

Markov Dynamics in a Spatial Ecological Model with Dispersion and Competition

Dmitri Finkelshtein* Yuri Kondratiev[†] Yuri Kozitsky[‡]
Oleksandr Kutoviy[§]

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Abstract

The evolution of an individual-based spatial ecological model with dispersion and competition is studied. In the model, an infinite number of individuals – point particles in \mathbb{R}^d – reproduce themselves, compete, and die at random. These events are described by a Markov generator, which determines the evolution of states understood as probability measures on the space of particle configurations. The main result is a statement that the corresponding correlation functions evolve in a scale of Banach spaces and remain sub-Poissonian, and hence no clustering occurs, if the dispersion is subordinate to the competition.

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1 Introduction and Overview

1.1 Introduction

Individual-based models of large ecological systems evolving in continuous space and time attract considerable attention of both mathematicians and theoretical biologists, see, e.g., [15–17] and [4–8, 24], respectively. In this paper, we study the model introduced and discussed in [5–8, 24] which describes a population of individuals (e.g., perennial plants) distributed over an infinite habitat, \mathbb{R}^d in our case. The individuals are point particles which reproduce themselves,

*Institute of Mathematics, National Academy of Sciences of Ukraine, 01601 Kiev-4, Ukraine, e-mail: fdl@imath.kiev.ua

[†]Fakultät für Mathematik, Universität Bielefeld, Postfach 110 131, 33501 Bielefeld, Germany, e-mail: kondrat@math.uni-bielefeld.de

[‡]Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, 20-031 Lublin, Poland, e-mail: jkozi@hektor.umcs.lublin.pl

[§]Fakultät für Mathematik, Universität Bielefeld, Postfach 110 131, 33501 Bielefeld, Germany, e-mail: kutoviy@math.uni-bielefeld.de

compete, and die at random, independently with constant (density-independent) mortality rate $m \geq 0$. The reproduction consists in random (independent) sending by a particle located at x a seed to point y , which immediately after that becomes a population member. This process is described by a *dispersal kernel*, $a_+(x, y) \geq 0$. In addition to the independent death each particle can die under the influence of the rest of population, so called density dependent mortality. This process is described by a *competition kernel*, $a_-(x, y) \geq 0$. The kernels a_{\pm} determine the corresponding rates in an additive way. For instance, for the particle located at x , the overall density dependent mortality rate is $\sum_y a_-(x, y)$, where the sum is taken over all other population members.

In theoretical biology, the mentioned model has appeared in the form of ‘spatial moment equations’, see [5, 6]. These are chains of (linear) differential equations describing the time evolution of densities and higher order moments¹, representing ‘spatial covariances’. These equations involve the dispersal and mortality rates just mentioned. The main difficulty encountered by the authors of those and similar works is that the mentioned chains are not closed, e.g., the time derivative of the density is expressed through the two-point moment, whereas the time derivative of the two-point moment is expressed through the moments of higher order, etc. Typically, this difficulty is circumvented by means of a ‘moment closure’ ansatz, in which one approximates moments of order higher than a certain value by the products of lower order moments, e.g., the two-point moment is set to be the product of two densities, and thus the two-point covariances are neglected. The equations obtained in this way are closed but nonlinear. An example can be the Lotka–Volterra equations with spatial dependence derived in [6]. At the same time, dozens of such equations of population biology are introduced heuristically, without employing individual-based models, see, e.g., Chapter 9 in [31] or [11], where nonlocal models of this type are considered. As is well-understood now, such equations play here the role of kinetic equations known in the statistical theory of Hamiltonian dynamical systems. Nowadays, for Hamiltonian systems such equations are derived from microscopic equations by means of a scaling procedure, see the discussion in [9] for more detail. We believe that also in population biology, the use of individual-based models in the appropriate mathematical framework will provide a more reliable theory of considered systems, and hence will result in deeper understanding their behavior. In that we mean the following program realized in part in the present paper. The model dynamics is described on micro- and mesoscopic levels. On the microscopic level, the dynamics is described as the evolution of states understood as probability measures on the corresponding phase space. As in the case of Hamiltonian systems, this evolution can be constructed indirectly as the evolution of the corresponding correlation functions. The mesoscopic description is then obtained by a scaling procedure, in which the correlation functions converge to ‘mesoscopic’ correlation functions. By virtue of the scaling procedure, the evolution of the latter functions ‘preserves chaos’, which means that, at each moment of time, such functions are the products of

¹These moments correspond to correlation functions used in this article.

density functions if the initial correlation functions possess this property. The corresponding kinetic equation is the equation for the density function. Typically, this is a nonlinear and nonlocal equation. We refer the reader to [12, 14] for more detail. Generally speaking, the aim of the present paper is to go further in developing the microscopic description of the model mentioned above comparing to what was done in [13, 15]. A more specific presentation of our aims and the results obtained in this article is given in the next subsection, see also the concluding remarks in Section 6.

As was suggested already in [5], the right mathematical context for studying individual-based models of ecological systems is the theory of random point fields in \mathbb{R}^d . Herein, populations appear as particle configurations constituting the configuration space

$$\Gamma \equiv \Gamma(\mathbb{R}^d) := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}, \quad (1.1)$$

where $|A|$ stands for the cardinality of A . In the terminology used in the theory of dynamical systems, Γ is a phase space. Noteworthy, along with finite ones Γ contains also infinite configurations, which allows for describing ‘bulk’ properties of the model ignoring boundary and size effects. If the initial configuration γ_0 is fixed, the system dynamics might be described as a map $t \mapsto \gamma_t \in \Gamma$, which in view of the random character of the events mentioned above ought to be a random process. However, for the model considered here this way can be realized only if γ_0 is finite. [17] In the present article, we follow the way known in the statistical theory of large systems [9] in which the system states are described in terms of probability measures rather than point-wise. Thus, in our case the states are probability measures on Γ . To characterize them one employs *observables*, which are appropriate functions $F : \Gamma \rightarrow \mathbb{R}$. The quantity

$$\langle\langle F, \mu \rangle\rangle = \int_{\Gamma} F(\gamma) \mu(d\gamma)$$

is called the value of observable F in state μ . Then the system evolution might be described as the evolution of observables obtained from the Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \quad F_t|_{t=0} = F_0, \quad t > 0, \quad (1.2)$$

where the ‘generator’ L specifies the model. The evolution of states is obtained from the Fokker–Planck equation

$$\frac{d}{dt} \mu_t = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0, \quad (1.3)$$

related to (1.2) by the duality

$$\langle\langle F_0, \mu_t \rangle\rangle = \langle\langle F_t, \mu_0 \rangle\rangle.$$

However, for the model considered in this article, even the mere definition of equations (1.2) and (1.3) in appropriate spaces is rather impossible as the phase

space Γ contains infinite configurations. Following classical works on the Hamiltonian dynamics [3, 9, 28] one can try to study the evolution $\mu_0 \mapsto \mu_t$ via the evolution of the corresponding correlation functions. For a measure μ and a bounded measurable $\Lambda \subset \mathbb{R}^d$, the probability that in state μ there is m particles in Λ can be expressed through the n -point correlation functions $k_\mu^{(n)}$ with $n \geq m$, see, e.g., [27]. In particular, for the Poisson measure π_\varkappa , see subsection 2.1.2 below, $k_{\pi_\varkappa}^{(n)}(x_1, \dots, x_n) = \varkappa^n$ for all $n \in \mathbb{N}_0$.

