal{A}^h(R)$, we assume that the natural map

$$A \otimes_R A^\circ \longrightarrow \mathrm{End}_R(A)$$

is an isomorphism. We are mostly interested in the intersection

$$\mathcal{A}^h_f(R) = \mathcal{A}^f(R) \cap \mathcal{A}^h(R)$$

which consists precisely of the Azumaya R-algebras in the sense of [AG60].

While f and h refer to restrictions on the objects, the prefix s will indicate that we are considering only those functors which are R-linear equivalences of categories: Morita equivalences. By Morita theory, these are – up to natural isomorphism – of the form $X \longmapsto X \otimes_A M$ for some invertible R-symmetric (A,B)-bimodule M. Similarly, the higher simplices in $s\mathcal{A}^h_f(R)$ codify natural isomorphisms rather than all natural transformations. This ends the description of the simplicial category $s\mathcal{A}^h_f(R)$.

Remark 2.2. It seems tempting to work in the 'bimodular' setting with R-algebras and bimodules throughout, as in [Mac98, XII.7]. Then composition and identities are given only up to choices, and we would have been forced to work in a setting for higher categories which is less rigid than simplicial categories. For example, it is rather straightforward to model this in Segal categories. With the present approach, this is unnecessary.

A symmetric monoidal structure on $\mathcal{A}(R)$ and its subcategories is induced by the tensor product

$$(A,B) \mapsto A \otimes_R B$$

of R-algebras with neutral element R. Note that this is not the categorical sum, as the morphisms in that category are not just the algebra maps.

Proposition 2.3. *With the induced multiplication, the abelian monoid*

$$\pi_0|s\mathcal{A}^h_f(R)|$$

of isomorphism classes of objects is an abelian group which is isomorphic to the Brauer group $\mathrm{Br}(R)$ *of the commutative ring R in the sense of* [AG60].

Proof. The elements of the monoid $\pi_0|s\mathcal{A}^h_f(R)|$ are represented by the objects of the category $s\mathcal{A}^h_f(R)$, and we have already noted that these are just the Azumaya algebras in the sense of [AG60]. Therefore, the elements in both structures have the same representatives, and the multiplications and units are also agree on those.

We now show that the abelian monoid $\pi_0|s\mathcal{A}^h_f(R)|$ is a group. As Azumaya algebras satisfy $A \otimes_R A^\circ \cong \mathrm{End}_R(A)$, we have $[A] + [A^\circ] = [\mathrm{End}_R(A)]$ in $\pi_0|s\mathcal{A}^h_f(R)|$, so that $[A^\circ]$ is an inverse to $[A]$ if there is a path from $\mathrm{End}_R(A)$ to R in the category $|s\mathcal{A}^h_f(R)|$. But by Proposition 2.1, we know that A is a finitely generated projective generator in the category of R-modules, so that the R-algebras $\mathrm{End}_R(A)$

and R are Morita equivalent, and an R-linear equivalence gives rise to a 1-simplex which connects the two. This shows that the monoid $\pi_0|s\mathcal{A}^h_f(R)|$ is in fact a group.

Since the elements of both groups have the same representatives, it suffices to show that the equivalence relations agree for both of them. The equivalence relation in the Brauer group is generated by the simple extensions, and these are Morita equivalences by the same argument as in the preceding paragraph. Conversely, if an algebra A is Morita equivalent to R, then A is isomorphic to $\mathrm{End}_R(P)$ for some finitely generated projective generator P of the category of R-modules, so that A is a simple extension of R, up to isomorphism. □

As explained in the previous section, the symmetric monoidal structure on $s\mathcal{A}^h_f(R)$ allows to construct $sN_\bullet\mathcal{A}^h_f(R)$ which provides for a delooping.

Proposition 2.4. *There is a natural homotopy equivalence*

$$|s\mathcal{A}^h_f(R)| \simeq \Omega|sN_\bullet\mathcal{A}^h_f(R)|.$$

Proof. Again, this is true because the canonical map $|w\mathcal{C}| \to \Omega|wN_\bullet\mathcal{C}|$ is a group completion in general, and the abelian monoid $\pi_0|s\mathcal{A}^h_f(R)|$ is a group by Proposition 2.3. □

The following is now motivated by Definition 1.4.

