Population dynamics in the stationary random environment: asymptotic analysis

Yuri Kondratiev* Oleksandr Kutoviy† Stanislav Molchanov‡

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Abstract

We study the long time behavior of some continuous population systems in the stationary random medium. The main results are stated in terms of density of the corresponding population. The quenched and annealed asymptotic analysis is given for the particles field as time tends to infinity. We show the difference in asymptotics for a non-local and local Anderson operators.

Keywords: Markov evolution, continuous system, random environment, parabolic Anderson-model, quenched asymptotics, annealed asymptotics.

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1 Introduction

The central result of our previous paper [5] is the statement about the absence of a limiting ergodic state for some classes of population models in time independent random environments. It was proven that under the mild conditions such a population either asymptotically degenerates or exponentially grows. In this situation, it is of interest to give an asymptotic analysis of the particles field for $t \to \infty$.

*Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany (kondrat@math.uni-bielefeld.de)

†Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany (kutoviy@math.uni-bielefeld.de)

‡Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA (smolchan@uncc.edu).

In the present paper we investigate the long time behavior of some continuous population systems in a random medium. The time evolution transforms the initial point field into a configuration (integer valued measure) on the probability space $(\Omega_m, \mathcal{F}_m, P_m) \times (\Omega, \mathcal{F}, P)_m$, where $(\Omega_m, \mathcal{F}_m, P_m)$ is related to the stationary in space random environment, which includes the description of branching and mortality mechanisms. For a fixed environment ω_m the conditional probability space $(\Omega, \mathcal{F}, P)_m$ contains information about random motion of the particles and their "reactions" (annihilation, splitting, etc). Our attention in this paper will be mainly concentrated on the study of the so-called first moment of the system (the density of a population). It is known that for a fixed environment $\omega_m \in \Omega_m$ the density $\rho(t, x, \omega_m)$, $t \geq 0$, $x \in \mathbb{R}^d$ will be solution to the following parabolic Anderson problem

$$\frac{\partial \rho(t, x, \omega_m)}{\partial t} = L\rho(t, x, \omega_m) + V(x, \omega_m)\rho(t, x, \omega_m), \quad \rho(0, x, \omega_m) \equiv \rho_0. \quad (1)$$

Here L is the generator of the underlying Markov process $x(s)$, $s \geq 0$ appearing effectively from the evolution of individuals and

$$V(x, \omega_m) = b(x, \omega_m) - m(x, \omega_m),$$

where $b(x, \omega_m)$ is the rate of production of new particles and $m(x, \omega_m)$ is the mortality rate (see [5] and details therein). The density of the initial population of particles ρ_0 is given by

$$\rho_0 = \frac{1}{|A|}\mathbb{E}[n(0, A, \omega_m)], \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Here $|A|$ denotes the Lebesgue measure of A, the symbol $\mathbb{E}$ stands for the mathematical expectation with respect to $(\Omega, \mathcal{F}, P)_m$ for a fixed random environment ω_m, and $n(0, A, \omega_m)$ is the number of particles in the set A at the moment of time $t = 0$.

The asymptotic analysis of the equation (1) as $t \to \infty$ will be the main task of our investigation in this paper. The quenched and annealed asymptotics for $\rho(t, x, \omega_m)$ in the case of the *local* generator L (e.g. Laplacian $\triangle$, generating the Brownian motion $b(s), s \geq 0$ in $\mathbb{R}^d$ or the lattice Laplacian, generating the symmetric random walk on $\mathbb{Z}^d$ with continuous time) are widely discussed in the literature, see e.g [1, 7, 9] and references therein forming almost complete bibliography of this problem up to now.

We would like to mention below two groups of results related to the continuous Anderson Hamiltonian $H = -(\Delta - V(\cdot, \omega_m))$, $V \geq 0$.

I. Assume that

$$V(x, \omega_m) = \sum_{y \in \omega_m} \varphi(x - y),$$

where φ is a positive elementary potential and the points of configuration ω_m are Poisson distributed with intensity λ. It is clear that such a situation corresponds to zero birth rate. Let $\rho(t,x)$ be the solution to the equation

$$\frac{\partial \rho}{\partial t} = \Delta\rho - V\rho, \quad \rho(0,x) = \rho_0.$$

It is well-known that this solution is a homogeneous field which can be represented by the Feynman-Kac formula

$$\begin{aligned}\rho(t,x,\omega) =& \mathbb{E}_x\left[\exp\left(-\int_0^t V(b(s),\omega_m)ds\right)\right]\\ =& \mathbb{E}_0\left[\exp\left(-\int_0^t V(x+b(s),\omega_m)ds\right)\right].\end{aligned}$$

Here and subsequently, $\mathbb{E}_x$ denotes the integration over the distribution of the underlying process $x(s)$, $s \geq 0$, $x(0) = x \in \mathbb{R}^d$ (in the situation described above, it is integration over the Brownian motion b) and the symbol $\langle\cdot\rangle$ stands for the integration over the environment law.

The following result is proved under some mild conditions in the case of $L = \triangle$.

