DERIVED EQUIVALENCES OF ACTIONS OF A CATEGORY

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ABSTRACT. Let $\Bbbk$ be a commutative ring and I a category. As a generalization of a $\Bbbk$-category with a (pseudo) action of a group we consider a family of $\Bbbk$-categories with a (pseudo, lax, or oplax) action of I, namely an oplax functor from I to the 2-category of small $\Bbbk$-categories. We investigate derived equivalences of those oplax functors, and establish a Morita type theorem for them. This gives a base of investigations of derived equivalences of Grothendieck constructions of those oplax functors.

1. INTRODUCTION

We fix a category I and a commutative ring $\Bbbk$ and denote by $\Bbbk$-**Cat** (resp. $\Bbbk$-**Ab**, $\Bbbk$-**Tri**) the 2-category of small $\Bbbk$-categories (resp. samll abelian $\Bbbk$-categories, small triangulated $\Bbbk$-categories). For a $\Bbbk$-category $\mathcal{C}$ a (right) $\mathcal{C}$-*module* is a contravariant functor from $\mathcal{C}$ to the category $\operatorname{Mod}\Bbbk$ of $\Bbbk$-modules, and we denote by $\operatorname{Mod}\mathcal{C}$ (resp. $\operatorname{Prj}\mathcal{C}$, $\operatorname{prj}\mathcal{C}$) the category of $\mathcal{C}$-modules (resp. projective $\mathcal{C}$-modules, finitely generated projective $\mathcal{C}$-modules).

A $\Bbbk$-category $\mathcal{C}$ with an action of a group G have been well investigated in connection with a so-called covering technique in representation theory of algebras (see e.g., [6]). The orbit category $\mathcal{C}/G$ and the canonical functor $\mathcal{C} \to \mathcal{C}/G$ are naturally constructed from these data, and one studied relationships between $\operatorname{Mod}\mathcal{C}$ and $\operatorname{Mod}\mathcal{C}/G$. We brought this point of view to the derived equivalence classification problem of algebras in [1], and a main tool obtained there was fully used in the derived equivalence classifications in [2, 3]. The main tool was extended in [4] in the following form:

Theorem 1.1. *Let G be a group acting on categories $\mathcal{C}$ and $\mathcal{C}'$. Assume the condition*

(∗) *There exists a G-stable tilting subcategory E of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$ such that there is a G-equivariant equivalence $\mathcal{C}' \to E$.*

Then the orbit categories $\mathcal{C}/G$ and $\mathcal{C}'/G$ are derived equivalent.

(In the above, E is said to be *G-stable* if the set of objects in E is stable under the G-action on $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}\mathcal{C})$ induced from that on $\mathcal{C}$.) Observe that if we regard G as a category with a single object $*$, then a G-action on a category $\mathcal{C}$ is nothing but a functor $X : G \to \Bbbk$-**Cat** with $X(*) = \mathcal{C}$; and the orbit category $\mathcal{C}/G$ coincides with (the $\Bbbk$-linear version of) the Grothendieck construction $\operatorname{Gr}(X)$ of X defined in [7].

In a subsequent paper [5] we will generalize this theorem to an arbitrary category I and to any *oplax functors* $X, X' \colon I \to \Bbbk$-**Cat** (roughly speaking an oplax functor X is a family $(X(i))_{i \in I_0}$ of $\Bbbk$-categories indexed by the objects i of I with an action of

I, the precise definition is given in Definition 2.1). In this paper before doing it we first investigate the meaning of the condition $(*)$. Recall the following theorem due to Rickard [11]:

Theorem 1.2. *For rings R and S the following are equivalent:*

(1) *R and S are derived equivalent.*
(2) *There exists a tilting complex T in $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,R)$ such that $\mathrm{End}_R(T)$ is isomorphic to S.*

Then the condition $(*)$ can be regarded as a generalized version of the condition (2). Therefore in this paper we first give a definition of derived equivalences of oplax functors and generalize the theorem above in the setting of oplax functors.

Recall also that if $\mathcal{C}$ is a category with an action of a group G, then the module category $\mathrm{Mod}\,\mathcal{C}$ (resp. the derived category $\mathcal{D}(\mathrm{Mod}\,\mathcal{C})$) has the induced G-action; thus both of them are again categories with G-actions. Hence for an oplax functor X the "module category" $\mathrm{Mod}\,X$ (resp. the "derived category" $\mathcal{D}(\mathrm{Mod}\,X)$) should again be a family of categories with an I-action, i.e., an oplax functor from I to $\Bbbk$-**Ab** (resp. to $\Bbbk$-**Tri**). An oplax functor $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X)$ is also defined as an oplax subfunctor of $\mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ by the family $(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X(i)))_{i\in I_0}$, which plays the same role as $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,R)$ in Theorem 1.2.

We need a notion of equivalences between oplax functors for two purposes:

(a) to generalize the statement $(*)$; and
(b) to define a derived equivalence of oplax functors X, X' by an existence of an equivalence between oplax functors $\mathcal{D}(\mathrm{Mod}\,X)$ and $\mathcal{D}(\mathrm{Mod}\,X')$.

To define equivalences of objects we need notions of 1-morphisms and 2-morphisms, thus we need a 2-categorical structure on the collection of oplax functors. We will define a 2-category $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$ of all oplax functors from I to a 2-category $\mathbf{C}$, which can be used for both (a) and (b) of the above. We have the following as a corollary of our main theorem (see Theorem 5.6 for detail), which generalizes Theorem 1.2 in the field case.

Theorem 1.3. *Let $X, X' \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Assume that $\Bbbk$ is a field. Then the following are equivalent.*

(1) *X and X' are derived equivalent.*
(2) *There exists a tilting oplax functor $\mathcal{T}$ for X such that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.*

The paper is organized as follows. In section 2 we define a 2-category $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$ of the oplax functors from I to a 2-category $\mathbf{C}$. In section 3 we define the "module category" $\mathrm{Mod}\,X$ of an oplax functor $X\colon I \to \Bbbk$-**Cat** as an oplax functor $I \to \Bbbk$-**Ab**. In section 4 we define the "derived category" $\mathcal{D}(\mathrm{Mod}\,X)$ of the oplax functor $\mathrm{Mod}\,X$ as an oplax functor $I \to \Bbbk$-**Tri**. In section 5 we state and prove our main result, which gives a characterization of derived equivalences of oplax functors by tilting oplax subfunctors. In section 6 as an appendix we include Keller's proof of a categorical version of the lifting theorem in [9], which is used in the proof of the main result in section 5.

2. The 2-category of oplax functors

The 2-category G-**Cat** of $\Bbbk$-categories with G-actions for a group G was generalized by D. Tamaki in [12] to the 2-category $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$ of oplax functors from a category I to a 2-category $\mathbf{C}$ of V-enriched categories for a monoidal category V, which we use in this paper for $V = \mathrm{Mod}\,\Bbbk$, the category of $\Bbbk$-modules.

Definition 2.1. Let $\mathbf{C}$ be a 2-category.
  (1) An *oplax functor* from I to $\mathbf{C}$ is a triple (X, η, θ) of data:
   - a quiver morphism $X \colon I \to \mathbf{C}$, where I and $\mathbf{C}$ are regarded as quivers by forgetting additional data such as 2-morphisms or compositions;
   - a family $\eta := (\eta_i)_{i \in I_0}$ of 2-morphisms $\eta_i \colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in I_0$; and
   - a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a} \colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b, a) \in \mathrm{com}(I) := \{(b, a) \in I_1 \times I_1 \mid ba \text{ is defined}\}$

satisfying the axioms:
  (a) For each $a \colon i \to j$ in I the following are commutative:

$$
\begin{array}{ccc}
X(a\mathbb{1}_i) \xRightarrow{\theta_{a,1_i}} X(a)X(\mathbb{1}_i) & & \\
\quad \searrow \qquad \Big\Downarrow{\scriptstyle X(a)\eta_i} & \text{and} & \\
X(a)\mathbb{1}_{X(i)} & &
\end{array}
\qquad
\begin{array}{c}
X(\mathbb{1}_j a) \xRightarrow{\theta_{1_j,a}} X(\mathbb{1}_j)X(a) \\
\quad \searrow \qquad \Big\Downarrow{\scriptstyle \eta_j X(a)} \\
\mathbb{1}_{X(j)}X(a)
\end{array}
\quad ; \text{ and}
$$

  (b) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in I the following is commutative:

$$
\begin{array}{ccc}
X(cba) & \xRightarrow{\theta_{c,ba}} & X(c)X(ba) \\
{\scriptstyle \theta_{cb,a}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle X(c)\theta_{b,a}} \\
X(cb)X(a) & \xRightarrow[\theta_{c,b}X(a)]{} & X(c)X(b)X(a).
\end{array}
$$

  (2) A *lax functor* from I to $\mathbf{C}$ is an oplax functor from I to $\mathbf{C}^{\mathrm{co}}$, where $\mathbf{C}^{\mathrm{co}}$ denotes the 2-category obtained from $\mathbf{C}$ by reversing the 2-morphisms.
  (3) A *pseudofunctor* from I to $\mathbf{C}$ is an oplax functor (X, η, θ) with all η_i and $\theta_{b,a}$ 2-isomorphisms.

Remark 2.2. A functor from I to a 2-category $\mathbf{C}$ is an oplax functor (X, η, θ) with all η_i and $\theta_{b,a}$ identities.

Definition 2.3. Let $\mathbf{C}$ be a 2-category and $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be oplax functors from I to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from X to X' is a pair (F, ψ) of data

- a family $F := (F(i))_{i \in I_0}$ of 1-morphisms $F(i) \colon X(i) \to X'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- a family $\psi := (\psi(a))_{a \in I_1}$ of 2-morphisms $\psi(a) \colon X'(a)F(i) \Rightarrow F(j)X(a)$ in $\mathbf{C}$ indexed by $a \colon i \to j$ in I_1:

$$
\begin{array}{ccc}
X(i) & \xrightarrow{F(i)} & X'(i) \\
{\scriptstyle X(a)}\downarrow & {\scriptstyle \psi(a)} \Swarrow & \downarrow{\scriptstyle X'(a)} \\
X(j) & \xrightarrow[F(j)]{} & X'(j)
\end{array}
$$

satisfying the axioms

(a) For each $i \in I_0$ the following is commutative:

$$
\begin{array}{ccc}
X'(\mathbb{1}_i)F(i) & \xRightarrow{\psi(\mathbb{1}_i)} & F(i)X(\mathbb{1}_i) \\
{\scriptstyle \eta_i'F(i)}\| & & \|{\scriptstyle F(i)\eta_i} \\
\mathbb{1}_{X'(i)}F(i) & = & F(i)\mathbb{1}_{X(i)}
\end{array} \quad ; \text{ and }
$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I the following is commutative:

$$
\begin{array}{ccccc}
X'(ba)F(i) & \xRightarrow{\theta'_{b,a}F(i)} & X'(b)X'(a)F(i) & \xRightarrow{X'(b)\psi(a)} & X'(b)F(j)X(a) \\
{\scriptstyle \psi(ba)}\| & & & & \|{\scriptstyle \psi(b)X(a)} \\
F(k)X(ba) & & \xRightarrow[F(k)\,\theta_{b,a}]{} & & F(k)X(b)X(a).
\end{array}
$$

Definition 2.4. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be oplax functors from I to $\mathbf{C}$, and (F, ψ), (F', ψ') 1-morphisms from X to X'. A 2-morphism from (F, ψ) to (F', ψ') is a family $\zeta = (\zeta(i))_{i \in I_0}$ of 2-morphisms $\zeta(i) \colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$ such that the following is commutative for each $a \colon i \to j$ in I:

$$
\begin{array}{ccc}
X'(a)F(i) & \xRightarrow{X'(a)\zeta(i)} & X'(a)F'(i) \\
{\scriptstyle \psi(a)}\| & & \|{\scriptstyle \psi'(a)} \\
F(j)X(a) & \xRightarrow[\zeta(j)X(a)]{} & F'(j)X(a).
\end{array}
$$

Definition 2.5. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be oplax functors from I to $\mathbf{C}$, and let $(F, \psi) \colon X \to X'$, $(F', \psi') \colon X' \to$

X'' be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of (F, ψ) and (F', ψ') is a 1-morphism from X to X'' defined by

$$(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),$$

where $F'F := ((F'(i)F(i))_{i \in I_0}$ and for each $a \colon i \to j$ in I, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

The following is straightforward to verify.

