THE GABRIEL–ROITER FILTRATION OF THE ZIEGLER SPECTRUM

HENNING KRAUSE AND MIKE PREST

Abstract. Inclusion preserving maps from modules over an Artin algebra to complete partially ordered sets are studied. This yields a filtration of the Ziegler spectrum which is indexed by all Gabriel–Roiter measures. Another application is a compactness result for the set of subcategories of finitely presented modules that are closed under submodules.

1. Introduction

Let A be an Artin algebra. We work in the category $\mathsf{Mod}\,A$ of all A-modules and $\mathsf{mod}\,A$ denotes the full subcategory consisting of all finitely presented A-modules.

In this paper we combine two concepts from representation theory which have the following in common: they are powerful but also technically involved. Our motivation is to understand invariants of representations which reflect the inclusion relation. Thus we study maps $f\colon \mathsf{Mod}\,A \to \mathsf{S}$ where S is a partially ordered set and for each pair X, Y of A-modules

$$X \subseteq Y \quad \Longrightarrow \quad f(X) \le f(Y).$$

The Gabriel–Roiter measure $\mu\colon \mathsf{Mod}\,A \to \mathbf{2}^{\mathbb{N}}$ in the sense of Ringel [12] is an example of particular importance. Here, $\mathbf{2}^{\mathbb{N}}$ denotes the power set of the set of natural numbers, endowed with the lexicographical order.

In a recent paper [13], Ringel used the Gabriel–Roiter measure to establish the following somewhat surprising result. Here, an additive subcategory of $\mathsf{mod}\,A$ is said to be of *infinite type* if it contains infinitely many non-isomorphic indecomposable objects.

Theorem (Ringel)**.** *Each submodule closed additive subcategory of* $\mathsf{mod}\,A$ *that is of infinite type contains one which is minimal among all submodule closed additive subcategories of infinite type.* □

We give a new proof of this result which involves the Ziegler spectrum of A and uses its compactness [14]. A further analysis then leads to a filtration of the Ziegler spectrum which is indexed by the totally ordered set $\{\mu(X) \mid X \in \mathsf{Mod}\,A\}$ consisting of all Gabriel–Roiter measures.

2. From modules to partially ordered sets

In this section we study maps from the category of A-modules to complete partially ordered sets. From a categorical point of view this means we consider the subcategory $\mathsf{Mon}\,A$ of $\mathsf{Mod}\,A$ where the objects are the A-modules and the morphisms between two modules are the A-linear monomorphisms. Then we study functors $\mathsf{Mon}\,A \to \mathsf{S}$ where S is a partially ordered set, viewed as a category having at most one morphism between any two objects.

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Submodule closed subcategories. Let $\mathsf{S}(\mathsf{mod}\,A)$ denote the set of full additive subcategories of $\mathsf{mod}\,A$ that are closed under submodules. This set is partially ordered by inclusion and in fact complete.

Recall that a partially ordered set S is *complete* if every subset U of S has a supremum, which then is denoted by

$$\sup \mathsf{U} = \bigvee_{x\in\mathsf{U}} x.$$

Note that the supremum can be expressed as an infimum:

$$\sup \mathsf{U} = \inf\{y \in \mathsf{S} \mid x \le y \text{ for all } x \in \mathsf{U}\}.$$

Given an A-module X, let $\mathsf{sub}\,X$ denote the full subcategory of $\mathsf{mod}\,A$ consisting of all A-modules that are submodules of finite direct sums of copies of X.

Proposition 2.1. *The map* $\mathsf{Mod}\,A \to \mathsf{S}(\mathsf{mod}\,A)$ *taking a module* X *to* $\mathsf{sub}\,X$ *is the universal map* $f\colon \mathsf{Mod}\,A \to \mathsf{S}$ *to a complete partially ordered set* S *satisfying*

(1) $f(X) \le f(Y)$ *for* $X \subseteq Y$ *in* $\mathsf{Mod}\,A$*;*
(2) $f(X \oplus Y) = f(X) \vee f(Y)$ *for* X, Y *in* $\mathsf{Mod}\,A$*;*
(3) $f(\bigcup_\alpha X_\alpha) = \bigvee_\alpha f(X_\alpha)$ *for every directed union* $\bigcup_\alpha X_\alpha$ *in* $\mathsf{Mod}\,A$.

