

ESTIMATES OF HEAT KERNELS FOR NON-LOCAL REGULAR DIRICHLET FORMS

ALEXANDER GRIGOR'YAN, JIAXIN HU, AND KA-SING LAU

ABSTRACT. In this paper we present new heat kernel upper bounds for a certain class of non-local regular Dirichlet forms on metric measure spaces, including fractal spaces. We use a new purely analytic method where one of the main tools is the parabolic maximum principle. We deduce off-diagonal upper bound of the heat kernel from the on-diagonal one under the volume regularity hypothesis, restriction of the jump kernel and the survival hypothesis. As an application, we obtain two-sided estimates of heat kernels for non-local regular Dirichlet forms with finite effective resistance, including settings with the walk dimension greater than 2.

CONTENTS

1. Introduction	1
2. Terminology and main results	6
3. Tail estimates for quasi-local Dirichlet forms	10
4. Heat semigroup of the truncated Dirichlet form	12
5. Proof of Theorem 2.1	17
6. Heat kernel bounds using effective resistance	20
6.1. Condition (R_1)	21
6.2. Parabolicity	22
6.3. Condition (R_2) with $\beta < 2$	24
6.4. Condition (R_2) with general β	25
6.5. The Green function	27
6.6. Upper bounds of the heat kernel under (R_1) and (R_2)	30
6.7. Two sided estimates of the heat kernel	32
7. Appendix A: Parabolic maximum principle	36
8. Appendix B: list of lettered conditions	37
References	38

1. INTRODUCTION

We are concerned with heat kernel estimates for a class of non-local regular Dirichlet forms. Let (M, d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. The triple (M, d, μ) will be referred to as a metric measure space. Let $(\mathcal{E}, \mathcal{F})$ be a regular *Dirichlet form* in $L^2(M, \mu)$. As is well known (cf. [11]), any regular Dirichlet form has the generator \mathcal{L} that is a non-positive definite

Date: August 10, 2011.

AG was supported by SFB 701 of the German Research Council (DFG) and the Grants from the Department of Mathematics and IMS of CUHK.

JH was supported by NSFC (Grant No. 11071138), SFB 701 and the HKRGC Grant of CUHK.

KSL was supported by the HKRGC Grant and the Focus Investment Scheme of CUHK.

self-adjoint operator in $L^2(M, \mu)$, which in turn gives rise to the *heat semigroup* $P_t = e^{t\mathcal{L}}$ and an associated *Hunt process* $(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M})$ on M .

If the operator P_t is an integral operator, that is, has the integral kernel $p_t(x, y)$, then the latter is called the *heat kernel*. At the same time, $p_t(x, y)$ is the *transition density* of the process $\{X_t\}$. The question of the existence and the estimates of the heat kernel has attracted considerable attention in the literature. The estimates of the heat kernel could be then used for many applications, in particular, for obtaining information about the path properties of $\{X_t\}$ and the spectral properties of the operator \mathcal{L} .

Depending on the setting one expects different types of estimates of the heat kernel $p_t(x, y)$. For example, in \mathbb{R}^n with the classical Dirichlet form

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx,$$

whose generator is the classical Laplace operator Δ , the heat kernel is the Gauss-Weierstrass function

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

that is also the transition density of Brownian motion. For the Dirichlet form

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$$

where $(a_{ij}(x))$ is a symmetric uniformly elliptic matrix, the generator is the operator

$$\mathcal{L} = \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right),$$

and the heat kernel admits (cf. [1]) the two-sided Gaussian bounds

$$p_t(x, y) \asymp \frac{C}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{ct}\right).$$

Similar bounds hold on some classes of Riemannian manifolds (see [14], [33]). Note that in the above examples the Dirichlet form is *local* and, hence, the corresponding Hunt process is a diffusion.

Local Dirichlet forms may also be defined on some singular spaces like fractals, where the definition is implicit and uses the self-similarity structure (cf. [2], [3], [21], [27], [32]). On some classes of fractals the heat kernel is known to exist and to satisfy the following *sub-Gaussian* estimates:

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d(x, y)}{ct^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \quad (1.1)$$

where $\alpha > 0$ and $\beta > 1$ are some parameters that characterize the underlying space in question, and $d(x, y)$ is an appropriate distance function.

Consider the following example of a *non-local* Dirichlet form in \mathbb{R}^n :

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+\beta}} dx dy,$$

where $0 < \beta < 2$. The generator of this form is $\text{const}(-\Delta)^{\beta/2}$, and the heat kernel admits the bounds

$$p_t(x, y) \asymp \frac{1}{t^{n/\beta}} \left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

The associated Hunt process is the symmetric stable process of index β , that is a jump process.

Similarly, if \mathcal{L} denotes the generator of diffusion with the heat kernel (1.1), then the operator $-(-\mathcal{L})^\gamma$ for any $0 < \gamma < 1$ is the generator of a non-local Dirichlet form, and its heat kernel admits the estimate

$$p_t(x, y) \asymp \frac{1}{t^{\alpha/\beta'}} \left(1 + \frac{d(x, y)}{t^{1/\beta'}} \right)^{-(\alpha+\beta')}, \quad (1.2)$$

where $\beta' = \beta\gamma$ (see, for example, [13], [26], [36]). It was shown in [20] that (1.1) and (1.2) exhaust all possible two-sided estimates of heat kernels of self-similar type.

The purpose of this paper is to give equivalent conditions for upper bounds of the heat kernels of non-local type.

Let us return to the general setup of a metric measure space (M, d, μ) equipped with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. Assume in addition that $(\mathcal{E}, \mathcal{F})$ is conservative, that is, $P_t 1 \equiv 1$. Denote by $V(x, r)$ the measure of the metric ball $B(x, r)$. We assume throughout that all metric balls are precompact. In particular, this implies that $V(x, r)$ is finite.

Let α, β be fixed positive numbers, and let C denote positive constant that can be different at different occurrences. Let us state the following conditions that in general may be true or not.

(V $_{\leq}$): *Upper α -regularity*: For all $x \in M$ and all $r > 0$,

$$V(x, r) \leq Cr^\alpha.$$

(DUE): *On-diagonal upper estimate*: The heat kernel p_t exists and satisfies the on-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}},$$

for all $t > 0$ and μ -almost all $x, y \in M$.

(UE): *Upper estimate of non-local type*: The heat kernel p_t exists and satisfies the off-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)}$$

for all $t > 0$ and μ -almost all $x, y \in M$.

(UE $_{loc}$): *Upper estimate of local type*: The heat kernel p_t exists and satisfies the off-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \exp \left(- \left(\frac{d(x, y)}{ct^{1/\beta}} \right)^{\beta/(\beta-1)} \right)$$

for all $t > 0$ and μ -almost all $x, y \in M$, where $\beta > 1$.

(S): *Survival estimate*. There exist constants $\varepsilon, \delta \in (0, 1)$ such that, for all $x \in M$, all balls $B(x, r)$ and all $t^{1/\beta} \leq \delta r$,

$$\mathbb{P}_x(\tau_{B(x, r)} \leq t) \leq \varepsilon,$$

where τ_Ω is the first exit time of the process $\{X_t\}$ from a set Ω .

Note that $\mathbb{P}_x(\tau_{B(x, r)} \leq t)$ is the probability of $\{X_t\}$ to leave $B(x, r)$ before time t . The smallness of this probability as is stated in (S) means a high probability of the process staying in $B(x, r)$ up to time t . The latter can be considered as the probability of *survival* of the process up to the time t assuming that outside $B(x, r)$ the process gets killed.

The above definition of the condition (S) is a bit loose because in general \mathbb{P}_x may not be defined for all $x \in M$. A rigorous analytic definition of (S) in terms of the heat semigroup will be given in Section 2.

The condition (V_{\leq}) determines the value of the parameter α , while the condition (S) determines β that is a space/time scaling parameter. In the fractal literature β is called the *walk dimension*.

Under the standing assumption (V_{\leq}) , the following characterization of (UE_{loc}) is known

$$(UE_{loc}) \Leftrightarrow (DUE) + (S) + (\text{“locality”}) \quad (1.3)$$

where “locality” means that the form $(\mathcal{E}, \mathcal{F})$ is local (see [16] and Remark 2.5 below). In the present paper we prove a similar characterization of the non-local upper estimate (UE) by relaxing the condition of the locality of $(\mathcal{E}, \mathcal{F})$. Recall that by a theorem of Beurling and Deny, any regular conservative Dirichlet form admits a decomposition

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v) \quad (1.4)$$

where $\mathcal{E}^{(L)}$ is a *local part* and

$$\mathcal{E}^{(J)}(u, v) = \int \int_{M \times M \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y)) dj(x, y) \quad (1.5)$$

is a *jump part* with a jump measure j defined on $M \times M \setminus \text{diag}$. In our setting the jump measure j will have a density with respect to $\mu \times \mu$, which will be denoted by $J(x, y)$, and so the jump part $\mathcal{E}^{(J)}$ becomes

$$\mathcal{E}^{(J)}(u, v) = \int \int_{M \times M} (u(x) - u(y))(v(x) - v(y)) J(x, y) d\mu(y) d\mu(x). \quad (1.6)$$

Let us consider the following condition:

(\mathbf{J}_{\leq}) : The jump density exists and admits the estimate

$$J(x, y) \leq Cd(x, y)^{-(\alpha+\beta)},$$

for μ -almost all $x, y \in M$.

In the case of a local form $(\mathcal{E}, \mathcal{F})$, we have $J = 0$ so that (\mathbf{J}_{\leq}) is trivially satisfied. In general, (\mathbf{J}_{\leq}) restricts the long jumps and can be regarded as a measure of non-locality.

Our main result (Theorem 2.3 below) states that, under the standing assumption (V_{\leq}) , the following equivalence holds:

$$(UE) \Leftrightarrow (DUE) + (S) + (\mathbf{J}_{\leq}). \quad (1.7)$$

We would like to emphasize an analogy with (1.3) where the locality was used instead of (\mathbf{J}_{\leq}) .

Let us comment on the individual conditions in (1.7). It is well-known that (DUE) is equivalent to a certain Nash functional inequality for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ (cf. [8]). On the other hand, the Nash inequality is known to be true provided

$$J(x, y) \geq cd(x, y)^{-(\alpha+\beta)},$$

and $V(x, r) \geq cr^\alpha$ (see [24] and Corollary 2.6 below). Hence, under these conditions one has also (DUE) . It seems that the upper bound (\mathbf{J}_{\leq}) of the jump density alone does not imply (DUE) .

An equivalence somewhat similar to (1.7) was proved in [6], where instead of the condition (S) one used a stronger condition

$$\mathbb{P}_x(\tau_{B(x, r)} \leq t) \leq C \frac{t}{r^\beta}$$

for all $t, r > 0$, and instead of (\mathbf{J}_{\leq}) – a certain property of the sample paths that also restricts the long jumps, but in different terms.

Under the assumption $\beta < 2$ and some additional hypotheses, another result of [6] states that, for the pure jump Dirichlet form (1.5),

$$(UE) \Leftrightarrow (DUE) + (J_{\leq}).$$

Hence, in this case the hypothesis (S) in (1.7) can be dropped. Moreover, if both condition (V) (that is, $V(x, r) \asymp r^\alpha$), and condition (J) (that is, $J(x, y) \asymp d(x, y)^{-(\alpha+\beta)}$) hold, then (UE) is true without any further hypothesis – see [9, 10].

In general if $\beta \geq 2$ (as may happen on fractal spaces), it is not known whether (S) can be dropped.

However, conditions (S) and (DUE) in (1.7) can be verified under certain hypotheses about *effective resistance* $R(x, y)$ (cf. Definition in the beginning of Section 6 as well as [27], [28], [29]). Assume that $0 < \alpha < \beta$, and consider the following condition, named (R):

$$R(x, y) \asymp d(x, y)^{\beta-\alpha}.$$

If $(\mathcal{E}, \mathcal{F})$ is parabolic (cf. Definition 6.3) and condition (V) is satisfied, we prove in Theorem 6.16 that

$$(UE) + (NLE) \Leftrightarrow (R) + (J_{\leq}),$$

where (NLE) stands for the near diagonal lower estimate of heat kernel (cf. Section 6.7). Let us mention for comparison that, under the same standing assumptions,

$$(UE_{loc}) + (NLE) \Leftrightarrow (R) + (\text{“locality”})$$

(cf. [31, Theorem 3.1]) and Corollary 6.17).

The techniques for obtaining heat kernel bounds for non-local Dirichlet forms has been developed by a number of authors, see for example [4, 6, 7, 9, 10] and the references therein. The basic approach to obtaining heat kernel upper estimates used in these papers consists of the two steps. The first step is to obtain the heat kernel upper bounds for a *truncated* Dirichlet form, that is, in the case when the jump density $J(x, y)$ has a bounded range. In this case one uses the Davies method as it was presented in the seminal work [8] and where the cut-off functions of form $(\lambda - d(x_0, x))_+$ were used (where λ is a positive constant). This method can be used as long as the cut-off functions belong to the domain of the Dirichlet form, which is the case only when $\beta < 2$ (hence, if $\beta \geq 2$ then this method does not work).

The second step is to obtain heat kernel estimates for the original Dirichlet form by comparing the heat semigroup of the truncated Dirichlet form with the original heat semigroup. We remark that while the first step was done by purely analytic means, the second step in the above-mentioned papers used a probabilistic argument.

In this article, we develop an alternative approach to obtaining upper bounds that is new in the following two aspects:

- (1) We give a new method of obtaining heat kernel estimates for truncated Dirichlet forms without restriction on the walk dimension.
- (2) We prove new simple relations between the truncated and original heat semigroups, using the parabolic maximum principle developed in [15]. This argument is purely analytic.

The structure of the paper is as follows. In Section 2 we state the main results of this paper: Theorems 2.1 and 2.3. In Section 3 we obtain tail estimates for non-local Dirichlet forms, the main technical results here being Theorem 3.1 and Corollary 3.2. In Section 4, we obtain heat kernel estimates for truncated heat semigroup, see Theorem 4.9. In Section 5, we prove the main Theorem 2.1.

In Section 6 we apply Theorem 2.1 to obtain two-sided heat kernel bounds using effective resistance, where main results are Theorems 6.13 and 6.16.

In Appendix A we state for convenience of the reader the parabolic maximum principle that is the main technical tool used in this paper. In Appendix B we list all lettered hypotheses used.

NOTATION. The letters c, c_1, C_1 etc. denote positive constants whose values are unimportant and may differ at different occurrences. The relation $f \asymp g$ between two non-negative functions f, g means that there is a constant $C \geq 1$ such that $C^{-1}f \leq g \leq Cf$ for a specified range of the variables.

2. TERMINOLOGY AND MAIN RESULTS

Let (M, d, μ) be a metric measure space that is unbounded. Recall that a *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$ is a symmetric, non-negative definite, bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ defined on a dense subspace \mathcal{F} of $L^2(M, \mu)$, which satisfies in addition the following properties:

(1) Closedness: \mathcal{F} is a Hilbert space with respect to the following inner product:

$$\mathcal{E}_1(f, g) := \mathcal{E}(f, g) + (f, g).$$

(2) The Markov property: if $f \in \mathcal{F}$ then also $\tilde{f} := (f \wedge 1)_+$ belongs to \mathcal{F} and $\mathcal{E}(\tilde{f}) \leq \mathcal{E}(f)$, where $\mathcal{E}(f) := \mathcal{E}(f, f)$.

Then $(\mathcal{E}, \mathcal{F})$ has the *generator* \mathcal{L} that is a non-positive definite, self-adjoint operator on $L^2(M, \mu)$ with domain $\mathcal{D} \subset \mathcal{F}$ such that

$$\mathcal{E}(f, g) = (-\mathcal{L}f, g)$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{F}$. The generator \mathcal{L} determines the *heat semigroup* $\{P_t\}_{t \geq 0}$ by $P_t = e^{t\mathcal{L}}$ in the sense of functional calculus of self-adjoint operators. It is known that $\{P_t\}_{t \geq 0}$ is strongly continuous, contractive, symmetric semigroup in L^2 , and is *Markovian*, that is, $0 \leq P_t f \leq 1$ for any $t > 0$ if $0 \leq f \leq 1$.

The Markovian property of the heat semigroup implies that the operator P_t preserves the inequalities between functions, which allows to use monotone limits to extend P_t from L^2 to L^∞ (in fact, P_t extends to any L^q , $1 \leq q \leq \infty$ as a contraction – cf. [11, p.37]). In particular, $P_t 1$ is defined. The form $(\mathcal{E}, \mathcal{F})$ is called *conservative* if $P_t 1 = 1$ for every $t > 0$.

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ can be recovered from the heat semigroup as follows. For any $f \in L^2$, the function

$$t \mapsto \frac{1}{t} (f - P_t f, f)$$

is increasing as t is decreasing. In particular, it has the limit as $t \rightarrow 0$. It turns out that the limit is finite if and only if $f \in \mathcal{F}$, and

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (f - P_t f, f) = \mathcal{E}(f). \quad (2.8)$$

Fix some $\rho \in [0, +\infty)$. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to be ρ -*local* if $\mathcal{E}(f, g) = 0$ for any two functions $f, g \in \mathcal{F}$ with compact supports such that

$$\text{dist}(\text{supp } f, \text{supp } g) > \rho.$$

We call a form $(\mathcal{E}, \mathcal{F})$ *quasi-local* if it is ρ -local for some $0 < \rho < \infty$.

For a non-empty open $\Omega \subset M$, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in the norm of \mathcal{F} . It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega, \mu)$ since $\mathcal{F}(\Omega)$ is dense in $L^2(\Omega, \mu)$. Denote by P_t^Ω the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$.

