

INVARIANTS OF A MAXIMAL UNIPOTENT SUBGROUP AND EQUIDIMENSIONALITY

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INTRODUCTION

The ground field \mathbb{k} is algebraically closed and of characteristic zero. Let G be a semisimple algebraic group. Fix a maximal unipotent subgroup $U \subset G$ and a maximal torus $T \subset B = N_G(U)$. Let G act on an affine variety X . As was shown by Hadžiev, the algebra of U -invariants, $\mathbb{k}[X]^U$, is finitely generated. He constructed a finitely generated \mathbb{k} -algebra \mathcal{A} , equipped with a locally finite representation of G (“the universal algebra”), such that

$$(0.1) \quad \mathbb{k}[X]^U \simeq (\mathbb{k}[X] \otimes \mathcal{A})^G,$$

[6, Theorem 3.1]. Actually, \mathcal{A} is isomorphic to $\mathbb{k}[G]^U$ and using our notation, the variety $\text{Spec}(\mathcal{A})$ is denoted by $\mathcal{C}(\mathcal{X}_+)$, where \mathcal{X}_+ is the monoid of dominant weights with respect to (T, U) , see Section 1. This variety has a dense G -orbit and the stabiliser of any $x \in \mathcal{C}(\mathcal{X}_+)$ contains a maximal unipotent subgroup of G . Affine varieties having these two properties have been studied in [14]. They are called *\mathcal{S} -varieties*. There is a bijection between the G -isomorphism classes of \mathcal{S} -varieties and the finitely generated monoids in \mathcal{X}_+ . If $\mathfrak{S} \subset \mathcal{X}_+$ is a finitely generated monoid, then $\mathcal{C}(\mathfrak{S})$ stands for the corresponding \mathcal{S} -variety.

This note contains two contributions to the theory of U -invariants related to the equidimensionality property of quotient morphisms.

C-1) Set $Z = X \times \mathcal{C}(\mathcal{X}_+)$. Isomorphism (0.1) means that the corresponding affine varieties, $X//U$ and $Z//G$, are isomorphic. This suggests that there can be a relationship between fibres of two quotient morphisms $\pi_{X,U} : X \rightarrow X//U$ and $\pi_{Z,G} : Z \rightarrow Z//G$. Suppose that X contains a T -fixed point x_0 , and let $\mathbf{0}$ denote the unique G -fixed point in $\mathcal{C}(\mathcal{X}_+)$. Then $(x_0, \mathbf{0})$ is a T -fixed point in Z . We prove that

$$\pi_{Z,G}^{-1}(\pi_{Z,G}(x_0, \mathbf{0})) = G \cdot \left(\pi_{X,U}^{-1}(\pi_{X,U}(x_0)) \times \overline{B \cdot \mathbf{v}} \right),$$

where \mathbf{v} is a point in the dense G -orbit in $\mathcal{C}(\mathcal{X}_+)$ such that $G_{\mathbf{v}} = U$, see Theorem 2.6. Here $\overline{B \cdot \mathbf{v}}$ is an affine space of dimension $\text{rk } G$. This readily implies that

$$\dim \pi_{Z,G}^{-1}(\pi_{Z,G}(x_0, \mathbf{0})) - \dim \pi_{X,U}^{-1}(\pi_{X,U}(x_0)) = \dim B = \dim Z - \dim X.$$

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An important corollary is that if X is an irreducible conical variety (e.g. a G -module), then $\pi_{X,U}$ is equidimensional (i.e., all the fibres have the same dimension) if and only if $\pi_{Z,G}$ is.

We also give a description of the fibres of $\pi_{X,U}$ containing T -fixed points via certain 1-parameter subgroups of T , see Theorem 2.4. This generalises [2, Theorem 5].

C-2) Our second contribution concerns arbitrary \mathcal{S} -varieties. In [11, Sect. 5], we proved that the following two conditions are equivalent:

- (i) $\pi_{\mathcal{C}(\mathfrak{S}),U}$ is equidimensional and $\mathcal{C}(\mathfrak{S})//U$ is an affine space;
- (ii) the monoid \mathfrak{S} is generated by pairwise disjoint dominant weights. (Two dominant weights are said to be *disjoint* if they depend on different fundamental weights.)

Condition (ii) implies that \mathfrak{S} is a free monoid and the cone $K(\mathfrak{S})$ generated by \mathfrak{S} is simplicial. In [11, Thm. 5.5], we also proved that if X is a G -spherical affine variety such that $\mathbb{k}[X]^U$ is polynomial (i.e., $X//U$ is an affine space) and the corresponding monoid of highest weights, \mathfrak{S}_X , is generated by pairwise disjoint dominant weights, then $\pi_{X,U} : X \rightarrow X//U$ is equidimensional. Recently, a classification of affine G -homogeneous spaces X with such properties is obtained in [1].

In Section 3, we derive some general properties of equidimensional quotient morphisms $\pi_{\mathcal{C}(\mathfrak{S}),H} : \mathcal{C}(\mathfrak{S}) \rightarrow \mathcal{C}(\mathfrak{S})//H$, where H is any *Grosshans subgroup* of G . (Recall that H is Grosshans if G/H is quasiaffine and $\mathbb{k}[G]^H$ is finitely generated.) For instance, we prove that the property of being equidimensional depends only on the cone $K(\mathfrak{S})$. Then we characterise the cones $K(\mathfrak{S})$ such that $\pi_{\mathcal{C}(\mathfrak{S}),U}$ is equidimensional, without assuming that $\mathcal{C}(\mathfrak{S})//U$ is an affine space. Namely, the following two conditions are equivalent:

- (1) $\pi_{\mathcal{C}(\mathfrak{S}),U}$ is equidimensional;
- (2) the edges of the cone $K(\mathfrak{S})$ correspond to pairwise disjoint dominant weights.

(See Theorem 3.7.) A similar equidimensionality criterion for the action of the derived group of U on $\mathcal{C}(\mathfrak{S})$, which only refers to $K(\mathfrak{S})$, is given in [12, Theorem 4.4]. We also prove that if X is affine, G -spherical, and the cone $K(\mathfrak{S}_X)$ is generated by pairwise disjoint dominant weights, then $\pi_{X,U}$ is equidimensional, see Theorem 3.9.

Notation. If an algebraic group H acts regularly on an irreducible affine variety X , then

- $H_x = \{h \in H \mid h \cdot x = x\}$ is the *stabiliser* of $x \in X$;
- $\mathbb{k}[X]^H$ is the algebra of H -invariant polynomial functions on X . If $\mathbb{k}[X]^H$ is finitely generated, then $X//H := \text{Spec}(\mathbb{k}[X]^H)$, and the *quotient morphism* $\pi_{X,H} : X \rightarrow X//H$ is the mapping associated with the embedding $\mathbb{k}[X]^H \hookrightarrow \mathbb{k}[X]$.
- The fibre of $\pi_{X,H}$ that contains $x \in X$ is denoted by $\mathfrak{F}_{X,H}(x)$.

Throughout, G is a semisimple simply-connected algebraic group, $r = \text{rk } G$, $W = N_G(T)/T$ is the Weyl group, and $B = TU$. Then U^- is the opposite subgroup to U .

– Δ is the root system of (G, T) , Π is the set of simple roots corresponding to U , and $\varpi_1, \dots, \varpi_r$ are the corresponding fundamental weights.

– \mathfrak{X} is the character group of T . All roots and weights are regarded as elements of the r -dimensional vector space $\mathfrak{X}_{\mathbb{R}} := \mathfrak{X} \otimes \mathbb{R}$ and $(\ , \)$ is a W -invariant symmetric non-degenerate bilinear form on $\mathfrak{X}_{\mathbb{R}}$.