In general, the correlation function k_μ is a collection of symmetric functions $k_\mu^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$, and $k_\mu^{(1)}$ is the particle density. The evolution $k_0 \mapsto k_t$ is obtained from the equation

$$\frac{d}{dt}k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_0, \quad (1.4)$$

which, in fact, is a chain of equations² for particular $k_t^{(n)}$. Here L^Δ is constructed from L as in (1.2) by a certain procedure. According to our program, the microscopic description consists in proving the existence of solutions of (1.4) in appropriate Banach spaces, and in studying their properties. The next step is to show that these solutions are correlation functions, i.e., are positive definite in a certain sense, see Proposition 2.1. An important property of k_t is being *sub-Poissonian*, which means that, for some $C > 0$, each $k_t^{(n)}$ is bounded by C^n . This property can be guaranteed by the appropriate choice of the Banach space where one solves (1.4). Note that the increase of $k_t^{(n)}$ with n as $n!$ (see (1.7) below), would correspond to the cluster formation due to dispersion. In the present article, we especially address the question concerning the role of the competition in preventing such clustering.

1.2 The overview

For the considered model, the ‘generator’ in (1.2) reads

$$\begin{aligned} (LF)(\gamma) &= \sum_{x \in \gamma} [m + E^-(x, \gamma \setminus x)] [F(\gamma \setminus x) - F(\gamma)] \\ &+ \int_{\mathbb{R}^d} E^+(y, \gamma) [F(\gamma \cup y) - F(\gamma)] dy, \end{aligned} \quad (1.5)$$

where

$$E^\pm(x, \gamma) := \sum_{y \in \gamma} a_\pm(x, y). \quad (1.6)$$

This is a typical ‘birth-and-death’ generator, in which the first term corresponds to the death of the particle located at x occurring (a) independently with rate $m \geq 0$, and (b) under the influence of the other particles in γ with rate $E^-(x, \gamma \setminus x) \geq 0$. Here and in the sequel in the corresponding context, we treat each $x \in \mathbb{R}^d$ also as a single-point configuration $\{x\}$. Note that $E^-(x, \gamma \setminus x)$ describes

²In the Hamiltonian dynamics, the analog of (1.4) is known as the BBGKY hierarchy.

the interparticle competition. The second term in (1.5) describes the birth of a particle at $y \in \mathbb{R}^d$ given by the whole configuration γ with rate $E^+(y, \gamma) \geq 0$. In this article, the model described by (1.5) will be called a *spatial ecological model (SEM)*. A particular case of SEM is the continuous contact model [22,23] where $a_- \equiv 0$, and hence the competition is absent, see also [15].

As was mentioned above, the definition of L as an operator in an appropriate Banach space cannot be done directly. This, however, can be done for L^Δ which appears in (1.4). Thus, the basic equation by means of which we study the dynamics of our model is the Cauchy problem (1.4) in the corresponding Banach spaces. The main characteristic feature of such spaces is that they contain only sub-Poissonian correlation functions. Note that for the contact model, mentioned above, it is known that, [15]

$$\text{const} \cdot n! c_t^n \leq k_t^{(n)}(x_1, \dots, x_n) \leq \text{const} \cdot n! C_t^n, \quad (1.7)$$

where the left-hand inequality holds if all x_i belong to a ball of small enough radius. Hence, in spite of the fact that $C_t \rightarrow 0$ as $t \rightarrow +\infty$ if the mortality dominates the dispersion (as in (5.11) below), k_t are definitely not sub-Poissonian if $a_- \equiv 0$. According to Theorem 4.2 below, if, for some $\theta > 0$, we have that $a_+ \leq \theta a_-$ pointwise, cf. (3.12), then (1.4) has a unique (classical) sub-Poissonian solution on a bounded time interval. A solution of (1.4) is the correlation function of a unique probability measure on Γ if it possesses a certain positivity property. In Theorem 5.4, we show that the solution k_t , existing according to Theorem 4.2, has this property if m dominates a_+ in the sense of (5.11).

The rest of the paper is organized as follows. In Section 2, we introduce a necessary mathematical framework and give a formal definition of the model. The evolution of correlation functions is studied in Section 4. It is, however, preceded by the study of certain auxiliary objects, *quasi-observables*, the evolution of which is generated by \widehat{L} whose dual, in the sense of (2.25), is L^Δ . This is done in Section 3, where we use a combination of C_0 -semigroup techniques in ordered Banach spaces with an Ovchinnikov-type method, which yields the evolution of quasi-observables in a scale of Banach spaces on a bounded time-interval, see Theorem 3.4. The main peculiarity of the evolution $k_0 \mapsto k_t$ described in Theorem 4.2 is that the corresponding Banach spaces are of L^∞ -type, which forced us to use a combination of C_0 -semigroups, sun-dual to those from Section 3, with Ovchinnikov's method. In Section 4, in addition to the classical solutions of the Cauchy problem for correlation functions and quasi-observables, we also study the dual evolutions defined in (4.22) and (4.23). Similarly to the usual C_0 -semigroup framework, we obtain that the classical evolution of correlation functions coincides with the evolution which is dual to the evolution of quasi-observables, see Proposition 5.6. The results of Section 4 are used for proving Theorem 5.4 concerning the dynamics of states. Another ingredient of our study of the dynamics of states is Lemma 5.1 where the dynamics of local densities is described. Note that the latter evolution might be extended to the evolution of states supported on finite configurations, that provides an alternative way of

constructing the evolution $\gamma_0 \mapsto \gamma_t$ mentioned above. We expect to realize this approach in the forthcoming paper. The concluding remarks are presented in Section 6.

2 The Basic Notions and the Model

2.1 The notions

All the details of the mathematical framework of this paper can be found in [2, 15, 16, 18, 20, 22, 23, 25]. Recall that we consider an infinite system of point particles distributed over \mathbb{R}^d , $d \geq 1$. By $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_b(\mathbb{R}^d)$ we denote the set of all Borel and all bounded Borel subsets of \mathbb{R}^d , respectively.

2.1.1 The configuration spaces

The configuration space Γ has been defined in (1.1). Any $\gamma \in \Gamma$ can be identified with the following positive integer-valued Radon measure

$$\gamma(dx) = \sum_{y \in \gamma} \delta_y(dx) \in \mathcal{M}(\mathbb{R}^d),$$

where δ_y is the Dirac measure centered at y , and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on $\mathcal{B}(\mathbb{R}^d)$. This allows us to consider Γ as the subset of $\mathcal{M}(\mathbb{R}^d)$, and hence to endow it with the vague topology. By definition, the vague topology is the weakest topology in which all the maps

$$\Gamma \ni \gamma \mapsto \int_{\mathbb{R}^d} f(x) \gamma(dx) = \sum_{x \in \gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here $C_0(\mathbb{R}^d)$ stands for the set of all continuous functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ which have compact supports. The vague topology on Γ admits a metrization which turns it into a complete and separable metric (Polish) space, see, e.g., Theorem 3.5 in [20]. By $\mathcal{B}(\Gamma)$ we denote the corresponding Borel σ -algebra.

The set of n -particle configurations in \mathbb{R}^d , $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, is

$$\Gamma^{(0)} = \{\emptyset\}, \quad \Gamma^{(n)} = \{\eta \subset X : |\eta| = n\}, \quad n \in \mathbb{N}.$$

For $n \geq 2$, $\Gamma^{(n)}$ can be identified with the symmetrization of the set

$$\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j, \text{ for } i \neq j\},$$

which allows one to introduce the corresponding topology on $\Gamma^{(n)}$ and hence the Borel σ -algebra $\mathcal{B}(\Gamma^{(n)})$. The set of finite configurations Γ_0 is the disjoint union of $\Gamma^{(n)}$, that is,

$$\Gamma_0 = \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}.$$

We endow Γ_0 with the topology of the disjoint union and hence with the Borel σ -algebra $\mathcal{B}(\Gamma_0)$. Obviously, Γ_0 can also be considered as a subset of Γ . However, the topology just mentioned and that induced on Γ_0 from Γ do not coincide.

