On the population dynamics in the stationary random environment

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Abstract

We study the mathematical models of population dynamics with birth and death mechanisms. We show that there is no limiting distribution for such models in a time independent random environment. The different examples of birth and death rates are considered.

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1 Introduction

Convergence to statistical equilibrium is the central topic in the theory of Gibbs fields and in the whole statistical physics in general. Among huge amount of literature in this area one should mention the first monograph by V. Malyshev and R. Minlos [13]. This monograph, as well as the numerous publications of Dobrushin, Malyshev, Minlos and Sinai school (see also [4] and references therein), contains the purely mathematical treatment of Gibbs random fields and a huge class of examples. The main technical tools here

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are the cluster expansions and related analytical, combinatorial and algebraic methods.

The mathematical models of population dynamics, which analysis is an essential part of modern mathematical biology, have several important features and, first of all, birth and death mechanisms. It can be considered as a part of the theory of branching processes with random spatial dynamics, where typically, one can not expect the convergence to the statistical equilibrium. However, there are several models of population dynamics which have an ergodic state: for arbitrary non-degenerated initial configurations of particles the state of the system at the moment of time $t+s$ (i.e. the integer valued measure $n(t+s,G)$, $G \in \mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ - the Borel σ-algebra on $\mathbb{R}^d$) converges in law as $s \to \infty$ to the limiting field $n^*(t,G)$ which is invariant with respect to translations $t \to t+h$, $G \to G+x$ in time and space. A typical example of such a situation can be found in [9] (contact model in $\mathbb{R}^d$, $d \geq 3$). The results can be extended to $\mathbb{R}^2$ if the jump distribution of the underlying Markov process has heavy tails. For other references, see also [1, 2, 5, 10, 11, 12]. The main feature of all such results is that the parameters of mathematical models (rate of jumps, birth and death rates, etc) are constants and there exists a complete local equilibrium between the rate of production of new particles and their annihilation (critical processes). The central result of this paper can be formulated (at the physical level of rigor) in the following opposite way: *there is no limiting distribution for the population dynamics models in a time independent random environment.*

Mathematical formulation, of course, must include some restriction on the medium and the exact definitions of the class of models for the evolution of particles field. In the present paper we will consider general models which cover a wide range of population dynamics schemes, including the contact model (see e.g. [9, 10]), Fisher-Kolmogorov-Petrovski-Piskunov (FKPP) model for the diffusion of genes (see [8]), and similar reaction-diffusion type equations, in a random environment given by the probability space $(\Omega_m, \mathcal{F}_m, P_m)$. The realization of the random environment has label $\omega_m \in \Omega_m$. The initial configuration of particles (in the phase space $\mathbb{R}^d$) is an homogeneous point field $n(0,G)$ (number of particles at the set G at the moment of time $t=0$). It can be, for instance, the Poisson field with intensity $\rho_0 > 0$. In general,

$$\mathbb{E}[n(0,G)] = \rho_0 |G|,$$

where $|G|$ denotes the Lebesgue measure of $G \in \mathcal{B}(\mathbb{R}^d)$. The evolution transforms the initial point field into the configuration (integer-valued measure) $n(t,G,\omega,\omega_m)$ on the probability space $(\Omega_m, \mathcal{F}_m, P_m) \times (\Omega, \mathcal{F}, P)_m$. Here $(\Omega_m, \mathcal{F}_m, P_m)$ corresponds to the stationary in space random environment, which includes the description of branching and mortality mechanisms. For a fixed environment ω_m the conditional probability space $(\Omega, \mathcal{F}, P)_m$ contains the information about random motion of particles and their "reactions" (annihilation, splitting, etc). The density ρ of particles (for fixed ω_m) is given by the relation

$$\mathbb{E}[n(t, G, \omega_m, \omega)|\mathcal{F}_m] = \int_G \rho(t, x, \omega_m)dx,$$

where $\mathbb{E}[\cdot]$ denotes the expectation with respect to P for a fixed random environment ω_m. To simplify notations, we will write $\rho_t(x) := \rho(t, x, \omega_m)$ if ω_m is considered to be fixed. This density (or the first correlation function) satisfies the parabolic Anderson equation:

$$\begin{cases} \frac{\partial}{\partial t}\rho_t(x) = \mathcal{L}\rho_t(x) + V(x, \omega_m)\rho_t(x) \\ \rho_0(x) \equiv \rho_0. \end{cases} \tag{1.1}$$

here $\mathcal{L}$ is the generator of the underlying Markov process $X(s)$, $s \geq 0$, appearing effectively from the evolution of particles, and $V(x, \omega_m)$ is a random potential. Usually, $V(x, \omega_m) = b(x, \omega_m) - m(x, \omega_m)$, where $b(x, \omega_m)$ is the rate of birth of a new particle and $m(x, \omega_m)$ is the mortality rate. Of course, one may consider V of more general type.

It is worth noting that the evolution equation for densities in the case of the contact model considered with random mortality rate and random rate of birth has form (1.1) unless the corresponding generator $\mathcal{L}$ will be environment dependent. However, if birth rate of the contact model is independent of the random media then $\mathcal{L}$ would be also independent of it.

In the sequel, for simplicity, we assume that the generator $\mathcal{L}$ is independent of the environment. Several examples with different generators $\mathcal{L}$ and potentials V are discussed in Appendix 2.