– $R(\lambda)$ is the simple G -module with highest weight $\lambda \in \mathcal{X}_+$ and λ^* is the highest weight of the dual G -module, i.e., $R(\lambda)^* \simeq R(\lambda^*)$.

$[m] = \{1, 2, \dots, m\}$ if $m \in \mathbb{N}$.

Our main reference for Invariant Theory is [9].

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1. RECOLLECTIONS AND PRELIMINARY RESULTS

1.1. Conical varieties and equidimensional morphisms. Let $\pi : X \rightarrow Y$ be a dominant morphism of irreducible algebraic varieties. We say that π is *equidimensional at* $y \in Y$ if all irreducible components of $\pi^{-1}(y)$ are of dimension $\dim X - \dim Y$. Then π is said to be *equidimensional* if it is equidimensional at any $y \in \pi(X)$. By a result of Chevalley [3, Ch. 5, n.5, Prop. 3], if $y = \pi(x)$ is a normal point, π is equidimensional at y , and $\Omega \subset X$ is a neighbourhood of x , then $\pi(\Omega)$ is a neighbourhood of y . Consequently, an equidimensional morphism to a normal variety is open.

An affine variety X is said to be *conical* if the algebra $\mathbb{k}[X]$ is \mathbb{N} -graded, $\mathbb{k}[X] = \bigoplus_{n=0}^{\infty} \mathbb{k}[X]_n$, and $\mathbb{k}[X]_0 = \mathbb{k}$. The point x_0 corresponding to the maximal ideal $\bigoplus_{n \geq 1} \mathbb{k}[X]_n$ is called the *vertex* of X . Geometrically, this means that X is equipped with an action of the multiplicative group \mathbb{k}^* such that $\{x_0\}$ is the only closed \mathbb{k}^* -orbit in X .

Lemma 1.1. *Suppose that both X and Y are conical and the morphism $\pi : X \rightarrow Y$ is dominant and \mathbb{k}^* -equivariant. (Then $\pi(x_0) =: y_0$ is the vertex of Y .) If π is equidimensional at y_0 , then π is onto and equidimensional.*

Proof. If Y is normal, then this readily follows from the above-mentioned result of Chevalley and standard semicontinuity results for dimension of fibres [10, § 3A]. Otherwise, one can consider the induced morphism between the normalisations $\tilde{\pi} : \tilde{X} \rightarrow \tilde{Y}$, and use the fact that the normalisation of a conical variety is again conical [9, Anhang I, 4.4]. \square

Suppose that X is a *G -conical variety*, i.e., X is conical and the actions of G and \mathbb{k}^* on X commute. Then the vertex is a G -fixed point. If H is an algebraic subgroup of G and $\mathbb{k}[X]^H$ is finitely generated, then $X//H$ is also conical, the vertex of $X//H$ being $\pi_H(x_0)$. The fibre

of $\pi_{X,H} : X \rightarrow X//H$ that contains the vertex x_0 is called the *null-cone*; it is also denoted by $\mathfrak{N}_H(X)$. It follows from Lemma 1.1 that if X is G -conical, then $\pi_{X,H}$ is equidimensional if and only if $\dim \mathfrak{N}_H(X) = \dim X - \dim X//H$.

1.2. Prehomogeneous horospherical varieties. A variety X acted upon by G is called *horospherical* if G_x contains a maximal unipotent subgroup of G for any $x \in X$. Affine horospherical varieties with a dense G -orbit have thoroughly been studied in [14]. Following that article, they will be referred to as *S-varieties*. Let \mathfrak{S} be a finitely generated monoid in \mathcal{X}_+ and $\{\lambda_1, \dots, \lambda_m\}$ the minimal set of generators of \mathfrak{S} . Let $v_{-\lambda_i} \in \mathbb{R}(\lambda_i^*)^{U^-}$ be a lowest weight vector. Set $\mathbf{v}^- = (v_{-\lambda_1}, \dots, v_{-\lambda_m})$ and consider

$$(1.1) \quad \mathcal{C}(\mathfrak{S}) := \overline{G \cdot \mathbf{v}^-} \subset \mathbb{R}(\lambda_1^*) \oplus \dots \oplus \mathbb{R}(\lambda_m^*).$$

Then $\mathcal{C}(\mathfrak{S})$ is an \mathcal{S} -variety and each \mathcal{S} -variety is obtained in this way [14]. It follows from (1.1) that $G_{\mathbf{v}^-} \supset U^-$ and $B \cdot \mathbf{v}^-$ is dense in $\mathcal{C}(\mathfrak{S})$. Write $\langle \mathfrak{S} \rangle$ for the linear span of \mathfrak{S} in $\mathcal{X}_{\mathbb{R}}$ and set $\text{rk } \mathfrak{S} := \dim_{\mathbb{R}} \langle \mathfrak{S} \rangle$. Let P^- denote the stabiliser of the lines $\langle v_{-\lambda_i} \rangle$, $i \in [m]$. Clearly, $B^- \subset P^-$. Let L denote the standard Levi subgroup of P^- ($L \supset T$ and the roots of L are those orthogonal to $\lambda_1, \dots, \lambda_m$). Write $P = LN$ for the opposite parabolic subgroup, where $N \subset U$ is the unipotent radical of P .

Theorem 1.2 ([14]). *The affine variety $\mathcal{C}(\mathfrak{S})$ has the following properties:*

1. *The algebra $\mathbb{k}[\mathcal{C}(\mathfrak{S})]$ is a multiplicity free G -module. Namely, $\mathbb{k}[\mathcal{C}(\mathfrak{S})] = \bigoplus_{\lambda \in \mathfrak{S}} \mathbb{R}(\lambda)$ and this decomposition is a multigrading, i.e., the product in $\mathbb{k}[\mathcal{C}(\mathfrak{S})]$ satisfies the relation $\mathbb{R}(\lambda)\mathbb{R}(\mu) = \mathbb{R}(\lambda + \mu)$;*
2. *The G -orbits in $\mathcal{C}(\mathfrak{S})$ are in a one-to-one correspondence with the faces of the polyhedral cone in $\mathcal{X}_{\mathbb{R}}$ generated by \mathfrak{S} ;*
3. *$\mathcal{C}(\mathfrak{S})$ is normal and $\mathbb{k}[\mathcal{C}(\mathfrak{S})] = \mathbb{k}[G/G_{\mathbf{v}^-}]$ if and only if $\mathbb{Z}\mathfrak{S} \cap \mathcal{X}_+ = \mathfrak{S}$;*
4. *$\dim \mathcal{C}(\mathfrak{S}) = \dim G/P + \text{rk } \mathfrak{S} = \dim N + \text{rk } \mathfrak{S}$.*

Recall that an irreducible affine variety X acted upon by G is *spherical*, if $\mathbb{k}[X]$ is a multiplicity free G -module. It follows that each $\mathcal{C}(\mathfrak{S})$ is spherical. It is easily seen that $U \cdot \mathbf{v}^- = N \cdot \mathbf{v}^-$ and $\dim U \cdot \mathbf{v}^- = \dim N = \max_{\xi \in \mathcal{C}(\mathfrak{S})} \dim U \cdot \xi$. Therefore, the Krull dimension of $\mathbb{k}[\mathcal{C}(\mathfrak{S})]^U$ equals $\text{rk } \mathfrak{S}$; in other words, $\dim \mathcal{C}(\mathfrak{S})//U = \text{rk } \mathfrak{S}$. For any $\lambda \in \mathcal{X}_+$, let v_{λ} be a nonzero vector in the one-dimensional space $\mathbb{R}(\lambda)^U$. By Theorem 1.2(1), each v_{λ} , $\lambda \in \mathfrak{S}$, can be regarded as a U -invariant function on $\mathcal{C}(\mathfrak{S})$ and $\mathbb{k}[\mathcal{C}(\mathfrak{S})]^U$ is generated by $v_{\lambda_1}, \dots, v_{\lambda_m}$.

