PRESENTATIONS OF GROTHENDIECK CONSTRUCTIONS

HIDETO ASASHIBA AND MAYUMI KIMURA

Abstract. We will give quiver presentations of the Grothendieck constructions of functors from a small category to the 2-category of $\Bbbk$-categories for a commutative ring $\Bbbk$.

Introduction

Throughout this paper I is a small category, $\Bbbk$ is a commutative ring, and $\Bbbk$-**Cat** denotes the the 2-category of all $\Bbbk$-categories, $\Bbbk$-functors between them and natural transformations between $\Bbbk$-functors.

The Grothendieck construction is a way to form a single category $\mathrm{Gr}(X)$ from a diagram X of small categories indexed by a small category I, which first appeared in [4, §8 of Exposé VI]. As is exposed by Tamaki [7] this construction has been used as a useful tool in homotopy theory (e.g., [8]) or topological combinatorics (e.g., [9]). This can be also regarded as a generalization of orbit category construction from a category with a group action.

In [2] we defined a notion of derived equivalences of (oplax) functors from I to $\Bbbk$-**Cat**, and in [3] we have shown that if (oplax) functors $X, X' : I \to \Bbbk\text{-}\mathbf{Cat}$ are derived equivalent, then so are their Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. An easy example of a derived equivalent pair of functors is given by using diagonal functors: For a category $\mathcal{C}$ define the *diagonal* functor $\Delta(\mathcal{C}) : I \to \Bbbk\text{-}\mathbf{Cat}$ to be a functor sending all objects of I to $\mathcal{C}$ and all morphisms in I to the identity functor of $\mathcal{C}$. Then if categories $\mathcal{C}$ and $\mathcal{C}'$ are derived equivalent, then so are their diagonal functors $\Delta(\mathcal{C})$ and $\Delta(\mathcal{C}')$. Therefore, to compute examples of derived equivalent pairs using this result, it will be useful to present Grothendieck constructions of functors by quivers with relations. We already have computations in two special cases. First for a $\Bbbk$-algebra A, which we regard as a $\Bbbk$-category with a single object, we noted in [3] that if I is a semigroup G, a poset S, or the free category $\mathbb{P}Q$ of a quiver Q, then the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ is isomorphic to the semigroup algebra AG, the incidence algebra AS, or the path-algebra AQ, respectively. Second in [1] we gave a quiver presentation of the orbit category $\mathcal{C}/G$ for each $\Bbbk$-category $\mathcal{C}$ with an action of a semigroup G in the case that $\Bbbk$ is a field, which can be seen as a computation of a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ of each functor $X : G \to \Bbbk\text{-}\mathbf{Cat}$.

In this paper we generalize these two results as follows:

(1) We compute the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ for each $\Bbbk$-algebra A and each small category I.

Key words and phrases. Grothendieck construction, functors, quivers.
This work is partially supported by Grant-in-Aid for Scientific Research (C) 21540036 from JSPS.

(2) We give a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ for each functor $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ and each small category I when $\Bbbk$ is a field.

In section 1 we give necessary definitions and recall the fact that all categories can be presented by quivers and relations. Sections 2 and 3 are devoted to the computation (1) and a quiver presentation (2) above, respectively. Finally in section 4 we give some examples.

1. Preliminaries

Throughout this paper $Q = (Q_0, Q_1, t, h)$ is a quiver, where $t(\alpha) \in Q_0$ is the *tail* and $h(\alpha) \in Q_0$ is the *head* of each arrow α of Q. For each path μ of Q, the tail and the head of μ is denoted by $t(\mu)$ and $h(\mu)$, respectively. For each non-negative integer n the set of all paths of Q of length at least n is denoted by $Q_{\geq n}$. In particular $Q_{\geq 0}$ denotes the set of all paths of Q.

A category $\mathcal{C}$ is called a $\Bbbk$-*category* if for each $x, y \in \mathcal{C}$, $\mathcal{C}(x, y)$ is a $\Bbbk$-module and the compositions are $\Bbbk$-bilinear.

Definition 1.1. Let Q be a quiver.

(1) The *free* category $\mathbb{P}Q$ of Q is the category whose underlying quiver is $(Q_0, Q_{\geq 0}, t, h)$ with the usual composition of paths.

(2) The *path* $\Bbbk$-category of Q is the $\Bbbk$-linearization of $\mathbb{P}Q$ and is denoted by $\Bbbk Q$.

Definition 1.2. Let $\mathcal{C}$ be a category. We set

$$\mathrm{Rel}(\mathcal{C}) := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \mathcal{C}(i,j) \times \mathcal{C}(i,j),$$

elements of which are called *relations* of $\mathcal{C}$. Let $R \subseteq \mathrm{Rel}(\mathcal{C})$. For each $i, j \in \mathcal{C}_0$ we set

$$R(i,j) := R \cap (\mathcal{C}(i,j) \times \mathcal{C}(i,j)).$$

(1) The smallest congruence relation

$$R^c := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \{(dac, dbc) \mid c \in \mathcal{C}(-, i), d \in \mathcal{C}(j, -), (a, b) \in R(i,j)\}$$

containing R is called the *congruence relation* generated by R.

(2) For each $i, j \in \mathcal{C}_0$ we set

$$\begin{aligned}
R^{-1}(i,j) &:= \{(g, f) \in \mathcal{C}(i,j) \times \mathcal{C}(i,j) \mid (f, g) \in R(i,j)\}\\
1_{\mathcal{C}(i,j)} &:= \{(f, f) \mid f \in \mathcal{C}(i,j)\}\\
S(i,j) &:= R(i,j) \cup R^{-1}(i,j) \cup 1_{\mathcal{C}(i,j)}\\
S(i,j)^1 &:= S(i,j)\\
S(i,j)^n &:= \{(h, f) \mid \exists g \in \mathcal{C}(i,j), (g, f) \in S(i,j), (h, g) \in S(i,j)^{n-1}\} \quad (\text{for all } n \geq 2)\\
S(i,j)^\infty &:= \bigcup_{n\geq 1} S(i,j)^n, \text{ and set}\\
R^e &:= \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} S(i,j)^\infty.
\end{aligned}$$

R^e is called the *equivalence relation* generated by R.

(3) We set $R^\# := (R^c)^e$ (cf. [5]).

Remark 1.3. In the statement (2) above, $S(i,j)^\infty$ is the smallest equivalence relation on $\mathcal{C}(i,j)$ containing $R(i,j)$ for each $i,j \in \mathcal{C}_0$.

Definition 1.4. Let $\mathcal{C}$ be a category and $R \subseteq \mathrm{Rel}(\mathcal{C})$. Then a category $\mathcal{C}/R^\#$ is defined as follows:

(i) $(\mathcal{C}/R^\#)_0 := \mathcal{C}_0$.

(ii) For $i,j \in (\mathcal{C}/R^\#)_0$, $(\mathcal{C}/R^\#)(i,j) := \mathcal{C}(i,j)/R^\#(i,j)$.
For each $f \in (\mathcal{C}/R^\#)(i,j)$, we set $\overline{f}$ the equivalence class of f in $R^\#$.