Proposition 2.6. *Let $\mathbf{C}$ be a 2-category. Then oplax functors $I \to \mathbf{C}$, 1-morphisms between them, and 2-morphisms between 1-morphisms (defined above) define a 2-category, which we denote by $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$.*

Example 2.7. We give a tiny example of an oplax functor that is not a pseudofunctor. Let I be the free category of the quiver $1 \xrightarrow{a} 2$. Define an oplax functor $X = (X, \eta, \theta) \colon I \to \Bbbk\text{-}\mathbf{Cat}$ as follows.

- $X(1)$ is the category given by the quiver $x \underset{\beta}{\overset{\alpha}{\rightleftarrows}} y$ with a relation $\alpha\beta = \mathbb{1}_y$;

- $X(2)$ is the category given by the quiver $x' \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} y'$ with a relation $\alpha'\beta' = \mathbb{1}_{y'}$;

- $X(\mathbb{1}_1)(z) := y$, $\forall z = x, y$,
  $X(\mathbb{1}_1)(\gamma) := \mathbb{1}_y$, $\forall \gamma = \alpha, \beta, \mathbb{1}_x, \mathbb{1}_y$;
- $X(\mathbb{1}_2)(z) := y'$, $\forall z = x', y'$,
  $X(\mathbb{1}_2)(\gamma) := \mathbb{1}_{y'}$, $\forall \gamma = \alpha', \beta', \mathbb{1}_{x'}, \mathbb{1}_{y'}$;
- $X(a)(x) := x'$, $X(a)(y) := y'$,
  $X(a)(\alpha) := \alpha'$, $X(a)(\beta) := \beta'$, $X(a)(\mathbb{1}_x) := \mathbb{1}_{x'}$, $X(a)(\mathbb{1}_y) := \mathbb{1}_{y'}$.

Define $\eta_1 \colon X(\mathbb{1}_1) \Rightarrow \mathbb{1}_{X(1)}$ by

- $\eta_1 x := \beta \colon X(\mathbb{1}_1)x \to x$ and $\eta_1 y := \alpha\beta \colon X(\mathbb{1}_1)y \to y$.

Define $\eta_2 \colon X(\mathbb{1}_2) \Rightarrow \mathbb{1}_{X(2)}$ by

- $\eta_2 x' := \beta' \colon X(\mathbb{1}_2)x' \to x'$ and $\eta_2 y' := \alpha'\beta' \colon X(\mathbb{1}_2)y' \to y'$.

Define $\theta_{b,a} \colon X(ba) \Rightarrow X(b)X(a)$, $\forall a, b \in I_1$ (with ba defined) by

- $\theta_{a,\mathbb{1}_1}x := \theta_{\mathbb{1}_2,a}x := \alpha'$ and $\theta_{a,\mathbb{1}_1}y := \theta_{\mathbb{1}_2,a}y := \mathbb{1}_{y'}$.

Then it is easy to check that X is an oplax functor, and X is not a pseudofunctor by definitions of η and θ above.

3. The Module oplax functor

Let $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ be an oplax functor. In this section we define the "module category" $\operatorname{Mod} X$ of X as an oplax functor $I \to \Bbbk\text{-}\mathbf{Cat}$. Recall that the *module category* $\operatorname{Mod}\mathcal{C}$ of a category $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ is defined to be the functor category $\Bbbk\text{-}\mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, \operatorname{Mod}\Bbbk)$, where $\operatorname{Mod}\Bbbk$ denotes the category of $\Bbbk$-modules. As is stated in Proposition 3.2 the composite $\operatorname{Mod}' \circ X$ turns out to be a contravariant lax functor $I \to \Bbbk\text{-}\mathbf{Ab}$. When X is a group action, namely when I is a group G and $X\colon G \to \Bbbk\text{-}\mathbf{Cat}$ is a functor, the usual module category $\operatorname{Mod} X$ with a G-action of X was defined to be the composite functor $\operatorname{Mod} X := \operatorname{Mod}' \circ X \circ i$, where $i\colon G \to G$ is the group anti-isomorphism defined by $x \mapsto x^{-1}$ for all $x \in G$. In this way we can change $\operatorname{Mod}' \circ X$ to a covariant one. But in general we cannot assume the existence of such an isomorphism i. Regarding $(\operatorname{Mod}' \circ X)(a^{-1})$ as a left adjoint to $(\operatorname{Mod}' \circ X)(a)$ for each $a \in G$ in the group action case, we define $\operatorname{Mod} X$ by using a left adjoint $(\operatorname{Mod} X)(a)$ to $(\operatorname{Mod}' \circ X)(a)$ for each $a \in I_1$ in the general case.

Definition 3.1. Let $X = (X, \eta, \theta) \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. We define a lax functor $\operatorname{Mod}' X = (\operatorname{Mod}' X, \operatorname{Mod}' \eta, \operatorname{Mod}' \theta)\colon I^{\mathrm{op}} \to \Bbbk\text{-}\mathbf{Cat}$ as follows.

- For each $i \in I_0$, $(\operatorname{Mod}' X)(i) := \operatorname{Mod}(X(i))$.
- For each $a\colon i \to j$ in I, $(\operatorname{Mod}' X)(a) := (\text{-}) \circ X(a)\colon (\operatorname{Mod}' X)(j) \to (\operatorname{Mod}' X)(i)$, the restriction functor. Namely, each $f\colon M \to N$ in $(\operatorname{Mod}' X)(j)$ is sent by $(\operatorname{Mod}' X)(a)$ to $f \circ X(a)\colon M \circ X(a) \to N \circ X(a)$ in $(\operatorname{Mod}' X)(i)$, where
$$\begin{cases} (M \circ X(a))(x) := M(X(a)x) \text{ and} \\ (f \circ X(a))(x) := f(X(a)x) \end{cases}$$
  for all $x \in X(i)_0$.
- For each $i \in I_0$, $(\operatorname{Mod}' \eta)_i\colon \mathbb{1}_{(\operatorname{Mod}' X)(i)} \Rightarrow (\operatorname{Mod}' X)(\mathbb{1}_i)$ is defined by
$$((\operatorname{Mod}' \eta)_i M)(x) := M(\eta_i x)\colon M(x) \to M(X(\mathbb{1}_i)x)$$
  for all $M \in (\operatorname{Mod}' X)(i)$ and $x \in X(i)_0$.
- For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I,
  $(\operatorname{Mod}' \theta)_{b,a}\colon (\operatorname{Mod}' X)(a) \circ (\operatorname{Mod}' X)(b) \Rightarrow (\operatorname{Mod}' X)(ba)$ is defined by
$$((\operatorname{Mod}' \theta)_{b,a} M)(x) := M(\theta_{b,a}(x))\colon M(X(b)(X(a)x)) \to M(X(ba)x)$$
  for all $M \in (\operatorname{Mod}' X)(k)$ and $x \in X(i)_0$.

Proposition 3.2. *In the above* $\operatorname{Mod}' X$ *is well-defined as a lax functor* $I^{\mathrm{op}} \to \Bbbk\text{-}\mathbf{Cat}$.

Proof. It is straightforward to check that both $(\operatorname{Mod}' \eta)_i$ and $(\operatorname{Mod}' \theta)_{b,a}$ are natural transformations for all $i \in I_0$ and $i \xrightarrow{a} j \xrightarrow{b} k$ in I.

Each axiom for $\operatorname{Mod}' X$ to be a lax functor at a module M follows from the corresponding axiom for X to be an oplax functor by applying M. $\qquad\square$

Proposition 3.3. *Let* $Y' = (Y', \eta', \theta')\colon I^{\mathrm{op}} \to \Bbbk\text{-}\mathbf{Cat}$ *be a lax functor. Assume that for each* $a\colon i \to j$ *in* I *there exists a left adjoint* $Y(a)\colon Y'(i) \to Y'(j)$ *to* $Y'(a)\colon Y'(j) \to Y'(i)$ *with a unit* $\varepsilon_a\colon \mathbb{1}_{Y'(i)} \Rightarrow Y'(a)Y(a)$ *and a counit* $\zeta_a\colon Y(a)Y'(a) \Rightarrow \mathbb{1}_{Y'(j)}$. *We set* $Y(i) := Y'(i)$ *for each* $i \in I_0$ *to define a quiver morphism* $Y\colon I \to \Bbbk\text{-}\mathbf{Cat}$. *If we define*

$\eta := (\eta_i)_{i \in I_0}$ *and* $\theta := (\theta_{b,a})_{b,a}$ *as follows, then* $Y := (Y, \eta, \theta) \colon I \to \Bbbk\text{-}\mathbf{Cat}$ *turns out to be an oplax functor.*

(1) *For each* $i \in I_0$ *define a natural transformation* $\eta_i \colon Y(\mathbb{1}_i) \Rightarrow \mathbb{1}_{Y(\mathbb{1}_i)}$ *as the composite*

$$Y(\mathbb{1}_i) \xrightarrow{Y(\mathbb{1}_i)\eta'_i} Y(\mathbb{1}_i)Y'(\mathbb{1}_i) \xrightarrow{\zeta_{\mathbb{1}_i}} \mathbb{1}_{Y'(i)}.$$

(2) *For each pair of composable morphisms* $i \xrightarrow{a} j \xrightarrow{b} k$ *in* I *define a natural transformation* $\theta_{b,a} \colon Y(ba) \Rightarrow Y(b)Y(a)$ *as the composite*

$$Y(ba) \xrightarrow{Y(ba)\varepsilon_a} Y(ba)Y'(a)Y(a) \xrightarrow{Y(ba)Y'(a)\varepsilon_b Y(a)} Y(ba)Y'(a)Y'(b)Y(b)Y(a)$$
$$\xrightarrow{Y(ba)\theta'_{b,a}Y(b)Y(a)} Y(ba)Y'(ba)Y(b)Y(a) \xrightarrow{\zeta_{ba}Y(b)Y(a)} Y(b)Y(a).$$