Let us give a probabilistic interpretation of the heat semigroups P_t and P_t^Ω . For any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, there is an associated *Hunt process*. Denote by $X_t, t \geq 0$, the trajectories of a process and by $\mathbb{P}_x, x \in M$, the probability measure in the space of

trajectories emanating from the point x . Denote by \mathbb{E}_x the expectation of the probability measure \mathbb{P}_x . Then the relation between the Dirichlet form and the associated Hunt process is given by the following identity:

$$P_t f(x) = \mathbb{E}_x [f(X_t)] \quad (2.9)$$

for all bounded (or non-negative) Borel functions f and all $t > 0$, and for μ -almost all $x \in M$ (note that $P_t f$ is a function from L^∞ and, hence, is defined up to a set of measure zero, whereas $\mathbb{E}_x f(X_t)$ is defined pointwise for all $x \in M$). By [11, Theorem 7.2.1, p.380], such a process always exists but, in general, is not unique. For a non-empty subset Ω of M , one has the following identity (see [11, p.153, eq. (4.1.2)]): for all bounded (or non-negative) Borel functions f and for all $t > 0$,

$$P_t^\Omega f(x) = \mathbb{E}_x [\mathbf{1}_{\{t < \tau_\Omega\}} f(X_t)] \quad (2.10)$$

where τ_Ω is the *first exit time* τ_Ω defined by

$$\tau_\Omega = \inf \{t > 0 : X_t \notin \Omega\}. \quad (2.11)$$

In particular, if $f = \mathbf{1}_\Omega$, we see from (2.10) that

$$P_t^\Omega \mathbf{1}_\Omega(x) = \mathbb{P}_x(t < \tau_\Omega). \quad (2.12)$$

A family $\{p_t\}_{t>0}$ of $\mu \times \mu$ -measurable functions on $M \times M$ is called the *heat kernel* of the form $(\mathcal{E}, \mathcal{F})$ if p_t is the integral kernel of the operator P_t , that is, for any $t > 0$ and for any $f \in L^2(M, \mu)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \quad (2.13)$$

for μ -almost all $x \in M$, where $\{P_t\}_{t \geq 0}$ is the heat semigroup of $(\mathcal{E}, \mathcal{F})$ as mentioned above.

For any $x \in M$ and $r > 0$, denote by

$$B(x, r) = \{y \in M : d(y, x) < r\},$$

a *metric ball* in M with center x and radius r . We assume throughout that all balls are precompact. Set $V(x, r) := \mu(B(x, r))$.

In addition to the conditions *(DUE)* and *(UE)* defined in Introduction, consider the following condition:

(UE Φ): The heat kernel p_t exists and there exist $C, \alpha > 0, \beta > 0$ such that

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right) \quad (2.14)$$

for all $t > 0$ and μ -almost all $x, y \in M$, where $\Phi : [0, \infty) \rightarrow [0, \infty)$ is continuous, non-increasing function such that

$$\int_0^\infty s^{\alpha-1} \Phi(s) ds < \infty. \quad (2.15)$$

Let us restate in analytic terms the survival condition mentioned in Introduction:

(S): *Survival estimate*. There exist constants $\varepsilon, \delta \in (0, 1)$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t^{1/\beta} \leq \delta r$,

$$1 - P_t^B \mathbf{1}_B(x) \leq \varepsilon \quad \text{for } \mu\text{-almost all } x \in \frac{1}{4}B, \quad (2.16)$$

where $\lambda B = B(x_0, \lambda r)$.

By (2.12), we have that

$$\begin{aligned} 1 - P_t^B \mathbf{1}_B(x) &= 1 - \mathbb{P}_x(t < \tau_B) \\ &= \mathbb{P}_x(\tau_B \leq t), \end{aligned}$$

which implies the equivalence of the two definitions of (S).

Finally, consider two more conditions.

(T) : *Tail estimate.* There exist constants $\varepsilon, \delta \in (0, 1)$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t^{1/\beta} \leq \delta r$,

$$P_t \mathbf{1}_{B^c}(x) \leq \varepsilon \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

(T_{strong}) : *Strong tail estimate.* There exist constants $c > 0$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t > 0$,

$$P_t \mathbf{1}_{B^c}(x) \leq \frac{ct}{r^\beta} \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

Clearly, we have that (T_{strong}) \Rightarrow (T).

We now state the main technical result of this paper.

Theorem 2.1. *Let (M, d, μ) be a metric measure space with precompact balls, and let $(\mathcal{E}, \mathcal{F})$ be a regular conservative Dirichlet form in $L^2(M, \mu)$ with jump density J . Then the following implication holds:*

$$(V_{\leq}) + (DUE) + (J_{\leq}) + (S) \Rightarrow (UE). \quad (2.17)$$

The proof of Theorem 2.1 will be given in Section 5.

Remark 2.2. Define also the condition

(V) : There exist constants $c, C, \alpha > 0$ such that, for all $x \in M$ and all $r > 0$,

$$cr^\alpha \leq V(x, r) \leq Cr^\alpha.$$

If the measure μ satisfies (V) then μ is called α -regular.

By [17, Theorem 3.2], if $(\mathcal{E}, \mathcal{F})$ is conservative then

$$(V_{\leq}) + (UE) \Rightarrow (V).$$

Hence, the hypotheses of Theorem 2.1 imply that μ is α -regular.

The following main theorem provides convenient equivalent conditions for (UE) and is a combination of Theorem 2.1 with previously known results.

Theorem 2.3. *Let (M, d, μ) be a metric measure space with precompact balls, and let $(\mathcal{E}, \mathcal{F})$ be a regular conservative Dirichlet form in $L^2(M, \mu)$ with jump density J . If (V_{\leq}) holds, then the following equivalences are true:*

$$\begin{aligned} (UE) &\Leftrightarrow (UE\Phi) + (J_{\leq}) \\ &\Leftrightarrow (DUE) + (J_{\leq}) + (T) \\ &\Leftrightarrow (DUE) + (J_{\leq}) + (S) \\ &\Leftrightarrow (DUE) + (J_{\leq}) + (T_{strong}). \end{aligned} \quad (2.18)$$

Proof. Observe that the implication $(UE) \Rightarrow (J_{\leq})$ holds by [6, p. 150], and $(UE) \Rightarrow (UE\Phi)$ is trivial by taking $\Phi(s) = (1 + s)^{-(\alpha+\beta)}$. The implication $(UE\Phi) \Rightarrow (DUE)$ is obvious. The implication $(UE\Phi) \Rightarrow (T)$ was proved in [17, formula (3.6), p.2072]. Since $(\mathcal{E}, \mathcal{F})$ is conservative, the equivalence $(T) \Leftrightarrow (S)$ holds by [15, Theorem 3.1, p.96]. By Theorem 2.1 we have

$$(DUE) + (J_{\leq}) + (S) \Rightarrow (UE),$$

which closes the cycle of implications, thus proving the first three equivalences.

Finally, the implication $(UE) \Rightarrow (T_{strong})$ is true (see also [17, formula (3.6),p.2072]), and hence

$$\begin{aligned} (UE) &\Rightarrow (DUE) + (J_{\leq}) + (T_{strong}) \\ &\Rightarrow (DUE) + (J_{\leq}) + (T) \Rightarrow (UE), \end{aligned}$$

which finishes the proof. \square

Remark 2.4. If $(\mathcal{E}, \mathcal{F})$ has only a jump part and $0 < \beta < 2$, then (V_{\leq}) and (J_{\leq}) imply that $(\mathcal{E}, \mathcal{F})$ is conservative (see [35, 19]).

Remark 2.5. Under the standing assumptions of Theorem 2.3 also the following equivalence is true

$$(UE_{loc}) \Leftrightarrow (DUE) + (\text{“locality”}) + (S).$$

Indeed, since (UE_{loc}) is stronger than (UE) , it implies (DUE) and (S) by Theorem 2.3. Next, $(UE_{loc}) \Rightarrow (\text{“locality”})$ by [20, Lemma 3.1]. The opposite implication

$$(DUE) + (\text{“locality”}) + (S) \Rightarrow (UE_{loc})$$

was proved in [16].

In order to state some consequence of Theorem 2.3, we need the following Proposition. Define first the following condition:

(\mathbf{J}_{\geq}) : There exist constants $C, \alpha, \beta > 0$ such that, for μ -almost all $x \neq y$,

$$J(x, y) \geq C^{-1}d(x, y)^{-(\alpha+\beta)}.$$

Proposition 2.6. *Let (M, d, μ) be a metric measure space, and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ with jump density J . Then*

$$(V) + (J_{\geq}) \Rightarrow (DUE). \quad (2.19)$$

Proof. As was proved in [24, Theorem 3.1], under (V) the following inequality holds for all non-zero functions $u \in L^1 \cap L^2$:

$$\int_M \int_M \frac{(u(x) - u(y))^2}{d(x, y)^{\alpha+\beta}} d\mu(x) d\mu(y) \geq c \|u\|_2^{2(1+\beta/\alpha)} \|u\|_1^{-2\beta/\alpha},$$

where c is a positive constant. Using (1.4), (1.6) and (J_{\geq}) we obtain

$$\begin{aligned} \mathcal{E}(u) &= \mathcal{E}^{(L)}(u) + \mathcal{E}^{(J)}(u) \\ &\geq C \int_M \int_M \frac{(u(x) - u(y))^2}{d(x, y)^{\alpha+\beta}} d\mu(x) d\mu(y) \\ &\geq c \|u\|_2^{2(1+\beta/\alpha)} \|u\|_1^{-2\beta/\alpha} \end{aligned}$$

for all $u \in \mathcal{F} \cap L^1$. Hence, (DUE) follows by [8]. \square

Corollary 2.7. *Let (M, d, μ) be a metric measure space with precompact balls, and let $(\mathcal{E}, \mathcal{F})$ be a regular conservative Dirichlet form in $L^2(M, \mu)$ with jump density J . If (V) holds and $J(x, y) \asymp d(x, y)^{-(\alpha+\beta)}$, then*

$$(UE) \Leftrightarrow (S). \quad (2.20)$$

Proof. Let us show that $(S) \Rightarrow (UE)$. Indeed, (DUE) holds by Proposition 2.6. Hence, (UE) is satisfied by Theorem 2.3. The opposite implication $(UE) \Rightarrow (S)$ holds also by Theorem 2.3. \square

Therefore, if (V) holds and $J(x, y) \asymp d(x, y)^{-(\alpha+\beta)}$, then in order to obtain off-diagonal upper bounds of heat kernels, one needs only to verify the survival condition (S). We will show in Section 6 that the survival condition (S) holds for a class of measure spaces with effective resistance metrics.

Remark 2.8. The upper estimate (UE) is best possible for non-local forms in the following sense: if the heat kernel p_t satisfies the estimate

$$p_t(x, y) \leq \frac{1}{t^{\alpha/\beta}} \Phi \left(\frac{d(x, y)}{t^{1/\beta}} \right)$$

for all $t > 0$ and μ -almost all $x, y \in M$, where Φ is a continuous decreasing function on $[0, +\infty)$ then necessarily

$$\Phi(s) \geq c(1+s)^{-(\alpha+\beta)}$$

for some $c > 0$ (see [20, Lemma 3.1]).

3. TAIL ESTIMATES FOR QUASI-LOCAL DIRICHLET FORMS

In this section we give the tail estimate for an arbitrary ρ -local regular Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ in $L^2(M, \mu)$. We use the superscript (ρ) in the notation of this form in order to emphasize that the results of this section will be applied in Sections 4 and 5 for the Dirichlet form, also denoted by $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$, that is obtained by ρ -truncation of the jump density of the original Dirichlet form $(\mathcal{E}, \mathcal{F})$. However, in this section $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ stands for any ρ -local regular Dirichlet form.

Denote by $\{Q_t^\Omega\}$ the heat semigroup of the form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ restricted to a non-empty open set $\Omega \subset M$. Denote by

$$U_\rho = \{x : d(x, U) < \rho\}$$

the ρ -neighborhood of a set $U \subset M$ if $\rho > 0$, and let $U_\rho = U$ if $\rho = 0$. We will use the following inequality obtained in [18, Corollary 4.8, Remark 4.10]. Let Ω, U be two open subsets of M such that U_ρ is precompact and $U_\rho \subset \Omega$. Then, for any $0 \leq f \in L^\infty(M)$ such that $f|_U \equiv 0$ and for all $t > 0$ and μ -almost all $x \in U_\rho$,

$$Q_t^\Omega f(x) \leq (1 - Q_t^U \mathbf{1}_U(x)) \sup_{0 < s \leq t} \|Q_s^\Omega f\|_{L^\infty(U_\rho)}. \quad (3.1)$$

Theorem 3.1. *Let $\phi(r, \cdot)$ be a non-decreasing function in $(0, \infty)$ for any $r \in (0, \infty)$. Assume that, for any ball $B = B(x, r)$ and for any $t \in (0, T_0)$ where $T_0 \in (0, \infty]$,*

$$1 - Q_t^B \mathbf{1}_B \leq \phi(r, t) \quad \text{in } \frac{1}{4}B. \quad (3.2)$$

Then, for any ball $B = B(x, r)$, $t \in (0, T_0)$ and any integer $k \geq 1$,

$$Q_t \mathbf{1}_{B(x, k(r+\rho))^c} \leq \phi(r, t)^k \quad \text{in } \frac{1}{4}B. \quad (3.3)$$

Consequently, for any ball $B(x, R)$ with $R > \rho$ and $t \in (0, T_0)$, for any integer $k \geq 1$,

$$Q_t \mathbf{1}_{B(x, kR)^c} \leq \phi(R - \rho, t)^{k-1} \quad \text{in } B(x, R). \quad (3.4)$$

Proof. We will prove (3.3) by induction in k . Indeed, let $B_k = B(x, k(r+\rho))$. Then (3.3) holds for $k = 1$, since by (3.2) we have in $\frac{1}{4}B$

$$Q_t \mathbf{1}_{B_1^c} \leq 1 - Q_t \mathbf{1}_{B_1} \leq 1 - Q_t^B \mathbf{1}_B \leq \phi(r, t).$$

For the inductive step from k to $k+1$, consider the function

$$u(t, \cdot) = Q_t \mathbf{1}_{B_{k+1}^c}.$$

Applying (3.1) with $\Omega = M$, $U = B$ and $f = \mathbf{1}_{B_{k+1}^c}$, we obtain the following inequality in $U_\rho = B_1$:

$$u(t, \cdot) \leq (1 - Q_t^B \mathbf{1}_B) \sup_{0 < s \leq t} \|u(s, \cdot)\|_{L^\infty(B_1)}. \quad (3.5)$$

Note that, for every $z \in B_1$,

$$B_{k+1}^c \subset B(z, k(r + \rho))^c,$$

and that, for $0 < s \leq t$,

$$\phi(r, s) \leq \phi(r, t).$$

Applying the inductive hypothesis for the ball $B(z, k(r + \rho))$, we obtain the following inequality in $B(z, \frac{1}{4}r)$:

$$u(s, \cdot) = Q_s \mathbf{1}_{B_{k+1}^c} \leq Q_s \mathbf{1}_{B(z, k(r + \rho))^c} \leq \phi(r, s)^k \leq \phi(r, t)^k.$$

Covering the ball B_1 by a countable family of balls like $B(z, \frac{1}{4}r)$, we obtain that

$$\sup_{0 < s \leq t} \|u(s, \cdot)\|_{L^\infty(B_1)} \leq \phi(r, t)^k.$$

Therefore, it follows from (3.5) and (3.2) that

$$u(t, \cdot) \leq \phi(r, t)^{k+1} \text{ in } \frac{1}{4}B,$$

proving (3.3).

Finally, in order to show (3.4), let $r := R - \rho > 0$. It follows from (3.3) that, for any $y \in M$ and $k \geq 1$,

$$Q_t \mathbf{1}_{B(y, kR)^c} \leq \phi(R - \rho, t)^k \text{ in } B(y, r/4). \quad (3.6)$$

Fixing $x \in M$, using (3.6) for any $y \in B(x, R)$, and noticing that

$$B(y, kR) \subset B(x, (k + 1)R)$$

we obtain

$$Q_t \mathbf{1}_{B(x, (k+1)R)^c} \leq Q_t \mathbf{1}_{B(y, kR)^c} \leq \phi(R - \rho, t)^k \text{ in } B(y, r/4).$$

Covering $B(x, R)$ by a countable family of balls like $B(y, r/4)$, we obtain

$$Q_t \mathbf{1}_{B(x, (k+1)R)^c} \leq \phi(R - \rho, t)^k,$$

whence (3.8) follows by renaming k to $k - 1$. \square

Corollary 3.2. *Assume that there exist constants $\varepsilon, \delta \in (0, 1)$ such that, for all balls B of radius r and for all $t \in (0, T_0)$ where $T_0 \in (0, \infty]$ provided that $t^{1/\beta} \leq \delta(r \wedge \rho)$,*

$$1 - Q_t^B \mathbf{1}_B \leq \varepsilon \text{ in } \frac{1}{4}B. \quad (3.7)$$

Then, for any ball $B(x, r)$ and $t \in (0, T_0)$,

$$Q_t \mathbf{1}_{B(x, r)^c} \leq C \exp\left(-c \left(\frac{r}{\rho} \wedge \frac{\rho}{t^{1/\beta}}\right)\right) \text{ in } B(x, \rho), \quad (3.8)$$

where the constants $C, c > 0$ depend only on ε, δ .