Theorem 1.1 (Donsker-Varadhan, [2])**.**

$$\ln\langle\rho(t,0,\omega_m)\rangle \backsim -c(d)t^{\frac{d}{d+2}}, \quad t \to \infty,$$

where $c(d)$ is a particular constant which is independent of the elementary potential φ.

This kind of moment asymptotics is called *annealed*. The almost sure (a.s.) asymptotics (*quenched*) is established in the next theorem.

Theorem 1.2 (Sznitman, [9])**.**

$$\ln\rho(t,0,\omega_m) \backsim c_1(d)\frac{t}{\ln^{2/d}(t)}, \quad t \to \infty, \qquad P_m \text{ - a.s.},$$

where $c_1(d)$ is a particular constant which is independent of the elementary potential φ.

Remark 1.3. *This Poissonian model, which is going back to Smolukhowski, describes the absorption process. Note that for a periodic absorption potential*

$$V(x) = \sum_{\vec{n}\in\mathbb{Z}^d} \varphi\left(x - \vec{n}\right)$$

the concentration $\rho(t,\cdot)$ is decreasing exponentially:

$$\ln \rho(t,\cdot) \backsim -c_2(\varphi)t, \qquad c_2(\varphi) > 0$$

and the constant $c_2(\varphi)$ is the ground energy of $-\Delta + \varphi$ on the cell of periodicity. It is worth pointing out that absorption in a random environment is "slowing down" for $t \to \infty$. The difference between annealed and quenched asymptotics is significant in this case. The deviation from the exponential decay of $\rho(t,\cdot)$ is related to the presence of "large clearings" in the realization of the potential $V(\cdot)$ (i.e. areas where $V = 0$).

II. Two other quenched and annealed asymptotics describe the case of the Gaussian potential where the density is increasing super-exponentially.

Let $V(x,\omega_m)$ be the homogeneous Gaussian field,

$$\langle V(x,\cdot)\rangle = 0, \quad \langle V(x)V(y)\rangle = B(x-y) \quad \text{and} \quad B \in C^{4+\delta}(\mathbb{R}^d),\ \delta > 0.$$

Then, according to [1], the realization of $V(\cdot,\omega_m) \in C^{2+\delta}_{\text{loc}}$ and

$$\max_{|x|\le R} V(x,\omega_m) \backsim \sqrt{2dB(0)\ln R}, \quad R \to \infty, \qquad P_m - \text{a.s.}$$

Moreover, one can give a precise description of all "high peaks" of the potential. The annealed asymptotics in this case is described in the next theorem.

Theorem 1.4 (Carmona-Molchanov, [1]; Gärtner-Molchanov, [3]).
For any $p \ge 1$

$$\ln\langle \rho^p(t,0,\omega_m)\rangle = \frac{B(0)p^2t^2}{2} - c_1(p,d)t + o(t), \quad t \to \infty.$$

Let us stress that the moments increase progressively

$$\frac{\langle \rho^2(t,0)\rangle}{\langle \rho(t,0)\rangle^2} \sim e^{ct^2}, \quad t \to \infty.$$

The last fact is the manifestation of "*intermittency*" (high irregularity of the density field), see [3], [7], [4].

Theorem 1.5 (Carmona-Molchanov, [1]).

$$\ln \rho(t,0,\cdot) = c_1 t\sqrt{\ln t} - c_2\frac{t}{\sqrt{\ln t}} + o\left(\frac{t}{\sqrt{\ln t}}\right), \quad t \to \infty, \quad P_m - \text{a.s.}$$

The goal of the present paper is to prove results similar to Theorems 1.1, 1.2, 1.4, and 1.5 for a *non-local* basic Hamiltonian. Instead of the Laplacian $\triangle$ we will deal with the generator L of jump process $x(s)$, $s \geq 0$:

$$Lf(x) = \kappa \int_{\mathbb{R}^d} [f(x+z) - f(x)] a(z) dz. \tag{2}$$

The process $x(s)$ has the following description. It spends random time τ_0 at the initial point x_0 with the exponential law $\mathrm{Exp}(\kappa)$; at the moment of time $\tau_0 + 0$ it jumps from x_0 to $x_0 + z + dz$ with the probability $a(z)dz$ and spends at $x_1 = x_0 + z$ the time τ_1 with the law $\mathrm{Exp}(k)$, etc. We will assume that the distribution density a of the single jumps satisfies the following conditions

1. $a(z) = a(|z|)$ (it implies $L = L^*$);
2. $a(z) \backsim \frac{C}{|z|^{d+\alpha}}$, as $z \to \infty$, $\alpha > 0$.

We would like to emphasize that it is *not necessary* to assume that $\alpha < 2$, i.e., the random walk whose jumps are distributed according to $a(z)$ can be in the domain of attraction of a stable isotropic processes in $\mathbb{R}^d$ (for $\alpha < 2$) or in the domain of attraction of Brownian motion (for $\alpha \geq 2$). Later, we will use this fact, discussing Vlasov scaling for population models.