Proof. It is enough to verify the commutativity of the diagrams (a_1), (a_2) below for each $a \colon i \to j$ in I and of the diagram (b) below for each triple $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ of morphisms in I:

$$
\begin{array}{ccc}
Y(a\mathbb{1}_i) & \xRightarrow{\theta_{a,\mathbb{1}_i}} & Y(a)Y(\mathbb{1}_i) \\
& \searrow & \big\Downarrow{\scriptstyle Y(a)\eta_i} \\
& & Y(a)\mathbb{1}_{Y(i)}
\end{array}
\qquad (a_1)
$$

$$
\begin{array}{ccc}
Y(\mathbb{1}_j a) & \xRightarrow{\theta_{\mathbb{1}_j,a}} & Y(\mathbb{1}_j)Y(a) \\
& \searrow & \big\Downarrow{\scriptstyle \eta_j Y(a)} \\
& & \mathbb{1}_{Y(j)}Y(a)
\end{array}
\qquad (a_2)
$$

$$
\begin{array}{ccc}
Y(cba) & \xRightarrow{\theta_{c,ba}} & Y(c)Y(ba) \\
\Big\Downarrow{\scriptstyle \theta_{cb,a}} & & \Big\Downarrow{\scriptstyle Y(c)\theta_{b,a}} \\
Y(cb)Y(a) & \xRightarrow[\theta_{c,b}Y(a)]{} & Y(c)Y(b)Y(a)
\end{array}
\qquad (b)
$$

(a_1) This follows from the following two commutative diagrams ("$\sim$" stands for a suitable functor that is uniquely determined in the diagram):

$$
\begin{array}{ccccccc}
Y(a\mathbb{1}_i) & \xrightarrow{\sim\varepsilon_{\mathbb{1}_i}} & Y(a\mathbb{1}_i)Y'(\mathbb{1}_i)Y(\mathbb{1}_i) & \xrightarrow{\sim\varepsilon_a\sim} & Y(a\mathbb{1}_i)Y'(\mathbb{1}_i)Y'(a)Y(a)Y(\mathbb{1}_i) & \xrightarrow{\sim\theta'_{a,\mathbb{1}_i}\sim} & Y(a\mathbb{1}_i)Y'(a\mathbb{1}_i)Y(a)Y(\mathbb{1}_i) \\
\Big\downarrow{\sim\eta'_i} & & & & \Big\downarrow{\sim\eta'_i} & & \Big\downarrow{\zeta_{a\mathbb{1}_i}\sim} \\
Y(a\mathbb{1}_i)Y'(\mathbb{1}_i) & \xrightarrow{\sim\varepsilon_{\mathbb{1}_i}\sim} & Y(a\mathbb{1}_i)Y'(\mathbb{1}_i)Y(\mathbb{1}_i)Y'(\mathbb{1}_i) & \xrightarrow{\sim\varepsilon_a\sim} & Y(a\mathbb{1}_i)Y'(\mathbb{1}_i)Y'(a)Y(a)Y(\mathbb{1}_i)Y'(\mathbb{1}_i) & & Y(a)Y(\mathbb{1}_i) \\
& & \Big\downarrow{\sim\zeta_{\mathbb{1}_i}} & & \Big\downarrow{\sim\zeta_{\mathbb{1}_i}} & & \Big\downarrow{\sim\eta'_i} \\
& & Y(a\mathbb{1}_i)Y(\mathbb{1}_i) & \xrightarrow{\sim\varepsilon_a} & Y(a\mathbb{1}_i)Y(\mathbb{1}_i)Y'(a)Y(a) & & Y(a)Y(\mathbb{1}_i)Y'(\mathbb{1}_i) \\
& & & & \Big\downarrow{\sim\theta'_{a,\mathbb{1}_i}\sim} & & \Big\downarrow{\sim\zeta_{\mathbb{1}_i}} \\
& & & & Y(a\mathbb{1}_i)Y'(a\mathbb{1}_i)Y(a) & \xrightarrow{\zeta_{a\mathbb{1}_i}\sim} & Y(a)
\end{array}
$$

and

$$
\begin{array}{ccc}
Y(a) & =\!=\!= \; Y(a\mathbb{1}_i) & \xrightarrow{\sim\eta'_i} \; Y(a\mathbb{1}_i)Y'(\mathbb{1}_i) \\
{}_{\zeta_a\sim}\nwarrow \quad \Big\downarrow{\sim\varepsilon_a} & & \Big\downarrow{\sim\varepsilon_a} \\
Y(a\mathbb{1}_i)Y'(a)Y(a) & \xrightarrow{\sim\eta'_i\sim} & Y(a\mathbb{1}_i)Y'(\mathbb{1}_i)Y'(a)Y(a) \\
& & \Big\downarrow{\sim\theta'_{a,\mathbb{1}_i}\sim} \\
& & Y(a\mathbb{1}_i)Y'(a\mathbb{1}_i)Y(a).
\end{array}
$$

(a_2) This follows similarly.

(b) Glue the following two commutative diagrams together at the common column.

$$
\begin{array}{ccccc}
Y(cba) & \xrightarrow{\sim\varepsilon_{ba}} & Y(cba)Y'(ba)Y(ba) & \xrightarrow{\sim\varepsilon_c\sim} & Y(cba)Y'(ba)Y'(c)Y(c)Y(ba) \\
\Big\downarrow{\sim\varepsilon_a} & & & & \Big\downarrow{\sim\varepsilon_a} \\
Y(cba)Y'(a)Y(a) & & & & Y(cba)Y'(ba)Y'(c)Y(c)Y(ba)Y'(a)Y(a) \\
\Big\downarrow{\sim\varepsilon_b\sim} & & & & \Big\downarrow{\sim\varepsilon_b\sim} \\
Y(cba)Y'(a)Y'(b)Y(b)Y(a) & \xrightarrow{\sim\varepsilon_{ba}\sim} & Y(cba)Y'(ba)Y(ba)Y'(a)Y'(b)Y(b)Y(a) & \xrightarrow{\sim\varepsilon_c\sim} & Y(cba)Y'(ba)Y'(c)Y(c)Y(ba)Y'(a)Y'(b)Y(b)Y(a) \\
\Big\downarrow{\sim\theta'_{b,a}\sim} & & \Big\downarrow{\sim\theta'_{b,a}\sim} & & \Big\downarrow{\sim\theta'_{b,a}\sim} \\
Y(cba)Y'(ba)Y(b)Y(a) & \xrightarrow{\sim\varepsilon_{ba}\sim} & Y(cba)Y'(ba)Y(ba)Y'(ba)Y(b)Y(a) & & Y(cba)Y'(ba)Y'(c)Y(c)Y(ba)Y'(ba)Y(b)Y(a) \\
& & \Big\downarrow{\sim\zeta_{ba}\sim} & & \Big\downarrow{\sim\zeta_{ba}\sim} \\
& & Y(cba)Y'(ba)Y(b)Y(a) & \xrightarrow{\sim\varepsilon_c\sim} & Y(cba)Y'(ba)Y'(c)Y(c)Y(b)Y(a)
\end{array}
$$

$$Y(cba)Y'(ba)Y'(c)Y(c)Y(ba) \xmapsto{\ \sim\theta'_{c,ba}\sim\ } Y(cba)Y'(cba)Y(c)Y(ba) \xmapsto{\ \zeta_{cba}\sim\ } Y(c)Y(ba)$$

$$\Big\downarrow {\sim\varepsilon_a} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow {\sim\varepsilon_a}$$

$$Y(cba)Y'(ba)Y'(c)Y(c)Y(ba)Y'(a)Y(a) \qquad\qquad\qquad Y(c)Y(ba)Y'(a)Y(a)$$

$$\Big\downarrow {\sim\varepsilon_b\sim} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow {\sim\varepsilon_b\sim}$$

$$Y(cba)Y'(ba)Y'(c)Y(c)Y(ba)Y'(a)Y'(b)Y(b)Y(a) \qquad\qquad Y(c)Y(ba)Y'(a)Y'(b)Y(b)Y(a)$$

$$\Big\downarrow {\sim\theta'_{b,a}\sim} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow {\sim\theta'_{b,a}\sim}$$

$$Y(cba)Y'(ba)Y'(c)Y(c)Y(ba)Y'(ba)Y(b)Y(a) \qquad\qquad Y(c)Y(ba)Y'(ba)Y(b)Y(a)$$

$$\Big\downarrow {\sim\zeta_{ba}\sim} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow {\sim\zeta_{ba}\sim}$$

$$Y(cba)Y'(ba)Y'(c)Y(c)Y(b)Y(a) \xmapsto{\ \sim\theta'_{c,ba}\sim\ } Y(cba)Y'(cba)Y(c)Y(b)Y(a) \xmapsto{\ \zeta_{cba}\sim\ } Y(c)Y(b)Y(a)$$

Then the composite of the path consisting of the top row and the right most column gives the clockwise composite of (b). The glued diagram and the commutative diagram

$$Y(cba)Y'(a)Y'(b)Y(b)Y(a) \xmapsto{\ \sim\theta'_{b,a}\sim\ } Y(cba)Y'(ba)Y(b)Y(a)$$

$$\Big\downarrow {\sim\varepsilon_c\sim} \qquad\qquad\qquad\qquad\qquad\qquad \Big\downarrow {\sim\varepsilon_c\sim}$$

$$Y(cba)Y'(a)Y'(b)Y'(c)Y(c)Y(b)Y(a) \xmapsto{\ \sim\theta'_{b,a}\sim\ } Y(cba)Y'(ba)Y'(c)Y(c)Y(b)Y(a)$$

show that the clockwise composite of (b) is given by

$$Y(c)\theta_{b,a} \circ \theta_{c,ba} = (\zeta_{cba}\sim) \circ (\sim\theta'_{c,ba}\sim) \circ (\sim\theta'_{b,a}\sim) \circ (\sim\varepsilon_c\sim) \circ (\sim\varepsilon_b\sim) \circ (\sim\varepsilon_a).$$

Similarly the anti-clockwise composite of (b) is given by

$$\theta_{c,b}Y(a) \circ \theta_{cb,a} = (\zeta_{cba}\sim) \circ (\sim\theta'_{cb,a}\sim) \circ (\sim\theta'_{c,b}\sim) \circ (\sim\varepsilon_c\sim) \circ (\sim\varepsilon_b\sim) \circ (\sim\varepsilon_a).$$

Hence they coincide because Y' is a lax functor and

$$(\sim\theta'_{c,ba}\sim) \circ (\sim\theta'_{b,a}\sim) = (\sim\theta'_{cb,a}\sim) \circ (\sim\theta'_{c,b}\sim).$$

$\square$

Definition 3.4. Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. We define the *module oplax functor* $\mathrm{Mod}\, X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Ab})$ of X as follows.