Proof. If $r \leq 2\rho$ or if $\delta\rho \leq t^{1/\beta}$, then (3.8) is trivially satisfied by choosing large enough constant C , because $Q_t \mathbf{1}_{B(x, r)^c} \leq 1$ and

$$\frac{r}{\rho} \wedge \frac{\rho}{t^{1/\beta}} \leq \max\{2, \delta^{-1}\}.$$

Assume that $\frac{r}{\rho} > 2$ and $t^{1/\beta} < \delta\rho$. Define the function $\phi(r, t)$ as follows:

$$\phi(r, t) = \begin{cases} \varepsilon, & \text{if } t^{1/\beta} \leq \delta\rho, \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $\phi(r, t)$ is increasing in t , and (3.7) implies that (3.2) holds with this function ϕ . Choose an integer $k \geq 1$ such that

$$k < \frac{r}{2\rho} \leq k + 1.$$

Applying the inequality (3.4) of Theorem 3.1 with $R = 2\rho$ and noticing that $\phi(\rho, t) = \varepsilon$ because $t^{1/\beta} < \delta\rho$, we obtain that in $B(x, 2\rho)$,

$$\begin{aligned} Q_t \mathbf{1}_{B(x, r)^c} &\leq Q_t \mathbf{1}_{B(x, kR)^c} \leq \phi(\rho, t)^{k-1} \\ &= \varepsilon^{k-1} = \exp\left(- (k-1) \log \frac{1}{\varepsilon}\right) \\ &\leq \exp\left(- \left(\frac{r}{2\rho} - 2\right) \log \frac{1}{\varepsilon}\right) = C \exp\left(-c \frac{r}{\rho}\right), \end{aligned}$$

whence (3.8) follows. \square

4. HEAT SEMIGROUP OF THE TRUNCATED DIRICHLET FORM

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 that allows a jump density $J(x, y)$ in the jump part. Fix $\rho \in (0, \infty)$ and define the bilinear form $\mathcal{E}^{(\rho)}(u, v)$ by

$$\mathcal{E}^{(\rho)}(u, v) = \mathcal{E}^{(L)}(u, v) + \int_M \int_{B(x, \rho)} (u(x) - u(y))(v(x) - v(y)) J(x, y) d\mu(y) d\mu(x). \quad (4.1)$$

Clearly, the form $\mathcal{E}^{(\rho)}(u, v)$ is well-defined for $u, v \in \mathcal{F}$ and

$$\mathcal{E}^{(\rho)}(u) \leq \mathcal{E}(u) \text{ for all } u \in \mathcal{F}. \quad (4.2)$$

In this section, we prove that the form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ can be extended to a regular Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$, and obtain upper estimates of the heat kernel q_t of the *truncated* Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$.

Proposition 4.1. *For all $u \in \mathcal{F}$, we have*

$$\mathcal{E}(u) \leq \mathcal{E}^{(\rho)}(u) + 4 \|u\|_2^2 \operatorname{esup}_{x \in M} \int_{B(x, \rho)^c} J(x, y) d\mu(y). \quad (4.3)$$

Proof. It follows from (1.4), (1.5) and (4.1) that

$$\begin{aligned} \mathcal{E}(u, u) - \mathcal{E}^{(\rho)}(u, u) &= \int_M \int_{B(x, \rho)^c} (u(x) - u(y))^2 J(x, y) d\mu(y) d\mu(x) \\ &\leq 2 \int_M \int_{B(x, \rho)^c} (u(x)^2 + u(y)^2) J(x, y) d\mu(y) d\mu(x) \\ &\leq 4 \int_M \int_{B(x, \rho)^c} u(x)^2 J(x, y) d\mu(y) d\mu(x) \\ &\leq 4 \|u\|_2^2 \operatorname{esup}_{x \in M} \int_{B(x, \rho)^c} J(x, y) d\mu(y), \end{aligned}$$

which was to be proved. \square

Proposition 4.2. *Assume that*

$$\operatorname{esup}_{x \in M} \int_{B(x, \rho)^c} J(x, y) d\mu(y) < \infty. \quad (4.4)$$

Then the form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is closable, and its closure $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ in L^2 is a regular, ρ -local Dirichlet form in L^2 .

Remark 4.3. Under the hypotheses (V_{\leq}) and (J_{\leq}) the condition (4.4) is satisfied (see the proof of Proposition 4.7).

Proof. It follows easily from (4.1) that the form $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is symmetric, non-negative definite, and Markovian. In order to prove that it is closable, it suffices to show that if $\{u_n\}_{n=1}^{\infty}$ is a sequence from \mathcal{F} such that

$$u_n \xrightarrow{L^2} 0 \text{ and } \mathcal{E}^{(\rho)}(u_n - u_m) \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

then $\mathcal{E}^{(\rho)}(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Indeed, by (4.3) we have

$$\mathcal{E}(u_n - u_m) \leq \mathcal{E}^{(\rho)}(u_n - u_m) + \text{const} \|u_n - u_m\|_2^2 \rightarrow 0$$

as $n, m \rightarrow \infty$. By the closedness of $(\mathcal{E}, \mathcal{F})$, we conclude that $\mathcal{E}(u_n) \rightarrow 0$ whence the claim follows from (4.2).

Let $\mathcal{F}^{(\rho)}$ be the domain of the closure of $(\mathcal{E}^{(\rho)}, \mathcal{F})$. Then $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is a Dirichlet form. Let us prove that it is regular, that is, $\mathcal{F}^{(\rho)} \cap C_0$ is dense both in C_0 and $\mathcal{F}^{(\rho)}$. The former follows from the fact that $\mathcal{F} \cap C_0$ is dense in C_0 and $\mathcal{F} \subset \mathcal{F}^{(\rho)}$. To prove the latter, observe that, by construction of $\mathcal{F}^{(\rho)}$, for any $u \in \mathcal{F}^{(\rho)}$ and any $\varepsilon > 0$ there exists $v \in \mathcal{F}$ such that

$$\mathcal{E}^{(\rho)}(u - v)^{1/2} + \|u - v\|_2 < \varepsilon.$$

By the regularity of $(\mathcal{E}, \mathcal{F})$, there exists $f \in \mathcal{F} \cap C_0$, such that

$$\mathcal{E}(v - f)^{1/2} + \|v - f\|_2 < \varepsilon$$

Adding up these two inequalities, using (4.2) and the triangle inequalities, we obtain

$$\mathcal{E}^{(\rho)}(u - f)^{1/2} + \|u - f\|_2 < \text{const } \varepsilon,$$

which proved the density of $\mathcal{F} \cap C_0$ in $\mathcal{F}^{(\rho)}$.

To show that $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is ρ -local, consider functions $u, v \in \mathcal{F}$ with compact supports and such that

$$\text{dist}(\text{supp } u, \text{supp } v) > \rho.$$

If both points x, y are outside one of the supports $\text{supp } u, \text{supp } v$ then the integrand function in (4.1) vanishes. If they belong to different supports then $d(x, y) > \rho$ so that the couple (x, y) is outside the domain of integration in (4.1). Hence, the integral in (4.1) vanishes. Since $\mathcal{E}^{(L)}$ is local, we see that $\mathcal{E}^{(L)}(u, v) = 0$, and hence, $\mathcal{E}^{(\rho)}(u, v) = 0$, which finishes the proof. \square

We now have two regular Dirichlet forms $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$. For any non-empty open subset $\Omega \subset M$, denote by $\{P_t^\Omega\}_{t>0}$ and $\{Q_t^\Omega\}_{t>0}$ the heat semigroups of $(\mathcal{E}, \mathcal{F}(\Omega))$ and $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ respectively. We investigate here the relationship between these two semigroups.

Proposition 4.4. *For any non-empty open subset Ω of M and any non-negative $f \in L^2 \cap L^1(M, \mu)$, we have that, for all $t > 0$ and almost all $x \in \Omega$,*

$$P_t^\Omega f(x) \leq Q_t^\Omega f(x) + 2t \|f\|_1 \text{ esup}_{x \in M, y \in B(x, \rho)^c} J(x, y). \quad (4.5)$$

Proof. Without loss of generality we assume that Ω is precompact; otherwise we exhaust Ω by a sequence of precompact open subsets and then pass to the limit. It suffices to assume that $\|f\|_1 \leq 1$. Note that the function

$$v(t, \cdot) = Q_t^\Omega f$$

is a weak solution of the heat equation associated with the form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ that is, for any $\psi \in \mathcal{F}^{(\rho)}(\Omega)$ and $t > 0$,

$$\left(\frac{\partial v}{\partial t}, \psi \right) + \mathcal{E}^{(\rho)}(v, \psi) = 0 \quad (4.6)$$

(see Appendix 7 for details).

Let φ be a cut-off function of the pair (Ω, M) , that is, $\varphi \in \mathcal{F} \cap C_0$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in an open neighborhood of $\bar{\Omega}$. Consider the function

$$u(t, x) = v(t, x) + 2w(\rho)t\varphi(x), \quad (4.7)$$

where

$$w(\rho) := \operatorname{esup}_{x \in M, y \in B(x, \rho)^c} J(x, y).$$

We claim that u is a weak supersolution of the heat equation, associated with $(\mathcal{E}, \mathcal{F}(\Omega))$, that is,

$$\left(\frac{\partial u}{\partial t}, \psi \right) + \mathcal{E}(u, \psi) \geq 0 \quad (4.8)$$

for all non-negative $\psi \in \mathcal{F}(\Omega)$ and $t > 0$. Fix such a function ψ and note that it satisfies (4.6) because $\mathcal{F}(\Omega) \subset \mathcal{F}^{(\rho)}(\Omega)$. Since $\varphi = 1$ in Ω , we see that $(\varphi, \psi) = \|\psi\|_1$ and

$$\mathcal{E}(\varphi, \psi) = \lim_{t \rightarrow 0} t^{-1} (\varphi - P_t \varphi, \psi) = \lim_{t \rightarrow 0} t^{-1} (1 - P_t \varphi, \psi) \geq 0. \quad (4.9)$$

Therefore, it follows from (4.7), (4.6) and (4.9) that

$$\begin{aligned} \left(\frac{\partial u}{\partial t}, \psi \right) + \mathcal{E}(u, \psi) &= \left(\frac{\partial v}{\partial t} + 2w(\rho)\varphi, \psi \right) + \mathcal{E}(v + 2w(\rho)t\varphi, \psi) \\ &= \left(\frac{\partial v}{\partial t}, \psi \right) + 2w(\rho)\|\psi\|_1 + \mathcal{E}(v, \psi) + 2w(\rho)t\mathcal{E}(\varphi, \psi) \\ &\geq -\mathcal{E}^{(\rho)}(v, \psi) + 2w(\rho)\|\psi\|_1 + \mathcal{E}(v, \psi). \end{aligned} \quad (4.10)$$

On the other hand, using the facts that $v, \psi \geq 0$ and that

$$\|v(t, \cdot)\|_1 = \|Q_t^\Omega f\|_1 \leq \|f\|_1 \leq 1,$$

we have that

$$\begin{aligned} \mathcal{E}(v, \psi) - \mathcal{E}^{(\rho)}(v, \psi) &= \int_M \int_{B(x, \rho)^c} (v(t, x) - v(t, y)) (\psi(x) - \psi(y)) J(x, y) d\mu(y) d\mu(x) \\ &\geq - \int_M \int_{B(x, \rho)^c} (v(t, x)\psi(y) + v(t, y)\psi(x)) J(x, y) d\mu(y) d\mu(x) \\ &\geq -2w(\rho) \int_M \psi(x) d\mu(x) \int_{B(x, \rho)^c} v(t, y) d\mu(y) \geq -2w(\rho)\|\psi\|_1. \end{aligned} \quad (4.11)$$

Combining (4.10) and (4.11), we obtain (4.8).

Finally, observing that

$$u(t, \cdot) \xrightarrow{L^2(\Omega)} f \text{ as } t \rightarrow 0,$$

we conclude that by the parabolic comparison principle that

$$u(t, \cdot) \geq P_t^\Omega f(\cdot)$$

in $(0, \infty) \times \Omega$ (cf. [15, Lemma 4.16], or Lemma 7.3 in Appendix). Substituting u from (4.7) and using that $\varphi \equiv 1$ in Ω , we obtain (4.5). \square

Corollary 4.5. *For an open set $\Omega \subset M$, let p_t^Ω and q_t^Ω be the heat kernels of $(\mathcal{E}, \mathcal{F}(\Omega))$ and $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$, respectively. Then*

$$p_t^\Omega(x, y) \leq q_t^\Omega(x, y) + 2t \operatorname{esup}_{x \in M, y \in B(x, \rho)^c} J(x, y) \quad (4.12)$$

for all $t > 0$ and almost all $x, y \in \Omega$. In particular, for $\Omega = M$,

$$p_t(x, y) \leq q_t(x, y) + 2t \operatorname{esup}_{x \in M, y \in B(x, \rho)^c} J(x, y) \quad (4.13)$$

for all $t > 0$ and almost all $x, y \in M$.

Proof. Inequality (4.12) immediately follows from (4.5). \square

Proposition 4.6. *For any open subset Ω of M and any non-negative $f \in L^2 \cap L^\infty(M, \mu)$, we have that, for all $t > 0$ and almost all $x \in \Omega$,*

$$|P_t^\Omega f(x) - Q_t^\Omega f(x)| \leq 2t \|f\|_\infty \operatorname{esup}_{x \in M} \int_{B(x, \rho)^c} J(x, y) d\mu(y). \quad (4.14)$$

Proof. The proof is similar to that of Proposition 4.4. It suffices to assume that $\|f\|_\infty \leq 1$. As in the proof of Proposition 4.4, set $v(t, \cdot) = Q_t^\Omega f$ and

$$u(t, x) = v(t, x) + 2w(\rho)t\varphi(x), \quad (4.15)$$

where φ as before is a cut-off function of the pair (Ω, M) , but $w(\rho)$ is defined now by

$$w(\rho) := \operatorname{esup}_{x \in M} \int_{B(x, \rho)^c} J(x, y) d\mu(y). \quad (4.16)$$

Then u satisfies (4.8) because similarly to (4.11) we have

$$\begin{aligned} & \mathcal{E}(v, \psi) - \mathcal{E}^{(\rho)}(v, \psi) \\ & \geq - \int_M \int_{B(x, \rho)^c} (v(t, x)\psi(y) + v(t, y)\psi(x)) J(x, y) d\mu(y) d\mu(x) \\ & \geq -2 \|v(t, \cdot)\|_\infty \int_M \psi(x) d\mu(x) \int_{B(x, \rho)^c} J(x, y) d\mu(y) \\ & \geq -2w(\rho) \|\psi\|_1, \end{aligned}$$

where we have used that

$$\|v(t, \cdot)\|_\infty = \|Q_t^\Omega f\|_\infty \leq \|f\|_\infty \leq 1.$$

Repeating the end of the proof of Proposition 4.4, we obtain that, for all $t > 0$ and almost all $x \in \Omega$,

$$P_t^\Omega f(x) \leq Q_t^\Omega f(x) + 2w(\rho)t. \quad (4.17)$$

In order to show the inequality

$$P_t^\Omega f(x) \geq Q_t^\Omega f(x) - 2w(\rho)t, \quad (4.18)$$

we instead consider the function

$$u = P_t^\Omega f(x) + 2w(\rho)t\varphi,$$

In a similar way, one can verify that u is a supersolution of (4.6). Hence, by the comparison principle of Lemma 7.3, we obtain that $u \geq Q_t^\Omega f$ in $(0, \infty) \times \Omega$, thus finishing the proof. \square

Next we show the existence of the heat kernel q_t of $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ and derive its on-diagonal upper bound.

Proposition 4.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 with a jump density $J(x, y)$ and let $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ be the truncated form as in (4.1). If $(\mathcal{E}, \mathcal{F})$ satisfies the conditions (DUE) , (V_{\leq}) and (J_{\leq}) , then the heat kernel q_t of $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ exists, and satisfies the on-diagonal upper estimate:*

$$q_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \quad (4.19)$$

for all $t > 0$ and μ -almost all $x, y \in M$, where $C > 0$ depends on the constants in the hypotheses but is independent of ρ .

Proof. First observe that (DUE) implies the Nash inequality:

$$\|u\|_2^{2(1+\frac{2\beta}{\alpha})} \leq C\mathcal{E}(u) \quad (4.20)$$

for all $u \in \mathcal{F} \cap L^1$ with $\|u\|_1 = 1$, for $C > 0$ (cf. [8, Theorem (2.1)]). By conditions (V_{\leq}) and (J_{\leq}) , we have that

$$\begin{aligned} \int_{B(x, \rho)^c} J(x, y) d\mu(y) &\leq C \int_{B(x, \rho)^c} d(x, y)^{-(\alpha+\beta)} d\mu(y) \\ &= C \int_{\rho}^{\infty} s^{-(\alpha+\beta)} dV(x, s) \\ &\leq c\rho^{-\beta}, \end{aligned}$$

and hence, there exists $c > 0$ independent of ρ such that

$$w(\rho) := \operatorname{esup}_{x \in M} \int_{B(x, \rho)^c} J(x, y) d\mu(y) \leq c\rho^{-\beta}. \quad (4.21)$$

Therefore, we obtain from (4.20) and (4.3) that for all $u \in \mathcal{F}^{(\rho)} \cap L^1$ with $\|u\|_1 = 1$,

$$\|u\|_2^{2(1+\frac{2\beta}{\alpha})} \leq C \left(\mathcal{E}^{(\rho)}(u) + 4\rho^{-\beta} \|u\|_2^2 \right)$$

Hence, by [8, Theorem (2.1)], the heat kernel q_t of $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ exists and satisfies (4.19). \square

The following proposition gives a survival estimate for the Dirichlet heat semigroup $\{Q_t^B\}_{t>0}$ associated with $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ for any ball B .