2 Description of the model. The main result

Let $X(s), s \geq 0$ be a Markov process in $\mathbb{R}^d$ with the generator $\mathcal{L}$ and the positive and continuous density $p(t, x, y)$:

$$\int_A p(t, x, y)dy = P_x[X(t) \in A], \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Assume that $X(\cdot)$ is shift-invariant, i.e. $p(t, x, y) = p(t, 0, y - x)$. We also assume that the generator $\mathcal{L}$ of $X(s)$ considered in $L^2(\mathbb{R}^d, dx)$ is a self-adjoint operator. The latter assumption means that $p(t, x, y) = p(t, y, x)$ or $p(t, 0, z) = p(t, 0, -z)$.

Below we introduce some examples of Markov process $X(\cdot)$ which will be discussed in the present paper.

Examples:

(a) $\mathcal{L} = \Delta$, i.e. $X(s) = b(s)$, $s \geq 0$ is a Brownian motion.

(b)

$$\mathcal{L}f(x) = \kappa \int_{\mathbb{R}^d} a(z)[f(x+z) - f(x)]dz,$$

where $a(z) = a(-z)$, $\int_{\mathbb{R}^d} a(z)dz = 1$. This is the generator of a jump process with the jump rate κ and the distribution density of jumps $a(z)$, i.e.

$$P[X(t+dt) = x + z + dz | X(t) = x] = \kappa a(z) dz dt.$$

(c) Lattice models similar to **(a)** and **(b)**, where $X(\cdot)$ is a random walk with independent increments on the lattice $\mathbb{Z}^d$.

Now we introduce a model which is similar to (FKPP) model describing the diffusion of genes in a random environment ω_m and cover the contact model with particular random birth and death mechanisms. In the sequel, this model will be called (FKKP) scheme.

Let $b(x, \omega_m)$ and $m(x, \omega_m)$ be two non-negative, continuous, homogeneous and ergodic fields on $\mathbb{R}^d$. The first one is the rate of duplications ($A \to A+A$) and the second one is the mortality rate ($A \to \emptyset$). At the moment of time $t = 0$ we start from the initial Poisson configuration of the particles with some intensity $\rho_0 > 0$ (initial density). Each particle produces (independently of others) its own generation due to the following random dynamics: starting from the initial position $x \in \mathbb{R}^d$, the A-particle performs a random motion (realization of the underlying process $X(\cdot)$). During the time $(s, s+ds)$ it can either disappear with the probability $m(X(s))ds$ or produce a new particle (splitting) with the probability $b(X(s))ds$. The distribution of the random time τ_1 of the first reaction ($A \to A + A$ or $A \to \emptyset$) along the trajectory $X(s)$, $s \in [0, t]$ is given by

$$P_x[\tau_1 > t] = \prod_{s=0}^{t} (1 - m(X(s))ds - b(X(s))ds) = \exp\left\{ - \int_0^t (m+b)(X(s))ds \right\}.$$

In the case of the reaction $A \to A+A$ the offsprings start their own evolution at the splitting point (independently of each other and the remaining particles). We have defined the evolution of the particles formally. There exists a possibility of an explosion in a finite time interval of our system. The following well-known lemma gives sufficient conditions for the global existence of the particle field $n(t, G, \omega, \omega_m)$.

Lemma 2.1 (cf. [3], [6]). *Assume that for any $t > 0$, $x \in \mathbb{R}^d$*

$$\langle e^{tb(x,\,\cdot)}\rangle = \langle e^{tb(0,\,\cdot)}\rangle = A(t) < \infty, \tag{2.1}$$

where $\langle\cdot\rangle$ is the expectation over the environment ω_m. If $n(0, G)$ is a Poisson initial point field, then the population dynamics, described above, defines uniquely the point field $n(t, G)$ and all statistical moments of this field are finite:

$$\big\langle \mathbb{E}.[n^k(t, G)]\big\rangle < \infty, \quad k \in \mathbb{N}, \quad G \in \mathcal{B}(\mathbb{R}^d), \quad |G| < \infty.$$

Here, as before, $\mathbb{E}.[\cdot]$ denotes expectation with respect to the law of the underlying Markov processes $X(s), s \geq 0$ with different starting points from the configuration $n(0, \cdot)$ and fixed random environment ω_m.

Proof. The scheme of the proof is proposed in Remark 3.4 of Appendix 1. □

Remark 2.2. Condition (2.1) means that the Laplace transform of $b(\cdot, \omega_m)$ is an entire function of t.

Now we will formulate two specific conditions on the process $X(t)$ and the fields b and m which will be used in the sequel.

(A) For any $L \geq 1$ one can find $\delta = \delta(L) > 0$ such that $\delta(L) \to 0$ as $L \to \infty$ and

$$P_0[\max_{s \leq t}\{|X(s)|\} \leq L] \geq e^{-\delta(L)t}.$$

Due to the shift invariance of $X(\cdot)$, for any $x \in \mathbb{R}^d$ holds

$$P_x[\max_{s \leq t}\{|X(s) - x|\} \leq L] \geq e^{-\delta(L)t}.$$

Remark 2.3. We will see later that for the typical examples the following holds

$$P_0[\max_{s \leq t}\{|X(s)|\} \leq L] \backsim e^{-t\frac{c}{L^{\varkappa}}}$$

fore some appropriate $\varkappa > 0$.

Remark 2.4. Condition **(A)** is a version of the self similarity for the process $X(\cdot)$.

