MORPHISMS DETERMINED BY OBJECTS IN TRIANGULATED CATEGORIES

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Abstract. The concept of a *morphism determined by an object* provides a method to construct or classify morphisms in a fixed category. We show that this works particularly well for triangulated categories having Serre duality. Another application of this concept arises from a reformulation of Freyd's generating hypothesis.

Contents

1. Introduction

Given a category, one may ask for a classification of all morphisms ending in a fixed object, where two such morphisms $\alpha_i\colon X_i \to Y$ $(i = 1, 2)$ are *isomorphic* if there exists an isomorphism $\phi\colon X_1 \to X_2$ such that $\alpha_1 = \alpha_2\phi$. In this note we dicuss an approach which is based on the concept of a *morphism determined by an object.* Originally, this concept was introduced by Auslander [1] in order to give a conceptual explanation for the existence of left and right almost split morphisms introduced before in joint work with Reiten [4]. Here, we show that some of Auslander's results have an analogue for triangulated categories. In a somewhat different direction, we reformulate a conjecture of Freyd from stable homotopy theory in terms of morphisms determined by objects.

2. Morphisms determined by objects: Auslander's work

We give a quick review of Auslander's work on morphisms determined by objects. This leads then to a precise formulation of the classification problem for morphisms in terms of morphisms determined by objects.

Definition 2.1 (Auslander [1])**.** A morphism $\alpha\colon X \to Y$ in some fixed category is said to be *right determined* by an object C if for every morphism $\alpha'\colon X' \to Y$ the following conditions are equivalent:

(1) The morphism α' factors through α.
(2) For every morphism $\phi\colon C \to X'$ the composite $\alpha'\phi$ factors through α.

A morphism is *left determined* by C if it is right determined by C when viewed as morphism in the opposite category.

Fix a morphism $\alpha\colon X \to Y$. We denote by $\operatorname{Im}\operatorname{Hom}(C,\alpha)$ the image of the induced morphism $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$ and observe that condition (2) means

$$\operatorname{Im}\operatorname{Hom}(C,\alpha') \subseteq \operatorname{Im}\operatorname{Hom}(C,\alpha).$$

The morphism α is called *right minimal* if every morphism $\phi\colon X \to X$ satisfying $\alpha = \alpha\phi$ is an isomorphism.

The following elementary observation yields a reformulation of the classification problem for morphisms ending in a fixed object.

Lemma 2.2. *Let $\alpha_i\colon X_i \to Y$ $(i = 1, 2)$ be morphisms that are right minimal and right C-determined. Then α_1 and α_2 are isomorphic if and only if* $\operatorname{Im}\operatorname{Hom}(C,\alpha_1) = \operatorname{Im}\operatorname{Hom}(C,\alpha_2)$. $\square$

Right almost split morphisms provide an important class of examples of right determined morphisms.

Example 2.3 (Auslander [1, §II.2])**.** A morphism $\alpha\colon X \to Y$ in some additive category is *right almost split* (that is, α is not a retraction and every morphism $X' \to Y$ that is not a retraction factors through α) if and only if $\Gamma = \operatorname{End}(Y)$ is a local ring, α is right determined by Y, and $\operatorname{Im}\operatorname{Hom}(Y,\alpha) = \operatorname{rad}\Gamma$.

Let us recall two of the main results from [1, 2]. Fix a ring Λ and denote by $\mathsf{Mod}\,\Lambda$ the category of Λ-modules. The full subcategory formed by all finitely presented Λ-modules is denoted by $\mathsf{mod}\,\Lambda$.

Theorem 2.4 (Auslander [1, Theorem I.3.19])**.** *Let C and Y be Λ-modules and suppose that C is finitely presented. Given an $\operatorname{End}_\Lambda(C)$-submodule $H \subseteq \operatorname{Hom}_\Lambda(C,Y)$, there exists, up to isomorphism, a unique right minimal morphism $\alpha\colon X \to Y$ in* $\mathsf{Mod}\,\Lambda$ *which is right C-determined and satisfies* $\operatorname{Im}\operatorname{Hom}_\Lambda(C,\alpha) = H$. $\square$