Proposition 4.8. *If conditions (S) , (J_{\leq}) and (V_{\leq}) are satisfied, then we have that, for any $\varepsilon' \in (\varepsilon, 1)$, there exists $\delta' > 0$ independent of ρ such that, for all $t > 0$ and all balls B of radius r ,*

$$1 - Q_t^B \mathbf{1}_B(x) \leq \varepsilon' \text{ in } \frac{1}{4}B, \quad (4.22)$$

provided that $t^{1/\beta} \leq \delta'(r \wedge \rho)$.

Proof. We show (4.22) by using Proposition 4.6. Indeed, let $B = B(x_0, r)$ and choose $\Omega = B$ and $f = \mathbf{1}_B$ in (4.14). It follows, using (4.21), that for all $t > 0$ and almost all $x \in B$,

$$P_t^B \mathbf{1}_B(x) \leq Q_t^B \mathbf{1}_B(x) + 2C\rho^{-\beta}t \|\mathbf{1}_B\|_{\infty} = Q_t^B \mathbf{1}_B(x) + c\rho^{-\beta}t.$$

Hence, we have from condition (S) that

$$\begin{aligned} 1 - Q_t^B \mathbf{1}_B(x) &\leq 1 - P_t^B \mathbf{1}_B(x) + c\rho^{-\beta}t \\ &\leq \varepsilon + c\rho^{-\beta}t \leq \varepsilon' \text{ in } \frac{1}{4}B \end{aligned} \quad (4.23)$$

if $t^{1/\beta} \leq \delta r$ and if $c\rho^{-\beta}t \leq \varepsilon' - \varepsilon$, for any $\varepsilon' \in (\varepsilon, 1)$. This finishes the proof. \square

The following theorem gives the upper estimate of the heat kernel q_t of $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$.

Theorem 4.9. *Let $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ be the truncated Dirichlet form as above. If $(\mathcal{E}, \mathcal{F})$ satisfies the conditions (DUE) , (J_{\leq}) , (S) and (V_{\leq}) then, for all $t > 0$ and μ -almost all $x, y \in M$,*

$$q_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \exp\left(-c\left(\frac{d(x, y)}{\rho} \wedge \frac{\rho}{t^{1/\beta}}\right)\right), \quad (4.24)$$

where the constants $C, c > 0$ depend on the constants in the hypotheses but are independent of ρ .

Proof. Fix $x_0, y_0 \in M$ and $t > 0$. Set $r = \frac{1}{2}d(x_0, y_0)$. By the semigroup property, we have that

$$\begin{aligned} q_{2t}(x, y) &= \int_M q_t(x, z)q_t(z, y)d\mu(z) \\ &\leq \int_{B(x_0, r)^c} q_t(x, z)q_t(z, y)d\mu(z) \\ &\quad + \int_{B(y_0, r)^c} q_t(x, z)q_t(z, y)d\mu(z). \end{aligned} \quad (4.25)$$

Using (4.19) and (3.8) with $T_0 = \infty$, we obtain that

$$\begin{aligned} \int_{B(x_0, r)^c} q_t(x, z)q_t(z, y)d\mu(z) &\leq \frac{C}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \int_{B(x_0, r)^c} q_t(x, z)d\mu(z) \\ &\leq \frac{C}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \exp\left(-c\left(\frac{r}{\rho} \wedge \frac{\rho}{t^{1/\beta}}\right)\right) \end{aligned}$$

for μ -almost all $x \in B(x_0, \rho)$ and $y \in M$. Similarly,

$$\int_{B(y_0, r)^c} q_t(x, z)q_t(z, y)d\mu(z) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \exp\left(-c\left(\frac{r}{\rho} \wedge \frac{\rho}{t^{1/\beta}}\right)\right)$$

for μ -almost all $y \in B(y_0, \rho)$ and $x \in M$. Hence, inequality (4.24) follows. \square

5. PROOF OF THEOREM 2.1

In this section, we prove Theorem 2.1. The proof of the main implication (2.17) will be done in two steps. In the first step we show that, for any real $n \geq 0$,

$$p_t(x, y) \leq \frac{c(n)}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\frac{(\alpha+\beta)n}{n+\alpha+\beta}} \quad (5.1)$$

for almost all $x, y \in M$ and all $t > 0$. In the second step, we obtain the desired estimate (UE) by a self-improvement of (5.1).

Lemma 5.1. *Assume that all the hypotheses of Theorem 2.1 hold. Then*

$$(V_{\leq}) + (DUE) + (J_{\leq}) + (S) \Rightarrow (5.1).$$

Proof. Since the estimate (5.1) becomes stronger for larger n , it suffices to assume that $n \geq \alpha + \beta$. Fix some $t > 0$, $x, y \in M$ and set $r = d(x, y)$. If $r < t^{1/\beta}$ then (5.1) follows from (DUE) . Hence, we assume in the sequel that

$$r \geq t^{1/\beta}. \quad (5.2)$$

Let $\rho > 0$ be such that

$$\frac{\rho}{t^{1/\beta}} \geq 1 \quad \text{and} \quad \frac{\rho}{t^{1/\beta}} \geq \frac{r}{\rho} \quad (5.3)$$

(we specify the value of ρ below). Using Theorem 4.9 and the elementary inequality

$$\exp(-cs) \leq c_1(n)s^{-n} \quad \text{for all } s > 0,$$

we obtain from (4.24) and (5.3) that

$$q_t(x, y) \leq \frac{C_1}{t^{\alpha/\beta}} \exp\left(-c_1 \frac{r}{\rho}\right) \leq \frac{C(n)}{t^{\alpha/\beta}} \left(\frac{r}{\rho}\right)^{-n}.$$

Therefore, it follows from (4.13) and (J_{\leq}) that

$$\begin{aligned} p_t(x, y) &\leq q_t(x, y) + c\rho^{-(\alpha+\beta)t} \\ &\leq \frac{C(n)}{t^{\alpha/\beta}} \left[\left(\frac{\rho}{r}\right)^n + \rho^{-(\alpha+\beta)t^{1+\alpha/\beta}} \right]. \end{aligned} \quad (5.4)$$

Now choose ρ such that

$$\left(\frac{\rho}{r}\right)^n = \rho^{-(\alpha+\beta)t^{1+\alpha/\beta}},$$

that is,

$$\rho = r^{\frac{n}{n+\alpha+\beta}} \left(t^{1/\beta}\right)^{\frac{\alpha+\beta}{n+\alpha+\beta}}.$$

This ρ satisfies (5.3), since by (5.2) and $n \geq \alpha + \beta$,

$$\begin{aligned} \frac{\rho}{t^{1/\beta}} &= \left(\frac{r}{t^{1/\beta}}\right)^{\frac{n}{n+\alpha+\beta}} \geq 1, \\ \frac{\rho}{t^{1/\beta}} &= \left(\frac{r}{t^{1/\beta}}\right)^{\frac{n}{n+\alpha+\beta}} \geq \left(\frac{r}{t^{1/\beta}}\right)^{\frac{\alpha+\beta}{n+\alpha+\beta}} = \frac{r}{\rho}. \end{aligned}$$

Substituting the value of ρ to (5.4), we obtain

$$p_t(x, y) \leq \frac{2C(n)}{t^{\alpha/\beta}} \left(\frac{\rho}{r}\right)^n = \frac{c(n)}{t^{\alpha/\beta}} \left(\frac{r}{t^{1/\beta}}\right)^{-\frac{(\alpha+\beta)n}{n+\alpha+\beta}}, \quad (5.5)$$

which finishes the proof. \square

As a second step, we now prove Theorem 2.1.

Proof of Theorem 2.1. To show (2.17), we divide the proof into four steps.

Step 1. For any ball $B(x, r)$, it follows from (5.1) (cf. [17, Inequality (3.7), p.2072]) that

$$\begin{aligned} \int_{B(x, r)^c} p_t(x, y) d\mu(y) &\leq \int_{B(x, r)^c} \frac{c(n)}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\frac{(\alpha+\beta)n}{n+\alpha+\beta}} d\mu(y) \\ &\leq c(n) \int_{\frac{1}{2}rt^{-1/\beta}}^{\infty} s^{\alpha-1} (1+s)^{-\frac{(\alpha+\beta)n}{n+\alpha+\beta}} ds \\ &\leq C(n) \left(rt^{-1/\beta}\right)^{-\theta}, \end{aligned} \quad (5.6)$$

where

$$\theta = \frac{n\beta - \alpha(\alpha + \beta)}{n + \alpha + \beta} \in (0, \beta), \quad (5.7)$$

provided that

$$n > \alpha(1 + \alpha/\beta). \quad (5.8)$$

Step 2. Let B be any ball of radius r . Using [18, Lemma 6.1], we obtain from (5.6) that there is $c > 0$ such that

$$1 - P_t^B \mathbf{1}_B(x) \leq c \left(rt^{-1/\beta}\right)^{-\theta} \quad \text{for } \mu\text{-a.a. } x \in \frac{1}{4}B. \quad (5.9)$$

Note that the conservativeness of $(\mathcal{E}, \mathcal{F})$ is required in [18, Lemma 6.1]. Therefore, using (4.23),

$$1 - Q_t^B \mathbf{1}_B(x) \leq c_1 \left[\left(rt^{-1/\beta} \right)^{-\theta} + \rho^{-\beta} t \right] := \phi(r, t) \quad (5.10)$$

for all $t > 0$ and μ -almost all $x \in \frac{1}{4}B$, where $c_1 > 0$ is independent of the ball B and ρ .

Step 3. We prove the following improvement of estimate (4.24): for all $t > 0, k \geq 1$, and all $x_0, y_0 \in M$ with $d(x_0, y_0) > 4k\rho$,

$$q_t(x, y) \leq \frac{C(k)}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \left(1 + \rho t^{-1/\beta}\right)^{-\theta(k-1)}, \quad (5.11)$$

for μ -almost all $x \in B(x_0, \rho), y \in B(y_0, \rho)$, where the constant $C(k)$ is independent of ρ . The argument is similar to that of Theorem 4.9 but we use the sharper estimate (5.10) instead of survival estimate (S). Indeed, fix $k \geq 1, t > 0$ and fix $x_0, y_0 \in M$. Set

$$r = \frac{1}{2}d(x_0, y_0) > 2k\rho.$$

Assume that

$$\rho t^{-1/\beta} > 1;$$

otherwise (5.11) follows directly from (4.19).

Applying inequality (3.4) in Theorem 3.1 with $R = 2\rho$ and with function $\phi(r, t)$ from (5.10), we obtain that in $B(x_0, \rho)$,

$$\begin{aligned} Q_t \mathbf{1}_{B(x_0, r)^c} &\leq Q_t \mathbf{1}_{B(x_0, 2k\rho)^c} \leq \phi(\rho, t)^{k-1} \\ &= c_1^{k-1} \left[\left(\rho t^{-1/\beta} \right)^{-\theta} + \rho^{-\beta} t \right]^{k-1} \\ &\leq C(k) \left(\rho t^{-1/\beta} \right)^{-\theta(k-1)}, \end{aligned}$$

since

$$\rho^{-\beta} t = \left(\rho t^{-1/\beta} \right)^{-\beta} \leq \left(\rho t^{-1/\beta} \right)^{-\theta}.$$

Similarly, we have that in $B(y_0, \rho)$

$$Q_t \mathbf{1}_{B(y_0, r)^c} \leq C(k) \left(\rho t^{-1/\beta} \right)^{-\theta(k-1)}.$$

Therefore, it follows from (4.25) and (4.19) that

$$\begin{aligned} q_{2t}(x, y) &\leq \|q_t(\cdot, y)\|_{L^\infty(M)} Q_t \mathbf{1}_{B(x_0, r)^c}(x) + \|q_t(x, \cdot)\|_{L^\infty(M)} Q_t \mathbf{1}_{B(y_0, r)^c}(y) \\ &\leq \frac{C}{t^{\alpha/\beta}} \exp\left(4\rho^{-\beta}t\right) \left(\rho t^{-1/\beta} \right)^{-\theta(k-1)} \end{aligned}$$

for all $t > 0$ and μ -almost all $x \in B(x_0, \rho), y \in B(y_0, \rho)$, thus proving (5.11).

Step 4. Finally, we prove (UE). Fix $x_0, y_0 \in M$, and let $r = \frac{1}{2}d(x_0, y_0)$ as in the above step. We show that

$$p_t(x, y) \leq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}} \right)^{-(\alpha+\beta)} \quad (5.12)$$

for all $t > 0$ and μ -almost all $x \in B(x_0, \rho), y \in B(y_0, \rho)$, for all small enough $\rho > 0$. Indeed, it suffices to assume that $rt^{-1/\beta}$ is sufficiently large. Fix an integer n such that (5.8) holds, for example, $n = 2\alpha(1 + \alpha/\beta)$, and hence, the number θ from (5.7) is also fixed. Let

$$k = 2 + [(\alpha + \beta)/\theta] \quad (5.13)$$

so that

$$\theta(k-1) \geq \alpha + \beta,$$

and let $\rho = \frac{r}{8k}$. Then

$$\rho^{-\beta} t = \left(\frac{rt^{-1/\beta}}{8k} \right)^{-\beta} \leq C,$$

and hence, using (4.13) again and (5.11),

$$\begin{aligned} p_t(x, y) &\leq q_t(x, y) + c\rho^{-(\alpha+\beta)}t \\ &\leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{r}{8kt^{1/\beta}} \right)^{-\theta(k-1)} + c \left(\frac{r}{8k} \right)^{-(\alpha+\beta)} t \\ &\leq \frac{c}{t^{\alpha/\beta}} \left(\frac{r}{t^{1/\beta}} \right)^{-(\alpha+\beta)} \end{aligned}$$

for all $t > 0$ and μ -almost all $x \in B(x_0, \rho), y \in B(y_0, \rho)$, thus proving (5.12). Therefore, (UE) follows. \square

6. HEAT KERNEL BOUNDS USING EFFECTIVE RESISTANCE

In this section we show how Theorem 2.3 can be applied for a certain class of metric measure spaces with effective resistance.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ as before. Recall that the *effective resistance* $R(A, B)$ between two disjoint non-empty closed subsets A and B of M is defined by

$$R(A, B)^{-1} = \inf \{ \mathcal{E}(u) : u \in \mathcal{F} \cap C_0, u|_A = 1 \text{ and } u|_B = 0 \}. \quad (6.1)$$

It follows from (6.1) that, for any fixed A , $R(A, B)$ is a non-increasing function of B . Denote by

$$R(x, B) := R(\{x\}, B) \text{ and } R(x, y) := R(\{x\}, \{y\}).$$

In general, it may happen that $R(x, y) = \infty$ for some points $x, y \in M$. In this section we will exclude this case. We assume that the following two conditions hold:

(R₁) : There exist constants $C, \gamma > 0$ such that

$$|u(x) - u(y)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(u) \quad (6.2)$$

for all $u \in \mathcal{F} \cap C_0$ and all $x, y \in M$.

(R₂) : There exist constants $C, \gamma > 0$ such that, for all balls $B = B(x, r)$,

$$R(x, B(x, r)^c) \geq C^{-1}r^\gamma. \quad (6.3)$$

Observe that condition (R₁) is the *Morrey-Sobolev type inequality* in the framework of Dirichlet forms. As we will see, it implies that all functions in \mathcal{F} are Hölder continuous.

Consider also the following estimates of the effective resistance:

(R_≤) : There exist constants $C, \gamma > 0$ such that, for all $x, y \in M$,

$$R(x, y) \leq Cd(x, y)^\gamma. \quad (6.4)$$

(R_≥) : There exist constants $C, \gamma > 0$ such that, for all $x, y \in M$,

$$R(x, y) \geq C^{-1}d(x, y)^\gamma. \quad (6.5)$$

Applying the definition (6.1) to $A = \{x\}$ and $B = \{y\}$ we see that

$$(R_1) \Rightarrow (R_{\leq}).$$

Observe that also

$$(R_2) \Rightarrow (R_{\geq}).$$

Indeed, for any two points $x, y \in M$, taking $r = d(x, y)$ in (6.3) and noticing that $y \in B(x, r)^c$, we obtain

$$R(x, y) \geq R(x, B(x, r)^c) \geq C^{-1}r^\gamma = C^{-1}d(x, y)^\gamma.$$

In the next subsections, we will investigate various sufficient conditions for (R_1) and (R_2) , as well as their consequences. Then we will prove Theorem 6.13 below that states the equivalence $(UE) \Leftrightarrow (J_{\leq})$ under the standing hypotheses (R_1) and (R_2) .