More precisely, given such a map $f\colon \mathsf{Mod}\,A \to \mathsf{S}$*, there exists a unique map* $\bar f\colon \mathsf{S}(\mathsf{mod}\,A) \to \mathsf{S}$ *satisfying* $f(X) = \bar f(\mathsf{sub}\,X)$ *for all* $X \in \mathsf{Mod}\,A$*. The map* $\bar f$ *is order preserving and*

$$\bar f(\bigvee_\alpha \mathsf{C}_\alpha) = \bigvee_\alpha \bar f(\mathsf{C}_\alpha)$$

for every set of elements $\mathsf{C}_\alpha \in \mathsf{S}(\mathsf{mod}\,A)$.

Proof. It is clear that the assignment $X \mapsto \mathsf{sub}\,X$ satisfies (1)–(3). Now fix an arbitrary map $f\colon \mathsf{Mod}\,A \to \mathsf{S}$ with these properties. Then $\mathsf{sub}\,X \subseteq \mathsf{sub}\,Y$ implies that X is a submodule of a finite direct sum of copies of Y, and therefore $f(X) \le f(Y)$. Thus $\bar f\colon \mathsf{S}(\mathsf{mod}\,A) \to \mathsf{S}$ taking $\mathsf{sub}\,X$ to $f(X)$ is well-defined and order preserving. Note that any C in $\mathsf{S}(\mathsf{mod}\,A)$ is of the form $\mathsf{C} = \mathsf{sub}\,X_\mathsf{C}$, where $X_\mathsf{C} = \bigoplus_{X\in\mathsf{C}} X$. Finally, we compute

$$\bar f(\bigvee_\alpha \mathsf{C}_\alpha) = \bar f(\mathsf{sub}\bigoplus_\alpha X_{\mathsf{C}_\alpha}) = f(\bigoplus_\alpha X_{\mathsf{C}_\alpha}) = \bigvee_\alpha f(X_{\mathsf{C}_\alpha}) = \bigvee_\alpha \bar f(\mathsf{C}_\alpha). \qquad \square$$

The Ziegler spectrum. We write $\mathsf{Ind}\,A$ for the set of isomorphism classes of indecomposable pure-injective A-modules. A subset of $\mathsf{Ind}\,A$ is *Ziegler closed* if it is of the form $\mathsf{C} \cap \mathsf{Ind}\,A$ for some definable subcategory $\mathsf{C} \subseteq \mathsf{Mod}\,A$. Following [2], a subcategory is *definable* if it is closed under filtered colimits, products and pure submodules. The Ziegler closed subsets provide the closed subsets of a topology on $\mathsf{Ind}\,A$; see [2, 14]. For each class C of A-modules, we denote by $\mathsf{Def}\,\mathsf{C}$ the smallest definable subcategory containing C and let $\mathsf{Zg}\,\mathsf{C} = \mathsf{Def}\,\mathsf{C} \cap \mathsf{Ind}\,A$. Note that

$$\mathsf{Zg}\,\mathsf{Def}\,\mathsf{C} = \mathsf{Zg}\,\mathsf{C} \quad\text{and}\quad \mathsf{Def}\,\mathsf{Zg}\,\mathsf{C} = \mathsf{Def}\,\mathsf{C}. \tag{2.1}$$

The first equality is clear from the definition; for the second one, see [6, §2.3] or [14, Corollary 6.9].

Given an additive subcategory C of $\mathsf{mod}\,A$, let $\varinjlim \mathsf{C}$ denote the full subcategory consisting of all A-modules that are filtered colimits of modules in C.

Proposition 2.2. *Let* C *be an additive subcategory of* $\mathsf{Mod}\,A$ *that is closed under submodules. Then*

$$\mathsf{Def}\,\mathsf{C} = \varinjlim(\mathsf{C} \cap \mathsf{mod}\,A) = \{X \in \mathsf{Mod}\,A \mid \mathsf{sub}\,X \subseteq \mathsf{C}\}.$$