Theorem 2.5 (Auslander [2, Theorem 2.6])**.** *Suppose that Λ is an Artin algebra and let $\alpha\colon X \to Y$ be a morphism between finitely presented Λ-modules. Denote by $\operatorname{Tr} D(\operatorname{Ker}\alpha)$ the transpose of the dual of the kernel of α and by $P(\operatorname{Coker}\alpha)$ a projective cover of the cokernel of α. Then α is right determined as a morphism in* $\mathsf{mod}\,\Lambda$ *by* $\operatorname{Tr} D(\operatorname{Ker}\alpha) \amalg P(\operatorname{Coker}\alpha)$.[1] $\square$

Note that the proofs of both theorems are based on the *Auslander–Reiten formula*

$$D\,\underline{\operatorname{Hom}}_\Lambda(C,-) \cong \operatorname{Ext}^1_\Lambda(-, D\operatorname{Tr} C),$$

where C is supposed to be a finitely presented Λ-module and $D = \operatorname{Hom}_\Gamma(-, I)$ with $\Gamma = \operatorname{End}_\Lambda(C)$ and I an injective Γ-module; see [1, Proposition I.3.4]. For instance, a C-determined epimorphism corresponding to a Γ-submodule $H \subseteq \operatorname{Hom}_\Lambda(C,Y)$ is obtained by choosing an injective envelope $\underline{\operatorname{Hom}}_\Lambda(C,Y)/H \to I$ over Γ and taking the morphism ending at Y from the corresponding extension $0 \to D\operatorname{Tr} C \to X \to Y \to 0$. A modification of this construction takes care of morphisms that are not epimorphisms.

In the following section, we move from module categories to dualising varieties and obtain similar results using an analogue of the Auslander–Reiten formula.

Motivated by Auslander's results, the classification problem for morphisms ending in a fixed object may be formulated more precisely in terms of the following definition.

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[1] This result is not correct as stated; the term $P(\operatorname{Coker}\alpha)$ needs to be modified, as pointed out by Ringel in [11].

Definition 2.6. An additive category C is said *to have right determined morphisms* if for every object $Y \in \mathsf{C}$ the following holds:

(1) Given an object $C \in \mathsf{C}$ and an $\operatorname{End}_{\mathsf{C}}(C)$-submodule $H \subseteq \operatorname{Hom}_{\mathsf{C}}(C, Y)$, there is a right C-determined morphism $\alpha\colon X \to Y$ with $\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(C, \alpha) = H$.
(2) Every morphism ending in Y is right determined by an object in C.

In categorical terms, this definition formulates properties of the *slice category* C/Y over a fixed object $Y \in \mathsf{C}$. Let us illustrate this by looking at categories having few morphisms.

Example 2.7. Let C be a partially ordered set, viewed as a category, and fix a morphism $\alpha\colon x \to y$, which means that $x \leq y$. If $x = y$, then α is right determined by every object of C. If $x \neq y$, then α is right determined by an object $c \in \mathsf{C}$ if and only if there exists a unique minimal element in

$$\mathsf{C}_\alpha = \{c \in \mathsf{C} \mid c \not\leq x, c \leq y\}.$$

In that case $c = \inf \mathsf{C}_\alpha$. Thus in $(\mathbb{Z}, \leq)$ all morphisms are determined by objects, while in $(\mathbb{Q}, \leq)$ only identity morphisms are determined by some object.

3. Dualising varieties

Dualising varieties were introduced by Auslander and Reiten in [3]. They form a convenient setting for studying morphisms determined by objects. We will see that such categories have right determined morphisms in the sense of Definition 2.6.

Throughout this work k denotes a commutative artinian ring with radical $\mathfrak{r}$. We fix a k-linear additive category C which is *Hom-finite*, that is, the k-module $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ has finite length for all objects X, Y in C. Suppose also that C is essentially small and idempotent complete.