Definition 2.5. The space

$$\mathbf{Br}(R) = |sN_\bullet\mathcal{A}^h_f(R)|$$

is called the *Brauer space* of R.

As in Remark 1.5, we see that the Brauer space is part of an infinite delooping of $|s\mathcal{A}^h_f(R)|$, and $\mathbf{br}(R)$ will denote the corresponding spectrum, so that

$$\Omega^\infty\mathbf{br}(R) = \mathbf{Br}(R).$$

and

$$\Omega^{\infty+1}\mathbf{br}(R) \simeq |s\mathcal{A}^h_f(R)|$$

by Proposition 2.4.

The Brauer space $\mathbf{Br}(R)$ of R is a path connected (infinite loop) space whose fundamental group is naturally isomorphic to the Brauer group $\mathrm{Br}(R)$ of R. Let us now turn our attention to the higher homotopy groups of it. Recall that an R-module M is *invertible* if there is another R-module L such that there is an isomorphism $L \otimes_R M \cong R$ of R-modules. (Later on, we will have occasion to consider integrally graded R-modules, but then we will explicitly say so.) The *Picard groupoid* of a commutative ring R is the groupoid of invertible R-modules and their isomorphisms. Thought of as a space, the Picard groupoid may have only two non-trivial homotopy groups: the group of components is the Picard group $\mathrm{Pic}(R)$ of R, and the fundamental groups are all isomorphic to the group of automorphisms of the R-module R, which is the group $\mathbb{G}_{\mathrm{m}}(R)$ of units in R. As the Picard groupoid is in particular a monoidal subcategory of the category of R-modules, it has a classifying space. Its homotopy groups are isomorphic to those of the Picard groupoid, but shifted up by one.

Theorem 2.6. *The components in $\Omega\mathbf{Br}(R)$ are all equivalent to the classifying space of the Picard groupoid of R.*

Proof. All components of an infinite loop space are homotopy equivalent, so that we only have to deal with the component of the unit R. As we have an equivalence $\Omega\mathbf{Br}(R) \simeq |s\mathcal{A}^h_f(R)|$, we may equally well try to understand the corresponding component of $s\mathcal{A}^h_f(R)$. But that component is the space of R-linear self-equivalences of the category (mod-R) and their natural isomorphisms. It remains to be verified that the space of R-linear self-equivalences of the category (mod-R) and their natural isomorphisms is naturally equivalent to the Picard groupoid of invertible R-modules and their isomorphisms.

On the level of components, this follows from Morita theory, see [Bas68]. On the level of spaces, the equivalence is given by evaluation at the symmetric monoidal unit R. In more detail, if F is an R-linear equivalence from (mod-R) to itself, then $F(R)$ is an invertible R-symmetric (R,R)-bimodule, and these are just the invertible R-modules. If $F \to G$ is a natural isomorphism between two R-linear self-equivalences, this gives in particular an isomorphism $F(R) \to G(R)$ betweem the corresponding two invertible R-modules. This map induces the classical isomorphism on components, and the natural automorphisms of the identity are given by the units (of the center) of R, which are precisely the automorphisms of R as an R-module. $\square$

The preceding result implies the calculation of all other homotopy groups of the Brauer space as a corollary. We note that a similar description and computation of the higher Whitehead groups of spaces is out of reach at the moment.

Corollary 2.7. *If R is a commutative ring, then the Brauer space $\mathbf{Br}(R)$ is a reduced Kan complex with at most three non-trivial homotopy groups:*

$$\pi_1\mathbf{Br}(R) \cong \mathrm{Br}(R),$$
$$\pi_2\mathbf{Br}(R) \cong \mathrm{Pic}(R),$$
$$\pi_3\mathbf{Br}(R) \cong \mathbb{G}_m(R).$$

3 The relation to the Azumaya complex

In this section, we will first make the lower-dimensional simplices in the Brauer spaces explicit. This will then lead to a comparison with earlier work of Duskin and Street.

