Negative eigenvalues of two-dimensional Schrödinger operators

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Abstract

We prove an upper bound for the number Neg (H) of negative eigenvalues of the Schrödinger operator $H = -\Delta - V$ in $\mathbb{R}^2$, in terms of a weighted L^p-norm of the potential V, for any $1 < p < \infty$. This estimate scales correctly (linearly) in α under the transformation $V \mapsto \alpha V$ of the potential. In $\mathbb{R}^n$, $n \geq 3$, an upper estimate of Neg (H) with a correct scaling in α has been known since 1970s and is due to Cwickel-Lieb-Rosenblum.

Contents

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1 Introduction

Given a non-negative L^1_{loc} function $V(x)$ on $\mathbb{R}^2$, consider the Schrödinger type operator

$$H_V = -\Delta - V$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{1.1}$$

is the classical Laplace operator. More precisely, H_V is defined as a form sum of $-\Delta$ and $-V$, so that, under certain assumptions about V, the operator H_V is self-adjoint in $L^2(\mathbb{R}^2)$. If its spectrum in $(-\infty, 0]$ is discrete, then denote by $\operatorname{Neg}(H_V)$ the number of non-negative eigenvalues of H_V, counted with multiplicity.

Our main result is the following theorem.

Theorem 1.1 *Fix $p > 1$ and assume that $\mathcal{W}(r)$ is a positive monotone increasing function on $(0, +\infty)$ that satisfies the following Dini type condition both at 0 and at ∞:*

$$\int_0^\infty \frac{r \left|\log r\right|^{\frac{p}{p-1}} dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} < \infty. \tag{1.2}$$

Then, for any potential V in $\mathbb{R}^2$, such that

$$\int_{\mathbb{R}^2} V^p(x)\, \mathcal{W}(|x|)\, dx < \infty, \tag{1.3}$$

the operator H_V is well-defined as a self-adjoint operator in $L^2(\mathbb{R}^2)$, its spectrum in $(-\infty, 0]$ is discrete, and

$$\operatorname{Neg}(H_V) \leq 1 + C \left(\int_{\mathbb{R}^2} V^p(x)\, \mathcal{W}(|x|)\, dx \right)^{1/p}, \tag{1.4}$$

where the constant C depends on p and $\mathcal{W}$.

This result extends standardly to signed potentials V, using in the integrals (1.3) and (1.4) instead of function V its positive part V_+. Hence, we restrict ourself to the case $V \geq 0$.

Here is an example of a weight function $\mathcal{W}(r)$ that satisfies (1.2):

$$\mathcal{W}(r) = r^{2(p-1)} \langle \log r \rangle^{2p-1} \log^{p-1+\varepsilon} \langle \log r \rangle, \tag{1.5}$$

where $\varepsilon > 0$ and $\langle \cdot \rangle \equiv 2 + |\cdot|$. In particular, for $p = 2$, we obtain the following estimate:

$$\operatorname{Neg}(H_V) \leq 1 + C \left(\int_{\mathbb{R}^2} V^2(x) \left|x\right|^2 \langle \log |x| \rangle^3 \log^{1+\varepsilon} \langle \log |x| \rangle dx \right)^{1/2}. \tag{1.6}$$

More generally, if a monotone increasing function $\mathcal{W}(r)$ satisfies the relations

$$\mathcal{W}(r) \geq \begin{cases} r^{2(p-1)} (\log r)^{2p-1} (\log \log r)^{p-1+\varepsilon}, & r \gg 1, \\ r^{2(p-1)} \left(\log \frac{1}{r}\right)^{2p-1} \left(\log \log \frac{1}{r}\right)^{p-1+\varepsilon}, & r \ll 1, \end{cases}$$

where $\varepsilon > 0$, then $\mathcal{W}(r)$ satisfies (1.2).

Note that the value of ε in (1.5) (or (1.6)) cannot be taken to be negative (see Section 2.2). The estimate (1.4) is not valid with $p = 1$, no matter which weight function to choose. In fact, in the case $p = 1$ the following opposite inequality is known:

$$\operatorname{Neg}(H_V) \geq c \int_{\mathbb{R}^2} V(x)\, dx \tag{1.7}$$

(see [6]).

An important feature of our estimate (1.4), that distinguishes it from other known results in $\mathbb{R}^2$ is that the right hand side of (1.4) scales linearly in a scalar parameter $\alpha \to \infty$ under the transformation $V \mapsto \alpha V$, which agrees with the semi-classical asymptotics

$$\operatorname{Neg}(H_{\alpha V}) \simeq \alpha^{n/2} \text{ as } \alpha \to \infty,$$

known to be true in $\mathbb{R}^n$ for all $n \geq 1$ (cf. [1]). For the operator H_V in $\mathbb{R}^n$ with $n \geq 3$ a celebrated inequality of Cwikel-Lieb-Rozenblum says that

$$\operatorname{Neg}(H_V) \leq C_n \int_{\mathbb{R}^n} V(x)^{n/2}\, dx, \tag{1.8}$$

where the right hand side scales correctly as $\alpha^{n/2}$. This estimate was proved independently by the above named authors in 1972-1977 (see [3], [12], [16]), and the question of obtaining a similar estimate in $\mathbb{R}^2$ was open since that time (cf. [15]). We hope that our result fills this gap.

The additive term 1 in (1.4) reflects a special feature of $\mathbb{R}^2$: for any non-trivial potential V, there is at least 1 negative eigenvalue of H_V, no matter how small is the integral in (1.4). In $\mathbb{R}^n$ with $n \geq 3$, $\operatorname{Neg}(H_V)$ can be 0 provided the integral in (1.8) is small enough.

Let us recall that a crucial role in the proof of Lieb is played by the heat kernel $p_t(x, y)$ of the Laplace operator Δ. Namely, one uses the long time estimate $p_t(x, x) \leq \text{const}\, t^{-n/2}$ and the fact that $n/2 > 1$, which allows to define the function

$$s \mapsto \int_s^\infty p_t(x, x)\, dt, \tag{1.9}$$

that determines the shape of the upper bound (1.8) (see [7], [9], [10], [11], [12], [13]). In the case $n = 2$ the integral (1.9) diverges, which makes this method (and other known methods) not applicable.

To circumvent this difficulty, Molchanov and Vainberg [13] represented H_V in the form

$$H_V = (-\Delta + V_0) - (V + V_0)$$

and regarded H_V not as perturbation of $-\Delta$, but as that of $-\Delta + V_0$, where $V_0 \not\equiv 0$ is a fixed non-negative compactly supported function. One can show that the heat kernel $p_t(x, x)$ of the operator $-\Delta + V_0$ has the following long time asymptotics

$$p_t(x, x) \simeq \frac{\log^2(2 + |x|)}{t \log^2 t}$$

(cf. [5], [13], [14]), so that the integral (1.9) converges. Applying Lieb's general approach to this heat kernel, Molchanov and Vainberg [13] obtained the following estimate in $\mathbb{R}^2$:

$$\operatorname{Neg}(H_V) \leq 1 + C\int_{\mathbb{R}^2} V(x)\log^2(2+|x|)\,dx + C\int_{\mathbb{R}^2} V(x)\log\left(2+V(x)\left(1+|x|^2\right)\right)dx. \tag{1.10}$$

However, this estimate fails to scale linearly with respect to the transformation $V \mapsto \alpha V$, as one can see from the second integral in (1.10).

In the case when $V(x)$ is a radial function, that is, $V(x) = V(|x|)$, a sharp upper bound for $\operatorname{Neg}(H_V)$ in $\mathbb{R}^2$ was obtained in [2] and [8]:

$$\operatorname{Neg}(H_V) \leq 1 + \int_0^\infty V(r)\langle \log r\rangle r\,dr. \tag{1.11}$$

However, the method of (1.11) is essentially one dimensional and cannot handle non-radial potential.

Our method of the proof of Theorem 1.1 is significantly different from other existing methods of estimating $\operatorname{Neg}(H_V)$ and uses the advantages of $\mathbb{R}^2$ such as the presence of a large class of conformal mappings preserving the Dirichlet integral. Let us briefly describe the structure of the proof that is reflected in the structure of the paper.

In Section 2 we define the quantity $\operatorname{Neg}(V,\Omega)$ as the Morse index of the quadratic form $\int_\Omega \left(|\nabla u|^2 - Vu^2\right)dx$, and prove its various properties including subadditivity and the behavior under conformal and bilipschitz mappings.

The main result of Section 3 is Lemma 3.3 that provides the following estimate for a unit square Q:

$$\operatorname{Neg}(V,Q) \leq 1 + C\|V\|_{L^p(Q)}. \tag{1.12}$$

The proof involves a careful partitioning of Q into pieces $\Omega_1,...,\Omega_N$ with small enough $\|V\|_{L^p(\Omega_n)}$ so that $\operatorname{Neg}(V,\Omega_n) = 1$. The main difficulty is to control the number N of the elements of the partition, which yields then (1.12). While the number of those Ω_n where $\|V\|_{L^p(\Omega_n)}$ admits a certain lower bound can be controlled via $\|V\|_{L^p(Q)}$, the pieces Ω_n with very small values of $\|V\|_{L^p(\Omega_n)}$ are controlled inductively using special features of the partitioning.

The main result of Section 4 is Proposition 4.4 that claims that $\operatorname{Neg}\left(V,\mathbb{R}^2\right) = 1$ provided the following quantity is small enough:

$$\int_{\mathbb{R}^2} \log\langle x\rangle V(x)\,dx + \sup_{y\in\mathbb{R}^2}\int_{\mathbb{R}^2}\log_+\frac{1}{|x-y|}V(x)\,dx. \tag{1.13}$$

The proof uses the estimate (4.6) of the Green function of the operator $-\Delta + V_0$, that was obtained in [5]. Using the Hölder inequality, one obtains from Proposition 4.4 a particular case of Theorem 1.1 when the integral in (1.4) is small enough (Proposition 4.7). It is tempting to think that the quantity (1.13) could be used to bound $\operatorname{Neg}\left(V,\mathbb{R}^2\right)$ also in general, but this is not the case as it is shown in Section 2.2.

Section 5 contains the proof of Theorem 1.1 for a particular weight function

$$\mathcal{W}(r) = \begin{cases} r^q, & r \leq 1, \\ r^{q'}, & r > 1, \end{cases} \tag{1.14}$$

where $0 < q < 2(p-1)$ and $q' > 2(p-1)$ (Proposition 5.3). We use the estimate (1.12) in a sector $A_\alpha = \{z \in \mathbb{C} : |z| < 1, 0 < \arg z < \pi\alpha\}$ and a biholomorphic mapping $z \mapsto z^\alpha$

from a half-disk onto A_α, which results in a *weighted* L^p-estimate in the half-disk, where the weight function arrises from the Jacobian of the mapping. Using then the inversion $z \mapsto \frac{1}{z}$, one obtains an estimate of $\mathrm{Neg}\,(V,\Omega)$ in the exterior of the disk and then in the entire $\mathbb{R}^2$.

Although the method of the proof of Theorem 1.1 in full generality does not use this particular case and, moreover, is based on a different approach, the proof in the particular case of a power weight function (1.14) seems to be interesting on its own merit.

Section 6 contains the proof of Theorem 1.1 in full generality. We use the mappings $z \mapsto \ln \frac{1}{z}$ and $z \mapsto \ln z$ to map respectively a half-disk and a half of the exterior of a disk onto a strip $\widetilde{S} = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \ \ 0 < \mathrm{Im}\, z < \pi\}$ and then to pull back the estimate of $\mathrm{Neg}(V, \widetilde{S})$. Estimating of $\mathrm{Neg}(V, \widetilde{S})$ is itself a multistage procedure that requires a careful partitioning of a strip into rectangles and considering two types of potentials: sparse and dense, which is done in Lemma 6.6.

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2 Generalities of negative eigenvalues

2.1 Index of quadratic forms

By a *potential* in an open set $\Omega \subset \mathbb{R}^n$ we mean always a non-negative function from $L^1_{loc}(\Omega)$. Given a potential V in Ω, define the energy form for functions on Ω by

$$\mathcal{E}_{V,\Omega}(f) = \int_\Omega |\nabla f|^2 \, dx - \int_\Omega V f^2 dx \tag{2.1}$$

in the domain

$$\mathcal{F}_{V,\Omega} = \left\{ f \in L^2_{loc}(\Omega) : \int_\Omega |\nabla f|^2 \, dx < \infty, \ \int_\Omega V f^2 dx < \infty \right\}.$$

Clearly, $\mathcal{F}_{V,\Omega}$ is a linear space. Let us emphasize that $\mathcal{F}_{V,\Omega}$ is not a subspace of $L^2(\Omega)$, but that of $L^2_{loc}(\Omega)$.

Set

$$\mathrm{Neg}\,(V,\Omega) := \sup\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{F}_{V,\Omega} : \mathcal{E}_{V,\Omega}(f) \le 0 \text{ for all } f \in \mathcal{V}\}, \tag{2.2}$$

where $\mathcal{V} \prec \mathcal{F}_{V,\Omega}$ means that $\mathcal{V}$ is a linear subspace of $\mathcal{F}_{V,\Omega}$, and the supremum of $\dim \mathcal{V}$ is taken over all subspaces $\mathcal{V}$ such that $\mathcal{E}_{V,\Omega}(\cdot) \le 0$ on $\mathcal{V}$. In other words, $\mathrm{Neg}\,(V,\Omega)$ is the Morse index of the quadratic form $\mathcal{E}_{V,\Omega}$ in $\mathcal{F}_{V,\Omega}$.

If $\Omega = \mathbb{R}^n$ then we use the abbreviations

$$\mathcal{E}_V \equiv \mathcal{E}_{V,\mathbb{R}^n}, \quad \mathcal{F}_V \equiv \mathcal{F}_{V,\mathbb{R}^n}, \quad \mathrm{Neg}\,(V) \equiv \mathrm{Neg}\,(V, \mathbb{R}^n).$$

The operator

$$H_V = -\Delta - V$$

is defined as a self-adjoint operator in $L^2(\mathbb{R}^n)$ using the following standard procedure. First observe that the quadratic form

$$\mathcal{E}(u) = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx$$

is closed in the Sobolev space $W^{1,2}$, and the quadratic form

$$u \mapsto \int_{\mathbb{R}^n} V u^2 dx$$

associated with the multiplication operator $u \mapsto Vu$, is closed in the domain $L^2(dx) \cap L^2(Vdx)$. Clearly, the form $\mathcal{E}_V$ is well-defined in the domain

$$\mathcal{D}_V = W^{1,2} \cap L^2(Vdx),$$

that is a subspace of $\mathcal{F}_V$. Under certain assumptions about V (in particular, if V satisfies (1.3)), the form $(\mathcal{E}_V, \mathcal{D}_V)$ is closed in L^2 (and, in fact, $\mathcal{D}_V = W^{1,2}$). Consequently, its generator, denoted by H_V, is a self-adjoint, semi-bounded below operator in L^2, whose domain is a subspace of $\mathcal{D}_V$.

For any self-adjoint operator A, denote by $\operatorname{Neg}(A)$ the number of non-positive eigenvalues of A counted with multiplicities. If the spectrum of A in $(-\infty, 0]$ has a continuous component then set $\operatorname{Neg}(A) := \infty$.