(iii) For $i,j,k \in (\mathcal{C}/R^\#)_0$ and $\overline{f} \in (\mathcal{C}/R^\#)(i,j)$, $\overline{g} \in (\mathcal{C}/R^\#)(j,k)$, $\overline{g} \circ \overline{f} := \overline{g \circ f}$.

(iv) A functor $F : \mathcal{C} \to \mathcal{C}/R^\#$ is defined as follows:
  (a) For $i \in \mathcal{C}_0$, $F(i) = i$.
  (b) For $i,j \in \mathcal{C}(i,j)$ and $f \in \mathcal{C}(i,j)$, $F(f) = \overline{f}$.

Remark 1.5. In definition1.4, $R^\#$ is a congruence relation, therefore the composition in (iii) is well-defined.

The following is well known (cf. [6]).

Proposition 1.6. *Let $\mathcal{C}$ be a category, and $R \subseteq \mathrm{Rel}(\mathcal{C})$. Then the category $\mathcal{C}/R^\#$ and the functor $F : \mathcal{C} \to \mathcal{C}/R^\#$ defined above satisfy the following conditions.*

(i) *For each $i,j \in \mathcal{C}_0$ and each $(f,f') \in R(i,j)$ we have $Ff = Ff'$.*

(ii) *If a functor $G : \mathcal{C} \to \mathcal{D}$ satisfies $Gf = Gf'$ for all $f,f' \in \mathcal{C}(i,j)$ and all $i,j \in \mathcal{C}_0$ with $(f,f') \in R(i,j)$, then there exists a unique functor $G' : \mathcal{C}/R^\# \to \mathcal{D}$ such that $G' \circ F = G$.*

Definition 1.7. Let Q be a quiver and $R \subseteq \mathrm{Rel}(\mathbb{P}Q)$. We set

$$\langle Q \mid R \rangle := \mathbb{P}Q/R^\#.$$

The following is straightforward.

Proposition 1.8. *Let $\mathcal{C}$ be a category, Q the underlying quiver of $\mathcal{C}$, and set*

$$R := \{(e_i, \mathbb{1}_i), (\mu, [\mu]) \mid i \in Q_0, \mu \in Q_{\geq 2}\} \subseteq \mathrm{Rel}(\mathbb{P}Q),$$

where e_i is the path of length 0 at each vertex $i \in Q_0$, and $[\mu] := \alpha_n \circ \cdots \circ \alpha_1$ (the composite in $\mathcal{C}$) for all paths $\mu = \alpha_n \ldots \alpha_1 \in Q_{\geq 2}$ with $\alpha_1, \ldots, \alpha_n \in Q_1$. Then

$$\mathcal{C} \cong \langle Q \mid R \rangle.$$

By this statement, an arbitrary category is presented by a quiver and relations. Throughout the rest of this paper I is a small category with a presentation $I = \langle Q \mid R \rangle$.

2. Grothendieck constructions of Diagonal functors

Definition 2.1. Let $X : I \to \Bbbk\text{-}\mathbf{Cat}$ be a functor. Then a category $\mathrm{Gr}(X)$, called the *Grothendieck construction* of X, is defined as follows:

(i) $(\mathrm{Gr}(X))_0 := \bigcup_{i \in I_0} \{(i,x) \mid x \in X(i)_0\}$

(ii) For $(i,x),(j,y) \in (\mathrm{Gr}(X))_0$

$$\mathrm{Gr}(X)((i,x),(j,y)) := \bigoplus_{a\in I(i,j)} X(j)(X(a)x,y)$$

(iii) For $f = (f_a)_{a\in I(i,j)} \in \mathrm{Gr}(X)((i,x),(j,y))$ and $g = (g_b)_{b\in I(j,k)} \in \mathrm{Gr}(X)((j,y),(k,z))$

$$g \circ f := \left(\sum_{\substack{c=ba \\ a\in I(i,j) \\ b\in I(j,k)}} g_b X(b) f_a \right)_{c\in I(i,k)}$$

Definition 2.2. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$. Then the *diagonal functor* $\Delta(\mathcal{C})$ of $\mathcal{C}$ is a functor from I to $\Bbbk$-**Cat** sending each arrow $a\colon i \to j$ in I to $\mathbb{1}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ in $\Bbbk$-**Cat**.

In this section, we fix a $\Bbbk$-algebra A which we regard as a $\Bbbk$-category with a single object $*$ and with $A(*,*) = A$. The *quiver algebra* AQ of Q over A is the A-linearization of $\mathbb{P}Q$, namely $AQ := A \otimes_{\Bbbk} \Bbbk Q$.

Definition 2.3. The ideal of AQ generated by the elements $g - h$ with $(g,h) \in R$ is denoted by $\langle R\rangle_A$:

$$\langle R\rangle_A := AQ\{g - h \mid (g,h) \in R\}AQ.$$

The purpose of this section is to prove the following theorem which computes the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of $\Delta(A) : I \to \Bbbk\text{-}\mathbf{Cat}$.

Theorem 2.4. $\mathrm{Gr}(\Delta(A)) \cong AQ/\langle R\rangle_A$.

To prove this theorem, we use the following two lemmas.

Lemma 2.5. *Let S be a set, $E \subseteq S \times S$ an equivalence relation on S. Then*

$$(\bigoplus_{x\in S} Ax)/(\sum_{(g,h)\in E} A(g-h)) \cong \bigoplus_{\overline{x}\in S/E} A\overline{x}$$

Proof. Let $\varepsilon : \bigoplus_{x\in S} Ax \to \bigoplus_{\overline{x}\in S/E} A\overline{x}$ be a homomorphism of A-modules defined by $x \mapsto \overline{x}$ $(x \in S)$. Then the sequence

$$0 \to \sum_{(g,h)\in E} A(g-h) \hookrightarrow \bigoplus_{x\in S} Ax \xrightarrow{\varepsilon} \bigoplus_{\overline{x}\in S/E} A\overline{x} \to 0$$

is exact. Indeed, since ε is obviously a surjection by definition, it is enough to show that $\operatorname{Ker}\varepsilon = \sum_{(g,h)\in E} A(g-h)$.

For each $(g,h) \in E$ we have

$$\varepsilon(g-h) = \overline{g-h} = \overline{g} - \overline{h} = 0,$$

and hence $\sum_{(g,h)\in E} A(g-h) \subseteq \operatorname{Ker}\varepsilon$.