In the sequel, $\Lambda \subset \mathbb{R}^d$ will always denote a bounded measurable subset, i.e., $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. For such Λ , we set

$$\Gamma_\Lambda = \{\gamma \in \Gamma : \gamma = \gamma \cap \Lambda\}.$$

Clearly, Γ_Λ is a measurable subset of Γ_0 and the following holds

$$\Gamma_\Lambda = \bigsqcup_{n \in \mathbb{N}_0} \Gamma_\Lambda^{(n)}, \quad \Gamma_\Lambda^{(n)} := \Gamma^{(n)} \cap \Gamma_\Lambda,$$

which allows one to equip Γ_Λ with the topology induced by that of Γ_0 . Let $\mathcal{B}(\Gamma_\Lambda)$ be the corresponding Borel σ -algebra. It can be proven, see Lemma 1.1 and Proposition 1.3 in [25], that

$$\mathcal{B}(\Gamma_\Lambda) = \{\Gamma_\Lambda \cap \Upsilon : \Upsilon \in \mathcal{B}(\Gamma)\}.$$

Next, we define the projection

$$\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda := \gamma \cap \Lambda, \quad \Lambda \in \mathcal{B}_b(\mathbb{R}^d). \quad (2.1)$$

It is known, cf. page 451 in [2], that $\mathcal{B}(\Gamma)$ is the smallest σ -algebra of subsets of Γ such that the maps p_Λ with all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ are $\mathcal{B}(\Gamma)/\mathcal{B}(\Gamma_\Lambda)$ measurable. This means that $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

2.1.2 Measures and functions on configuration spaces

The basic examples of measures on Γ and Γ_0 are the Poisson measure π and the Lebesgue–Poisson measure λ , respectively, cf. Section 2.2 in [2].

The image of the Lebesgue product measure $dx_1 dx_2 \cdots dx_n$ in $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$ is denoted by $\sigma^{(n)}$. For $\varkappa > 0$, the Lebesgue–Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is

$$\lambda_\varkappa := \delta_\emptyset + \sum_{n=1}^{\infty} \frac{\varkappa^n}{n!} \sigma^{(n)}. \quad (2.2)$$

For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, the restriction of λ_\varkappa to Γ_Λ will be denoted by $\lambda_\varkappa^\Lambda$. However, we shall drop the superscript if no ambiguity arises. Clearly, $\lambda_\varkappa^\Lambda$ is a finite measure on $\mathcal{B}(\Gamma_\Lambda)$ such that $\lambda_\varkappa^\Lambda(\Gamma_\Lambda) = e^{\varkappa \ell(\Lambda)}$, where $\ell(\Lambda)$ is the Lebesgue measure of Λ . Then

$$\pi_\varkappa^\Lambda := \exp(-\varkappa \ell(\Lambda)) \lambda_\varkappa^\Lambda \quad (2.3)$$

is a probability measure on $\mathcal{B}(\Gamma_\Lambda)$. It can be shown [2] that the family $\{\pi_\varkappa^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$ is consistent, and hence there exists a unique probability measure, π_\varkappa , on $\mathcal{B}(\Gamma)$ such that

$$\pi_\varkappa^\Lambda = \pi_\varkappa \circ p_\Lambda^{-1}, \quad \Lambda \in \mathcal{B}_b(\mathbb{R}^d),$$

where p_Λ is as in (2.1). This π_\varkappa is called the Poisson measure with intensity $\varkappa > 0$. If $\varkappa = 1$ we shall drop the subscript and consider the Lebesgue–Poisson measure λ and the Poisson measure π .

Now we turn to functions on Γ_0 and Γ . In fact, any measurable $G : \Gamma_0 \rightarrow \mathbb{R}$ is a sequence of measurable symmetric functions $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$. A measurable $F : \Gamma \rightarrow \mathbb{R}$ is called a cylinder function if there exist $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and a measurable $G : \Gamma_\Lambda \rightarrow \mathbb{R}$ such that, cf. (2.1), $F(\gamma) = G(\gamma_\Lambda)$ for all $\gamma \in \Gamma$. By $\mathcal{F}_{\text{cyl}}(\Gamma)$ we denote the set of all cylinder functions.

A set $\Upsilon \in \mathcal{B}(\Gamma_0)$ is said to be bounded if

$$\Upsilon \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)} \quad (2.4)$$

for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$. By $B_{\text{bs}}(\Gamma_0)$ we denote the set of all bounded measurable functions $G : \Gamma_0 \rightarrow \mathbb{R}$, which have bounded supports. That is, each such G is the zero function on $\Gamma_0 \setminus \Upsilon$ for some bounded Υ . For $\gamma \in \Gamma$, by writing $\eta \Subset \gamma$ we mean that $\eta \subset \gamma$ and η is finite, i.e., $\eta \in \Gamma_0$. For $G \in B_{\text{bs}}(\Gamma_0)$, we set

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma. \quad (2.5)$$

Obviously, K is a linear and positivity preserving map, which maps $B_{\text{bs}}(\Gamma_0)$ into $\mathcal{F}_{\text{cyl}}(\Gamma)$, see, e.g., [18]. In the sequel, we use the following set

$$B_{\text{bs}}^+(\Gamma_0) := \{G \in B_{\text{bs}}(\Gamma_0) : KG \neq 0, \quad (KG)(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma\}. \quad (2.6)$$

By $\mathcal{M}_{\text{fm}}^1(\Gamma)$ we denote the set of all probability measures on $\mathcal{B}(\Gamma)$ that have finite local moments, that is, for which

$$\int_{\Gamma} |\gamma_\Lambda|^n \mu(d\gamma) < \infty, \quad \text{for all } n \in \mathbb{N} \text{ and } \Lambda \in \mathcal{B}_b(\mathbb{R}^d).$$

A measure $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ is said to be *locally absolutely continuous* with respect to the Poisson measure π if, for every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to π^Λ , cf. (2.3). A measure ρ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is said to be *locally finite* if $\rho(\Upsilon) < \infty$ for every bounded $\Upsilon \subset \Gamma_0$. By $\mathcal{M}_{\text{lf}}(\Gamma_0)$ we denote the set of all such measures. For a bounded $\Upsilon \subset \Gamma_0$, let \mathbb{I}_Υ be its indicator function. Then \mathbb{I}_Υ is in $B_{\text{bs}}(\Gamma_0)$ and hence one can apply (2.5). For $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$, the representation

$$\rho_\mu(\Upsilon) = \int_{\Gamma} (K\mathbb{I}_\Upsilon)(\gamma) \mu(d\gamma) \quad (2.7)$$

determines a unique measure $\rho_\mu \in \mathcal{M}_{\text{lf}}(\Gamma_0)$. It is called the *correlation measure* for μ . Then (2.7) defines the map $K^* : \mathcal{M}_{\text{fm}}^1(\Gamma) \rightarrow \mathcal{M}_{\text{lf}}(\Gamma_0)$ such that $K^*\mu = \rho_\mu$. In particular, $K^*\pi = \lambda$. It is known, see Proposition 4.14 in [18], that ρ_μ is absolutely continuous with respect to λ if μ is locally absolutely continuous with

respect to π . In this case, for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, we have that

$$k_\mu(\eta) = \frac{d\rho_\mu}{d\lambda}(\eta) = \int_{\Gamma_\Lambda} \frac{d\mu^\Lambda}{d\pi^\Lambda}(\eta \cup \gamma) \pi^\Lambda(d\gamma) \quad (2.8)$$

$$= \int_{\Gamma_\Lambda} \frac{d\mu^\Lambda}{d\lambda^\Lambda}(\eta \cup \gamma) \lambda^\Lambda(d\gamma). \quad (2.9)$$

The Radon–Nikodym derivative k_μ is called the *correlation function* corresponding to the measure μ . In the sequel, λ will be the basic measure on $\mathcal{B}(\Gamma_0)$, and we shall tacitly assume that the equalities or inequalities, like (2.9) or (2.11), hold for λ -almost all $\eta \in \Gamma_0$. The following fact is known, see Theorems 6.1 and 6.2 and Remark 6.3 in [18].