- For each $i \in I_0$, we set $(\mathrm{Mod}\, X)(i) := (\mathrm{Mod}'\, X)(i) := \mathrm{Mod}(X(i))$.
- For each $a \colon i \to j$ in I, define an $X(i)$-$X(j)$-bimodule $\overline{X(a)}$ by

$$\overline{X(a)}(x, y) := X(j)(y, X(a)(x))$$

  for all $x \in X(i)_0$ and $y \in X(j)_0$. Using this bimodule we define a functor $(\mathrm{Mod}\, X)(a) \colon (\mathrm{Mod}\, X)(i) \to (\mathrm{Mod}\, X)(j)$ by $(\mathrm{Mod}\, X)(a) := \text{-} \otimes_{X(i)} \overline{X(a)}$ that is a left adjoint to $(\mathrm{Mod}'\, X)(a)$.
- By applying Proposition 4.8 to the lax functor $\mathrm{Mod}'\, X$ and $(\mathrm{Mod}\, X)(a)$ $(a \in I_1)$, we can define an oplax functor $\mathrm{Mod}\, X = (\mathrm{Mod}\, X, \mathrm{Mod}\, \eta, \mathrm{Mod}\, \theta) \colon I \to \Bbbk\text{-}\mathbf{Ab}$.

4. The derived oplax functor

Definition 4.1. Let $Y = (Y, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Ab})$. Then we define an oplax functor $\mathcal{C}(Y) = (\mathcal{C}(Y), \mathcal{C}(\eta), \mathcal{C}(\theta)) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and an oplax functor $\mathcal{K}(Y) = (\mathcal{K}(Y), \mathcal{K}(\eta), \mathcal{K}(\theta)) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$ as follows.

(1) $\mathcal{C}(Y)$

- For each $i \in I_0$, $\mathcal{C}(Y)(i) := \mathcal{C}(Y(i))$, the category of complexes in $Y(i)$.
- For each $a \colon i \to j$ in I, $\mathcal{C}(Y)(a) \colon \mathcal{C}(Y)(i) \to \mathcal{C}(Y)(j)$ is a functor defined as follows:
    - For each $(x^p, d_x^p)_{p \in \mathbb{Z}} \in \mathcal{C}(Y)(i)_0$,
    $$\mathcal{C}(Y)(a)(x^p, d_x^p)_{p \in \mathbb{Z}} := (Y(a)x^p, Y(a)d_x^p)_{p \in \mathbb{Z}}.$$
    - For each $(f^p)_{p \in \mathbb{Z}} \colon (x^p, d_x^p)_{p \in \mathbb{Z}} \to (y^p, d_y^p)_{p \in \mathbb{Z}}$ in $\mathcal{C}(Y)(i)$,
    $$\mathcal{C}(Y)(a)(f^p)_{p \in \mathbb{Z}} := (Y(a)f^p)_{p \in \mathbb{Z}}.$$
- For each $i \in I_0$, $\mathcal{C}(\eta)_i \colon \mathcal{C}(Y)(\mathbb{1}_i) \Rightarrow \mathbb{1}_{\mathcal{C}(Y)(i)}$ is defined by
$$\mathcal{C}(\eta)_i(x^p, d_x^p)_{p \in \mathbb{Z}} := (\eta_i(x^p))_{p \in \mathbb{Z}} \colon (Y(\mathbb{1}_i)x^p, Y(\mathbb{1}_i)d_x^p)_{p \in \mathbb{Z}} \to (x^p, d_x^p)_{p \in \mathbb{Z}}$$
    for all $(x^p, d_x^p)_{p \in \mathbb{Z}} \in \mathcal{C}(Y)(i)_0$.
- For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I, $\mathcal{C}(\theta)_{b,a} \colon \mathcal{C}(Y)(ba) \Rightarrow \mathcal{C}(b)\mathcal{C}(a)$ is defined by
$$\mathcal{C}(\theta)_{b,a}(x^p, d_x^p)_{p \in \mathbb{Z}} := (\theta_{b,a}(x^p))_{p \in \mathbb{Z}} \colon (Y(ba)x^p, Y(ba)d^p)_{p \in \mathbb{Z}} \to (Y(b)Y(a)x^p, Y(b)Y(a)d^p)_{p \in \mathbb{Z}}$$
    for all $(x^p, d_x^p)_{p \in \mathbb{Z}} \in \mathcal{K}(Y)(i)_0$.

(2) $\mathcal{K}(Y)$ (underlines denote the homotopy classes below)

- For each $i \in I_0$, $\mathcal{K}(Y)(i) := \mathcal{K}(Y(i))$, the homotopy category of $Y(i)$.
- For each $a \colon i \to j$ in I, $\mathcal{K}(Y)(a) \colon \mathcal{K}(Y)(i) \to \mathcal{K}(Y)(j)$ is a functor defined as follows:
    - For each $(x^p, d_x^p)_{p \in \mathbb{Z}} \in \mathcal{K}(Y)(i)_0$,
    $$\mathcal{K}(Y)(a)(x^p, d_x^p)_{p \in \mathbb{Z}} := (Y(a)x^p, Y(a)d_x^p)_{p \in \mathbb{Z}}.$$
    - For each $\underline{(f^p)}_{p \in \mathbb{Z}} \colon (x^p, d_x^p)_{p \in \mathbb{Z}} \to (y^p, d_y^p)_{p \in \mathbb{Z}}$ in $\mathcal{K}(Y)(i)$,
    $$\mathcal{K}(Y)(a)\underline{(f^p)}_{p \in \mathbb{Z}} := \underline{(Y(a)f^p)}_{p \in \mathbb{Z}}.$$
- For each $i \in I_0$, $\mathcal{K}(\eta)_i \colon \mathcal{K}(Y)(\mathbb{1}_i) \Rightarrow \mathbb{1}_{\mathcal{K}(Y)(i)}$ is defined by
$$\mathcal{K}(\eta)_i(x^p, d_x^p)_{p \in \mathbb{Z}} := \underline{(\eta_i(x^p))}_{p \in \mathbb{Z}} \colon (Y(\mathbb{1}_i)x^p, Y(\mathbb{1}_i)d_x^p)_{p \in \mathbb{Z}} \to (x^p, d_x^p)_{p \in \mathbb{Z}}$$
    for all $(x^p, d_x^p)_{p \in \mathbb{Z}} \in \mathcal{K}(Y)(i)_0$.
- For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I, $\mathcal{K}(\theta)_{b,a} \colon \mathcal{K}(Y)(ba) \Rightarrow \mathcal{K}(b)\mathcal{K}(a)$ is defined by
$$\mathcal{K}(\theta)_{b,a}(x^p, d_x^p)_{p \in \mathbb{Z}} := \underline{(\theta_{b,a}(x^p))}_{p \in \mathbb{Z}} \colon (Y(ba)x^p, Y(ba)d^p)_{p \in \mathbb{Z}} \to (Y(b)Y(a)x^p, Y(b)Y(a)d^p)_{p \in \mathbb{Z}}$$
    for all $(x^p, d_x^p)_{p \in \mathbb{Z}} \in \mathcal{K}(Y)(i)_0$.

Proposition 4.2. *In the above, $\mathcal{C}(Y)$ and $\mathcal{K}(Y)$ are well-defined.*

Proof. Straightforward. $\qquad\square$

Example 4.3. Let $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then applying the proposition above to $Y := \mathrm{Mod}\, X$ we obtain the definition of $\mathcal{K}(\mathrm{Mod}\, X)$. Note that for each $a\colon i \to j$ in I, $\mathcal{K}(\mathrm{Mod}\, X)(i) \xrightarrow{\mathcal{K}(\mathrm{Mod}\, X)(a)} \mathcal{K}(\mathrm{Mod}\, X)(j)$ is equal to

$$\mathcal{K}(\mathrm{Mod}\, X(i)) \xrightarrow{-\otimes_{X(i)} \overline{X(a)}} \mathcal{K}(\mathrm{Mod}\, X(j)).$$

The following is obvious.

Lemma 4.4. *Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and $Y(i)$ be a subcategory of $X(i)$ for each $i \in I_0$. Assume that $X(a)(Y(i)_0) \subseteq Y(j)_0$ and $X(a)(Y(i)_1) \subseteq Y(j)_1$ for each $a\colon i \to j$ in I. Then we can define a functor $Y(a)\colon Y(i) \to Y(j)$ as the restriction $Y(a) := X(a)|_{Y(i)}$ of $X(a)$ to $Y(i)$ for each $a\colon i \to j$ in I. If we define $\eta' := (\eta'_i)_{i \in I_0}$ and $\theta' := (\theta'_{b,a})$ as follows, then $Y = (Y, \eta', \theta')$ turns out to be an oplax functor $I \to \Bbbk\text{-}\mathbf{Cat}$:*

(1) *For each $i \in I_0$ and each $y \in Y(i)_0$, we set*

$$\eta'_i(y) := \eta_i(y)\colon Y(\mathbb{1}_i)y \to y;$$

(2) *For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I and $y \in Y(i)_0$, we set*

$$\theta'_{b,a}(y) := \theta_{b,a}(y)\colon Y(ba)y \to Y(b)Y(a)y.$$

This Y is called the oplax subfunctor *of X defined by $Y(i)$ $(i \in I_0)$.* $\qquad\square$

Definition 4.5. Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Using the lemma above we define the oplax subfunctor $\mathrm{prj}\, X$ of $\mathrm{Mod}\, X$ by $\mathrm{prj}(X(i))$ $(i \in I_0)$ and the oplax subfunctor $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ of $\mathcal{K}(\mathrm{Mod}\, X)$ by $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(X(i)))$ $(i \in I_0)$.

Lemma 4.6. *Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then for each $a\colon i \to j$ in I, $\mathcal{K}(\mathrm{Mod}\, X)(a)(\mathcal{K}_p(\mathrm{Mod}\, X(i)) \subseteq \mathcal{K}_p(\mathrm{Mod}\, X(j))$.*

Proof. Let A be an acyclic complex in $\mathcal{K}(\mathrm{Mod}\, X(j))$ and let $P \in \mathcal{K}_p(\mathrm{Mod}\, X(i))$. Then $\mathcal{K}(\mathrm{Mod}\, X(j))(\mathcal{K}(\mathrm{Mod}\, X)(a)(P), A) \cong \mathcal{K}(\mathrm{Mod}\, X(i))(P, A \circ X(a)) = 0$. $\qquad\square$

This enables us to define the following by Lemma 4.4.

Definition 4.7. Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then we define an oplax functor $\mathcal{K}_p(\mathrm{Mod}\, X) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$ as the oplax subfunctor of $\mathcal{K}(\mathrm{Mod}\, X)$ defined by $\mathcal{K}_p(\mathrm{Mod}\, X(i))$ $(i \in I_0)$.

Lemma 4.8. *Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. For each $i \in I_0$, let $Y(i) \in \Bbbk\text{-}\mathbf{Cat}_0$ and assume that we have an adjoint pair $L_i \dashv R_i$ of functors $L_i\colon X(i) \to Y(i)$ and $R_i\colon Y(i) \to X(i)$ with a unit $\varepsilon_i\colon \mathbb{1}_{X(i)} \Rightarrow R_i L_i$ and a counit $\zeta_i\colon L_i R_i \Rightarrow \mathbb{1}_{Y(i)}$.*

Define (Y, η', θ') as follows.