To prove the reverse inclusion, let $\sum_{x\in S} a_x x \in \operatorname{Ker}\varepsilon$ $(a_x \in A)$. Then since

$$0 = \varepsilon\left(\sum_{x\in S} a_x x\right) = \sum_{x\in S} a_x \overline{x} = \sum_{\overline{x}\in S/E} \sum_{x'\in\overline{x}} a_{x'} \overline{x},$$

we have $\sum_{x'\in\overline{x}} a_{x'} = 0$ for each $\overline{x} \in S/E$, and hence for each $x \in S$ we have

$$a_x = -\sum_{x'\in\overline{x}\setminus\{x\}} a_{x'}$$

and

$$\sum_{x'\in\overline{x}} a_{x'}x' = a_x x + \sum_{x'\in\overline{x}\setminus\{x\}} a_{x'}x' = \sum_{x'\in\overline{x}\setminus\{x\}} a_{x'}(x'-x).$$

Let L be a complete set of representatives in S/E. Then we have

$$\sum_{x\in S} a_x x = \sum_{x\in L} \sum_{(x,x')\in E\setminus\{(x,x)\}} a_{x'}(x'-x).$$

Hence $\operatorname{Ker}\varepsilon \subseteq \sum_{(g,h)\in E} A(g-h)$ and we have $\operatorname{Ker}\varepsilon = \sum_{(g,h)\in E} A(g-h)$. □

We will gives an explicit form of $\langle R\rangle_A$ as follows.

Lemma 2.6. *For each $i, j \in Q_0$,*

$$\langle R\rangle_A(i,j) = \sum_{(g,h)\in R^\#(i,j)} A(g-h)$$

Proof. We set $I(i,j) := \sum_{(g,h)\in R^\#(i,j)} A(g-h)$. First, we prove that $I(i,j)$ is an ideal of AQ. It is obvious that $I(i,j)$ is closed under addition. Let $a \in AQ(i',i)$, $b \in AQ(j,j')$, $c \in I(i,j)$. Then there exist $a_\alpha, b_\beta, c_{g,h} \in A$ such that

$$\begin{aligned} a &= \sum_{\alpha\in\mathbb{P}Q(i'i)} a_\alpha \alpha \\ b &= \sum_{\beta\in\mathbb{P}Q(j,j')} b_\beta \beta \\ c &= \sum_{(g,h)\in R^\#(i,j)} c_{g,h}(g-h) \end{aligned}$$

and

$$\begin{aligned} bca &= \left(\sum_{\beta\in\mathbb{P}Q(j,j')} b_\beta\beta\right)\left(\sum_{(g,h)\in R^\#(i,j)} c_{g,h}(g-h)\right)\left(\sum_{\alpha\in\mathbb{P}Q(i'i)} a_\alpha\alpha\right) \\ &= \sum_{\delta\in\mathbb{P}Q(i',j')} \sum_{\substack{\delta=\beta\gamma \\ \gamma\in\mathbb{P}Q(i',j) \\ \beta\in\mathbb{P}Q(j,j')}} \sum_{\substack{\gamma=(g-h)\alpha \\ \alpha\in\mathbb{P}Q(i',i) \\ (g,h)\in R^\#(i,j)}} b_\beta c_{g,h} a_\alpha \delta. \end{aligned}$$

By $(g,h)\in R^\#$, $(\beta g\alpha, \beta h\alpha)\in R^\#$. Hence $bca\in AQ(i',j')$ as desired.

Next, we prove that $\langle R\rangle_A(i,j) = I(i,j)$. Since $R\subseteq R^\#$, for each $(g,h)\in R(i,j)$ we have

$$g-h\in I(i,j).$$

Hence $\langle R\rangle_A(i,j)\subseteq I(i,j)$. Further for each $(g,h)\in R^c(i,j)$, there exist $(g',h')\in R(i',j')$, $e\in\mathbb{P}Q(i,i')$ and $f\in\mathbb{P}Q(j',j)$ such that

$$(g,h) = (fg'e, fh'e).$$

Then

$$g-h = fg'e - fh'e = f(g'-h')e\in\langle R\rangle_A(i,j).$$

Hence also for each $(g,h)\in R^\#(i,j)$ we have $g-h\in I(i,j)$ because $I(i,j)$ is closed under addition. Therefore $I(i,j)\subseteq\langle R\rangle_A(i,j)$, and hence $\langle R\rangle_A(i,j) = I(i,j)$. □

Proof of Theorem2.4. The object classes and the morphism spaces of $\mathrm{Gr}(\Delta(A))$ and $AQ/\langle R\rangle_A$ are given as follows.

$\mathrm{Gr}(\Delta(A))$:

(i) $\mathrm{Gr}(\Delta(A))_0 = \{(i,*)\mid i\in Q_0\}$.

(ii) For $(i,*),(j,*)\in\mathrm{Gr}(\Delta(A))_0$

$$\begin{aligned} \mathrm{Gr}(\Delta(A))((i,*),(j,*)) &= \bigoplus_{a\in I(i,j)} \Delta(A)(j)(\Delta(A)(a)(*),*) \\ &= \bigoplus_{a\in I(i,j)} A(*,*) = A^{(I(i,j))} \end{aligned}$$

$AQ/\langle R\rangle_A$:

(i) $(AQ/\langle R\rangle_A)_0 = Q_0$.

(ii) For $i,j\in(AQ/\langle R\rangle_A)_0$

$$\begin{aligned} (AQ/\langle R\rangle_A)(i,j) &= (\bigoplus_{a\in\mathbb{P}Q(i,j)} Aa)/\langle R\rangle_A(i,j) \\ &= (\bigoplus_{a\in\mathbb{P}Q(i,j)} Aa)/\sum_{(g,h)\in R^\#(i,j)} A(g-h) \\ &= \bigoplus_{a\in I(i,j)} Aa \end{aligned}$$

by Lemma 2.6 and the last equality is given by the isomorphism in Lemma 2.5. We define a functor $F : \mathrm{Gr}(\Delta(A)) \to AQ/\langle R\rangle_A$ by

$$(i,*) \mapsto i$$

$$(f_a)_{a\in I(i,j)} \mapsto \sum_{a\in I(i,j)} f_a a$$

for each $(f_a)_{a\in I(i,j)} : (i,*) \to (j,*)$ in $\mathrm{Gr}(\Delta(A))$. We check that F is well-defined as a $\Bbbk$-linear functor. For each $(i,*) \in \mathrm{Gr}(\Delta(A))_0$ we have

$$\begin{aligned} F(\mathbb{1}_{(i,*)}) &= F((\delta_{1_i a})_{a\in I(i,i)}) \\ &= \sum_{a\in I(i,i)} \delta_{1_i a} a \\ &= 1_i \end{aligned}$$

For each $f \in \mathrm{Gr}(\Delta(A))((i,*,),(j,*))$ and $g \in \mathrm{Gr}(\Delta(A))((j,*,),(k,*))$, there exist $f_a, g_b \in A$ $(a \in I(i,j), b \in I(j,k))$ such that

$$\begin{aligned} f &= (f_a)_{a\in I(i,j)} \\ g &= (g_b)_{b\in I(j,k)}. \end{aligned}$$