6.1. Condition (R_1) .

Lemma 6.1. *If (R_1) and $(V \geq)$ are satisfied then any function $u \in \mathcal{F}$ admits a continuous version that satisfies (6.2). Consequently, all functions in \mathcal{F} have Hölder continuous versions with the Hölder exponent $\gamma/2$.*

Proof. By the regularity of $(\mathcal{E}, \mathcal{F})$, for every $u \in \mathcal{F}$ there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap C_0$ such that $\|u_n - u\|_2 \rightarrow 0$ and $\mathcal{E}(u - u_n) \rightarrow 0$ as $n \rightarrow \infty$. By (R_1) we have

$$|u_n(y) - u_n(x)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(u_n) \quad (6.6)$$

for all $x, y \in M$ and $n \geq 1$. Since the sequence $\{\mathcal{E}(u_n)\}$ is uniformly bounded, it follows from (6.6) that the sequence $\{u_n\}_{n=1}^{\infty}$ is equicontinuous in (M, d) . Let us prove that the sequence $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded in M in sup-norm. Indeed, it follows from (6.6) that

$$|u_n(y) - u_n(x)| \leq cd(x, y)^{\gamma/2},$$

with some $c > 0$, and hence,

$$u_n(x)^2 \geq \left(u_n(y) - cd(x, y)^{\gamma/2}\right)_+^2.$$

Integrating this inequality with respect to $x \in M$ with a fixed y and setting $m = |u_n(y)|$, we obtain

$$\|u_n\|_2^2 \geq \int_M \left(m - cd(x, y)^{\gamma/2}\right)_+^2 d\mu(x) \geq \int_0^R \left(m - cs^{\gamma/2}\right)_+^2 dV(x, s)$$

where $R > 0$ is chosen so that $\frac{1}{2}m = cR^{\gamma/2}$. It follows that $m - cs^{\gamma/2} \geq \frac{1}{2}m$ and

$$\|u_n\|_2^2 \geq \int_0^R \frac{1}{4}m^2 dV(x, s) = \frac{1}{4}m^2 V(x, R) \geq c_1 m^2 R^\alpha = c_2 m^{2+2\frac{\gamma}{\alpha}},$$

where we have also used $(V \geq)$. It follows that

$$|u_n(y)| \leq C \|u_n\|_2^{\frac{\alpha}{\alpha+\gamma}}.$$

Since the sequence $\{\|u_n\|_2\}$ is uniformly bounded and the constant C is independent of n and y , we conclude that the sequence $\{\sup |u_n|\}$ is uniformly bounded as well.

By the Arzela-Ascoli theorem, the sequence $\{u_n\}$ has a subsequence that converges locally uniformly to a continuous function \tilde{u} , that is a continuous version of u . Obviously, \tilde{u} satisfies (6.2) that proves the Hölder continuity. \square

The following proposition is useful for verifying (R_1) .

Proposition 6.2. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$. If (V) and (J_{\geq}) are satisfied with $0 < \alpha < \beta$, then (R_1) holds with $\gamma = \beta - \alpha$. Consequently, all functions in \mathcal{F} are Hölder continuous with the Hölder exponent $\frac{1}{2}(\beta - \alpha)$.*

Proof. We use in the proof the following fact: if μ is α -regular and $0 < \alpha < \beta$, then there exists a constant $C > 0$ such that for all functions $u \in L^2(M, \mu)$ the following inequality holds:

$$|u(y) - u(x)|^2 \leq Cd(x, y)^{\beta-\alpha} \sup_{0 < r < r_0} \int_M \int_{B(x, r)} \frac{1}{r^{\alpha+\beta}} |u(y) - u(x)|^2 d\mu(y) d\mu(x) \quad (6.7)$$

for all Lebesgue points $x, y \in M$ of u with $d(x, y) < r_0/3$, where $r_0 \in (0, +\infty]$ is fixed but arbitrary (cf. [17, the proof of Theorem 4.11(iii)]).

Since by hypothesis μ is α -regular and $J(x, y) \geq cd(x, y)^{-(\alpha+\beta)}$, we have, for all $u \in \mathcal{F} \cap C_0$ and for all $r > 0$,

$$\begin{aligned} \mathcal{E}^{(J)}(u) &= \int \int_{M \times M} |u(y) - u(x)|^2 J(x, y) d\mu(y) d\mu(x) \\ &\geq c \int \int_{M \times M} |u(y) - u(x)|^2 d(x, y)^{-(\alpha+\beta)} d\mu(y) d\mu(x) \\ &\geq c \int_M \int_{B(x, r)} \frac{1}{r^{\alpha+\beta}} |u(y) - u(x)|^2 d\mu(y) d\mu(x). \end{aligned}$$

Combining this with (6.7), we obtain

$$|u(y) - u(x)|^2 \leq Cd(x, y)^{\beta-\alpha} \mathcal{E}(u) \quad (6.8)$$

which yields (R_1) with $\gamma = \beta - \alpha$.

The second claim follows from Lemma 6.1. \square

6.2. Parabolicity. Condition (R_1) can also be obtained from (R_{\leq}) assuming in addition the *parabolicity* of $(\mathcal{E}, \mathcal{F})$.

For any precompact set K define its capacity by

$$\text{cap}(K) = \inf \{ \mathcal{E}(u) : u \in \mathcal{F} \cap C_0, u|_K = 1 \}, \quad (6.9)$$

that is, $\text{cap}(K) = R(K, \{\infty\})^{-1}$. Obviously, $\text{cap}(K)$ is a monotone increasing function of K .

Definition 6.3. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *parabolic*¹ if $\text{cap}(K) = 0$ for any compact subset of M .

A trivial sufficient condition for the parabolicity is that $1 \in \mathcal{F}$ and $\mathcal{E}(1, 1) = 0$. Indeed, in this case the constant function 1 can be used as a test function u in (6.9) that yields $\text{cap}(K) = 0$.

Lemma 6.4. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 . Then the following statements hold:*

$$\text{parabolicity} \Rightarrow \text{conservativeness}.$$

Proof. Indeed, if $(\mathcal{E}, \mathcal{F})$ is parabolic, then there exists a sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ from \mathcal{F} such that $0 \leq \varphi_n \uparrow 1$ and

$$\mathcal{E}(\varphi_n) \rightarrow 0$$

as $n \uparrow \infty$. In particular, we have that

$$\mathcal{E}(\varphi_n, v) \rightarrow 0$$

as $n \uparrow \infty$, for any $v \in \mathcal{F} \cap L^1$. Hence, $(\mathcal{E}, \mathcal{F})$ is conservative (cf. [11, Theorem 1.6.6, p.63]). \square

The following gives a criterion of verifying the parabolicity.

¹The term ‘‘parabolic’’ comes from classification theory of Riemann surfaces. A simply connected non-compact Riemann surface is conformally equivalent either to the Euclidean plane \mathbb{R}^2 or to the hyperbolic plane \mathbb{H}^2 . In the first case the surface is called parabolic, whereas in the second case – hyperbolic. It is known that the parabolicity of the Riemann surface is equivalent to vanishing of the capacity of any compact subset (cf. [12])

Proposition 6.5. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in $L^2(M, \mu)$ and (M, d) be unbounded. Then the following implication holds:*

$$(R_1) + (R_{\geq}) \Rightarrow \text{parabolicity of } (\mathcal{E}, \mathcal{F}).$$

Proof. Let K be a compact subset of M and fix $x_0 \in K$. Fix a number $\eta \in (0, 1)$ to be determined later on, and choose a point $x \notin K$ such that $K \subset B(x_0, \eta r)$ where $r = d(x_0, x)$. Let $\psi_x \in \mathcal{F}$ such that $\psi_x(x_0) = 1$, $\psi_x(x) = 0$, $0 \leq \psi_x \leq 1$ in M , and

$$\mathcal{E}(\psi_x) = R(x_0, x)^{-1}.$$

It follows from $(R_1), (R_{\geq})$ that, for all $y \in B(x_0, \eta r)$,

$$\begin{aligned} (1 - \psi_x(y))^2 &= (\psi_x(x_0) - \psi_x(y))^2 \leq C d(x_0, y)^\gamma \mathcal{E}(\psi_x) \\ &\leq C \frac{(\eta r)^\gamma}{R(x_0, x)} \leq c\eta^\gamma. \end{aligned}$$

Now choose η so small that $c\eta^\gamma \leq \frac{1}{4}$. Hence, we have $\psi_x \geq \frac{1}{2}$ in $B(x_0, \eta r) \supset K$. It follows that $f := 2\psi_x \geq 1$ on K , and

$$\begin{aligned} \text{cap}(K) &\leq \mathcal{E}(f) = 4\mathcal{E}(\psi_x) \\ &= 4R(x_0, x)^{-1} \leq Cr^{-\gamma} \end{aligned}$$

by using condition (R_{\geq}) . Letting $r = d(x_0, x) \rightarrow \infty$, we obtain that $\text{cap}(K) = 0$, showing that $(\mathcal{E}, \mathcal{F})$ is parabolic. \square

Proposition 6.6. *Assume that $(\mathcal{E}, \mathcal{F})$ is parabolic. Then $(R_{\leq}) \Rightarrow (R_1)$.*

Proof. We need to prove (6.2) for any $u \in \mathcal{F} \cap C_0$ and all $x, y \in M$. If $u(x) = u(y)$ then (6.2) is trivially satisfied, so that we assume in the sequel that $u(x) \neq u(y)$.

Let $\{\Omega_n\}_{n=1}^\infty$ be an increasing sequence of precompact open sets that exhausts M . We can assume that all Ω_n contain x and y . Since $\text{cap}(\Omega_n) = 0$, there exists $\varphi_n \in \mathcal{F} \cap C_0$ such that $\varphi_n|_{\Omega_n} = 1$ and $\mathcal{E}(\varphi_n) < \frac{1}{n}$, in particular, $\mathcal{E}(\varphi_n) \rightarrow 0$ as $n \rightarrow \infty$.

For fixed x, y as above, define the function \tilde{u} on M by

$$\tilde{u}_n = \frac{u - u(y)\varphi_n}{u(x) - u(y)}.$$

It is obvious that $\tilde{u}_n \in \mathcal{F} \cap C_0$ and $\tilde{u}_n(x) = 1$, $\tilde{u}_n(y) = 0$. Hence, by definition,

$$\begin{aligned} R(x, y)^{-1} &\leq \mathcal{E}(\tilde{u}_n) \\ &= \frac{\mathcal{E}(u) - 2u(y)\mathcal{E}(u, \varphi_n) + u(y)^2\mathcal{E}(\varphi_n)}{(u(x) - u(y))^2}. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we obtain

$$(u(x) - u(y))^2 \leq R(x, y)\mathcal{E}(u). \quad (6.10)$$

Substituting here the estimate (R_{\leq}) we obtain (6.2). \square

Define the *condition* (R) as the conjunction of (R_{\leq}) and (R_{\geq}) , that is,

(R) : For all $x, y \in M$,

$$R(x, y) \asymp d(x, y)^\gamma.$$

We see from Propositions 6.5 and 6.6 that, under condition (R) ,

$$\text{parabolicity} \Leftrightarrow (R_1).$$

6.3. Condition (R_2) with $\beta < 2$.

Proposition 6.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 with $\mathcal{E}^{(L)} \equiv 0$ and the jump density $J(x, y)$. Assume that (V_{\leq}) and (J_{\leq}) hold with some $0 < \alpha < \beta < 2$. Assume also that \mathcal{F} contains all Lipschitz functions with compact supports. Then condition (R_2) holds with $\gamma = \beta - \alpha$. Moreover, $(\mathcal{E}, \mathcal{F})$ is parabolic.*

Remark 6.8. By hypothesis we have

$$\mathcal{E}(\varphi) = \mathcal{E}^{(J)}(\varphi) = \int \int_{M \times M} (\varphi(x) - \varphi(y))^2 J(x, y) d\mu(y) d\mu(x).$$

As we will see from the proof, the right hand side here is finite for any Lipschitz function ϕ with compact support, provided (V_{\leq}) and (J_{\leq}) hold. Therefore, the condition that \mathcal{F} must contain all such functions is reasonable and not restrictive. In fact, one can define \mathcal{F} as the closure of the space of Lipschitz functions with compact supports.

The condition $\beta < 2$ is closely related to the use of Lipschitz functions and cannot be dropped. It is not known if a similar statement is true for $\beta \geq 2$.

Proof. Set $B = B(x_0, r)$. Let ϕ be a Lipschitz function on M with the Lipschitz constant L , such that $\phi \equiv 0$ in B^c . Then $\phi \in \mathcal{F}$. Since the function $|\phi(x) - \phi(y)|$ is symmetric in x, y and vanishes if $(x, y) \in B^c \times B^c$, in particular, if $(x, y) \in (2B)^c \times (2B)^c$, it follows that

$$\begin{aligned} \mathcal{E}(\phi) &= \mathcal{E}^{(J)}(\phi) \leq C \int_M \int_M \frac{(\phi(x) - \phi(y))^2}{d(x, y)^{\alpha+\beta}} d\mu(y) d\mu(x) \\ &\leq C \left(\int_{2B} \int_{2B} + 2 \int_{2B} \int_{(2B)^c} \right) \frac{(\phi(x) - \phi(y))^2}{d(x, y)^{\alpha+\beta}} d\mu(y) d\mu(x). \end{aligned} \quad (6.11)$$

Using (V_{\leq}) , the Lipschitz condition, and $2B \subset B(x, 4r)$, the first integral in (6.11) can be estimated by

$$\begin{aligned} L^2 \int_{2B} \left(\int_{B(x, 4r)} \frac{d\mu(y)}{d(x, y)^{\alpha+\beta-2}} \right) d\mu(x) &\leq L^2 \mu(2B) \int_0^{4r} s^{2-(\alpha+\beta)} \cdot s^{\alpha-1} ds \\ &\leq CL^2 r^{\alpha-\beta+2}, \end{aligned}$$

where we have also used $\beta < 2$.

In the second integral in (6.11) we have $\phi(y) = 0$. Since $\phi(x) = 0$ outside B , the integration in x can be reduced to B instead of $2B$. Hence, the second integral is estimated by

$$\begin{aligned} \int_B \phi(x)^2 \left(\int_{B(x, r)^c} \frac{d\mu(y)}{d(x, y)^{\alpha+\beta}} \right) d\mu(x) &\leq C \|\phi\|_{\infty}^2 \mu(B) \int_r^{\infty} s^{-(\alpha+\beta)} \cdot s^{\alpha-1} ds \\ &\leq C \|\phi\|_{\infty}^2 r^{\alpha-\beta}. \end{aligned} \quad (6.12)$$

Hence, we obtain

$$\mathcal{E}(\phi) \leq CL^2 r^{\alpha-\beta+2} + C \|\phi\|_{\infty}^2 r^{\alpha-\beta}. \quad (6.13)$$

Specify the function ϕ by

$$\phi(x) = \left(1 - \frac{d(x_0, x)}{r} \right)_+$$

and observe that it vanishes in B^c , has the Lipschitz constant $L = \frac{1}{r}$, and $\|\phi\|_{\infty} = \varphi(x_0) = 1$. It follows from (6.1) and (6.13) that

$$R(x_0, B^c)^{-1} \leq \mathcal{E}(\phi) \leq Cr^{\alpha-\beta},$$

thus proving (R_2) .

Finally, for any compact subset K of M , let $r_n \uparrow \infty$ as $n \rightarrow \infty$ so that $K \subset B(x_0, r_1/2)$. Set

$$\phi_n(x) = \left(1 - \frac{d(x_0, x)}{r_n}\right)_+.$$

Then $\phi_n \geq \frac{1}{2}$ in K . Set $u_n = (2\phi_n) \wedge 1$. We have

$$\text{cap}(K) \leq \mathcal{E}(u_n) \leq 4\mathcal{E}(\phi_n) \leq \frac{C}{r_n^{\beta-\alpha}} \rightarrow 0$$

as $n \rightarrow \infty$, showing that $\text{cap}(K) = 0$. Hence, $(\mathcal{E}, \mathcal{F})$ is parabolic. \square

6.4. Condition (R_2) with general β . Fix some $0 < \alpha < \beta$ and set $\gamma = \beta - \alpha$. In this section we obtain (R_2) for arbitrary value of β at the expense of using more hypotheses.

Proposition 6.9. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 . Then*

$$(V_{\leq}) + (J_{\leq}) + (R_{\geq}) + (R_1) \Rightarrow (R_2).$$

Proof. We first claim that, for any $u_1, u_2 \in \mathcal{F}$,

$$\mathcal{E}(u_1 \wedge u_2) \leq \mathcal{E}(u_1) + \mathcal{E}(u_2) \tag{6.14}$$

Indeed, set $v_1 = u_1 - u_2$ and $v_2 = u_1 + u_2$ so that

$$u_1 \wedge u_2 = \frac{1}{2}(v_2 - |v_1|)$$

and

$$\begin{aligned} \mathcal{E}(u_1 \wedge u_2) &= \frac{1}{4}\mathcal{E}(v_2 - |v_1|) = \frac{1}{4}(\mathcal{E}(v_2) + \mathcal{E}(|v_1|) - 2\mathcal{E}(v_2, |v_1|)) \\ &\leq \frac{1}{2}(\mathcal{E}(v_2) + \mathcal{E}(|v_1|)) \\ &\leq \frac{1}{2}(\mathcal{E}(v_2) + \mathcal{E}(v_1)) \\ &= \frac{1}{2}(\mathcal{E}(u_1 + u_2) + \mathcal{E}(u_1 - u_2)) \\ &= \mathcal{E}(u_1) + \mathcal{E}(u_2), \end{aligned}$$

which proves (6.14).