The lattice model with long jumps was studied recently in [8]. In the next two subsections we study Poisson and Gaussian environments for jump evolution. About the contact model, closely related to the operator (2), see [6] and our previous paper [5].

2 Jumps with birth and death

2.1 Poisson environment

Let us fix an arbitrary function $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ such that

1. $\varphi(x) = 0$, $|x| \geq r_0$ for some $r_0 > 0$;
2. $\max_{x \in \mathbb{R}^d} \varphi(x) = \varphi(0)$;
3. $\varphi \in C^2(\mathbb{R}^d)$ and $|\det(\mathrm{Hess}\varphi(0))| > 0$.

In this subsection we study the following parabolic problem for $\omega_m \in \Omega_m$

$$\frac{\partial u(t,x,\omega_m)}{\partial t} = Lu(t,x,\omega_m) + V(x,\omega_m)u(t,x,\omega_m), \quad t \geq 0,\ x \in \mathbb{R}^d \tag{3}$$

$$u(0,x) \equiv \rho_0. \tag{4}$$

Here L is a convolution operator given by (2) and

$$V(x, \omega_m) = \sum_{y \in \omega_m} \varphi(x - y).$$

Noteworthy that ω_m is locally finite subset (configuration) of $\mathbb{R}^d$. Its points are distributed by Poisson with intensity $\lambda > 0$.

The solution to the problem (4) can be represented by the Feynman-Kac formula. Namely,

$$u(t, x, \omega_m) = \rho_0 \, \mathbb{E}_x \left[e^{\int_0^t V(x(s), \omega_m) ds} \right],$$

where $x(s)$, $s \geq 0$ is a jump process with the generator L.

Annealed asymptotics.

Upper estimate. Let us consider the mean of $u(t, x, \omega_m)$ with respect to P_m:

$$\langle u(t, x, \omega_m) \rangle = \rho_0 \left\langle \mathbb{E}_x \left[e^{\frac{1}{t} \int_0^t t \, V(x(s), \omega_m) ds} \right] \right\rangle.$$

Using the Jensen inequality the latter expression can be estimated by

$$\frac{\rho_0}{t} \, \mathbb{E}_0 \left[\int_0^t \left\langle e^{tV(x(s)+x, \omega_m)} \right\rangle ds \right] = \rho_0 \langle e^{tV(0, \omega_m)} \rangle.$$

Note, that $\langle e^{tV(0, \omega_m)} \rangle$ is the Lapace transform of the Poisson measure. Hence,

$$\langle e^{tV(0, \omega_m)} \rangle = \int_{\Omega_m} \exp \left\{ t \sum_{x \in \omega_m} \varphi(x) \right\} dP_m(\omega_m) = \exp \left\{ \kappa \int_{\mathbb{R}^d} (e^{t\varphi(x)} - 1) dx \right\}.$$

Using Laplace method for integrals and taking into account the properties of φ we obtain

$$\int_{\mathbb{R}^d} (e^{t\varphi(x)} - 1) dx$$

$$= \int_{|x| \leq a} (e^{t\varphi(x)} - 1) dx \sim \frac{e^{t\varphi(0)}}{t^{d/2}} \left[\frac{(2\pi)^d}{|\det(\text{Hess}\, \varphi(0))|} \right]^{\frac{1}{2}}, \quad t \to \infty.$$

Hence,

$$\ln \langle u(t, x, \omega_m) \rangle \lesssim \ln \rho_0 + \kappa \frac{e^{t\varphi(0)}}{t^{d/2}} \left[\frac{(2\pi)^d}{\det(\text{Hess}\, \varphi(0))} \right]^{\frac{1}{2}}, \quad t \to \infty.$$

Lower estimate. To prove the estimate below it is enough to see that

$$\langle u(t,x,\omega_m)\rangle \geq \rho_0 \left\langle \mathbb{E}_x \left[e^{\int_0^t V(x(s),\,\omega_m)ds} \cdot \mathbb{I}_{\{x(s)\equiv x,\, s\in[0,\,t]\}} \right] \right\rangle$$

$$= \rho_0 \langle e^{tV(x,\,\omega_m)} P_x[x(s)\equiv x,\, s\in[0,\,t]]\rangle = \rho_0\, e^{-\kappa t} \langle e^{tV(0,\,\omega_m)}\rangle.$$

As a result, there exists $C>0$ such that

$$\ln\langle u(t,x,\omega_m)\rangle \gtrsim zC\frac{e^{t\varphi(0)}}{t^{d/2}} - \kappa t, \quad t\to\infty.$$

Quenched asymptotics.
Lower estimate. According to [1]

$$\max_{|x|\leq R} V(x,\,\omega_m) \sim \frac{d\,\varphi(0)\ln R}{\ln\ln R}, \quad R\to\infty, \quad P_{\mathrm{m}}\, -\, \text{a.s.}.$$

Moreover, near the point of maximum $x_0(R)$, for any $\delta>0$ there exists $\varepsilon(\delta)>0$ such that for $R>R_0(\omega_m)$ and any $x:\ |x-x_0(R)|\leq\varepsilon(\delta)$

$$V(x,\,\omega_m) \geq (1-\delta)\frac{d\,\varphi(0)\ln R}{\ln\ln R}.$$

Consider the following strategy ($E_t\subset\Omega$): the random walk spends time $\tau_0\leq 1$ at the initial point $x_0=0$ and then jumps to the $\varepsilon=\varepsilon(\delta)$ neighborhood of the point $x_0(R)$. After that it stays in the neighborhood of $x_0(R)$ until the moment t.