Then

$$\begin{aligned} F(g\circ f) &= F\left(\left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} g_b f_a\right)_{c\in I(i,k)}\right) \\ &= \sum_{c\in I(i,k)} \left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} g_b f_a\right) c \end{aligned}$$

$$\begin{aligned} F(g)F(f) &= \left(\sum_{b\in I(j,k)} g_b b\right)\left(\sum_{a\in I(i,j)} f_a a\right) \\ &= \sum_{c\in I(i,k)} \left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} g_b f_a\right) c \\ &= F(g\circ f). \end{aligned}$$

Hence F is a functor. Obviously F is $\Bbbk$-linear. It is clear that F is bijective on objects and that for each $i, j \in Q_0$, F induces an isomorphism

$$\mathrm{Gr}(\Delta(A))((i,*),(j,*)) \to (AQ/\langle R\rangle_A)(i,j)$$

by the definition of F. Therefore $\mathrm{Gr}(\Delta(A)) \cong AQ/\langle R\rangle_A$. □

Remark 2.7. Theorem 2.4 can be written in the form

$$\mathrm{Gr}(\Delta(A)) \cong A \otimes_{\Bbbk} (\Bbbk Q/\langle R\rangle_{\Bbbk}).$$

3. The quiver presentation of Grothendieck constructions

In this section we give a quiver presentation of the Grothendieck construction of an arbitrary functor $I \to \Bbbk$-**Cat**. Throughout this section we assume that $\Bbbk$ is a field.

Theorem 3.1. *Let $X : I \to \Bbbk$-**Cat** be a functor, and for each $i \in I$ set $X(i) = \Bbbk Q^{(i)}/\langle R^{(i)}\rangle$ with $\Phi^{(i)}\colon \Bbbk Q^{(i)} \to X(i)$ the canonical morphism, where $R^{(i)} \subseteq \Bbbk Q^{(i)}$, $\langle R^{(i)}\rangle \cap \{e_x \mid x \in Q(i)_0\} = \emptyset$. Then Grothendieck construction is presented by the quiver with relations (Q, R') defined as follows.*

Quiver: $Q' = (Q'_0, Q'_1, t', h')$, where

(i) $Q'_0 := \bigcup_{i\in I}\{{}_ix \mid x \in Q_0^{(i)}\}$.

(ii) $Q'_1 := \bigcup_{i\in I}\{\{{}_i\alpha \mid \alpha \in Q_1^{(i)}\} \cup \{(a, {}_ix) : {}_ix \to {}_j(ax) \mid a : i \to j \in Q_1, x \in Q_0^{(i)}, ax \neq 0\}\}$,
where we set $ax := X(\overline{a})(x)$.

(iii) *For $\alpha \in Q_1^{(i)}$, $t'({}_i\alpha) = t^{(i)}(\alpha)$ and $h'({}_i\alpha) = h^{(i)}(\alpha)$.*

(iv) *For $a : i \to j \in Q_1, x \in Q_0^{(i)}$, $t'(a, {}_ix) = {}_ix$ and $h'(a, {}_ix) = {}_j(ax)$.*

Relations: $R' := R'_1 \cup R'_2 \cup R'_3$, where

(i) $R'_1 := \{\sigma^{(i)}(\mu) \mid i \in Q_0, \mu \in R^{(i)}\}$,
where we set $\sigma^{(i)} : \Bbbk Q^{(i)} \hookrightarrow \Bbbk Q'$.

(ii) $R'_2 := \{\pi(g, {}_ix) - \pi(h, {}_ix) \mid i, j \in Q_0, (g, h) \in R(i, j), x \in Q_0^{(i)}\}$, *where for each path a in Q we set*

$$\pi(a, {}_ix) := (a_n, {}_{i_{n-1}}(a_{n-1}a_{n-2}\dots a_1x))\dots(a_2, {}_{i_1}(a_1x))(a_1, {}_ix)$$

if $a = a_n \dots a_2a_1$ for some $a_1, \dots, a_n$ arrows in Q, and

$$\pi(a, {}_ix) := e_{{}_ix}$$

if $a = e_i$ for some $i \in Q_0$.

(iii) $R'_3 := \{(a, {}_iy)_i\alpha - {}_j(a\alpha)(a, {}_ix) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\}$, *where we take $a\alpha : ax \to ay$ so that $\Phi^{(j)}(a\alpha) \in X(\overline{a})\Phi^{(i)}(\alpha)$:*

$$\begin{array}{ccc} \alpha \in \Bbbk Q^{(i)} & \xrightarrow{\Phi^{(i)}} & X(i) \\ & & \downarrow{\scriptstyle X(\overline{a})} \\ a\alpha \in \Bbbk Q^{(j)} & \xrightarrow{\Phi^{(j)}} & X(j). \end{array}$$

Note that the ideal $\langle R' \rangle$ is independent of the choice of $a\alpha$ because $R'_1 \subseteq R'$.

Proof. We define a $\Bbbk$-functor $\Phi : \Bbbk Q' \to \mathrm{Gr}(X)$ by:

(i) for ${}_ix \in Q'_0$, $\Phi({}_ix) = (i, x)$;
(ii) for ${}_i\alpha : {}_ix \to {}_iy \in Q_1^{(i)}$, $\Phi({}_i\alpha) = (\delta_{\mathbf{1}_i a}\Phi^{(i)}(\alpha))_{a\in I(i,i)}$;
(iii) for $(a, {}_ix) : {}_ix \to {}_j(ax) \in Q'_1$, $\Phi((a, {}_ix)) = (\delta_{\overline{a}b}\mathbb{1}_{X(\overline{a})(x)})_{b\in I(i,j)}$;
(iv) for $\alpha_n\alpha_{n-1}\dots\alpha_1 \in \mathbb{P}Q'$ $(\alpha_1, \dots, \alpha_n \in Q')$

$$\Phi(\alpha_n\alpha_{n-1}\dots\alpha_1) := \Phi(\alpha_n)\Phi(\alpha_{n-1})\dots\Phi(\alpha_1); \text{ and}$$

(v) for $f := \sum_{\alpha\in\mathbb{P}Q'({}_ix,{}_jy)} f_\alpha\alpha \in \Bbbk Q'({}_ix, {}_jy)$ $(f_\alpha \in \Bbbk)$

$$\Phi(f) := \sum_{\alpha\in\mathbb{P}Q'({}_ix,{}_jy)} f_\alpha\Phi(\alpha).$$

Claim 1. Φ *is well-defined as a $\Bbbk$-functor, and is bijective on objects.*

Indeed, this is clear by noting that for each ${}_ix \in Q'_0$ we have

$$\begin{aligned}\Phi(e_{{}_ix}) &= (\delta_{\mathbf{1}_i,a}\Phi^{(i)}(e_x))_{a\in I(i,i)} \\ &= \mathbb{1}_{(i,x)}.\end{aligned}$$

Claim 2. $\Phi(R') = 0$.