(B) There exists a (small) $\epsilon_0 > 0$ such that for any $L > 0$

$$P[V(z, \omega_m) \geq \epsilon_0, |z| \leq L] > 0.$$

Here $V(z, \omega_m) = b(z, \omega_m) - m(z, \omega_m)$ is the potential in the Anderson parabolic problem for the "quenched" density $\rho(t, x, \omega_m)$, given by (1.1).

Due to the ergodicity of the pair (b, m), condition **(B)** means that for any $L \geq 1$ there exists infinitely many disjoint spherical islands

$$I_j(\omega_m) = I_j := \{x : |x - x_j(\omega_m)| \leq L\}, \quad j \geq 1$$

such that $V(x, \omega_m) \geq \epsilon_0$ on $I_j(\omega_m)$ for any $j \geq 1$. Noteworthy that due to the independence of $n(0, I_j(\omega_m))$ for different j, there are P - almost surely (a.s.) infinitely many islands $I_{j'}$ such that $n(0, I_{j'}) \geq 1$.

The following main theorem gives the precise form of the "physical" statement claimed in Introduction:

Theorem 2.5. *Let $b(x, \omega_m)$ and $m(x, \omega_m)$ be continuous, ergodic, homogeneous, and non-negative fields such that $\langle e^{tb(0,\omega_m)} \rangle = A(t) < \infty$. Suppose, additionally, that the underlying process $X(s)$, $s \geq 0$ satisfies condition* ***(A)*** *and $V(x, \omega_m) = (b - m)(x, \omega_m)$ satisfies condition* ***(B)****. Then for any open domain D, $|D| < \infty$:*

$$n(t, D) \to \infty, \quad t \to \infty, \quad P - a.s.$$

and, even more,

$$\liminf_{t \to \infty} \frac{\ln n(t, D)}{t} > 0, \quad P - a.s.$$

Remark 2.6. It is worth noting that the central condition **(B)** on the potential $V(x, \omega_m) = (b - m)(x, \omega_m)$ does not exclude the case $\langle V \rangle < 0$. Indeed, consider the following example: assume that $b(x, \cdot)$ and $m(x, \cdot)$ are two independent Bernoulli fields with constant values on the cubes

$$Q_{\vec{n}} = \left\{ x : \|x - \vec{n}\|_\infty \leq \frac{1}{2} \right\}, \quad \vec{n} \in \mathbb{Z}^d$$

such that on $Q_{\vec{n}}$ (independently for different $\vec{n}$)

$$b(x, \omega_m) = \begin{cases} \epsilon \text{ with probability } \epsilon_1 > 0 \\ 0 \text{ with probability } 1 - \epsilon_1 \end{cases}$$

and

$$m(x, \omega_m) = \begin{cases} 0 \text{ with probability } \epsilon_2 > 0 \\ M \text{ with probability } 1 - \epsilon_2. \end{cases} \tag{2.2}$$

Here $\epsilon, \epsilon_1, \epsilon_2$ are small parameters and the parameter M is large enough. In this example $V(x, \omega_m) \geq \epsilon$, $x \in Q_{\vec{n}}$ with probability $\epsilon_1 \cdot \epsilon_2$ and, hence, condition **(B)** is satisfied.

Proof of the main theorem. Let us fix $L_0 \geq 1$ such that $\delta(L_0) < \frac{\epsilon_0}{2}$. If $D = I_{L_0}(x_0)$ is the ball of radius L_0 with its center at x_0 such that $V(x) = (b-m)(x) \geq \epsilon_0$, $x \in I_{L_0}(x_0)$ then directly from the representation

$$q_D(t,x,y) = \sum_{k=0}^{\infty} e^{\lambda_k t}\psi_k(x)\psi_k(y)$$

of the fundamental solution to the parabolic Anderson problem

$$\frac{\partial q_D}{\partial t} = \mathcal{L}q_D + Vq_D, \quad q_D(0,x,y) = \delta_y(x), \quad q_D(t,x,y) \equiv 0,\ x \notin D$$

one can deduce that $\lambda_0 = \lambda_0(D) \geq \frac{\epsilon_0}{2} > 0$ (cf. Khasminskii lemma [7]). Here, as usual, $\{\lambda_k,\ k \geq 0\}$ are the eigenvalues for the problem $\mathcal{L}\psi + V\psi = \lambda\psi$, with $\psi = 0$ on $\tilde{D} = \mathbb{R}^d \setminus D$, and $\{\psi_k, k \geq 0\}$ are the corresponding eigenfunctions. It is important to emphasize that the principal eigenvalue λ_0 is positive and $\psi_0 > 0$ on D. Moreover, $\lambda_k \to -\infty$ as $k \to \infty$.