Taking a suitable specialisation of the multigrading of $\mathbb{k}[\mathcal{C}(\mathfrak{S})]$, we see that $\mathcal{C}(\mathfrak{S})$ is a G -conical variety. The vertex is $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}(\lambda_1^*) \oplus \dots \oplus \mathbb{R}(\lambda_m^*)$. This implies that $\mathcal{C}(\mathfrak{S})//U$ is conical.

Remark 1.3. For fixed subgroups $T, U \subset G$, there are two natural choices of a point in the dense G -orbit in $\mathcal{C}(\mathfrak{S})$. One possibility is to use \mathbf{v}^- related to the *lowest* weight vectors in the G -modules $R(\lambda_i^*)$, as above. Another possibility is to use the *highest* weight vectors in all these G -modules, i.e., the point $\mathbf{v} = (v_{\lambda_1^*}, \dots, v_{\lambda_m^*})$ with $G_{\mathbf{v}} \supset U$. The second possibility is used in Section 2, while the first one is used in Section 3.

2. COMPARING FIBRES OF QUOTIENTS BY G AND U

The biggest \mathfrak{S} -variety corresponds to the monoid $\mathfrak{S} = \mathcal{X}_+$ generated by the fundamental weights. By Theorem 1.2, here

$$\mathbb{k}[\mathcal{C}(\mathcal{X}_+)] = \mathbb{k}[G/U] = \bigoplus_{\lambda \in \mathcal{X}_+} R(\lambda),$$

and $\mathcal{C}(\mathcal{X}_+)$ is a normal affine variety. In this section, it will be more convenient for us to work with representatives of G -orbits whose stabilisers contain U . We will write v_i in place of $v_{\varpi_i} \in R(\varpi_i)^U$. Set $\mathbf{v} = (v_1, \dots, v_r) \in R(\varpi_1) \oplus \dots \oplus R(\varpi_r)$. Then $G_{\mathbf{v}} = U$ and $\overline{G \cdot \mathbf{v}} = \mathcal{C}(\mathcal{X}_+)$. It is clear that

$$(2.1) \quad \overline{B \cdot \mathbf{v}} = \overline{T \cdot \mathbf{v}} = \langle v_1 \rangle \oplus \dots \oplus \langle v_r \rangle \simeq \mathbb{A}^r.$$

For any subset $\mathcal{J} \subset [r]$, define $\mathbf{v}_{\mathcal{J}} = (v_1^{\mathcal{J}}, \dots, v_r^{\mathcal{J}})$ as follows. If $i \in \mathcal{J}$, then $v_i^{\mathcal{J}} = 0$; if $i \notin \mathcal{J}$, then $v_i^{\mathcal{J}} = v_i$. In particular, $\mathbf{v}_{\emptyset} = \mathbf{v}$ and $\mathbf{v}_{[r]} = (0, \dots, 0)$. It follows from (2.1) that $\{\mathbf{v}_{\mathcal{J}} \mid \mathcal{J} \subset [r]\}$ is a complete set of representatives of B -orbits in $\overline{B \cdot \mathbf{v}}$. Since $\overline{B \cdot \mathbf{v}}$ meets all G -orbits in $\mathcal{C}(\mathcal{X}_+)$, this is also a complete set of representatives of G -orbits in $\mathcal{C}(\mathcal{X}_+)$. Let $P_{\mathcal{J}}$ be the standard parabolic subgroup of G such that the set of simple roots of the standard Levi subgroup of $P_{\mathcal{J}}$ is $\{\alpha_j \mid j \in \mathcal{J}\}$. Write $P_{\mathcal{J}} = L_{\mathcal{J}}N_{\mathcal{J}}$, where $N_{\mathcal{J}}$ is the unipotent radical of $P_{\mathcal{J}}$. Then $G_{\mathbf{v}_{\mathcal{J}}} = (P_{\mathcal{J}}, P_{\mathcal{J}}) = (L_{\mathcal{J}}, L_{\mathcal{J}})N_{\mathcal{J}}$. The closure of $G \cdot \mathbf{v}_{\mathcal{J}}$ is the \mathfrak{S} -variety corresponding to the monoid $\sum_{i \notin \mathcal{J}} n_i \varpi_i^*$. Therefore, $\overline{G \cdot \mathbf{v}_{\mathcal{J}}}$ is normal and

$$(2.2) \quad \mathbb{k}[\overline{G \cdot \mathbf{v}_{\mathcal{J}}}] = \mathbb{k}[G/G_{\mathbf{v}_{\mathcal{J}}}].$$

Theorem 2.1 (Hadžiev [6], Grosshans [4]). *Let X be an affine variety acted upon by G . The map $\Psi : \mathbb{k}[X \times \mathcal{C}(\mathcal{X}_+)]^G \rightarrow \mathbb{k}[X]^U$ such that $\Psi(f)(x) = f(x, \mathbf{v})$ is an isomorphism of algebras.*

Remark 2.2. This result remains true for all Grosshans subgroups of G (see [5, Ch. I] for generalities on Grosshans subgroups). Namely, letting $\overline{G/H} := \text{Spec}(\mathbb{k}[G]^H)$, one has the natural isomorphism $\Psi : \mathbb{k}[X \times \overline{G/H}]^G \rightarrow \mathbb{k}[X]^H$. However, there is no good structure theory for $\overline{G/H}$ in case of arbitrary Grosshans subgroups, and our main results cannot be generalised.

Corollary 2.3. *Let Y be a G -stable closed subvariety of X and H a Grosshans subgroup of G . Then the restriction homomorphism $\mathbb{k}[X]^H \rightarrow \mathbb{k}[Y]^H$ is onto. Consequently, $Y//H$ can be regarded as a closed subvariety of $X//U$ and $\pi_{X,H}|_Y = \pi_{Y,H}$.*

Proof. We have the commutative diagram

$$\begin{array}{ccc} \mathbb{k}[X \times \overline{G/H}]^G & \simeq & \mathbb{k}[X]^H \\ \downarrow & & \downarrow \\ \mathbb{k}[Y \times \overline{G/H}]^G & \simeq & \mathbb{k}[Y]^H \end{array}$$

and the leftmost vertical arrow is onto, since G is reductive [9, II.3.5]. \square

In [2, Sect. IV], Brion gave a description of fibres of $\pi_{X,U}$ containing G -fixed points via certain 1-parameter subgroups (= 1-PS) of G . We note that a similar description, with similar proofs, applies to the fibres containing T -fixed points. Let $\tau : \mathbb{k}^* \rightarrow T$ be a 1-PS. Using the canonical pairing between \mathfrak{X} and the set of 1-PS of T , we will regard τ as an element of $\mathfrak{X}_{\mathbb{R}}$. Let us say that τ is *admissible*, if $(\tau, \alpha) > 0$ for all $\alpha \in \Pi$. As in the introduction, we set $Z = X \times \mathcal{C}(\mathcal{X}_+)$.