Dualising varieties. Let $\mathsf{mod}\, k$ denote the category of finitely presented k-modules and fix an injective envelope $E = E(k/\mathfrak{r})$ over k. This provides the duality

$$D = \operatorname{Hom}_k(-, E)\colon \ \mathsf{mod}\, k \longrightarrow \mathsf{mod}\, k.$$

Denote by $(\mathsf{C}, \mathsf{mod}\, k)$ the category of k-linear functors $\mathsf{C} \to \mathsf{mod}\, k$. The basic tools are the fully faithful *Yoneda functor*

$$\mathsf{C} \longrightarrow (\mathsf{C}^{\mathrm{op}}, \mathsf{mod}\, k), \quad X \mapsto \operatorname{Hom}_{\mathsf{C}}(-, X),$$

and the duality

$$(\mathsf{C}, \mathsf{mod}\, k)^{\mathrm{op}} \xrightarrow{\sim} (\mathsf{C}^{\mathrm{op}}, \mathsf{mod}\, k), \quad F \mapsto DF.$$

Recall that an additive functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{mod}\, k$ is *finitely presented* if it fits into an exact sequence

$$\operatorname{Hom}_{\mathsf{C}}(-, X) \longrightarrow \operatorname{Hom}_{\mathsf{C}}(-, Y) \longrightarrow F \longrightarrow 0.$$

We denote by $\mathsf{mod}\, \mathsf{C}$ the full subcategory of $(\mathsf{C}^{\mathrm{op}}, \mathsf{mod}\, k)$ formed by all finitely presented functors.

Definition 3.1 (Auslander–Reiten [3])**.** A k-linear additive Hom-finite essentially small and idempotent complete category C is called *dualising k-variety* if the assignment $F \mapsto DF$ induces an equivalence

$$(\mathsf{mod}\, \mathsf{C})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{mod}(\mathsf{C}^{\mathrm{op}}).$$

A morphism $X \to Y$ is a *weak kernel* of a morphism $Y \to Z$ in C if it induces an exact sequence

$$\operatorname{Hom}_{\mathsf{C}}(-,X) \longrightarrow \operatorname{Hom}_{\mathsf{C}}(-,Y) \longrightarrow \operatorname{Hom}_{\mathsf{C}}(-,Z).$$

A *weak cokernel* is defined analogously. The following lemma is well-known and easily proved.

Lemma 3.2. *The category* $\operatorname{\mathsf{mod}} \mathsf{C}$ *is an additive category with cokernels; it is abelian if and only if* C *has weak kernels.* □

This yields the following reformulation of the definition of a dualising variety.

Lemma 3.3. *Let* C *be a* k*-linear additive Hom-finite essentially small and idempotent complete category. Then* C *is a dualising* k*-variety if and only if the following holds:*

1. *The category* C *has weak kernels and weak cokernels.*
2. *The functors* $D\operatorname{Hom}_{\mathsf{C}}(-,C)$ *and* $D\operatorname{Hom}_{\mathsf{C}}(C,-)$ *are finitely presented for all objects* C *in* C. □

Example 3.4 (Auslander–Reiten [3, §2])**.** Let Λ be an Artin k-algebra. Then the category $\operatorname{\mathsf{proj}}\Lambda$ of finitely generated projective Λ-modules is a dualising k-variety.

If C is a dualising k-variety, then $\operatorname{\mathsf{mod}}\mathsf{C}$ is a dualising k-variety. In particular, $\operatorname{\mathsf{mod}}\Lambda = \operatorname{\mathsf{mod}}(\operatorname{\mathsf{proj}}\Lambda)$ is a dualising k-variety.

Restriction. For an object C in C and $\Gamma = \operatorname{End}_{\mathsf{C}}(C)$, consider the restriction functor

$$(\mathsf{C}^{\mathrm{op}}, \operatorname{\mathsf{mod}} k) \longrightarrow \operatorname{\mathsf{mod}}\Gamma, \quad F \mapsto F(C)$$

and its right adjoint

$$\operatorname{coind}_C \colon \operatorname{\mathsf{mod}}\Gamma \longrightarrow (\mathsf{C}^{\mathrm{op}}, \operatorname{\mathsf{mod}} k), \quad I \mapsto \operatorname{Hom}_\Gamma(\operatorname{Hom}_{\mathsf{C}}(C,-), I).$$