The following arguments are motivated by [5, the proof of Lemma 2.4]. Fix a ball $B = B(x_0, r)$. For $x \in B \setminus \frac{1}{2}B$, let $\psi_x \in \mathcal{F}$ such that $\psi_x(x_0) = 1$, $\psi_x(x) = 0$, $0 \leq \psi_x \leq 1$ in M , and

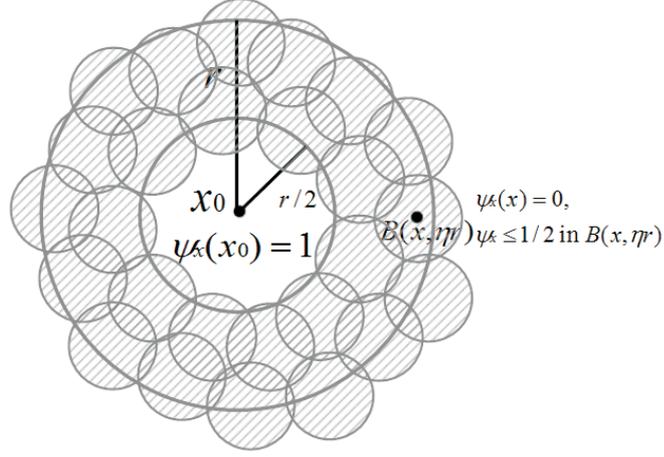
$$\mathcal{E}(\psi_x) = R(x_0, x)^{-1}.$$

Fix some small enough $\eta > 0$. It follows from (R_1) , (R_{\geq}) that, for all $y \in B(x, \eta r)$,

$$\begin{aligned} \psi_x(y)^2 &= |\psi_x(y) - \psi_x(x)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(\psi_x) \\ &\leq C \frac{(\eta r)^\gamma}{R(x_0, x)} \leq c\eta^\gamma \leq \frac{1}{4}, \end{aligned}$$

provided η is small enough. Hence, we have $\psi_x \leq \frac{1}{2}$ in $B(x, \eta r)$. See Figure 1. μ is α -regular, there exists an integer N independent of x_0, r such that the annulus $B \setminus \frac{1}{2}B$ can be covered by N balls $\{B(x_i, \eta r)\}_{i=1}^N$. Define the function

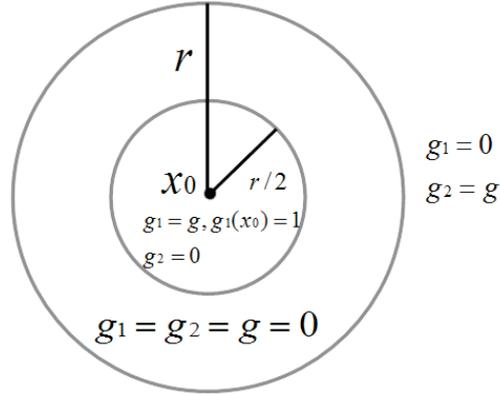
$$f = \psi_{x_1} \wedge \psi_{x_2} \wedge \cdots \wedge \psi_{x_N},$$

FIGURE 1. Function ψ_x and covering for $B \setminus \frac{1}{2}B$

and let $g = 2(f - \frac{1}{2})_+$. We have $f(x_0) = 1$ and, by the above argument, $f \leq \frac{1}{2}$ in $B \setminus \frac{1}{2}B$. It follows that $g(x_0) = 1$, $g \equiv 0$ in $B \setminus \frac{1}{2}B$, and $0 \leq g \leq 1$ in M . Using the Markov property of $(\mathcal{E}, \mathcal{F})$, (6.14), and (R_{\geq}) , we have

$$\mathcal{E}(g) \leq 4\mathcal{E}(f) \leq 4 \sum_{i=1}^N \mathcal{E}(\psi_{x_i}) = 4 \sum_{i=1}^N R(x_0, x_i)^{-1} \leq cr^{\alpha-\beta}. \quad (6.15)$$

Let $g_1 = g\phi$, $g_2 = g\mathbf{1}_{B^c}$, where ϕ is a cut-off function of the pair $(\frac{1}{2}B, B)$. See Figure 2. that $\text{supp } g_1 \subset \frac{1}{2}B$, $g_1(x_0) = 1$ and $g = g_1 + g_2$. Also, $g_1 \in \mathcal{F}$ by construction and $g_2 \in \mathcal{F}$

FIGURE 2. Functions g_1 and g_2

because $g_2 = g - g_1$. Since the supports of g_1 and g_2 are disjoint, we obtain

$$\mathcal{E}^{(L)}(g) = \mathcal{E}^{(L)}(g_1 + g_2) = \mathcal{E}^{(L)}(g_1) + \mathcal{E}^{(L)}(g_2) \geq \mathcal{E}^{(L)}(g_1). \quad (6.16)$$

On the other hand, using (J_{\leq}) and (V_{\leq}) and arguing as in (6.12), we obtain

$$\begin{aligned} \mathcal{E}^{(J)}(g_1, g_2) &= \int \int_{M \times M} (g_1(x) - g_1(y))(g_2(x) - g_2(y)) J(x, y) d\mu(y) d\mu(x) \\ &= -2 \int_{\frac{1}{2}B} \int_{B^c} g_1(x) g_2(y) J(x, y) d\mu(y) d\mu(x) \\ &\geq -2 \int_{\frac{1}{2}B} \int_{B(x, r/2)^c} g_1(x) g_2(y) J(x, y) d\mu(y) d\mu(x) \\ &\geq -cr^{\alpha-\beta}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{E}^{(J)}(g) &= \mathcal{E}^{(J)}(g_1 + g_2) \\ &= \mathcal{E}^{(J)}(g_1) + \mathcal{E}^{(J)}(g_2) + 2\mathcal{E}^{(J)}(g_1, g_2) \\ &\geq \mathcal{E}^{(J)}(g_1) - cr^{\alpha-\beta}. \end{aligned} \tag{6.17}$$

Combining (6.16), (6.17), and (6.15), we obtain

$$\begin{aligned} \mathcal{E}(g_1) &= \mathcal{E}^{(L)}(g_1) + \mathcal{E}^{(J)}(g_1) \\ &\leq \mathcal{E}^{(L)}(g) + \mathcal{E}^{(J)}(g) + cr^{\alpha-\beta} \\ &= \mathcal{E}(g) + cr^{\alpha-\beta} \\ &\leq Cr^{\alpha-\beta}. \end{aligned}$$

Since g_1 is a test function for the resistance $R(x_0, B^c)$, we conclude that

$$R(x_0, B^c)^{-1} \leq \mathcal{E}(g_1) \leq Cr^{\alpha-\beta},$$

which was to be proved. \square

Corollary 6.10. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 . Then*

$$\text{parabolicity} + (V_{\leq}) + (J_{\leq}) + (R) \Rightarrow (R_1) + (R_2).$$

Proof. Indeed, by Proposition 6.6 we have

$$\text{parabolicity} + (R_{\leq}) \Rightarrow (R_1),$$

and the rest follows directly from Proposition 6.9. \square

6.5. The Green function. Assuming that (M, d) is non-compact and the condition (R_1) is satisfied, we construct the Dirichlet Green function, vanishing outside a precompact open set $\Omega \subset M$. It turns out that it can be expressed in terms of effective resistance and harmonic function in Ω .

By condition (R_1) , for any precompact open subset Ω of M , the space $\mathcal{F}(\Omega)$ is a Hilbert space under norm $\mathcal{E}(u)^{1/2}$ because, for $u \in \mathcal{F}(\Omega)$ and for $x \in \Omega, y \in M \setminus \Omega$, we have from (R_1) that

$$|u(x)|^2 = |u(y) - u(x)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(u),$$

and thus the integration over Ω gives that $\|u\|_2^2 \leq c(\Omega)\mathcal{E}(u)$. Therefore, by the Arzelà-Ascoli theorem, the embedding

$$\mathcal{F}(\Omega) \hookrightarrow C(\overline{\Omega})$$

is compact.

By the standard argument of PDE, it follows that, for any fixed precompact open subset Ω and any fixed $x \in \Omega$, the variational problem

$$\inf \{ \mathcal{E}(u) : u \in \mathcal{F}(\Omega), u(x) = 1 \} \tag{6.18}$$

possesses a solution, that is, there exists a function $\psi_\Omega(x, \cdot) \in \mathcal{F}(\Omega)$ satisfying that $\psi_\Omega(x, x) = 1$, $0 \leq \psi_\Omega(x, \cdot) \leq 1$ on M , such that

$$R(x, \Omega^c)^{-1} = \mathcal{E}(\psi_\Omega(x, \cdot)). \quad (6.19)$$

For any $x \in \Omega$, the function $\psi_{\Omega, x}(\cdot) := \psi_\Omega(x, \cdot)$ is *harmonic* in $M \setminus (\{x\} \cup \Omega^c)$ in the sense that

$$\mathcal{E}(\psi_{\Omega, x}, \varphi) = 0 \quad (6.20)$$

for any test function $\varphi \in \mathcal{F}(\Omega)$ with $\varphi(x) = 0$, because the function $f := \psi_{\Omega, x} + \lambda\varphi \in \mathcal{F}(\Omega)$, $f(x) = 1$ for any real λ , and

$$\begin{aligned} \mathcal{E}(\psi_{\Omega, x}) &\leq \mathcal{E}(f) = \mathcal{E}(\psi_{\Omega, x} + \lambda\varphi) \\ &= \mathcal{E}(\psi_{\Omega, x}) + 2\lambda\mathcal{E}(\psi_{\Omega, x}, \varphi) + \lambda^2\mathcal{E}(\varphi), \end{aligned}$$

showing that $\mathcal{E}(\psi_{\Omega, x}, \varphi) = 0$.

The effective resistance $R(x, \Omega^c)$ together with the harmonic function $\psi_\Omega(x, \cdot)$ gives rise to the *Green function* as follows

$$g_\Omega(x, y) := \begin{cases} R(x, \Omega^c)\psi_\Omega(x, y), & \text{if } x, y \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \quad (6.21)$$

Such a way of defining a Green function was first addressed in [28, Section 4] if Ω (instead of a ball) is a finite subset of M , and then was extended to infinite subsets Ω in [29, Section 4]. See also [23, Section 4] follows the same pattern as in [28] when Ω is infinite.

Clearly, it follows from (6.21) that $g_\Omega(x, y) \geq 0$ for any $x, y \in M$, and $g_\Omega(x, \cdot) \in \mathcal{F}(\Omega)$. Moreover, for any $\varphi \in \mathcal{F}(\Omega)$ and any fixed $x \in \Omega$,

$$\mathcal{E}(g_\Omega(x, \cdot), \varphi) = \varphi(x), \quad (6.22)$$

because, if $\varphi(x) = 0$ then (6.22) follows directly from (6.20), and if $\varphi(x) \neq 0$ then (6.22) follows by replacing the function $\varphi(\cdot)$ in (6.20) by $\psi_\Omega(x, \cdot) - \varphi(\cdot)/\varphi(x)$ and then using (6.19) and (6.21). For any $x, y \in \Omega$, letting $\varphi(\cdot) = g_\Omega(y, \cdot)$ in (6.22), we obtain that

$$\begin{aligned} g_\Omega(y, x) &= \mathcal{E}(g_\Omega(x, \cdot), g_\Omega(y, \cdot)) \\ &= \mathcal{E}(g_\Omega(y, \cdot), g_\Omega(x, \cdot)) = g_\Omega(x, y), \end{aligned} \quad (6.23)$$

that is, the Green function $g_\Omega(x, y)$ is symmetric in $x, y \in \Omega$.

We estimate the Green function $g_B(x, y)$ for any ball B .

Proposition 6.11. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 that satisfies (R_1) and (R_2) . Assume that there exists a constant $N > 1$ such that the annuli $B(x, Nr) \setminus B(x, r)$ are non-empty for all $x \in M$ and $r > 0$. Then there exist constants $C > 0$, $\eta \in (0, 1)$ such that, for all balls $B = B(x_0, r)$,*

$$g_B(x, y) \leq Cr^\gamma \text{ for all } x, y \in B, \quad (6.24)$$

$$g_B(x_0, y) \geq C^{-1}r^\gamma \text{ for all } y \in B(x_0, \eta r). \quad (6.25)$$

Proof. Set $\Omega = B = B(x_0, r)$ and let $\psi_B(x, \cdot)$ be as in (6.19). Choosing a point $z \in 2B \setminus B$ and using (R_1) , we obtain, for any $u \in \mathcal{F} \cap C_0$ such that $u(x_0) = 1$ and $u|_{B^c} = 0$, that

$$1 = (u(x_0) - u(z))^2 \leq Cd(x, z)^\gamma \mathcal{E}(u) \leq Cr^\gamma \mathcal{E}(u),$$

whence

$$R(x_0, B^c)^{-1} = \inf \mathcal{E}(u) \geq C^{-1}r^{-\gamma}$$

and $R(x_0, B^c) \leq Cr^\gamma$. Applying the latter inequality to the ball $B(x, 2r)$ instead of $B(x_0, r)$, we obtain, for any $x \in B$,

$$R(x, B^c) \leq R(x, B(x, 2r)^c) \leq Cr^\gamma.$$

Substituting this estimate into (6.21) and using $0 \leq \psi_B(x, \cdot) \leq 1$ we obtain (6.24).

To show (6.25) observe that by (6.19) and (R₂)

$$\mathcal{E}(\psi_B(x_0, \cdot)) = R(x_0, B^c)^{-1} \leq cr^{-\gamma}.$$

Then, using (R₁), we obtain, for any $y \in B(x_0, \eta r)$,

$$\begin{aligned} (1 - \psi_B(x_0, y))^2 &= (\psi_B(x_0, x_0) - \psi_B(x_0, y))^2 \\ &\leq Cd(x_0, y)^\gamma \mathcal{E}(\psi_B(x_0, \cdot)) \\ &\leq C \left(\frac{d(x_0, y)}{r} \right)^\gamma \\ &\leq C\eta^\gamma. \end{aligned}$$

If η is small enough then $C\eta^\gamma < \frac{1}{4}$ whence it follows

$$\psi_B(x_0, y) \geq \frac{1}{2} \text{ in } B(x_0, \eta r).$$

Using again (6.21) and (R₂), we obtain, for all $y \in B(x_0, \eta r)$,

$$g_B(x_0, y) = R(x_0, B^c)\psi_B(x_0, y) \geq \frac{1}{2}cr^\gamma,$$

which proves (6.25). \square

Let \mathcal{L}_B be the generator of $(\mathcal{E}, \mathcal{F}(B))$. Given $f \in L^2(B)$, consider the function

$$u(x) = \int_B g_B(x, y)f(y)d\mu(y), \quad (6.26)$$

where $g_B(x, y)$ is the Green function defined by (6.21). Then $u \in \mathcal{F}(B)$ and u is a *weak* solution to the following *Poisson-type* equation

$$-\mathcal{L}_B u = f \text{ in } B, \quad (6.27)$$

that is, for any $\varphi \in \mathcal{F}(B)$,

$$\mathcal{E}(u, \varphi) = \int_B f(x)\varphi(x)d\mu(x). \quad (6.28)$$

Indeed, by (6.22) and the dominated convergence theorem, we obtain that, for any $\varphi \in \mathcal{F}(B)$,

$$\begin{aligned} \mathcal{E}(u, \varphi) &= \lim_{t \rightarrow 0} \frac{1}{t} (u - P_t u, \varphi) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_B (g_B(\cdot, y) - P_t(g_B(\cdot, y)), \varphi) f(y)d\mu(y) \\ &= \int_B \mathcal{E}(g_B(\cdot, y), \varphi) f(y)d\mu(y) \\ &= \int_B \varphi(y)f(y)d\mu(y), \end{aligned}$$

that proves (6.28).

Proposition 6.12. *Assume that μ is α -regular and $(\mathcal{E}, \mathcal{F})$ satisfies the conditions (R₁) and (R₂). Let f be any cut-off function for the pair $(\frac{1}{2}B, B)$ where $B = B(x_0, r)$ is an arbitrary ball. Then the solution u defined by (6.26) satisfies the following estimates:*

$$u(x) \leq Cr^{\gamma+\alpha} \text{ in } B(x_0, r) \quad (6.29)$$

and

$$u(x) \geq cr^{\gamma+\alpha} \text{ in } B(x_0, \eta r), \quad (6.30)$$

where $C, c > 0$ and $\eta \in (0, 1/2)$ are constants that depend only on the constants in the hypotheses (and do not depend on the ball B).

Proof. Observe that the α -regularity of μ implies that the annuli $B(x, Nr) \setminus B(x, r)$ are non-empty provided N is a large enough constant. Hence, Proposition 6.11 holds in the present setting. By reducing the constant η from the statement of this proposition, we can assume that $\eta < \frac{1}{2}$.

It follows from (6.24) and (6.26) that, using the fact that $0 \leq f \leq 1$ in M ,

$$\begin{aligned} u(x) &= \int_B g_B(x, y) f(y) d\mu(y) \leq \int_B g_B(x, y) d\mu(y) \\ &\leq Cr^\gamma \mu(B) \leq Cr^{\gamma+\alpha}. \end{aligned} \quad (6.31)$$

Similarly, observing that $f = 1$ in ηB and using (6.25), we obtain

$$\begin{aligned} u(x_0) &= \int_B g_B(x_0, y) f(y) d\mu(y) \\ &\geq \int_{B(x_0, \eta r)} g_B(x_0, y) d\mu(y) \\ &\geq cr^\gamma \mu(B(x_0, \eta r)) \geq c_1 r^{\gamma+\alpha}. \end{aligned} \quad (6.32)$$

Taking $\varphi = u$ in (6.28) and using (6.31) we obtain

$$\mathcal{E}(u) = \int_B f(x) u(x) d\mu(x) \leq Cr^{\gamma+\alpha} \mu(B) \leq cr^{\gamma+2\alpha}.$$

By Lemma 6.1, function u satisfies (6.2), so that for any $x \in B(x_0, \eta r)$,

$$\begin{aligned} |u(x) - u(x_0)|^2 &\leq cd(x, x_0)^\gamma \mathcal{E}(u) \\ &\leq Cr^{\gamma+2\alpha} d(x, x_0)^\gamma \\ &\leq C\eta^\gamma r^{2(\gamma+\alpha)}. \end{aligned}$$

Choosing η small enough and comparing with (6.32), we obtain that

$$|u(x) - u(x_0)| < \frac{1}{2} u(x_0)$$

whence $u(x) \geq \frac{1}{2} u(x_0) \geq cr^{\gamma+\alpha}$ follows. \square

6.6. Upper bounds of the heat kernel under (R_1) and (R_2) . We now use the estimates in (6.29)-(6.30) to obtain the survival estimate (S) and (DUE).