- *Define $Y(a)\colon Y(i) \to Y(j)$ as $Y(a) := L_j X(a) R_i$ for each $a\colon i \to j$ in I.*
- *For each $i \in I_0$, define $\eta'_i\colon Y(\mathbb{1}_i) \Rightarrow \mathbb{1}_{Y(i)}$ as the composite*

$$Y(\mathbb{1}_i) = L_i X(\mathbb{1}_i) R_i \xRightarrow{L_i \eta_i R_i} L_i R_i \xRightarrow{\zeta_i} \mathbb{1}_{Y(i)}$$

and set $\eta' := (\eta'_i)_{i \in I_0}$;

- *For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I, define $\theta'_{b,a} \colon Y(ba) \Rightarrow Y(b)Y(a)$ as the composite*

$$Y(ba) = L_k X(ba) R_i \xRightarrow{L_k \theta_{b,a} R_i} L_k X(b) X(a) R_i$$
$$\xRightarrow{L_k X(b) \varepsilon_j X(a) R_j} L_k X(b) R_j L_j X(a) R_i = Y(b)Y(a).$$

  and set $\theta' := (\theta'_{b,a})$.

Then

(1) $Y = (Y, \eta', \theta') \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. *We say that this Y is the oplax functor induced from X by* adjoint pairs L_i, R_i $(i \in I_0)$.

(2) *The family $(R_i)_{i \in I_0}$ is extended to a morphism*

$$R = (R, \phi^R) \colon Y \to X \quad in \ \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$$

  by defining $R(i) := R_i$ for all $i \in I_0$ and $\phi^R(a) := \varepsilon_j X(a) R_i$ for all $a \colon i \to j$ in I.

(3) *Assume further that ε_i is an isomorphism for each $i \in I_0$. Then*

  (a) $\phi^R(a)$ *is an isomorphism for each $a \in I_1$, i.e., R is I-equivariant; and*

  (b) *the family $(L_i)_{i \in I_0}$ is extended to an I-equivariant morphism*

$$L = (L, \phi^L) \colon X \to Y \quad in \ \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$$

  by defining $L(i) := L_i$ for all $i \in I_0$ and $\phi^L(a) := L_j X(a) \varepsilon_i^{-1}$ for all $a \colon i \to j$ in I.

Proof. (1) For each $a \colon i \to j$ in I, the axiom (a$_1$) follows from the following commutative diagram:

$$
\begin{array}{ccccc}
L_j X(a \mathbb{1}_i) R_i & \xRightarrow{\ \sim \theta_{a,1_i} \sim\ } & L_j X(a) X(\mathbb{1}_i) R_i & \xRightarrow{\ \sim \varepsilon_i \sim\ } & L_j X(a) R_i L_i X(\mathbb{1}_i) R_i \\
 & \searrow & \Big\Vert {\scriptstyle \sim \eta_i \sim} & & \Big\Vert {\scriptstyle \sim \eta_i \sim} \\
 & & L_j X(a) R_i & \xRightarrow{\ \sim \varepsilon_i \sim\ } & L_j X(a) R_i L_i R_i \\
 & & & \searrow & \Big\Vert {\scriptstyle \sim \zeta_i} \\
 & & & & L_j X(a) R_i
\end{array}
$$

The axiom (a$_2$) follows similarly.

For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in I, the axiom (b) follows from the following commutative diagram:

$$
\begin{array}{ccccc}
L_l X(cba) R_i & \xRightarrow{\ \sim \theta_{c,ba} \sim\ } & L_l X(c) X(ba) R_i & \xRightarrow{\ \sim \varepsilon_k \sim\ } & L_l X(c) R_k L_k X(ba) R_i \\
{\scriptstyle \sim \theta_{cb,a} \sim} \Big\Vert & & \Big\Vert {\scriptstyle \sim \theta_{b,a} \sim} & & \Big\Vert {\scriptstyle \sim \theta_{b,a} \sim} \\
L_l X(cb) X(a) R_i & \xRightarrow{\ \sim \theta_{c,b} \sim\ } & L_l X(c) X(b) X(a) R_i & \xRightarrow{\ \sim \varepsilon_k \sim\ } & L_l X(c) R_k L_k X(b) X(a) R_i \\
{\scriptstyle \sim \varepsilon_j \sim} \Big\Vert & & \Big\Vert {\scriptstyle \sim \varepsilon_j \sim} & & \Big\Vert {\scriptstyle \sim \varepsilon_j \sim} \\
L_l X(cb) R_j L_j X(a) R_i & \xRightarrow{\ \sim \theta_{c,b} \sim\ } & L_l X(c) X(b) R_j L_j X(a) R_i & \xRightarrow{\ \sim \varepsilon_k \sim\ } & L_l X(c) R_k L_k X(b) R_j L_j X(a) R_i.
\end{array}
$$

(2) and (3) Straightforward. □

Definition 4.9. Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.

(1) For each $i \in I_0$ let L_i be the composite $\mathcal{K}_p(\mathrm{Mod}\, X(i)) \hookrightarrow \mathcal{K}(\mathrm{Mod}\, X(i)) \to \mathcal{D}(\mathrm{Mod}\, X(i))$ of the embedding and the quotient functor. Then it is well-known that L_i is a triangle equivalence with a quasi-inverse $R_i \colon \mathcal{D}(\mathrm{Mod}\, X(i)) \to \mathcal{K}_p(\mathrm{Mod}\, X(i))$. Then we define the *derived oplax functor*

$$\mathcal{D}(\mathrm{Mod}\, X) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$$

of X as the oplax functor induced from the oplax functor $\mathcal{K}_p(\mathrm{Mod}\, X)$ by the adjoint pairs L_i, R_i $(i \in I_0)$.

(2) We define an oplax functor per $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$ as the oplax subfunctor of $\mathcal{D}(\mathrm{Mod}\, X)$ defined by the subcategories per $X(i)$ of $\mathcal{D}(\mathrm{Mod}\, X(i))$ consisting of the perfect complexes $(i \in I_0)$.

Remark 4.10. In the above, we have the following by definition.

(1) For each $a \colon i \to j$ in I, $\mathcal{D}(\mathrm{Mod}\, X)(i) \xrightarrow{\mathcal{D}(\mathrm{Mod}\, X)(a)} \mathcal{D}(\mathrm{Mod}\, X)(j)$ is equal to

$$\mathcal{D}(\mathrm{Mod}\, X(i)) \xrightarrow{\text{-}\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{X(a)}} \mathcal{D}(\mathrm{Mod}\, X(j)).$$

(2) per X and $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$.

5. Derived equivalences of oplax functors

Recall that a 1-morphism $f \colon x \to y$ in a 2-category $\mathbf{C}$ is called an *equivalence* if there exist 1-morphism $g \colon y \to x$ in $\mathbf{C}$ such that there exist 2-isomorphisms $gf \Rightarrow \mathbb{1}_x$ and $fg \Rightarrow \mathbb{1}_y$ in $\mathbf{C}$; and that two objects x and y in $\mathbf{C}$ are called *equivalent* in $\mathbf{C}$ if there exists an equivalence $f \colon x \to y$.

Lemma 5.1. *Let $\mathbf{C}$ be a 2-category and $(F, \psi) \colon X \to X'$ a 1-morphism in the 2-category $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$. Then (F, ψ) is an equivalence in $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$ if and only if*

(1) *For each $i \in I_0$, $F(i)$ is an equivalence in $\mathbf{C}$; and*
(2) *For each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ (namely, (F, ψ) is I-equivariant).*

Proof. Set $X = (X, \eta, \theta)$ and $X' = (X', \eta', \theta')$.

($\Rightarrow$). Assume that (F, ψ) is an equivalence in $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$. Then there exists a 1-morphism $(E, \phi) \colon X' \to X$ and 2-isomorphisms

$$\zeta \colon (\mathbb{1}_X, (\mathbb{1}_{X(a)})_a) \Longrightarrow (E, \phi) \circ (F, \psi) = (EF, (E(j)\psi(a) \circ \phi(a)F(i))_{a:i\to j}) \text{ and}$$

$$\zeta' \colon (\mathbb{1}'_X, (\mathbb{1}_{X'(a)})_a) \Longrightarrow (F, \psi) \circ (E, \phi) = (FE, (F(j)\phi(a) \circ \psi(a)E(i))_{a:i\to j})$$

in $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$. Then the assumption first shows that for each $i \in I_0$ we have 2-isomorphisms $\zeta(i) \colon \mathbb{1}_{X(i)} \Longrightarrow E(i)F(i)$ and $\zeta'(i) \colon \mathbb{1}_{X'(i)} \Longrightarrow F(i)E(i)$ in $\mathbf{C}$. Thus $F(i)$ is an equivalence in $\mathbf{C}$ with a quasi-inverse $E(i)$.

Next the assumption shows that for each $a\colon i \to j$ in I we have the following commutative diagrams of 2-morphisms in $\mathbf{C}$:

$$
\begin{array}{ccc}
X(a)\mathbb{1}_{X(i)} \overset{X(a)\zeta(i)}{\underset{\sim}{\Longrightarrow}} X(a)E(i)F(i) & \qquad & X'(a)\mathbb{1}_{X'(i)} \overset{X'(a)\zeta'(i)}{\underset{\sim}{\Longrightarrow}} X'(a)F(i)E(i) \\
\mathbf{1}_{X(a)}\Big\Downarrow \qquad\qquad \Big\Downarrow E(j)\psi(a)\circ\phi(a)F(i) & & \mathbf{1}_{X'(a)}\Big\Downarrow \qquad\qquad \Big\Downarrow F(j)\phi(a)\circ\psi(a)E(i) \\
\mathbb{1}_{X(j)}X(a) \overset{\sim}{\underset{\zeta(j)X(a)}{\Longrightarrow}} E(j)F(j)X(a) & & \mathbb{1}_{X'(j)}X'(a) \overset{\sim}{\underset{\zeta'(j)X'(a)}{\Longrightarrow}} F(j)E(j)X'(a)
\end{array}
$$

$$\tag{5.1}$$

Hence both $E(j)\psi(a) \circ \phi(a)F(i)$ and $F(j)\phi(a) \circ \psi(a)E(i)$ are 2-isomorphisms in $\mathbf{C}$, and hence so are both $F(j)E(j)\psi(a) \circ F(j)\phi(a)F(i)$ and $F(j)\phi(a)F(i) \circ \psi(a)E(i)F(i)$. Thus $F(j)E(j)\psi(a)$ is a 2-retraction and $\psi(a)E(i)F(i)$ is a 2-section. Hence $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ because both $\zeta(i)$ and $\zeta'(j)$ are 2-isomorphisms.