Assume that this domain D contains at least one initial particle at the point $x_1 \in D$ which will be fixed in the sequel. Let $n_D(t,G)$ be the (FKPP) scheme inside D which corresponds to the random dynamics $X(\cdot,\omega)$ with the same birth-and-death rates $b(x,\omega_m)$, $m(x,\omega_m)$, $x \in D$ and total annihilation on $\tilde{D}$. Consider the initial configuration for $n_D(0,G)$ of the following type: $n_D(0,G) = \mathbb{I}_G(x_1)$, where $\mathbb{I}_G(\cdot)$ is the indicator function of $G \in \mathcal{B}(\mathbb{R}^d)$. It means that the initial configuration consists of a single particle localized at $x_1 \in D$. One can realize $n(t,G)$, $G \in \mathcal{B}(\mathbb{R}^d)$ and $n_D(t,G)$, $G \subset D$ on the same probability space in such a way that $n_D(t,G) \leq n(t,G)$ for any $t \geq 0$ and $G \in \mathcal{B}(\mathbb{R}^d)$. Let $\rho_D(t,x_1,x)$ (of course, it depends also on ω_m) be the density of $n_D(t,G)$, i.e.

$$\mathbb{E}_{x_1}[n_D(t,G)] = \int_G \rho_D(t,x_1,x)dx.$$

Then it satisfies

$$\begin{cases} \frac{\partial}{\partial t}\rho_D = \mathcal{L}\rho_D + V\rho_D, \quad \rho_D(t,x_1,x) \equiv 0, \quad x \in \tilde{D}, \\ \rho_D(0,x_1,x) = \delta_{x_1}(x). \end{cases}$$

The Fourier method gives (we already formulated this fact):

$$\rho_D(t,x_1,x) = \sum_{k=0}^{\infty} e^{\lambda_k t}\psi_k(x_1)\psi_k(x) \sim e^{\lambda_0(D)t}\psi_0(x_1)\psi_0(x), \quad x \in D, \quad t \to \infty.$$

In particular

$$m_1(t,x_1) = \mathbb{E}_{x_1}[n_D(t,D)]$$

$$= \int_D \rho_D(t, x_1, x)dx \sim e^{\lambda_0(D)t}\psi_0(x_1)(1, \psi_0), \quad t \to \infty.$$

The following important result is known in the mathematical community, but we can give the corresponding reference only for the particular case $\mathcal{L} = \Delta$ (see e.g. [14] for the similar fact).

Lemma 2.7. *If* $\lambda_0 = \lambda_0(D) > 0$ *then*

$$\frac{n_D(t, D)}{e^{\lambda_0 t}} \to n^*(D), \quad t \to \infty \quad \textit{in} \quad \textit{law.}$$

The limiting distribution is non-degenerate

$$q(x_1) := P_{x_1}[n^*(D) > 0] > 0.$$

The sketch of the proof of this lemma will be given in Appendix 1.

Consider now an infinite set of "islands" $I_{L_0}(x_j(\omega_m))$, $j \geq 1$ for which $\lambda_0(I_{L_0}(x_j(\omega_m))) \geq \frac{\epsilon_0}{2}$. Since the populations generated by different initial particles are independent one can find such a particular "island" $I_{L_0}(x_0(\omega_m)) = D_0$ that

$$n_{D_0}(t, D_0) \to \infty, \quad t \to \infty$$

exponentially, i.e.

$$\lim_{t \to \infty} \frac{\ln n_{D_0}(t, D_0)}{t} = \lambda_0(D_0) = \lambda_0$$

i.e. for any $\delta > 0$ and $t \geq t(\delta, \omega_m)$

$$e^{(\lambda_0 - \delta)t} \leq n_{D_0}(t, D_0) \leq e^{t(\lambda_0 + \delta)}.$$

For an arbitrary domain $D \subset \mathbb{R}^d$ we would like to estimate $n(t, D)$ in terms of the population on the island D_0. Assume that $t > 1$ and $\omega_m \in \Omega_m$ are fixed. The cardinality of the configuration $n_{D_0}(t-1, \cdot)$ at the moment of time $t - 1$ will be not less than $e^{(\lambda_0 - \delta)(t-1)}$. Elementary compactness arguments show that one can find $\varepsilon = \varepsilon(D_0, D, \omega_m) > 0$ such that for any particle of $n_{D_0}(t - 1, \cdot)$ located at some point $z \in D_0$ its generation $n_z(1, D)$ in $\mathbb{R}^d$ for the remaining unite time satisfies inequality

$$P[n_z(1, D) > 0] \geq \varepsilon > 0.$$

But

$$n(t, D) \geq \sum_{z \in n(t-1, D_0)} \mathbb{I}_{n_z(1,D)>0}$$

i.e. $\mathbb{E}[n(t,D)] \geq \varepsilon e^{(\lambda_0-\delta)(t-1)}$. At the same time

$$\mathrm{Var}\,[n(t,D)] \leq \frac{1}{4} n_{D_0}(t-1,D_0) \leq \frac{1}{4} e^{(\lambda_0+\delta)(t-1)}.$$

The Borel-Cantelli lemma together with the Chebyshev inequality gives:

$$n(t,D) \geq \varepsilon e^{-(\lambda_0-\delta)t}, \quad t \geq t_0(\omega_m), \quad P_m \ - \ a.s.$$

It proves the main theorem. ■

3 Appendix 1. The limit theorem for the branching process with the random space evolution in the supercritical regime (the compact phase space)