Note that Yoneda's lemma gives for each $Y \in \mathsf{C}$ the adjointness isomorphism

$$\operatorname{Hom}_{\mathsf{C}}(\operatorname{Hom}_{\mathsf{C}}(-,Y), \operatorname{coind}_C I) \xrightarrow{\sim} \operatorname{Hom}_\Gamma(\operatorname{Hom}_{\mathsf{C}}(C,Y), I), \quad \eta \mapsto \eta_* \tag{3.1}$$

with

$$\eta_X(\alpha) = \eta_* \operatorname{Hom}_{\mathsf{C}}(C,\alpha) \quad \text{for all} \quad X \in \mathsf{C},\ \alpha \in \operatorname{Hom}_{\mathsf{C}}(X,Y). \tag{3.2}$$

In particular,

$$\eta \operatorname{Hom}_{\mathsf{C}}(-,\alpha) = 0 \quad \iff \quad \eta_* \operatorname{Hom}_{\mathsf{C}}(C,\alpha) = 0. \tag{3.3}$$

Lemma 3.5. *Let* $I = D\Gamma$. *Then* $\operatorname{coind}_C I \cong D\operatorname{Hom}_{\mathsf{C}}(C,-)$.

Proof. One computes

$$\begin{aligned} \operatorname{Hom}_\Gamma(\operatorname{Hom}_{\mathsf{C}}(C,-), \operatorname{Hom}_k(\Gamma,k)) &\cong \operatorname{Hom}_k(\operatorname{Hom}_{\mathsf{C}}(C,-) \otimes_\Gamma \Gamma, k) \\ &\cong D\operatorname{Hom}_{\mathsf{C}}(C,-). \end{aligned}$$

□

Finding a determinator of a morphism. Following Ringel [11], an object that determines a morphism is called a *determinator*. Our first aim is to find for each morphism in C a determinator.

Lemma 3.6. *Fix an object* $C \in \mathsf{C}$ *and set* $\Gamma = \operatorname{End}_{\mathsf{C}}(C)$. *Let* $\alpha \colon X \to Y$ *be a morphism in* C *and suppose there is an exact sequence*

$$\operatorname{Hom}_{\mathsf{C}}(-,X) \xrightarrow{(-,\alpha)} \operatorname{Hom}_{\mathsf{C}}(-,Y) \xrightarrow{\eta} \operatorname{coind}_C I$$

for some $I \in \operatorname{\mathsf{mod}}\Gamma$. *Then* α *is right* C*-determined.*

Proof. Fix a morphism $\alpha'\colon X' \to Y$ such that for every morphism $\phi\colon C \to X'$ the composite $\alpha'\phi$ factors through α. This means

$$\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(C,\alpha') \subseteq \operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(C,\alpha).$$

It follows from (3.3) that $\eta\operatorname{Hom}_{\mathsf{C}}(-,\alpha') = 0$. Thus α' factors through α. □

Proposition 3.7. *Let $\alpha\colon X \to Y$ be a morphism in C and suppose there is an exact sequence*

$$\operatorname{Hom}_{\mathsf{C}}(-,X) \xrightarrow{(-,\alpha)} \operatorname{Hom}_{\mathsf{C}}(-,Y) \longrightarrow D\operatorname{Hom}_{\mathsf{C}}(C,-)$$

for some object $C \in \mathsf{C}$. Then α is right C-determined.

Proof. Observe that $D\operatorname{Hom}_{\mathsf{C}}(C,-) = \operatorname{coind}_C I$ for $I = D\Gamma$ and $\Gamma = \operatorname{End}_{\mathsf{C}}(C)$, by Lemma 3.5. Now apply Lemma 3.6 to see that α is right determined by C. □

Corollary 3.8. *Suppose that C has weak cokernels and $D\operatorname{Hom}_{\mathsf{C}}(-,C)$ is finitely presented for each $C \in \mathsf{C}$. Then every morphism in C is right determined by an object in C.*

Proof. Fix a morphism α in C. The assumptions on C ensure that the functor $D\operatorname{Coker}\operatorname{Hom}_{\mathsf{C}}(-,\alpha)$ is finitely presented; see Lemma 3.2. This means that there is a monomorphism $\operatorname{Coker}\operatorname{Hom}_{\mathsf{C}}(-,\alpha) \to D\operatorname{Hom}_{\mathsf{C}}(C,-)$ for some $C \in \mathsf{C}$. Thus α is right C-determined by Proposition 3.7. □

Finding morphisms determined by an object. We construct morphisms that are determined by a fixed object.