Proposition 2.1. *Let $\rho \in \mathcal{M}_{\text{lf}}(\Gamma_0)$ have the following properties:*

$$\rho(\emptyset) = 1, \quad \int_{\Gamma_0} G(\eta) \rho(d\eta) \geq 0 \quad \text{for all } G \in B_{\text{bs}}^+(\Gamma_0). \quad (2.10)$$

Then there exist $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ such that $K^ \mu = \rho$. Such μ is unique if*

$$\frac{d\rho}{d\lambda}(\eta) \leq \prod_{x \in \eta} C(x), \quad \eta \in \Gamma_0, \quad (2.11)$$

for some locally integrable $C : \mathbb{R}^d \rightarrow \mathbb{R}_+$.

Here and below we use the conventions

$$\sum_{a \in \emptyset} \phi_a := 0, \quad \prod_{a \in \emptyset} \psi_a := 1.$$

Finally, we mention the following integration rule, see, e.g., [15],

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta) \lambda(d\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) \lambda(d\xi) \lambda(d\eta), \quad (2.12)$$

which holds for any appropriate function H if both sides are finite.

2.2 The model

An informal generator corresponding to the SEM is given in (1.5). The competition and dispersion rates $E^\pm(x, \gamma)$ are supposed to be additive, and the corresponding kernels a_\pm are translation invariant, see [5]. In view of the latter assumption, we write them as

$$a_\pm(x, y) = a^\pm(x - y),$$

and hence, cf. (1.6),

$$E^\pm(x, \gamma) = \sum_{y \in \gamma} a^\pm(x - y). \quad (2.13)$$

We suppose that

$$a^\pm \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \quad a^\pm(x) = a^\pm(-x) \geq 0, \quad (2.14)$$

and thus set

$$\langle a^\pm \rangle = \int_{\mathbb{R}^d} a^\pm(x) dx, \quad \|a^\pm\| = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} a^\pm(x), \quad (2.15)$$

and

$$E^\pm(\eta) = \sum_{x \in \eta} E^\pm(x, \eta \setminus x) = \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^\pm(x - y), \quad \eta \in \Gamma_0. \quad (2.16)$$

By (2.14), we have

$$E^\pm(\eta) \leq \|a^\pm\| |\eta|^2. \quad (2.17)$$

For the sake of brevity, we also denote

$$E(\eta) = \sum_{x \in \eta} (m + E^-(x, \eta \setminus x)) = m|\eta| + E^-(\eta), \quad (2.18)$$

where m is the same as in (1.5).

Following the general scheme developed in [21] one constructs the evolution of correlation functions as a dual evolution to that of *quasi-observables*, which are functions $G : \Gamma_0 \rightarrow \mathbb{R}$. This latter evolution is obtained from the following Cauchy problem

$$\frac{d}{dt} G_t(\eta) = \widehat{L} G_t(\eta), \quad G_t|_{t=0} = G_0, \quad (2.19)$$

where

$$\widehat{L} = K^{-1} L K \quad (2.20)$$

is the so called *symbol* of L , which has the form, cf. [15],

$$\widehat{L} = A + B \quad (2.21)$$

with

$$A = A_1 + A_2 \quad (2.22)$$

$$(A_1 G)(\eta) = -E(\eta) G(\eta), \quad (A_2 G)(\eta) = \int_{\mathbb{R}^d} E^+(y, \eta) G(\eta \cup y) dy, \quad (2.23)$$

and

$$B = B_1 + B_2, \quad (2.24)$$

$$(B_1 G)(\eta) = \sum_{x \in \eta} E^-(x, \eta \setminus x) G(\eta \setminus x),$$

$$(B_2 G)(\eta) = \int_{\mathbb{R}^d} \sum_{x \in \eta} a_+(x, y) G(\eta \setminus x \cup y) dy.$$

Clearly, the action of \widehat{L} on $G \in B_{\text{bs}}(\Gamma_0)$ is well-defined. Its extension to wider classes of G will be done in Section 3 below.

For a measurable locally integrable function $k : \Gamma_0 \rightarrow \mathbb{R}$ and $G \in B_{\text{bs}}(\Gamma_0)$, we define

$$\langle\langle G, k \rangle\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta). \quad (2.25)$$

This pairing can be extended to appropriate classes of G and k . Then the Cauchy problem ‘dual’ to (2.19) takes the form

$$\frac{dk_t}{dt} = L^\Delta k_t, \quad k_t|_{t=0} = k_0, \quad (2.26)$$

where the action of L^Δ is obtained by means of (2.12) according to the rule

$$\langle\langle \widehat{L}G, k \rangle\rangle = \langle\langle G, L^\Delta k \rangle\rangle,$$

as well as from (2.25) and (2.21)–(2.24). It thus has the form, cf. [15],

$$L^\Delta = A^\Delta + B^\Delta \quad (2.27)$$

with

$$A^\Delta = A_1^\Delta + A_2^\Delta \quad (2.28)$$

$$(A_1 k)(\eta) = -E(\eta)k(\eta), \quad (A_2 k)(\eta) = \sum_{x \in \eta} E^+(x, \eta \setminus x)k(\eta \setminus x),$$

and

$$B^\Delta = B_1^\Delta + B_2^\Delta, \quad (2.29)$$

$$(B_1^\Delta k)(\eta) = - \int_{\mathbb{R}^d} E^-(y, \eta)k(\eta \cup y)dy,$$

$$(B_2^\Delta k)(\eta) = \int_{\mathbb{R}^d} \sum_{x \in \eta} a^+(x-y)k(\eta \setminus x \cup y)dy.$$

Of course, like \widehat{L} the above introduced L^Δ is well-defined only for ‘good enough’ k . In the next sections, we define both operators in the corresponding Banach spaces.