Example 3.1. Since $N_0\mathcal{A}_f^h(R)$ is the category with one object and one morphism, there is only one 0-simplex in $\mathrm{Br}(R)$. It does not need a name, but it can be thought of as the commutative ring R.

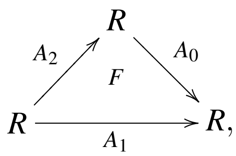
Example 3.2. Since $N_1\mathcal{A}_f^h(R)$ is the category $\mathcal{A}_f^h(R)$ itself, the 1-simplices in the Brauer space $\mathrm{Br}(R)$ are the Azumaya R-algebras A, denoted by

$$R \xrightarrow{A} R.$$

The degenerate 1-simplex is given by the R-algebra R itself. We emphasize that, despite the suggestive notation, these are just 1-simplices. Yet, for each pair of these, there is a candidate for a composition: the tensor product $A \otimes_R B$, which should be read as 'A followed by B.' Note that this is only well-defined up to isomorphism, which indicates that in order to make this precise, the situation needs to be either placed in a higher categorical context or otherwise rigidified. (Recall

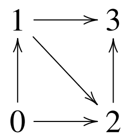
Remark 2.2.) Likewise, associativity and identities hold only up to isomorphism. In any case, as the next example shows, we only consider 1-simplices A such that the opposite algebra A° is an inverse up to homotopy.

Example 3.3. The idea behind the definition of next category $N_2\mathcal{A}^h_f(R)$ is to build in the choices needed to get a well-defined composition. In more detail, it is defined to be the category of triples (A_0, A_1, A_2) of R-algebras, together with a chosen R-linear equivalence $F\colon (\text{mod-}A_1) \to (\text{mod-}A_2 \otimes_R A_0)$. If F is such a 2-simplex with boundaries $A_j = \mathrm{d}_j F$,

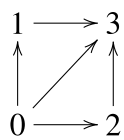


then evaluation at A_1 gives an R-symmetric $(A_1, A_2 \otimes_R A_0)$-bimodule which is invertible. A suggestive notation is $F\colon A_1 \Rightarrow A_2 \otimes_R A_0$. The faces are as indicated, and the degeneracies are given by the R-linear equivalences induced by the isomorphisms $A \otimes_R R \cong R \cong R \otimes_R A$.

Example 3.4. As for $N_3\mathcal{A}^h_f(R)$, the additional data here yield two potentially different R-linear equivalences, and a natural isomorphism between both of them. In more detail, if



is the front side of a 3-simplex u, given by equivalences $F_{012}\colon A_{02} \Rightarrow A_{01} \otimes_R A_{12}$ as well as $F_{123}\colon A_{13} \Rightarrow A_{12} \otimes_R A_{23}$, and



is the back side of the 3-simplex u, given by equivalences $F_{013} \colon A_{03} \Rightarrow A_{01} \otimes_R A_{13}$ as well as $F_{023} \colon A_{03} \Rightarrow A_{02} \otimes_R A_{23}$, then u is a natural isomorphism

$$F_{013} \otimes (A_{01} \otimes F_{123}) \cong F_{023} \otimes (F_{012} \otimes A_{23}),$$

where there is only one way to interpret this.

We may now discuss some related earlier work of Duskin and Street. Let again R be an ordinary commutative ring. In this case, Duskin, in [Dus88], has built a reduced Kan complex $\mathbf{Az}(R)$ with $\pi_1 \mathbf{Az}(R)$ isomorphic to the Brauer group of R, with the group $\pi_2 \mathbf{Az}(R)$ isomorphic to the Picard group of R, and with the group $\pi_3 \mathbf{Az}(R)$ isomorphic to the multiplicative group of units in R. (As $\mathbf{Az}(R)$ is reduced, we may omit the base-point from the notation.) In fact, he hand-crafts the 4-truncation so that the homotopy groups work out as stated, and then he takes its 4-co-skeleton.