Proposition 3.9. *Suppose that C has weak kernels. Fix two objects C, Y in C and an $\operatorname{End}_{\mathsf{C}}(C)$-submodule $H \subseteq \operatorname{Hom}_{\mathsf{C}}(C,Y)$. Suppose also that the functor $D\operatorname{Hom}_{\mathsf{C}}(C,-)$ is finitely presented. Then there exists a right C-determined morphism $\alpha\colon X \to Y$ satisfying $\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(C,\alpha) = H$.*

Remark 3.10. The morphism $X \to Y$ in Proposition 3.9 can be chosen to be right minimal. This follows from the subsequent remark. With this choice, the morphism is unique up to isomorphism, by Lemma 2.2.

Remark 3.11. Given a morphism $\alpha\colon X \to Y$ in C, there is a decomposition $X = X' \amalg X''$ such that $\alpha|_{X'}$ is right minimal and $\alpha|_{X''} = 0$. This follows from the fact that the endomorphism ring of every object in C is semiperfect.

Proof of Proposition 3.9. Choose an injective envelope $\operatorname{Hom}_{\mathsf{C}}(C,Y)/H \to I$ over $\Gamma = \operatorname{End}_{\mathsf{C}}(C)$. The composite

$$\operatorname{Hom}_{\mathsf{C}}(C,Y) \twoheadrightarrow \operatorname{Hom}_{\mathsf{C}}(C,Y)/H \to I$$

corresponds under the isomorphism (3.1) to a morphism

$$\eta\colon \operatorname{Hom}_{\mathsf{C}}(-,Y) \to \operatorname{coind}_C I.$$

Next observe that $\operatorname{coind}_C I$ is finitely presented since $\operatorname{coind}_C(D\Gamma)$ is finitely presented, by the assumption on C and Lemma 3.5. It follows that the kernel of η is finitely presented since $\operatorname{mod}\mathsf{C}$ is abelian. Thus there is a morphism $\alpha\colon X \to Y$ which yields an exact sequence

$$\operatorname{Hom}_{\mathsf{C}}(-,X) \xrightarrow{(-,\alpha)} \operatorname{Hom}_{\mathsf{C}}(-,Y) \xrightarrow{\eta} \operatorname{coind}_C I.$$

Evaluating this sequence at C shows that $\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(C,\alpha) = H$, and Lemma 3.6 shows that α is determined by C. □

Corollary 3.12. *Every dualising variety has right determined morphisms.* □

Minimal determinators. Suppose a morphism is determined by two objects C and C'. What is then the relationship between these objects? The following proposition gives a precise answer. For each $X \in \mathsf{C}$ let $\operatorname{\mathsf{add}} X$ denote the full subcategory consisting of the direct summands of finite directs sums of copies of X.

Proposition 3.13. *Let C be a dualising variety and $\alpha \colon X \to Y$ a morphism in C. Then there exists in $\operatorname{\mathsf{mod}} \mathsf{C}$ an injective envelope of the form*

$$\operatorname{Coker} \operatorname{Hom}_{\mathsf{C}}(-, \alpha) \longrightarrow D \operatorname{Hom}_{\mathsf{C}}(C, -)$$

for some object $C \in \mathsf{C}$. Given an object C' in C, the morphism α is right C'-determined if and only if $\operatorname{\mathsf{add}} C \subseteq \operatorname{\mathsf{add}} C'$.

Proof. The category $\operatorname{\mathsf{mod}} \mathsf{C}$ has projective covers since the endomorphism ring of each object in C is semiperfect; see [9, Proposition A.1]. Applying the duality, it follows that each object F has an injective envelope of the form $F \to D \operatorname{Hom}_{\mathsf{C}}(C, -)$ for some $C \in \mathsf{C}$. Now set $F = \operatorname{Coker} \operatorname{Hom}_{\mathsf{C}}(-, \alpha)$. Then Proposition 3.7 shows that α is right C-determined. Given an object $C' \in \mathsf{C}$, it follows that α is right C'-determined if $\operatorname{\mathsf{add}} C \subseteq \operatorname{\mathsf{add}} C'$.