It is obvious that

$$u(t,0,\omega_m) \geq \mathbb{E}_0\left[e^{\int_0^t V(x(s),\,\omega_m)ds}\mathbb{I}_{E_t}\right].$$

Now, we calculate the probabilities of all events which take place in E_t during the time t. The probability to spend the time less then 1 at zero $P[\tau_0<1] = 1-e^{-\kappa}$. In its turn, the probability to jump from zero to the ball $B_{\varepsilon(\delta)}(x_0(R))$ is equal asymptotically $\frac{\kappa C\varepsilon^d}{R^{d+\alpha}}$, as $R\to\infty$. Hence

$$u(t,0,\omega_m) \geq \frac{\kappa(1-e^{-\kappa})C\varepsilon^d}{R^{d+\alpha}} e^{V(0,\,\omega_m)}\left[e^{(1-\delta)\frac{d\,\varphi(0)\ln R}{\ln\ln R}(t-1)}\right]$$

$$\geq (1-e^{-\kappa})\kappa C\varepsilon^d e^{-(d+\alpha)\ln R}\left[e^{(1-\delta)\frac{d\,\varphi(0)\ln R}{\ln\ln R}t}\right]$$

The parameter R can be used for optimization. A trivial computation shows that there exists $c>0$ such that

$$u(t,0,\omega_m) \geq (1-e^{-\kappa})\kappa C\varepsilon^d e^{-ct+e^{ct}/ct}$$

and consequently

$$u(t,0,\omega_m) \gtrsim const\, e^{-ct+e^{ct}/ct}, \quad t\to\infty$$

(double exponential growth in time of the quenched first moment).

Let us stress that the annealed moment also has the double-exponential growth, which is remarkable if one compares with the similar results for the Gaussian potential (see Theorems 1.4 and 1.5).

2.2 Gaussian environment

In this subsection we will study the Gaussian case, i.e. the equations

$$\frac{\partial u}{\partial t} = Lu + V(x,\omega_m)u, \quad u(0,x)=\rho_0$$
$$u(t,x,\omega_m) = \rho_0\,\mathbb{E}_x\left[\exp\left(\int_0^t V(x(s),\omega_m)ds\right)\right]$$

where $V(x,\omega_m)$ is the Gaussian field with correlation $B(x-y)=\langle V(x)V(y)\rangle$, $\langle V\rangle = 0$. To control the high peaks we will assume as in [1] that $B\in C^{4+\delta}$ and $V(\cdot,\omega_m)\in C^{2+\delta_1}$ P_m - a.s.

Theorem 2.1. *For any $p\in\mathbb{N}$*

$$\langle u^p(t,0,\omega_m)\rangle = \frac{B(0)t^2p^2}{2} - p\kappa t + o(t).$$

Proof. Lower estimate.

$$\begin{aligned}
u(t,0,\omega_m) &= \rho_0\,\mathbb{E}_0\left[\exp\left(\int_0^t V(x(s),\omega_m)ds\right)\right]\\
&\ge \rho_0\,\mathbb{E}_0\left[\exp\left(\int_0^t V(x(s),\omega_m)ds\right)\mathbb{I}_{x(s)\equiv 0, s\le t}\right]\\
&= \rho_0 \exp\left(tV(0,\omega_m)\right)\cdot P\left[x(s)=0, s\le t\right] = \rho_0\,\exp\left(tV(0,\omega_m)\right)e^{-\kappa t}
\end{aligned}$$

Now we apply to both parts of the inequality the averaging $\langle\cdot\rangle$ over the potential V:

$$\langle u(t,0,\omega_m)\rangle \ge \rho_0 e^{-\kappa t} e^{\frac{1}{2}B(0)t^2}.$$

For the higher moments one can use the standard trick:

$$u^p(t,0,\omega_m) = \rho_0^p\,\mathbb{E}_0 \exp\left(\int_0^t [V(x_1(s),\omega_m)+\cdots+V(x_p(s),\omega_m)]ds\right) \quad (5)$$

where $x_1(s), \ldots, x_p(s)$ are independent copies of the random walk $x(s)$. Then

$$\langle u^p(t, 0, \omega_m)\rangle \geq \rho_0^p \langle e^{tpV(0,\omega_m)}\rangle P\left[\{x_1(s) \equiv \cdots \equiv x_p(s) \equiv 0, s \in [0, t]\}\right]$$
$$= \rho_0^p e^{\frac{B(0)p^2t^2}{2}} e^{-p\kappa t}.$$