His definition can be sketched as follows. The 0-simplices and 1 simplices are as indicated in our Examples 3.1 and 3.2 above. The 2-simplices are as in Example 3.3, except for the fact that he works with bimodules rather than equivalences. Similarly, the 3-simplices are as in Example 3.4, except for the fact that he works with isomorphisms of bimodules rather than natural isomorphisms. As for the 4-simplices in $\mathbf{Az}(R)$, let us just note that they do not involve additional structure, but their existence depends only on a coherence condition for the isomorphisms on the faces. We will not recall further details here, but state Duskin's result for later reference.

Proposition 3.5. [Dus88] *The simplicial set $\mathbf{Az}(R)$ is a reduced Kan complex with at most three non-trivial homotopy groups:*

$$\begin{aligned} \pi_1 \mathbf{Az}(R) &\cong \mathrm{Br}(R), \\ \pi_2 \mathbf{Az}(R) &\cong \mathrm{Pic}(R), \\ \pi_3 \mathbf{Az}(R) &\cong \mathbb{G}_m(R). \end{aligned}$$

Remark 3.6. As already mentioned in [Dus88], Street has described some categorical structures underlying Duskin's construction. However, these were published only much later, in [Str04]. Street considers the bicategory whose objects are R-algebras, whose morphism $M \colon A \to B$ are R-symmetric (A,B)-bimodules, and whose 2-cells $f\colon M \Rightarrow N$ are bimodule morphisms; vertical composition is composition of functions and horizontal composition of modules $M\colon A \to B$ and $N\colon B \to C$ is given by tensor product $M \otimes_B N\colon A \to C$ over B. The tensor product $A \otimes_R B$ of algebras is again an algebra, and this makes this category a monoidal bicategory. He then passes to its suspension, the one-object tricategory whose morphism bicategory is the category described before and whose composition is the tensor product of algebras. While this cannot, in general, be rigidified to a 3-category, there is a 3-equivalent Gray category. The Gray subcategory of invertibles consists of the arrows A which are biequivalences, the 2-cells M which are equivalences, and the 3-cells f which are isomorphisms, so that the morphisms A are the Azumaya algebras, and the 2-cells are the Morita equivalences. The nerve of this is Duskin's complex $\mathbf{Az}(R)$.

We can now show that Duskin's hand-crafted Azumaya complex is equivalent to the Brauer space obtained by the general K-theory machine.

Theorem 3.7. *There is a natural equivalence*

$$\mathbf{Br}(R) \simeq \mathbf{Az}(R)$$

between the Brauer space, as defined in Section 2, and Duskin's Azumaya complex $\mathbf{Az}(R)$.

Proof. As the co-skeleton is right-adjoint to the truncation, in order to define a map into it, we only need to specify it on the latter. As explained above, both spaces are reduced and have the same set of 1-simplices. On 2-simplices and 3-simplices, the map is given by evaluation at ground rings, as in the proof of Theorem 2.6, so that an R-linear equivalence $F\colon (\text{mod-}A_1) \to (\text{mod-}A_2 \otimes_R A_0)$ is sent to the R-symmetric $(A_1, A_2 \otimes_R A_0)$-bimodule $F(A_1)$. The coherence condition for the 4-simplices is automatically satisfied in this case. This defines the map. It then follows from Corollary 2.7 and Proposition 3.5 that both spaces have the same homotopy groups, and it is readily verified that the map induces an isomorphism on homotopy groups. The result follows from the Whitehead theorem. □

Note that in contrast to the work of Duskin and Street, the present approach makes it immediately clear that the Brauer space constructed here by means of the K-theory delooping machines are infinite loop spaces.

4 Brauer spectra for commutative $\mathbb{S}$-algebras

We will now transfer the preceding theory from the context of commutative rings to the context of structured ring spectra. There are many equivalent models for this, such as symmetric spectra [HSS00] or $\mathbb{S}$-modules [EKMM97], and we will choose the latter for the sake of concordance with [BRS]. In Section 2, we have defined a Brauer space $\mathbf{Br}(R)$ and a Brauer spectrum $\mathbf{br}(R)$ for each commutative ring R, starting from a category $\mathcal{A}(R)$ and its subcategory $s\mathcal{A}^h_f(R)$. If now R denotes a commutative $\mathbb{S}$-algebra, we may proceed similarly. Let us see how to define the corresponding categories.