Now suppose that α is right C'-determined. The proof of Proposition 3.9 yields a monomorphism $F \to \operatorname{coind}_{C'} I$ for some injective $\operatorname{End}_{\mathsf{C}}(C')$-module I. Here, we use the uniqueness of a right determined morphism; see Remark 3.10. From Lemma 3.5 it follows that $\operatorname{coind}_{C'} I$ is a direct summand of a finite direct sum of copies of $D \operatorname{Hom}_{\mathsf{C}}(C', -)$. On the other hand, the assumption on C implies that $D \operatorname{Hom}_{\mathsf{C}}(C, -)$ is a direct summand of $\operatorname{coind}_{C'} I$, since $\operatorname{coind}_{C'} I$ is an injective object. Thus C is a direct summand of a finite direct sum of copies of C'. □

4. TRIANGULATED CATEGORIES WITH SERRE DUALITY

Fix a k-linear triangulated category C which is Hom-finite, essentially small, and idempotent complete.

Recall from [10] that a *right Serre functor* is an additive functor $S \colon \mathsf{C} \to \mathsf{C}$ together with a natural isomorphism

$$\eta_X \colon D \operatorname{Hom}_{\mathsf{C}}(X, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{C}}(-, SX)$$

for all $X \in \mathsf{C}$, where $D = \operatorname{Hom}_k(-, E(k/\mathfrak{r}))$. A right Serre functor is called a *Serre functor* if it is an equivalence.

Proposition 4.1. *Consider for the category C the following conditions:*

1. *The category C admits a right Serre functor $S \colon \mathsf{C} \to \mathsf{C}$.*
2. *Given objects $C, Y \in \mathsf{C}$ and an $\operatorname{End}_{\mathsf{C}}(C)$-submodule $H \subseteq \operatorname{Hom}_{\mathsf{C}}(C, Y)$, there is a right C-determined morphism $\alpha \colon X \to Y$ with $\operatorname{Im} \operatorname{Hom}_{\mathsf{C}}(C, \alpha) = H$.*
3. *Every morphism in C is left determind by an object in C.*

Then $(1) \iff (2) \implies (3)$.

Proof. Observe first that a triangulated category has weak kernels and cokernels.

That (1) implies (2) follows from Proposition 3.9, and that (1) implies (3) follows from Corollary 3.8.

It remains to show that (2) implies (1). From [10, Proposition I.2.3] it follows that C has a right Serre functor if and only if there is an Auslander–Reiten triangle ending at each indecomposable object of C. An exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to$ is by definition an *Auslander–Reiten triangle* ending at Z, if α is left almost split and β is right almost split. This is equivalent to β being right minimal and right almost split, by [8, Lemma 2.6].

Now fix an indecomposable object Z in C. Then $\Gamma = \operatorname{End}_{\mathsf{C}}(Z)$ is local and there exists a right Z-determined morphism $\beta\colon Y \to Z$ such that $\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(Z,\beta) = \operatorname{rad}\Gamma$. We may assume that β is right minimal by Remark 3.11, and it follws from Example 2.3 that β is right almost split. Completing β to an exact triangle then gives an Auslander–Reiten triangle ending at Z. It follows that C has a Serre functor. ☐

Theorem 4.2. *For a Hom-finite essentially small and idempotent complete k-linear triangulated category C the following are equivalent:*

1. *The category C admits a Serre functor $S\colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$.*
2. *The category C is a dualising variety.*
3. *The category C has right determined morphisms.*

In this case, every morphism in C with cone C is right determined by $S^{-1}C$.

Proof. (1) ⇒ (2): A triangulated category has weak kernels and cokernels. From the definition of a Serre functor, it follows that $D\operatorname{Hom}_{\mathsf{C}}(-,C)$ and $D\operatorname{Hom}_{\mathsf{C}}(C,-)$ are finitely presented for each $C \in \mathsf{C}$. Thus C is a dualising variety by Lemma 3.3.

(2) ⇒ (3): Apply Corollary 3.12.

