# FINITE INJECTIVE DIMENSION OVER RINGS WITH NOETHERIAN COHOMOLOGY

JESSE BURKE

Abstract. We study rings that have Noetherian cohomology over a ring of cohomology operators. Examples of such rings include commutative complete intersection rings and finite dimensional cocommutative Hopf algebras. The main result is a criterion for a complex of modules over a ring with Noetherian cohomology to have finite injective dimension. The criterion implies in particular that for any module over such a ring, if all higher self-extensions of the module vanish, then it must have finite injective dimension. This generalizes a theorem of Avramov and Buchweitz for complete intersection rings, and a well-known theorem in the representation theory of finite groups.

## 1. Introduction

Let $R$ be an associative ring and $S$ a ring of cohomology operators on $R$. Thus $S$ is a commutative graded ring and there is a family of homogeneous maps of graded rings indexed by all complexes of $R$-modules $M$:

$$\zeta_M : S \to \operatorname{Ext}^*_R(M, M),$$

that satisfies a commutativity condition. See Section 3 for the full definition. We say $R$ has *Noetherian cohomology* over $S$ if $\operatorname{Ext}^*_R(M, M)$ is a Noetherian $S$-module via $\zeta_M$ for all $M$ with Noetherian cohomology over $R$.

In this paper we prove the following:

**Theorem.** *Let $R$ be a ring with Noetherian cohomology over a ring of cohomology operators $S$, and let $M$ be a complex of $R$-modules with $H^n(M) = 0$ for $n \gg 0$. Let $S^+$ be the ideal $\oplus_{i \geq 1} S^i$. If the $S$-module $\operatorname{Ext}^*_R(M, M)$ is $S^+$-torsion, then $M$ has finite injective dimension.*

Recall that $\operatorname{Ext}^*_R(M, M)$ is $S^+$-torsion if for every $x \in \operatorname{Ext}^*_R(M, M)$ there exists an integer $n$ such that $(S^+)^n x = 0$. There is, for instance, an integer $l$ depending on the degrees of the generators of $S^+$, such that if $\operatorname{Ext}^{nl}_R(M, M) = 0$ for some $n \geq 1$, then $\operatorname{Ext}^*_R(M, M)$ is $S^+$-torsion; see 4.5. The notions of Ext and injective dimension for a complex generalize the usual ones for a module. In particular, if $M$ is an $R$-module with $\operatorname{Ext}^n_R(M, M) = 0$ for $n \gg 0$, then $M$ has finite injective dimension.

There is a wide class of rings with Noetherian cohomology and hence to which the above result applies. First, assume that $R$ is a ring of the form $Q/(f_1, \ldots, f_c)$, where $Q$ is a commutative Noetherian regular ring of finite Krull dimension and $f_1, \ldots, f_c$ is a $Q$-regular sequence. The graded polynomial ring $S = R[\chi_1, \ldots, \chi_c]$, where the degree of each $\chi_i$ is 2, is a ring of cohomology operators for $R$ and $R$ has Noetherian cohomology over $S$ by [Gul74]. In this context the Theorem generalizes a key instance of [AB00, Theorem 4.2] from finitely generated modules to a large class of complexes, including all modules:

**Corollary A.** *Let $R = Q/(f_1, \ldots, f_c)$, where $Q$ is a commutative Noetherian regular ring of finite Krull dimension and $f_1, \ldots, f_c$ is a $Q$-regular sequence. Let $M$ be a complex of $R$-modules with $H^n(M) = 0$ for $n \gg 0$. If $\operatorname{Ext}^{2n}_R(M, M) = 0$ for some $n \geq 1$, then $M$ has finite injective dimension.*

Indeed, if $\operatorname{Ext}^{2n}_R(M, M) = 0$ for some $n$, then $\operatorname{Ext}^*_R(M, M)$ must be $S^+$-torsion since the degree of $\chi_i$ is 2. Thus $l = 2$ in the notation above. See 5.1 for further details.

If $R$ is an algebra, which is possibly non-commutative, over a field and $S$ is a commutative graded subring of the Hochschild cohomology ring of $R$, then $S$ is a ring of cohomology operators for $R$; see 5.5 for the construction. Let $\mathfrak{r}$ be the Jacobson radical of $R$. If we further assume that $R$ is finite dimensional over the field, $S$ is Noetherian, and $\mathrm{Ext}^*_R(R/\mathfrak{r}, R/\mathfrak{r})$ is a Noetherian $S$-module, then by [EHT$^+$04, 2.4], $R$ has Noetherian cohomology over $S$. In this context, the Theorem generalizes [EHT$^+$04, Theorem 2.5] from finite dimensional modules to a large class of complexes, including all modules:

**Corollary B.** *Let $R$ be a finite dimensional algebra over a field and let $\mathfrak{r}$ be the Jacobson radical of $R$. Assume that $\mathrm{Ext}^*_R(R/\mathfrak{r}, R/\mathfrak{r})$ is a Noetherian module over some commutative Noetherian subring of the Hochschild cohomology of $R$. There exists an integer $l \geq 1$ such that for any complex of $R$-modules $M$ with $H^n(M) = 0$ for $n \gg 0$, if $\mathrm{Ext}^{nl}_R(M, M) = 0$ for some $n \geq 1$, then $M$ has finite injective dimension.*

By the main result of [FS97] every finite dimensional cocommutative Hopf algebra satisfies the hypotheses of Corollary B. In particular the result applies to the group ring of a finite group over a field.