Theorem 2.4. *Suppose that $x_0 \in X^T$. Then the following conditions are equivalent:*

- (i) $x \in \mathfrak{F}_{X,U}(x_0)$;
- (ii) *there are $u \in U$ and an admissible 1-PS $\tau : \mathbb{k}^* \rightarrow T$ such that $\lim_{t \rightarrow 0} \tau(t)u \cdot x \in G \cdot x_0$.*

Proof. (i) \Rightarrow (ii). By the hypothesis, $f(x) = f(x_0)$ for all $f \in \mathbb{k}[X]^U$. In view of Theorem 2.1, this is equivalent to the fact that $\tilde{f}(x, \mathbf{v}) = \tilde{f}(x_0, \mathbf{v})$ for all $\tilde{f} \in \mathbb{k}[Z]^G$. Since $T \cdot x_0 = x_0$ and $\overline{T \cdot \mathbf{v}} \ni \mathbf{0}$, we have $\tilde{f}(x_0, \mathbf{v}) = \tilde{f}(x_0, \mathbf{0})$ for all $\tilde{f} \in \mathbb{k}[Z]^G$. Since $x_0 \in X^T$, the orbit $G \cdot x_0$ is closed [8, Remark 11]. Therefore, by the Hilbert–Mumford criterion for G (see [9, III.2.4]), there is a 1-PS $\nu : \mathbb{k}^* \rightarrow G$ such that $\lim_{t \rightarrow 0} \nu(t) \cdot x \in G \cdot x_0$ and $\lim_{t \rightarrow 0} \nu(t) \cdot \mathbf{v} = \mathbf{0}$.

The rest is the same as the proof of [2, Theorem 5], cf. also proof of [12, Theorem 5.1]. First, we may reduce to the case in which $\text{Im}(\nu) \subset B$. Then we can write $\nu = u^{-1}\tau u$ for some $u \in U$ and 1-PS $\tau : \mathbb{k}^* \rightarrow T$. Hence $\lim_{t \rightarrow 0} \tau(t)u \cdot \mathbf{v} = \lim_{t \rightarrow 0} \tau(t)\mathbf{v} = \mathbf{0}$, which implies that τ is admissible.

(ii) \Rightarrow (i). We have $\lim_{t \rightarrow 0} \tau(t)u \cdot (x, \mathbf{v}) \in G \cdot x_0 \times \{\mathbf{0}\}$, i.e., $(x, \mathbf{v}) \in \mathfrak{F}_{Z,G}(x_0, \mathbf{0})$. Hence

$$\tilde{f}(x, \mathbf{v}) = \tilde{f}(x_0, \mathbf{0}) = \tilde{f}(x_0, \mathbf{v}) \quad \forall \tilde{f} \in \mathbb{k}[Z]^G.$$

Therefore $f(x) = f(x_0)$ for all $f \in \mathbb{k}[X]^U$, i.e., $x \in \mathfrak{F}_{X,U}(x_0)$. \square

Corollary 2.5. *If $x_0 \in X$ is a T -fixed point, then $\mathfrak{F}_{X,G}(x_0) = G \cdot \mathfrak{F}_{X,U}(x_0)$.*

Taking into account Theorem 2.4, the proof is basically the same as the proof of [2, Theorem 6(ii)].

The main result of this section is

Theorem 2.6. *Suppose that X is an affine variety acted upon by G and $x_0 \in X^T$. Then $\mathfrak{F}_{Z,G}(x_0, \mathbf{0}) = G \cdot (\mathfrak{F}_{X,U}(x_0) \times \overline{B \cdot \mathbf{v}})$.*

Proof. 1°. By the very definition of fibres,

$$\mathfrak{F}_{Z,G}(x_0, \mathbf{0}) \cap (X \times \{\mathbf{v}\}) = \{(x, \mathbf{v}) \mid \tilde{f}(x, \mathbf{v}) = \tilde{f}(x_0, \mathbf{0}) \ \forall \tilde{f} \in \mathbb{k}[Z]^G\}.$$

Since $T \cdot x_0 = x_0$ and $\overline{T \cdot \mathbf{v}} \ni \mathbf{0}$, we have $\tilde{f}(x_0, \mathbf{v}) = \tilde{f}(x_0, \mathbf{0})$ for all $\tilde{f} \in \mathbb{k}[Z]^G$. Combining this observation with the isomorphism in Theorem 2.1, we obtain

$$\mathfrak{F}_{Z,G}(x_0, \mathbf{0}) \cap (X \times \{\mathbf{v}\}) = \mathfrak{F}_{X,U}(x_0) \times \{\mathbf{v}\}.$$

Since B normalises U and $T \cdot x_0 = x_0$, the fibre $\mathfrak{F}_{X,U}(x_0) \subset X$ is B -stable. Therefore, $\overline{B \cdot (\mathfrak{F}_{X,U}(x_0) \times \{\mathbf{v}\})} = \mathfrak{F}_{X,U}(x_0) \times \overline{B \cdot \mathbf{v}}$ is also contained in $\mathfrak{F}_{Z,G}(x_0, \mathbf{0})$.

2°. Conversely, suppose that $(x, p) \in \mathfrak{F}_{Z,G}(x_0, \mathbf{0})$. Without loss of generality, we may assume that $p = \mathbf{v}_j$ for some $j \in [r]$. Recall that $G_{\mathbf{v}_j} = (P_j, P_j) = S_j N_j$, where $S_j = (L_j, L_j)$ is semisimple. In the rest of the proof, we suppress j from our notation for subgroups of P_j and merely write $(P, P) = SN = G_{\mathbf{v}_j}$, etc. Then $U(S) := U \cap S$ is a maximal unipotent subgroup of S , and $U/N \simeq U(S)$.

Since G is reductive and $Z_j := X \times \overline{G \cdot \mathbf{v}_j}$ is a closed G -stable subvariety of $Z = X \times \mathcal{C}(\mathcal{X}_+)$, we have $\mathfrak{F}_{Z,G}(x_0, \mathbf{0}) \cap Z_j = \mathfrak{F}_{Z_j,G}(x_0, \mathbf{0})$ [9, II.3.2]. Now, using (2.2) and the canonical isomorphism $\mathbb{k}[X]^{G_j} = (\mathbb{k}[X] \otimes \mathbb{k}[G/G_j])^G$ shows that if $(x, \mathbf{v}_j) \in \mathfrak{F}_{Z,G}(x_0, \mathbf{0})$, then $x \in \mathfrak{F}_{X,G_j}(x_0)$. (Use the same argument as in part 1°, where \mathbf{v} and U are replaced with \mathbf{v}_j and G_j .)

Consider the commutative diagram (we write π_N in place of $\pi_{X,N}$, etc.)

$$\begin{array}{ccccc} X & \xrightarrow{\pi_N} & X//N & & \\ & \searrow \pi_U & \downarrow \pi_{U(S)} & \searrow \pi_S & \\ & & X//U & \longrightarrow & X//SN = X//G_j \end{array}$$

Set $Y = X//N$. The variety Y is acted upon by the semisimple group S , and $Y//S \simeq X//SN$. Inside Y , we consider the fibres of π_S and $\pi_{U(S)}$ containing $\bar{x}_0 = \pi_N(x_0) \in Y$, i.e., $\mathfrak{F}_{Y,S}(\bar{x}_0)$ and $\mathfrak{F}_{Y,U(S)}(\bar{x}_0)$, respectively. They satisfy the following properties:

- (1) $S \cdot \mathfrak{F}_{Y,U(S)}(\bar{x}_0) = \mathfrak{F}_{Y,S}(\bar{x}_0)$ (Corollary 2.5)
- (2) $\pi_N^{-1}(\mathfrak{F}_{Y,U(S)}(\bar{x}_0)) = \mathfrak{F}_{X,U}(x_0)$;
- (3) $\pi_N^{-1}(\mathfrak{F}_{Y,S}(\bar{x}_0)) = \mathfrak{F}_{X,G_j}(x_0)$;