Consider the (FKPP) scheme in the bounded domain $D \subset \mathbb{R}^d$ (e.g., the ball $B_L(x_0) = \{x : |x - x_0| \leq L\}$). The underlying Markov process has the self-adjoint generator $\mathcal{L} = \mathcal{L}^*$ in $L^2(D,\, dx)$ with the boundary condition of annihilation on $\tilde{D} = \mathbb{R}^d \backslash D$. Assume that the fundamental solution $p_D(t,x,y)$ of

$$\frac{\partial p_D}{\partial t} = \mathcal{L} p_D, \quad p_D(t,x,y) \equiv 0, \quad y \in \tilde{D}, \quad x \in D,$$
$$p_D(0,x,y) = \delta_y(x), \quad x,\, y \in D.$$

is continuous in x and y for any $t > 0$ in the closure $\bar{D}$ of D and strictly positive on D. Let also $b(x)$, $m(x)$, $x \in \bar{D}$ be the rates of the duplication and annihilation for the particles with, as usual, independent evolution after the branching. Suppose that we start from a single particle at the point $x \in D$ at the moment $t = 0$. If $n(t,D)$ is the total number of the particles in our model at the moment $t > 0$ then for the generating function $u_z(t,x) = \mathbb{E}_x z^{n(t,D)}$ we have the usual Skorokhod equation

$$\frac{\partial u_z}{\partial t} = \mathcal{L} u_z + b u_z^2 - (b+m) u_z + m,$$
$$u_z(0,x) = z \mathbb{I}_D(x),$$
$$u_z(t,x) = 1, \quad x \notin D.$$

Differentiating the latter equation with respect to z in $z = 1$ one obtains the sequence of moment equations for the particle field

$$m_k(t,\, x) = \mathbb{E}_x \left[n(t,D)(n(t,D)-1) \ldots (n(t,D)-k+1) \right].$$

They have the following forms:

- $k = 1$

$$\frac{\partial m_1}{\partial t} = \mathcal{L} m_1 + V m_1,$$
$$m_1(0,\, x) = \mathbb{I}_D(x).$$

- $k = 2$

$$\frac{\partial m_2}{\partial t} = \mathcal{L} m_2 + V m_2 + 2b m_1^2,$$
$$m_2(0, x) = 0.$$

- $k \geq 2$

$$\frac{\partial m_k}{\partial t} = \mathcal{L} m_k + V m_k + b \sum_{i=1}^{k-1} C_k^i m_i m_{k-i},$$
$$m_k(0, x) = 0, \quad k \geq 2.$$

Let $\lambda_0 > \lambda_1 \geq \ldots \geq \lambda_k > \ldots$, with $\lambda_k \to -\infty$ be the spectrum of the self-adjoint operator

$$H_V \psi = \mathcal{L}\psi + V\psi,$$
$$\psi(x) = 0, \quad x \in \tilde{D}$$

and the corresponding orthonormal eigenbasis is $\{\psi_k,\, k \geq 1\} \subset L^2(D)$.

The central assumption in this section will be $\lambda_0 > 0$. In this case the corresponding eigenfunction ψ_0 is positive on D due Perrount-Frobenius theorem. In terms of branching processes it means that $n(t, G)$ is a supercritical process. Below we are giving the proof of Lemma 2.7 reformulating it once again.

Lemma 3.1. *(cf. Lemma 2.7) Under the assumption $\lambda_0 > 0$, the normalized population $e^{-\lambda_0 t} n_{x_1}(t,\, D)$ converges in law to $n^*(D)$ as t tends to ∞. Moreover,*

$$P_{x_1}\left[n^*(D) > 0\right] = q(x_1) > 0.$$

Proof. The monotonicity arguments show that one can consider the case when $b(x, \omega_m) \geq \varepsilon_0$, $x \in D$ and $\tilde{m}(x, \omega) = b(x, \omega_m) - \varepsilon_0$, i.e. $V(x, \omega_m) = \varepsilon_0$ (in particular, it gives $q(x_1) \geq q_{\varepsilon_0}(x_1)$, where $q_{\varepsilon_0}(x_1)$ corresponds to the case described above). Using the method of moments we are going to prove that for any $k \geq 1$

$$\mathbb{E}_{x_1}\left[\frac{(n_{x_1}(t, D))^k}{e^{\lambda_0 t}}\right] \to c_k k! \psi_0(x_1), \quad t \to \infty \tag{3.1}$$

where $c_k \le A^k$ for some fixed $A \ge 1$, i.e.

$$\left\langle \mathbb{E}_{\cdot} \left[n_{\cdot}^k(t, D) \right] \right\rangle \le c A^k k!.$$

The Carleman estimate above will provide us the limit theorem.

In its turn the proof of (3.1) is based on induction. As we have already shown

$$\mathbb{E}_{x_1} \left[n_{x_1}(t, D) \right] \sim \psi_0(x_1) e^{\lambda_0(D)t}(1, \psi_0).$$

This means that (3.1) is fulfilled for $k = 1$. Moreover, $c_1 = (1, \psi_0)$. The transition to the higher moments will be based on the following elementary proposition

Proposition 3.2. *Consider the problem*

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + V(x)u(t,x) + F(t,x),$$

$$u(0,x) = 0,$$

where

$$F(t,x) \sim f(x)e^{\varkappa t}, \quad t \to \infty, \quad \varkappa > \lambda_0 > 0.$$

Then,

$$u(t,x) \sim \psi_0(x) \frac{e^{\varkappa t}}{\varkappa - \lambda_0}(f, \psi_0), \quad t \to \infty.$$

Proof. In fact

$$F(t,x) = \sum_k \psi_k(x)(F(t,x), \psi_k) \sim \psi_0(x)e^{\varkappa t}(f, \psi_0), \quad t \to \infty, \quad \varkappa > \lambda_0.$$

The latter means that

$$u(t,x) \sim \tilde{u}(t,x), \quad t \to \infty,$$

where $\tilde{u}$ is a solution to the following problem

$$\frac{\partial \tilde{u}(t,x)}{\partial t} = \mathcal{L}\tilde{u}(t,x) + V(x)\tilde{u}(t,x) + \psi_0(x)e^{\varkappa t}(f, \psi_0), \quad \tilde{u}(0,x) = 0.$$