(3) ⇒ (1): From Proposition 4.1 we know that there is a right Serre functor $S\colon \mathsf{C} \to \mathsf{C}$, and it remains to show that S is an equivalence. In fact, it suffices to show that S is essentially surjective on objects, since S is automatically fully faithful. Choose an object Y and suppose the morphism $0 \to Y$ is right determined by some object C. The proof of Proposition 3.9 yields a monomorphism $\eta\colon \operatorname{Hom}_{\mathsf{C}}(-Y) \to \operatorname{coind}_C I$ for some injective $\operatorname{End}_{\mathsf{C}}(C)$-module I. Here, we use the uniqueness of a right determined morphism; see Remark 3.10. From Lemma 3.5 it follows that $\operatorname{coind}_C I$ is a direct summand of a finite direct sum of copies of $D\operatorname{Hom}_{\mathsf{C}}(C,-) \cong \operatorname{Hom}_{\mathsf{C}}(-,SC)$. Now one uses that $\operatorname{Hom}_{\mathsf{C}}(-Y)$ is an injective object in $\mathsf{mod}\,\mathsf{C}$; this follows from a direct argument [7, Lemma 1.6] or the fact that for every triangulated category C the assignment $\operatorname{Hom}_{\mathsf{C}}(-,X) \mapsto \operatorname{Hom}_{\mathsf{C}}(X,-)$ induces an equivalence

$$(\mathsf{mod}\,\mathsf{C})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{mod}(\mathsf{C}^{\mathrm{op}}).$$

Thus η is a split monomorphism, and it follows that Y is a direct summand of some object in the image of S. Using that C is idempotent complete, it follows that $Y \cong SX$ for some $X \in \mathsf{C}$.

To complete the proof, we fix an exact triangle $X \xrightarrow{\alpha} Y \to SC \to$ and claim that α is right determined by C. This an immediate consequence of Proposition 3.7, since the triangle induces an exact sequence

$$\operatorname{Hom}_{\mathsf{C}}(-,X) \xrightarrow{(-,\alpha)} \operatorname{Hom}_{\mathsf{C}}(-,Y) \longrightarrow D\operatorname{Hom}_{\mathsf{C}}(C,-). \qquad \square$$

5. A GENERALISATION

In this section we generalise Auslander's definition of a right determined morphism as follows; see also [8, §4].

Definition 5.1. A morphism $\alpha\colon X \to Y$ in a category C is said to be *right determined* by a class D of objects of C if for every morphism $\alpha'\colon X' \to Y$ the following conditions are equivalent:

1. The morphism α' factors through α.
2. For every morphism $\phi\colon C \to X'$ with $C \in \mathsf{D}$ the composite $\alpha'\phi$ factors through α.

Let C be an essentially small additive category. We denote by $\operatorname{\mathsf{Mod}}\mathsf{C}$ the category of additive functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. Given a full additive subcategory D of C, we consider the restriction functor $\operatorname{res}_{\mathsf{D}}\colon \operatorname{\mathsf{Mod}}\mathsf{C} \to \operatorname{\mathsf{Mod}}\mathsf{D}$ and its right adjoint $\operatorname{coind}_{\mathsf{D}}\colon \operatorname{\mathsf{Mod}}\mathsf{D} \to \operatorname{\mathsf{Mod}}\mathsf{C}$ with

$$(\operatorname{coind}_{\mathsf{D}} F)(X) = \operatorname{Hom}_{\mathsf{D}}(\operatorname{res}_{\mathsf{D}} \operatorname{Hom}_{\mathsf{C}}(-,X), F)$$

for $F \in \operatorname{\mathsf{Mod}}\mathsf{D}$ and $X \in \mathsf{C}$.

Fix a morphism $\alpha\colon X \to Y$ in C. Then a morphism $\alpha'\colon X' \to Y$ factors through α if and only if

$$\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\alpha') \subseteq \operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\alpha). \tag{5.1}$$

This condition implies

$$\operatorname{res}_{\mathsf{D}}\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\alpha') \subseteq \operatorname{res}_{\mathsf{D}}\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\alpha). \tag{5.2}$$

Reformulating the above definition, the morphism α is determined by D if and only if (5.1) and (5.2) are equivalent for all $\alpha'\colon X' \to Y$.

The following proposition characterises the morphisms that are right determined by a fixed class of objects. This provides a conceptual explanation for some of the previous results.