Let $\mathcal{A}(R)$ denote the category of cofibrant R-algebras and R-functors between their categories of modules. This is slightly more subtle than the situation for ordinary rings, as the categories of modules are not just categories, but come with homotopy theories. In order to take this into account, the model categories of modules will first be replaced by the simplicial categories obtained from them by Dwyer-Kan localization. Note that the categories of modules are enriched in the symmetric monoidal model category of R-modules in this situation. This allows us to use the model structure from [Lur09, Appendix A.3] on these. Then $\mathcal{A}(R)$ is again a simplicial category: The class of objects A is still discrete, and the space of morphisms $A \to B$ is the derived mapping space of R-functors $(\text{mod-}A) \to (\text{mod-}B)$.

There is the full subcategory $\mathcal{A}_f(R)$, where A is assumed to satisfy the finiteness condition used in [BRS]: it has to be faithful and dualizable as an R-module.

Remark 4.1. Dualizability is a finiteness condition which refers to the monoidal structure on the category of R-modules. There are other natural finiteness conditions, such as compactness or perfection. These all turn out to be equivalent in the present context. See [B-ZFN10, Lemma 3.1]. As for ordinary rings, the faithfulness assumption is again necessary, not least to rule out contractibility.

Also, there is the full subcategory $\mathcal{A}^h(R)$, where we assume that the natural map

$$A \wedge_R A^\circ \longrightarrow \mathrm{End}_R(A)$$

is an equivalence. We are mostly interested in the intersection

$$\mathcal{A}^h_f(R) = \mathcal{A}^f(R) \cap \mathcal{A}^h(R),$$

which consists precisely the Azumaya R-algebras in the sense of [BRS]. While f and h again refer to restrictions on the objects, the prefix s will indicate that we are considering only those R-functors which are equivalences of simplicial categories, and their natural equivalences. The standard references for Morita theory in this context, are [SS03] as well as the expositions [Sch04] and [Shi07]. Up to natural equivalence, the R-equivalences are of the form $X \longmapsto X \wedge_A M$ for some invertible R-symmetric (A,B)-bimodule M. Similarly, the higher simplices codify natural equivalences rather than all natural transformations. This ends the description of the simplicial category $s\mathcal{A}^h_f(R)$.

Proposition 4.2. *The space of auto-equivalences of the category of R-modules is naturally equivalent to the space of invertible R-modules.*

Proof. This is formally the same as the corresponding result in the proof of Theorem 2.6. A map in one direction is given by evaluation, which sends a functor F to its value $F(R)$ on R. On the other hand, if M is an invertible R-module, it can be used to define an equivalence $? \mapsto ? \wedge_A F$. □

Remark 4.3. More generally, if A and B are associative R-algebras, then the space of R-equivalences $(\text{mod-}A) \to (\text{mod-}B)$ is naturally equivalent to the space of invertible R-symmetric (A,B)-bimodules.

A symmetric monoidal structure on $\mathcal{A}(R)$ and its subcategories is induced by the smash product

$$(A,B) \mapsto A \wedge_R B$$

of R-algebras with neutral element R. Note that this is not the categorical sum, as the morphisms in these categories are not just the algebra maps.

Proposition 4.4. *With the induced multiplication, the abelian monoid*

$$\pi_0|s\mathcal{A}^h_f(R)|$$

of isomorphism classes of objects is an abelian group which is isomorphic to the Brauer group $\mathrm{Br}(R)$ *of the commutative* $\mathbb{S}$*-algebra R in the sense of* [BRS].

Proof. This is formally the same as the proof of Proposition 2.3. □

As explained in Section 1, the (symmetric) monoidal structure on $s\mathcal{A}_f^h(R)$ allows to construct $sN_\bullet\mathcal{A}_f^h(R)$ which provides a delooping.