We will be looking for the solution to the previous equation in the form $\tilde{u}(t,x) = a(t)\psi_0(x)$, $a(0) = 0$. Then

$$a'(t) = \lambda_0 a(t) + e^{\varkappa t}(f, \psi_0), \quad a(0) = 0.$$

This equation can be also written in the following way

$$\left(a(t)e^{-\lambda_0 t}\right)' = e^{(\varkappa - \lambda_0)t}(f, \psi_0), \quad a(0) = 0.$$

Integrating the last equality from 0 to t we get

$$a(t)e^{-\lambda_0 t} = (f,\psi_0)\frac{e^{(\varkappa-\varkappa_0)t}-1}{\varkappa-\lambda_0}.$$

Finally,

$$u(t,x) \sim \frac{(f,\psi_0)e^{\varkappa t}\psi_0(x)}{\varkappa-\lambda_0}, \quad t\to\infty.$$

□

The induction completes the proof of Lemma 2.7. □

Remark 3.3. Let us consider the corresponding equation for the second moments

$$\frac{\partial m_2(t,x)}{\partial t} = \mathcal{L}m_2(t,x) + V(x)m_2(t,x) + 2b(x)m_1^2(t,x),$$
$$m_2(0,x) = 0.$$

Since

$$2b(x)m_1^2(t,x) \sim 2b(x)\psi_0^2(x)e^{2\lambda_0 t}, \quad t\to\infty$$

in view of Proposition 3.2 we have

$$m_2(t,x) \sim \frac{e^{2\lambda_0 t}(b\psi_0^2,\psi_0)}{\lambda_0}, \quad t\to\infty.$$

In this case $c_2 = \frac{(b\psi_0^2,\psi_0)}{\lambda_0}$. In fact one can calculate exactly all moments of the limiting distribution.

Remark 3.4. *(Proof of Lemma 2.1)* Let us consider the case $k=1$. Let $G\subset D$ be arbitrary and fixed. Due to Feynmann-Kac formula

$$\mathbb{E}_{x_1}\left[n_D(t,G)\right] = \mathbb{E}_{x_1}\left[e^{\int_0^{t\wedge\tau_D}V(X_s)ds}\mathbb{I}_G(X_t)\right]$$
$$\leq \mathbb{E}_{x_1}\left[e^{\frac{1}{t}\int_0^t tV(X_s)ds}\mathbb{I}_G(X_t)\right],$$

where τ_D is the first exit time of the process from D. Using Jensen inequality we can estimate the latter expression by

$$\frac{1}{t}\int_0^t \mathbb{E}_{x_1}\left[e^{tV(X_s)}\mathbb{I}_G(X_t)\right]ds.$$

As result

$$\langle\mathbb{E}_{x_1}\left[n(t,G)\right]\rangle = \left\langle e^{tV(0)}\right\rangle P_{x_1}\left[X_t\in G\right]$$

Here $x_1 \in D$ is one of the initial points inside D. But the number $\nu(D)$ of such points has Poissonian law with intensity $\rho(D)$ and each point x_1 is uniformly distributed on D. Then

$$\left\langle \mathbb{E}_{n(0,\cdot)\cap D}\ [n_D(t,G)] \right\rangle \leq E\left\langle \sum_{i=0}^{\nu(D)} \frac{1}{|D|} \left[\int_D p(t,x_1,G)dx_1\right] \left\langle e^{tV(0)} \right\rangle \right\rangle =$$

$$E\nu(D)\frac{|G|}{|D|} = \rho_0|G| \left\langle e^{tV(0)} \right\rangle .$$

If D tends to $\mathbb{R}^d$ then $\langle E\left[n(t,G)\right]\rangle \leq \rho_0|G| \left\langle e^{tV(0)} \right\rangle$.

Similar calculations can be done for higher moments.

4 Appendix 2. Examples.

We will present three examples of the "typical" underlying Markov process and the fields $b(x,\cdot)$, $m(x,\cdot)$ for which one can check the basic assumptions (A), (B) presented in Section 2.

1a. $b(t)$, $t \geq 0$ is a Brownian motion on $\mathbb{R}^d$ with the generator $\mathcal{L} = \triangle$. Its transition probabilities are

$$p(t,x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}$$

and $P\left\{\max_{s\leq t}|b(s)| \leq L\right\} \sim \exp\left(-\frac{\lambda_0 t}{L^2}\right)$, $\quad \frac{t}{L^2} \to \infty$, where $-\lambda_0$ is the principle eigenvalue of $\triangle$ in the unite ball B_1 with the Dirichlet boundary condition on ∂B_1 (due to self-similarity of b).