Indeed, for each $i \in Q_0$, $\alpha, \beta \in Q_1^{(i)}$ we have

$$\begin{aligned}\Phi({}_i\beta{}_i\alpha) &= \Phi({}_i\beta)\Phi({}_i\alpha) \\ &= (\delta_{1_i,b}\Phi^{(i)}(\beta))_{b\in I(i,i)}(\delta_{1_i,a}\Phi^{(i)}(\alpha))_{a\in I(i,i)} \\ &= \left(\sum_{\substack{c=ba\\ a\in I(i,i)\\ b\in I(i,i)}} \delta_{1_i,b}\Phi^{(i)}(\beta)X(b)(\delta_{1_i,a}\Phi^{(i)}(\alpha))\right)_{c\in I(i,i)} \\ &= (\delta_{1_i,c}\Phi^{(i)}(\beta\alpha))_{c\in I(i,i)},\end{aligned}$$

which shows that $\Phi(\sigma^{(i)}(\mu)) = (\delta_{1_i,c}\Phi^{(i)}(\mu))_{c\in I(i,i)}$ for each $\mu \in \mathbb{P}Q^{(i)}$, and that for each $\mu \in R^{(i)}$,

$$\Phi(\sigma^{(i)}(\mu)) = (\delta_{1_i,a}\Phi^{(i)}(\mu))_{a\in I(i,i)} = (\delta_{1_i,a}0)_{a\in I(i,i)} = 0.$$

Thus $\Phi(R'_1) = 0$.

For each $g_1 : i \to j, g_2 : j \to k \in Q_1$, ${}_ix \in Q'$,

$$\begin{aligned}
\Phi(\pi(g_2g_1, {}_ix)) &= \Phi(({g_2}, {}_j(g_1x)))\Phi((g_1, {}_ix)) \\
&= (\delta_{\overline{g_2},b}\mathbb{1}_{X(\overline{g_2})(g_1x)})_{b\in I(j,k)}(\delta_{\overline{g_1},a}\mathbb{1}_{X(\overline{g_1})(x)})_{a\in I(i,j)} \\
&= \left(\sum_{\substack{c=ba\\ a\in I(i,j)\\ b\in I(j,k)}} \delta_{\overline{g_2},b}\mathbb{1}_{X(\overline{g_2})(g_1x)}X(b)(\delta_{\overline{g_1},a}\mathbb{1}_{X(\overline{g_1})(x)})\right)_{c\in I(i,k)} \\
&= (\delta_{\overline{g_2g_1},c}\mathbb{1}_{X(\overline{g_2})(g_1x)}\mathbb{1}_{X(\overline{g_1})(x)})_{c\in I(i,k)} \\
&= (\delta_{\overline{g_2g_1},c}\mathbb{1}_{X(\overline{g_2g_1})(x)})_{c\in I(i,k)},
\end{aligned}$$

which shows that $\Phi(\pi(g, {}_ix)) = (\delta_{\overline{g},b}\mathbb{1}_{X(\overline{g})(x)})_{b\in I(i,j)}$ for each $g \in \mathbb{P}Q$. Therefore

$$\begin{aligned}
\Phi(\pi(g, {}_ix) - \pi(h, {}_ix)) &= \Phi(\pi(g, {}_ix)) - \Phi(\pi(h, {}_ix)) \\
&= (\delta_{\overline{g},b}\mathbb{1}_{X(\overline{g})(x)})_{b\in I(i,j)} - (\delta_{\overline{h},a}\mathbb{1}_{X(\overline{h})(x)})_{a\in I(i,j)} \\
&= 0
\end{aligned}$$

because $\overline{g} = \overline{h}$ for each $(g, h) \in R(i, j)$. Thus $\Phi(R'_2) = 0$.

For $a : i \to j \in Q_1$, $\alpha : x \to y \in Q_1^{(i)}$

$$\begin{aligned}
\Phi((a, {}_iy){}_i\alpha) &= \Phi((a, {}_iy))\Phi({}_i\alpha) \\
&= (\delta_{\overline{a},c}\mathbb{1}_{X(\overline{a})(y)})_{c\in I(i,j)}(\delta_{1_i,b}\Phi^{(i)}(\alpha))_{b\in I(i,i)} \\
&= \left(\sum_{\substack{d=cb\\ b\in I(i,i)\\ c\in I(i,j)}} \delta_{\overline{a},c}\mathbb{1}_{X(\overline{a})(y)}X(c)(\delta_{1_i,b}\Phi^{(i)}(\alpha))\right)_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}\mathbb{1}_{X(\overline{a})(y)}X(\overline{a})(\Phi^{(i)}(\alpha)))_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}X(\overline{a})(\Phi^{(i)}(\alpha)))_{d\in I(i,j)},
\end{aligned}$$

$$\begin{aligned}
\Phi({}_j(a\alpha)(a, {}_ix)) &= \Phi({}_j(a\alpha))\Phi((a, {}_ix)) \\
&= (\delta_{1_j,c}\Phi^{(j)}(a\alpha))_{c\in I(j,j)}(\delta_{\overline{a},b}\mathbb{1}_{X(\overline{a})(x)})_{b\in I(i,j)} \\
&= \left(\sum_{\substack{d=cb\\ b\in I(i,j)\\ c\in I(j,j)}} \delta_{1_j,c}\Phi^{(j)}(a\alpha)X(c)(\delta_{\overline{a},b}\mathbb{1}_{X(\overline{a})(x)})\right)_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}\Phi^{(j)}(a\alpha)X(1_j)(\mathbb{1}_{X(\overline{a})(x)}))_{d\in I(i,j)} \\
&= (\delta_{\overline{a},d}\Phi^{(j)}(a\alpha))_{d\in I(i,j)}.
\end{aligned}$$

Since $X(\overline{a})(\Phi^{(i)}(\alpha)) = \Phi^{(j)}(a\alpha)$ by the choice of $a\alpha$, we have

$$\Phi((a, {}_iy)_i\alpha) = \Phi({}_j(a\alpha)(a, {}_ix)).$$

Hence $\Phi(R'_3) = 0$, and finally $\Phi(R') = 0$.

By the claim above we see that Φ induces a functor $\overline{\Phi} : \Bbbk Q'/\langle R'\rangle \to \mathrm{Gr}(X)$. We prove that $\overline{\Phi}$ is an isomorphism. To this end, we first consider a basis of $(\Bbbk Q'/\langle R'\rangle)({}_ix, {}_jy)$ for each ${}_ix, {}_jy \in Q'_0$.

Claim 3. *For each $(g, h) \in R^{\#}(i, j)$ and $x \in Q^{(i)}$, $\overline{\pi(g, {}_ix)} = \overline{\pi(h, {}_ix)}$.*

Indeed, there exist some $(a, b) \in R(i', j')$, $c \in \mathbb{P}Q(i, i')$ and $d \in \mathbb{P}Q(j', j)$ such that

$$(g, h) = (dac, dbc).$$

Then

$$\begin{aligned}\pi(g, {}_ix) - \pi(h, {}_ix) &= \pi(dac, {}_ix) - \pi(dbc, {}_ix)\\ &= \pi(d, {}_{j'}(acx))\pi(a, {}_{i'}(cx))\pi(c, {}_ix) - \pi(d, {}_{j'}(bcx))\pi(b, {}_{i'}(cx))\pi(c, {}_ix)\\ &= \pi(d, {}_{j'}(acx))(\pi(a, {}_{i'}(cx)) - \pi(b, {}_{i'}(cx)))\pi(c, {}_ix).\end{aligned}$$

Therefore since $\pi(a, {}_{i'}(cx)) - \pi(b, {}_{i'}(cx)) \in R'$, we have $\pi(g, {}_ix) - \pi(h, {}_ix) \in R'$. Hence $\overline{\pi(g, {}_ix)} = \overline{\pi(h, {}_ix)}$.