Proposition 4.5. *There is a natural homotopy equivalence*

$$|s\mathcal{A}_f^h(R)| \simeq \Omega|sN_\bullet\mathcal{A}_f^h(R)|.$$

Proof. Again, this is true because the canonical map $|w\mathcal{C}| \to \Omega|wN_\bullet\mathcal{C}|$ is a group completion in general, and the abelian monoid $\pi_0|s\mathcal{A}_f^h(R)|$ is a group by Proposition 4.4. □

Definition 4.6. The space

$$\mathbf{Br}(R) = |sN_\bullet\mathcal{A}_f^h(R)|$$

is called the *Brauer space* of the commutative $\mathbb{S}$-algebra R. It is part of an infinite delooping of $|s\mathcal{A}_f^h(R)|$, and $\mathbf{br}(R)$ will denote the corresponding spectrum, so that $\Omega^\infty\mathbf{br}(R) = \mathbf{Br}(R)$.

Thus, the Brauer space $\mathbf{Br}(R)$ of R is a path connected (infinite loop) space whose fundamental group is naturally isomorphic to the Brauer group $\mathrm{Br}(R)$ of R.

As for the Brauer space of a commutative ring, also in the context of structured ring spectra, there is a relation to the Picard groupoid of R, and this will be discussed now. We will first review here parts of the work of Ando, Blumberg, Gepner, Hopkins, and Rezk, see [ABG10], [ABGHR], and [AHR], so that we can then compare this to the present work.

Let R be a cofibrant commutative $\mathbb{S}$-algebra, and let (R-mod) be the quasi-category of cofibrant left R-modules. In [ABGHR], the authors define quasi-groupoids (R-line) and (R-triv) as the quasi-subgroupoids of (R-mod) consisting of the R-modules equivalent to R and their equivalences, and the category of R-modules with a chosen equivalence to R, i.e. the slice category $(R\text{-line})/R$, respectively. The forgetful map

$$(R\text{-triv}) \longrightarrow (R\text{-line})$$

models the universal $\mathrm{GL}_1(R)$-principal bundle $\mathrm{EGL}_1(R) \to \mathrm{BGL}_1(R)$. In particular, the quasi-groupoid (R-line) is a connected delooping of the space $\mathrm{GL}_1(R)$ of units of R. It is only natural to make the following definition, compare [ABG10, Remark 1.3].

Definition 4.7. The *Picard space*

$$\mathbf{Pic}(R) = (R\text{-inv})$$

of R is the Kan subset of the quasi-category (R-mod) consisting of the invertible R-modules and their equivalences: those R-modules L such that there is an R-module M with $L \wedge_R M \simeq R$.

Note that $\mathbf{Pic}(R)$ is grouplike symmetric monoidal. The group of components is isomorphic to Hopkins' Picard group $\mathrm{Pic}(R)$ of R:

$$\pi_0(\mathbf{Pic}(R)) \cong \mathrm{Pic}(R).$$

See for example [Str92], [HMS94], and [BR05] for more information about this group.

Remark 4.8. Units in ring spectra are related to Thom spectra, because Thom spectra can be defined from maps $X \to \mathrm{BGL}_1(R)$. This is classical for $R = \mathbb{S}$, but otherwise needs a modern version of the Thom spectrum construction, such as given in [ABGHR]. See also [AHR]. If R is an $\mathbb{S}$-algebra, and $f\colon X \to \mathrm{BGL}_1(R)$ is a map, there is an R-module Thom spectrum $\mathrm{M}(f)$. The *f-twisted R-homology of X* is by definition

$$R_k^f(X) = \pi_0((R\text{-mod})(\Sigma^k R, \mathrm{M}(f))),$$

while the *f-twisted R-cohomology of X* is

$$R^k_f(X) = \pi_0((R\text{-mod})(\mathrm{M}(f), \Sigma^k R)).$$

If f factors through $\mathrm{BGL}_1(\mathbb{S}) \to \mathrm{BGL}_1(R)$, the R-module Thom spectrum is just the base change to R of the ordinary $\mathbb{S}$-module Thom spectrum, so that in this case, the f-twisted homology and cohomology of X coincides with the untwisted R-homology and cohomology of the ordinary $\mathbb{S}$-module Thom spectrum of the spherical fibration. Otherwise, the f-twisted generalized cohomology is the cohomology of the more general R-module Thom spectrum. More generally, one may now use maps $X \to \mathbf{Pic}(R) = (R\text{-inv})$ instead of those into $\mathrm{BGL}_1(R) = (R\text{-line})$ throughout this story.