Upper estimate. We will use here the Jensen inequality and start from the rough bound

$$u(t, 0, \omega_m) = \rho_0 \mathbb{E}_0 \left[\exp\left(\frac{1}{t}\int_0^t tV(x(s), \omega_m)ds\right)\right]$$
$$\leq \rho_0 \frac{1}{t}\int_0^t \mathbb{E}_0\left[\exp\left(tV(x(s), \omega_m)\right)\right] ds.$$

As result

$$\langle u(t, 0, \omega_m)\rangle \leq \rho_0 \frac{1}{t}\int_0^t \exp\left(\frac{t^2B(0)}{2}\right) ds = \rho_0 e^{\frac{t^2B(0)}{2}}.$$

Similarly, from the representation (5) we will get

$$\langle u^p(t, 0, \omega_m)\rangle \leq \rho_0{}^p \exp\left(\frac{B(0)p^2t^2}{2}\right).$$

We have already proved that

$$\ln\langle u^p(t, 0, \omega_m)\rangle \backsim \frac{p^2t^2}{2}, \quad t \to \infty.$$

The proof of the sharper estimate (mentioned in Theorem 2.1):

$$\ln\langle u^p(t, 0, \omega_m)\rangle = \frac{p^2t^2}{2} - pkt + o(t), \quad t \to \infty$$

is much more difficult. We will discuss in detail only the case $p = 1$. Transition to the general case $p \geq 2$ requires to introduce, instead of one trajectory $x(s)$, $s \in [0, t]$, p independent copies of such trajectories. Since for any fixed piece-wise continuous trajectory $\gamma(s)$, $s \in [0, t]$

$$\int_0^t V(\gamma(s))ds \sim N\left(0, \int_0^t\int_0^t B(\gamma(s_1) - \gamma(s_2))ds_1ds_2\right)$$

we have

$$\left\langle \mathbb{E}_0\left[\exp\left(\int_0^t V(x(s))ds\right)\right]\right\rangle = \mathbb{E}_0\left[e^{\frac{1}{2}\int_0^t\int_0^t B(x(s_1)-x(s_2))ds_1ds_2}\right].$$

The number ν_t of jumps of $x(s)$, $s \in [0, t]$ has a Poissonian law with parameter $\kappa t > 0$, i.e.

$$P\left[\{\nu_t = m\}\right] = (\kappa t)^m e^{-\kappa t}, \quad m \geq 0.$$

Then

$$I(t) := \mathbb{E}_0 \left[e^{\frac{1}{2}\int_0^t \int_0^t B(x(s_1)-x(s_2))ds_1 ds_2} \right]$$

$$= \exp\left\{ \frac{B(0)t^2}{2} - \frac{1}{2}\int_0^t \int_0^t \tilde{B}(x(s_1) - x(s_2)) ds_1 ds_2 \right\}.$$

Here $\tilde{B}(z) = B(0) - B(z) \geq 0$. It is easy to see that $\tilde{B}(z) \geq c_0 > 0$ if $|z| \geq 1$ and $B(z) \geq c_1 |z|^2$, if $|z| \leq 1$. Using total expectation formula for the hypothesis $H_r = \{\nu_t = r\}$, $r \geq 0$ we get

$$I(t) = \sum_{r=0}^{\infty} \frac{e^{-\kappa t}(\kappa t)^r}{r!} \mathbb{E}_0 \left[e^{\frac{1}{2}\int_0^t \int_0^t \tilde{B}(x(s_1)-x(s_2))ds_1 ds_2} \mid \nu_t = r \right] = \sum_{r=0}^{\infty} I_r(t).$$

It is worth pointing out that $I_0(t) = e^{-\kappa t}$. Hence, the statement of our theorem will be proved once we check that

$$\sum_{r=1}^{\infty} I_r(t) = o(e^{-\kappa t}).$$

Due to the well-known property of the Poisson process

$$I_r(t) = e^{-\kappa t} \frac{(\kappa t)^r}{r!} \frac{r!}{t^r} \underset{s_1+\dots+s_{r+1}=t}{\int \cdots \int} e^{\frac{1}{2}\int_0^t \int_0^t \tilde{B}(x(s_1)-x(s_2))ds_1 ds_2} d\sigma_r =$$

$$e^{-\kappa t} \kappa^r \underset{s_1+\dots+s_{r+1}=t}{\int \cdots \int} e^{\frac{1}{2}\int_0^t \int_0^t \tilde{B}(x(s_1)-x(s_2))ds_1 ds_2} d\sigma_r,$$

where $d\sigma_r$ is the volume element on the tetrahedron $s_1 + \dots + s_{r+1} = t$ and

$$\frac{1}{2}\int_0^t \int_0^t \tilde{B}(x(s_1) - x(s_2)) ds_1 ds_2 = s_1 s_2 B(x_1) + s_1 s_3 B(x_1 + x_2) + \dots s_1 s_{r+1} B(x_1 + \dots + x_r)$$

$$+ s_2 s_3 B(x_1 + x_2) + \ldots + s_2 s_{r+1} B(x_1 + \ldots + x_r)$$

$$+ \ldots + s_r s_{r+1} B(x_1 + \ldots + x_r),$$

where $x_1, \dots, x_r$ are i.i.d. random variables with the density $a(z)$. The number of terms in the sum above is equal to $A(r) = \frac{r(r+1)}{2}$. Using the inequality

$$\prod_{i=1}^{A(r)} a_i \leq \frac{\sum_{i=1}^{A(r)} a_i^{A(r)}}{A(r)}$$

we obtain

$$\tilde{I}_r(t) := e^{\kappa t}\kappa^{-r} I_r(t) \le \int_{s_1+\cdots+s_{r+1}=t}\cdots\int \frac{1}{A(r)}\sum_{i<j} e^{-A(r)s_is_jB(x_i-x_j)}d\sigma_r.$$