Theorem 6.13. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Assume that μ is α -regular and conditions (R_1) , (R_2) are satisfied with some $\gamma > 0$. Then the survival estimate (S) and (DUE) hold with $\beta = \alpha + \gamma$. Consequently, we have*

$$(UE) \Leftrightarrow (J_\leq). \quad (6.33)$$

Proof. Let us first prove (S). Fix a ball $B = B(x_0, r)$. We claim that in $(0, \infty) \times B$,

$$P_t^B 1_B(x) \geq \frac{u(x) - t}{\|u\|_\infty}, \quad (6.34)$$

where u is given by (6.26) with f being a cut-off function of the pair $(\frac{1}{2}B, B)$. Indeed, fix a cut-off function ϕ of the pair (B, M) and consider the function

$$w(t, x) = u(x) - t\phi(x) - \|u\|_\infty P_t^B 1_B(x).$$

Clearly, (6.34) is equivalent to $w \leq 0$. Let us first show that $w(t, x)$ is a weak subsolution of the heat equation in $(0, \infty) \times B$. We use the following facts:

(1) $P_t^B 1_B$ is a weak solution of the heat equation in $(0, \infty) \times B$; in particular,

$$\left(\frac{\partial}{\partial t} P_t^B 1_B, \varphi \right) + \mathcal{E}(P_t^B 1_B, \varphi) = 0$$

for any $\varphi \in \mathcal{F}(B)$.

(2) $\mathcal{E}(\phi, \varphi) \geq 0$ for any $0 \leq \varphi \in \mathcal{F}(B)$ (see (4.9)).

Using these as well as (6.28) we obtain for any $0 \leq \varphi \in \mathcal{F}(B)$ and $t > 0$

$$\begin{aligned} \left(\frac{\partial w}{\partial t}, \varphi \right) + \mathcal{E}(w, \varphi) &= \left(-\phi - \|u\|_\infty \frac{\partial}{\partial t} P_t^B 1_B, \varphi \right) \\ &\quad + \mathcal{E}(u - t\phi - \|u\|_\infty P_t^B 1_B, \varphi) \\ &= -(\phi, \varphi) + \mathcal{E}(u, \varphi) - t\mathcal{E}(\phi, \varphi) \\ &\leq (f - \phi, \varphi) \leq 0, \end{aligned}$$

which proves that $w(t, x)$ is a weak subsolution of the heat equation in $(0, \infty) \times B$.

On the other hand, we have as $t \rightarrow 0+$

$$w(t, \cdot) \xrightarrow{L^2(B)} u - \|u\|_\infty 1_B \leq 0$$

and hence, w_+ satisfies the initial condition

$$w_+(t, \cdot) \xrightarrow{L^2(B)} 0.$$

Therefore, by the parabolic maximum principle of Proposition 7.1 we conclude that $w \leq 0$ in $(0, \infty) \times B$, thus proving (6.34).

Using (6.34) and the estimates (6.29)-(6.30) of Proposition 6.12, we obtain that, for all $x \in B(x_0, \eta r)$,

$$\begin{aligned} P_t^B 1_B(x) &\geq \frac{u(x) - t}{\|u\|_\infty} \\ &\geq c - c_1 t r^{-\beta} \\ &\geq \frac{c}{2}, \end{aligned}$$

provided $t r^{-\beta}$ is small enough. This proves (S) although with constant η in (2.16) instead of $\frac{1}{4}$. Using ball covering argument as in the proof of Theorem 3.1 one can replace a small value of η by any value < 1 , in particular, by $\frac{1}{4}$.

Next we show that condition (R_1) implies (DUE) . It suffices to show that condition (R_1) implies the *Faber-Krahn inequality*

$$\lambda_{\min}(\Omega) := \inf_{f \in \mathcal{F} \cap C_0(\Omega)} \frac{\mathcal{E}(f)}{\|f\|_2^2} \geq c \mu(\Omega)^{-\beta/\alpha} \quad (6.35)$$

for all non-empty bounded² open subsets Ω of M , which in turn implies (DUE) (see for example [16, Lemmas 5.4 and 5.5]). Motivated by the argument in [22, Lemma 4.2], consider a function $f \in \mathcal{F} \cap C_0(\Omega)$ normalized so that $\sup |f| = 1$, and let $x \in \Omega$ be a point such that $|f(x)| = 1$. Let r be the largest radius such that $B(x, r) \subset \Omega$. Then the ball $B(x, 2r)$ is not covered by Ω so that there exists a point $y \in B(x, 2r) \setminus \Omega$ (note that M is unbounded by condition (V)). In particular, $y \notin \text{supp } f$ (see Figure 3).

Noting that $\mathcal{E}^{(J)}(f) \leq \mathcal{E}(f)$ and by the α -regularity of μ

$$r \leq C [\mu(B(x, r))]^{1/\alpha} \leq C [\mu(\Omega)]^{1/\alpha},$$

²If (6.35) is satisfied for bounded Ω , then (6.35) is also satisfied for all open Ω by the exhaustion argument.

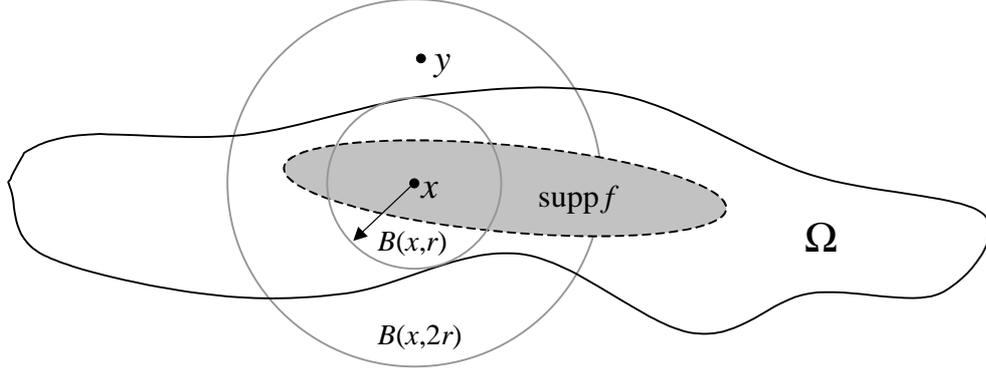


FIGURE 3.

we obtain from (R_1) that

$$\begin{aligned} 1 &= |f(y) - f(x)|^2 \\ &\leq C d(y, x)^{\beta-\alpha} \mathcal{E}^{(J)}(f) \\ &\leq C (2r)^{\beta-\alpha} \mathcal{E}(f) \leq C 2^{\beta-\alpha} [\mu(\Omega)]^{\beta/\alpha-1} \mathcal{E}(f). \end{aligned}$$

Since $\|f\|_2^2 \leq \mu(\Omega)$, it follows that

$$\frac{\mathcal{E}(f)}{\|f\|_2^2} \geq c [\mu(\Omega)]^{-\beta/\alpha},$$

for some $c > 0$, thus proving the Faber-Krahn inequality.

Finally, observe that $(\mathcal{E}, \mathcal{F})$ is parabolic by Proposition 6.5, and hence, is conservative by Lemma 6.4. Therefore, the equivalence (6.33) follows directly from Theorem 2.3. \square

Corollary 6.14. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 with a jump density $J(x, y)$ and let $0 < \alpha < \beta$. Then*

$$(V) + (J) + (R_2) \Rightarrow (UE). \quad (6.36)$$

Proof. Note that $(V) + (J) \Rightarrow (R_1)$ by Proposition 6.2. Hence, the conclusion (6.36) follows from Theorem 6.13. \square

Remark 6.15. *Besides the hypotheses of Corollary 6.14, if in addition $\mathcal{E}^{(L)} \equiv 0$ and $\beta < 2$, and if \mathcal{F} contains all Lipschitz functions with compact supports, then condition (R_2) also holds by Proposition 6.7. Therefore, it follows from (6.36) that*

$$(V) + (J) \Rightarrow (UE). \quad (6.37)$$

Note that the conclusion (6.37) was addressed in [6, 9].

6.7. Two sided estimates of the heat kernel. Let us introduce the following condition: **(NLE)**: *(The near diagonal lower estimate)* There exist $\delta \in (0, 1)$ and $c > 0$ such that, for all $t > 0$ and almost all $x, y \in M$ such that $d(x, y) \leq \delta t^{1/\beta}$,

$$p_t(x, y) \geq ct^{-\alpha/\beta}.$$

Theorem 6.16. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Fix parameters $0 < \alpha < \beta$ and set $\gamma = \beta - \alpha$. Then the following equivalences are true:*

$$\begin{aligned} \text{parabolicity} + (V) + (J_{\leq}) + (R) &\Leftrightarrow (UE) + (NLE) \\ &\Leftrightarrow (UE) + (NLE) + \text{continuity} \end{aligned}$$

where “continuity” means that for any $t > 0$ the heat kernel $p_t(x, y)$ is continuous in $(x, y) \in M \times M$.

Proof. We first show the implication

$$\text{parabolicity} + (V) + (J_{\leq}) + (R) \Rightarrow (UE) + (NLE) + \text{continuity}.$$

By Corollary 6.10 both conditions (R_1) and (R_2) are satisfied. Hence, by Theorem 6.13, we obtain (UE) .

Let us show that $p_t(x, y)$ as a function of x, y admits a continuous version. By (R_1) and Lemma 6.1, all functions from \mathcal{F} are Hölder continuous and, moreover, satisfy (6.2). By (UE) we have, for any $f \in L^2$

$$\|P_t f\|_{L^\infty} \leq C t^{-\alpha/\beta} \|f\|_2.$$

Since $P_t f \in \mathcal{F}$, it admits a continuous version that also will be denoted by $P_t f$. Then we can write, for all $t > 0$ and $x \in M$,

$$|P_t f(x)| \leq C t^{-\alpha/\beta} \|f\|_2.$$

By the Riesz representation theorem, there is a function $p_t(x, \cdot) \in L^2$ such that

$$P_t f(x) = (p_t(x, \cdot), f). \quad (6.38)$$

The function $p_t(x, y)$ as a function of x, y is symmetric in the following sense: $p_t(x, y) = p_t(y, x)$ for almost all $x, y \in M$. Using the semigroup identity $P_{t+s} = P_t P_s$ and the symmetry of the heat kernel, we obtain

$$\begin{aligned} (p_{t+s}(x, \cdot), f) &= \int_M p_t(x, z) \left(\int_M p_s(z, y) f(y) d\mu(y) \right) d\mu(z) \\ &= \int_M \left(\int_M p_t(x, z) p_s(y, z) d\mu(z) \right) f(y) d\mu(y) \end{aligned}$$

whence

$$p_{t+s}(x, y) = \int_M p_t(x, z) p_s(y, z) d\mu(z) = (p_t(x, \cdot), p_s(y, \cdot)) \quad (6.39)$$

for almost all y . The right hand side is well-defined for all $x, y \in M$ and is continuous separately in x and in y because it has the form (6.38). Therefore, the left hand side, that is, $p_{t+s}(x, y)$ has a pointwise version that is continuous in x and y separately. Switching to this version we obtain, in particular, that the identity (6.39) is true pointwise.

On the other hand, using the spectral resolution $\{E_\lambda\}$ of the generator \mathcal{L} and the elementary inequality $\lambda e^{-2\lambda t} \leq \frac{1}{2et}$ that is true for all $\lambda \geq 0$ and $t > 0$, we obtain

$$\mathcal{E}(P_t f) = \int_0^\infty \lambda e^{-2\lambda t} d(E_\lambda f, f) \leq \frac{1}{2et} \|f\|_2^2 \quad \text{for } f \in L^2.$$

Applying this inequality with $f = p_t(x, \cdot)$ and using (UE) and (6.39) yields

$$\begin{aligned} \mathcal{E}(p_{2t}(x, \cdot)) &\leq \frac{1}{2et} \|p_t(x, \cdot)\|_2^2 \\ &= \frac{1}{2et} p_{2t}(x, x) \leq C t^{-1-\alpha/\beta}, \end{aligned} \quad (6.40)$$

for all $t > 0$ and $x \in M$.

Since by (6.39) $p_{t+s}(x, \cdot) \in \mathcal{F}$, we obtain by (6.2) and (6.40)

$$|p_t(x, y_1) - p_t(x, y_2)|^2 \leq C d(y_1, y_2)^\gamma \mathcal{E}(p_t(x, \cdot)) \leq C t^{-1-\alpha/\beta} d(y_1, y_2)^\gamma$$

so that the function $p_t(x, y)$ is continuous in y uniformly in x . Hence, $p_t(x, y)$ is continuous jointly in x, y .

To verify (NLE) , we use a standard argument. It follows from (UE) and the α -regularity of μ that, for all $x \in M$ and $t, r > 0$,

$$\int_{B(x,r)^c} p_t(x,y) d\mu(y) \leq c \int_{rt^{-1/\beta}/2}^{\infty} s^{\alpha-1} (1+s)^{-(\alpha+\beta)} ds \leq \frac{1}{2}$$

provided $rt^{-1/\beta} \geq C$ for some large C . Choosing $r = Ct^{1/\beta}$, we obtain

$$\int_{B(x,r)} p_t(x,y) d\mu(y) = 1 - \int_{B(x,r)^c} p_t(x,y) d\mu(y) \geq \frac{1}{2}.$$

By the semigroup property, the symmetry of p_t and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} p_{2t}(x,x) &= \int_M p_t(x,y)^2 d\mu(y) \\ &\geq \frac{1}{V(x,r)} \left[\int_{B(x,r)} p_t(x,y) d\mu(y) \right]^2 \\ &\geq cr^{-\alpha} = c_1 t^{-\alpha/\beta}. \end{aligned} \tag{6.41}$$

Combining (6.40) with (R_1) we obtain, for all $y \in B(x, \delta t^{1/\beta})$,

$$\begin{aligned} |p_{2t}(x,x) - p_{2t}(x,y)|^2 &\leq Cd(x,y)^{\beta-\alpha} \mathcal{E}(p_{2t}(x,\cdot)) \\ &\leq Cd(x,y)^{\beta-\alpha} t^{-1-\alpha/\beta} \\ &\leq C\delta^{\beta-\alpha} t^{-2\alpha/\beta}. \end{aligned} \tag{6.42}$$

If δ is small enough then $C\delta^{\beta-\alpha} < \frac{1}{4}c_1^2$ where c_1 is the constant in (6.41), whence it follows that

$$p_{2t}(x,y) \geq p_{2t}(x,x) - \frac{c_1 t^{-\alpha/\beta}}{2} \geq \frac{c_1 t^{-\alpha/\beta}}{2},$$

thus proving (NLE) .

Next we show the opposite implication

$$(UE) + (NLE) \Rightarrow \text{parabolicity} + (V) + (J_{\leq}) + (R).$$

Note that $(\mathcal{E}, \mathcal{F})$ is conservative by Lemma 6.4. By [17, Theorem 3.2], $(UE) + (NLE) \Rightarrow (V)$. By Theorem 2.3, we have $(UE) \Rightarrow (J_{\leq})$. To show (R_1) , we use the following inequality

$$\mathcal{E}(u) \geq \frac{1}{2t} \int_M \int_M (u(x) - u(y))^2 p_t(x,y) d\mu(y) d\mu(x)$$

that is true for any $u \in \mathcal{F}$ and for any $t > 0$ (see [17]). Fix some $r > 0$ and choose t from the identity $r = \delta t^{1/\beta}$. It follows from (NLE) that, for any $u \in \mathcal{F}$,

$$\begin{aligned} \mathcal{E}(u) &\geq \frac{1}{2t} \int_M \int_{B(x,r)} (u(x) - u(y))^2 p_t(x,y) d\mu(y) d\mu(x) \\ &\geq \frac{c}{2t^{1+\alpha/\beta}} \int_M \int_{B(x,r)} (u(x) - u(y))^2 d\mu(y) d\mu(x) \\ &= Cr^{-(\alpha+\beta)} \int_M \int_{B(x,r)} (u(x) - u(y))^2 d\mu(y) d\mu(x). \end{aligned}$$

Combining this with (6.7), we obtain (R_1) , which implies also (R_{\leq}) .

As in the previous part of the proof, (UE) and (R_1) imply that the heat kernel is continuous. Using (NLE) and $\alpha < \beta$ we have

$$\int_0^{\infty} p_t(x,y) dt \geq \int_{(d(x,y)/\delta)^\beta}^{\infty} ct^{-\alpha/\beta} dt = \infty, \tag{6.43}$$

which implies that $(\mathcal{E}, \mathcal{F})$ is recurrent (cf. [11, formula (1.6.2), p. 55]), that is (cf. [11, Theorem 1.6.3, p.58]), there exists a sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{F}$ such that

$$\lim_{n \rightarrow \infty} u_n = 1 \text{ } \mu\text{-a.e. and } \lim_{n \rightarrow \infty} \mathcal{E}(u_n) = 0.$$

Let K be a compact subset of M and let $x_0 \in K$. Then there exists an integer N_{x_0} such that $u_n(x_0) \geq \frac{3}{4}$ for all $n > N_{x_0}$. It follows from (R_1) that there exists η small enough such that $u_n(x) \geq \frac{1}{2}$ for all $x \in B(x_0, \eta)$ and all $n > N_{x_0}$. Covering K by a finite number of the balls $\{B(x_i, \eta)\}$, we obtain that $u_n(x) \geq \frac{1}{2}$ for all $x \in K$ and all $n \geq N$, for some large integer N . Set $v_n := (2u_n) \wedge 1$. We see that $v_n \in \mathcal{F}$, $v_n = 1$ in K for all $n \geq N$, and

$$\lim_{n \rightarrow \infty} \mathcal{E}(v_n) \leq 4 \lim_{n \rightarrow \infty} \mathcal{E}(u_n) = 0,$$

showing that $\text{cap}(K) \leq \lim_{n \rightarrow \infty} \mathcal{E}(v_n) = 0$, and hence, $(\mathcal{E}, \mathcal{F})$ is parabolic.