Since π_N is an S -equivariant morphism, it follows from these properties that $S \cdot \mathfrak{F}_{X,U}(x_0) = \mathfrak{F}_{X,G_j}(x_0)$, i.e., if $x \in \mathfrak{F}_{X,G_j}(x_0)$, then it is S -conjugate to a point $x' \in \mathfrak{F}_{X,U}(x_0)$. Hence (x, \mathbf{v}_j) is S -conjugate to (x', \mathbf{v}_j) . This yields the inclusion $\mathfrak{F}_{Z,G}(x_0, \mathbf{0}) \subset G \cdot (\mathfrak{F}_{X,U}(x_0) \times \overline{B \cdot \mathbf{v}})$. \square

Corollary 2.7. *There is a natural bijection between the irreducible components of $\mathfrak{F}_{X,U}(x_0)$ and $\mathfrak{F}_{Z,G}(x_0, \mathbf{0})$. Namely, if Y_i is an irreducible component of $\mathfrak{F}_{X,U}(x_0)$, then $G \cdot (Y_i \times \overline{B \cdot \mathbf{v}})$ is an*

irreducible component of $\mathfrak{F}_{Z,G}(x_0, \mathbf{0})$. Furthermore,

$$(2.3) \quad \dim(G \cdot (Y_i \times \overline{B \cdot \mathbf{v}})) = \dim Y_i + \dim B.$$

Proof. The first assertion is obvious. The dimension formula stems from the fact that each Y_i is B -stable and if $y \in \overline{B \cdot \mathbf{v}} \simeq \mathbb{A}^r$ is generic and $g \cdot y \in \overline{B \cdot \mathbf{v}}$, then $g \in B$. Therefore, $\dim(G \cdot (Y_i \times \overline{B \cdot \mathbf{v}})) = \dim G + \dim Y_i + \dim \overline{B \cdot \mathbf{v}} - \dim B = \dim Y_i + \dim B$. \square

Corollary 2.8. *Suppose that X is G -conical. Then $\pi_{X,U} : X \rightarrow X//U$ is equidimensional if and only if $\pi_{Z,G} : Z \rightarrow Z//G \simeq X//U$ is.*

Proof. Let $x_0 \in X$ be the vertex. Since $\dim \mathcal{C}(\mathcal{X}_+) = \dim B$, equality (2.3) shows that $\dim \mathfrak{F}_{X,U}(x_0) = \dim X - \dim X//U$ if and only if $\dim \mathfrak{F}_{Z,G}(x_0, \mathbf{0}) = \dim Z - \dim Z//G$. Clearly, $Z = X \times \mathcal{C}(\mathcal{X}_+)$ is also conical, with the vertex $(x_0, \mathbf{0})$. Hence the conclusion follows from Lemma 1.1. \square

Corollary 2.9. *Suppose that X is G -conical and Cohen-Macaulay, and $\mathbb{k}[X]^U$ is a polynomial algebra. Then*

$$\mathbb{k}[X] \text{ is a free } \mathbb{k}[X]^U\text{-module} \iff \mathbb{k}[Z] \text{ is a free } \mathbb{k}[Z]^G\text{-module}.$$

Proof. By [13, Theorem 10]), a normal spherical variety has rational singularities and therefore is Cohen-Macaulay. In particular, $\mathcal{C}(\mathcal{X}_+)$ is Cohen-Macaulay. Hence Z is Cohen-Macaulay if and only if X is. The rest follows from the geometric property of Cohen-Macaulay rings: If Y is conical and \mathcal{B} is a graded polynomial subalgebra of $\mathbb{k}[Y]$, then $\mathbb{k}[Y]$ is a free \mathcal{B} -module if and only if $Y \rightarrow \text{Spec } \mathcal{B}$ is equidimensional. \square

The irreducible representations V of simple algebraic groups G such that $V//U$ is an affine space and $\pi_{V,U}$ is equidimensional are classified in [11, Appendix]. This provides interesting examples of singular varieties $Z = V \times \mathcal{C}(\mathcal{X}_+)$ such that $Z//G$ is an affine space and $\mathbb{k}[Z]$ is a free $\mathbb{k}[Z]^G$ -module.

3. EQUIDIMENSIONAL QUOTIENTS OF \mathcal{S} -VARIETIES BY GROSSHANS SUBGROUPS

In this section, we study the quotient morphism $\pi_{\mathcal{C}(\mathfrak{S}),H} : \mathcal{C}(\mathfrak{S}) \rightarrow \mathcal{C}(\mathfrak{S})//H$, where H is a Grosshans subgroup. Preliminary results apply to all such H , but the complete equidimensionality criterion is obtained for $H = U$.

For any $S \subset \mathcal{X}$, let $\mathsf{K}(S)$ denote the closed cone in $\mathcal{X}_{\mathbb{R}}$ generated by S . For brevity, we write below $\pi_{\mathfrak{S}}$ in place of $\pi_{\mathcal{C}(\mathfrak{S}),H}$.

Lemma 3.1. *Let \mathfrak{S}_1 and \mathfrak{S}_2 be finitely generated monoids in \mathcal{X}_+ such that $\mathsf{K}(\mathfrak{S}_1) = \mathsf{K}(\mathfrak{S}_2)$. Then $\pi_{\mathfrak{S}_1}$ is equidimensional if and only if $\pi_{\mathfrak{S}_2}$ is.*

Proof. It suffices to treat the case in which $\mathfrak{S}_2 = \mathsf{K}(\mathfrak{S}_1) \cap \mathcal{X}_+$. Then $\mathfrak{S}_2 \supset \mathfrak{S}_1$ and $\mathbb{k}[\mathcal{C}(\mathfrak{S}_2)]$ is a finite $\mathbb{k}[\mathcal{C}(\mathfrak{S}_1)]$ -module [14, Prop. 4]. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{S}_2) & \xrightarrow{\psi} & \mathcal{C}(\mathfrak{S}_1) \\ \downarrow \pi_{\mathfrak{S}_2} & & \downarrow \pi_{\mathfrak{S}_1} \\ \mathcal{C}(\mathfrak{S}_2) // H & \xrightarrow{\psi // U} & \mathcal{C}(\mathfrak{S}_1) // H. \end{array}$$

Here ψ is finite, and it is sufficient to prove that $\psi // H$ is also finite, i.e., that $\mathbb{k}[\mathcal{C}(\mathfrak{S}_2)]^H$ is a finite $\mathbb{k}[\mathcal{C}(\mathfrak{S}_1)]^H$ -module. Set $\overline{G/H} = \text{Spec}(\mathbb{k}[G]^H)$. By Theorem 2.1, with H in place of U , an equivalent assertion is that $\mathbb{k}[\mathcal{C}(\mathfrak{S}_2) \times \overline{G/H}]^G$ is a finite $\mathbb{k}[\mathcal{C}(\mathfrak{S}_1) \times \overline{G/H}]^G$ -module. This readily follows from the fact that $\mathbb{k}[\mathcal{C}(\mathfrak{S}_2) \times \overline{G/H}]$ is a finite $\mathbb{k}[\mathcal{C}(\mathfrak{S}_1) \times \overline{G/H}]$ -module and the existence of the G -equivariant projection to the subalgebra of G -invariants (the Reynolds operator). \square

Remark 3.2. As the equidimensionality property for $\pi_{\mathfrak{S}}$ depends only on $\mathsf{K}(\mathfrak{S})$, we may work with any monoid generating the cone $\mathsf{K}(\mathfrak{S})$. In many cases, it is convenient to assume that \mathfrak{S} is generated by the extreme elements of $\mathsf{K}(\mathfrak{S})$, i.e., those lying on the edges of $\mathsf{K}(\mathfrak{S})$. That is, we may assume that \mathfrak{S} is a free monoid whenever $\mathsf{K}(\mathfrak{S})$ is simplicial.