Lemma 2.1 *If the form $(\mathcal{E}_V, \mathcal{D}_V)$ is closed and, hence, H_V is well-defined, then*

$$\operatorname{Neg}(H_V) \leq \operatorname{Neg}(V). \tag{2.3}$$

Proof. It is well-known that

$$\operatorname{Neg}(H_V) = \sup\{\dim \mathcal{V} : \mathcal{V} \prec \mathcal{D}_V \text{ and } \mathcal{E}_V(f) \leq 0\ \forall f \in \mathcal{V}\}$$

(cf. [6, Lemma 2.7]). Since $\mathcal{D}_V \subset \mathcal{F}_V$, (2.3) holds by monotonicity argument. ■

In the proof of Theorem 1.1 (cf. Section 6.3), we obtain the upper bound for $\operatorname{Neg}(V)$, which implies then by Lemma 2.1 the same bound for $\operatorname{Neg}(H_V)$. If this method were engaged in $\mathbb{R}^n$ with $n \geq 3$ then the resulting estimate would not have been satisfactory, because $\operatorname{Neg}(H_V)$ can be 0 (as follows, for example, from (1.8)), whereas $\operatorname{Neg}(V) \geq 1$ for all potentials $V \in L^1(\mathbb{R}^2)$. Indeed, the latter follows from $1 \in \mathcal{F}_V$ and $\mathcal{E}_V(1) \leq 0$. However, our aim is $\mathbb{R}^2$, where $\operatorname{Neg}(H_V) \geq 1$ for non-zero V, so that we do not loose 1 in the estimate.

In the rest of this section we prove some general properties of $\operatorname{Neg}(V, \Omega)$ that will be used in the next sections. For bounded domains Ω, the form $\mathcal{E}_{V,\Omega}$ could be associated with the operator $\Delta + V$ in Ω with the Neumann boundary condition on $\partial\Omega$. This understanding helps the intuition, but technically we never need this operator itself. Nor the closability of the form $\mathcal{E}_{V,\Omega}$ is needed, except for Lemma 2.1.

Lemma 2.2 *Let Ω be any open subset of $\mathbb{R}^n$, and K be a closed subset of $\mathbb{R}^n$ of measure 0. Set $\Omega' = \Omega \setminus K$. Then we have*

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(V, \Omega'). \tag{2.4}$$

Moreover, if $n \geq 2$ and K is a singe point set, then

$$\operatorname{Neg}(V, \Omega) = \operatorname{Neg}(V, \Omega'). \tag{2.5}$$

Proof. Every function $u \in \mathcal{F}_{V,\Omega}$ can be considered as an element of $\mathcal{F}_{V,\Omega'}$ simply by restriction of u to Ω'. Since the difference $\Omega \setminus \Omega'$ has measure 0, we obtain $\mathcal{E}_{V,\Omega}(u) = \mathcal{E}_{V,\Omega'}(u)$. Then Lemma 2.5 implies (2.4).

Let K be a single point set. If u and ∇u belong to $L^2_{loc}(\Omega')$ then they both belong also to $L^2_{loc}(\Omega)$ because K is a polar set. It follows that the spaces $\mathcal{F}_{V,\Omega}$ and $\mathcal{F}_{V,\Omega'}$ coincide, whence the claim follows. ■

Definition 2.3 We say that a sequence $\{\Omega_k\}$ of open sets $\Omega_k \subset \mathbb{R}^n$ is a *partition* of an open set $\Omega \subset \mathbb{R}^n$ if all the sets Ω_k are disjoint, $\Omega_k \subset \Omega$, and $\overline{\Omega} \setminus \bigcup_k \Omega_k$ has measure 0.

Lemma 2.4 *If $\{\Omega_k\}$ is a partition of Ω, then*

$$\operatorname{Neg}(V, \Omega) \leq \sum_k \operatorname{Neg}(V, \Omega_k). \tag{2.6}$$

Proof. Set $\widetilde{\Omega} = \bigcup_k \Omega_k$ and $K = \overline{\Omega} \backslash \widetilde{\Omega}$. Since K is closed, has measure 0 and $\widetilde{\Omega} = \Omega \backslash K$, we obtain by Lemma 2.2 that

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(V, \widetilde{\Omega}).$$

Next, we claim that

$$\operatorname{Neg}(V, \widetilde{\Omega}) \leq \sum_k \operatorname{Neg}(V, \Omega_k). \tag{2.7}$$

Denoting $\mathcal{F}_k = \mathcal{F}_{V,\Omega_k}$ and $\mathcal{F} = \mathcal{F}_{V,\widetilde{\Omega}}$ we see that $\mathcal{F} = \bigoplus \mathcal{F}_k$ and, for all $f_k \in \mathcal{F}_k$ and $f = \sum f_k$ we have

$$\mathcal{E}_{V,\widetilde{\Omega}}(f) = \sum_k \mathcal{E}_{V,\Omega_k}(f_k).$$

Let $\mathcal{V}$ be a finite dimensional subspace of $\mathcal{F}$ where $\mathcal{E}_{V,\widetilde{\Omega}} \leq 0$. Denote by $\mathcal{V}_k$ the projection of $\mathcal{V}$ onto $\mathcal{F}_k$, and set $\mathcal{U} = \bigoplus \mathcal{V}_k$, so that $\mathcal{V} \subset \mathcal{U}$. The quadratic form $\mathcal{E}_{V,\Omega_k}$ is diagonalizable on the finite dimensional space $\mathcal{V}_k$, and the number N_k of the non-positive terms in the signature of $\mathcal{E}_{V,\Omega_k}$ on $\mathcal{V}_k$ is clearly bounded by $\operatorname{Neg}(V, \Omega_k)$. Hence, the number N of the non-positive terms in the signature of $\mathcal{E}_{V,\widetilde{\Omega}}$ on $\mathcal{U}$ is bounded as follows:

$$N = \sum_k N_k \leq \sum_k \operatorname{Neg}(V, \Omega_k).$$

If $\dim \mathcal{V} > N$ then $\mathcal{V}$ intersects the subspace of $\mathcal{U}$ where $\mathcal{E}_{V,\widetilde{\Omega}}$ is positive definite, which contradicts the definition of $\mathcal{V}$. Therefore, $\dim \mathcal{V} \leq N$, whence (2.7) follows. ■

It is easy to see that in (2.7) also the opposite inequality is true, but we do not use this.

Lemma 2.5 *Let $\Omega, \widetilde{\Omega}$ be open subsets of $\mathbb{R}^n$ and V and $\widetilde{V}$ be potentials in Ω and $\widetilde{\Omega}$, respectively. Let $\mathcal{L} : \mathcal{F}_{V,\Omega} \to \mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$ be a linear injective mapping.*

(*a*) *If $\mathcal{E}_{V,\Omega}(u) \leq 0$ implies $\mathcal{E}_{\widetilde{V},\widetilde{\Omega}}(\widetilde{u}) \leq 0$ for $\widetilde{u} = \mathcal{L}(u)$ then*

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}(\widetilde{V}, \widetilde{\Omega}). \tag{2.8}$$

(*b*) *Assume that there are positive constants c_1, c_2, such that, for any $u \in \mathcal{F}_{V,\Omega}$, the function $\widetilde{u} = \mathcal{L}(u)$ satisfies*

$$\int_{\widetilde{\Omega}} |\nabla \widetilde{u}|^2 \, dx \leq c_1 \int_{\Omega} |\nabla u|^2 \, dx \tag{2.9}$$

and

$$\int_{\widetilde{\Omega}} \widetilde{V} \widetilde{u}^2 dx \geq c_2 \int_{\Omega} V u^2 dx. \tag{2.10}$$

Then

$$\operatorname{Neg}(V, \Omega) \leq \operatorname{Neg}\left(\frac{c_1}{c_2} \widetilde{V}, \widetilde{\Omega}\right). \tag{2.11}$$

Proof. Let $\mathcal{V}$ be a finitely dimensional linear subspace of $\mathcal{F}_{V,\Omega}$ where $\mathcal{E}_{V,\Omega} \leq 0$. Then $\widetilde{\mathcal{V}} := \mathcal{L}(\mathcal{V})$ is a linear subspace of $\mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$ of the same dimension.

(a) For any $\widetilde{u} \in \widetilde{\mathcal{V}}$ we have $\mathcal{E}_{\widetilde{V},\widetilde{\Omega}}(\widetilde{u}) \leq 0$, which implies $\dim \widetilde{\mathcal{V}} \leq \mathrm{Neg}(\widetilde{V}, \widetilde{\Omega})$. Since $\dim \mathcal{V} = \dim \widetilde{\mathcal{V}}$, we have also $\dim \mathcal{V} \leq \mathrm{Neg}(\widetilde{V}, \widetilde{\Omega})$, whence (2.8) follows.

(b) Observe that, for any $\widetilde{u} \in \widetilde{\mathcal{V}}$, there is $u \in \mathcal{V}$ such that (2.9) and (2.10) are satisfied. It follows that

$$\begin{aligned}
\mathcal{E}_{\frac{c_1}{c_2}\widetilde{V},\widetilde{\Omega}}(\widetilde{u}) &= \int_{\widetilde{\Omega}} |\nabla \widetilde{u}|^2 \, dx - \frac{c_1}{c_2} \int_{\widetilde{\Omega}} \widetilde{V} \widetilde{u}^2 dx \\
&\leq c_1 \int_{\Omega} |\nabla u|^2 \, dx - c_1 \int V u^2 dx = c_1 \mathcal{E}_{V,\Omega}(u) \leq 0.
\end{aligned}$$

Applying part (a) with $\frac{c_1}{c_2}\widetilde{V}$ instead of $\widetilde{V}$, we obtain (2.11). ■

2.2 Examples

We give here two examples to show that the weight function $W(x)$ in (1.5) is optimal in the following sense: $p > 1$ cannot be replaced by $p = 1$ and $\varepsilon > 0$ cannot be replaced by $\varepsilon < 0$, without violating the estimate (1.4).

Example 2.6 Fix some $R > 10$ and consider the following potential on $\mathbb{R}^2$

$$V(x) = \begin{cases} \frac{1}{2|x|^2 \log^2 |x|}, & \text{if } 9 < |x| < R, \\ 0, & \text{otherwise.} \end{cases}$$

The following function

$$f(x) = \sqrt{\log |x|} \sin\left(\frac{1}{2} \log\log |x|\right)$$

satisfies in the region $\{9 < |x| < R\}$ the differential equation

$$\Delta f + V(x) f = 0. \tag{2.12}$$

Indeed, in the polar coordinates $r = |x|$ this equation becomes

$$f''(r) + \frac{1}{r} f'(r) + \frac{1}{2r^2 \log^2 r} f(r) = 0,$$

and that f satisfies it can be verified by a direct computation. Another proof of the latter, using the Liouville transformation of this equation, can be found in [2].

The function $f(x)$ is sign changing. More precisely, for any positive integer k, function f has constant sign in the ring

$$A_k := \left\{x \in \mathbb{R}^2 : 2\pi k < \log\log |x| < 2\pi (k+1)\right\},$$

and vanishes on ∂A_k, assuming that $A_k \subset \{9 < |x| < R\}$. Integrating (2.12) over A_k yields

$$\int_{A_k} |\nabla f|^2 \, dx = \int_{A_k} V f^2 dx.$$

Hence, if N is the number of the rings A_k inside $\{9 < |x| < R\}$ then, using the space $\mathcal{V}$ spanned by the functions $f\mathbf{1}_{A_k}$, we obtain $\mathrm{Neg}(V) \geq N$. Clearly, we have $N \geq c \log\log R$, whence it follows that

$$\mathrm{Neg}(V) \geq c \log\log R. \tag{2.13}$$

Consider now a monotone increasing function $\mathcal{W}(r)$ such that

$$\mathcal{W}(r) = r^{2(p-1)} (\log r)^{2p-1} (\log\log r)^{p-1+\varepsilon} \tag{2.14}$$

for $r > 9$. For this function we have

$$\begin{aligned} \int_{\mathbb{R}^2} V^p(x)\,\mathcal{W}(|x|)\,dx &= 2\pi \int_9^R \frac{r^{2(p-1)} (\log r)^{2p-1} (\log\log r)^{p-1+\varepsilon}}{r^{2p}\log^{2p} r} r dr \\ &= 2\pi \int_9^R \frac{(\log\log r)^{p-1+\varepsilon}\,dr}{r\log r} \\ &\le C(\log\log R)^{p+\varepsilon} \end{aligned}$$

so that

$$\left(\int_{\mathbb{R}^2} V^p(x)\,\mathcal{W}(|x|)\,dx\right)^{1/p} \le C(\log\log R)^{1+\varepsilon/p}.$$

Comparing with (2.13), we see that, for $\varepsilon < 0$, the inequality (1.4) of Theorem 1.1 breaks down for large R.

The question whether (1.4) remains true for $\varepsilon = 0$ remains still open. We conjecture that there exists a counterexample – perhaps, some elaboration of the above one.

Example 2.7 Let us show that, for $p = 1$, no estimate of the type

$$\mathrm{Neg}(V) \le \mathrm{const} + \int_{\mathbb{R}^2} VW\,dx$$

can be true, provided a weight function W is bounded in a neighborhood of at least one point. Indeed, assume without loss of generality that $W(x) \le C$ for $|x| < \varepsilon$. We will construct a potential V supported in $\{|x| < \varepsilon\}$ such that $\int_{\mathbb{R}^2} V dx < \infty$ while $\mathrm{Neg}(V) = \infty$.

It will be easier to construct V as a measure but then it can be routinely approximated by a L^1_{loc}-function. For any $r > 0$, let S_r be the circle $\{|x| = r\}$. We will use the measure δ_{S_r} supported on S_r. Given two sequences $\{a_n\}$ and $\{b_n\}$ of reals such that $0 < a_n < b_n$, consider the (generalized) functions

$$V_n = \frac{1}{a_n \ln\frac{b_n}{a_n}} \delta_{S_{a_n}}$$

and test functions

$$\varphi_n(x) = \begin{cases} 1, & |x| < a_n, \\ \frac{\ln\frac{b_n}{|x|}}{\ln\frac{b_n}{a_n}}, & a_n \le |x| \le b_n, \\ 0, & |x| > b_n. \end{cases} \tag{2.15}$$

An easy computation shows that

$$\int_{\mathbb{R}^2} |\nabla\varphi_n|^2\,dx = \frac{2\pi}{\ln\frac{b_n}{a_n}} \tag{2.16}$$

and

$$\int_{\mathbb{R}^2} \varphi_n^2 V_n dx = \int_{\mathbb{R}^2} V_n dx = \frac{2\pi}{\ln\frac{b_n}{a_n}},$$

whence it follows that $\mathcal{E}_{Vn}(\varphi_n) = 0$.

Let us now specify $a_n = 4^{-n^3}$ and $b_n = 2^{-n^3}$. Consider also the following sequence of points in $\mathbb{R}^2$: $y_n = (4^{-n}, 0)$. Then all disks $D_{b_n}(y_n)$ with large enough n are disjoint and

$$\sum_{n=1}^{\infty} \frac{2\pi}{\ln \frac{b_n}{a_n}} < \infty. \tag{2.17}$$

Consider the generalized function

$$V = \sum_{n=N}^{\infty} V(\cdot - y_n). \tag{2.18}$$

The functions $\psi_n = \varphi_n(\cdot - y_n)$ have disjoint supports and satisfy $\mathcal{E}_V(\psi_n) = 0$ for all $n \ge N$, whence it follows that $\operatorname{Neg}(V) = \infty$. On the other hand, by (2.17) we have

$$\int_{\mathbb{R}^2} V dx < \infty.$$

By taking N large enough, one can make $\int_{\mathbb{R}^2} V dx$ arbitrarily small and $\operatorname{supp} V$ to be located in an arbitrarily small neighborhood of the origin, while still having $\operatorname{Neg}(V) = \infty$.