For the proof of the Theorem, we work in an "infinite completion" of the bounded derived category of Noetherian $R$-modules. This allows us to avoid finiteness conditions on the complexes to which the criterion is applied. By [Kra05], such a completion is given by the homotopy category of injective $R$-modules. We recall relevant facts about this category in Section 2. In Section 3 we give the precise definition of a ring of cohomology operators and prove a preliminary result. The proof of the Theorem occupies Section 4 and in Section 5 we apply it to the cases discussed above.

The techniques in this paper are inspired by [BIK08]. We have minimized the use of machinery from that paper to make this one more self-contained.

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## 2. Background

Throughout $R$ denotes an associative ring. By the word "module" we mean a left-module. An $R$-complex is a complex of $R$-modules.

In this section we briefly recall some definitions and results on triangulated categories. We then review the homological algebra of complexes that we will need.

**2.1.** Let $M$ be an $R$-complex. We write $H^n(M)$ for the $n$th cohomology group of $M$ and $H(M)$ for the graded $R$-module which in degree $n$ is $H^n(M)$. We say $M$ has *finite cohomology* if $H(M)$ is a Noetherian $R$-module; this implies in particular that $H^n(M) = 0$ for $|n| \gg 0$. The complex $M$ is *acyclic* if $H(M) = 0$.

Let $N$ be another $R$-complex. We denote the *Hom-complex* between $M$ and $N$ by $\mathrm{Hom}_R(M, N)$; this has components and differential given by

$$\mathrm{Hom}_R(M, N)^n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M^i, N^{i+n}) \qquad \partial(f) = \partial^N \circ f - (-1)^{|f|} f \circ \partial^M,$$

where $|f|$ is the degree of $f$. A *morphism* $f : M \to N$ is a degree zero cycle of $\mathrm{Hom}_R(M, N)$, i.e. $\partial(f) = 0$. It is a *quasi-isomorphism* when $H(f) : H(M) \to H(N)$ is an isomorphism.

**2.2.** The *homotopy category of injective $R$-modules*, denoted by $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$, has as objects complexes of injective $R$-modules. The morphisms between objects $X, Y$ are given by

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y) := H^0(\mathrm{Hom}_R(X, Y)).$$

In other words, morphisms in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ are homotopy equivalence classes of morphisms of complexes.

The standard shift functor on $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is denoted $\Sigma$. Thus for a complex

$$X = \ldots \to X^{n-1} \to X^n \to X^{n+1} \to \ldots$$

we have that $(\Sigma X)^n = X^{n+1}$ and $\partial_{\Sigma X} = -\partial_X$. By $\operatorname{Hom}^*_{\mathsf{K}}(X,Y)$ we denote the $\mathbb{Z}$-graded abelian group which in degree $n$ is $\operatorname{Hom}_{\mathsf{K}}(X,\Sigma^n Y)$. With multiplication given by composition $\operatorname{Hom}^*_{\mathsf{K}}(X,X)$ is a graded ring while $\operatorname{Hom}^*_{\mathsf{K}}(X,Y)$ is a bimodule with left action by $\operatorname{Hom}^*_{\mathsf{K}}(Y,Y)$ and right action by $\operatorname{Hom}^*_{\mathsf{K}}(X,X)$.

**2.3.** The category $\mathsf{K}(\mathsf{Inj\,R})$ is triangulated. For a proof and reference on triangulated categories see e.g. [Ver96]. A triangulated subcategory of $\mathsf{K}(\mathsf{Inj\,R})$ is *thick* if it is closed under direct summands; it is *localizing* when it is closed under set-indexed direct sums. Every localizing subcategory in $\mathsf{K}(\mathsf{Inj\,R})$ is automatically thick, see e.g. the proof of [HPS97, 1.4.8].

For a subclass of objects $\mathsf{C}$ in $\mathsf{K}(\mathsf{Inj\,R})$, we denote by $\mathsf{thick}_{\mathsf{K}}(\mathsf{C})$, respectively $\mathsf{loc}_{\mathsf{K}}(\mathsf{C})$, the smallest thick, respectively localizing, subcategory containing $\mathsf{C}$. One may realize these by taking the intersection of all thick, respectively localizing, subcategories containing $\mathsf{C}$.

An object $C \in \mathsf{K}(\mathsf{Inj\,R})$ is *compact* if the natural map

$$\bigoplus_{i\in I} \operatorname{Hom}_{\mathsf{K}(\mathsf{Inj\,R})}(C, X_i) \to \operatorname{Hom}_{\mathsf{K}(\mathsf{Inj\,R})}(C, \bigoplus_{i\in I} X_i)$$

is an isomorphism for any set of objects $\{X_i\}_{i\in I}$ of $\mathsf{K}(\mathsf{Inj\,R})$. We denote the collection of compact objects of $\mathsf{K}(\mathsf{Inj\,R})$ by $\mathsf{K}(\mathsf{Inj\,R})^{\mathsf{c}}$.

When $R$ is left-Noetherian, [Kra05, 2.3.1] shows that $\mathsf{K}(\mathsf{Inj\,R})$ is *compactly generated*, i.e. an object $X \in \mathsf{K}(\mathsf{Inj\,R})$ is nonzero if and only if there exists a compact object $C \in \mathsf{K}(\mathsf{Inj\,R})$ such that $\operatorname{Hom}_{\mathsf{K}(\mathsf{Inj\,R})}(C,X) \neq 0$.