For each $a : i \to j$ in I we define a functor $\tilde{X}(a)\colon \Bbbk Q^{(i)} \to \Bbbk Q^{(j)}$ as follows:

- For each $x \in Q^{(i)}_0$, $\tilde{X}(a)(x) := X(\overline{a})(x)$.
- For each arrow $\alpha : x \to y$ in $Q^{(i)}$, $\tilde{X}(a)(\alpha) := a\alpha$.
- For each path $\mu := \alpha_n \dots \alpha_1$ $(n \geq 2)$ in $Q^{(i)}$, $\tilde{X}(a)(\mu) := \tilde{X}(a)(\alpha_n) \dots \tilde{X}(a)(\alpha_1)$.

Claim 4. *For each ${}_ix, {}_jy \in Q'_0$ and $\mu \in \mathbb{P}Q'({}_ix, {}_jy)$, there exist some $a \in I(i, j)$ and $\nu \in \Bbbk Q^{(j)}({}_j(ax), {}_jy)$ such that $\overline{\mu} = \overline{\nu\pi(a, {}_ix)}$.*

Indeed, since $(b, {}_kv)_k\alpha - {}_l(b\alpha)(b, {}_ku) \in R'$ for each $b : k \to l$ in Q_1 and $\alpha : u \to v$ in $Q^{(k)}_1$, we have

$$\overline{(b, {}_kv)_k\alpha} = \overline{{}_l(b\alpha)(b, {}_ku)},$$

which implies

$$\overline{(b, {}_kv)\sigma^{(k)}(\lambda)} = \overline{\sigma^{(l)}\tilde{X}(b)(\lambda)(b, {}_ku)}$$

for each $\lambda \in \Bbbk Q^{(k)}({}_ku, {}_kv)$. By using this formula in the path μ we can move factors of the form $\overline{(b, {}_kv)}$ to the right, and finally we have

$$\overline{\mu} = \overline{\nu(a_t, x_t) \cdots (a_1, x_1)}$$

for some $0 \leq t \in \mathbb{Z}$, $\nu \in \Bbbk Q^{(j)}$, $x_1, \cdots, x_t \in Q'_0$, $a_1, \cdots, a_t \in Q_1$, where $(a_t, x_t) \cdots (a_1, x_1)$ is a path of length t in Q', and hence we have $\overline{(a_t, x_t) \cdots (a_1, x_1)} = \pi(a, x_1)$ $(a := a_t \cdots a_1)$. Hence we have $\nu \in \Bbbk Q^{(j)}({}_j(ax), {}_jy)$ and $\overline{\mu} = \overline{\nu\pi(a, {}_ix)}$.

Claim 5. *$\mathcal{M} := \{\overline{\alpha\pi(a, {}_ix)} | a \in I(i, j), \alpha \in \mathcal{M}_j(ax, y)\}$ is a basis of $(\Bbbk Q'/\langle R'\rangle)({}_ix, {}_jy)$, where $\mathcal{M}_j(ax, y)$ is a basis of $(\Bbbk Q^{(j)}/\langle R^{(j)}\rangle)(ax, y)$.*

Indeed, assume $\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}\overline{\alpha\pi(a,{}_ix)} = 0$. Then

$$
\begin{aligned}
\overline{\Phi}\left(\sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}\overline{\alpha\pi(a,{}_ix)}\right) &= \sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}\overline{\Phi}(\overline{\alpha})\overline{\Phi}(\overline{\pi(a,{}_ix)})\\
&= \sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\delta_{1_j,c}\Phi^{(j)}(\alpha))_{c\in I(j,j)}(\delta_{a,b}\mathbb{1}_{X(a)(x)})_{b\in I(i,j)}\\
&= \sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\sum_{\substack{d=cb\\ b\in I(i,j)\\ c\in I(j,j)}} \delta_{1_j,c}\Phi^{(j)}(\alpha)X(c)(\delta_{a,b}\mathbb{1}_{X(a)(x)}))_{d\in I(i,j)}\\
&= \sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\delta_{a,d}\Phi^{(j)}(\alpha)X(1_j)(\mathbb{1}_{X(a)(x)}))_{d\in I(i,j)}\\
&= \sum_{\substack{a\in I(i,j)\\ \alpha\in\mathbb{P}Q^{(j)}(ax,y)}} k_{a,\alpha}(\delta_{a,d}\Phi^{(j)}(\alpha))_{d\in I(i,j)}\\
&= \left(\Phi^{(j)}\left(\sum_{\alpha\in\mathbb{P}Q^{(j)}(ax,y)} k_{d,\alpha}\alpha\right)\right)_{d\in I(i,j)}\\
&= 0
\end{aligned}
$$

Since $\alpha\in\mathcal{M}_j(ax,y)$, we have $k_{d,\alpha}=0$. Therefore $\mathcal{M}$ is a basis of $(\Bbbk Q'/\langle R'\rangle)({}_ix,{}_jy)$.

Here we define $\sigma_a\colon X(j)(X(a)(x),y)\hookrightarrow \bigoplus_{a\in I(i,j)} X(j)(X(a)(x),y)$ by $\mu\mapsto(\delta_{b,a}\mu)_{b\in I(i,j)}$ for each $\mu\in X(j)(X(a)(x),y)$. Then a basis of $\mathrm{Gr}(X)((i,x),(j,y))$ is written by $\bigcup_{a\in I(i,j)}\sigma_a(\Phi^{(j)}(\mathcal{M}_j(ax,y)))$, and for each $\overline{\alpha\pi(a,{}_ix)}\in\mathcal{M}$ we have

$$
\begin{aligned}
\overline{\Phi}(\overline{\alpha\pi(a,{}_ix)}) &= (\delta_{a,d}\Phi^{(j)}(\alpha))_{d\in I(i,j)}\\
&= \sigma_a\Phi^{(j)}(\alpha).
\end{aligned}
$$

Hence $\overline{\Phi}$ induces an isomorphism $(\Bbbk Q'/\langle R'\rangle)({}_ix,{}_jy)\xrightarrow{\sim}\mathrm{Gr}(X)((i,x),(j,y))$.