($\Leftarrow$). Conversely, assume that (F,ψ) satisfies the conditions (1) and (2). Then by (1), for each $i \in I_0$ there exists a quasi-inverse $E(i)$ of $F(i)$, thus there exist 2-isomorphisms $\zeta_i\colon \mathbb{1}_{X(i)} \Longrightarrow E(i)F(i)$ and $\zeta'_i\colon \mathbb{1}_{X'(i)} \Longrightarrow F(i)E(i)$ satisfying the following equations:

$$E(i)\zeta'_i = \zeta_i E(i) \tag{5.2}$$

$$F(i)\zeta_i = \zeta'_i F(i) \tag{5.3}$$

By (2) we can construct a $\phi := (\phi(a))_{a \in I_1}$ by the following commutative diagram for each $a\colon i \to j$ in I:

$$
\begin{array}{ccc}
X(a)E(i) & \overset{\phi(a)}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\Longrightarrow} & E(j)X'(a) \\
\zeta(j)X(a)E(i)\Big\Downarrow & & \Big\Uparrow E(j)X'(a)\zeta'(i)^{-1} \\
E(j)F(j)X(a)E(i) & \underset{E(j)\psi(a)^{-1}E(i)}{=\!=\!=\!=\!=\!=\!=\!=\!=\Longrightarrow} & E(j)X'(a)F(i)E(i).
\end{array}
$$

It is enough to show the following

$$(E,\phi)\colon (X',\eta',\theta') \to (X,\eta,\theta) \text{ is in } \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat}); \tag{5.4}$$

$$\zeta := (\zeta(i))_{i \in I_0}\colon (\mathbb{1}_X, (\mathbb{1}_{X(a)})_a) \Longrightarrow (E,\phi) \circ (F,\psi) \text{ is a 2-isomorphism; and} \tag{5.5}$$

$$\zeta' := (\zeta'(i))_{i \in I_0}\colon (\mathbb{1}_{X'}, (\mathbb{1}_{X'(a)})_a) \Longrightarrow (F,\psi) \circ (E,\phi) \text{ is a 2-isomorphism.} \tag{5.6}$$

Using (5.2) it is not hard to verify the statement (5.4). The statements (5.5) and (5.6) are equivalent to the commutativity of the left diagram and of the right diagram in (5.1), respectively, and both follow from (5.3). $\qquad\square$

Definition 5.2. Let $X, X' \in \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})$. Then X and X' are said to be *derived equivalent* if $\mathcal{D}(\mathrm{Mod}\,X)$ and $\mathcal{D}(\mathrm{Mod}\,X')$ are equivalent in the 2-category $\overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Tri})$.

By Lemma 5.1 we obtain the following.

Proposition 5.3. *Let $X, X' \in \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})$. Then X and X' are derived equivalent if and only if there exists a 1-morphism $(F,\psi)\colon \mathcal{D}(\mathrm{Mod}\,X) \to \mathcal{D}(\mathrm{Mod}\,X')$ in $\overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Tri})$ such that*

(1) *For each $i \in I_0$, $F(i)$ is a triangle equivalence; and*
(2) *For each $a \in I_1$, $\psi(a)$ is a natural isomorphism (i.e., (F, ψ) is I-equivariant).*

A $\Bbbk$-category $\mathcal{A}$ is called $\Bbbk$-*projective* if $\mathcal{A}(x, y)$ are projective $\Bbbk$-modules for all $x, y \in \mathcal{A}_0$. We formulate a categorical version of Keller's lifting theorem [9, Theorem 2.1] (in the $\Bbbk$-projective case) as follows, a proof of which is given by B. Keller in Appendix.

Theorem 5.4 (Keller)**.** *Let $\mathcal{A}, \mathcal{B} \in \Bbbk$-**Cat**, and let $T \colon \mathcal{A} \to \mathcal{K}(\operatorname{Prj} \mathcal{B})$ be a $\Bbbk$-functor factoring through the inclusion $\mathcal{K}^-(\operatorname{Prj} \mathcal{B}) \to \mathcal{K}(\operatorname{Prj} \mathcal{B})$. Set $\nu \colon \mathcal{C}(\operatorname{Prj} \mathcal{B}) \to \mathcal{K}(\operatorname{Prj} \mathcal{B})$ and $Q \colon \mathcal{K}(\operatorname{Prj} \mathcal{B}) \to \mathcal{D}(\operatorname{Mod} \mathcal{B})$ to be the canonical functors, and put $\pi := Q\nu$. Assume that $\mathcal{A}$ is $\Bbbk$-projective and that T satisfies the* Toda condition*:*

$$\mathcal{K}(\operatorname{Mod} \mathcal{B})(T(x), T(y)[n]) = 0, \forall n < 0, \forall x, y \in \mathcal{A}_0.$$

Then the following hold.

(a) *There exists a $\Bbbk$-functor $B \colon \mathcal{A} \to \mathcal{C}(\operatorname{Prj} \mathcal{B})$ factoring through the inclusion $\mathcal{C}^-(\operatorname{Prj} \mathcal{B}) \to \mathcal{C}(\operatorname{Prj} \mathcal{B})$ and a natural transformation $q \colon T \implies \nu B$ such that $q_x \colon T(x) \to \nu B(x)$ are quasi-isomorphisms for all $x \in \mathcal{A}_0$.*
(b) *If $B' \colon \mathcal{A} \to \mathcal{C}(\operatorname{Prj} \mathcal{B})$ is a $\Bbbk$-functor factoring through the inclusion $\mathcal{C}^-(\operatorname{Prj} \mathcal{B}) \to \mathcal{C}(\operatorname{Prj} \mathcal{B})$ and $q' \colon T \implies \nu B'$ is a natural transformation such that $q'_x \colon T(x) \to \nu B'(x)$ are quasi-isomorphisms for all $x \in \mathcal{A}_0$, then there exists a natural transformation $p \colon B \to B'$ such that $q' = (\nu p) \circ q$ and that $\pi p \colon \pi B \to \pi B'$ is an isomorphism in $\operatorname{Fun}(\mathcal{A}, \mathcal{D}(\operatorname{Mod} \mathcal{B}))$. Further if a natural transformation $p' \colon B \to B'$ has the same property as p, then $\nu p = \nu p'$, i.e., p is unique up to homotopy.*

Definition 5.5. Let $X \colon I \to \Bbbk$-**Cat** be an oplax functor.

(1) X is called $\Bbbk$-*projective* if $X(i)$ are $\Bbbk$-projective for all $i \in I_0$.
(2) An oplax subfunctor $\mathcal{T}$ of of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$ is called *tilting* if for each $i \in I_0$, $\mathcal{T}(i)$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$, namely,
 - $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))(U, V[n]) = 0$ for all $U, V \in \mathcal{T}(i)_0$ and $0 \neq n \in \mathbb{Z}$; and
 - the smallest thick subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$ containing $\mathcal{T}(i)$ is equal to $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$.
(3) A tilting oplax subfunctor $\mathcal{T}$ of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$ with an I-equivariant inclusion $(\sigma, \rho) \colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$ is called a *tilting oplax functor* for X.

The following is our main result in this paper that gives a generalization of the Morita type theorem characterizing derived equivalences of categories by Rickard [11] and Keller [8] in our setting.

Theorem 5.6. *Let $X, X' \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Consider the following conditions.*

(1) X *and* X' *are derived equivalent.*
(2) $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$ *and* $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X')$ *are equivalent in* $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$.
(3) *There exists a tilting oplax functor $\mathcal{T}$ for X such that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.*

Then

(a) (1) *implies* (2).

(b) (2) *implies* (3).

(c) *If X' is $\Bbbk$-projective, then* (3) *implies* (1).

Proof. (a) Assume the statement (1). Then there exists an equivalence

$$(F, \psi) \colon \mathcal{D}(\operatorname{Mod} X) \to \mathcal{D}(\operatorname{Mod} X')$$

in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$. Since $F(i)$ sends compact objects to compact objects, we have $F(i)(\operatorname{per} X(i)) \subseteq \operatorname{per} X'(i)$ for each $i \in I_0$. This shows that (F, ψ) induces an equivalence $\operatorname{per} X \to \operatorname{per} X'$. Hence by Remark 4.10(2) the statement (2) follows.

(b) Assume the statement (2). Then we have an equivalence

$$(F, \psi) \colon \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X') \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$$

in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$. We define an I-equivariant morphism

$$(H, \phi^H) \colon X' \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X')$$

as follows. For each $i \in I_0$, let $H(i) \colon X'(i) \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X'(i))$ be the Yoneda embedding, namely it is defined by sending each morphism $f \colon x \to y$ in $X'(i)$ to the morphism

$$X'(i)(?, x) \xrightarrow{X'(i)(?, f)} X'(i)(?, y)$$

of complexes concentrated in degree zero. For each $a \colon i \to j$ in I, let

$$\phi^H(a)x \colon X'(i)(?, x) \otimes_{X'(i)} X'(j)(\text{-}, X'(a)(?)) \to X'(j)(\text{-}, X'(a)x)$$

be the canonical isomorphism. Then $\phi^H(a) := (\phi^H(a)x)_{x \in X'(i)_0}$ is a natural isomorphism:

$$
\begin{array}{ccc}
X'(i) & \xrightarrow{\ H(i)\ } & \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X'(i)) \\
{\scriptstyle X'(a)}\downarrow & {\scriptstyle \phi^H(a)} \nearrow & \downarrow{\scriptstyle \text{-}\otimes_{X'(i)}\overline{X'(a)}} \\
X'(j) & \xrightarrow[\ H(j)\ ]{} & \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X'(j)),
\end{array}
$$

and it is easy to check that $(H, \phi^H) := ((H(i))_{i \in I_0}, (\phi^H(a))_{a \in I_1})$ is a morphism in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.

Let $i \in I_0$. We set $\mathcal{T}(i)$ to be the full subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$ consisting of the objects $F(i)H(i)x$ with $x \in X'(i)_0$, and $\sigma(i) \colon \mathcal{T}(i) \to \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$ to be the inclusion functor. Then $\mathcal{T}(i)$ turns out to be a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i))$ because the full subcategory of $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X'(i))$ consisting of the objects $H(i)x$ with $x \in X'(i)_0$ is tilting and $F(i)$ is a triangle equivalence. Since $F(i)$ is an equivalence, $F(i)$ restricts to an equivalence $R_i \colon X'(i) \to \mathcal{T}(i)$ with a quasi-inverse L_i:

$$
\begin{array}{ccc}
\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X'(i)) & \xrightarrow{\ F(i)\ } & \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X(i)) \\
{\scriptstyle H(i)}\uparrow & & \uparrow{\scriptstyle \sigma(i)} \\
X'(i) & \dashrightarrow[R_i] & \mathcal{T}(i),
\end{array}
$$

where we can take L_i as a section of R_i to have $R_i L_i = \mathbb{1}_{\mathcal{T}(i)}$.

By Lemma 4.8 $(\mathcal{T}(i))_{i \in I_0}$ extends to an oplax functor $\mathcal{T} \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and both $(R_i)_{i \in I_0}$ and $(L_i)_{i \in I_0}$ extend to I-equivariant morphisms $(R, \phi^R)\colon X' \to \mathcal{T}$ and $(L, \phi^L)\colon \mathcal{T} \to X'$, respectively. Then (R, ϕ^R) is an equivalence, and hence $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. We extend the family of inclusions $(\sigma(i))_{i \in I_0}$ to an I-equivariant morphism $(\sigma, \rho)\colon \mathcal{T} \to \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$. Since $F(i)H(i)L_i = \sigma(i)R_iL_i = \sigma(i)$ for each $i \in I_0$, we can define $(\sigma, \rho)\colon \mathcal{T} \to \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ by $(\sigma, \rho) := (F, \psi) \circ (H, \phi^H) \circ (L, \phi^L)$. Since $\phi^L(a)$, $\phi^H(a)$ and $\psi(a)$ are isomorphisms, $\rho(a)$ is also an isomorphism for each $a \in I_1$. Thus (σ, ρ) is an I-equivariant morphism, which shows that $\mathcal{T}$ is a tilting oplax functor for X.