**2.4.** A complex of injective modules $I$ is *semi-injective* if for all acyclic complexes $A$, the complex $\operatorname{Hom}_R(A,I)$ is acyclic. When $I$ is semi-injective it has the following lifting property: for every morphism $\alpha : M \to I$ and every quasi-isomorphism $\beta : M \to N$ there exists a unique up to homotopy map $\gamma : N \to I$ making the following diagram commute

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{\beta} & N \\ {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\gamma} & \\ I & & \end{array} .$$

A *semi-injective resolution* of a complex $M$ is a quasi-isomorphism $\eta_M : M \to \mathsf{i}M$, where $\mathsf{i}M$ is semi-injective. Every complex has a semi-injective resolution; this was first proven in [Spa88]. Moreover, by the lifting property, a semi-injective resolution is unique up to isomorphism in $\mathsf{K}(\mathsf{Inj\,R})$.

When $M$ is a module viewed as a complex concentrated in degree 0, a semi-injective resolution of $M$ is just an injective resolution in the usual sense.

**2.5.** Let $\mathsf{i}M, \mathsf{i}N$ be semi-injective resolutions of complexes $M, N$, respectively. Define the derived Hom functors as

$$\operatorname{Ext}^n_R(M,N) := \operatorname{Hom}_{\mathsf{K}}(\mathsf{i}M, \Sigma^n \mathsf{i}N) \cong H^n \operatorname{Hom}_R(\mathsf{i}M, \mathsf{i}N).$$

The lifting property of semi-injective complexes shows that $\operatorname{Ext}^*_R(M,N)$ is independent of the choice of resolutions, up to isomorphism.

If there exists a semi-injective resolution $\eta_M : M \to \mathsf{i}M$ such that $(\mathsf{i}M)^n = 0$ for all $n \gg 0$ then we say $M$ has *finite injective dimension* and write $\operatorname{inj\,dim}_R M < \infty$.

**2.6.** Let $\mathsf{D}(R)$ be the unbounded derived category of $R$-modules, see e.g. [Ver96] for the defintion. We denote by $Q$ the localization functor $Q : \mathsf{K}(\mathsf{Inj\,R}) \to \mathsf{D}(R)$ which sends a complex to its image in the derived category. When $R$ is left-Noetherian [Kra05, 2.3.2] shows that $Q$ restricts to an equivalence

$$Q : \mathsf{K}(\mathsf{Inj\,R})^{\mathsf{c}} \xrightarrow{\cong} \mathsf{D}^{\mathsf{f}}(R),$$

where $\mathsf{D}^{\mathsf{f}}(R)$ is the full subcategory of $\mathsf{D}(R)$ of objects with finite cohomology. The functor $Q$ has a right adjoint, denoted by $Q_\rho$, which when restricted to $\mathsf{D}^{\mathsf{f}}(R)$ gives an inverse to the equivalence above; moreover, for any $M \in \mathsf{D}^{\mathsf{f}}(R)$, the object $Q_\rho M$ is semi-injective [Kra05, 3.6]. Thus *the compact objects of* $\mathsf{K}$ *are exactly the semi-injective resolutions of objects in* $\mathsf{D}^{\mathsf{f}}(R)$.

The following construction is a key part of the proof of the main Theorem.

**2.7.** Let $\mathsf{S} = \mathsf{loc}_{\mathsf{K}}(\mathsf{C})$, for a set of compact objects $\mathsf{C}$ in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. For any object $X$ in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is a triangle

$$\Gamma X \to X \to \mathsf{L}X \to$$

such that $\Gamma X \in \mathsf{S}$ and $\mathsf{L}X \in \mathsf{S}^{\perp}$, where

$$\mathsf{S}^{\perp} = \{\, Y \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R}) \mid \operatorname{Hom}_{\mathsf{K}}(Z, Y) = 0 \text{ for all } Z \in \mathsf{S}\}.$$

This is a form of *Bousfield localization*; see [Nee92, 1.7] for a proof.

## 3. Cohomology operators

Throughout this section $S = \oplus_{i \geq 0} S^i$ denotes a commutative graded ring .

**3.1.** We say $S$ is a *ring of cohomology operators* for $R$ if for every $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is a map of graded rings

$$\zeta_X : S \to \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X)$$

such that the two $S$-module structures on $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y)$ via $\zeta_X$ and $\zeta_Y$ agree. Thus for each $\alpha \in \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y)$, and all homogeneous $s \in S$, we require

$$\zeta_Y(s) \cdot \alpha = (-1)^{|s|} \alpha \cdot \zeta_X(s). \tag{3.1.1}$$

We say $R$ has *Noetherian cohomology* over $S$ if $S$ is Noetherian, has finite Krull dimension, and $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C, C)$ is a Noetherian $S$-module for all compact objects $C$ in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$.

**Remark 3.2.** A ring of cohomology operators for $R$ has been defined previously in [AI07] to be maps $S \to \operatorname{Ext}^*_R(M, M)$ for all complexes $M$ that satisfy the relations (3.1.1) for all $M, N$ in $\mathsf{D}(R)$. If $S$ is a ring of cohomology operators for $R$ in our sense and $M$ is a complex of $R$-modules, then by setting $X := \mathsf{i}M$ to be a semi-injective resolution of $M$ there is a graded ring map

$$\zeta'_M : S \to \operatorname{Ext}^*_R(M, M) \cong \operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}M, \mathsf{i}M)$$

and this family of maps evidently satisfies the relations 3.1.1. Thus a ring of cohomology operators as defined here is a ring of cohomology operators as defined in [AI07].

In the rest of the section we assume that $S$ is Noetherian, has finite Krull dimension, and is a ring of cohomology operators on $R$. We set $S^+ = \oplus_{i \geq 1} S^i$.