Therefore $\overline{\Phi}$ is an isomorphism. □

Remark 3.2. The description of the proof of Claim 5 in the proof of Theorem 8.1 in [1] is not complete. This corresponds to Claim 4 above, and the formula (8.4) in [1] should be replaced by a linear combination

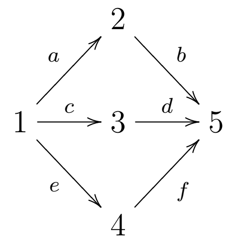
$$\overline{\eta}=\sum t_{y,\alpha_s,\dots}\overline{e_y\alpha_s\dots\alpha_1(g_t,x_t)\dots(g_1,x_1)}$$

with $t_{y,\alpha_s,\dots} \in \Bbbk$. Correspondingly, we must remove "$\overline{\eta} =$" in the last formula in Claim 5 there. The earlier version arXiv:0807.4706v6 of the paper records the correct proof.

4. Examples

In this section, we illustrate Theorems 2.4 and 3.1 by some examples.

Example 4.1. Let Q be the quiver

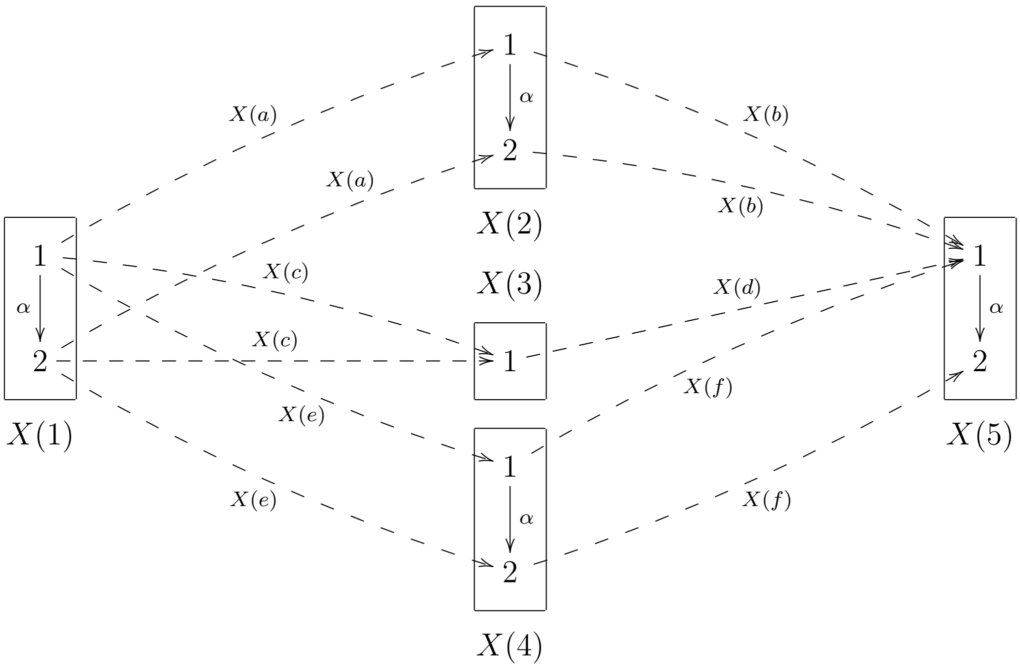


and let $R = \{(ba, dc)\}$. Then the category $I := \langle Q \mid R \rangle$ is not given as a semigroup, as a poset or as the free category of a quiver. For any algebra A consider the diagonal functor $\Delta(A)\colon I \to \Bbbk\text{-}\mathbf{Cat}$. Then by Theorem 2.4 the category $\mathrm{Gr}(\Delta(A))$ is given by

$$AQ/\langle ba - dc \rangle.$$

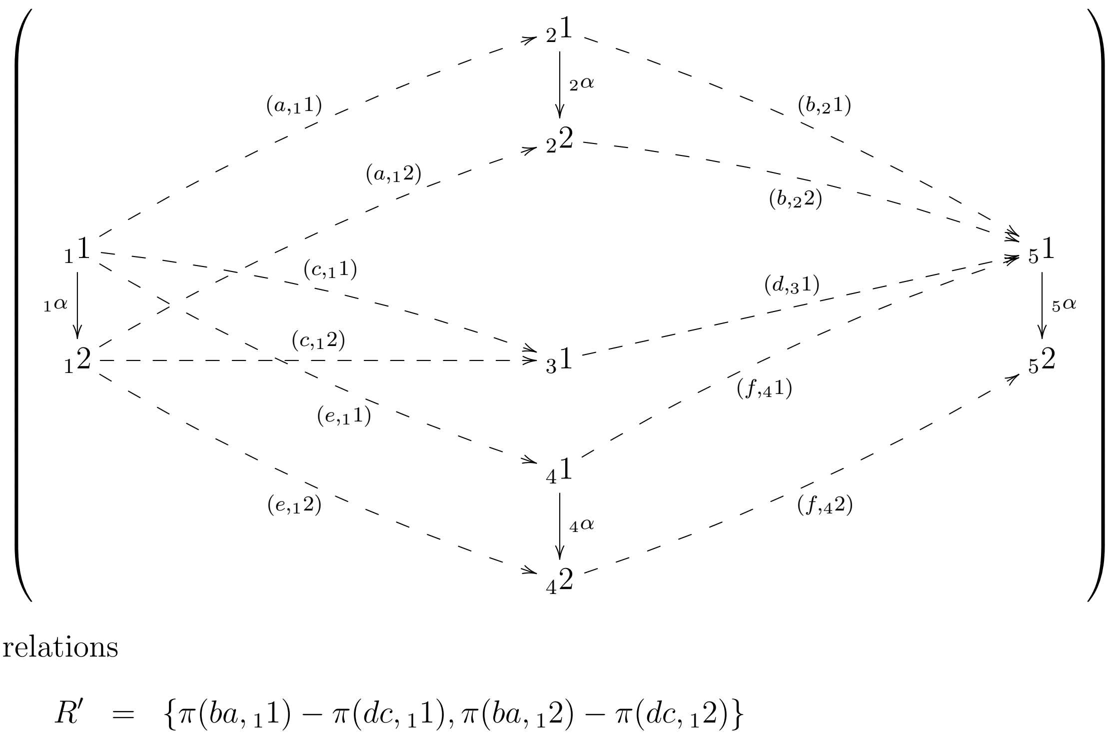
Remark 4.2. Let Q and Q' be quivers having neither double arrows nor loops, and let $f\colon Q_0 \to Q'_0$ be a map (a *vertex map* between Q and Q'). If $Q(x, y) \neq \emptyset$ $(x, y \in Q_0)$ implies $Q'(f(x), f(y)) \neq \emptyset$ or $f(x) = f(y)$, then f induces a $\Bbbk$-functor $\hat{f}\colon \Bbbk P \to \Bbbk P'$ defined by the following correspondence: For each $x \in Q_0$, $\hat{f}(e_x) := e_{f(x)}$, and for each arrow $a\colon x \to y$ in Q, $f(a)$ is the unique arrow $f(x) \to f(y)$ (resp. $e_{f(x)}$) if $f(x) \neq f(y)$ (resp. if $f(x) = f(y)$).