1b. $x_\alpha(t)$ is the isotropic stable process in $\mathbb{R}^d$ with the generator $\mathcal{L} = -(-\triangle)^{\alpha/2}$, $0 < \alpha < 2$. The process $x_\alpha(t)$ is self similar, i.e.

$$\frac{x_\alpha(st)}{t^{1/\alpha}} = x_\alpha(s) \quad \text{in} \quad \text{law}$$

and $P\left\{\max_{s\leq t}|x_\alpha(s)| \leq L\right\} \sim e^{-\lambda_\alpha \frac{t}{L^\alpha}}$, $\frac{t}{L^\alpha} \to \infty$. The process $x(t)$ is discontinuous and the Dirichlet boundary conditions for $\mathcal{L}$ have the form

$$\psi(x) = 0, \quad |x| > L.$$

1c. Jumping process with the generator

$$\mathcal{L}\psi(x) = \kappa \int_{\mathbb{R}^d} a(z)[\psi(x+z) - \psi(x)]dz.$$

Assume that $a(z) \sim \frac{c_0}{|z|^{d+\alpha}}$, $|z| \to \infty$, $0 < \alpha < 2$. Due to the central limit theorem for stable distributions and appropriate c_0

$$\frac{x_1 + \ldots + x_k}{c_0 k^{1/\alpha}} \to St_\alpha, \quad \text{in} \quad \text{law}.$$

It is worth noting that $St_\alpha = x_\alpha(1)$ from the previous example. These facts imply the property (A) with $\delta(L) \sim \frac{C_1}{L^\alpha}$

The last example provides the asymptotic formula for $\lambda_0(R) \sim \frac{c_{\alpha,\,d}}{R^\alpha}$ but under the strong regularity conditions on the density $a(z)$. The next theorem covers arbitrary $a(z)$, whereas the estimate for the $\lambda_0(R)$ is very rough.

Theorem 4.1. *Consider the non-negative operator*

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))a(z)dz,$$

where $a(z) = a(-z) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and the spectral problem in the ball* $B_R(0) = \{x \,|\, |x| \le R\}$*:*

$$\mathcal{L}\psi = \lambda\psi, \quad x \in B_R(0),$$

$$\psi \equiv 0, \quad x \notin B_R(0).$$

Let $\lambda_a(R)$ *be the minimal eigenvalue of the above problem. Then* $\lambda_a(R) \to 0,\ R \to \infty$.

Proof. Let us note that the operator $\mathcal{L}$ is a bounded symmetric operator in $L^2(\mathbb{R}^d)$. In Fourier representation it acts as a multiplication, by $1 - \hat{a}(k)$. It is also compact in $L^2(B_R(0))$ since $a \in L^2(B_R(0) \times B_R(0))$.

To estimate $\lambda_a(R)$ let us select the test function $\psi_0 = \sqrt{c_d}\frac{\mathbb{I}_{\{|x| \le R\}}}{R^{d/2}}$, where c_d is the volume of $B_1(0)$. It is well known that

$$\lambda_a(R) = \min_{\psi \in L^2(B_R(0)),\, ||\psi||_{L^2}=1} (Lf, f) \le (L\psi_0, \psi_0)$$

$$= \frac{c_d}{R^d}\int_{B_R(0)} (\mathbb{I}(x) - \mathbb{I}(x+z))\psi_0(x)a(z)dxdz = \frac{c_d}{R^d}\iint_{|x| \le R,\, |x+z| > R} a(z)dxdz$$

$$= \frac{c_d}{R^d}\left[\iint_{R - \sqrt{R} \le |x| \le R} a(z)dxdz + \iint_{|x| < R - \sqrt{R},\, |x+z| \ge R} a(z)dxdz\right]$$

$$\le \frac{c_d}{R^d} Vol\{R - \sqrt{R} \le |x| \le R\} + \frac{c_d}{R^d}\iint_{|x| < R - \sqrt{R},\, |z| \ge \sqrt{R}} a(z)dz.$$

Finally,

$$\lambda_a(R) \le \frac{c_1(d)\sqrt{R}R^{d-1}}{R^d} + c_2(d)\int_{|z|\ge\sqrt{R}} a(z)dz\frac{(R-\sqrt{R})^d}{R^d}$$
$$\le \frac{c_1(d)}{\sqrt{R}} + c_2\int_{|z|\ge\sqrt{R}} a(z)dz \to 0, \quad R\to\infty.$$

□

Remark 4.2. There are situations when our basic condition **(B)** does not hold, but one can use it in the integral form. Consider the Hamiltonian

$$H = \mathcal{L} + V(x,\omega_m), \quad \mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))a(z)dz,$$
$$V(x,\,\omega_m) = \sum_{n\in\mathbb{Z}^d} \varphi(x-n)X_n - \mu, \quad \mu > 0$$

where the function $\varphi(z) \ge 0$ is compactly supported ($\mathrm{supp}\varphi(\cdot) \subset B_{r_0}(0)$, $r_0 < \frac{1}{2}$). The random variables $X_n \ge 0$, $n\in\mathbb{Z}^d$ are i.i.d. and $\sup_{n\in\mathbb{Z}^d} X_n = A < +\infty$ P_m - a.s. Then the following results holds.

Proposition 4.3. *If*

$$\int_{|x|_\infty\le\frac{1}{2}} A\varphi(z)dz - \mu = C_0 > 0$$

then the spectrum of H contains energies $\lambda \ge (1-\varepsilon)C_0$ for any $\varepsilon > 0$, i.e. the corresponding point field $n(t,G)$, $G\in\mathcal{B}(\mathbb{R}^d)$ is supercritical P_m - a.s.

Proof. Note that in any unite cube $Q_n = \{x\,:\,||x-n||_\infty \le \frac{1}{2}\}$ the potential $V(x) = -\mu < 0$ on $Q_n \setminus \mathrm{supp}\,\varphi(\cdot - n)$.