3 The Evolution of Quasi-observables

3.1 Setting

For $\alpha \in \mathbb{R}$ and the measure λ as in (2.2), we consider the Banach space

$$\mathcal{G}_\alpha := L^1(\Gamma_0, e^{-\alpha|\cdot|}d\lambda), \quad (3.1)$$

in which the norm is

$$\|G\|_\alpha = \int_{\Gamma_0} |G(\eta)| \exp(-\alpha|\eta|) \lambda(d\eta).$$

Clearly, $\|G\|_{\alpha'} \leq \|G\|_{\alpha''}$ for $\alpha'' < \alpha'$; hence, we have that

$$\mathcal{G}_{\alpha''} \hookrightarrow \mathcal{G}_{\alpha'}, \quad \text{for } \alpha'' < \alpha', \quad (3.2)$$

where the embedding is dense and continuous. Now we fix $\alpha \in \mathbb{R}$ and turn to the definition of \widehat{L} in \mathcal{G}_α , see (2.21)–(2.24). Set

$$\begin{aligned} \mathcal{D}(A_1) &= \{G \in \mathcal{G}_\alpha : E(\cdot)G(\cdot) \in \mathcal{G}_\alpha\}, \\ \mathcal{D}(A_2) &= \{G \in \mathcal{G}_\alpha : E^+(\cdot)G(\cdot) \in \mathcal{G}_\alpha\}, \end{aligned}$$

where $E^\pm(\eta)$ are as in (2.16). As a multiplication operator, A_1 with $\text{Dom}(A_1) = \mathcal{D}(A_1)$ is closed. By (2.12), for an appropriate G , we get

$$\begin{aligned} \|A_2 G\|_\alpha &\leq \int_{\Gamma_0} \int_{\mathbb{R}^d} E^+(y, \eta) |G(\eta \cup y)| e^{-\alpha|\eta|} dy \lambda(d\eta) \\ &= e^\alpha \int_{\Gamma_0} |G(\eta)| e^{-\alpha|\eta|} \left(\sum_{x \in \eta} E^+(x, \eta \setminus x) \right) \lambda(d\eta) \\ &= e^\alpha \|E^+(\cdot)G(\cdot)\|_\alpha. \end{aligned} \quad (3.3)$$

Hence, A_2 with $\text{Dom}(A_2) = \mathcal{D}(A_2)$ is well-defined. Further, we set

$$\mathcal{D}(B) = \{G \in \mathcal{G}_\alpha : |\cdot|G(\cdot) \in \mathcal{G}_\alpha\}.$$

Like in (3.3), for an appropriate G , we obtain

$$\begin{aligned} \|B_1 G\|_\alpha &\leq \int_{\Gamma_0} \sum_{x \in \eta} E^-(x, \eta \setminus x) |G(\eta \setminus x)| e^{-\alpha|\eta|} \lambda(d\eta) \\ &= e^{-\alpha} \int_{\Gamma_0} \left(\int_{\mathbb{R}^d} E^-(y, \eta) dy \right) |G(\eta)| e^{-\alpha|\eta|} \lambda(d\eta) \\ &= e^{-\alpha} \langle a^- \rangle \int_{\Gamma_0} |\eta| |G(\eta)| e^{-\alpha|\eta|} \lambda(d\eta), \end{aligned} \quad (3.4)$$

where we have used (2.15). In a similar way, we get

$$\|B_2 G\|_\alpha \leq \langle a^+ \rangle \int_{\Gamma_0} |\eta| |G(\eta)| e^{-\alpha|\eta|} \lambda(d\eta). \quad (3.5)$$

Thus, the operator B as in (2.24) with $\text{Dom}(B) = \mathcal{D}(B)$ is also well-defined. Thereafter, we set

$$\text{Dom}(\widehat{L}) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2) \cap \mathcal{D}(B). \quad (3.6)$$

For $\varkappa > 0$ and any $\eta \in \Gamma_0$, we have that

$$|\eta|e^{-\varkappa|\eta|} \leq \frac{1}{e\varkappa}, \quad |\eta|^2e^{-\varkappa|\eta|} \leq \left(\frac{2}{e\varkappa}\right)^2. \quad (3.7)$$

Then by (3.4) and (3.5)

$$\|BG\|_\alpha \leq \frac{\langle a^+ \rangle + \langle a^- \rangle e^{-\alpha}}{e(\alpha - \alpha')} \|G\|_{\alpha'}, \quad (3.8)$$

which holds for any $\alpha' < \alpha$. By the second estimate in (3.7), and by (2.17) and (2.18), we also get

$$\operatorname{ess\,sup}_{\eta \in \Gamma_0} E(\eta) \exp(-\varkappa|\eta|) \leq M'/\varkappa^2, \quad \operatorname{ess\,sup}_{\eta \in \Gamma_0} E^+(\eta) \exp(-\varkappa|\eta|) \leq M''/\varkappa^2, \quad (3.9)$$

which holds for any $\varkappa > 0$ and some positive M' and M'' . Thus, we have proven the following

Lemma 3.1. *For each $\alpha' < \alpha$, the expressions (2.21)–(2.24) define a bounded linear operator acting from $\mathcal{G}_{\alpha'}$ into \mathcal{G}_α , which we also denote by \widehat{L} , such that the corresponding operator norm obeys the estimate*

$$\|\widehat{L}\|_{\alpha'\alpha} \leq M/(\alpha - \alpha')^2, \quad (3.10)$$

for some $M > 0$. Furthermore, the same expressions and (3.6) define an unbounded operator on \mathcal{G}_α such that, for any $\alpha' < \alpha$,

$$\mathcal{G}_{\alpha'} \subset \operatorname{Dom}(\widehat{L}). \quad (3.11)$$

Definition 3.2. By a classical solution of the problem (2.19), in the space \mathcal{G}_α and on the time interval $[0, T)$, we understand a map $[0, T) \ni t \mapsto G_t \in \operatorname{Dom}(\widehat{L}) \subset \mathcal{G}_\alpha$, continuous on $[0, T)$ and continuously differentiable on $(0, T)$, such that (2.19) is satisfied for $t \in [0, T)$.

Remark 3.3. In view of (3.11), the condition $G_t \in \operatorname{Dom}(\widehat{L})$ can be verified by showing that the solution G_t belongs to \mathcal{G}_{α_t} for some $\alpha_t < \alpha$.

3.2 The statement

The basic assumption regarding the model properties which we need is the following: there exists $\theta > 0$ such that, for almost all $x \in \mathbb{R}^d$,

$$a^+(x) \leq \theta a^-(x). \quad (3.12)$$

For $\alpha^* \in \mathbb{R}$ and $\alpha_* < \alpha^*$, we set

$$T_* = \frac{\alpha^* - \alpha_*}{\langle a^+ \rangle + \langle a^- \rangle e^{-\alpha_*}}. \quad (3.13)$$

Theorem 3.4. *Let (3.12) be satisfied. Then, for every $\alpha^* \in \mathbb{R}$ such that*

$$e^{\alpha^* \theta} < 1, \quad (3.14)$$

and any $\alpha_ < \alpha^*$, the problem (2.19) with $G_0 \in \mathcal{G}_{\alpha_*}$ has a unique classical solution in \mathcal{G}_{α^*} on the time interval $[0, T_*)$ with T_* given in (3.13).*

The main idea of the proof is to obtain the solution as the limit in \mathcal{G}_{α^*} of the sequence $\{G_t^{(n)}\}_{n \in \mathbb{N}_0}$ which we obtain recursively by solving the following Cauchy problems

$$\frac{d}{dt} G_t^{(n)} = A G_t^{(n)} + B G_t^{(n-1)}, \quad G_t^{(n)}|_{t=0} = G_0, \quad n \in \mathbb{N}, \quad (3.15)$$

and

$$\frac{d}{dt} G_t^{(0)} = A G_t^{(0)}, \quad G_t^{(0)}|_{t=0} = G_0, \quad (3.16)$$

where A and B are given in (2.22) and (2.24), respectively. The reason to split \widehat{L} as in (2.21) is the following. In view of (3.8), B acts continuously from a smaller $\mathcal{G}_{\alpha''}$ into a bigger $\mathcal{G}_{\alpha'}$, cf. (3.2). The fact that the denominator in (3.8) contains the difference $\alpha - \alpha'$ in the power one allows us to employ Ovcyannikov's type arguments, see e.g., [30]. However, this is true only for B but not for A , cf. (3.9) and (3.10). In Lemma 3.6 below, we prove that, under the assumption (3.14), A is the generator of a substochastic analytic semigroup³. Combining these facts and employing standard results of the theory of inhomogeneous differential equations in Banach spaces, we prove the existence of $G_t^{(n)}$, $n \in \mathbb{N}_0$, and then the convergence $G_t^{(n)} \rightarrow G_t$. The uniqueness is proven by showing that the only classical solution of the problem (2.19) with the zero initial condition is $G_t \equiv 0$.