(c) Assume the statement (3). Then we have an I-equivariant morphism

$$(E, \phi)\colon X' \to T \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X) \hookrightarrow \mathcal{K}^{-}(\mathrm{Prj}\, X)$$

such that $E(i)\colon X'(i) \to \mathcal{K}^{-}(\mathrm{Prj}\, X(i))$ is fully faithful for each $i \in I_0$ and that $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))(E(i)x, E(i)y[n]) = 0$ for each $x, y \in X'(i)$ and each $n \neq 0$. By Theorem 5.4 (a) there exist a $\Bbbk$-functor $B_i\colon X'(i) \to \mathcal{C}(\mathrm{Mod}\, X(i))$ and a quasi-isomorphism $q_i\colon E(i) \Longrightarrow B_i$. For each $a\colon i \to j$ in I we set $\chi(a)$ to be the composite

$$E(i) \otimes_{X(i)} \overline{X(a)} \overset{\rho(a)E(i)}{\Longrightarrow} T(a)E(i) \overset{\phi(a)}{\Longrightarrow} E(j)X'(a) = \overline{X'(a)} \otimes_{X'(j)} E(j)$$

of natural isomorphisms. Then we have the following diagram with solid arrows.

$$
\begin{array}{ccc}
E(i) \otimes_{X(i)} \overline{X(a)} & \overset{\chi(a)}{\Longrightarrow} & \overline{X'(a)} \otimes_{X'(j)} E(j) \\
{\scriptstyle q_i \otimes_{X(i)} \overline{X(a)}} \Big\Vert & & \Big\Vert {\scriptstyle \overline{X'(a)} \otimes_{X'(j)} q_j} \\
\nu B_i \otimes_{X(i)} \overline{X(a)} & \!\!=\!=\!\Rightarrow\!\! & \overline{X'(a)} \otimes_{X'(j)} \nu B_j
\end{array}
$$

We show that this is completed to a commutative diagram. Let $x \in X'(i)_0$.

Claim 1. $\chi(a)x$ is a quasi-isomorphism.

Indeed, $E(i)x \in \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$ implies $E(i)x \otimes_{X(i)} \overline{X(a)} \in \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(j))$, and hence $\chi(a)x$ is given by a genuine morphism and is an isomorphism in $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(j))$.

Claim 2. $\overline{X'(a)} \otimes_{X'(j)} q_j(x)$ is a quasi-isomorphism.

This is obvious because $q_j(x)$ is a quasi-isomorphism in $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(j))$ and $\overline{X'(a)}_{X'(j)}$ is projective.

Claim 3. $q_i(x) \otimes_{X(i)} \overline{X(a)}$ is a quasi-isomorphism.

Indeed, since $q_i(x)$ is a quasi-isomorphism in $\mathcal{K}^{-}(\mathrm{Prj}\, X(i))$ the mapping cone $C(q_i(x))$ is acyclic, and hence so is $C(q_i(x)) \otimes_{X(i)} \overline{X(a)}$, from which the claim follows.

Claim 4. $E(i) \otimes_{X(i)} \overline{X(a)}$ satisfies the Toda condition.

Indeed, let y be another object of $X'(i)$ and $n \neq 0$. Then

$$\mathcal{K}^{-}(\mathrm{Prj}\, X(j)) \left(E(i)x \otimes_{X(i)} \overline{X(a)}, \left(E(i)y \otimes_{X(i)} \overline{X(a)} \right)[n] \right)$$
$$\cong \mathcal{K}^{-}(\mathrm{Prj}\, X(j))(E(j)(X'(a)x), E(j)(X'(a)y)[n]) = 0.$$

By Theorem 5.4 (b), it follows from these claims that there exists a natural transformation $\psi(a)\colon B_i \otimes_{X(i)} \overline{X(a)} \implies \overline{X'(a)} \otimes_{X'(j)} B_j$ such that $\nu\psi(a)$ completes the commutative diagram above and $\pi\psi(a)$ is an isomorphism. Thus we have the following diagram

$$
\begin{array}{ccc}
\mathcal{D}(\mathrm{Mod}\,X'(i)) & \xrightarrow{\ \overline{B}(i)\ } & \mathcal{D}(\mathrm{Mod}\,X(i)) \\
{\scriptstyle -\overset{\mathbf{L}}{\otimes}_{X'(i)}\overline{X'(a)}}\Big\downarrow & \overset{\overline{\psi}(a)}{\Longleftarrow} & \Big\downarrow{\scriptstyle -\overset{\mathbf{L}}{\otimes}_{X(i)}\overline{X(a)}} \\
\mathcal{D}(\mathrm{Mod}\,X'(j)) & \xrightarrow[\ \overline{B}(j)\ ]{} & \mathcal{D}(\mathrm{Mod}\,X(j)),
\end{array}
$$

where $\overline{B}(i) := -\overset{\mathbf{L}}{\otimes}_{X'(i)}\pi B_i$ are triangle equivalences because πB_i are tilting bimodule complexes, and $\overline{\psi}(a) := -\overset{\mathbf{L}}{\otimes}_{X'(i)}\pi\psi(a)$ are natural isomorphisms. We set $\overline{B} := (\overline{B}(i))_{i\in I_0}$ and $\overline{\psi} := (\overline{\psi}(a))_{a\in I_1}$. It remains to show that the pair

$$
(\overline{B},\overline{\psi})\colon \mathcal{D}(\mathrm{Mod}\,X') \to \mathcal{D}(\mathrm{Mod}\,X)
$$

is a 1-morphism in $\overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Tri})$ because when this is proved, $(\overline{B},\overline{\psi})$ becomes an equivalence in $\overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Tri})$ by Proposition 5.3, and we see that X and X' are derived equivalent.

It is enough to show that the diagram

$$
\begin{array}{ccc}
\pi B_i \otimes_{X(i)} \overline{X(\mathbb{1}_i)} & \overset{\psi(\mathbf{1}_i)}{=\!=\!=\!\Longrightarrow} & \pi B_i X'(\mathbb{1}_i) \\
{\scriptstyle \overline{\eta}_i B_i}\Big\| & & \Big\|{\scriptstyle B_i\eta_i} \\
B_i & =\!=\!=\!=\!=\!=\!= & B_i
\end{array}
\tag{5.7}
$$

for each $i \in I_0$ and the diagram

$$
\begin{array}{ccc}
\pi B_i \otimes_{X(i)} \overline{X(ba)} \overset{\pi B_i\overline{\theta}_{b,a}}{\Longrightarrow} \pi B_i \otimes_{X(i)} \overline{X(a)} \otimes_{X(j)} \overline{X(b)} & \overset{\sim\psi(a)\sim}{\Longrightarrow} & \overline{X'(a)} \otimes_{X'(j)} \pi B_j \otimes_{X(j)} \overline{X(b)} \\
{\scriptstyle \psi(ba)}\Big\| & & \Big\|{\scriptstyle \sim\psi(b)\sim} \\
\overline{X'(ba)} \otimes_{X'(k)} \pi B_k & \xrightarrow[\ \sim\overline{\theta}_{b,a}\sim\ ]{} & \overline{X'(a)} \otimes_{X'(j)} \overline{X'(b)} \otimes_{X'(k)} \pi B_k
\end{array}
\tag{5.8}
$$

for each $i \xrightarrow{a} j \xrightarrow{b} k$ in I commute, where we put $X = (X,\eta,\theta)$ and $X' = (X',\eta',\theta')$, and $\overline{\eta}_i$, $\overline{\theta}_{b,a}$ denote the morphisms induced by η_i, $\theta_{b,a}$, respectively.

The commutativity of the diagram (5.7) follows from the following commutative diagram by using the fact that $q_i \otimes_{X(i)} \overline{X(\mathbb{1}_i)}$ is a quasi-isomorphism:

$$
\begin{array}{ccc}
\nu B_i \otimes_{X(i)} \overline{X(\mathbb{1}_i)} & \xrightarrow{\ \nu\psi(\mathbf{1}_i)\ } & \nu B_i X'(\mathbb{1}_i) \\
\end{array}
$$

with intermediate maps $q_i \otimes_{X(i)} \overline{X(\mathbf{1}_i)}$, $T(i) \otimes_{X(i)} \overline{X(\mathbb{1}_i)} \Longrightarrow T(i)X'(\mathbb{1}_i)$, $q_i X'(\mathbf{1}_i)$, $T(i)\eta_i$, $\nu B_i \eta_i$, $T(i) = T(i)$, q_i, q_i, $\nu B_i = \nu B_i$.

The commutativity of the diagram (5.8) follows from the following commutative diagram by using the fact that $T(ba)q_i$ is a quasi-isomorphism:

$$
T(ba)\nu B_i \xrightarrow{\ \overline{\theta}_{b,a}\nu B_i\ } T(b)T(a)\nu B_i \xrightarrow{\ T(b)\psi(a)\ } T(b)\nu B_j X'(a)
$$

with maps $T(ba)q_i$, $T(b)T(a)q_i$, $T(b)q_j X'(a)$, $T(ba)E(i) \Longrightarrow_{\overline{\theta}_{b,a}E(i)} T(b)T(a)E(i) \Longrightarrow_{T(b)\chi(a)} T(b)E(j)X'(a)$, $\psi(ba)$, $\chi(ba)$, $\chi(b)X'(a)$, $\psi(b)X'(a)$, $E(k)X'(ba) \xrightarrow{\ E(k)\theta'_{b,a}\ } E(k)X'(b)X'(a)$, $q_k X'(ba)$, $q_k X'(b)X'(a)$, $\nu B_k X'(ba) \xrightarrow{\ \nu B_k \theta'_{b,a}\ } \nu B_k X'(b)X'(a)$,

where we regard $T(a) = \text{-} \otimes_{X(i)} \overline{X(a)}\colon T(i) \to T(j)$. $\square$

Definition 5.7. Regard a group G as a category with a unique object $*$. Then a $\Bbbk$-*category with a pseudo-action* of G is a pair $(\mathcal{C}, X)$ of a $\Bbbk$-category $\mathcal{C}$ and a pseudo-functor $X\colon G \to \Bbbk\text{-}\mathbf{Cat}$ with $\mathcal{C} = X(*)$.

As a special case of Theorem 5.6 we obtain the following.