We will need the following result on the structure of a ring with Noetherian cohomology:

**3.3.** Assume $R$ has Noetherian cohomology over $S$. Then the following hold:

(1) $R$ is left-Noetherian;
(2) $\operatorname{inj\,dim}_R R < \infty$;
(3) An $R$-complex with finite cohomology $M$ has finite projective dimension if and only if $\operatorname{Ext}^n_R(M, M) = 0$ for all $n \gg 0$ if and only if $M$ has finite injective dimension.

This is contained in [AI], where less assumptions are placed on $S$. For all the rings in Section 5 to which we apply the Theorem, the properties above are well-known.

The following construction was introduced in [BIK08].

**3.4.** Let $s$ be a homogeneous element of $S$ of degree $n$ and let $X$ be an object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. The *Koszul object* of $s$ on $X$, denoted $X /\!\!/ s$, is the mapping cone of $\zeta_X(s) \in \operatorname{Hom}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, \Sigma^n X)$. Thus there is an exact triangle

$$X \xrightarrow{\zeta_X(s)} \Sigma^n X \to X /\!\!/ s \to, \tag{3.4.1}$$

and $X/\!\!/s$ is unique up to isomorphism. For $\mathsf{s} = s_1, \ldots, s_r$ a sequence of homogeneous elements of $S$, the Koszul object of $\mathsf{s}$ on $X$, denoted $X/\!\!/\mathsf{s}$, is defined inductively as the Koszul object of $s_r$ on $X/\!\!/(s_1, \ldots, s_{r-1})$.

Let $Y$ be another object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. We need the following properties of Koszul objects:

(1) If $X$ is compact then so is $X/\!\!/\mathsf{s}$; this follows by induction and the triangle (3.4.1) above.
(2) There exists an integer $n \geq 0$, independent of $X$ and $Y$, such that

$$(\mathsf{s})^n \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(Y, X/\!\!/\mathsf{s}) = 0 = (\mathsf{s})^n \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X/\!\!/\mathsf{s}, Y).$$

(3) If $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X/\!\!/\mathsf{s}, Y) = 0$ and the $S$-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y)$ is $\mathsf{s}$-torsion then

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, Y) = 0.$$

The last two results are contained in [BIK08, 5.11].

The next result shows that every compact object of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ can be cut down to an object with finite projective dimension using the above construction.

**Proposition 3.5.** *Assume $R$ has Noetherian cohomology over $S$. Let $\mathsf{s} = s_1, \ldots, s_r$ be a set of generators of the ideal $S^+ = \oplus_{i>0} S^i$ and let $\mathsf{i}R \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ be an injective resolution of $R$. For every compact object $C$ of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ the object $C/\!\!/\mathsf{s}$ is in $\mathsf{thick}_{\mathsf{K}}(\mathsf{i}R)$. In particular there is an inclusion of subcategories:*

$$\mathsf{thick}_{\mathsf{K}}(C/\!\!/\mathsf{s} \,|\, C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}}) \subseteq \mathsf{thick}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}R).$$

*Proof.* By 3.4(2) there exists $n \geq 1$ such that $(\mathsf{s})^n \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}) = 0$. Since $C/\!\!/\mathsf{s}$ is compact, the $S$-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s})$ is finitely generated by the definition of Noetherian cohomology. A standard argument now shows that

$$\operatorname{Hom}^m_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}) = 0 \text{ for } m \gg 0. \tag{3.5.1}$$

Since $C/\!\!/\mathsf{s}$ is compact, by 2.6, the complex $C/\!\!/\mathsf{s}$ is semi-injective. Thus

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}) \cong \operatorname{Ext}^*_R(C/\!\!/\mathsf{s}, C/\!\!/\mathsf{s}).$$

Now 3.5.1 and 3.3(3) show that $C/\!\!/\mathsf{s}$ has finite projective dimension. One checks, by induction on projective dimension for instance, that this implies that $C/\!\!/\mathsf{s} \in \mathsf{thick}_{\mathsf{D}(R)}(R)$. Since triangulated functors preserve thick subcategories we have that

$$Q_\rho(C/\!\!/\mathsf{s}) \in \mathsf{thick}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(Q_\rho R).$$

As semi-injective resolutions are unique in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ and $C/\!\!/\mathsf{s}$ and $Q_\rho(C/\!\!/\mathsf{s})$ are semi-injective, we have that $Q_\rho(C/\!\!/\mathsf{s}) \cong C/\!\!/\mathsf{s}$ and $Q_\rho R \cong \mathsf{i}R$. Stringing together the above shows that $C/\!\!/\mathsf{s}$ is in $\mathsf{thick}_{\mathsf{K}}(\mathsf{i}R)$. □

Since every localizing subcategory in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ is thick there is a containment of subcategories $\mathsf{thick}_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}R) \subseteq \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$. This immediately implies the following

**Corollary 3.6.** *Under the assumptions of 3.5, there is an inclusion of subcategories*

$$\mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \,|\, C \in \mathsf{K}^{\mathsf{c}}) \subseteq \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R). \quad \square$$

In 4.4 we show that the above is actually an equality.

## 4. Finite Injective Dimension

In this section we prove the Theorem in the introduction. To do this we need the following:

**Proposition 4.1.** *Let $R$ be a left-Noetherian ring which has finite injective dimension as a left $R$-module and $M$ be an $R$-complex with $H^n(M) = 0$ for $n \gg 0$. Let $\mathsf{i}R$ and $\mathsf{i}M$ be semi-injective resolutions of $R$ and $M$ respectively. If $\mathsf{i}M$ is in $\mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$, then $M$ has finite injective dimension.*

*Proof.* The hypotheses guarantee that $M$ has finite Gorenstein injective dimension, i.e. there exists a short exact sequence of complexes

$$0 \to L \to (\mathsf{i}M)' \xrightarrow{v} T \to 0$$

where $(\mathsf{i}M)'$ is a semi-injective resolution of $M$, $T$ is an acyclic complex of injective modules, and $L$ is such that $L^n = 0$ for all $n \gg 0$. See Theorem 3.2, and Definitions 2.1 and 2.2 of [AS06]. We have isomorphisms

$$\operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}R, T) \cong \operatorname{Hom}^*_{\mathsf{K}(R)}(R, T) \cong H^*(T) = 0.$$

The first is [Kra05, 2.1], the second is clear, and the third is the fact that $T$ is acyclic.