For any $\varepsilon_1 > 0$

$$P[X_n > A-\varepsilon_1] > 0,$$

i.e. one can find a sequence of arbitrary large disjoint cubes $Q_{L_N}(X_N) = \{x\,:\,|x-X_N| \le L_N\}$, $L_N\to\infty$ such that $X_n > A-\varepsilon$ for $n\in Q_{L_N}$. Consider the test function

$$\psi_N(x) = \frac{\mathbb{I}_{Q_{L_n}(x_N)}(x)}{|Q_{L_n}(x_N)|^{i/2}}.$$

Due to the previous construction

$$(H\psi_N,\psi_N) = o(1) + \frac{1}{|Q_{L_n}(x_N)|}\int_{Q_N} V dz \to C_0.$$

The well known Glazman lemma gives that intersection of $[C_0-\delta,\,C_0+\delta]$ with the spectrum of H on Q_N with Dirichlet boundary conditions $\psi_N\equiv 0$ on $\tilde{Q}_N$ contains at least one eigenvalue. □

In general the following theorem is true.

Theorem 4.4. *Let probability density $a \in L^\infty$. Suppose there exists $C_0 > 0$ and a sequence $Q_{L_N}(x_N) = \{x : |x - x_N| \le L_N\}$ such that P_m - a.s.*

$$\int_{Q_{L_N}} V(x, \omega_m)dx = \int_{Q_{L_N}} (b - m)(x, \omega_m)dx \ge C_0 L_N^d$$

if $L_N \to \infty$. Then, the process $(n(t, G))_{t \ge 0}$ is supercritical.

The next group contains examples of the rates $[b(x, \omega_m),\, m(x, \omega_m)]$.

2a. Consider the partition of $\mathbb{R}^d$ onto unit cubes

$$\mathbb{R}^d = \bigcup_{\vec{n} \in \mathbb{Z}^d} Q_{\vec{n}}, \quad Q_{\vec{n}} = \{y \,|\, |x - \vec{n}|_\infty \le \frac{1}{2}\}.$$

Take $[b(x, \omega_m),\, m(x, \omega_m)] = [\beta_i, \mu_i]_i$ with probabilities $p_i,\ i = 1, 2, \dots, N$ on $Q_{\vec{n}}$ and for different $\vec{n}$ these random vectors are independent (Bernoulli fields). If $\beta_i > \mu_i$ at least for one $i \in \{1, 2, \dots, N\}$, then the potential $V = b - m$ satisfies the condition (B).

2b. Let $\xi(x)$ be a continuous homogeneous ergodic Gaussian field on $\mathbb{R}^d$ (the ergodicity, due to the well-known Fomin-Marujama theorem is equivalent to the continuity of the spectral measure of $\xi(\cdot)$).

Consider $b(x, \omega_m) = \xi^+ = \max\{\xi(x), 0\}$, $m(x, \omega_m) = \xi^- = \max\{-\xi(x), 0\}$, $V(x, \omega_m) = \xi(x, \omega)$. It is well-known that for a "good" domain $D \subset \mathbb{R}^d$ (ball, cube, etc) the support of law of $\xi(x),\ x \in D$ is equal to $C(D)$. Due to the ergodicity, the property (B) is satisfied P - a.s. Also, $\left\langle e^{tV(0)} \right\rangle = e^{\frac{ct^2}{2}} < \infty$.

2c. Poissonian fields. Let $\{x_i\}$, $\{y_i\}$ be two independent Poissonian point fields on $\mathbb{R}^d$ and $\phi_1(x)$, $\phi_2(x)$ be two compactly supported nonnegative continuous functions. Let $\phi_1(x) = \phi_2(x) = 0$ for any $|x| \ge a$ and $\phi_1(x), \phi_2(x) \ge \varepsilon_0 > 0$, $|x| \le \frac{a}{2}$.

Take $b(x, \omega_m) = \sum_{\{x_i\}} \phi_1(x - x_i)$ and $m(x, \omega_m) = \sum_{\{y_i\}} \phi_2(x - y_j)$. The fields b, m are non-negative ergodic and independent. It is well-known that

$$\left\langle e^{tb(0,\cdot)} \right\rangle = e^{\lambda_1 \int_{\mathbb{R}^d} \left(e^{t\phi_1(x)} - 1\right)dx} < \infty.$$

Let us check the condition (B). Consider the cube $[-L, L]^d = Q_L$ and the partition of Q_L into disjoint cubes $Q_{i,a}$ with the side a. Let us introduce the following two events

$$A_{L+\frac{a}{2}} = \{\text{there is no points } \{y_j\} \text{ in } Q_{L+\frac{a}{2}}\},$$

$B_L = \{$each subcube $Q_{i,a}$ of Q_L contains at least one point of $\omega_m\}$.

Then $P(A_{L+\frac{a}{2}}) = e^{-\lambda_2(2L+a)^d} > 0$, $P(B_L) = (1 - e^{-a^d\lambda_1})^{\left(\frac{L}{a}\right)^d} > 0$, and $P(A_{L+\frac{a}{2}}, B_L) = P(A_{L+\frac{a}{2}})P(B_L) > 0$. But events $A_{L+\frac{a}{2}}$, B_L imply that $b(x) \geq \varepsilon_0$ and $m(x, \omega_m) = 0$ on Q_L. Hence, $V(x) = (b - m)(x) \geq \varepsilon_0$ on Q_L. From the ergodicity of b and m it follows that for any $L > 1$ there are (P-almost surely) infinitely many cubes $Q_L + z_j = Q_{L,j}$, where $V \geq \varepsilon_0$ (condition (B)).

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