After this recollection, let us now see how the Picard space relates to the Brauer space and spectrum defined above.

Theorem 4.9. *If R is a commutative $\mathbb{S}$-algebra, then the component of the neutral element R in $|s\mathcal{A}^h_f(R)|$ is naturally equivalent (as an infinite loop space) to the classifying space of the Picard groupoid* $\mathbf{Pic}(R)$.

Proof. We will see that both spaces are naturally equivalent to the classifying space of the simplicial groupoid of self-equivalences of (R-mod). This is rather obvious for the component of the neutral element R in $|s\mathcal{A}^h_f(R)|$, as the 0-simplex R is just the symbol for the category (R-mod), and the 1-simplices which begin and end there are the self-equivalences of it. By Proposition 4.2, the space of self-equivalences of (R-mod) and the Picard groupoid $\mathbf{Pic}(R)$ are equivalent to each other. □

Corollary 4.10. *There are natural isomorphisms*

$$\pi_n\mathbf{Br}(R) \cong \pi_{n-2}\mathbf{Pic}(R)$$

for $n \geqslant 2$,

$$\pi_n\mathbf{Br}(R) \cong \pi_{n-3}\mathrm{GL}_1(R)$$

for $n \geqslant 3$, *and*

$$\pi_n\mathbf{Br}(R) \cong \pi_{n-3}(R).$$

for $n \geqslant 4$.

Proof. The first statement is an immediate consequence of the preceding theorem. The second follows from the first and the fact that the Picard space deloops the units, and the last statement follows from the second and $\pi_n\mathrm{GL}_1(R) \cong \pi_n(R)$ for $n \geqslant 1$. □

Note in particular that the Brauer space is 3-truncated in the case of an Eilenberg-Mac Lane spectrum. This will be used in Section 6.

5 Relative invariants

In this section, we define relative Brauer spectra, as these are likely to be easier to compute than their absolute counterparts. We will focus on the case of extensions of commutative $\mathbb{S}$-algebras, but the case of ordinary commutative rings is formally identical.

To start with, let us first convince ourselves that the construction of the Brauer space (or spectrum) is sufficiently natural.

Proposition 5.1. *If $R \to S$ is a map of commutative $\mathbb{S}$-algebras, then there is a natural map*

$$\mathbf{br}(R) \longrightarrow \mathbf{br}(S)$$

of Brauer spectra. It induces similarly a natural map of Brauer spaces.

Proof. The map is induced by $A \mapsto S \wedge_R A$. By [BRS, Proposition 1.5], it maps Azumaya algebras to Azumaya algebras. It therefore induces functors between the categories used to define the Brauer spectra. The details are omitted. □

If $R \to S$ is a map of commutative $\mathbb{S}$-algebras, then $\mathbf{br}(S/R)$ and $\mathbf{Br}(S/R)$ will denote the homotopy fibers of the natural maps in Proposition 5.1. Note that there is an equivalence $\mathbf{Br}(S/R) \simeq \Omega^\infty \mathbf{br}(S/R)$ of infinite loop spaces.

Remark 5.2. The general theory of homotopy colimits of symmetric monoidal categories ([Tho79b] and [Tho82]) might provide a first step to obtain a more manageable description of these relative terms.

The defining homotopy fibre sequences lead to exact sequences of homotopy groups. Together with the identifications from Proposition 4.9 and Corollary 4.10, these read

$$\begin{gathered}\cdots \to \pi_3 \mathbf{Br}(S/R) \to \pi_0 \mathrm{GL}_1(R) \to \pi_0 \mathrm{GL}_1(S) \to \pi_2 \mathbf{Br}(S/R) \to \mathrm{Pic}(R) \to \\ \to \mathrm{Pic}(S) \to \pi_1 \mathbf{Br}(S/R) \to \mathrm{Br}(R) \to \mathrm{Br}(S) \to \pi_0 \mathbf{Br}(S/R) \to 0.\end{gathered}$$

In [BRS, Definition 2.5], the relative Brauer group $\mathrm{Br}(S/R)$ is defined as the kernel of the natural homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(S)$.