Assume that $\alpha = \max_{z\in\mathbb{R}^d} a(z)$. Then for any r the density $a_r(z)$ of the sum $x_1+\ldots+x_r$ has the same estimate $a_r(z)\le\alpha$. Then

$$\tilde{I}_r(t) \le \frac{1}{A(r)}\sum_{i<j}\int_{s_1+\cdots+s_{r+1}=t}\cdots\int d\sigma_r\left[\alpha\int_{|z|\le 1} e^{-A(r)c_1s_is_j|z|^2}dz\right.$$

$$\left. + \int_{|z|>1} a(z)dz e^{-A(r)c_0s_is_j}\right]. \tag{6}$$

The summands in the first term of (6) are the same for different $i<j$, i.e.

$$\tilde{I}_r(t) \le \iint_{s_1+s_2\le t} d\sigma_3\frac{(t-s_1-s_2)^{r-1}}{(r-1)!}\left[\alpha\int_{|z|\le 1} e^{-A(r)c_1s_1s_1|z|^2}dz\right.$$

$$\left. + \iint_{s_1+s_2\le t} d\sigma_3\frac{(t-s_1-s_2)^{r-1}}{(r-1)!}e^{-A(r)c_0s_1s_2}\right]$$

$$\le \frac{t^{r-1}}{(r-1)!}\left[\iint_{s_1+s_2\le t} e^{-A(r)c_0s_1s_2}ds_1ds_2 + \right.$$

$$\left.\iint_{s_1+s_2\le t} ds_1ds_2\int_{|z|\le 1} e^{-A(r)c_1s_1s_1}|z|^2dz\right].$$

It follows from either the Laplace method or direct calculation that the expression in the square brackets is $o(t)$ if $r\to\infty$ which implies that

$$\sum_{r\ge 1} I_r(t) = o(e^{-\kappa t}).$$

□

Let us find the quenched asymptotics in the same Gaussian model. According to [1] we have

$$\max_{|x|\le R} V(x) \sim \sqrt{2dB(0)\ln R}, \quad R\to\infty, \quad P_{\rm m} - \text{a.s.}$$

Moreover, near the point of maximum $x_0(R)$ the potential has second derivatives of the order $\sqrt{\ln R}$, i.e. there exists $\varepsilon(\delta) > 0$ such that for $R > R_0(\omega_m)$ and any $x: \ |x - x_0| \le \varepsilon(\delta)$

$$V(x) \ge (1-\delta)\sqrt{2dB(0)\ln R}.$$

Consider the following strategy: the random walk spends time (less 1) at the initial point $x_0 = 0$ and jumps to the $\varepsilon(\delta)$ neighborhood of the point $x_0(R)$. After that it stays at $x_0(R)$ until the moment t. The parameter R can be used for optimization. As a result, using equality $P\left[\{\tau_0 < 1\}\right] = 1 - e^{-k}$ we obtain

$$u(t,0,\omega_m) \ge (1-e^{-k}) \max_R \left[e^{(1-\delta)\sqrt{2dB(0)\ln R}(t-1)} \frac{c_1 \varepsilon^d}{R^{d+\alpha}} \right].$$

A trivial computation shows that

$$u(t,0,\omega_m) \ge e^{\frac{(1-\delta)^2 t^2}{4(d+\alpha)}}$$

and consequently

$$\ln u(t,0,\omega_m) \ge c(\alpha) t^2.$$

It is worth noting (in contrast to [1], [3]) that $\ln \langle u(t,0)\rangle$ and $\ln u(t,0,\omega_m)$ have P_m - a.s. the scale $O(t^2)$. We will see later that it is the general property of the jump operators with heavy tails. Let us stress that $\alpha > 0$ is an arbitrary positive constant. In particular, it is possible that $\alpha > 2$, i.e. $\int_{\mathbb{R}^d} z^2 a(z) dz < \infty$. In this situation the process $x(s)$, $s \ge 0$ after rescaling

$$x_t(s) = \frac{x(st)}{\sigma\sqrt{t}}, \quad t \to \infty$$

on each interval $s \in [0,s]$ converges weakly in $C([0,s])$ to the standart brownian motion (for the appropriate $\sigma > 0$).