One can also show that for this V

$$\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|} V(x)\, dx < \text{const} \sum_{n=1}^{\infty} \frac{\ln \frac{4}{|y_n|}}{\ln \frac{b_n}{a_n}} \simeq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

(cf. Remark 4.5 below).

2.3 Transformation of potentials and weights

Given a 2×2 matrix $A = (a_{ij})$, denote by $\|A\|$ the norm of A as an linear operator in $\mathbb{R}^2$ with the Euclidean norm. Denote also

$$\|A\|_2 := \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2}.$$

It is easy to see that

$$\frac{1}{\sqrt{2}} \|A\|_2 \le \|A\| \le \|A\|_2$$

Assuming further that A is non-singular, define the quantities

$$M(A) := \frac{\|A\|^2}{\det A} \quad \text{and} \quad M_2(A) := \frac{\|A\|_2^2}{\det A}$$

For example, if A is a conformal matrix, that is, $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ or $\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$, then

$$\det A = \alpha^2 + \beta^2 = \|A\|^2,$$

whence $M(A) = 1$.

For a general non-singular matrix A, the following identity holds:

$$M_2(A) = M_2\left(A^{-1}\right). \tag{2.19}$$

Indeed, denoting $a = \det A$, we obtain

$$A^{-1} = \frac{1}{a}\begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix},$$

whence $\left\|A^{-1}\right\|_2^2 = \frac{1}{a^2}\left\|A\right\|_2^2$, which implies (2.19). Consequently, we obtain that, for any non-singular matrix A,

$$\frac{1}{2}M(A) \le M\left(A^{-1}\right) \le 2M(A). \tag{2.20}$$

Let Ω and $\widetilde{\Omega}$ be two open subsets of $\mathbb{R}^2$ and $\Phi : \widetilde{\Omega} \to \Omega$ be a C^1-diffeomorphism. Denote by Φ' its Jacobi matrix and by J_Φ - its Jacobian, that it $J_\Phi = \det \Phi'$. Set

$$M_\Phi := \sup_{x\in\widetilde{\Omega}} M\left(\Phi'(x)\right) = \sup_{x\in\widetilde{\Omega}} \frac{\left\|\Phi'(x)\right\|^2}{\left|J_\Phi(x)\right|}.$$

We will use two types of mappings Φ : bilipschitz and conformal. If Φ is conformal then we have $M_\Phi = 1$. Moreover, if Φ is holomorphic then

$$J_\Phi(z) = \left|\Phi'(z)\right|^2, \tag{2.21}$$

where now $\Phi' = \frac{d\Phi}{dz}$ is a complex derivative in $z \in \mathbb{C}$.

If Φ is bilipschitz and with bilipschitz constant L then an easy calculation shows that $\left\|\Phi'(x)\right\|^2 \le 4L^2$ and that both $|J_\Phi|$ and $|J_{\Phi^{-1}}|$ are bounded by $2L^2$ whence $M_\Phi \le 8L^4$.

By (2.20), we always have

$$\frac{1}{2}M_\Phi \le M_{\Phi^{-1}} \le 2M_\Phi \tag{2.22}$$

Given a domain $\Omega \subset \mathbb{R}^2$, one can ask for which weight function W, the index $\operatorname{Neg}(V,\Omega)$ can be estimated via $\int_\Omega V^p W dx$. The next lemma establishes the behavior of such estimates under transformations of Ω.

Lemma 2.8 *Assume that, for any potential $\widetilde{V}$ in a domain $\widetilde{\Omega} \subset \mathbb{R}^2$,*

$$\operatorname{Neg}(\widetilde{V},\widetilde{\Omega}) \le F\left(\int_{\widetilde{\Omega}} \widetilde{V}(y)^p\, \widetilde{W}(y)\, dy\right), \tag{2.23}$$

where $F : [0,+\infty) \to [0,+\infty]$ is a monotone increasing function and $\widetilde{W}$ is some weight on $\widetilde{\Omega}$. Let

$$\Psi : \Omega \to \widetilde{\Omega}$$

be a C^1 diffeomorphism with a finite M_Ψ. Then, for any potential V in Ω,

$$\operatorname{Neg}(V,\Omega) \le F\left(\int_\Omega V(x)^p\, W(x)\, dx\right), \tag{2.24}$$

where

$$W(x) = M_{\Psi^{-1}}^p \widetilde{W}(\Psi(x))\left|J_\Psi(x)\right|^{1-p}. \tag{2.25}$$

We say that the weight $W(x)$ defined by (2.25) is a pullback of $\widetilde{W}$ under the mapping Ψ. The pullback weight exists whenever M_Ψ is finite. For holomorphic Ψ the pullback weight (2.25) can be rewritten in the form

$$W(z) = \frac{\widetilde{W}(\Psi(z))}{|\Psi'(z)|^{2(p-1)}}. \tag{2.26}$$

Proof. Set $\Phi = \Psi^{-1}$ and, for a given potential V on Ω, defined a pullback potential $\widetilde{V}$ on $\widetilde{\Omega}$ by

$$\widetilde{V}(y) = M_\Phi V(\Phi(y)) |J_\Phi(y)|. \tag{2.27}$$

Let us prove that

$$\mathrm{Neg}(V,\Omega) \le \mathrm{Neg}(\widetilde{V},\widetilde{\Omega}). \tag{2.28}$$

Let $\mathcal{V}$ be a subspace of $\mathcal{F}_{V,\Omega}$ as in (2.2). Define $\widetilde{\mathcal{V}}$ as the pullback of $\mathcal{V}$ under the mapping Φ, that is, any function $\widetilde{f} \in \widetilde{\mathcal{V}}$ has the form

$$\widetilde{f}(y) = f(\Phi(y))$$

for some $f \in \mathcal{V}$. Let us show that $\widetilde{f} \in \mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$. Indeed, using the change $y = \Psi(x)$ (or $x = \Phi(y)$), we obtain

$$\begin{aligned}
\int_{\widetilde{\Omega}} \left|\widetilde{f}(y)\right|^2 \widetilde{V}(y)\, dy &= \int_{\Omega} \left|\widetilde{f}(y)\right|^2 \widetilde{V}(y) |J_\Psi(x)|\, dx \\
&= \int_{\Omega} |f(x)|^2 M_\Phi V(x) |J_\Phi(y)| |J_\Psi(x)|\, dx \\
&= M_\Phi \int_\Omega |f(x)|^2 V(x)\, dx.
\end{aligned} \tag{2.29}$$

Similarly, we have

$$\begin{aligned}
\int_{\widetilde{\Omega}} \left|\nabla\widetilde{f}(y)\right|^2 dy &= \int_{\widetilde{\Omega}} \left|(\nabla f)(\Phi(y)) \cdot \Phi'(y)\right|^2 dy \\
&\le \int_{\widetilde{\Omega}} \left\|\Phi'(y)\right\|^2 |\nabla f|^2 (\Phi(y))\, dy \\
&\le M_\Phi \int_{\widetilde{\Omega}} |J_\Phi(y)| |\nabla f|^2 (\Phi(y))\, dy \\
&= M_\Phi \int_\Omega |\nabla f(x)|^2\, dx.
\end{aligned} \tag{2.30}$$

It follows that $\widetilde{f} \in \mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$. The mapping $f \mapsto \widetilde{f}$ is obviously a linear injective mapping from $\mathcal{F}_{V,\Omega}$ to $\mathcal{F}_{\widetilde{V},\widetilde{\Omega}}$. By Lemma 2.5, (2.29) and (2.30) imply (2.28).

By hypothesis, we have (2.23). Using the change $y = \Psi(x)$ in the integral in (2.23), we obtain

$$\begin{aligned}
\int_{\widetilde{\Omega}} \widetilde{V}(y)^p \widetilde{W}(y)\, dy &= \int_\Omega \widetilde{V}(\Psi(x))^p \widetilde{W}(\Psi(x)) |J_\Psi(x)|\, dx \\
& \int_\Omega (M_\Phi V(x) |J_\Phi(y)|)^p \frac{W(x)}{M_\Phi^p |J_\Psi(x)|^{1-p}} |J_\Psi(x)|\, dx \\
&= \int_\Omega V(x)^p W(x)\, dx.
\end{aligned} \tag{2.31}$$

Combining (2.23), (2.28), and (2.31), we obtain (2.24). ■

Remark 2.9 It follows from the proof that if Ψ is conformal then

$$\operatorname{Neg}(V, \Omega) = \operatorname{Neg}(\widetilde{V}, \widetilde{\Omega}), \tag{2.32}$$

where

$$\widetilde{V}(y) = V(\Phi(y)) |J_\Phi(y)|.$$

Furthermore, if Ψ is holomorphic then

$$\widetilde{V}(z) = V(\Phi(z)) \left|\Phi'(z)\right|^2.$$

For comparison let us rewrite (2.26) in this case in the form

$$\widetilde{W}(z) = \frac{W(\Phi(z))}{|\Phi'(z)|^{2(p-1)}}.$$

3 L^p-estimate in bounded domains

Let Ω be a domain in $\mathbb{R}^n$ and V be a potential in Ω such that $V \in L^1(\Omega)$. Then $1 \in \mathcal{F}_{V,\Omega}$ and $\mathcal{E}_{V,\Omega}(1) \le 0$, whence it follows that $\operatorname{Neg}(V, \Omega) \ge 1$.

In this section we obtain upper bound for $\operatorname{Neg}(V, \Omega)$ for certain bounded domains in $\mathbb{R}^2$.

3.1 One negative eigenvalue in a disc

Denote by D the unit open disk in $\mathbb{R}^2$, that is,

$$D = \left\{x \in \mathbb{R}^2 : |x| < 1\right\},$$

and by D_r the disk of radius r, that is,

$$D_r = \left\{x \in \mathbb{R}^2 : |x| < r\right\}.$$

Lemma 3.1 *For any $p > 1$ there is $\varepsilon > 0$ such that, for any potential V in D,*

$$\|V\|_{L^p(D)} \le \varepsilon \Rightarrow \operatorname{Neg}(V, D) = 1.$$

Proof. As was mentioned above, we have always $\operatorname{Neg}(V, D) \ge 1$. Hence, we need to prove that $\operatorname{Neg}(V, D) \le 1$.

Given a function $u \in \mathcal{F}_{V,D}$, extend u to the entire $\mathbb{R}^2$ using the inversion $\Phi(x) = \frac{x}{|x|^2}$: for any $|x| > 1$, set

$$u(x) := u(\Phi(x)).$$

Choose a cutoff function φ such that $\varphi|_{D_2} \equiv 1$, $\varphi|_{\mathbb{R}^2 \setminus D_3} = 0$ and $\varphi = \varphi(|x|)$ is linear in $|x|$ in $D_3 \setminus D_2$. Define a function u^* by

$$u^* = u\varphi.$$

Clearly, we have $u^* \in W^{1,2}(\mathbb{R}^2)$ and u^* is supported in D_4. Let us prove some integral estimates for the function u^*.

Claim 1.. *We have*

$$\int_{D_4} |\nabla u^*|^2 \, dx \le 4 \int_D |\nabla u|^2 \, dx + 162 \int_D u^2 dx. \tag{3.1}$$

Indeed, since $\nabla u^* = \varphi \nabla u + u \nabla \varphi$, we have

$$\begin{aligned} \int_{D_4} |\nabla u^*|^2 \, dx &\leq 2 \int_{D_4} \varphi^2 |\nabla u|^2 \, dx + 2 \int_{D_4} u^2 |\nabla \varphi|^2 \, dx \\ &\leq 2 \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + 2 \int_{D_3 \setminus D_2} u^2 dx, \end{aligned}$$

where we have used that $|\nabla \varphi| = 1$ in $D_3 \setminus D_2$ and $\nabla \varphi = 0$ otherwise. Next, use the change $y = \Phi(x)$ to map $D_3 \setminus D_2$ to $D_{1/2} \setminus D_{1/3}$. Since $|J_{\Phi^{-1}}(y)| = \frac{1}{|y|^4}$, we obtain

$$\int_{D_3 \setminus D_2} u^2(x) \, dx = \int_{D_{1/2} \setminus D_{1/3}} u^2(y) \frac{1}{|y|^4} dy \leq 3^4 \int_D u^2 dy.$$

Using also the conformal invariance of the Dirichlet integral

$$\int_{\mathbb{R}^2 \setminus D} |\nabla u|^2 \, dx = \int_D |\nabla u|^2 \, dx,$$

and combining the above estimates, we obtain (3.1).

Given a potential V in D, extend V to entire $\mathbb{R}^2$ by setting $V(x) = 0$ for all $|x| > 1$.

Claim 2.. *If $u \perp 1$ in $L^2(D)$ and $\mathcal{E}_{V,D}(u) \leq 0$ then*

$$\int_{D_4} |\nabla u^*|^2 \, dx \leq C \int_D V u^2 dx, \tag{3.2}$$

with some absolute constant C.

Indeed, the assumption $u \perp 1$ implies by the Poincaré inequality

$$\int_D u^2 dx \leq c \int_D |\nabla u|^2 \, dx,$$

which together with (3.1) yields

$$\int_{D_4} |\nabla u^*|^2 \, dx \leq (4 + 162c) \int_D |\nabla u|^2 \, dx.$$

Combining this with the hypothesis $\mathcal{E}_V(u) \leq 0$, that is,

$$\int_D |\nabla u|^2 \, dx \leq \int_D V u^2 dx, \tag{3.3}$$

we obtain (3.2).

Applying the Hölder inequality to the right hand side of (3.2), we obtain

$$\int_D V u^2 dx \leq \left(\int_D V^p dx \right)^{1/p} \left(\int_D |u|^{\frac{2p}{p-1}} \, dx \right)^{1-1/p} \leq \left(\int_D V^p dx \right)^{1/p} \left(\int_{D_4} |u^*|^{\frac{2p}{p-1}} \, dx \right)^{1-1/p}. \tag{3.4}$$

Next, let us use Sobolev inequality for functions $f \in W^{1,1}(\mathbb{R}^2)$ supported in D_4:

$$\left(\int_{D_4} |f|^\alpha \, dx \right)^{1/\alpha} \leq C \int_{D_4} |\nabla f| \, dx$$

where $\alpha \in (1,2]$ is arbitrary and $C = C(\alpha)$. Replacing f by f^β (where $\beta > 1$), we obtain

$$\left(\int_{D_4} |f|^{\alpha\beta}\, dx\right)^{1/\alpha} \le C \int_{D_4} |\nabla f|\, |f|^{\beta-1}\, dx \le C \left(\int_{D_4} |\nabla f|^2\, dx\right)^{1/2} \left(\int_{D_4} |f|^{2(\beta-1)}\, dx\right)^{1/2}.$$

Choosing β to satisfy the identity $\alpha\beta = 2(\beta - 1)$, that is, $\beta = \frac{2}{2-\alpha}$ (for that assume $\alpha < 2$), we obtain

$$\left(\int_{D_4} |f|^{\frac{2\alpha}{2-\alpha}}\, dx\right)^{\frac{2-\alpha}{\alpha}} \le C \int_{D_4} |\nabla f|^2\, dx. \tag{3.5}$$

Choosing $\alpha = \frac{2p}{2p-1}$, we obtain $\frac{2\alpha}{2-\alpha} = \frac{2p}{p-1}$ and $\frac{2-\alpha}{\alpha} = 1 - \frac{1}{p}$. Applying (3.5) with this α for $f = u^*$ we obtain

$$\left(\int_{D_4} |u^*|^{\frac{2p}{p-1}}\, dx\right)^{1-1/p} \le C \int_{D_4} |\nabla u^*|^2\, dx,$$

which together with (3.2), (3.4) yields

$$\int_{D_4} |\nabla u^*|^2\, dx \le C \left(\int_D V^p dx\right)^{1/p} \int_{D_4} |\nabla u^*|^2\, dx. \tag{3.6}$$

Assuming that

$$\|V\|_{L^p(D)} \le \varepsilon := \frac{1}{2C}, \tag{3.7}$$

we see that (3.6) is only possible if $u^* = \text{const}$. Since $u \perp 1$ in $L^2(D)$, it follows that $u \equiv 0$.