Proposition 5.2. *Let C be an essentially small additive category and D a full additive subcategory. For a morphism $\alpha\colon X \to Y$ the following are equivalent:*

1. *The morphism α is right determined by D.*
2. *For $F = \operatorname{Coker}\operatorname{Hom}_{\mathsf{C}}(-,\alpha)$ the canonical morphism $F \to \operatorname{coind}_{\mathsf{D}}\operatorname{res}_{\mathsf{D}} F$ is a monomorphism.*
3. *For some $I \in \operatorname{\mathsf{Mod}}\mathsf{D}$ there is an exact sequence*

$$\operatorname{Hom}_{\mathsf{C}}(-,X) \xrightarrow{(-,\alpha)} \operatorname{Hom}_{\mathsf{C}}(-,Y) \longrightarrow \operatorname{coind}_{\mathsf{D}} I.$$

Proof. (1) ⇔ (2): A morphism $\phi\colon C \to Y$ in C yields an element $\bar\phi \in F(C)$, and $\bar\phi = 0$ if and only if

$$\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\phi) \subseteq \operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\alpha).$$

Now consider the canonical morphism $\eta\colon F \to \operatorname{coind}_{\mathsf{D}}\operatorname{res}_{\mathsf{D}} F$ and observe that $\eta_C(\bar\phi) = 0$ if and only if

$$\operatorname{res}_{\mathsf{D}}\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\phi) \subseteq \operatorname{res}_{\mathsf{D}}\operatorname{Im}\operatorname{Hom}_{\mathsf{C}}(-,\alpha).$$

Thus α is right determined by D if and only if η_C is a monomorphism for all $C \in \mathsf{C}$.

(2) ⇒ (3): Take $I = \operatorname{res}_{\mathsf{D}} F$.

(3) ⇒ (2): Every morphism $\theta\colon F \to \operatorname{coind}_{\mathsf{D}} I$ factors through the canonical morphism $\eta\colon F \to \operatorname{coind}_{\mathsf{D}}\operatorname{res}_{\mathsf{D}} F$. It follows that η is a monomorphism if θ is a monomorphism. □

The general definition of a morphism determined by a class of objects suggests the following question.

Question 5.3. Given a morphism in some category C, is there a *minimal* class D of objects of C such that the morphism is right D-determined?

We have seen in Proposition 3.13 that such minimal determinators always exist for dualising varieties. Next we discuss a classical problem from stable homotopy theory. It turns out that Freyd's generating hypothesis predicts a determinator for a particular class of morphisms.

6. Freyd's generating hypothesis

We consider the stable homotopy category of spectra and the set $\mathsf{S} = \{\Sigma^n S \mid n \in \mathbb{Z}\}$ formed by the suspensions of the sphere spectrum S.

Theorem 6.1. *The following conditions are equivalent:*

1. *Freyd's generating hypothesis holds, that is, for every finite spectrum Y, the morphism $0 \to Y$ is right S-determined as morphism in the category of finite spectra* [5, §9] .
2. *For every finite torsion spectrum Y, the morphism $0 \to Y$ is right S-determined as morphism in the category of finite spectra.*
3. *For every finite torsion spectrum Y, the morphism $0 \to Y$ is right S-determined as morphism in the category of all spectra.*

Proof. (1) ⇔ (2): One direction is clear, and the other follows from the fact that the torsion spectra cogenerate the category of finite spectra; see [5, Proposition 6.8].

(2) ⇔ (3): One needs to show that $0 \to Y$ is a right determined morphism in the category of all spectra if it is right determined in the category of finite spectra. Observe first that every finite torsion spectrum Y is *endofinite* [6, §1]. More precisely, $\operatorname{Hom}(F, Y)$ has finite length as an $\operatorname{End}(Y)$-module for each finite spectrum F. It follows from [6, Theorem 1.2] that for each non-zero morphism $X \to Y$ the induced map $\operatorname{Hom}(F, X) \to \operatorname{Hom}(F, Y)$ is non-zero for some finite spectrum F. Thus there is some non-zero morphism $F \to X \to Y$. If (2) holds, this implies that for some $n \in \mathbb{Z}$ there is a morphism $\Sigma^n S \to F$ such that the composite $\Sigma^n S \to F \to X \to Y$ is non-zero. Thus the morphism $0 \to Y$ is determined by S. ☐

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