If Γ is a face of the cone $\mathsf{K}(\mathfrak{S})$, then $\mathfrak{T} = \Gamma \cap \mathfrak{S}$ is a finitely generated monoid and one can consider the \mathcal{S} -variety $\mathcal{C}(\mathfrak{T})$.

Proposition 3.3. *If $\pi_{\mathfrak{S}} : \mathcal{C}(\mathfrak{S}) \rightarrow \mathcal{C}(\mathfrak{S}) // H$ is equidimensional, then so is $\pi_{\mathfrak{T}}$.*

Proof. By [14, Prop. 7], $\mathcal{C}(\mathfrak{T})$ is a subvariety of $\mathcal{C}(\mathfrak{S})$. On the other hand, $\mathbb{k}[\mathcal{C}(\mathfrak{T})] = \bigoplus_{\mu \in \mathfrak{T}} \mathbb{R}(\mu)$ is also a G -stable subalgebra of $\mathbb{k}[\mathcal{C}(\mathfrak{S})] = \bigoplus_{\nu \in \mathfrak{S}} \mathbb{R}(\nu)$. This yields the chain of G -equivariant maps

$$\mathcal{C}(\mathfrak{T}) \hookrightarrow \mathcal{C}(\mathfrak{S}) \xrightarrow{p} \mathcal{C}(\mathfrak{T}).$$

Here the composite map is the identity, i.e., p is a G -equivariant retraction. Furthermore, passage to the quotients by H (= subalgebras of H -invariants) yields the maps

$$\mathcal{C}(\mathfrak{T}) // H \hookrightarrow \mathcal{C}(\mathfrak{S}) // H \xrightarrow{\bar{p}} \mathcal{C}(\mathfrak{T}) // H,$$

which shows that \bar{p} is a retraction, too. This also shows that both p and \bar{p} are onto. These data can be organised in the commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}(\mathfrak{T}) & \hookrightarrow & \mathcal{C}(\mathfrak{S}) & \xrightarrow{p} & \mathcal{C}(\mathfrak{T}) \\ \pi_{\mathfrak{T}} \downarrow & & \pi_{\mathfrak{S}} \downarrow & & \pi_{\mathfrak{T}} \downarrow \\ \mathcal{C}(\mathfrak{T}) // H & \hookrightarrow & \mathcal{C}(\mathfrak{S}) // H & \xrightarrow{\bar{p}} & \mathcal{C}(\mathfrak{T}) // H \end{array}$$

Since $\pi_{\mathfrak{S}}$ is equidimensional and both $\mathcal{C}(\mathfrak{S})$ and $\mathcal{C}(\mathfrak{S}) // H$ are conical, it follows from Lemma 1.1 that $\pi_{\mathfrak{S}}$ is onto. Therefore, $\pi_{\mathfrak{T}}$ is onto as well. Furthermore, $\pi_{\mathfrak{T}} = \pi_{\mathfrak{S}}|_{\mathcal{C}(\mathfrak{T})}$,

since $\mathcal{C}(\mathfrak{T})$ is a G -stable subvariety of $\mathcal{C}(\mathfrak{S})$ (Corollary 2.3). This shows that $\pi_{\mathfrak{S}}(\mathcal{C}(\mathfrak{T}))$ is a closed subset of $\mathcal{C}(\mathfrak{S})//H$.

Let $Y \subset \mathcal{C}(\mathfrak{S})$ be an irreducible component of $\pi_{\mathfrak{S}}^{-1}(\pi_{\mathfrak{S}}(\mathcal{C}(\mathfrak{T})))$ that contains $\mathcal{C}(\mathfrak{T})$ and maps dominantly to $\pi_{\mathfrak{S}}(\mathcal{C}(\mathfrak{T}))$. Consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{p|_Y} & \mathcal{C}(\mathfrak{T}) \\ & \searrow \pi_{\mathfrak{S}|_Y} & \swarrow \pi_{\mathfrak{S}|\mathcal{C}(\mathfrak{T})} \\ & \pi_{\mathfrak{S}}(\mathcal{C}(\mathfrak{T})) & \end{array}$$

By the very construction of Y , the morphism $p|_Y$ is onto and $\pi_{\mathfrak{S}|_Y}$ is equidimensional. It follows that $\pi_{\mathfrak{S}|\mathcal{C}(\mathfrak{T})}$ is also equidimensional. Consequently, $\pi_{\mathfrak{T}} = \pi_{\mathfrak{S}|\mathcal{C}(\mathfrak{T})}$ is equidimensional. \square

Remark 3.4. It is not clear how to derive an explicit equidimensionality criterion for any H in terms of $\mathcal{K}(\mathfrak{S})$. For $H = (U, U)$, such a criterion is found in [12, Theorem 4.4], and below we present a criterion for $H = U$.

From now on, $\pi_{\mathfrak{S}}$ stands for $\pi_{\mathcal{C}(\mathfrak{S}), U}$. We say that two weights $\lambda_1, \lambda_2 \in \mathcal{X}_+$ are *disjoint* if they depend on different fundamental weights. In other words, if $\lambda_1 = \sum_{i \in I_1} a_i \varpi_i$ and $\lambda_2 = \sum_{i \in I_2} b_i \varpi_i$, then $I_1 \cap I_2 = \emptyset$.

In [11, Sect. 5], we proved that the following two conditions are equivalent:

- $\mathcal{C}(\mathfrak{S})//U$ is an affine space and $\pi_{\mathfrak{S}}$ is equidimensional;
- the monoid \mathfrak{S} is generated by pairwise disjoint dominant weights.

The second condition guarantee us that the generators of \mathfrak{S} are linearly independent, i.e., \mathfrak{S} is a free monoid.

Below, we prove a general equidimensionality criterion for $\pi_{\mathfrak{S}}$, i.e., without the assumption that $\mathcal{C}(\mathfrak{S})//U$ is an affine space. To this end, we need more notation and an auxiliary result on subvarieties of the null-cone. Recall that $\mathbf{0}$ is the only G -fixed point of $\mathcal{C}(\mathfrak{S})$ and $\mathfrak{N}_U(\mathcal{C}(\mathfrak{S})) = \mathfrak{F}_{\mathcal{C}(\mathfrak{S}), U}(\mathbf{0})$ is the null-cone. For any $\mathcal{I} \subset \Pi$, let $P_{\mathcal{I}} = L_{\mathcal{I}}N_{\mathcal{I}}$ be the standard parabolic subgroup of G . Here $L_{\mathcal{I}}$ is the Levi subgroup whose set of simple roots is \mathcal{I} and $N_{\mathcal{I}} \subset U$ is the unipotent radical of $P_{\mathcal{I}}$. Then $P_{\mathcal{I}}^- = L_{\mathcal{I}}N_{\mathcal{I}}^-$ is the opposite parabolic subalgebra. We also need the factorisation

$$W = W^{\mathcal{I}} \times W_{\mathcal{I}},$$

where $W_{\mathcal{I}}$ is the subgroup generated by $\{s_{\alpha} \mid \alpha \in \mathcal{I}\}$ and $W^{\mathcal{I}}$ is the set of representatives of minimal length for $W/W_{\mathcal{I}}$ [7, 1.10]. It is also true that $W^{\mathcal{I}} = \{w \in W \mid w(\alpha) \in \Delta^+ \ \forall \alpha \in \mathcal{I}\}$ [7, 5.4]. For each $w \in W$, we fix a representative, \dot{w} , in $N_G(T)$. As is well-known, the U -orbits in $G/P_{\mathcal{I}}^-$ can be parametrised by $W^{\mathcal{I}}$, and letting $\mathcal{O}(w) = U\dot{w}P_{\mathcal{I}}^- \subset G/P_{\mathcal{I}}^-$

($w \in W^I$), we have $G/P_{\mathcal{I}}^- = \sqcup_{w \in W^I} \mathcal{O}(w)$ and $\text{codim } \mathcal{O}(w) = \ell(w)$, where ℓ is the usual length function on W associated with the simple reflections $s_\alpha, \alpha \in \Pi$.