Note that since semi-injective resolutions are unique up to isomorphism in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$, we have that $(\mathsf{i}M)' \cong \mathsf{i}M \in \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$. Thus we may assume that $\mathsf{i}M = (\mathsf{i}M)'$.

The class of complexes

$$\{\, X \mid \operatorname{Hom}^*_{\mathsf{K}}(X, T) = 0 \text{ is localizing } \}$$

is a localizing subcategory of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. Since $\mathsf{i}R$ is in this class so is the subcategory $\mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$. In particular $\mathsf{i}M \in \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$, and thus $\operatorname{Hom}^*_{\mathsf{K}}(\mathsf{i}M, T) = 0$. This shows that the map $v$ above is nulhomotopic. We will show that this forces $\mathsf{i}M$ to have an injective cokernel in a high degree.

Since $v$ is nullhomotopic there exists a map $s : \mathsf{i}M \to T$ such that $\partial s + s\partial = v$. Let $k$ be an integer such that $L^n = 0$ for all $n \geq k$, which exists by assumption. Thus $v^n$ is bijective for all $n \geq k$ and we have that $(v^n)^{-1}\partial s + (v^n)^{-1} s\partial = 1_{\mathsf{i}M^n}$. One checks that $v^{-1}$ commutes with the differentials in the degrees for which it is defined; this gives

$$\partial (v^{n-1})^{-1} s + (v^n)^{-1} s\partial = 1_{\mathsf{i}M^n}.$$

Thus $v^{-1}s$ is a contracting homotopy of $1_{\mathsf{i}M}$ in high degrees. A simple diagram chase now shows that $\operatorname{Im}(\partial^k)$ splits as a submodule of $(\mathsf{i}M)^{k+1}$ and hence is injective.

Since $v$ is a bijection in degrees $n \geq k$ and $T$ is acylic, this implies that $H^n(\mathsf{i}M) = 0$ for $n \geq k$. Thus $\mathsf{i}M$ has an injective cokernel in a degree higher than its last nonzero cohomology; by [AF91, 2.4.I] this implies that $M$ has finite injective dimension. □

**Theorem 4.2.** *Let $R$ be an associative ring and $S$ a Noetherian graded ring of finite Krull dimension. Assume that $S$ is a ring of cohomology operators on $R$ and that $R$ has Noetherian cohomology over $S$. For an $R$-complex $M$ with $H^n(M) = 0$ for $n \gg 0$, if the $S$-module* $\operatorname{Ext}^*_R(M, M)$ *is $S^+ = \oplus_{i\geq 1} S^i$-torsion, then $M$ has finite injective dimension.*

*Proof.* Let $X = \mathsf{i}M$ be a semi-injective resolution of $M$. Then

$$\operatorname{Ext}^*_R(M, M) \cong \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X).$$

Let $\mathsf{s}$ be a finite set of generators of the ideal $S^+$. By 3.3 $R$ is left-Noetherian and has finite injective dimension over itself. Thus by 4.1 it is enough to show that $\mathsf{i}M \in \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R)$. Set

$$\mathsf{C} = \mathsf{loc}_{\mathsf{K}}(C /\!\!/ \mathsf{s} \mid C \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})^{\mathsf{c}}).$$

By Corollary 3.6 it is enough to show that $X \in \mathsf{C}$.

Fix a compact object $D$. By hypothesis $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X)$ is $S^+$-torsion. By the definition of cohomology operators, the action of $S$ on $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, X)$ factors through $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X)$ and hence $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, X)$ is also $S^+$-torsion.

Now consider the full subcategory $\mathsf{T}$ of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ with objects those $Z \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ such that $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, Z)$ is $S^+$-torsion. It is closed under suspension; given a triangle $Y \to Z \to W \to \Sigma Y$ in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is an exact sequence of $S$-modules:

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, Y) \to \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, Z) \to \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, W).$$

From this we see that if $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, Y)$ and $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, W)$ are $S^+$-torsion then $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, Z)$ is as well. This shows that $\mathsf{T}$ is triangulated. For a family of objects $\{Z_i\}$ in $\mathsf{T}$, we have that

$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, \bigoplus_i Z_i) \cong \bigoplus_i \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(D, Z_i)$$

since $D$ is compact. Thus $\mathsf{T}$ is closed under direct sums and hence is localizing. By 3.4(2), for every object $C$ the module $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, C/\!\!/\mathsf{s})$ is $S^+$-torsion. Thus

$$\mathsf{C} = \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}}) \subseteq \mathsf{T}$$

since $\mathsf{T}$ is localizing and every object $C/\!\!/\mathsf{s}$ is in $\mathsf{T}$.

Since $\mathsf{C}$ is compactly generated there is a triangle

$$\Gamma X \to X \to \mathsf{L}X \to \tag{4.2.1}$$

with $\Gamma X \in \mathsf{C}$ and $\mathsf{L}X \in \mathsf{C}^{\perp}$; see 2.7.