Proposition 5.3. *The relative Brauer group $\mathrm{Br}(S/R)$ is naturally isomorphic to the cokernel of the natural boundary map $\mathrm{Pic}(S) \to \pi_1 \mathbf{Br}(S/R)$.*

Proof. This is an immediate consequence of the definition of $\mathrm{Br}(S/R)$ as the kernel of the natural homomorphism $\mathrm{Br}(R) \to \mathrm{Br}(S)$ and the long exact sequence above. □

Remark 5.4. We note that the theory of Brauer spaces presented here also has Bousfield local variants, building on [BRS, Definition 1.6], and that it might be similarly interesting to study the behavior of the Brauer spaces under variation of the localizing homology theory.

6 Eilenberg-Mac Lane spectra

Let us finally see how the definitions of the present paper work out in the case of Eilenberg-Mac Lane spectra. Let R be an ordinary commutative ring, and let $\mathrm{H}R$ denote its Eilenberg-Mac Lane spectrum. This means that we have two Brauer groups to compare: $\mathrm{Br}(R)$ as defined in [AG60], and $\mathrm{Br}(\mathrm{H}R)$ as defined in [BRS], where there is also produced a natural homomorphism

$$\mathrm{Br}(R) \longrightarrow \mathrm{Br}(\mathrm{H}R) \tag{6.1}$$

of groups, see [BRS, Proposition 5.2].

The homomorphism (6.1) can be refined using the spaces and spectra defined in Section 2 for R and in Section 4 for $\mathrm{H}R$.

Proposition 6.1. *There is a natural map*

$$\mathbf{br}(R) \longrightarrow \mathbf{br}(\mathrm{H}R) \tag{6.2}$$

of spectra which induces the homomorphism (6.1) *on* π_1.

Proof. This map is induced by the Eilenberg-Mac Lane functor H. It induced functors between the categories used to define the Brauer spectra. Again, the details are omitted. □

Theorem 6.2. *The homotopy fibre of the map* (6.2) *is a* 1*-truncated connective spectrum. Its only possibly non-trivial homotopy groups are the kernel and cokernel of the map* (6.1).

Proof. If R is a commutative ring, then the natural equivalence

$$\mathrm{gl}_1(\mathrm{H}R) \simeq \mathrm{H}\mathbb{G}_m(R)$$

describes the spectrum of units of the Eilenberg-Mac Lane spectrum. It follows that $\mathbf{br}(\mathrm{H}R)$ is 3-truncated. As $\mathbf{br}(R)$ is always 3-truncated by Corollary 2.7, so is the homotopy fibre. On π_3, the map (6.2) is an isomorphism between two groups both isomorphic to the group of units of R. On π_2, the map (6.2) is the map

$$\mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(\mathrm{H}R) \tag{6.3}$$

induced by the Eilenberg-Mac Lane functor H. A more general map is studied in [BR05], where the left hand side is replaced by the Picard group of graded R-modules, and then the map is shown to be an isomorphism. For the present situation, this means that (6.3) is a monomorphism with cokernel isomorphic to the group $\mathbb{Z}$ of integral grades. The result follows. □

Remark 6.3. The homomorphism (6.1) is an isomorphism if R is a separably closed field, where both sides are trivial, again see [BRS, Proposition 5.2]. It is generally believed that it is an isomorphism for all fields K, although a proof has not been published yet. From what has been proven here, this would follow from a natural isomorphism $\mathrm{Br}(\mathrm{H}K_{\mathrm{sep}}/\mathrm{H}K) \cong \mathrm{Br}(K_{\mathrm{sep}}/K)$ between the relative groups. As the latter is known to be describable in term of Galois cohomology, this is what is need for the former.

The preceding theorem implies that there is a Picard stack in the sense of [Del73, 1.4] which describes the homotopy fibre of the map (6.2). As it is an invariant of R, it would be very interesting to obtain a purely algebraical way to define it. Remark 5.2 applies.

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