Hence, $\mathcal{E}_{V,D}(u) \le 0$ and $u \perp 1$ imply $u \equiv 0$, whence $\mathrm{Neg}(V,D) \le 1$ follows. ■

Corollary 3.2 *Let Ω be a bounded domain in $\mathbb{R}^2$ and $\Phi : D \to \Omega$ be a C^1-diffeomorphism with finite M_Φ and $\sup |J_\Phi|$. Then there is $\varepsilon_\Omega > 0$ such that*

$$\|V\|_{L^p(\Omega)} \le \varepsilon_\Omega \Rightarrow \mathrm{Neg}(V,\Omega) = 1,$$

where ε_Ω depends on p, M_Φ and $\sup |J_\Phi|$.

Consequently, if Ω is bilipschitz equivalent to D_r then

$$\|V\|_{L^p(\Omega)} \le c r^{2/p-2} \Rightarrow \mathrm{Neg}(V,\Omega) = 1 \tag{3.8}$$

where $c > 0$ depends on p and on the bilipschitz constant of mapping between D_r and Ω.

We say will that an open set $\Omega \subset \mathbb{R}^2$ has the *size* r if it is bilipschitz equivalent to D_r with some fixed bound on the bilipschitz constant.

Proof. By the proof of Lemma 2.8, we have

$$\mathrm{Neg}(V,\Omega) \le \mathrm{Neg}(\widetilde{V}, D),$$

where $\widetilde{V}$ is given by (2.27). By Lemma 3.1,

$$\left\|\widetilde{V}\right\|_{L^p(D)} \le \varepsilon \Rightarrow \mathrm{Neg}(\widetilde{V}, D) = 1.$$

Using the notation of Lemma 2.8, set $\Psi = \Phi^{-1}$, $\widetilde{W} \equiv 1$ and define a function $W(x)$ on Ω by (2.25), that is,

$$W(x) = M_\Phi^p\, |J_\Psi(x)|^{1-p} \le M_\Phi^p \sup |J_\Phi|^{p-1}.$$

Then by (2.31) we have

$$\int_D \widetilde{V}(y)^p\,dy = \int_\Omega V(x)^p\,W(x)\,dx \le M_\Phi^p \sup|J_\Phi|^{p-1}\int_\Omega V(x)^p\,dx,$$

whence

$$\left\|\widetilde{V}\right\|_{L^p(D)} \le M_\Phi \sup|J_\Phi|^{\frac{p-1}{p}}\,\|V\|_{L^p(\Omega)}\,.$$

Therefore, if

$$\|V\|_{L^p(\Omega)} \le \varepsilon_\Omega := \frac{\varepsilon}{M_\Phi \sup|J_\Phi|^{\frac{p-1}{p}}}, \tag{3.9}$$

then $\left\|\widetilde{V}\right\|_{L^p(D)} \le \varepsilon$, which implies by the above argument $\operatorname{Neg}(V,\Omega) = 1$.

Let $\Omega = D_r$. Then, for the mapping $\Phi(x) = rx$. we have $M_\Phi = 1$ and $|J_\Phi| = r^2$ whence we obtain

$$\varepsilon_{D_r} = \varepsilon r^{2/p-2}. \tag{3.10}$$

More generally, let Ω be of size r, that is, there is a bilipschitz mapping $\Phi: D_r \to \Omega$ with a bilipschitz constant L. Arguing as in the first part of the proof but using D_r instead of D, we obtain similarly to (3.9) that ε_Ω can be determined by

$$\varepsilon_\Omega = \frac{\varepsilon_{D_r}}{M_\Phi \sup|J_\Phi|^{\frac{p-1}{p}}} \ge c r^{2/p-2},$$

where $c > 0$ depends on p and L, which was to be proved. ■

3.2 Negative eigenvalues in a square

Denote by Q the unit square in $\mathbb{R}^2$, that is,

$$Q = \left\{(x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1\right\}.$$

Lemma 3.3 (1st Main Lemma) *For any $p > 1$ and for any potential V in Q,*

$$\operatorname{Neg}(V,Q) \le 1 + C\,\|V\|_{L^p(Q)}\,, \tag{3.11}$$

where C depends only on p.

Proof. It suffices to construct a partition $\mathcal{P}$ of Q into a family of N disjoint subsets such that

1. $\|V\|_{L^p(\Omega)} \le \varepsilon_\Omega$ for any $\Omega \in \mathcal{P}$;
2. $N \le 1 + C\,\|V\|_{L^p(Q)}$.

Indeed, if such a partition exists then we obtain by Lemma 2.4 and Corollary 3.2

$$\operatorname{Neg}(V,Q) \le \sum_{\Omega\in\mathcal{P}} \operatorname{Neg}(V,\Omega) = N, \tag{3.12}$$

and (3.11) follows from the above bound of N.

The elements of a partition will be of two shapes: it is either a square of the side length $0 < l \le 1$ or a *step*, that is, a set of the form $\Omega = A \setminus B$ where A is a square of the side

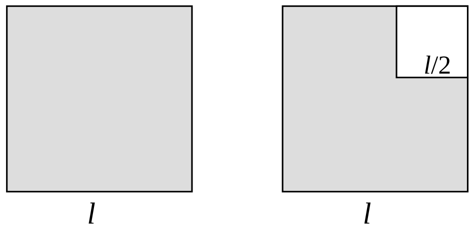

Figure 1: A square and a step of size l

length l, and B is a square of the side length $\leq l/2$ that is attached to one of corners of A (see Fig. 1).

In the both cases we refer to l as the size of Ω. By Corollary 3.2, the condition 1 for such a set Ω will follow from

$$\int_\Omega V^p dx \leq c l^{2-2p}, \tag{3.13}$$

with some constant $c > 0$.

Apart from the shape, we will distinguish also the *type* of a set $\Omega \in \mathcal{P}$ of size l as follows: we say that

- Ω is of a large type, if
$$\int_\Omega V^p dx > c l^{2-2p},$$
- Ω is of a medium type if
$$c' l^{2-2p} < \int_\Omega V^p dx \leq c l^{2-2p}, \tag{3.14}$$
- and Ω is of small type if
$$\int_\Omega V^p dx \leq c' l^{2-2p}. \tag{3.15}$$

Here c is the constant from (3.13) and $c' \in (0, c)$ is to be chosen below.

The construction of the partition $\mathcal{P}$ will be done by induction. At each step $i \geq 1$ of induction we will have a partition $\mathcal{P}^{(i)}$ of Q such that

1. each $\Omega \in \mathcal{P}^{(i)}$ is either a square or a step;
2. If $\Omega \in \mathcal{P}^{(i)}$ is a step then Ω is of a medium type.

At step 1 we have just one set: $\mathcal{P}^{(1)} = \{Q\}$. At any step $i \geq 1$, partition $\mathcal{P}^{(i+1)}$ is obtained from $\mathcal{P}^{(i)}$ as follows. If $\Omega \in \mathcal{P}^{(i)}$ is of small or medium type then Ω becomes one of the elements of the partition $\mathcal{P}^{(i+1)}$. If $\Omega \in \mathcal{P}^{(i)}$ is of large type, then it is a square, and it will be further partitioned into a few smaller sets that will become elements of $\mathcal{P}^{(i+1)}$. Denoting by l the side length of the square Ω, let us first split Ω into four equal squares $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ of side length $l/2$ and consider the following cases (see Fig. 2).

Case 1. If among $\Omega_1, ..., \Omega_4$ the number of small type squares is not equal to 3, then all sets $\Omega_1, ..., \Omega_4$ become elements of $\mathcal{P}^{(i+1)}$ (as we will see below, in this case the number of small type squares among $\Omega_1, ..., \Omega_4$ is at most 2).

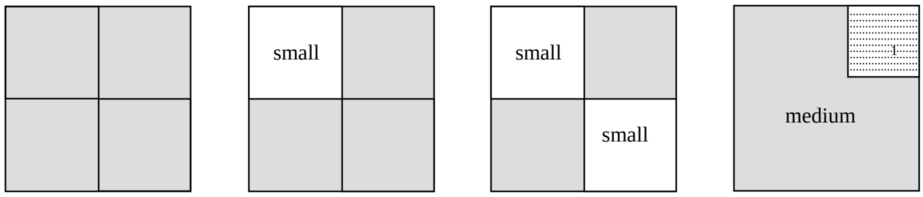

Figure 2: Various possibilities of partitioning of a square Ω (the shaded shapes are of medium or large type, the hatched shape Ω_1 can be of any type)

Case 2. If among $\Omega_1, ..., \Omega_4$ there are exactly 3 small type squares, say, $\Omega_2, \Omega_3, \Omega_4$, then we have

$$\int_{\Omega\setminus\Omega_1} V^p dx = \int_{\Omega_2\cup\Omega_3\cup\Omega_4} V^p dx \leq 3c' \left(\frac{l}{2}\right)^{2-2p}$$

whereas

$$\int_{\Omega} V^p dx > cl^{2-2p}.$$

Choose c' so that $3c'2^{2p-2} < c$. Then by reducing the size of Ω_1 (but keeping Ω_1 attached to the corner of Ω) one can achieve the equality

$$\int_{\Omega\setminus\Omega_1} V^p dx = cl^{2-2p}.$$

Hence, we obtain a partition of Ω into two sets Ω_1 and $\Omega \setminus \overline{\Omega_1}$, where the step $\Omega \setminus \overline{\Omega_1}$ is of medium type, while the square Ω_1 can be of any type. Both Ω_1 and $\Omega \setminus \overline{\Omega_1}$ become elements of $\mathcal{P}^{(i+1)}$.

As we see from construction, at each step i only large type squares get partitioned further, and the size of the large type squares in $\mathcal{P}^{(i+1)}$ reduces at least by a factor 2. If the size of a square is small enough then it is necessarily of small type, because the right hand side of (3.15) goes to ∞ as $l \to 0$. Hence, the process will stop after finitely many steps. After sufficiently many steps we obtain a partition $\mathcal{P}$ where all elements are either of small or medium types (see Fig. 3).

Let N be a number of elements of $\mathcal{P}$. We need to show that

$$N \leq 1 + C \|V\|_{L^p(Q)} . \tag{3.16}$$

Let us come back to the case 1 in the induction step and show that the case when all $\Omega_1, ..., \Omega_4$ are small cannot occur. Indeed, in this case we would have

$$\int_{\Omega} V^p dx = \sum_{k=1}^{4} \int_{\Omega_k} V^p dx \leq 4c' \left(\frac{l}{2}\right)^{2-2p} .$$

Let us choose c' so small that $4c'2^{2p-2} < c$. Then the above estimate contradicts the assumption that Ω is of large type.

Combining with the case 2, we see that at each step of partitioning of a large square Ω, we obtain

1. either 4 squares where at most 2 squares are small;

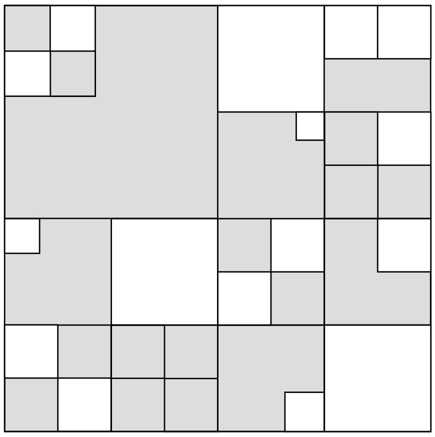

Figure 3: An example of a final partition $\mathcal{P}$. The shaded shapes are of medium type, the white squares are of small type.

2. or 1 square and 1 step, where the latter is of medium type (in particular, at most 1 small set is obtained).

At each step of construction, denote by L the number of large elements, by M the number of medium elements, and by S the number of small elements, and let us show that the quantity $2L+3M-S$ is non-decreasing during the construction. Indeed, at each step we split one large square Ω, so that by removing this square, L decreases by 1. However, we add new elements of partitions, which contribute to the quantity $2L+3M-S$ as follows.

1. If Ω is split into $s \le 2$ small and $4-s$ medium/large squares then the value of $2L+3M-S$ has the increment at least
$$-2+2(4-s)-s=6-3s\ge 0.$$
2. If Ω is split into 1 square and 1 step, then one obtains at least 1 medium set and at most 1 small, so that $2L+3M-S$ has the increment at least
$$-2+3-1=0.$$

(Note that we have avoided situations when Ω is split into ≥ 3 small and ≤ 1 large squares. In that case L and M would not have increased, whereas S would have increased at least by 3, so that no quantity of the type C_1L+C_2M-S would have been monotone increasing).

Since for the partition $\mathcal{P}^{(1)}$ we have $2L+3M-S\ge -1$, this inequality will remain true at all steps of construction and, in particular, it is satisfied for the final partition $\mathcal{P}$. For the final partition we have $L=0$, whence it follows that $S\le 1+3M$ and, hence,
$$N=S+M\le 1+4M. \tag{3.17}$$

Let us estimate M. Let $\Omega_1,...,\Omega_M$ be the medium type elements of $\mathcal{P}$ and let l_k be the size of Ω_k. Then each Ω_k satisfies (3.14). If all Ω_k were squares, we would have
$$\sum_{k=1}^{M} l_k^2 \le 1.$$

However, some Ω_k may be steps. In all cases, there is a square $\Omega_k' \subset \Omega_k$ of the size $l_k/2$, and all squares $\{\Omega_k'\}_{k=1}^M$ are disjoint, which implies that

$$\sum_{k=1}^{M} l_k^2 \leq 4. \tag{3.18}$$

Using the Hölder inequality, (3.18) and (3.14), we obtain

$$\begin{aligned} M &= \sum_{k=1}^{M} l_k^{-\frac{2}{p'}} l_k^{\frac{2}{p'}} \\ &\leq \left(\sum_{k=1}^{M} l_k^{-\frac{2p}{p'}}\right)^{1/p} \left(\sum_{k=1}^{M} l_k^2\right)^{1/p'} \\ &\leq 4 \left(\sum_{k=1}^{M} l_k^{2-2p}\right)^{1/p} \\ &\leq C \left(\sum_{k=1}^{M} \int_{\Omega_k} V^p dx\right)^{1/p} \\ &\leq C \|V\|_{L^p(Q)}. \end{aligned}$$

Combining this with (3.17), we obtain (3.16), thus finishing the proof. ∎

Corollary 3.4 *If Ω is bilipschitz equivalent to Q, then*

$$\operatorname{Neg}(V, \Omega) \leq 1 + C \|V\|_{L^p(\Omega)}, \tag{3.19}$$

where the constant C depends on p and on the bilipschitz constant.