We have that $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, \Gamma X)$ is $S^+$-torsion since $\Gamma X \in \mathsf{C} \subseteq \mathsf{T}$. We have shown above that $X \in \mathsf{T}$. Thus $\mathsf{L}X \in \mathsf{T}$ since $\mathsf{T}$ is triangulated. By definition this means $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, \mathsf{L}X)$ is $S^+$-torsion. Since $D/\!\!/\mathsf{s} \in \mathsf{C}$ and $\mathsf{L}X \in \mathsf{C}^{\perp}$, we have that

$$\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D/\!\!/\mathsf{s}, \mathsf{L}X) = 0.$$

By 3.4(3) this implies that $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(D, \mathsf{L}X) = 0$. Since $D$ was an arbitrary compact object and $\mathsf{K(Inj\,R)}$ is compactly generated, see 2.3, this shows that $\mathsf{L}X = 0$. By the triangle (4.2.1) we see that $\Gamma X \cong X$ and hence $X$ is an object of $\mathsf{C} = \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}})$. □

**Remark 4.3.** The hypothesis that $H^n(M) = 0$ for $n \gg 0$ are necessary. Indeed, from the definition 2.4, one can see that if a complex $M$ has finite injective dimension then $H^n(M) = 0$ for $n \gg 0$.

**Corollary 4.4.** *Under the assumptions and notation of Theorem 4.2, there is an equality*

$$\mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}}) = \mathsf{loc}_{\mathsf{K}}(\mathsf{i}R).$$

*Proof.* Since the $S$-module $\mathrm{Hom}^*_{\mathsf{K(Inj\,R)}}(\mathsf{i}R, \mathsf{i}R) \cong \mathrm{Ext}^*_R(R, R)$ is clearly $S^+$-torsion, the proof of Theorem 4.2 above shows that $\mathsf{i}R \in \mathsf{loc}_{\mathsf{K}}(C/\!\!/\mathsf{s} \mid C \in \mathsf{K(Inj\,R)}^{\mathsf{c}})$. This gives the opposite inclusion of 3.6 and hence it is equality. □

**Corollary 4.5.** *Let $R, S$ and $M$ be as in 4.2. Let $s_1, \dots, s_r$ be a finite set of homogeneous generators of the ideal $S^+$. Set*

$$d := \max\{\deg s_i \mid 1 \le i \le r\} \text{ and } l := \mathrm{lcm}\{\deg s_i \mid 1 \le i \le r\}.$$

*Then* $\mathrm{inj\,dim}_R M < \infty$ *if one of the following holds:*

(1) *there exists an integer $n \geq 0$ such that $\mathrm{Ext}^j_R(M, M) = 0$ for all $n \le j \le n + d - 1$; or*
(2) *there exists an integer $m \ge 0$ such that $\mathrm{Ext}^{ml}_R(M, M) = 0$.*

*Proof.* Either condition forces the $S$-module $\mathrm{Ext}^*_R(M, M)$ to be $S^+$-torsion. Indeed, assume that there exists an integer $n$ such that (1) holds. For every $i$ there exists an integer $k_i$ such that

$$n \le k_i(\deg s_i) \le n + d - 1.$$

One way to see this is by induction on $n$. Consider the ideal $(S^+)^{k_1+\dots+k_r} = (s_1, \dots, s_r)^{k_1+\dots+k_r}$ in $S$. It is generated by monomials in the $s_i$ of the form $s_1^{n_1} \dots s_r^{n_r}$ for positive integers $n_i$ with $\sum n_i = \sum k_i$. For each such monomial there exists an $i$ such that $n_i \ge k_i$, else $\sum n_i < \sum k_i$; applying $\eta_M$ to the monomial, and using that $\eta_M$ is a map of rings, we see that

$$\begin{aligned}\eta_M(s_1^{n_1} \dots s_r^{n_r}) &= \eta_M(s_1^{n_1}) \dots \eta_M(s_i^{n_i}) \dots \eta_M(s_r^{n_r}) \\ &= \eta_M(s_1^{n_1}) \dots \eta_M(s_i^{k_i}) \eta_M(s_i^{n_i - k_i}) \dots \eta_M(s_r^{n_r}) = 0\end{aligned}$$

since $\eta_M(s_i^{k_i}) \in \mathrm{Ext}^{k_i(\deg s_i)}_R(M, M) = 0$. Thus

$$(S^+)^{k_1+\dots+k_r} \mathrm{Ext}^*_R(M, M) = \eta_M((S^+)^{k_1+\dots+k_r}) \mathrm{Ext}^*_R(M, M) = 0$$

and hence $\mathrm{Ext}^*_R(M, M)$ is $S^+$-torsion. By Theorem 4.2 this shows that $\mathrm{inj\,dim}_R M < \infty$.

To prove (2) assume that such an $m$ exists. For every $i = 1, \dots, r$, there exists an integer $d_i$ such that $d_i(\deg s_i) = l$. Letting $\alpha = m(\sum_i d_i)$, a similar proof as above shows that $(s_1, \dots, s_r)^{\alpha} \mathrm{Ext}^*_R(M, M) = 0$. □

## 5. Applications

In this section we apply Theorem 4.2 in the two contexts discussed in the introduction.

**5.1.** Let $R$ be a commutative ring with a presentation
$$R \cong Q/(\mathbf{f}),$$
where $Q$ is a commutative Noetherian regular ring of finite Krull dimension and $(\mathbf{f}) = (f_1, \dots, f_c)$ is a $Q$-regular sequence.