Finally, let us now prove (R_{\geq}) . Indeed, fix $x, y \in M$. By (6.10) and (6.40) we have, for $x, y \in M$ and $t > 0$,

$$\begin{aligned} p_t(x, x) - p_t(x, y) &\leq R(x, y)^{1/2} \mathcal{E}(p_t(x, \cdot))^{1/2} \\ &\leq CR(x, y)^{1/2} t^{-1/2 - \alpha/(2\beta)}. \end{aligned}$$

On the other hand, it follows from (NLE) , (UE) , that

$$\begin{aligned} p_t(x, x) - p_t(x, y) &\geq c_1 t^{-\alpha/\beta} - C_1 t^{-\alpha/\beta} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \\ &\geq \frac{1}{2} c_1 t^{-\alpha/\beta} \end{aligned}$$

provided $d(x, y) = ct^{1/\beta}$ with a large enough constant c . It follows from above two estimates that

$$CR(x, y)^{1/2} t^{-1/2 - \alpha/(2\beta)} \geq \frac{1}{2} c_1 t^{-\alpha/\beta}$$

whence

$$R(x, y) \geq \left(\frac{c_1}{2C}\right)^2 t^{\frac{\beta-\alpha}{\beta}} = cd(x, y)^{\beta-\alpha},$$

which proves (R_{\geq}) . \square

Corollary 6.17. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Fix parameters $0 < \alpha < \beta$ and set $\gamma = \beta - \alpha$. Then the following equivalence is true:*

$$\text{parabolicity} + (V) + \text{“locality”} + (R) \Leftrightarrow (UE_{loc}) + (NLE). \quad (6.44)$$

Proof. Since $(UE_{loc}) \Rightarrow (UE)$, we obtain that by Theorem 6.16

$$(UE_{loc}) + (NLE) \Rightarrow \text{parabolicity} + (V) + (R),$$

whereas $(UE_{loc}) \Rightarrow \text{“locality”}$ by [20, Lemma 3.1]. Since the locality is a stronger assumption than (J_{\leq}) , by Theorems 6.16 and 2.3 we have

$$\text{parabolicity} + (V) + \text{“locality”} + (R) \Rightarrow (UE) + (NLE) \Rightarrow (DUE) + (S).$$

Finally, by (1.3) we obtain (UE_{loc}) . \square

Remark 6.18. *If $(\mathcal{E}, \mathcal{F})$ is parabolic, then $R(x, y)$ is symmetric in x, y . Assume in addition that R is a metric in M (called the effective resistance metric). Setting $d(x, y) = R(x, y)$, we see that conditions (R_1) , (R_2) and (R) are satisfied with $\gamma = 1$. Therefore, Theorem 6.16 implies that*

$$(V) + (J_{\leq}) \Leftrightarrow (UE) + (NLE),$$

and Corollary 6.17 implies that

$$(V) + (\textit{locality}) \Leftrightarrow (UE_{loc}) + (NLE).$$

The latter was also proved in [31, Theorem 3.1] by a probabilistic method.

Example 6.19. Let M be a nested fractal in \mathbb{R}^n (cf. [34]) that is generated by an iterated function system $\{f_i\}_{i=1}^N$, $N \geq 2$ with contraction ratio $0 < s < 1$:

$$|f_i(x) - f_i(y)| = s|x - y| \text{ for all } x, y \in \mathbb{R}^n.$$

Let $(\mathcal{E}, \mathcal{F})$ be a local regular Dirichlet form introduced in [27] that is self-similar:

$$\mathcal{E}(u) = r^{-1} \sum_{k=1}^N \mathcal{E}(u \circ f_k),$$

where $0 < r < 1$. It was shown that the associated effective resistance R is a metric (cf [27]). Moreover,

$$R(x, y) \asymp |x - y|^{\frac{\log r}{\log s}} \text{ for all } x, y \in M,$$

see [25, p.163]. If the open set condition holds for $\{f_i\}_{i=1}^N$, then the Hausdorff dimension of M is

$$\alpha = \frac{\log N}{\log(1/s)},$$

and the α -dimensional Hausdorff measure μ is α -regular. In this case, we see that both conditions (V) and (R) are true. Noting that $(\mathcal{E}, \mathcal{F})$ is parabolic because $1 \in \mathcal{F}$, we obtain (UE_{loc}) and (NLE) by using Corollary 6.17. Kumagai [30] obtained the same result by a probabilistic method, see also [21].

7. APPENDIX A: PARABOLIC MAXIMUM PRINCIPLE

Let (M, d, μ) be a metric measure space. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in $L^2(M, \mu)$ and \mathcal{L} be its generator. Let I be an interval in \mathbb{R} , and let Ω be an open subset of M . A function $u : I \rightarrow \mathcal{F}$ is said to be a *weak subsolution* (resp. a *weak supersolution*) of the heat equation

$$\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \tag{7.1}$$

in $I \times \Omega$, if the Frechet derivative $\frac{\partial u}{\partial t}$ of u in L^2 exists in I in the norm of $L^2(\Omega)$ and, for any $t \in I$ and any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$\left(\frac{\partial u}{\partial t}(t, \cdot), \psi \right) + \mathcal{E}(u(t, \cdot), \psi) \leq 0 \quad (\text{resp. } \geq 0). \tag{7.2}$$

If the inequality in (7.2) is replaced by equality, then u is called a *weak solution* of the heat equation (7.1) in $I \times \Omega$. It is known that $P_t f$ is a weak solution in $(0, \infty) \times \Omega$ for any open $\Omega \subset M$ (cf. [15, Example 4.10]).

Proposition 7.1 (parabolic maximum principle [15]). *Let u be a weak subsolution of the heat equation in $(0, T) \times \Omega$, where $T \in (0, +\infty]$ and Ω is an open subset of M . Assume in addition that u satisfies the following boundary and initial conditions:*

- $u_+(t, \cdot) \in \mathcal{F}(\Omega)$ for any $t \in (0, T)$;
- $u_+(t, \cdot) \xrightarrow{L^2(\Omega)} 0$ as $t \rightarrow 0$.

Then $u(t, x) \leq 0$ for any $t \in (0, T)$ and μ -almost all $x \in \Omega$.

Remark 7.2. It was shown in [15, Lemma 4.4] that, for a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, if $u \in \mathcal{F}$ and if $u \leq v$ for some $v \in \mathcal{F}(\Omega)$, then $u_+ \in \mathcal{F}(\Omega)$.

Lemma 7.3 (Lemma 4.16, p.122, [15]). *Let U be an open subset of M , and $0 \leq f \in L^2(U)$. If $u : \mathbb{R}_+ \rightarrow \mathcal{F}$ is a weak non-negative supersolution to the heat equation in $\mathbb{R}_+ \times U$ and*

$$u(t, \cdot) \xrightarrow{L^2(U)} f \text{ as } t \rightarrow 0 \quad (7.3)$$

then, for all $t > 0$,

$$u(t, \cdot) \geq P_t^U f \text{ in } M. \quad (7.4)$$

8. APPENDIX B: LIST OF LETTERED CONDITIONS

Here we list all the lettered conditions used in the text.

Replacing the sign \asymp in condition, say **(X)** by \leq and \geq , one obtains conditions **(X_≤)** and **(X_≥)**, respectively. The letters α, β denote always fixed positive reals. The constant γ is always equal to $\beta - \alpha$ and appears only in the case when $\beta > \alpha$.

(V) : There exist constants $c, C, \alpha > 0$ such that, for all $x \in M$ and all $r > 0$,

$$V(x, r) \asymp Cr^\alpha.$$

(J) : The jump density exists and admits the estimate

$$J(x, y) \asymp Cd(x, y)^{-(\alpha+\beta)},$$

for μ -almost all $x, y \in M$.

(DUE) : *On-diagonal upper estimate*: the heat kernel p_t exists and satisfies the on-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}},$$

for all $t > 0$ and μ -almost all $x, y \in M$.

(UE) : *Upper estimate of non-local type*: the heat kernel p_t exists and satisfies the off-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}} \right)^{-(\alpha+\beta)}$$

for all $t > 0$ and μ -almost all $x, y \in M$.

(UE_{loc}) : *Upper estimate of a local type*: the heat kernel p_t exists and satisfies the off-diagonal upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \exp \left(- \left(\frac{d(x, y)}{ct^{1/\beta}} \right)^{\beta/(\beta-1)} \right)$$

for all $t > 0$ and μ -almost all $x, y \in M$ (here $\beta > 1$).

(UE Φ) : The heat kernel p_t exists and satisfies the estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \Phi \left(\frac{d(x, y)}{t^{1/\beta}} \right)$$

for all $t > 0$ and μ -almost all $x, y \in M$, where $\Phi : [0, \infty) \rightarrow [0, \infty)$ is continuous, non-increasing function such that

$$\int_0^\infty s^{\alpha-1} \Phi(s) ds < \infty.$$

(NLE) : (*The near diagonal lower estimate*) There exist $\delta \in (0, 1)$ such that, for all $t > 0$ and almost all $x, y \in M$ such that $d(x, y) \leq \delta t^{1/\beta}$,

$$p_t(x, y) \geq ct^{-\alpha/\beta}.$$

(**S**) : *Survival estimate*. There exist constants $\varepsilon, \delta \in (0, 1)$ such that, for all balls $B = B(x_0, r)$ and for all $t^{1/\beta} \leq \delta r$,

$$1 - P_t^B \mathbf{1}_B(x) \leq \varepsilon \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

(**T**) : *Tail estimate*. There exist constants $\varepsilon, \delta \in (0, 1)$ such that, for all balls $B = B(x_0, r)$ and for all $t^{1/\beta} \leq \delta r$,

$$P_t \mathbf{1}_{B^c}(x) \leq \varepsilon \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

(**T**_{strong}) : *Strong tail estimate*. There exist constants $c > 0$ and $\beta > 0$ such that, for all balls $B = B(x_0, r)$ and for all $t > 0$,

$$P_t \mathbf{1}_{B^c}(x) \leq \frac{ct}{r^\beta} \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

(**R**₁) : For all $u \in \mathcal{F} \cap C_0$ and all $x, y \in M$, the following inequality holds:

$$|u(x) - u(y)|^2 \leq Cd(x, y)^\gamma \mathcal{E}(u).$$

(**R**₂) : For all $x \in M$ and $r > 0$,

$$R(x, B(x, r)^c) \geq C^{-1}r^\gamma.$$

(**R**) : For all $x, y \in M$,

$$R(x, y) \asymp Cd(x, y)^\gamma.$$

REFERENCES

- [1] D.G. Aronson, Non-negative solutions of linear parabolic equations Ann. Scuola Norm. Sup. Pisa. Cl. Sci. (4) **22** (1968), 607–694. Addendum **25** (1971), 221–228.
- [2] M.T. Barlow, *Diffusions on fractals*, Lect. Notes Math. **1690**, Springer, 1998, 1–121.
- [3] M.T. Barlow and R.F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets, Canad. J. Math. (4) **51** (1999), 673–744.
- [4] M.T. Barlow, R.F. Bass, Z.-Q. Chen and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes, Trans. Amer. Math. Soc. **361** (2009), 1963–1999.
- [5] M. Barlow, T. Coulhon and T. Kumagai, Characterization of sub-Gaussian heat kernel estimates on strongly recurrent graphs, Comm. Pure Appl. Math. **58** (2005), 1642–1677.
- [6] M. Barlow, A. Grigor'yan and T. Kumagai, Heat kernel upper bounds for jump processes and the first exit time, J. Reine Angew. Math. **626** (2009), 135–157.
- [7] R.F. Bass and D.A. Levin, Transition probabilities for symmetric jump processes, Trans. Amer. Math. Soc. **7** (2002), 2933–2953.
- [8] E.A. Carlen and S. Kusuoka and D.W. Stroock, Upper bounds for symmetric Markov transition functions, Ann. Inst. Henri Poincaré-Probab. Statist. **23**(1987), 245–287.
- [9] Z.Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d -sets, Stochastic Process. Appl. **108** (2003), 27–62.
- [10] Z.Q. Chen and T. Kumagai, Heat kernel estimates for jump processes of mixed types on metric measure spaces, Prob. Th. Rel. Fields **140** (2008), 277–317.
- [11] M. Fukushima and Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Second revised and extended edition, De Gruyter, Studies in Mathematics, **19**, 2011.
- [12] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. **36** (1999), 135–249.
- [13] A. Grigor'yan, Heat kernels and function theory on metric measure spaces, Contemporary Mathematics, **338** (2003), 143–172.
- [14] A. Grigor'yan, Heat kernels on weighted manifolds and applications, Contemporary Mathematics **398** (2006), 93–191.
- [15] A. Grigor'yan and J. Hu, Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces, Invent. Math. **174** 2008, 81–126.
- [16] A. Grigor'yan and J. Hu, Upper bounds of heat kernels on doubling spaces, preprint.
- [17] A. Grigor'yan, J. Hu and K.-S. Lau, Heat kernels on metric-measure spaces and an application to semilinear elliptic equations, Trans. Amer. Math. Soc. **355** (2003), 2065–2095.

- [18] A. Grigor'yan, J. Hu and K.-S. Lau, Comparison inequalities for heat semigroups and heat kernels on metric measure spaces, *J. Funct. Anal.* **259** (2010), 2613-2641.
- [19] A. Grigor'yan, X. Huang and J. Masamune, On stochastic completeness of jump processes, *Math. Z.* (2011)
- [20] A. Grigor'yan and T. Kumagai, On the dichotomy in the heat kernel two sided estimates. In: *Analysis on Graphs and its Applications* (P. Exner et al. (eds.)), *Proc. of Symposia in Pure Math.* **77**, pp. 199–210, Amer. Math. Soc. 2008.
- [21] B.M. Hambly and T. Kumagai, Transition density estimates for diffusion processes on post critically finite self-similar fractals, *Proc. London Math. Soc.* **78** (1999), 431–458.
- [22] B.M. Hambly and T. Kumagai, Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries. In: *Fractal geometry and applications: A Jubilee of B. Mandelbrot* (M.L. Lapidus and M. van Frankenhuijsen (eds.)), *Proc. of Symposia in Pure Math.* **72**, Part 2, pp. 233–260, Amer. Math. Soc. 2004.
- [23] J. Hu, An analytical approach to heat kernel estimates on strongly recurrent metric spaces, *Proc. Edin. Math. Soc.* **51** (2008), 171-199.
- [24] J. Hu and T. Kumagai, Nash-type inequalities and heat kernels for non-local Dirichlet forms, *Kyushu J. Math.* **60** (2006), 245-265.
- [25] J. Hu and X.S. Wang, Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals, *Studia Math.* **177** (2006), 153–172.
- [26] J. Hu and M. Zähle, Generalized Bessel and Riesz potentials on metric measure spaces, *Potential Anal.* **30** (2009), 315–340.
- [27] J. Kigami, *Analysis on fractals*, Cambridge University Press, Cambridge, 2001.
- [28] J. Kigami, Harmonic analysis for resistance forms, *J. Funct. Anal.* **204** (2003), 399-444.
- [29] J. Kigami, *Volume doubling measures and heat kernel estimates on self-similar sets*, *Memoirs of AMS* **199** (2009) no. 932.
- [30] T. Kumagai, Estimates of the transition densities for Brownian motion on nested fractals, *Probab. Theory and Related Fields*, **96** (1993), 205-224.
- [31] T. Kumagai, Heat kernel estimates and parabolic Harnack inequalities on graphs and resistance forms, *Publ. Res. Inst. Math. Sci.* **40** (2004), 793-818.
- [32] S. Kusuoka, A diffusion process on a fractal, in “*Probabilistic methods in Mathematical Physics, Taniguchi Symp., Katana, 1985*”, ed. K.Ito and N.Ikeda, Kinokuniya-North Holland, Amsterdam, 1987. 251-274
- [33] P.Li and S.-T.Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** (1986), 153-201.
- [34] T. Lindstrøm, Brownian motion on nested fractals, *Mem. Amer. Math. Soc.* **420** (1990).
- [35] J. Masamune and T. Uemura, Conservation property of symmetric jump processes, *Ann. Inst. H. Poincaré Probab. Statist.*, **47** (2011), 650-662.
- [36] A. Stós, Symmetric α -stable processes on d -sets, *Bull. Pol. Acad. Sci. Math.* **48** (2000), 237-245.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, POSTFACH 100131, 33501 BIELEFELD, GERMANY.

E-mail address: grigor@math.uni-bielefeld.de

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA.

E-mail address: hujiaxin@mail.tsinghua.edu.cn

DEPARTMENT OF MATHEMATICS, CHINESE UNIVERSITY OF HONG KONG, SHATIN, N.T., HONG KONG.

E-mail address: kslau@math.cuhk.edu.hk