Proof. Indeed, Ω is bilipschitz equivalent to Q, so that (3.19) follows from (3.11) and Lemma 2.8. ∎

Corollary 3.5 *Let R be a rectangle in $\mathbb{R}^2$ with the side lengths a, b where $a \geq b$. Then*

$$\operatorname{Neg}(V, R) \leq 1 + C \left(a^{2p-1} b^{-1}\right)^{1/p} \|V\|_{L^p(R)}.$$

Proof. Without loss of generality, we can assume that

$$R = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < a,\ 0 < x_2 < b\right\}.$$

Consider a mapping

$$\Psi(x_1, x_2) = \left(\frac{x_1}{a}, \frac{x_2}{b}\right)$$

that maps R onto the unit square Q. We have

$$\Psi' = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix},$$

whence $\|\Psi'\| = \frac{1}{b}$, $J_\Psi = \frac{1}{ab}$, and $M_\Psi = \frac{a}{b}$. Using the estimate (3.11) in Q, we obtain by Lemma 2.8 with $\widetilde{W} = 1$ that

$$\begin{aligned} \operatorname{Neg}(V, R) &\leq 1 + C M_\Psi |J_\Psi|^{1/p-1} \|V\|_{L^p(R)} \\ &= 1 + C \left(a^{2p-1} b^{-1}\right)^{1/p} \|V\|_{L^p(R)}, \end{aligned}$$

which was to be proved. ∎

4 One negative eigenvalue in $\mathbb{R}^2$

4.1 Weighted L^1-estimate

Let us show that, for any potential V in $\mathbb{R}^2$, $\mathrm{Neg}\,(V) \geq 1$. If $V \in L^1$ then it follows from $1 \in \mathcal{F}_V$ and $\mathcal{E}_V\,(1) \leq 0$. For a general non-zero potential we argue as follows. There is a sequence of non-negative Lipschitz functions φ_n on $\mathbb{R}^2$ with compact supports such that $\varphi_n \uparrow 1$ as $n \to \infty$ and

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2\, dx \to 0. \tag{4.1}$$

For example, one can take φ_n as in (2.15) with $a_n = n$ and $b_n = n^2$, that is,

$$\varphi_n\,(x) = \min\left(1, \frac{1}{\ln n} \ln_+ \frac{n^2}{|x|}\right). \tag{4.2}$$

Clearly, $\varphi_n \in \mathcal{F}_V$. By (2.16) we have

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2\, dx = \frac{2\pi}{\ln n} \to 0 \text{ as } n \to \infty.$$

If n is large enough then $\int_{\mathbb{R}^2} V \varphi_n^2 dx$ is positive and it is increasing with n. It follows from (4.1) that, for large enough n,

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2\, dx < \int_{\mathbb{R}^2} V \varphi_n^2 dx, \tag{4.3}$$

whence $\mathrm{Neg}\,(V) \geq 1$. If the operator H_V is well-defined then (4.3) implies that also $\mathrm{Neg}\,(H_V) \geq 1$ since φ_n belongs to the domain $\mathcal{D}_V$ of the quadratic form of H_V.

On the other hand, the following is true.

Lemma 4.1 *There exists non-negative non-zero function $V_0 \in C_0^\infty\left(\mathbb{R}^2\right)$ such that*

$$\mathrm{Neg}\,(V_0) = 1.$$

Proof. Choose V_0 to be supported in the unit disk D and such that $\|V_0\|_{L^p(D)}$ is small enough as in Lemma 3.1. By that Lemma, we have $\mathrm{Neg}\,(V_0, D) = 1$. Let us show that also $\mathrm{Neg}\,(V_0) = 1$. Indeed, if $\mathrm{Neg}\,(V_0) \geq 2$ then there is a non-zero function $u \in \mathcal{F}_{V_0}$ such that $\mathcal{E}_{V_0}\,(u) \leq 0$ and $u \perp 1$ in $L^2\,(D)$. Then also $\mathcal{E}_{V_0,D}\,(u) \leq 0$, whence by the proof of Lemma 3.1, we have $u \equiv 0$ in D. Then

$$\mathcal{E}_{V_0}\,(u) = \int_{\mathbb{R}^2 \setminus D} |\nabla u|^2\, dx \leq 0$$

implies that $\nabla u \equiv 0$ in $\mathbb{R}^2 \setminus D$ and, hence, $u \equiv \text{const}$ in $\mathbb{R}^2 \setminus D$. Since $\nabla u \in L^2_{loc}$, it follows that $u \equiv 0$ in $\mathbb{R}^2$. This contradiction proves that $\mathrm{Neg}\,(V_0) = 1$. ■

From now on let us fix a potential V_0 as in Lemma 4.1, and consider the quadratic form $\mathcal{E}_0$

$$\mathcal{E}_0\,(u) := \int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} V_0 u^2 dx = \mathcal{E}_{-V_0}\,(u)\,,$$

defined on the space

$$\mathcal{F}_0 := \left\{u \in L^2_{loc}\left(\mathbb{R}^2\right) : \int_{\mathbb{R}^2} |\nabla u|^2\, dx < \infty\right\} = \mathcal{F}_{V_0}.$$

Note that $\int_{\mathbb{R}^2} V_0 u^2 dx$ is finite automatically for any $u \in \mathcal{F}_0$ because $V_0 \in L^\infty$. It follows that $\mathcal{F}_V \subset \mathcal{F}_0$ for any potential V.

Lemma 4.2 *If, for all* $u \in \mathcal{F}_V$,

$$\mathcal{E}_0(u) \geq 2 \int_{\mathbb{R}^2} V u^2 dx, \tag{4.4}$$

then $\operatorname{Neg}(V) = 1$.

Proof. If $\mathcal{E}_V(u) \leq 0$ that is, if

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \int_{\mathbb{R}^2} V u^2 dx,$$

then, combining this with (4.4), we obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx + \int_{\mathbb{R}^2} V_0 u^2 dx \geq 2 \int_{\mathbb{R}^2} |\nabla u|^2 dx$$

whence

$$\int_{\mathbb{R}^2} |\nabla u|^2 dx \leq \int_{\mathbb{R}^2} V_0 u^2 dx,$$

that is, $\mathcal{E}_{V_0}(u) \leq 0$. By Lemma 2.5 this implies $\operatorname{Neg}(V) \leq \operatorname{Neg}(V_0)$, whence the claim follows. ■

Lemma 4.2 provides the following method of proving that $\operatorname{Neg}(V) = 1$: it suffices to prove the inequality (4.4) for all $u \in \mathcal{F}_V$. For the latter, we will use the Green function of the operator

$$H_0 = -\Delta + V_0.$$

It was shown in [5, Example 10.14] that the operator H_0 has a symmetric positive Green function $g(x,y)$ and that the Green function satisfies the following estimate

$$g(x,y) \simeq \log\langle y\rangle + \frac{\log\langle y\rangle}{\log\langle x\rangle} \log_+ \frac{1}{|x-y|} \quad \text{if} \quad |y| \leq |x|, \tag{4.5}$$

and a symmetric estimate if $|y| \geq |x|$, where we use the notation

$$\langle x\rangle = 2 + |x|.$$

It follows that, for all $x, y \in \mathbb{R}^2$,

$$g(x,y) \leq C\left(\log\langle y\rangle + \log_+ \frac{1}{|x-y|}\right). \tag{4.6}$$

For comparison, let us recall that the operator $-\Delta$ in $\mathbb{R}^2$ has no positive Green function, so that adding a small perturbation V_0 changes this property.

Consider a measure ν on $\mathbb{R}^2$ given by

$$d\nu = V dx,$$

and the integral operator G on functions from $L^2(\mathbb{R}^2, \nu)$ that acts by the rule

$$Gf(x) = \int_{\mathbb{R}^2} g(x,y) f(y) d\nu(y).$$

Lemma 4.3 *Assume that* $V > 0$ *and, moreover, that* $\frac{1}{V} \in L^1_{loc}\left(\mathbb{R}^2\right)$. *Assume also, that* G *is a bounded operator in* $L^2(\nu)$, *and denote by* $\|G\|$ *its norm. Then the following inequality holds for all* $u \in \mathcal{F}_V$:

$$\mathcal{E}_0(u) \geq \frac{1}{\|G\|} \int_{\mathbb{R}^2} V u^2 dx. \tag{4.7}$$

Proof. Consider first the case $u \in C_0^\infty\left(\mathbb{R}^2\right)$. Set $f = \frac{1}{V} H_0 u$ so that $H_0 u = fV$. Then function u can be recovered using the Green function as follows:

$$u(x) = \int_{\mathbb{R}^2} g(x,y) f(y) V(y) dy = Gf(x).$$

Observe that $f \in L^2(\nu)$ because

$$\int_{\mathbb{R}^2} f^2 d\nu = \int_{\mathbb{R}^2} \frac{(H_0 u)^2}{V} dx \leq \sup |H_0 u|^2 \int_{\operatorname{supp} u} \frac{1}{V} dx < \infty.$$

It follows that

$$\mathcal{E}_0(u) = (H_0 u, u)_{L^2(dx)} = (fV, Gf)_{L^2(dx)} = (f, Gf)_{L^2(\nu)} \tag{4.8}$$

and

$$\int_{\mathbb{R}^2} V u^2 dx = (u,u)_{L^2(\nu)} = (Gf, Gf)_{L^2(\nu)}.$$

Inequality (4.7) will follows if we prove that, for all $f \in L^2(\nu)$,

$$(f, Gf) \geq \frac{1}{\|G\|} (Gf, Gf), \tag{4.9}$$

where both inner products are in $L^2(\nu)$.

Recall that G is a bounded symmetric (hence, self-adjoint) operator in $L^2(\nu)$. Observe that G is non-negative definite. Indeed, if $f \in C_0^\infty\left(\mathbb{R}^2\right)$ then, setting $u = Gf$, we obtain the identities (4.8) so that

$$(f, Gf) = \mathcal{E}_0(u) \geq 0.$$

Then $(f, Gf) \geq 0$ follows from the fact that $C_0^\infty\left(\mathbb{R}^2\right)$ is dense in $L^2(\nu)$.

Now, let us prove (4.9). For non-negative definite self-adjoint operators the following inequality holds, for all $f, h \in L^2(\nu)$:

$$(Gf, h)^2 \leq (Gf, f)(Gh, h).$$

Setting $h = Gf$, we obtain

$$(Gf, Gf)^2 \leq (Gf, f) \|G\| \|h\|^2 = \|G\| (Gf, f)(Gf, Gf).$$

Dividing by (Gf, Gf), we obtain (4.9).

Hence, we have proved (4.7) for $u \in C_0^\infty\left(\mathbb{R}^2\right)$. Let us extend this inequality to all $u \in \mathcal{F}_V$. If $u \in \mathcal{F}_V$ has a compact support then u can be approximated by a sequence of functions $u_n = u * \varphi_n \in C_0^\infty\left(\mathbb{R}^2\right)$ where φ_n is an approximation of identity. Since (4.7) holds for u_n and all the terms with u_n converge as $n \to \infty$ to the corresponding terms with u, we obtain (4.7) for u.

Let a function $u \in \mathcal{F}_V$ be essentially bounded. Consider the sequence $\{\varphi_n\}$ of functions defined by (4.2). Since (4.7) holds for the functions $u_n = u\varphi_n$ with compact support, it suffices to show that passing to the limit as $n \to \infty$, we obtain (4.7) for the function u. The terms $\int V_0 u_n^2 dx$ and $\int V u_n^2 dx$ are obviously survive under the monotone limit. We are left to prove that

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx \to \int_{\mathbb{R}^2} |\nabla u|^2 \, dx. \tag{4.10}$$

We have

$$\int_{\mathbb{R}^2} |\nabla (u\varphi_n)|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u|^2 \varphi_n^2 dx + 2 \int_{\mathbb{R}^2} \langle \nabla u, \nabla \varphi_n \rangle u \varphi_n dx + \int_{\mathbb{R}^2} u^2 |\nabla \varphi_n|^2 \, dx.$$

Observe that

$$\int_{\mathbb{R}^2} u^2 |\nabla \varphi_n|^2 \, dx \le \|u\|_{L^\infty}^2 \int_{\mathbb{R}^2} |\nabla \varphi_n|^2 \, dx \to 0 \text{ as } n \to \infty.$$

The middle term admits the estimate

$$\left| \int_{\mathbb{R}^2} \langle \nabla u, \nabla \varphi_n \rangle u \varphi_n dx \right| \le \left(\int_{\mathbb{R}^2} |\nabla u|^2 \varphi_n^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^2} u^2 |\nabla \varphi_n|^2 \, dx \right) \to 0$$

as $n \to \infty$. Combining the above three lines together, we obtain (4.10).

For a general function $u \in \mathcal{F}_V$, consider an approximating sequence

$$u_n = \max(\min(u, n), -n).$$

The function u_n is bounded so that (4.7) holds for u_n. Letting $n \to \infty$, we obtain (4.7) for u. ■

Now we can prove the main result of this section.

Proposition 4.4 *Consider the following function on* $\mathbb{R}^2 \times \mathbb{R}^2 \setminus \{\mathrm{diag}\}$

$$K(x, y) = \log\langle x \rangle + \log_+ \frac{1}{|x - y|}. \tag{4.11}$$

Then, for any potential V *and for all* $u \in \mathcal{F}_V$,

$$\mathcal{E}_0(u) \ge \frac{c}{b} \int_{\mathbb{R}^2} V u^2 dx, \tag{4.12}$$

where $c > 0$ *is an absolute constant and*

$$b := \sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} K(x, y) V(x) \, dx.$$

Consequently, there is a constant $\varepsilon > 0$ *such that*

$$\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} K(x, y) V(x) \, dx \le \varepsilon \Rightarrow \mathrm{Neg}(V) = 1.$$

Proof. Let us first show that

$$\mathcal{E}_0(u) \geq \frac{1}{a}\int_{\mathbb{R}^2} Vu^2 dx, \tag{4.13}$$

where

$$a = \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y)\,V(y)\,dy.$$

Assume first that $\frac{1}{V} \in L^1_{loc}(\mathbb{R}^2)$ so that Lemma 4.3 applies. Then (4.13) follows from (4.7) and the well-known fact that $\|G\|_{L^2(\nu)} \leq a$.

Let us show that (4.13) holds also for an arbitrary potential V, without the assumption $\frac{1}{V} \in L^1_{loc}$. Fix some function $u \in \mathcal{F}_V$ and define a function $\varphi(r)$ on $(0,+\infty)$ to be piecewise constant on intervals $[n, n+1)$ for integer values of n and to satisfy

$$0 < \varphi(r) \leq e^{-r} \quad \text{and} \quad \varphi(n)\int_{\{n\leq|x|<n+1\}} u^2 dx < e^{-n},$$

which implies

$$\int_{\mathbb{R}^2} u^2(x)\,\varphi(|x|)\,dx < \infty.$$

For any $\varepsilon > 0$, consider the potential

$$V_\varepsilon(x) = V(x) + \varepsilon\varphi(|x|).$$

By construction we have $u \in \mathcal{F}_{V_\varepsilon}$ and $\frac{1}{V_\varepsilon} \in L^1_{loc}$. Hence, by the above argument, we conclude that

$$\mathcal{E}_0(u) \geq \frac{1}{a_\varepsilon}\int_{\mathbb{R}^2} V_\varepsilon u^2 dx, \tag{4.14}$$

where

$$a_\varepsilon = \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y)\,V_\varepsilon(y)\,dy.$$

Using (4.6) we obtain

$$a_\varepsilon \leq a + \varepsilon \sup_{x\in\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y)\,e^{-|y|}dy \to a \text{ as } \varepsilon \to 0.$$

Hence, passing to the limit in (4.14) as $\varepsilon \to 0$, we obtain (4.13).