In the proof of Lemma 3.6 below we employ the perturbation theory for positive semigroups of operators in ordered Banach spaces developed in [29]. Before stating this lemma we present the relevant fragments of this theory in the special case of spaces of integrable functions. Let E be a measurable space with a σ -finite measure ν , and $X := L^1(E \rightarrow \mathbb{R}, d\nu)$ be the Banach space of ν -integrable real-valued functions on X with norm $\|\cdot\|$. Let X^+ be the cone in X consisting of all ν -a.e. nonnegative functions on E . Clearly, $\|f + g\| = \|f\| + \|g\|$ for any $f, g \in X^+$, and this cone is generating, that is, $X = X^+ - X^+$. Recall that a C_0 -semigroup $\{S(t)\}_{t \geq 0}$ of bounded linear operators on X is called *positive* if $S(t)f \in X^+$ for all $f \in X^+$. A positive semigroup is called *substochastic* (corr., *stochastic*) if $\|S(t)f\| \leq \|f\|$ (corr., $\|S(t)f\| = \|f\|$) for all $f \in X^+$. Let $(A_0, D(A_0))$ be the generator of a positive C_0 -semigroup $\{S_0(t)\}_{t \geq 0}$ on X . Set $D^+(A_0) = D(A_0) \cap X^+$. Then $D(A_0)$ is dense in X , and $D^+(A_0)$ is dense in X^+ . Let $P : D(A_0) \rightarrow X$ be a positive linear operator, namely, $Pf \in X^+$ for all $f \in D^+(A_0)$. The next statement is an adaptation of Theorem 2.2 in [29].

³Which is the only reason of imposing (3.14).

Proposition 3.5. *Suppose that, for any $f \in D^+(A_0)$,*

$$\int_E ((A_0 + P)f)(x) \nu(dx) \leq 0. \quad (3.17)$$

Then, for all $r \in [0, 1)$, the operator $(A_0 + rP, D(A_0))$ is the generator of a substochastic C_0 -semigroup in X .

Now we apply Proposition 3.5 to the operator (2.22).

Lemma 3.6. *Let θ and α^* be as in (3.12) and (3.14). Then, for any $\alpha \leq \alpha^*$, the operator A given by (2.22) with $\text{Dom}(A) = \mathcal{D}(A_1)$, is the generator of a substochastic analytic semigroup $\{S(t)\}_{t \geq 0}$ in \mathcal{G}_α .*

Proof. We apply Proposition 3.5 with $E = \Gamma_0$, $X = \mathcal{G}_\alpha$ as in (3.1), and $A_0 = A_1$. For $r > 0$ and A_2 as in (2.23), we set $P = r^{-1}A_2$. The cone \mathcal{G}_α^+ contains all nonnegative elements of \mathcal{G}_α . For such A_0 and P , and for $G \in \mathcal{G}_\alpha^+ \cap \mathcal{D}(A_1)$, the left-hand side of (3.17) takes the form, cf. (3.3),

$$\begin{aligned} & - \int_{\Gamma_0} E(\eta)G(\eta) \exp(-\alpha|\eta|)\lambda(d\eta) \\ & + r^{-1} \int_{\Gamma_0} \int_{\mathbb{R}^d} E^+(y, \eta)G(\eta \cup y) \exp(-\alpha|\eta|)dy\lambda(d\eta) \\ & = \int_{\Gamma_0} (-E(\eta) + r^{-1}e^\alpha E^+(\eta))G(\eta) \exp(-\alpha|\eta|)\lambda(d\eta). \end{aligned}$$

For a fixed $\alpha \leq \alpha^*$, pick $r \in (0, 1)$ such that $r^{-1}e^{\alpha\theta} < 1$, cf. (3.14). Then, for such α and r , we have

$$\int_{\Gamma_0} (-E(\eta) + r^{-1}e^\alpha E^+(\eta))G(\eta) \exp(-\alpha|\eta|)\lambda(d\eta) \leq 0,$$

which holds in view of (3.12) and (2.16), (2.18). By (3.3) and (3.12), we have

$$\|A_2G\|_\alpha \leq e^{\alpha\theta}\|A_1G\|_\alpha. \quad (3.18)$$

This means that $r^{-1}A_2 : \mathcal{D}(A_1) \rightarrow \mathcal{G}_\alpha$. Since $r^{-1}A_2$ is a positive operator, cf. (2.23), by Proposition 3.5 we have that $A = A_1 + A_2 = A_1 + r(r^{-1}A_2)$ generates a substochastic semigroup $\{S(t)\}_{t \geq 0}$. Let us prove that this semigroup is analytic.

For an appropriate $\zeta \in \mathbb{C}$ and the resolvents of A and A_1 , we have

$$R(\zeta, A) = R(\zeta, A_1) \sum_{n=0}^{\infty} Q^n(\zeta), \quad Q(\zeta) := A_2R(\zeta, A_1). \quad (3.19)$$

For $G \in \mathcal{G}_\alpha$,

$$(Q(\zeta)G)(\eta) = \int_{\mathbb{R}^d} \frac{E^+(y, \eta)}{\zeta + E(\eta \cup y)} G(\eta \cup y) dy.$$

Thus, for $\operatorname{Re} \zeta =: \sigma > 0$, by (2.12) we obtain

$$\begin{aligned}
\|(Q(\zeta)G)\|_\alpha &\leq \int_{\Gamma_0} \int_{\mathbb{R}^d} \frac{E^+(y, \eta)}{\sigma + E(\eta \cup y)} |G(\eta \cup y)| \exp(-\alpha|\eta|) dy \lambda(d\eta) \\
&= \int_{\Gamma_0} \frac{|G(\eta)|}{\sigma + E(\eta)} \exp(-\alpha|\eta| + \alpha) \left(\sum_{x \in \eta} E^+(x, \eta \setminus x) \right) \lambda(d\eta) \\
&\leq \theta e^\alpha \int_{\Gamma_0} \frac{|G(\eta)|}{\sigma + E(\eta)} E(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \\
&\leq \theta e^\alpha \|G\|_\alpha,
\end{aligned}$$

where we have taken into account (2.16) and (3.12). Note that the latter estimate is uniform in ζ . We use it in (3.19) and obtain

$$\|R(\zeta, A)\| \leq \frac{1}{1 - \theta e^\alpha} \|R(\zeta, A_1)\|. \quad (3.20)$$

For $\zeta = \sigma + i\tau$ with $\sigma > 0$ and $\tau \neq 0$, readily $\|R(\zeta, A_1)G\|_\alpha \leq |\tau|^{-1} \|G\|_\alpha$. Employing this estimate and (3.20) we get

$$\|R(\sigma + i\tau, A)\| \leq \frac{1}{|\tau|(1 - \theta e^\alpha)}.$$

Then we apply Theorem 4.6 of [10] page 101, and obtain the analyticity of $\{S(t)\}_{t \geq 0}$, which completes the proof. \square

As a corollary, we immediately get the solution of the problem (3.16) in the form

$$G_t^{(0)} = S(t)G_0, \quad t \geq 0,$$

from which we see that $G_t^{(0)} \in \mathcal{G}_{\alpha_*}$ since $G_0 \in \mathcal{G}_{\alpha_*}$, and the map $t \mapsto G_t^{(0)}$ is continuously differentiable on $(0, +\infty)$.