Let $S = R[\chi_1, \dots, \chi_c]$ be the polynomial ring in $c$ indeterminates over $R$, graded by setting $|\chi_i| = 2$. For every $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ there is a homomorphism of graded $R$-algebras
$$\zeta_X : S \to \operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X).$$
When $X = \mathsf{i}M$ is the injective resolution of a finitely generated $R$-module $M$, so that
$$\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(X, X) \cong \operatorname{Ext}^*_R(M, M),$$
such a map $\zeta_X$ may be constructed as in [Eis80, Section 1] using a free resolution of $M$. The process described in [Avr89, Section 1], which replaces free resolutions with injective resolutions, generalizes to arbitrary objects of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. The results of *loc. cit.* show that the maps $\zeta_X$ satisfy the conditions of a ring of cohomology operators.

By [AS98, 5.1] the $S$-module $\operatorname{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}M, \mathsf{i}M) \cong \operatorname{Ext}^*_R(M, M)$ is finitely generated when $M$ has finite cohomology over $R$. This was proved first by Gulliksen [Gul74] for modules. It follows 2.6 that $R$ has Noetherian cohomology over $S$. Restating Theorem 4.2 in this context, we have:

**Corollary 5.2.** *Let $Q$ be a commutative Noetherian regular ring of finite Krull dimension, $(\mathbf{f}) = (f_1, \dots, f_c)$ a $Q$-regular sequence and $R = Q/(\mathbf{f})$. For an $R$-complex $M$ with $H^n(M) = 0$ for all $n \gg 0$, if $\operatorname{Ext}^*_R(M, M)$ is $S^+$-torsion, then $M$ has finite injective dimension.* □

In the notation of Corollary 4.5 we see that $d = 2 = l$. Since $R$ is a Gorenstein ring of finite Krull dimension, a module has finite projective dimension if and only if it has finite injective dimension. This gives:

**Corollary 5.3.** *If $M$ is an arbitrary $R$-module such that $\operatorname{Ext}^{2n}_R(M, M) = 0$ for some $n \geq 1$ then $M$ has finite projective dimension.* □

**Remark 5.4.** In [AB00, 4.2] the same statement is proved for finitely generated modules of finite complete intersection dimension over a Noetherian ring. All finitely generated modules over the ring $R$ have finite complete intersection dimension. However, complete intersection dimension is not defined for non-finitely generated modules, so we have not generalized completely [AB00, 4.2].

**5.5.** Let $k$ be a field and $R$ a $k$-algebra. We set $R^{\mathsf{e}} = R \otimes_k R^{\mathsf{op}}$, where $R^{\mathsf{op}}$ is the opposite ring of $R$. Note that $R$ is naturally a left $R^{\mathsf{e}}$-module. The *Hochschild cohomology* of $R$ over $k$ is
$$\operatorname{HH}^*(R \,|\, k) = \operatorname{Ext}^*_{R^{\mathsf{e}}}(R, R).$$
Hochschild cohomology is a graded-commutative ring under the cup-product [Ger63, Section 7, Corollary 1], where recall that a graded-ring $\mathcal{T} = \bigoplus_i \mathcal{T}_i$ is *graded-commutative* if $rs = (-1)^{|r||s|} sr$ for all homogeneous elements $r, s$.

Any commutative graded subring $S$ of $\operatorname{HH}^*(R \,|\, k)$ is a ring of cohomology operators on $R$. To see this, let $\gamma : F \to R$ be a free resolution of $R$ over $R^{\mathsf{e}}$. Since $R^{\mathsf{e}}$ is a free $R$-module, $\gamma$ is a homotopy equivalence of chain complexes over $R$. Given an element
$$h \in S \subseteq \operatorname{HH}^n(R|k) = \operatorname{Ext}^n_{R^{\mathsf{e}}}(R, R) \cong H^n \operatorname{Hom}_R(F, F),$$
let $\tilde{h} : F \to \Sigma^n F$ be a morphism of complexes that represents $h$. For any complex $X$ in $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$, we define $\zeta_X(h) : X \to \Sigma^n X$ as the composition of

$$X \cong X \otimes_R R \xrightarrow{1 \otimes \gamma^{-1}} X \otimes_R F \xrightarrow{1 \otimes \tilde{h}} X \otimes_R \Sigma^n F \xrightarrow{1 \otimes \Sigma^n(\gamma)} X \otimes_R \Sigma^n R \cong \Sigma^n X. \qquad (5.5.1)$$

One can check that this is independent of the choice of $\tilde{h}$ and makes $S$ into a ring of cohomology operators on $R$.

The following is a restatement of Theorem 4.2 in this context.

**Corollary 5.6.** *Let $k$ be a field and $R$ be a $k$-algebra. Assume there exists a commutative Noetherian graded subring $S$ of $\mathrm{HH}^*(R\,|\,k)$ over which $R$ has Noetherian cohomology. For an $R$-complex $M$ with $H^n(M) = 0$ for all $n \gg 0$, if $\mathrm{Ext}^*_R(M, M)$ is $S^+$-torsion, then $M$ has finite injective dimension.* □

When $R$ is finite dimensional over $k$, to show that $R$ has Noetherian cohomology it is enough to check one module:

**Corollary 5.7.** *Let $R$ be a finite dimensional $k$-algebra and $S$ a graded Noetherian subring of $\mathrm{HH}^*(R\,|\,k)$. Let $\mathfrak{r}$ be the Jacobson radical of $R$, and assume that $\mathrm{Ext}^*_R(R/\mathfrak{r}, R/\mathfrak{r})$ is a finitely generated $S$-module via $\zeta_{R/\mathfrak{r}}$. For an $R$-complex $M$ with $H^n(M) = 0$ for all $n \gg 0$, if $\mathrm{Ext}^*_R(M, M)$ is $S^+$-torsion, then $M$ has finite injective dimension.*