Example 4.3. Let $I = \langle Q \mid R \rangle$ be as in the previous example. Define a functor $X : I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the 𝕜-functors induced by the vertex maps expressed by broken arrows between them:



Then by Theorem 3.1 Gr(X) is presented by the quiver

$Q' =$



with relations

$$\begin{aligned} R' &= \{\pi(ba, {}_1 1) - \pi(dc, {}_1 1), \pi(ba, {}_1 2) - \pi(dc, {}_1 2)\} \\ &\quad \cup \{(a, {}_i y)_i \alpha - {}_j(a\alpha)(a, {}_i x) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\}, \end{aligned}$$

where the new arrows are presented by broken arrows.

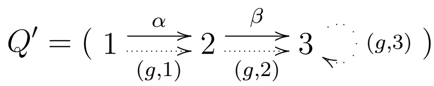
Example 4.4 (Semigroup case)**.** Define a category $I = \langle Q \mid R \rangle$ by setting

$$Q = \left(1 \circlearrowleft g \right), \quad R = \{(g^2, g^3)\}.$$

Then I can be regarded as a semigroup with the presentation $\langle g \mid g^2 = g^3 \rangle$. We define a functor $X : G \to \Bbbk\text{-}\mathbf{Cat}$ as follows. Let $Q^{(1)}$ be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3\ .$$

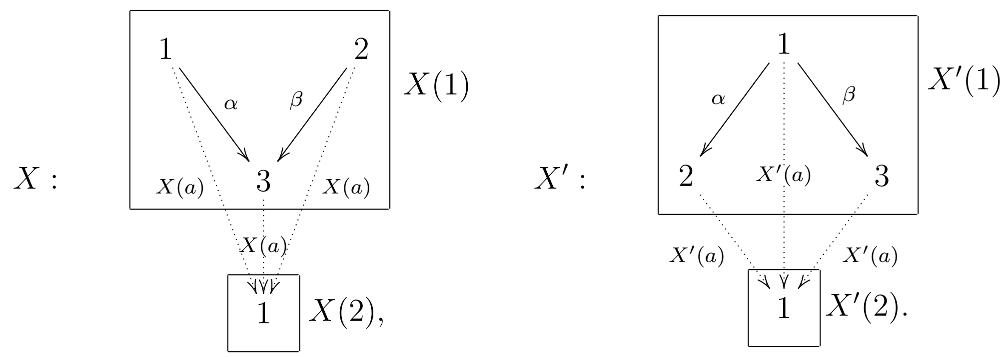
and set $X(1) := \Bbbk Q^{(1)}$, and define an endofunctor $X(g)$ of $X(1)$ as the $\Bbbk$-functor induced by the vertex map $X(g)(1) = 2, X(g)(2) = 3, X(g)(3) = 3$. Then by Theorem 3.1 $\mathrm{Gr}(X)$ is presented by the quiver



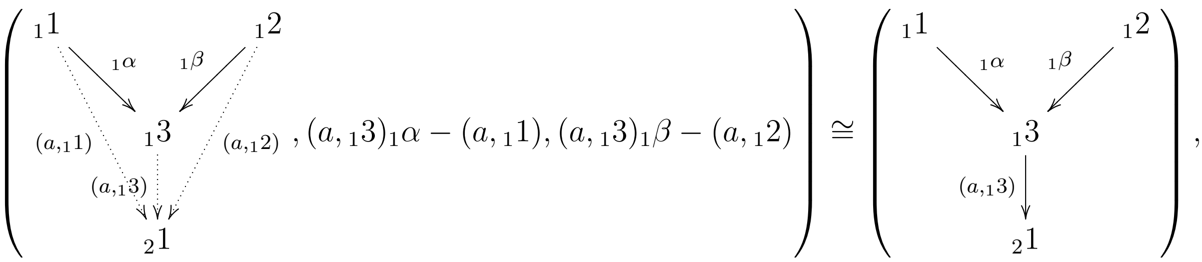
with relations

$$\begin{aligned} R' \;=\; \{&(g,3)(g,2)(g,1) - (g,2)(g,1), (g,3)(g,3)(g,2) - (g,3)(g,2), \\ &(g,3)(g,3)(g,3) - (g,3)(g,3), (g,2)\alpha - \beta(g,1), (g,3)\beta - (g,2)\}. \end{aligned}$$

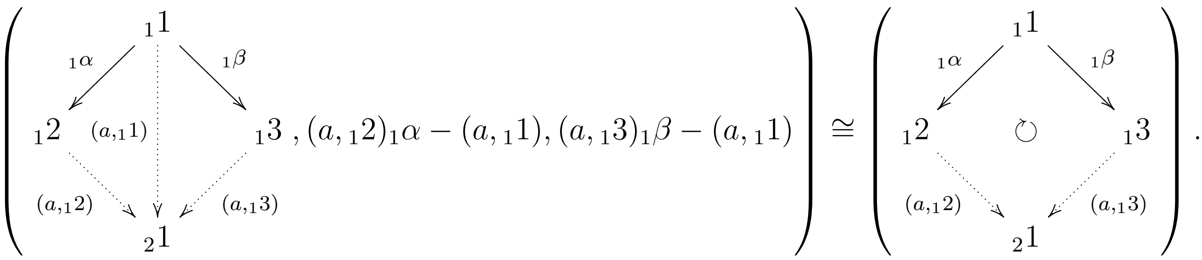
Example 4.5. Let $Q = (\ 1 \xrightarrow{a} 2\ )$ and $I := \langle Q \rangle$. Define functors $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by dotted arrows between them:



Then by Theorem 3.1 $\mathrm{Gr}(X)$ is given by the following quiver with no relations



and $\mathrm{Gr}(X')$ is given by the following quiver with a commutativity relation



By using the main theorem in [3] derived equivalences between $X(1)$ and $X'(1)$ and between $X(2)$ and $X'(2)$ are glued together to have a derived equivalence between $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.

References

[1] Asashiba, H.: *A generalization of Gabriel's Galois covering functors and derived equivalences*, J. Algebra **334** (2011), 109–149.

[2] ———: *Derived equivalences of actions of a category*, arXiv:1111.2239.

[3] ———: *Gluing derived equivalences together*, in preparation.

[4] Revêtements étales et groupe fondamental, Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. **224**.

[5] Howie, J. M.: *Fundamentals of Semigroup Theory*, London Mathematical Society Monographs New Series, **12**. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[6] Mac Lane, S.: *Categories for the Working Mathematician*, Graduate Texts in Mathematics, **5**.Springer-Verlag, New York,1998.

[7] Tamaki, D.: *The Grothendieck construction and gradings for enriched categories*, arXiv:0907.0061.

[8] Thomason, R. W.: *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc., **85**(1), 91–109, 1979.

[9] Welker, V., Ziegler, G. M. and Žîvaljević,R. T.: *Homotopy colimits—comparison lemmas for combinatorial applications*, J. Reine Angew. Math., **509**, 117–149, 1999.

Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka, 422-8529, Japan