Let $\mathfrak{S} \subset \mathcal{X}_+$ be a monoid, with the minimal set of generators $\{\lambda_1, \dots, \lambda_m\}$. Consider

$$\mathcal{I} = \{\alpha \in \Pi \mid (\alpha, \mu) = 0 \text{ for all } \mu \in \mathfrak{S}\} = \{\alpha \in \Pi \mid (\alpha, \lambda_i) = 0 \text{ for } i = 1, \dots, m\}.$$

As in Section 1, take $\mathbf{v}^- = (v_{-\lambda_1}, \dots, v_{-\lambda_m}) \in \mathcal{C}(\mathfrak{S})$, and consider the affine variety $\mathcal{V} = \overline{T \cdot \mathbf{v}^-} = \overline{P_{\mathcal{I}}^- \cdot \mathbf{v}^-}$. Clearly $\mathcal{V} \subset \langle v_{-\lambda_1} \rangle \oplus \dots \oplus \langle v_{-\lambda_m} \rangle \simeq \mathbb{k}^m$ and $\dim \mathcal{V} = \text{rk } \mathfrak{S}$. In fact, \mathcal{V} is the toric variety for T corresponding to the monoid \mathfrak{S} . Since \mathcal{V} is acted upon by $P_{\mathcal{I}}^-$, we may consider the homogeneous fibre bundle $G \times_{P_{\mathcal{I}}^-} \mathcal{V}$ and the natural morphism (collapsing)

$$\tau : G \times_{P_{\mathcal{I}}^-} \mathcal{V} \rightarrow G \cdot \mathcal{V} = \mathcal{C}(\mathfrak{S}).$$

Recall that $G \times_{P_{\mathcal{I}}^-} \mathcal{V}$ is the (geometric) quotient of $G \times \mathcal{V}$ by the action of $P_{\mathcal{I}}^-$ defined by $p \cdot (g, y) = (gp^{-1}, p \cdot y)$, where $g \in G, y \in \mathcal{V}$, and $p \in P_{\mathcal{I}}^-$. The image of (g, y) in the quotient is denoted by $g * y$ and τ is defined by $\tau(g * y) = g \cdot y$. The group G acts on $G \times_{P_{\mathcal{I}}^-} \mathcal{V}$ by $g' \cdot (g * y) = (g'g * y)$ and τ is G -equivariant. It is easily seen that τ is proper and birational. We also need the G -equivariant projection $\phi : G \times_{P_{\mathcal{I}}^-} \mathcal{V} \rightarrow G/P_{\mathcal{I}}^-$, $\phi(g * y) = gP_{\mathcal{I}}^-$. Recall that $\mathbb{k}[\mathcal{C}(\mathfrak{S})]^U$ is generated by $v_{\lambda_1}, \dots, v_{\lambda_m}$, where $v_{\lambda_i} \in R(\lambda_i)^U$.

Proposition 3.5. *Suppose that $w \in W^I$. Then the codimension of $\tau(\phi^{-1}(\mathcal{O}(w)))$ in $\mathcal{C}(\mathfrak{S})$ equals $\ell(w)$. Moreover, if $w(\lambda_i) \neq \lambda_i$ for all $i \in [m]$, then $\tau(\phi^{-1}(\mathcal{O}(w))) \subset \mathfrak{N}_U(\mathcal{C}(\mathfrak{S}))$.*

Proof. 1) Clearly, $\text{codim } \phi^{-1}(\mathcal{O}(w)) = \text{codim } \mathcal{O}(w) = \ell(w)$. Since $\phi^{-1}(\mathcal{O}(w))$ meets the dense G -orbit in $G \times_{P_{\mathcal{I}}^-} \mathcal{V}$ and τ is birational, we also have $\dim \tau(\phi^{-1}(\mathcal{O}(w))) = \dim \phi^{-1}(\mathcal{O}(w))$.

2) For any $n \in U$ and $w \in W^I$, we have

$$\tau(\phi^{-1}(n\dot{w}P_{\mathcal{I}}^-)) = \tau(n\dot{w} * \mathcal{V}) = n\dot{w} \cdot \mathcal{V}.$$

If $w(\lambda_i) \neq \lambda_i$, then $v_{\lambda_i}(\dot{w} \cdot v_{-\lambda_i}) = 0$. Hence, regarding v_{λ_i} as a U -invariant regular function on the whole of $\mathcal{C}(\mathfrak{S})$, we see that all v_{λ_i} ($i = 1, \dots, m$) vanish on $w \cdot \mathcal{V}$ and hence on $U\dot{w} \cdot \mathcal{V} = \tau(\phi^{-1}(\mathcal{O}(w)))$. Thus, $\tau(\phi^{-1}(\mathcal{O}(w))) \subset \mathfrak{N}_U(\mathcal{C}(\mathfrak{S}))$. \square

Remark 3.6. It is enough to require that $w(\lambda_i) \neq \lambda_i$ only for generators λ_i that belong to the edges of $K(\mathfrak{S})$. Then this property automatically holds for all other generators of \mathfrak{S} , if any.

Theorem 3.7. *The morphism $\pi_{\mathfrak{S}} : \mathcal{C}(\mathfrak{S}) \rightarrow \mathcal{C}(\mathfrak{S})//U$ is equidimensional if and only if the cone $K(\mathfrak{S}) \subset \mathfrak{X}_{\mathbb{R}}$ is generated by pairwise disjoint dominant weights.*

Proof. 1) Suppose that $K(\mathfrak{S})$ is generated by disjoint dominant weights. In particular, $K(\mathfrak{S})$ is simplicial.

In view of Lemma 3.1 and Remark 3.2, we may assume that \mathfrak{S} is generated by dominant

weights lying on the edges of $K(\mathfrak{S})$. Then \mathfrak{S} is a free monoid, and this case is already handled in [11, Theorem 5.1].

2) Suppose that $\pi_{\mathfrak{S}}$ is equidimensional. To prove that $K(\mathfrak{S})$ is generated by pairwise disjoint dominant weights, we argue by induction on $\text{rk } \mathfrak{S}$.

- The assertion is obvious, if $\text{rk } \mathfrak{S} = 1$.
- Suppose that $\text{rk } (\mathfrak{S}) = k \geq 2$ and the assertion is true for all equidimensional morphisms corresponding to the monoids of rank $\leq k - 1$.

Let Γ be a facet of $K(\mathfrak{S})$ and $\mathfrak{T} = \Gamma \cap \mathfrak{S}$. By Proposition 3.3, $\pi_{\mathfrak{T}}$ is equidimensional; and by the induction assumption, the cone $K(\mathfrak{T})$ is generated by disjoint dominant weights and hence is simplicial. Let $\lambda_1, \dots, \lambda_{k-1}$ be these weights. If $K(\mathfrak{S})$ itself is simplicial, then we may assume that \mathfrak{S} is a free monoid, and then the assertion follows from [11, Theorem 5.1].