The upper bound (4.6) of the Green function that can be rewritten in the form

$$g(x,y) \leq CK(y,x),$$

for all $x, y \in \mathbb{R}^2$, which implies $a \leq Cb$. Substituting into (4.13) we obtain (4.12) with $c = C^{-1}$.

Finally, set $\varepsilon = \frac{1}{2}c$. If $b \leq \varepsilon$ then (4.12) implies (4.4), whence $\operatorname{Neg}(V) = 1$ by Lemma 4.2. ■

Remark 4.5 In the view of Proposition 4.4 one could have hoped to estimate $\operatorname{Neg}(V)$ in terms of the quantity

$$\sup_{y\in\mathbb{R}^2}\int_{\mathbb{R}^2} K(x,y)\,V(x)\,dx.$$

However, this is not the case: as it was mentioned in Example 2.7, this quantity could be finite while $\operatorname{Neg}(V) = \infty$.

4.2 Weighted L^1-estimate in half-plane

Let $H_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ be an upper half-plane.

Corollary 4.6 *We have*

$$\sup_{y \in H_+} \int_{H_+} K(x, y) V(x) \, dx \leq \frac{\varepsilon}{2} \Rightarrow \operatorname{Neg}(V, H_+) = 1.$$

Proof. Any function $u \in \mathcal{F}_{V,H_+}$ can be extended to a function $u^* \in \mathcal{F}_V$ by the axial symmetry, that is, by

$$u^*(x) = \begin{cases} u(x_1, x_2), & x_2 > 0, \\ u(x_1, -x_2), & x_2 < 0. \end{cases}$$

Indeed, the resulting function u^* has a weak gradient in $\mathbb{R}^2$, whose restrictions to the domains $\{x_2 > 0\}$ and $\{x_2 < 0\}$ coincide with the weak gradient of $u(x_1, x_2)$ and $u(x_1, -x_2)$, respectively. It follows that

$$\int_{\mathbb{R}^2} |\nabla u^*|^2 \, dx = 2 \int_{H_+} |\nabla u|^2 \, dx.$$

Extending V to the domain $\{x_2 < 0\}$ by setting $V = 0$, we obtain

$$\int_{\mathbb{R}^2} V (u^*)^2 \, dx = \int_{H_+} V u^2 dx.$$

By Lemma 2.5 we conclude that

$$\operatorname{Neg}(V, H_+) \leq \operatorname{Neg}\left(2V, \mathbb{R}^2\right).$$

By Proposition 4.4 we obtain $\operatorname{Neg}\left(2V, \mathbb{R}^2\right) = 1$, whence the claim follows. ■

4.3 Weighted L^p-estimate

Proposition 4.7 *Let $\mathcal{W}(r)$ be as in Theorem 1.1.*

(*a*) *There is $\varepsilon > 0$ depending only on p and $\mathcal{W}$, such that*

$$\int_{\mathbb{R}^2} V^p(x) \mathcal{W}(|x|) \, dx \leq \varepsilon \Rightarrow \operatorname{Neg}(V) = 1.$$

(*b*) *If*

$$\int_{\mathbb{R}^2} V^p(x) \mathcal{W}(|x|) \, dx < \infty, \tag{4.15}$$

then the form $\left(\mathcal{E}_V, W^{1,2}\right)$ is closed in $L^2\left(\mathbb{R}^2\right)$. Consequently, its generator H_V is a self-adjoint, semi-bounded below operator in $L^2\left(\mathbb{R}^2\right)$.

Proof. (*a*) By Proposition 4.4, it suffices to show that

$$\sup_{y \in \mathbb{R}^2} \int_{\mathbb{R}^2} K(x, y) V(x) \, dx \leq C \left(\int_{\mathbb{R}^2} V^p(x) \mathcal{W}(|x|) \, dx\right)^{1/p}, \tag{4.16}$$

where C is a constant depending only on p and $\mathcal{W}$.

For any $y \in \mathbb{R}^2$, we have by the Hölder inequality

$$\int_{\mathbb{R}^2} K(x,y)\, V(x)\, dx \leq \left(\int_{\mathbb{R}^2} V^p(x)\, \mathcal{W}(|x|)\, dx \right)^{1/p} \left(\int_{\mathbb{R}^2} K(x,y)^{p'} \frac{dx}{\mathcal{W}(|x|)^{\frac{1}{p-1}}} \right)^{1/p'},$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. It suffices to show that the last integral is bounded by constant uniformly in $y \in \mathbb{R}^2$. We have by (4.11)

$$\begin{aligned} \int_{\mathbb{R}^2} K(x,y)^{p'} \frac{dx}{\mathcal{W}(|x|)^{\frac{1}{p-1}}} &\leq C \int_{\mathbb{R}^2} \frac{\log^{p'} \langle x \rangle}{\mathcal{W}(|x|)^{\frac{1}{p-1}}} dx \qquad (4.17) \\ &\quad + C \int_{\mathbb{R}^2} \frac{1}{\mathcal{W}(|x|)^{\frac{1}{p-1}}} \log_+^{p'} \frac{1}{|x-y|} dx. \qquad (4.18) \end{aligned}$$

The integral in the right hand side of (4.17) is equal to

$$C \int_0^\infty \frac{\log^{\frac{p}{p-1}}(2+r)}{\mathcal{W}(r)^{\frac{1}{p-1}}} r dr < \infty,$$

where the finiteness follows from (1.2).

By the rearrangement inequality, the integral (4.18) as a function of y takes the maximal value at $y = 0$. At $y = 0$ it is equal to

$$\int_{\{|x|<1\}} \frac{1}{\mathcal{W}(|x|)^{\frac{1}{p-1}}} \log^{p'} \frac{1}{|x|} dx = 2\pi \int_0^1 \frac{\left(\log \frac{1}{r}\right)^{\frac{p}{p-1}}}{\mathcal{W}(r)^{\frac{1}{p-1}}} r dr < \infty,$$

where the finiteness follows from (1.2). Combining all the above estimates, we obtain (4.16).

(b) It follows from (4.15) that V can be represented as a sum of two potentials $V = V_1 + V_2$ where

$$\int_{\mathbb{R}^2} V_1^p(x)\, \mathcal{W}(|x|)\, dx \leq \varepsilon$$

and $V_2 \in L^\infty(\mathbb{R}^2)$. If ε is sufficiently small (as in (a)), then it follows from (4.12) and (4.16) that,

$$\mathcal{E}_0(u) \geq \frac{c}{C\varepsilon^{1/p}} \int_{\mathbb{R}^2} V_1 u^2 dx \geq 2 \int_{\mathbb{R}^2} V_1 u^2 dx.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}^2} V u^2 dx &= \int_{\mathbb{R}^2} V_1 u^2 dx + \int_{\mathbb{R}^2} V_2 u^2 dx \\ &\leq \frac{1}{2} \mathcal{E}_0(u) + \|V_2\|_{L^\infty} \|u\|_{L^2}^2 \\ &\leq \frac{1}{2} \mathcal{E}(u) + K \|u\|_{L^2}^2, \end{aligned}$$

where

$$\mathcal{E}(u) = \int_{\mathbb{R}^2} |\nabla u|^2 dx$$

and

$$K = \|V_0\|_{L^\infty} + \|V_2\|_{L^\infty} < \infty.$$

Since $(\mathcal{E}, W^{1,2})$ is a closed form in $L^2(\mathbb{R}^2)$, we conclude by [4, Theorem 1.8.2] that the perturbed form $\mathcal{E}_V = \mathcal{E} - V$ with domain $W^{1,2}$ is closed and semi-bounded below. ■

5 L^p-estimate with a power weight

In this section we prove a particular case of Theorem 1.1 – Proposition 5.3, with a power weight function. This result is not used in consequent sections. In particular, the proof of Theorem 1.1 in full generality is independent of Proposition 5.3 and uses different argument. However, we feel that a simple argument in the proof of Proposition 5.3 deserves a publication on its own merit.

Denote by D_+ the upper half unit disk, that is,

$$D_+ = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x| < 1,\ x_2 > 0\right\}.$$

Lemma 5.1 *For any $p > 1$ and $0 < q < 2p - 2$, we have*

$$\operatorname{Neg}(V, D_+) \le 1 + C \left(\int_{D_+} V^p |x|^q \, dx\right)^{1/p},$$

where $C = C(p, q)$.

Proof. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the notation of theory of complex variables. In particular, a point on $\mathbb{R}^2$ will be denoted by z, so that

$$D_+ = \{z \in \mathbb{C} : 0 < |z| < 1,\ 0 < \arg z < \pi\}.$$

For any $\alpha \in (0,1)$ define a sector

$$A_\alpha := \{z \in \mathbb{C} : 0 < |z| < 1,\ 0 < \arg z < \pi\alpha\}$$

and consider a biholomorphic mapping $\Psi : D_+ \to A_\alpha$ given by $\Psi(z) = z^\alpha$ (see Fig. 4).

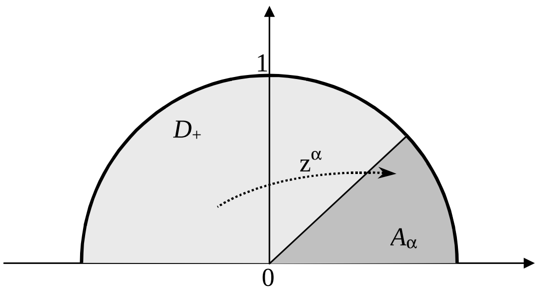

Figure 4: The mapping $z \mapsto z^\alpha$

The sector A_α is bilipschitz equivalent to the unit square, so that the estimate (3.19) applies to A_α, with the constant C depending on α. Hence, the hypothesis (2.23) of Lemma 2.8 is satisfied for $\widetilde{\Omega} = A_\alpha$ with $\widetilde{W} = 1$. By Lemma 2.8, we obtain for $\Omega = D_+$ the following estimate:

$$\operatorname{Neg}(V, D_+) \le 1 + C \left(\int_{D_+} |V(y)|^p |J_\Psi(y)|^{1-p} \, dy\right)^{1/p}.$$

Evaluating J_Ψ by (2.21), we obtain $J_\Psi(y) = \alpha^2 |y|^{2\alpha - 2}$, whence

$$\operatorname{Neg}(V, D_+) \le 1 + C \left(\int_{D_+} |V(y)|^p |y|^{2(1-\alpha)(p-1)} \, dy\right)^{1/p}.$$

We are left to choose α from the equation

$$2(1-\alpha)(p-1)=q.$$

Clearly, we have $0<\alpha<1$ if and only if $0<q<2p-2$. ■

Corollary 5.2 *For any $p>1$ and $q<2p-2$, we have*

$$\operatorname{Neg}(V,D)\leq 2+C\left(\int_D V^p|x|^q\,dx\right)^{1/p} \tag{5.1}$$

where $C=C(p,q)$.

Proof. Indeed, partitioning the disk D into upper half D_+ and lower half D_-, applying

$$\operatorname{Neg}(V,D)\leq \operatorname{Neg}(V,D_+)+\operatorname{Neg}(V,D_-)$$

and estimating each term on the right hand side by Lemma 5.1, we obtain (5.1) ■

Proposition 5.3 *Fix $p>1$ and choose $0<q<2(p-1)$ and $q'>2(p-1)$. Then*

$$\operatorname{Neg}(V)\leq 1+C\left(\int_{\mathbb{R}^2} V^pW(x)\,dx\right)^{1/p} \tag{5.2}$$

where

$$W(x)=\begin{cases}|x|^q, & |x|\leq 1,\\ |x|^{q'}, & |x|>1,\end{cases}$$

and the constant C depends only on p,q,q'.

Proof. It suffices to prove this with $q'<4(p-1)$, which will be assumed in what follows. Let E be the (open) exterior of the unit disk D. Then $\{D,E\}$ is a partition of $\mathbb{R}^2$, and by (2.6) we have

$$\operatorname{Neg}\left(V,\mathbb{R}^2\right)\leq \operatorname{Neg}(V,D)+\operatorname{Neg}(V,E).$$

Using (5.1), let us prove that

$$\operatorname{Neg}(V,E)\leq 2+C\left(\int_E V^p|x|^{q'}\,dx\right)^{1/p}. \tag{5.3}$$

Using again notation from complex variables, consider the biholomorphic mapping $\Psi: E\to D':=D\setminus\{0\}$ given by $\Psi(z)=\frac{1}{z}$. Since in D' we have by (2.5) and (5.1) the estimate

$$\operatorname{Neg}(\widetilde{V},D')=\operatorname{Neg}(\widetilde{V},D)\leq 2+C\left(\int_D \widetilde{V}(x)^p|x|^{\widetilde{q}}\,dx\right)^{1/p},$$

for any $\widetilde{q}<2(p-1)$, the hypotheses of Lemma 2.8 is satisfied in $\widetilde{\Omega}=D'$ with $\widetilde{W}(x)=|x|^{\widetilde{q}}$. Using Lemma 2.8 and $J_\Psi(y)=\frac{1}{|y|^4}$, we obtain for $\Omega=E$ the estimate

$$\begin{aligned}\operatorname{Neg}(V,E) &\leq 2+C\left(\int_E V(y)^p|\Psi(y)|^{\widetilde{q}}|J_\Psi(y)|^{1-p}\,dy\right)^{1/p}\\ &= 2+C\left(\int_E V(y)^p|y|^{4(p-1)-\widetilde{q}}\,dy\right)^{1/p}.\end{aligned}$$

Choosing $\widetilde{q} = 4(p-1) - q'$, we obtain (5.3). Combining it with (5.1), we obtain

$$\operatorname{Neg}(V) \leq 4 + C\left(\int_{\mathbb{R}^2} V^p W(x)\,dx\right)^{1/p}. \tag{5.4}$$

Combining (5.4) with Proposition 4.7 and adjusting the value of C, we obtain (5.2). ∎

6 Weighted L^p-estimate

6.1 Sparse potentials in strips

The main result of this section is Proposition 6.5 below.

Lemma 6.1 *Consider a strip*

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < 1\right\}$$

and a potential V in S. There is a positive constant η such that if

$$\int_S V(x)\langle x\rangle\,dx + \sup_{y\in S}\int_S V(x)\log_+\frac{1}{|x-y|}dx \leq \eta \tag{6.1}$$

then $\operatorname{Neg}(V, S) = 1$.