Proof of Theorem 3.4. Let α_* and α^* be as in the statement of the theorem, and then let T_* be as in (3.13). Now we fix $n \in \mathbb{N}$ in (3.15) and take $\alpha \in (\alpha_*, \alpha^*)$. Set

$$T = \frac{\alpha - \alpha_*}{\alpha^* - \alpha_*} T_*, \quad \epsilon = (\alpha - \alpha_*)/n, \quad \alpha_l = \alpha_* + l\epsilon, \quad l = 0, \dots, n. \quad (3.21)$$

By (3.8), we have

$$\|B\|_{\alpha_{l-1}\alpha_l} \leq \frac{n}{eT}, \quad l = 1, \dots, n, \quad (3.22)$$

where $\|B\|_{\alpha_{l-1}\alpha_l}$ stands for the norm in the space of all bounded linear operators from $\mathcal{G}_{\alpha_{l-1}}$ to \mathcal{G}_{α_l} . For $l = 1, \dots, n$, let us consider the Cauchy problem (3.15) in \mathcal{G}_{α_l} , i.e.,

$$\frac{d}{dt} G_t^{(l)} = A G_t^{(l)} + B G_t^{(l-1)}, \quad G_t^{(l)}|_{t=0} = G_0. \quad (3.23)$$

Assume that $G_t^{(l-1)} \in \mathcal{G}_{\alpha_{l-1}}$ is continuously differentiable on $(0, +\infty)$. Note that this assumption holds true for $l = 1$. Then, by (3.22), $BG_t^{(l-1)} \in \mathcal{G}_{\alpha_l}$ is continuously differentiable, and hence locally Hölder continuous on $(0, +\infty)$ and integrable on $[0, \tau]$, for any $\tau > 0$. By our Lemma 3.6 and Corollary 3.3, page 113 in [26], this yields that the problem (3.23) on the time interval $[0, +\infty)$ has a unique classical solution in \mathcal{G}_{α_l} , given by the formula

$$G_t^{(l)} = S(t)G_0 + \int_0^t S(t-s)BG_s^{(l-1)}ds. \quad (3.24)$$

By the very definition of a classical solution, it is continuously differentiable on $(0, +\infty)$, and hence we can proceed until $l = n$. Reiterating (3.24) we obtain

$$\begin{aligned} G_t^{(n)} &= S(t)G_0 + \sum_{l=1}^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} S(t-t_1)B \\ &\quad \times S(t_1-t_2)B \cdots S(t_{l-1}-t_l)BS(t_l)G_0 dt_1 \cdots dt_l. \end{aligned} \quad (3.25)$$

Note that $G_t^{(n)} \in \mathcal{G}_\alpha$ and $\alpha = \alpha_n$, $\alpha_* = \alpha_0$, see (3.21). From the latter representation we readily obtain

$$\begin{aligned} \|G_t^{(n)} - G_t^{(n-1)}\|_{\alpha_n} &\leq \|G_0\|_{\alpha_*} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|B\|_{\alpha_0\alpha_1} \\ &\quad \times \|B\|_{\alpha_1\alpha_2} \cdots \|B\|_{\alpha_{n-1}\alpha_n} dt_1 \cdots dt_n \\ &\leq \frac{1}{n!} \left(\frac{n}{e}\right)^n \left(\frac{t}{T}\right)^n \|G_0\|_{\alpha_*}, \end{aligned} \quad (3.26)$$

where we have used (3.22) and the fact that $\|S(t)\| \leq 1$ for all $t \geq 0$, see Lemma 3.6. For any $t \in [0, T)$, the right-hand side of the latter estimate is summable in n ; hence, $\{G_t^{(n)}\}_{n \in \mathbb{N}_0}$ is a Cauchy sequence in \mathcal{G}_α . Its limit G_t is an analytic function of t on the disc $\{t \in \mathbb{C} : |t| < T\}$, and thus is continuously differentiable there. Since $G_t \in \mathcal{G}_\alpha$, we have

$$G_t \in \text{Dom}(\widehat{L}) \subset \mathcal{G}_{\alpha^*},$$

see (3.6), (3.11), and also (3.18). For any $\alpha' \in (\alpha, \alpha^*]$, by (3.15) the sequence $\{dG_t^{(n)}/dt\}_{n \in \mathbb{N}_0}$ converges in $\mathcal{G}_{\alpha'}$ to $\widehat{L}G_t$, where we consider \widehat{L} as a bounded operator from \mathcal{G}_α to $\mathcal{G}_{\alpha'}$, cf. Lemma 3.1. Thus, G_t is a classical solution of (2.19).

Now we prove the stated uniqueness. Let $\widetilde{G}_t \in \mathcal{G}_{\alpha^*}$ be another solution of the problem (2.19) with the same initial $G_0 \in \mathcal{G}_{\alpha_*}$, which has the properties stated in the theorem, i.e., which exists for every $\alpha^* > \alpha_*$ on the corresponding time interval. Then, as above, one can show that \widetilde{G}_t is analytic at $t = 0$, and

$$\frac{d^n}{dt^n} \widetilde{G}_t|_{t=0} = \frac{d^n}{dt^n} G_t|_{t=0} = \widehat{L}^n G_0 \in \mathcal{G}_{\alpha^*},$$

where \widehat{L}^n is considered as a bounded operator from \mathcal{G}_{α_*} to \mathcal{G}_{α^*} , the norm of which can be estimated by (3.10). Since the above holds for all $n \in \mathbb{N}$, both solutions \widetilde{G}_t and G_t coincide. \square

Remark 3.7. From the proof given above one readily concludes that the evolution described by the problem (2.19) takes place in the scale of spaces $\{\mathcal{G}_\alpha\}_{\alpha \in [\alpha_*, \alpha^]}$ in the following sense. For every $t \in (0, T_+)$, there exists $\alpha_t \in (\alpha_*, \alpha^*)$ such that the solution G_t lies in $\mathcal{G}_{\alpha_t} \subset \mathcal{G}_{\alpha^*}$.

4 The Evolution of Correlation Functions

4.1 Setting

For the Banach space \mathcal{G}_α (3.1), the dual space with respect to (2.25) is

$$\mathcal{K}_\alpha = \{k : \Gamma_0 \rightarrow \mathbb{R} : \|k\|_\alpha < \infty\}, \quad (4.1)$$

with the norm, see (2.25),

$$\|k\|_\alpha = \operatorname{ess\,sup}_{\eta \in \Gamma_0} |k(\eta)| \exp(\alpha|\eta|). \quad (4.2)$$

For $\alpha'' < \alpha'$, we have $\|k\|_{\alpha''} \leq \|k\|_{\alpha'}$; and hence, cf. (3.2),

$$\mathcal{K}_{\alpha'} \hookrightarrow \mathcal{K}_{\alpha''}, \quad \text{for } \alpha'' < \alpha'. \quad (4.3)$$

The above embedding is continuous but not dense. In the sequel, we always suppose that (3.12) and (3.14) hold, and tacitly assume that $\alpha < \alpha^*$ for each α we are dealing with. For such an α , let A be defined on \mathcal{G}_α by (2.22), and let A^* be its adjoint in \mathcal{K}_α with

$$\operatorname{Dom}(A^*) = \{k \in \mathcal{K}_\alpha : \exists \tilde{k} \in \mathcal{K}_\alpha \quad \forall G \in \mathcal{D}(A) \quad \langle\langle AG, k \rangle\rangle = \langle\langle G, \tilde{k} \rangle\rangle\}.$$