In the so-called Vlasov limit one can try to approximate our initial problem (with the jumping process in the underground) by the problem with Brownian diffusion instead of jumping. The previous arguments indicate that this approximation gives wrong results in the large deviation region (that fact one could expect in advance).

3 Jump process with pure annihilation

In this part we will study the absorption phenomena with underlying jumping process. Our particular goal is the quenched and annealed asymptotics for the parabolic problem

$$\frac{\partial u(t,x,\omega_m)}{\partial t} = Lu(t,x,\omega_m) - V(x,\omega_m)u(t,x,\omega_m), \quad u(0,x) \equiv 1,$$

where

$$Lf(x) = \int_{\mathbb{R}^d} a(z)(f(x+z) - f(x))dz$$

and $a(-z) = a(z) \sim \frac{C_0}{|z|^{d+\alpha}}$, $|z| \to \infty$, $\alpha < 2$ (cf. the previous subsection). The potential $(-V)(x)$ is non-positive, i.e. there is no birth of new particles, only annihilation. Let

$$V(x,\omega_m) = \sum_{y \in \omega_m} \varphi(x-y)$$

Suppose that points of the configuration ω_m form the Poisson point field with the intensity λ and the function $\varphi \geq 0$ is compactly supported on the ball of the radius a.

3.1 Annealed asymptotics

The following lemma is related to the fact that the random variable $x_j \in \mathbb{R}^d$ belong to the domain of attraction of the isotropic stable process with the parameters $\alpha < 2$ and the generator $-(-\triangle)^{\alpha/2} = L_\alpha$

Lemma 3.1. *Consider the spectral problem*

$$-L_\alpha \psi = E\psi, \quad x \in B_r(0) = \{x \,|\, |x| \leq r\},$$

$$\psi(x) \equiv 0, \quad |x| > r.$$

then the principal (minimal) eigenvalue $E_0 = E_0(r)$ has the asymptotics $E_0(r) \sim \frac{C_{d,\alpha}}{r^\alpha}$, $r \to +\infty$ (see [5]).

Now we will deduce from Lemma 3.1 the following result.

Theorem 3.2. *Consider the following parabolic Anderson problem*

$$\frac{\partial u(t,x,\omega_m)}{\partial t} = -L_\alpha u(t,x,\omega_m) - V(x,\omega_m)u(t,x,\omega_m), \quad u(0,x) \equiv 1.$$

The solution to this problem has the following asymptotics

$$\ln \langle u^p(t,0,\omega_m) \rangle \asymp -(pt)^{\frac{d}{d+\alpha}}, \quad t \to \infty.$$

Proof. We give below only the idea of the proof.

Below estimate. As the first step we consider the case $p = 1$. Then

$$\langle u(t,0)\rangle = \mathbb{E}_0\langle e^{-\int_0^t V(x(s))ds}\rangle \geq \mathbb{E}_0\langle e^{-\int_0^t V(x(s))ds}\mathbb{I}_{\{\#(\omega_m\cap B_{r+a}(0))=0\}}\mathbb{I}_{\{|x(s)|\leq r,\, s\in[0,t]\}}\rangle$$
$$= P_0\left[|x(s)|\leq r,\, s\in[0,t]\right]P_m\left[\#(\omega_m\cap B_{r+a}(0))=0\right]$$
$$\sim e^{-\frac{c_\alpha}{r^\alpha}t}e^{-\lambda(r+a)^d\tilde{c}_d} \sim e^{-\frac{c_\alpha}{r^\alpha}t-\lambda r^d}, \quad r\to\infty.$$

The optimization of the exponent over parameter r gives

$$\langle u(t,0)\rangle \gtrsim e^{-\tilde{c}(\alpha,d,\lambda)t^{\frac{d}{d+\alpha}}}, \quad t\to\infty.$$

The optimal $r = r(t) \sim \tilde{C}_1 t^{\frac{1}{d+\alpha}}$, $t\to\infty$.

To estimate $\langle u^p(t,0)\rangle$ we have to consider p-independent copies of $x(s)$. The same calculations as above lead to the following inequality

$$\langle u^p(t,0)\rangle \gtrsim e^{-\tilde{c}(\alpha,d,\lambda)(pt)^{\frac{d}{d+\alpha}}}$$

Upper estimate. Upper estimation is the direct repetition of the large-deviation approach by Donsker-Varadhan (see [2]). The central moment in the construction of the action functional in [2] is an analog of Lemma 1 introduced above (stated for the case of the underlying brownian motion, i.e. for $\triangle$ instead of L) which leads to $E_0(r) \sim \frac{C_d}{r^2}$. □

In the case of jump process with heavy tails one can expect (the rigorous proof is omitted in this paper) the following result.

Theorem 3.3. *For* $\alpha < 2$

$$\ln\langle u(t,0,\omega_m)\rangle \sim \tilde{c}(\alpha,d,\lambda)t^{\frac{d}{d+\alpha}}, \quad t\to\infty.$$

Note that the proof of the annealed result is based only on the self-similarity arguments and it uses the "heavy tails" of a only implicitly.

3.2 Quenched asymptotics

The situation with the quenched asymptotics is described in the following theorem.