Corollary 5.8. *Let G be a group, $(\mathcal{C}, X)$ and $(\mathcal{C}', X')$ $\Bbbk$-categories with pseudo-actions of G. Assume that $\Bbbk$ is a field. Then the following are equivalent.*

(1) *$(\mathcal{C}, X)$ and $(\mathcal{C}', X')$ are derived equivalent.*
(2) *There exists a G-equivariant tilting subcategory $\mathcal{T}$ with a pseudo-action of G in $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(\mathcal{C}, X))$ such that $(\mathcal{C}', X')$ is equivalent to $\mathcal{T}$ in the 2-category $\overleftarrow{\mathrm{Oplax}}(G, \Bbbk\text{-}\mathbf{Cat})$.*

Example 5.9. Consider $\mathbb{Z}$ as an additive group. For a $\Bbbk$-algebra B and an automorphism λ of B denote by (B, λ) the category B with a $\mathbb{Z}$-action defined by sending 1 to λ, and by $\hat{B}$ the repetitive category of B. If A and A' are derived equivalent algebras,

then the categories $(\hat{A}, \nu^n)$ and $(\hat{A}', \nu'^n)$ with $\mathbb{Z}$-actions are derived equivalent for all $n \in \mathbb{N}$, where ν (resp. ν') are the Nakayama automorphism of $\hat{A}$ (resp. $\hat{A}'$).

By applying Theorem 5.6 to the free category of the quiver $1 \xrightarrow{a} 2$ we obtain the following.

Corollary 5.10. *Let $\lambda\colon A \to B$ and $\lambda'\colon A' \to B'$ be morphisms of $\mathbb{k}$-algebras, by which we regard B, B' as a left A-module and a left A'-module, respectively. Assume that $\mathbb{k}$ is a field. Then the following are equivalent.*

(1) *There exist equivalences F, G of triangulated categories such that the following diagram is commutative up to natural isomorphisms*

$$
\begin{array}{ccc}
\mathcal{D}(\mathrm{Mod}\,A) & \xrightarrow{\ F\ } & \mathcal{D}(\mathrm{Mod}\,A') \\
{\scriptstyle -\overset{\mathbf{L}}{\otimes}B}\big\downarrow & & \big\downarrow{\scriptstyle -\overset{\mathbf{L}}{\otimes}B'} \\
\mathcal{D}(\mathrm{Mod}\,B) & \xrightarrow[\ G\ ]{} & \mathcal{D}(\mathrm{Mod}\,B').
\end{array}
$$

(2) *There exist a tilting complex T for A with $T \otimes_A B$ a tilting complex for B, $\mathbb{k}$-algebra isomorphisms α, β and a $\mathbb{k}$-algebra morphism μ such that the following diagram is commutative up to natural isomorphism*

$$
\begin{array}{ccccc}
A' & \xrightarrow{\ \alpha\ } & \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,A)}(T) & \hookrightarrow & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,A) \\
{\scriptstyle \lambda'}\big\downarrow & & {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle -\otimes_A B} \\
B' & \xrightarrow[\ \beta\ ]{} & \mathrm{End}_{\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,B)}(T \otimes_A B) & \hookrightarrow & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,B).
\end{array}
$$

6. Appendix: A categorical version of Keller's lifting theorem

We prove Theorem 5.4. We will prove existence of the lifting following section 9 of [8]. Uniqueness follows easily. We use the notations of the main body of the paper. Moreover, for two complexes L and M of $\mathcal{B}$-modules, we write $\mathsf{Hom}^\bullet_{\mathcal{B}}(L, M)$ for the complex whose nth component is formed by the morphisms $f : L \to M$ homogeneous of degree n between the $\mathbb{Z}$-graded objects underlying L and M and whose differential takes f to $d(f) = d_M \circ f - (-1)^n f \circ d_L$. The class of complexes of $\mathcal{B}$-modules endowed with the assignment

$$(L, M) \mapsto \mathsf{Hom}^\bullet_{\mathcal{B}}(L, M)$$

naturally becomes a dg category [8] [10] denoted by $\mathcal{C}_{dg}(\mathsf{Mod}\,\mathcal{B})$ and we have

$$Z^0 \mathcal{C}_{dg}(\mathsf{Mod}\,\mathcal{B}) = \mathcal{C}(\mathsf{Mod}\,\mathcal{B}) \quad \text{and} \quad H^0 \mathcal{C}_{dg}(\mathsf{Mod}\,\mathcal{B}) = \mathcal{K}(\mathsf{Mod}\,\mathcal{B}).$$

6.1. **Existence.** Let $\mathcal{E}$ be the dg endomorphism category of T: Its objects are those of $\mathcal{A}$ and for two objects x, y of $\mathcal{A}$, we put

$$\mathcal{E}(x, y) = \mathsf{Hom}^\bullet_{\mathcal{B}}(T(x), T(y)).$$

Thus, for $n \in \mathbb{Z}$, we have

$$H^n \mathcal{E}(x, y) = \mathsf{Hom}_{\mathcal{K}(\mathrm{Prj}\,\mathcal{B})}(T(x), T(y)[n]).$$

and this group vanishes for $n < 0$ by the Toda condition. Let $\tau_{\leq 0}\mathcal{E}$ denote the dg subcategory of $\mathcal{E}$ with the same objects and with the morphism complexes

$$(\tau_{\leq 0}\mathcal{E})(x,y) = \tau_{\leq 0}(\mathcal{E}(x,y)).$$

By the Toda condition, the projection $\tau_{\leq 0}\mathcal{E} \to H^0\mathcal{E}$ is a quasi-isomorphism (i.e. a dg functor which induces a bijection on the objects and quasi-isomorphisms in the morphism complexes). Since we have

$$(H^0\mathcal{E})(x,y) = \mathsf{Hom}_{\mathcal{K}(\mathrm{Prj}\,\mathcal{B})}(T(x), T(y))\,,$$

the functor T yields a functor $F : \mathcal{A} \to H^0\mathcal{E}$ which is the identity on the objects and given by T on the morphisms. Thus, we obtain a chain of dg functors

$$\mathcal{A} \xrightarrow{\ F\ } H^0\mathcal{E} \xleftarrow{\ qis\ } \tau_{\leq 0}\mathcal{E} \rightarrowtail \mathcal{E} \longrightarrow \mathcal{C}_{dg}(\mathrm{Prj}\,\mathcal{B}).$$

To 'invert' the quasi-isomorphism, we now temporarily pass from functors to bimodules: Let X_1 be the $\mathcal{A}^{op} \otimes_{\mathbf{k}} H^0\mathcal{E}$–module given by

$$X_1(x,y) = (H^0\mathcal{E})(x, Fy).$$

Since $\mathcal{A}$ is projective over $\mathbf{k}$, the induced dg functor

$$\mathcal{A}^{op} \otimes_{\mathbf{k}} H^0\mathcal{E} \longleftarrow \mathcal{A}^{op} \otimes_{\mathbf{k}} \tau_{\leq 0}\mathcal{E}$$

is a quasi-isomorphism. The restriction of X_1 along this quasi-isomorphism is still denoted by X_1. Now let us denote by T' the $\mathcal{E}^{op} \otimes_{\mathbf{k}} \mathcal{B}$–module given by

$$T'(?, y) = T(y).$$

We put

$$X_2 = X_1 \otimes^{\mathsf{L}}_{\tau_{\leq 0}\mathcal{E}} T' = (\mathbf{p}X_1) \otimes_{\tau_{\leq 0}\mathcal{E}} T'\,,$$

where $\mathbf{p}X_1$ is a cofibrant resolution of the $\mathcal{A}^{op} \otimes_{\mathbf{k}} \tau_{\leq 0}\mathcal{E}$–module X_1. Notice that X_1 is right bounded so that $\mathbf{p}X_1$ may be chosen right bounded. Since T' is also right bounded, the tensor product X_1 is right bounded. Moreover, since $\mathcal{A}(x,y)$ is $\mathbf{k}$-projective for all objects x, y of $\mathcal{A}$, cofibrant modules over $\mathcal{A}^{op} \otimes_{\mathbf{k}} \tau_{\leq 0}\mathcal{E}$ tensored by right cofibrant $\tau_{\leq 0}\mathcal{E} \otimes_{\mathbf{k}} \mathcal{B}$-modules are cofibrant over $\mathcal{B}$. Thus, for each object x of $\mathcal{A}$, the complex $X_2(?, x)$ is a right bounded complex of projective $\mathcal{B}$-modules. We define $B : \mathcal{A} \to \mathcal{C}(\mathrm{Prj}\,\mathcal{B})$ by

$$B(x) = X_2(?, x).$$

Now let us construct the natural transformation $q : T \to \nu B$ of functors from $\mathcal{A}$ to $\mathcal{K}(\mathrm{Prj}\,\mathcal{B})$. For an object x of $\mathcal{A}$, the object $X_1(?, x)$ is isomorphic to $(H^0\mathcal{E})(?, x)$ and we have two quasi-isomorphisms with cofibrant $\tau_{\leq 0}\mathcal{E}$–modules given by

$$(\tau_{\leq 0}\mathcal{E})(?, x) \longrightarrow (H^0\mathcal{E})(?, x) \longleftarrow (\mathbf{p}X_1)(?, x)$$

Thus, there is a unique morphism $(\tau_{\leq 0}\mathcal{E})(?,x) \to (\mathbf{p}X_1)(?,x)$ in the homotopy category of $\tau_{\leq 0}\mathcal{E}$–modules which makes the following triangle commutative

$$(\tau_{\leq 0}\mathcal{E})(?,x) \longrightarrow (\mathbf{p}X_1)(?,x)$$
$$(H^0\mathcal{E})(?,x)$$

By tensoring this morphism with T' we obtain a morphism

$$T(x) = (\tau_{\leq 0}\mathcal{E})(?,x) \otimes_{\tau_{\leq 0}\mathcal{E}} T' \to (\mathbf{p}X_1(?,x)) \otimes_{\tau_{\leq 0}\mathcal{E}} T' = X_2(?,x) = \nu B(x) \,,$$

which we use to define q_x. It is straightforward to check that the q_x do yield a natural transformation $q : T \to \nu B$.

6.2. **Uniqueness.** Suppose we are given a morphism $q' : T \to \nu B'$ as in part b) of Theorem 8.4. Then clearly there exists a unique natural transformation $\bar{p} : \pi B \to \pi B'$ of functors from $\mathcal{A}$ to $\mathcal{D}(\mathsf{Mod}\,\mathcal{B})$ such that $Qq' = \bar{p} \circ Qq$. Since for each $x \in \mathcal{A}$, $\bar{p}x : B(x) \to B'(x)$ is in $\mathcal{D}^-(\mathrm{Prj}\,\mathcal{B})$, $\bar{p}$ lifts to a morphism $p : B \to B'$ of functors from $\mathcal{A}$ to $\mathcal{C}^-(\mathrm{Prj}\,\mathcal{B})$. Thus we have $\pi p = \bar{p}$. Then $Qq' = Q(\nu p \circ q)$ and we have $q' = \nu p \circ q$ because Q is fully faithful on $\mathcal{K}^-(\mathrm{Prj}\,\mathcal{B})$. If $q' = \nu p' \circ q$ for some $p' : B \to B'$, then $Q\nu p = Qq'(Qq)^{-1} = Q\nu p'$ shows $\nu p = \nu p'$ by the same reason. $\qquad\square$

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