Proof. We may assume that $\mathsf{C} \subseteq \operatorname{\mathsf{mod}} A$; the general case is then an immediate consequence. For each $X \in \operatorname{\mathsf{mod}} A$, let $X \to X_{\mathsf{C}}$ denote the universal morphism to an object of C. This is an epimorphism, since C is closed under submodules; take $X_{\mathsf{C}} = X/U$ where U denotes the minimal submodule with $X/U \in \mathsf{C}$. An A-module Y belongs to $\varinjlim \mathsf{C}$ if and only if each morphism $X \to Y$ with X finitely presented factors through the morphism $X \to X_{\mathsf{C}}$; see [7, Proposition 2.1]. It follows that an A-module Y belongs to $\varinjlim \mathsf{C}$ if and only if every finitely presented submodule belongs to C. From the same description, it is easily seen that $\varinjlim \mathsf{C}$ is closed under filtered colimits and products. Thus $\varinjlim \mathsf{C} = \operatorname{\mathsf{Def}} \mathsf{C}$. ☐

Corollary 2.3. *Let* $\mathsf{C} \subseteq \operatorname{\mathsf{Mod}} A$ *be a full additive subcategory closed under submodules. Then*

$$\operatorname{\mathsf{Zg}} \mathsf{C} = \{X \in \operatorname{\mathsf{Ind}} A \mid \operatorname{\mathsf{sub}} X \subseteq \mathsf{C}\}.$$

For a set of submodule closed full additive subcategories $\mathsf{C}_\alpha \subseteq \operatorname{\mathsf{Mod}} A$*, one has*

$$\operatorname{\mathsf{Zg}}(\bigcap_\alpha \mathsf{C}_\alpha) = \bigcap_\alpha \operatorname{\mathsf{Zg}} \mathsf{C}_\alpha.$$

Proof. The first part is clear from the preceding proposition. Now let $\mathsf{C} = \bigcap_\alpha \mathsf{C}_\alpha$. We need to check that $\operatorname{\mathsf{Zg}} \mathsf{C} \supseteq \bigcap_\alpha \operatorname{\mathsf{Zg}} \mathsf{C}_\alpha$ while the other inclusion is clear. Fix a module Y in $\bigcap_\alpha \operatorname{\mathsf{Zg}} \mathsf{C}_\alpha$. A finitely presented submodule of Y belongs to C_α for all α, and therefore it belongs to C. Thus Y is in $\operatorname{\mathsf{Zg}} \mathsf{C}$. ☐

The following example shows that in the preceding corollary the assumption on each C_α to be submodule closed is necessary.

Example 2.4. Let A be a tame hereditary algebra. Given any tube C of the AR-quiver, $\operatorname{\mathsf{Zg}} \mathsf{C}$ contains the unique generic A-module [5, Corollary 8.6]. Thus we have for two diferent tubes $\mathsf{C}_1, \mathsf{C}_2$ that $\operatorname{\mathsf{Zg}} \mathsf{C}_1 \cap \operatorname{\mathsf{Zg}} \mathsf{C}_2 \neq \varnothing$, while $\mathsf{C}_1 \cap \mathsf{C}_2 = \varnothing$.

For each class C of A-modules, let $\operatorname{\mathsf{sub}} \mathsf{C}$ denote the full subcategory consisting of all finitely presented submodules of finite direct sums of modules in C.

Corollary 2.5. *Let* C *be a class of* A*-modules. Then*

$$\operatorname{\mathsf{sub}} \mathsf{C} = \operatorname{\mathsf{sub}} \operatorname{\mathsf{Zg}} \mathsf{C} = \operatorname{\mathsf{sub}} \operatorname{\mathsf{Def}} \mathsf{C}.$$

Proof. We apply Proposition 2.2 and get

$$\operatorname{\mathsf{sub}} \mathsf{C} \subseteq \operatorname{\mathsf{sub}} \operatorname{\mathsf{Def}} \mathsf{C} \subseteq \operatorname{\mathsf{sub}} \varinjlim \operatorname{\mathsf{sub}} \mathsf{C} = \operatorname{\mathsf{sub}} \mathsf{C}.$$

Combining this identity with (2.1) gives

$$\operatorname{\mathsf{sub}} \operatorname{\mathsf{Zg}} \mathsf{C} = \operatorname{\mathsf{sub}} \operatorname{\mathsf{Def}} \operatorname{\mathsf{Zg}} \mathsf{C} = \operatorname{\mathsf{sub}} \operatorname{\mathsf{Def}} \mathsf{C} = \operatorname{\mathsf{sub}} \mathsf{C}. \qquad ☐$$