*Proof.* Since $\mathrm{Ext}^*_R(R/\mathfrak{r}, R/\mathfrak{r})$ is a Noetherian $S$-module via $\zeta_{R/\mathfrak{r}}$,

$$\mathrm{Ext}^*_R(M, M) \cong \mathrm{Hom}^*_{\mathsf{K}(\mathsf{Inj}\,\mathsf{R})}(\mathsf{i}M, \mathsf{i}M)$$

is a Noetherian $S$-module via $\zeta_M$ for all $M$ with finite cohomology by [Sol06, 10.3]. By the description of the compact objects of $\mathsf{K}(\mathsf{Inj}\,\mathsf{R})$, see 2.6, $R$ has Noetherian cohomology over $S$ and Theorem 5.6 applies. □

The following result generalizes [EHT+04, 2.5.b] from finitely generated to arbitrary modules.

**Corollary 5.8.** *Let $R$ be as above and $M$ an $R$-module. There exists an integer $l \geq 1$ such that if $\mathrm{Ext}^{nl}_R(M, M) = 0$ for some $n \geq 1$ then $M$ has finite projective dimension.*

*Proof.* The first statement follows from Corollary 4.5.(2). For the second, by [EHT+04, Theorem 2.5.a] $R$ is self-injective. Thus a module has finite projective dimension if and only if it has finite injective dimension. □

The following is one more context in which the Theorem applies.

**5.9.** Let $R$ be a Hopf algebra over a field $k$. For two $R$-modules $M, N$ we view $M \otimes_k N$ as an $R$-module via the diagonal map $\Delta : R \to R \otimes_k R$. When $M, N$ are injective then so is $M \otimes_k N$. For $X \in \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$ the functor $- \otimes_k X$ preserves homotopies of maps. Thus there is a functor $- \otimes_k X : \mathsf{K}(\mathsf{Inj}\,\mathsf{R}) \to \mathsf{K}(\mathsf{Inj}\,\mathsf{R})$. Viewing $k$ as an $R$-module via the augmentation there is an isomorphism

$$\varphi_X : \mathsf{i}k \otimes_k X \xrightarrow{\cong} X,$$

see [BK08, 5.3] which proof holds in our more general situation. Thus for each $X$ one gets a map

$$\eta_X : \ \mathrm{Hom}^*_{\mathsf{K}}(\mathsf{i}k, \mathsf{i}k) \to \mathrm{Hom}^*_{\mathsf{K}}(X, X)$$

which sends $\alpha : \mathsf{i}k \to \Sigma^n \mathsf{i}k$ to

$$\varphi_{\Sigma^n X}(\alpha \otimes_k X)(\varphi_X)^{-1} : X \to \Sigma^n X.$$

One can check that $\eta_X$ is a ring map. Let $S$ be the ring $\mathrm{Ext}^*_R(k, k) \cong \mathrm{Hom}^*_{\mathsf{K}}(\mathsf{i}k, \mathsf{i}k)$. By [ML63, (VIII.4.7), (VIII.4.3)] the ring $S$ is graded-commutative and the maps $\eta_X$ satisfy the commutativity relations (3.1.1). Thus setting

$$S^{\mathrm{even}} := \begin{cases} \bigoplus_{i \geq 0} \mathrm{Ext}^{2i}_R(k, k) & \text{if } \operatorname{char} k \neq 2 \\ \mathrm{Ext}^*_R(k, k) & \text{if } \operatorname{char} k = 2 \end{cases}$$

we see that $S^{\mathrm{even}}$ is commutative and is a ring of cohomology operators on $R$.

By the main result of [FS97], when $R$ is cocommutative and finite dimensional over $k$, the ring $S$ is Noetherian and $\mathrm{Ext}^*_R(M, N)$ is a Noetherian $S$-module (via $\eta_M$, or equivalently, $\eta_N$) for all complexes $M, N$ with finite cohomology. The ideal of odd degree elements in $S$ is nilpotent when $\operatorname{char} k \neq 2$. Thus when $R$ is a cocommutative finite dimensional Hopf algebra it has Noetherian cohomology over $S^{\text{even}}$.

Specializing Theorem 4.2 and Corollary 4.5 to this context, and using that $R$ is self-injective, we have:

**Corollary 5.10.** *Let $R$ be a finite dimensional cocommutative Hopf algebra and $S^{\text{even}}$ the commutative ring defined as above. For an $R$-complex $M$ with $H^n(M) = 0$ for all $n \gg 0$, if $\operatorname{Ext}^*_R(M, M)$ is $S^+$-torsion, then $M$ has finite injective dimension.*

**Corollary 5.11.** *Let $R$ be as above and $M$ an $R$-module. There exists an integer $l$ such that if $\operatorname{Ext}^{nl}_R(M, M) = 0$ for some $n \geq 1$ then $M$ has finite projective dimension.*

**Remark 5.12.** There is a map $S = \operatorname{Ext}^*_R(k, k) \to \operatorname{HH}^*(R \,|\, k)$ by which $\eta_M : S \to \operatorname{Ext}^*_R(M, M)$ factors through $\zeta_M : \operatorname{HH}^*(R \,|\, k) \to \operatorname{Ext}^*_R(M, M)$. Using this one could apply the previous subsection to the case of a cocommutative Hopf algebra. However due to the different module structures involved, verifying this commutativity is delicate and thus we have opted for a more direct approach. See [PW09, Appendix] for more details on this issue.

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Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany.
*E-mail address*: `jburke@math.uni-bielefeld.de`