Assume that $K(\mathfrak{S})$ is not simplicial. Then there exist at least two edges of $K(\mathfrak{S})$ that do not belong to $K(\mathfrak{T})$. Let μ_1, μ_2 be the dominant weights corresponding to an arbitrary pair of edges outside $K(\mathfrak{T})$. Since $\dim \langle \mathfrak{S} \rangle = k$, we have

$$(3.1) \quad a_1\mu_1 - a_2\mu_2 = \sum_{i=1}^{k-1} b_i\lambda_i$$

for some $a_1, a_2, b_1, \dots, b_{k-1} \in \mathbb{Z}$. As the $(k-1)$ -plane $\langle \lambda_1, \dots, \lambda_{k-1} \rangle$ does not separate μ_1 and μ_2 in the k -dimensional space $\langle \mathfrak{S} \rangle$, we have $a_1a_2 > 0$. We may assume that a_1 and a_2 are positive. Furthermore, the sum $\sum_{i=1}^{k-1} b_i\lambda_i$ must contain coefficients of both signs. (Otherwise, either μ_1 or μ_2 would be superficial.) Rearranging summands of Eq. (3.1), we can write

$$a_1\mu_1 + \sum_{i \in I} c_i\lambda_i = a_2\mu_2 + \sum_{j \in J} d_j\lambda_j,$$

where both I and J are non-empty, $I \cap J = \emptyset$, and $c_i, d_j > 0$. Since both sides represent a dominant weight and $\lambda_1, \dots, \lambda_{k-1}$ are pairwise disjoint, we must have

$$(3.2) \quad a_1\mu_1 - \sum_{i \in I} d_i\lambda_i = a_2\mu_2 - \sum_{j \in J} c_j\lambda_j \in \mathcal{X}_+.$$

For each λ_i , choose a simple reflection $s_i \in W$ such that $s_i(\lambda_i) \neq \lambda_i$. Since $\lambda_1, \dots, \lambda_{k-1}$ are pairwise disjoint, all s_i are different and $s_i(\lambda_j) = \lambda_j$ for $j \neq i$. Set $\mathbf{w} = s_1 \dots s_{k-1} \in W$. Then $\ell(\mathbf{w}) = k - 1$ and $\mathbf{w}(\lambda_i) \neq \lambda_i$ for all $i = 1, \dots, k - 1$. It follows from (3.2) that $s_i(\mu_1) \neq \mu_1$ for any $i \in I$. As $\mu_1 \in \mathcal{X}_+$, W_{μ_1} is generated by some simple reflections. Therefore, $\mathbf{w}(\mu_1) \neq \mu_1$. Clearly, the same conclusion is applicable to μ_2 and all other extreme elements (edges) of $K(\mathfrak{S})$, if any.

Thus, assuming that $K(\mathfrak{S})$ is not simplicial and using the induction assumption, we have proved that there exists $\mathbf{w} \in W$ such that $\ell(\mathbf{w}) = k - 1 < \text{rk } \mathfrak{S}$ and $\mathbf{w}(\mu) \neq \mu$ for all

extreme elements $\mu \in \mathcal{K}(\mathfrak{S})$. Let us prove that this contradicts the equidimensionality of $\pi_{\mathfrak{S}}$. Recall that $\mathcal{I} = \{\alpha \in \Pi \mid (\alpha, \mu) = 0 \text{ for all } \mu \in \mathfrak{S}\}$ and we have the following diagram:

$$\begin{array}{ccc} G \times_{P_{\mathcal{I}}^-} \mathcal{V} & \xrightarrow{\tau} & \mathcal{C}(\mathfrak{S}) \\ \downarrow \phi & & \downarrow \pi_{\mathfrak{S}} \\ G/P_{\mathcal{I}}^- & & \mathcal{C}(\mathfrak{S})//U \end{array}$$

where $\phi(g*y) := gP_{\mathcal{I}}^-$ and $\tau(g*y) := g*y$. By the very construction, $w \in W^{\mathcal{I}}$ and it follows from Proposition 3.5 that

$$\text{codim } \tau(\phi^{-1}(\mathcal{O}(w))) = \text{codim } \mathcal{O}(w) = k - 1 \text{ and } \tau(\phi^{-1}(\mathcal{O}(w))) \subset \mathfrak{N}_U(\mathcal{C}(\mathfrak{S})).$$

Since $\dim \mathcal{C}(\mathfrak{S})//U = k$, $\pi_{\mathfrak{S}}$ is not equidimensional. This contradiction shows that $\mathcal{K}(\mathfrak{S})$ must be simplicial, and then one completes the proof as explained above. \square

Our last goal is to give a *sufficient* condition of equidimensionality of $\pi_{X,U}$ for a wider class of varieties X . Any affine variety X acted upon by G has a canonical degeneration to a horospherical variety, which is denoted by $\text{gr } X$, see [13]. Recall some relevant properties of $\text{gr } X$. Let us endow the affine line \mathbb{A}^1 with the action of $G \times \mathbb{k}^*$ such that G acts trivially and \mathbb{k}^* acts by homotheties. There is an affine variety \mathcal{X} equipped with an action of $G \times \mathbb{k}^*$ and a $G \times \mathbb{k}^*$ -equivariant morphism $\varphi : \mathcal{X} \rightarrow \mathbb{A}^1$ such that $\varphi^{-1}(\xi) \simeq \begin{cases} X, & \text{if } \xi \neq 0; \\ \text{gr } X, & \text{if } \xi = 0. \end{cases}$

The \mathbb{k}^* -action has the property that, for any $x \in \mathcal{X} \setminus \varphi^{-1}(0)$, the closure of $\mathbb{k}^* \cdot x$ contains a point from $\text{gr } X = \varphi^{-1}(0)$. Another important fact is that the algebras $\mathbb{k}[X]^U$ and $\mathbb{k}[\text{gr } X]^U$ are naturally isomorphic.

Lemma 3.8. *Let X be an affine variety acted upon by G and $H \subset G$ is Grosshans.*

If $\pi_{\text{gr } X, H} : \text{gr } X \rightarrow (\text{gr } X)//H$ is equidimensional, then so is $\pi_{X, H}$.

Proof. Consider the $G \times \mathbb{k}^*$ -equivariant morphism $\varphi : \mathcal{X} \rightarrow \mathbb{A}^1$, as above, and the induced commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi_{X, H}} & \mathcal{X}//H \\ & \searrow \varphi & \swarrow \psi \\ & \mathbb{A}^1 & \end{array}$$

It follows from the G -equivariance of φ and Corollary 2.3 that $\varphi^{-1}(\xi)//H \simeq \psi^{-1}(\xi)$ and $\pi_{X, H}|_{\varphi^{-1}(\xi)} = \pi_{\varphi^{-1}(\xi), H}$ for all $\xi \in \mathbb{A}^1$. In particular, $\psi^{-1}(0) \simeq (\text{gr } X)//H$. By the semicontinuity of dimensions of fibres [10, §3A], the set

$$\Xi = \{x \in \mathcal{X} \mid \dim \mathfrak{F}_{x, H}(x) > \dim \mathcal{X} - \dim \mathcal{X}//H = \dim X - \dim X//H\}$$

is closed, and by the hypothesis, $\varphi^{-1}(0) \cap \Xi = \emptyset$. On the other hand, for any $x \in \mathcal{X} \setminus \varphi^{-1}(0)$, we have $\overline{\mathbb{k}^* \cdot x} \cap \varphi^{-1}(0) \neq \emptyset$. Hence $x \notin \Xi$ and $\Xi = \emptyset$. \square

If X is spherical, then $\text{gr } X$ is an \mathcal{S} -variety [13, Thm. 8]. More precisely, if $\mathbb{k}[X] = \bigoplus_{\lambda \in \mathfrak{S}_X} R(\lambda)$ is a vector space decomposition, then \mathfrak{S}_X is a finitely generated monoid and $\text{gr } X \simeq \mathcal{C}(\mathfrak{S}_X)$. Therefore, combining Theorem 3.7 and Lemma 3.8, we obtain

Theorem 3.9. *If X is an affine G -spherical variety and the cone $K(\mathfrak{S}_X)$ is generated by pairwise disjoint dominant weights, then $\pi_{X,U}$ is equidimensional.*

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