Corollary 2.6. *Let* $f\colon \operatorname{\mathsf{Mod}} A \to \mathsf{S}$ *be a map to a complete partially ordered set* S *satisfying the conditions (1)–(3) from Proposition 2.1. Then*

$$f(X) = \bigvee_{Y \in \operatorname{\mathsf{Zg}} X} f(Y) \qquad \textit{for all } X \in \operatorname{\mathsf{Mod}} A.$$

Proof. From Corollary 2.5 one has

$$\operatorname{\mathsf{sub}} X = \operatorname{\mathsf{sub}} \operatorname{\mathsf{Zg}} X = \bigvee_{Y \in \operatorname{\mathsf{Zg}} X} \operatorname{\mathsf{sub}} Y.$$

Using the map $\bar{f}\colon \mathsf{S}(\operatorname{\mathsf{mod}} A) \to \mathsf{S}$ from Proposition 2.1, one gets

$$f(X) = \bar{f}(\operatorname{\mathsf{sub}} X) = \bar{f}\Big(\bigvee_{Y \in \operatorname{\mathsf{Zg}} X} \operatorname{\mathsf{sub}} Y\Big) = \bigvee_{Y \in \operatorname{\mathsf{Zg}} X} \bar{f}(\operatorname{\mathsf{sub}} Y) = \bigvee_{Y \in \operatorname{\mathsf{Zg}} X} f(Y). \qquad ☐$$

3. The Gabriel–Roiter filtration

In this section we study a specific inclusion preserving map $\mathsf{Mod}\, A \to \mathsf{S}$, namely the Gabriel–Roiter measure. This map refines the usual length function $\mathsf{Mod}\, A \to \mathbb{N}$ and has the additional property that the set S is totally ordered.

The Gabriel–Roiter measure. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and denote by $\mathbf{2}^{\mathbb{N}}$ the set of all subsets of $\mathbb{N}$. We view this as a partially ordered set via the *lexicographical order*, given by

$$I \le J \quad \Longleftrightarrow \quad \inf(J \setminus I) \le \inf(I \setminus J) \qquad \text{for } I, J \in \mathbf{2}^{\mathbb{N}}.$$

Note that $\mathbf{2}^{\mathbb{N}}$ is totally ordered and complete.

Given an A-module X of finite length, let $\ell(X)$ denote the length of a composition series. Following [3, 12], the *Gabriel–Roiter measure* of an A-module X is

$$\mu(X) = \bigvee_{X_1 \subsetneq \dots \subsetneq X_r \subseteq X} \{\ell(X_1), \dots, \ell(X_r)\},$$

where $X_1 \subsetneq \dots \subsetneq X_r \subseteq X$ runs through all finite chains of submodules such that each X_i is indecomposable and of finite length. For a class C of A-modules, we write

$$\mu(\mathsf{C}) = \bigvee_{X \in \mathsf{C}} \mu(X).$$

The basic properties of the Gabriel–Roiter measure are summarised in the following statement. Note that these are precisely the properties appearing in Proposition 2.1.

Proposition 3.1. *Let X, Y be A-modules. Then*

1. $\mu(X) \le \mu(Y)$ *if* $X \subseteq Y$;
2. $\mu(X) = \bigvee_\alpha \mu(X_\alpha)$ *for every directed union* $X = \bigcup_\alpha X_\alpha$;
3. $\mu(X \oplus Y) = \mu(X) \vee \mu(Y)$.

Proof. (1) and (2) are clear from the definition of μ. (3) holds for finitely presented A-modules by [3, Corollary 5.3]. To prove the general case, write $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\beta Y_\beta$ as directed unions of finitely presented modules. Then

$$X \oplus Y = \bigcup_{(\alpha,\beta)} X_\alpha \oplus Y_\beta$$

and therefore

$$\begin{aligned} \mu(X \oplus Y) &= \bigvee_{(\alpha,\beta)} \mu(X_\alpha \oplus Y_\beta) \\ &= \bigvee_{(\alpha,\beta)} \mu(X_\alpha) \vee \mu(Y_\beta) \\ &= \big(\bigvee_\alpha \mu(X_\alpha)\big) \vee \big(\bigvee_\beta \mu(Y_\beta)\big) \\ &= \mu(X) \vee \mu(Y). \end{aligned}$$ □