Proof. For the sake of the proof it will be convenient to consider a strip of the width π rather than 1:

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R},\ 0 < x_2 < \pi\right\},$$

which will only affect the value of η. Consider the function $\Psi(z) = e^z$ on S. Clearly, Ψ is a biholomorphic mapping from S onto H_+, where H_+ is the upper half-plane. Set $\Phi = \Psi^{-1}$ so that $\Phi(z) = \ln z$. Using the argument in the proof of Lemma 2.8, we obtain that

$$\operatorname{Neg}(V, S) \leq \operatorname{Neg}(\widetilde{V}, H_+)$$

where

$$\widetilde{V}(z) = V(\Phi(z))\left|\Phi'(z)\right|^2.$$

On the other hand, we have

$$\int_{H_+}\widetilde{V}(y)\widetilde{W}(y)\,dy = \int_S V(x)W(x)\,dx$$

provided

$$W(z) = \widetilde{W}(\Psi(z)) = \widetilde{W}(e^z)$$

Applying this with

$$\widetilde{W}_1(x) = \log\langle x\rangle,$$

we obtain, for any $z \in S$,

$$W_1(z) = \log(2 + |e^z|) \leq 2 + |z| = \langle z\rangle.$$

Applying this with

$$\widetilde{W}_2(x, y) = \log_+\frac{1}{|x-y|},$$

where $y \in \mathbb{R}^2$, we obtain, denoting $w = \Phi(y) \in S$,

$$\begin{aligned} W_2(z,w) &= \log_+ \frac{1}{|e^z - e^w|} \\ &\le C + |z| + \log_+ \frac{1}{|z-w|}. \end{aligned}$$

In the last line we have used the following inequality

$$|e^z - e^w| \ge c e^{-|z|} \min(1, |z-w|) \quad \text{for all } z, w \in S,$$

that holds with the constant

$$c = \min\left(\inf_{\{z\in S:|z|\ge 1\}} |e^z - 1|, \inf_{\{z\in S:|z|<1\}} \frac{|e^z-1|}{|z|} \right) > 0.$$

It follows that

$$\begin{aligned} & \int_S V(z) W_1(z)\,dz + \sup_{w\in S} \int_S V(z) W_2(z,w)\,dz \\ \le\ & C \int_S V(z) \langle z \rangle\,dz + \sup_{w\in S} \int_S V(z) \log_+ \frac{1}{|z-w|} dz, \end{aligned}$$

If η is small enough, then (6.1) implies that

$$\int_S V(z) W_1(z)\,dz + \sup_{w\in S} \int_S V(z) W_2(z,w)\,dz \le \frac{\varepsilon}{2},$$

where ε is the constant from Corollary 4.6, whence also

$$\int_{H_+} \widetilde{V}(x) \widetilde{W}_1(x)\,dx + \sup_{y\in H_+} \int_{H_+} \widetilde{V}(x) \widetilde{W}_2(x,y)\,dx \le \frac{\varepsilon}{2}.$$

By Corollary 4.6 we conclude that $\operatorname{Neg}(\widetilde{V}, H_+) = 1$, whence also $\operatorname{Neg}(V,S) = 1$. ∎

Let $B(x,r)$ be a disk in $\mathbb{R}^2$ of the radius r centered at a point $x \in \mathbb{R}^2$.

Definition 6.2 Fix $p > 1$. We say that a potential V in a domain Ω is *sparse* if, for any $y \in \Omega$,

$$\int_{B(y,1)\cap\Omega} V^p(x)\,dx < \delta, \tag{6.2}$$

where δ is a small enough positive constant, depending only on p that will be specified below (cf. (6.6)).

For any $a \in \mathbb{R}$, denote

$$S_a = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > a,\ 0 < x_2 < 1 \right\}. \tag{6.3}$$

Corollary 6.3 *There is a constant $\eta_1 > 0$ such that if*

$$\int_{S_1} V(x)\, x_1 dx + \sup_{y\in S_1} \int_{S_1} V(x) \log_+ \frac{1}{|x-y|} dx \le \eta_1 \tag{6.4}$$

then $\operatorname{Neg}(V, S_1) = 0$. *Moreover, if V is a sparse potential in S_1 then*

$$\int_{S_1} V(x)\, x_1 dx \le \frac{\eta_1}{2} \Rightarrow \operatorname{Neg}(V, S_1) = 1.$$

Proof. It will be convenient to prove the first claim for the strip S_0 instead of S_1. In this case the term x_1 in (6.4) should be replaced by $1+x_1$. Since $1+x_1 \simeq \langle x\rangle$, it suffices to prove the following: if

$$\int_{S_0} V(x)\langle x\rangle\,dx+\sup_{y\in S_0}\int_{S_0} V(x)\log_+\frac{1}{|x-y|}dx\le\eta_0 \tag{6.5}$$

then $\mathrm{Neg}(V,S_0)=1$. Any function $u\in\mathcal{F}_{V,S_0}$ can be symmetrically extended to a function u^* in the unbounded strip

$$S=\left\{(x_1,x_2)\in\mathbb{R}^2 : x_1\in\mathbb{R},\ 0<x_2<1\right\}.$$

Extending also $V(x)$ for $x_1<0$ by zero, we obtain

$$\int_S |\nabla u^*|^2\,dx=2\int_{S_0}|\nabla u|^2\,dx \quad\text{and}\quad \int_S V(u^*)^2\,dx=\int_{S_0}Vu^2dx,$$

whence by Lemma 2.5

$$\mathrm{Neg}(V,S_0)\le\mathrm{Neg}(2V,S).$$

If η_0 in (6.5) is sufficiently small then $2V$ satisfies the condition of Lemma 6.1 whence $\mathrm{Neg}(2V,S)=1$ follows.

Let us prove the second claim. It follows from (6.2) that

$$\begin{aligned}\int_{S_1} V(x)\log_+\frac{1}{|x-y|}dx &\le \left(\int_{B(y,1)\cap S_1} V^p(x)\,dx\right)^{1/p}\\ &\quad\times\left(\int_{B(y,1)}\left(\log_+\frac{1}{|x-y|}\right)^{p'}dx\right)^{1/p'},\end{aligned}$$

where p' is the Hölder conjugate to p. The last integral is equal to a finite constant that is independent of y. Denoting it by C and assuming that the constant δ in (6.2) satisfies the bound

$$\delta<\left(\frac{\eta_1}{2C}\right)^p, \tag{6.6}$$

we obtain that, for a sparse potential V,

$$\sup_{y\in S_1}\int_{B(y,1)\cap S_1} V(x)\log_+\frac{1}{|x-y|}dx<\frac{\eta_1}{2}.$$

Hence, the hypothesis $\int_{S_1}V(x)\,x_1dx\le\frac{\eta_1}{2}$ implies that V satisfies (6.4), whence $\mathrm{Neg}(V,S_1)=1$. ■

For any $r\ge 2$ denote by P_r be the rectangle

$$P_r:=\left\{(x_1,x_2)\in\mathbb{R}^2 :\ 1<x_1<r,\ \ 0<x_2<1\right\}.$$

Corollary 6.4 *There exists a positive constant c, such that, for any $r\ge 2$ and for any sparse potential V in P_r,*

$$\int_{P_r} V(x)\,x_1dx\le c\Rightarrow \mathrm{Neg}(V,P_r)=1.$$

Proof. Consider a slightly modified domain

$$\widetilde{P}_r = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \quad |x| < r, \ 0 < x_2 < 1\right\}.$$

Since P_r and $\widetilde{P}_r$ are obviously bilipschitz equivalent uniformly in $r \geq 2$, by Lemma 2.8 it suffices to prove the claim for $\widetilde{P}_r$ instead of P_r, that is,

$$\int_{\widetilde{P}_r} V(x)(1 + x_1)\, dx < c \Rightarrow \operatorname{Neg}\left(V, \widetilde{P}_r\right) = 1.$$

Extend V to the strip S_0 by setting $V = 0$ outside $S_0 \setminus \widetilde{P}_r$. If c is small enough then by Corollary 6.3 we have $\operatorname{Neg}(2V, S_0) = 1$.

Consider the inversion Φ in the circle of radius r: $\Phi(x) = r\frac{x}{|x|^2}$ (see Fig. 5). Clearly, $x \in S_0 \setminus \widetilde{P}_r$ implies $\Phi(x) \in \widetilde{P}_r$ so that we can extend any function $f \in \mathcal{F}_{V,\widetilde{P}_r}$ to $S_0 \setminus \widetilde{P}_r$ by setting

$$f(x) = f(\Phi(x)) \quad \text{for all } x \in S_0 \setminus \widetilde{P}_r.$$

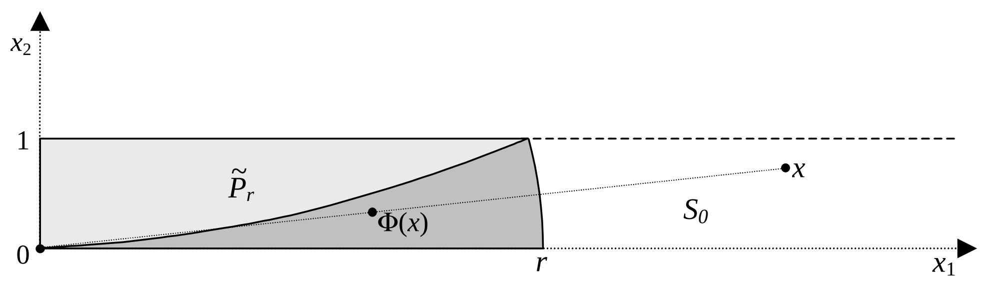

Figure 5: Extension of function f from $\widetilde{P}_r$ to S_0 (the dark shaded area is the image of $S_0 \setminus \widetilde{P}_r$ under the inversion Φ)

Since the inversion Φ preserves the Dirichlet integral, we obtain

$$\int_{S_0} |\nabla f|^2\, dx \leq 2\int_{\widetilde{P}_r} |\nabla f|^2\, dx.$$

By Lemma 2.5, we conclude that

$$\operatorname{Neg}\left(V, \widetilde{P}_r\right) \leq \operatorname{Neg}(2V, S_0),$$

whence the claim follows. ■

Proposition 6.5 (Bargmann type estimate) *Let V be a sparse potential in Ω, where Ω is either S_1 or P_r. Then*

$$\operatorname{Neg}(V, \Omega) \leq 1 + C\int_\Omega V(x)\, x_1 dx, \tag{6.7}$$

where the constant C depends only on p

Proof. Assume for certainty that $\Omega = S_1$. Choose some increasing sequence $\{r_k\}_{k=1}^{\infty}$ such that $r_1 = 1$ and $r_k \to \infty$ (where $r_k = \infty$ is also allowed), and partition S_1 into a sequence of rectangles

$$R_k = \{(x_1, x_2) : r_k < x_1 < r_{k+1}, \quad 0 < x_2 < 1\}.$$

The rectangle R_k is congruent to $P_{r_{k+1}-r_k+1}$ (or to the strip S_1 if $r_{k+1} = \infty$) and is obtained from the latter by shifting by $r_k - 1$ along the axis x_1. If

$$r_{k+1} - r_k \geq 1 \tag{6.8}$$

and

$$\int_{R_k} V(x)(x_1 - r_k + 1)\, dx \leq c \tag{6.9}$$

then it follows from Corollaries 6.3 and 6.4 that

$$\operatorname{Neg}(V, R_k) = 1.$$

Let us show how to define the sequence $\{r_k\}$ inductively so that (6.8) and (6.9) are satisfied. Namely, if r_k is already constructed, then choose r_{k+1} to be the maximal possible value (including ∞) so that (6.9) is satisfied. Let us show that (6.8) is then also satisfied. Indeed, if $r_{k+1} - r_k < 1$ then, using the equality in (6.9) (that holds by the maximality of r_{k+1}), we obtain

$$\int_{R_k} V(x)\, dx \geq \frac{c}{2}.$$

It follows by the Hölder inequality that also

$$\left(\int_{R_k} V^p dx\right)^{1/p} \geq \frac{c}{2}. \tag{6.10}$$

However, if the constant δ in the definition (6.2) of a space potential is small enough, then, noticing that R_k is covered by a unit square and, hence, by a unit disk, we obtain that (6.10) and (6.2) contradict each other, which proves (6.8).

As was already mentioned, by the maximality of r_{k+1} we always have equality in (6.9) except for the case $r_{k+1} = \infty$. So, if $r_{k+1} < \infty$ then we have

$$\int_{R_k} V(x)\, x_1 dx \geq c.$$

Assuming that

$$I := \int_{S_1} V(x)\, x_1 dx < \infty,$$

we see that the total number of bounded rectangles in our construction is bounded by I/c. Since an unbounded rectangle (a strip) can occur only once, the total number of elements of the partition $\{R_k\}$ is bounded by $1 + I/c$. It follows that

$$\operatorname{Neg}(V, S_1) \leq \sum_k \operatorname{Neg}(V, R_k) \leq 1 + I/c,$$

which finishes the proof of (6.7). ■

6.2 Negative eigenvalues in a strip

We continue using the notation (6.3) for strips in $\mathbb{R}^2$. As before, $p > 1$ is fixed.

Lemma 6.6 (2nd Main Lemma) *Let $\widetilde{\mathcal{W}}(r)$ be a positive monotone increasing function on $[1, +\infty)$ such that*

$$\int^{\infty} \frac{r^{\frac{p}{p-1}} dr}{\widetilde{\mathcal{W}}(r)^{\frac{1}{p-1}}} < \infty. \tag{6.11}$$

Then, for any potential V in the strip S_1, we have

$$\operatorname{Neg}(V, S_1) \leq 1 + C \left(\int_{S_1} V^p(x) \widetilde{\mathcal{W}}(x_1) \, dx \right)^{1/p}, \tag{6.12}$$

where the constant C depends only on p and $\widetilde{\mathcal{W}}$.

In particular, if for large r

$$\widetilde{\mathcal{W}}(r) = r^{2p-1} \log^{p-1+\varepsilon} r,$$

where $\varepsilon > 0$, then $\widetilde{\mathcal{W}}$ satisfies (6.11).

Proof. Let us extend the function $\widetilde{\mathcal{W}}(r)$ to $(0, +\infty)$ be setting $\widetilde{\mathcal{W}}(r) = \widetilde{\mathcal{W}}(1)$ for $r < 1$, which preserves the monotonicity of $\widetilde{\mathcal{W}}$.

For any subset $\Omega \subset S_1$, set

$$\begin{aligned} I(\Omega) &= \int_{\Omega} V^p(x) \, dx \\ J(\Omega) &= \int_{\Omega} V^p(x) \widetilde{\mathcal{W}}(x_1) \, dx \end{aligned}$$

and

$$w(\Omega) = \int_{\Omega} \frac{x_1^{p'}}{\widetilde{\mathcal{W}}(x_1)^{\frac{1}{p-1}}} dx,$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. In particular, for a rectangle

$$R = \{(x_1, x_2) : a < x_1 < b, \quad 0 < x_2 < 1\}$$

we have

$$w(R) = \int_a^b \frac{r^{p'} dr}{\widetilde{\mathcal{W}}(r)^{\frac{1}{p-1}}}.$$

The hypothesis (6.11) implies, in particular, that $w\left(S_{1/2}\right) < \infty$. We will prove that

$$\operatorname{Neg}(V, S_1) \leq 1 + C J(S_1)^{1/p} \, w\left(S_{1/2}\right)^{1/p'},$$

where C is a constant depending only on p, which will then imply (6.12).