Theorem 3.4.

$$\ln u(t,0,\omega_m) \asymp -t^{\frac{d}{d+\alpha}},$$

i.e. for appropriate constants C^+, $C^- > 0$

$$C^- t^{\frac{d}{d+\alpha}} \leq -\ln u(t,0,\omega_m) \leq C^+ t^{\frac{d}{d+\alpha}}$$

Proof. Note that $u(t,0,\omega_m)$ is a non-increasing function of t. Then for discrete moments of time $t = l \in \mathbb{N}$ and any $\delta > 0$ the Chebyshev inequality implies

$$P_m\left[u(l,0,\omega_m) > \langle u(l,0,\omega_m)\rangle^{1-\delta}\right] \le \frac{\langle u(l,0,\omega_m)\rangle}{\langle u(l,0,\omega_m)\rangle^{1-\delta}} \le e^{-\delta C l^{\frac{d}{d+\alpha}}}. \quad (7)$$

In the last inequality we have used the annealed asymptotics. The sum of all probabilities (7) is finite and Borel-Cantelli lemma gives

$$-\ln\left(u(t,0,\omega_m)\right) \le (1+\delta)\tilde{c}(\alpha,d,\lambda)t^{\frac{d}{d+\alpha}}$$

for $t \ge t_0(\omega_m)$.

To prove the lower estimate we will start from the following technical lemma.

Lemma 3.5. *(existence of big clearings). Let $\gamma > 1$ be arbitrary and fixed. Consider the following partition of $\mathbb{R}^d$:*

$$\mathbb{R}^d = \bigcup_{n=0}^{\infty} D_n, \quad D_0 = \{x \mid |x|_\infty \le 1\},$$

$$D_n = \{x \mid \gamma^{n-1} < |x|_\infty \le \gamma^n\}, \quad |D_n| = 2^d\gamma^{(n-1)d}(\gamma^d - 1) \quad n \ge 1.$$

Let ω_m be the Poisson point field with the intensity $\lambda > 0$. Then one can find a constant $\tilde{c} = \tilde{c}(\lambda,d,\gamma) > 0$ and a finite random variable $n_0(\omega_m)$ such that P_m - almost surely for any $n \ge n_0(\omega_m)$ there exist a ball $B_{R_n}(x_n) \subset D_n$ of radius $R_n = \tilde{c}n^{1/d} + a$ where $V(x,\omega_m) = 0$.

Proof. Consider the maximal family of disjoint balls of the radius R_n inside the ring D_n. The volume of such a ball is equal to $c(d)R_n^d$. The cardinality of such a family

$$N_n \ge \frac{|D_n|}{c(d)R_n^d} \ge c_1(d,\gamma)\frac{(2\gamma)^{nd}}{R_n^d}.$$

Consider the following event:
$A_n = \{\omega_m :$ all balls in this family contain at least one Poisson point of $\omega_m\}$. Then

$$P(A_n) \le \left(1 - e^{-\lambda c(d)R_n^d}\right)^{N_n} \le e^{-c_1\frac{(2\gamma)^{nd}}{R_n^d}e^{-\lambda c(d)R_n^d}}$$

$$\le e^{-\frac{c_2}{\tilde{c}^d n}\left[(2\gamma)^d e^{-\lambda c(d)\tilde{c}^d}\right]^n}.$$

One can select now sufficiently small $\tilde{c} = \tilde{c}(\lambda,d,\gamma)$ from the condition

$$(2\gamma)^d e^{-\lambda c(d)\tilde{c}^d} > 1.$$

Then, $\sum_n P(A_n) < \infty$. The application of Borel-Cantelli lemma completes the proof. □

Let us denote by $B_{R_n}(x_n)(\omega_m)$ one of the empty balls in the set D_n, which exists for $n \geq n_0(\omega_m)$. We consider the following strategy (the event E_t): the chain $x(t)$ spends time $\tau_1 \leq 1$ in the initial point $x_0 = 0$ and jumps to the central part of the ball $B_{R_n}(x_n)$, given by $|x - x_n| \leq R_n/2$, and stays in $B_{R_n}(x_n)$ at the time interval $[1, t]$. The number $n = n(t)$ is selected from the condition of maximization of the expression

$$\frac{1}{\gamma^{n(d+\alpha)}} e^{-\frac{c_1 t}{\tilde{c} n^{\alpha/d}}} = e^{-n \ln \gamma (d+\alpha) - \frac{c_1 t}{\tilde{c} n^{\alpha/d}}}.$$

Elementary calculations give

$$\min_n \left[n c_2 - \frac{c_3}{n^{\alpha/d}} \right] = c_4(d, \lambda, \gamma) t^{\frac{d}{d+\alpha}}.$$

But

$$u(t, 0, \omega_m) \geq \mathbb{E}_0 \left[e^{-\int_0^t V(x(s)) ds} \mathbb{I}_{E_t} \right] \geq c_5 e^{-c_4 t^{\frac{d}{d+\alpha}}}.$$

This finishes the proof of Theorem 3.4. □

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