Corollary 3.2. *Let C be a class of A-modules. Then*

$$\mu(\mathsf{C}) = \mu(\mathsf{sub}\, \mathsf{C}) = \mu(\mathsf{Zg}\, \mathsf{C}) = \mu(\mathsf{Def}\, \mathsf{C}).$$

Proof. The first identity follows from Proposition 3.1. The rest then follows by Corollary 2.5. □

It seems to be an interesting question to ask, whether each element $I = \mu(X)$ in the image of $\mu\colon \mathsf{Mod}\,A \to \mathbf{2}^{\mathbb{N}}$ is of the form $I = \mu(Y)$ for some indecomposable pure-injective A-module Y.

The Gabriel–Roiter filtration. The following proposition yields a collection of (not necessarily distinct) Ziegler closed subsets of $\mathsf{Ind}\,A$ which is indexed by the elements of $\mathbf{2}^{\mathbb{N}}$. For each $I \in \mathbf{2}^{\mathbb{N}}$, set

$$\mathsf{Zg}\,I = \{X \in \mathsf{Ind}\,A \mid \mu(X) \le I\} \quad \text{and} \quad \mathsf{sub}\,I = \{X \in \mathsf{mod}\,A \mid \mu(X) \le I\}.$$

Proposition 3.3. *Let $I \in \mathbf{2}^{\mathbb{N}}$.*

1. *The set $\mathsf{Zg}\,I$ is Ziegler closed and the subcategory $\mathsf{sub}\,I$ is additive and submodule closed.*
2. *If $I = \mu(X)$ for some A-module X, then*
$$\mu(\mathsf{Zg}\,I) = I \quad \text{and} \quad \mu(\mathsf{sub}\,I) = I.$$
3. *For each subset $\mathsf{U} \subseteq \mathsf{Ind}\,A$, one has*
$$\mu(\mathsf{U}) \le I \quad \iff \quad \mathsf{U} \subseteq \mathsf{Zg}\,I.$$
4. *For each subcategory $\mathsf{C} \subseteq \mathsf{mod}\,A$, one has*
$$\mu(\mathsf{C}) \le I \quad \iff \quad \mathsf{C} \subseteq \mathsf{sub}\,I.$$

Proof. The A-modules X satisfying $\mu(X) \le I$ form an additive subcategory of $\mathsf{Mod}\,A$ that is closed under submodules, by Proposition 3.1. In fact, these modules form a definable subcategory, by Proposition 2.2, and therefore $\mathsf{Zg}\,I$ is Ziegler closed. The rest is clear from the definitions of $\mathsf{Zg}\,I$ and $\mathsf{sub}\,I$. ☐

We shorten our notation and set $\mathsf{V}_I = \mathsf{Zg}\,I$ for each $I \in \mathbf{2}^{\mathbb{N}}$.

Corollary 3.4. *There is a filtration $(\mathsf{V}_I)_{I\in\mathbf{2}^{\mathbb{N}}}$ of $\mathsf{Ind}\,A$ consisting of Ziegler closed subsets such that the following holds:*

1. *$\mathsf{V}_I \subseteq \mathsf{V}_J$ for all $I \le J$ in $\mathbf{2}^{\mathbb{N}}$;*
2. *$\mathsf{V}_{\inf \mathsf{S}} = \bigcap_{I\in\mathsf{S}} \mathsf{V}_I$ for all $\mathsf{S} \subseteq \mathbf{2}^{\mathbb{N}}$;*
3. *$\mu(\mathsf{V}_I) \le I$ for all $I \in \mathbf{2}^{\mathbb{N}}$, and equality holds if and only if $I = \mu(X)$ for some A-module X.* ☐