Let us partition S_1 into a sequence of rectangles

$$R_n = \left\{(x_1, x_2) \in \mathbb{R}^2 : 2^n < x_1 < 2^{n+1}, 0 < x_2 < 1\right\},$$

where $n = 0, 1, 2, \ldots$. Clearly, we have

$$\widetilde{\mathcal{W}}(2^n) I(R_n) \le J(R_n) \le \widetilde{\mathcal{W}}\left(2^{n+1}\right) I(R_n). \tag{6.13}$$

By Corollary 3.5, we have

$$\operatorname{Neg}(V, R_n) \le 1 + C\left((2^n)^{2p-1}\int_{R_n} V^p dx\right)^{1/p} = 1 + C\left(2^{n(2p-1)} I(R_n)\right)^{1/p}. \tag{6.14}$$

We say that the rectangle R_n is *saturated* if

$$2^{n(2p-1)} I(R_n) \ge c \tag{6.15}$$

where c is a positive constant whose value will be determined below. Denote by σ the set of all indices n for which R_n is saturated. For any $n \in \sigma$ we obtain, by adjusting the value of the constant C in (6.14) and by using (6.13), that

$$\begin{aligned}
\operatorname{Neg}(V, R_n) &\le C\left(2^{n(2p-1)} I(R_n)\right)^{1/p} \\
&\le C J(R_n)^{1/p}\left(\frac{2^{n(2p-1)}}{\widetilde{\mathcal{W}}(2^n)}\right)^{1/p} \\
&\le C J(R_n)^{1/p}\left(\frac{2^{n\frac{2p-1}{p-1}}}{\widetilde{\mathcal{W}}(2^n)^{\frac{1}{p-1}}}\right)^{1/p'} \\
&\le C J(R_n)^{1/p}\left(2\int_{2^{n-1}}^{2^n} \frac{r^{\frac{p}{p-1}}}{\widetilde{\mathcal{W}}(r)^{\frac{1}{p-1}}} dr\right)^{1/p'} \\
&= C' J(R_n)^{1/p} w(R_{n-1})^{1/p'}.
\end{aligned} \tag{6.16}$$

For any $n \in \sigma$ denote by m the maximal element of σ preceding n (if n is the smallest element of σ then set $m = -1$) and consider the rectangle

$$P_n = \left\{(x_1, x_2) : 2^{m+1} < x_1 < 2^n, 0 < x_2 < 1\right\} \tag{6.17}$$

(see Fig. 6).

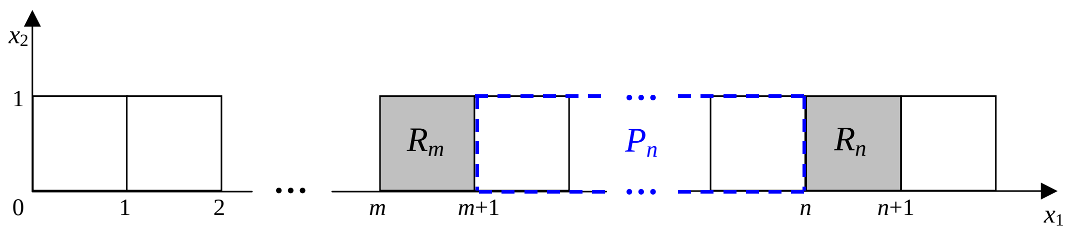

Figure 6: Saturated rectangles R_m and R_n, and a rectangle P_n where the potential V is sparse.

In particular, P_n is empty if R_{n-1} is also saturated. Similarly, define P_∞: choose the maximal element m from σ (or $m = -1$ if σ is empty) and set

$$P_\infty = \left\{(x_1, x_2) : 2^{m+1} < x_1 < \infty, 0 < x_2 < 1\right\}.$$

If the set σ is infinite, that is, σ does not have the maximal element, then set $P_\infty = \emptyset$. If σ is empty then we have $P_\infty = S_1$.

Clearly, we obtain a partition of S_1 into the following sets: $\{P_n, R_n\}_{n\in\sigma}$ and P_∞ (where some of P_n may be empty). By construction, the rectangle P_n from (6.17) admits a partition into non-saturated rectangles R_k with $k = m+1, ..., n-1$, which implies that the potential V is sparse in P_n (assuming that the constant c in (6.15) small enough compared to δ from (6.2)). In the same way, V is sparse in P_∞. By Proposition 6.5, we have

$$\begin{aligned}\operatorname{Neg}(V, P_n) &\leq 1 + C\int_{P_n} V(x)\, x_1 dx \\ &\leq 1 + C\left(\int_{P_n} V^p(x)\,\widetilde{\mathcal{W}}(x_1)\,dx\right)^{1/p}\left(\int_{P_n}\frac{x_1^{p'}}{\widetilde{\mathcal{W}}(x_1)^{\frac{1}{p-1}}}dx\right)^{1/p'} \\ &= 1 + CJ(P_n)^{1/p}\, w(P_n)^{1/p'}. \end{aligned} \tag{6.18}$$

Renaming the constant C' in (6.16) by C and combining this inequality with with (6.18), we obtain

$$\begin{aligned}\operatorname{Neg}(V, R_n) + \operatorname{Neg}(V, P_n) &\leq CJ(R_n)^{1/p}\, w(R_{n-1})^{1/p'} + \left[1 + CJ(P_n)^{1/p}\, w(P_n)^{1/p'}\right] \\ &\leq 2C\left[J(R_n)^{1/p}\, w(R_{n-1})^{1/p'} + J(P_n)^{1/p}\, w(P_n)^{1/p'}\right],\end{aligned}$$

where we have used that

$$CJ(R_n)^{1/p}\, w(R_{n-1})^{1/p'} \geq 1.$$

Renaming in what follows $2C$ back to C, we obtain

$$\begin{aligned}\operatorname{Neg}(V, S_1) &\leq \sum_{n\in\sigma}\left[\operatorname{Neg}(V, P_n) + \operatorname{Neg}(V, R_n)\right] + \operatorname{Neg}(V, P_\infty) \\ &\leq C\sum_{n\in\sigma}\left[J(R_n)^{1/p}\, w(R_{n-1})^{1/p'} + J(P_n)^{1/p}\, w(P_n)^{1/p'}\right] \\ &\quad + CJ(P_\infty)^{1/p}\, w(P_\infty)^{1/p'} + 1.\end{aligned}$$

Applying the Hölder inequality to the sum, we obtain

$$\begin{aligned}\operatorname{Neg}(V, S_1) &\leq 1 + C\left(\sum_{n\in\sigma}\left[J(R_n) + J(P_n)\right] + J(P_\infty)\right)^{1/p} \\ &\quad \times\left(\sum_{n\in\sigma}\left[w(R_{n-1}) + w(P_n)\right] + w(P_\infty)\right)^{1/p'} \\ &\leq 1 + 2CJ(S_1)^{1/p}\, w\left(S_{1/2}\right)^{1/p'},\end{aligned}$$

which was to be proved. ■

6.3 Negative eigenvalues in $\mathbb{R}^2$

Finally, we can prove the main result of this paper.

Proof of Theorem 1.1. Consider the upper half of the disk exterior

$$E_+ = \{z \in \mathbb{C} : |z| > 1,\ \operatorname{Im} z > 0\},$$

and the function $\Psi(z) = \ln z$ in E_+ that is a biholomorphic mapping from E_+ onto the strip

$$\widetilde{S} = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \quad 0 < x_2 < \pi\right\}$$

(see Fig. 7).

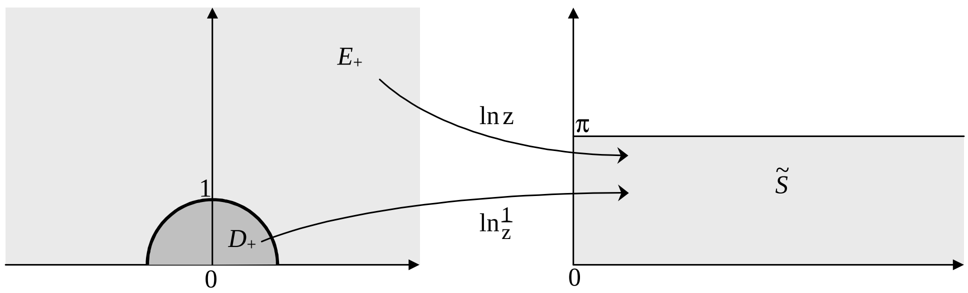

Figure 7: Conformal mappings from E_+ and D_+ to $\widetilde{S}$

Since the strips $\widetilde{S}$ and S_1 are bilipschitz equivalent, the estimate (6.12) of Lemma 6.6 holds also for $\widetilde{S}$, that is,

$$\operatorname{Neg}(V, \widetilde{S}) \leq 1 + C \left(\int_{\widetilde{S}} V^p(x)\, \widetilde{\mathcal{W}}(x_1)\, dx \right)^{1/p}, \tag{6.19}$$

provided $\widetilde{\mathcal{W}}(r)$ is any positive monotone increasing function on $[0, +\infty)$ that satisfies (6.11). Define function $\widetilde{\mathcal{W}}(r)$ for $r \geq 0$ by

$$\widetilde{\mathcal{W}}(r) = \mathcal{W}(e^r)\, e^{-2(p-1)r},$$

where $\mathcal{W}$ is a given function that satisfies (1.2). A change $r = e^t$ in (1.2) yields

$$\int_1^\infty \frac{r(\log r)^{\frac{p}{p-1}}\, dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} = \int_0^\infty \frac{e^{2t} t^{\frac{p}{p-1}}\, dt}{\mathcal{W}(e^t)^{\frac{1}{p-1}}} = \int_0^\infty \frac{t^{\frac{p}{p-1}}\, dt}{\widetilde{\mathcal{W}}(t)^{\frac{1}{p-1}}}.$$

Hence, the hypothesis (1.2) implies that $\widetilde{\mathcal{W}}$ satisfies (6.11) so that by Lemma 6.6 we obtain (6.19).

Denoting $\widetilde{W}(z) = \widetilde{\mathcal{W}}(\operatorname{Re} z)$ and observing that $J_\Psi = |\Psi'(z)|^2 = \frac{1}{|z|^2}$, we obtain from (6.19) and Lemma 2.8 that

$$\operatorname{Neg}(V, E_+) \leq 1 + C \left(\int_{E_+} V^p(x)\, W(x)\, dx \right)^{1/p},$$

where

$$\begin{aligned} W(z) &= \widetilde{W}(\Psi(z))\, |J_\Psi(z)|^{1-p} = \widetilde{W}(\ln z)\, |z|^{2(p-1)} \\ &= \widetilde{\mathcal{W}}(\operatorname{Re} \ln z)\, |z|^{2(p-1)} = \widetilde{\mathcal{W}}(\ln |z|)\, |z|^{2(p-1)} = \mathcal{W}(|z|). \end{aligned}$$

In other words, we have proved (1.4) for E_+ instead of $\mathbb{R}^2$.

Similarly, consider the upper half-disk

$$D_+ = \{z \in \mathbb{C} : |z| < 1, \ \mathrm{Im}\, z > 0\}$$

and the biholomorphic mapping $\Psi(z) = \ln \frac{1}{z}$ from D_+ onto $\widetilde{S}$. Define the weight function for $r \geq 0$ by

$$\widetilde{\mathcal{W}}(r) = \mathcal{W}\left(e^{-r}\right) e^{2(p-1)r}.$$

Changing $r = e^{-t}$ yields

$$\int_0^1 \frac{r\,|\log r|^{\frac{p}{p-1}}\, dr}{\mathcal{W}(r)^{\frac{1}{p-1}}} = \int_0^\infty \frac{e^{-2t} t^{\frac{p}{p-1}} dt}{\mathcal{W}(e^{-t})^{\frac{1}{p-1}}} = \int_0^\infty \frac{t^{\frac{p}{p-1}} dt}{\widetilde{\mathcal{W}}(t)^{\frac{1}{p-1}}},$$

so that $\widetilde{\mathcal{W}}$ satisfies (6.11). By Lemma 6.6, we obtain

$$\mathrm{Neg}\,(V, D_+) \leq 1 + C\left(\int_{D_+} V^p(x)\, W(x)\, dx\right)^{1/p},$$

where

$$\begin{aligned} W(z) &= \widetilde{W}\left(\ln \frac{1}{z}\right) |z|^{2(p-1)} \\ &= \widetilde{\mathcal{W}}\left(\mathrm{Re}\ln \frac{1}{z}\right) |z|^{2(p-1)} = \widetilde{\mathcal{W}}\left(\ln \frac{1}{|z|}\right) |z|^{2(p-1)} = \mathcal{W}(|z|). \end{aligned}$$

Similar estimates hold for the lower half-disk D_- and lower half of the disk exterior E_-. Partitioning $\mathbb{R}^2$ into four domains D_+, D_-, E_+, E_-, we obtain by Lemma 2.4

$$\mathrm{Neg}\,(V) \leq 4 + C\left(\int_{\mathbb{R}^2} V^p(x)\, \mathcal{W}(|x|)\, dx\right)^{1/p}.$$

Finally, combining this estimate with Proposition 4.7 and adjusting the constant C, we can replace 4 here by 1, that is

$$\mathrm{Neg}\,(V) \leq 1 + C\left(\int_{\mathbb{R}^2} V^p(x)\, \mathcal{W}(|x|)\, dx\right)^{1/p}.$$

By Proposition 4.7, the form $\left(\mathcal{E}_V, W^{1,2}\right)$ is closed, and, hence, its generator H_V is well-defined as a self-adjoint operator in L^2. By Lemma 2.1, $\mathrm{Neg}\,(H_V) \leq \mathrm{Neg}\,(V)$, whence (1.4) follows. ■

References

[1] **Birman M.Sh., Laptev A.**, The negative discrete spectrum of a two-dimensional Schrödinger operator, *Comm. Pure Appl. Math.*, **49** (1996) no.9, 967997.

[2] **Chadan K., Khuri N.N., Martin A., Wu Tai Tsun**, Bound states in one and two spatial dimensions, *J. Math. Physics*, **44** (2003) no.2, 406-422.

[3] **Cwikel W.**, Weak type estimates for singuar values and the number of bound states of Schrödinger operators, *Ann. Math.*, **106** (1977) 93–100.

[4] **Davies E.B.**, "Heat kernels and spectral theory", Cambridge University Press, 1989.

[5] **Grigor'yan A.**, Heat kernels on weighted manifolds and applications, *Contemporary Mathematics*, **398** (2006) 93-191.

[6] **Grigor'yan A., Netrusov Yu., Yau S.-T.**, Eigenvalues of elliptic operators and geometric applications, *in:* "Eigenvalues of Laplacians and other geometric operators", Surveys in Differential Geometry IX, (2004) 147-218.

[7] **Grigor'yan A., Yau S.-T.**, Isoperimetric properties of higher eigenvalues of elliptic operator, *Amer. J. Math*, **125** (2003) 893-940.

[8] **Khuri N.N., Martin A., Wu Tai Tsun**, Bound states in n dimensions (especially $n = 1$ and $n = 2$), *Few-Body Systems*, **31** (2002) 83-89.

[9] **Levin D., Solomyak M.**, The Rozenblum-Lieb-Cwikel inequality for Markov generators, *J. d'Analyse Math.*, **71** (1997) 173-193.

[10] **Li P., Yau S.-T.**, On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.*, **88** (1983) 309–318.

[11] **Lieb E.H.**, Bounds on the eigenvalues of the Laplace and Schrödinger operators, *Bull. Amer. Math. Soc.*, **82** (1976) 751-753.

[12] **Lieb E.H.**, The number of bound states of one-body Schrödinger operators and the Weyl problem, *Proc. Sym. Pure Math.*, **36** (1980) 241-252.

[13] **Molchanov S., Vainberg B.**, On negative eigenvalues of low-dimensional Schrödinger operators, preprint 2010.

[14] **Murata M.**, Positive solutions and large time behaviors of Schrödinger semigroups, Simon's problem, *J. Funct. Anal.*, **56** (1984) 300-310.

[15] **Rozenblum G., Solomyak M.**, Counting Schrödinger boundstates: semiclassics and beyond, *in:* "Sobolev spaces in mathematics II", 329353. Ed. V.Maz'ya, Int. Math. Ser. 9, Springer, New York, 2009.

[16] **Rozenblum G.V.**, The distribution of the discrete spectrum for singular differential operators, *Dokl. Akad. Nauk SSSR*, **202** (1972) 1012-1015.