The partially ordered set of Ziegler closed sets. We denote by $\mathsf{Cl}(\mathsf{Ind}\,A)$ the set of Ziegler closed subsets of $\mathsf{Ind}\,A$; they form a complete partially ordered set. Corollary 2.6 says that the map taking an A-module X to $\mathsf{Zg}\,X$ is universal in the sense that any map $f\colon \mathsf{Mod}\,A \to \mathsf{S}$ to a complete partially ordered set satisfying the conditions (1)–(3) from Proposition 2.1 satisfies

$$f(X) = \bigvee_{Y\in\mathsf{Zg}\,X} f(Y).$$

The basic examples of such assignments are $X \mapsto \mathsf{sub}\,X$ and $X \mapsto \mu(X)$. This yields the following diagram:

$$\begin{tikzcd}
& \mathsf{Mod}\,A \arrow[dl, "\mathsf{Zg}"'] \arrow[d, "\mathsf{sub}"] \arrow[dr, "\mu"] & \\
\mathsf{Cl}(\mathsf{Ind}\,A) \arrow[r, shift left, "\mathsf{sub}"] & \mathsf{S}(\mathsf{mod}\,A) \arrow[l, shift left, "\mathsf{Zg}"] \arrow[r, shift left, "\mu"] & \mathbf{2}^{\mathbb{N}} \arrow[l, shift left, "\mathsf{sub}"]
\end{tikzcd}$$

Here, we write

$$\begin{tikzcd}
\mathsf{S} \arrow[r, shift left, "f"] & \mathsf{T} \arrow[l, shift left, "g"]
\end{tikzcd}$$

for an *adjoint pair* of morphisms between partially ordered sets which means that

$$f(x) \leq y \quad \Longleftrightarrow \quad x \leq g(y) \qquad \text{for all } x \in \mathsf{S},\ y \in \mathsf{T}.$$

The adjointness of the pair $(\mathsf{sub}, \mathsf{Zg})$ follows from Corollary 2.3; for (μ, sub) it follows from Proposition 3.3.

We say that a morphism $f\colon \mathsf{S} \to \mathsf{T}$ is a *quotient map* if f induces an isomorphism $\mathsf{S}/_\sim \to \mathsf{T}$, where $x \sim y$ iff $f(x) = f(y)$ for $x, y \in \mathsf{S}$. An equivalent condition is that $fg = \mathrm{id}_\mathsf{T}$; see [4, Proposition I.1.3].

Let us denote by $\mathsf{GR}(A)$ the image of $\mu\colon \mathsf{Mod}\, A \to \mathbf{2}^\mathbb{N}$. This is a complete partially ordered set.

Proposition 3.5. *The morphisms*

$$\mathsf{sub}\colon \mathsf{Cl}(\mathsf{Ind}\, A) \longrightarrow \mathsf{S}(\mathsf{mod}\, A) \quad \textit{and} \quad \mu\colon \mathsf{S}(\mathsf{mod}\, A) \longrightarrow \mathsf{GR}(A)$$

are quotient maps.

Proof. We have $\mathsf{sub}\,\mathsf{Zg}\,\mathsf{C} = \mathsf{C}$ for each $\mathsf{C} \in \mathsf{S}(\mathsf{mod}\, A)$, by Corollary 2.5. On the other hand, $\mu(\mathsf{sub}\, I) = I$ for each $I \in \mathsf{GR}(A)$, by Proposition 3.3. □

Given a pair of Ziegler closed subsets U, V of $\mathsf{Ind}\, A$, when is $\mathsf{sub}\,\mathsf{U} = \mathsf{sub}\,\mathsf{V}$? This amounts to computing $\mathsf{Zg}\,\mathsf{sub}\,\mathsf{U}$, since

$$\mathsf{sub}\,\mathsf{U} = \mathsf{sub}\,\mathsf{V} \quad \Longleftrightarrow \quad \mathsf{Zg}\,\mathsf{sub}\,\mathsf{U} = \mathsf{Zg}\,\mathsf{sub}\,\mathsf{V}.$$

Note that

$$\mathsf{V} \subseteq \mathsf{Zg}\,\mathsf{sub}\,\mathsf{V}$$

holds automatically; we describe when equality holds.

Proposition 3.6. *Let* C *be a definable subcategory of* $\mathsf{Mod}\, A$ *and* $\mathsf{V} = \mathsf{C} \cap \mathsf{Ind}\, A$ *the corresponding Ziegler closed set. Then the following are equivalent:*

1. C *is closed under submodules;*
2. V *is closed under submodules:* $X \in \mathsf{V}$*,* $Y \in \mathsf{Ind}\, A$*, and* $Y \subseteq X^n$ *for some* $n \in \mathbb{N}$ *implies* $Y \in \mathsf{V}$*;*
3. $\mathsf{V} = \mathsf{Zg}\,\mathsf{sub}\,\mathsf{V}$.

Proof. (1) ⇒ (2): Clear.

(2) ⇒ (3): That V is closed under submodules implies $\mathsf{Def}\,\mathsf{V} = \mathsf{Def}\,\mathsf{sub}\,\mathsf{V}$. Using (2.1) then gives

$$\mathsf{V} = \mathsf{Zg}\,\mathsf{Def}\,\mathsf{V} = \mathsf{Zg}\,\mathsf{Def}\,\mathsf{sub}\,\mathsf{V} = \mathsf{Zg}\,\mathsf{sub}\,\mathsf{V}.$$

(3) ⇒ (1): The equality in (3) yields

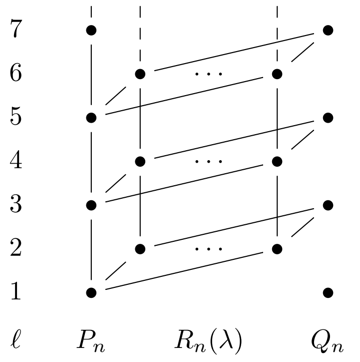
$$\mathsf{C} = \mathsf{Def}\,\mathsf{V} = \mathsf{Def}\,\mathsf{Zg}\,\mathsf{sub}\,\mathsf{V} = \mathsf{Def}\,\mathsf{sub}\,\mathsf{V} = \mathsf{Def}\,\mathsf{sub}\,\mathsf{Def}\,\mathsf{V} = \mathsf{Def}\,\mathsf{sub}\,\mathsf{C}.$$

Here, (2.1) and Corollary 2.5 are used. The equality $\mathsf{C} = \mathsf{Def}\,\mathsf{sub}\,\mathsf{C}$ implies that C is closed under submodules. □

The Kronecker algebra. Let $\Lambda = \left[\begin{smallmatrix} k & k^2 \\ 0 & k \end{smallmatrix}\right]$ be the Kronecker algebra over an algebraically closed field k. A complete list of the indecomposables in $\mathsf{mod}\,\Lambda$ is given by the preprojectives P_n, the regulars $R_n(\lambda)$, and the preinjectives Q_n; see [1, Thm. VIII.7.5]. More precisely,

$$\mathsf{Ind}\,\Lambda \cap \mathsf{mod}\,\Lambda = \{P_n \mid n \in \mathbb{N}\} \cup \{R_n(\lambda) \mid n \in \mathbb{N},\ \lambda \in \mathbb{P}^1(k)\} \cup \{Q_n \mid n \in \mathbb{N}\},$$

and the inclusion order is described by the following Hasse diagram.



From this, one computes

$$\begin{aligned}\mu(P_n) &= \{1,3,5,\ldots,2n-1\}\\ \mu(R_n) &= \{1,2,4,\ldots,2n\}\\ \mu(Q_n) &= \{1,2,4,\ldots,2n-2,2n-1\}\end{aligned}$$

where the Gabriel–Roiter measure of $R_n = R_n(\lambda)$ does not depend on λ. This gives the following order:

$$\mu(Q_1)=\mu(P_1)<\mu(P_2)<\mu(P_3)< \ \ldots\ <\mu(R_1)<\mu(R_2)<\mu(R_3)< \ \ldots\ <\mu(Q_4)<\mu(Q_3)<\mu(Q_2)$$

The indecomposable pure-injective Λ-modules which are not finitely presented are the Prüfer modules $R_\infty(\lambda) = \varinjlim R_n(\lambda)$, the adic modules $\widehat{R}(\lambda) = \varprojlim R_n(\lambda)$, and the generic module G; see [9, 11]. Thus

$$\mathsf{Ind}\,\Lambda \setminus \mathsf{mod}\,\Lambda = \{R_\infty(\lambda), \widehat{R}(\lambda) \mid \lambda \in \mathbb{P}^1(k)\} \cup \{G\}.$$

Now one computes

$$\begin{aligned}\mu(\widehat{R}(\lambda)) = \mu(G) = \{1,3,5,7,\ldots\} &= \bigvee_{n\geq 1} \mu(P_n)\\ \mu(R_\infty(\lambda)) = \{1,2,4,6,\ldots\} = \bigvee_{n\geq 1} \mu(R_n) &= \bigwedge_{n\geq 1} \mu(Q_n)\end{aligned}$$

and this completes the list of values of the Gabriel–Roiter measure; see also [12, Appendix B]. Note that this yields the description of the Gabriel–Roiter filtration of $\mathsf{Ind}\,\Lambda$.

4. Compactness

The collection of submodule closed additive subcategories of $\mathsf{mod}\,A$ enjoys a compactness property which we discuss in this section. A consequence is the existence of minimal submodule closed subcategories of infinite type. This is a somewhat surprising result from a recent article of Ringel [13]. Note that the proof given here is quite different from Ringel's. He uses the Gabriel–Roiter measure, while the compactness result is derived from the compactness of the Ziegler spectrum.

Let C be an additive subcategory of $\mathsf{mod}\,A$ which is closed under direct summands. We say that C is of *finite type* if C contains only finitely many pairwise non-isomorphic indecomposable modules. Note that a submodule closed subcategory C is of finite type if and only if the set

$$\{\mathsf{D} \in \mathsf{S}(\mathsf{mod}\,A) \mid \mathsf{D} \subseteq \mathsf{C}\}$$

is finite.

Theorem 4.1. *Let* $(\mathsf{C}_\alpha)_{\alpha\in\Lambda}$ *be a collection of additive subcategories* $\mathsf{C}_\alpha \subseteq \operatorname{mod} A$ *that are submodule closed. If* $\mathsf{C} = \bigcap_{\alpha\in\Lambda} \mathsf{C}_\alpha$ *is of finite type, then there is a finite subset* $\Lambda' \subseteq \Lambda$ *such that* $\mathsf{C} = \bigcap_{\alpha\in\Lambda'} \mathsf{C}_\alpha$.

The proof uses some properties of the Ziegler spectrum which are collected in the following proposition. For a general introduction, we refer the reader to [6, 10].

Proposition 4.2. *The space* $\operatorname{Ind} A$ *has the following properties.*

(1) *The space* $\operatorname{Ind} A$ *is quasi-compact.*
(2) *For* $X \in \operatorname{Ind} A \cap \operatorname{mod} A$*, the subset* $\{X\}$ *is open.*
(3) *An additive subcategory* $\mathsf{C} \subseteq \operatorname{mod} A$ *is of finite type iff* $\operatorname{Zg} \mathsf{C} \subseteq \operatorname{mod} A$.

Proof. (1) See [14, Theorem 4.9] or [2, §2.5].

(2) See [8, Proposition 13.1].

(3) If C is of finite type, then the direct sums of modules in C form a definable subcategory; see [2, §2.5]. Thus $\operatorname{Zg} \mathsf{C} \subseteq \operatorname{mod} A$. If C is of infinite type, then part (1) and (2) imply that $\operatorname{Zg} \mathsf{C}$ contains modules which are not finitely presented. □

Proof of Theorem 4.1. We have $\operatorname{Zg} \mathsf{C} = \bigcap_{\alpha\in\Lambda} \operatorname{Zg} \mathsf{C}_\alpha$ by Corollary 2.3. Using the properties of $\operatorname{Ind} A$ collected in Proposition 4.2, it follows that $\operatorname{Zg} \mathsf{C} = \bigcap_{\alpha\in\Lambda'} \operatorname{Zg} \mathsf{C}_\alpha$ for some finite subset $\Lambda' \subseteq \Lambda$. We have $\operatorname{sub} \operatorname{Zg} \mathsf{D} = \mathsf{D}$ for each submodule closed additive subcategory $\mathsf{D} \subseteq \operatorname{mod} A$, by Corollary 2.5. Thus $\mathsf{C} = \bigcap_{\alpha\in\Lambda'} \mathsf{C}_\alpha$. □

A combination of Theorem 4.1 with Zorn's lemma gives the following result, and Ringel's theorem [13] mentioned in the introduction is an immediate consequence.

Corollary 4.3. *Let* S *be a set of submodule closed additive subcategories of* $\operatorname{mod} A$ *that is closed under forming intersections. Then the subset of* S *consisting of all subcategories of infinite type is either empty or it has a minimal element.* □

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Henning Krause, Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany.
E-mail address: `hkrause@math.uni-bielefeld.de`

Mike Prest, School of Mathematics, Alan Turing Building, University of Manchester, Manchester M13 9PL, United Kingdom.
E-mail address: `mprest@manchester.ac.uk`