Then, for A^* and A^Δ defined by (2.28), we have

$$A^*k = A^\Delta k = A_1^\Delta k + A_2^\Delta k,$$

which holds for all $k \in \mathcal{K}_\alpha$ such that both A_1^Δ and A_2^Δ map into \mathcal{K}_α . Let \mathcal{Q}_α stand for the closure of $\operatorname{Dom}(A^*)$ in $\|\cdot\|_\alpha$. Then, cf. (3.11),

$$\mathcal{Q}_\alpha := \overline{\operatorname{Dom}(A^*)} \supset \operatorname{Dom}(A^*) \supset \mathcal{K}_{\alpha'}, \quad \text{for any } \alpha' > \alpha. \quad (4.4)$$

The latter inclusion in (4.4) follows from (3.9) and the next obvious estimates:

$$\begin{aligned} \|A_1^\Delta k\|_\alpha &\leq \|k\|_{\alpha'} \operatorname{ess\,sup}_{\eta \in \Gamma_0} E(\eta) \exp(-(\alpha' - \alpha)|\eta|), & (4.5) \\ \|A_2^\Delta k\|_\alpha &\leq \operatorname{ess\,sup}_{\eta \in \Gamma_0} e^{\alpha|\eta|} \sum_{x \in \eta} E^+(x, \eta \setminus x) |k(\eta \setminus x)| \\ &\leq \|k\|_{\alpha'} e^{\alpha'} \operatorname{ess\,sup}_{\eta \in \Gamma_0} E^+(\eta) \exp(-(\alpha' - \alpha)|\eta|). \end{aligned}$$

Noteworthy, \mathcal{Q}_α is a proper subspace of \mathcal{K}_α .

Let $\{S(t)\}_{t \geq 0}$ be the semigroup as in Lemma 3.6. For every $t > 0$, let $S^\circ(t)$ denote the restriction of $S(t)^*$ to \mathcal{Q}_α . Since $\{S(t)\}_{t \geq 0}$ is the semigroup of contractions, for $k \in \mathcal{Q}_\alpha$ we have that, for all $t \geq 0$,

$$\|S^\circ(t)k\|_\alpha = \|S^*(t)k\|_\alpha \leq \|k\|_\alpha. \quad (4.6)$$

For any $\alpha' > \alpha$ and $t \geq 0$, in view of (4.3) we can consider $S^\circ(t)$ as a bounded operator from $\mathcal{K}_{\alpha'}$ to \mathcal{K}_α , for which by (4.6) we have

$$\|S^\circ(t)\|_{\alpha'\alpha} \leq 1, \quad t \geq 0. \quad (4.7)$$

Proposition 4.1. *For every $\alpha' > \alpha$ and any $k \in \mathcal{K}_{\alpha'}$, the map*

$$[0, +\infty) \ni t \mapsto S^\circ(t)k \in \mathcal{K}_\alpha$$

is continuous.

Proof. By Theorem 10.4, page 39 in [26], the collection $\{S^\circ(t)\}_{t \geq 0}$ constitutes a C_0 -semigroup on \mathcal{Q}_α , which in view of (4.4) yields the proof. \square

By Theorem 10.4, page 39 in [26], the generator of the semigroup $\{S^\circ(t)\}_{t \geq 0}$ is the part of A^* in \mathcal{Q}_α , which we denote by A° . Hence, by Definition 10.3, page 39 in [26], A° is the restriction of A^* to the set

$$\text{Dom}(A^\circ) := \{k \in \text{Dom}(A^*) : A^*k \in \mathcal{Q}_\alpha\}. \quad (4.8)$$

For $\alpha' > \alpha$, we take $\alpha'' \in (\alpha, \alpha')$ and obtain by (4.5) that

$$A^* : \mathcal{K}_{\alpha'} \rightarrow \mathcal{K}_{\alpha''}.$$

Hence, for any $\alpha' > \alpha$,

$$\text{Dom}(A^\circ) \supset \mathcal{K}_{\alpha'}. \quad (4.9)$$

We recall that each k may be identified with a sequence $\{k^{(n)}\}_{n \in \mathbb{N}_0}$ of symmetric $k^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, $k^{(0)} \in \mathbb{R}$. Put $q^{(n)} = \|k^{(n)}\|_{L^\infty(\mathbb{R}^{nd})}$, $q^{(0)} = |k^{(0)}|$. Then (4.2) can be rewritten in the form

$$\|k\|_\alpha = \sup_{n \in \mathbb{N}_0} q^{(n)} e^{n\alpha}.$$

Set, cf. (2.29),

$$\mathcal{D}(B^\Delta) = \{k \in \mathcal{K}_\alpha : \sup_{n \in \mathbb{N}_0} nq^{(n)} e^{\alpha n} < \infty\}.$$

Then, see (2.13),

$$\begin{aligned} \|B_1^\Delta k\|_\alpha &\leq \sup_{n \in \mathbb{N}_0} q^{(n+1)} e^{\alpha n} \sup_{\eta \in \Gamma^{(n)}} \int_{\mathbb{R}^d} E^-(y, \eta) dy \\ &= \langle a^- \rangle \sup_{n \in \mathbb{N}_0} nq^{(n+1)} e^{\alpha n} \leq e^{-\alpha} \langle a^- \rangle \sup_{n \in \mathbb{N}_0} nq^{(n)} e^{\alpha n}. \end{aligned}$$

$\|B_2^\Delta k\|_\alpha$ can be estimated in the same way, which then yields

$$\|B^\Delta k\|_\alpha \leq \left(\langle a^+ \rangle + \langle a^- \rangle e^{-\alpha} \right) \sup_{n \in \mathbb{N}_0} n q^{(n)} e^{\alpha n}. \quad (4.10)$$

Hence, B^Δ maps $\mathcal{D}(B^\Delta)$ into \mathcal{K}_α . Let $(B^*, \text{Dom}(B^*))$ be the adjoint operator to $(B, \text{Dom}(B))$. Then $B^*k = B^\Delta k$ for $k \in \mathcal{D}(B^\Delta)$, and

$$\text{Dom}(B^*) \supset \mathcal{D}(B^\Delta) \supset \mathcal{K}_{\alpha'}, \quad \text{for any } \alpha' > \alpha. \quad (4.11)$$

The latter inclusion follows from the estimate, cf. (3.8) and (3.22),

$$\|B^*\|_{\alpha'\alpha} \leq \frac{\langle a^+ \rangle + \langle a^- \rangle e^{-\alpha}}{e(\alpha' - \alpha)}, \quad (4.12)$$

which can easily be obtained from (4.10). Now we can define L^Δ as an operator in \mathcal{K}_α . Namely, we set

$$L^\Delta = A^\odot + B^\Delta, \quad (4.13)$$

$$\text{Dom}(L^\Delta) = \text{Dom}(A^\odot) \cap \mathcal{D}(B^\Delta).$$

By (4.9) and (4.11), for any $\alpha' > \alpha$ we have

$$\text{Dom}(L^\Delta) \supset \mathcal{K}_{\alpha'}.$$

4.2 The statement

Theorem 4.2. *Let θ , α_* , α^* , and T_* be as in Theorem 3.4. Then for every $k_0 \in \mathcal{K}_{\alpha^*}$, the problem (2.26) has a unique classical solution in \mathcal{K}_{α_*} on the time interval $[0, T_*]$.*

Proof. Let $k_0 \in \mathcal{K}_{\alpha^*}$, $\alpha \in (\alpha_*, \alpha^*)$, $n \in \mathbb{N}$, and $l = 1, \dots, n$ be fixed. Consider

$$K_l(t, t_1, \dots, t_l) := S^\odot(t - t_1)B^*S^\odot(t_1 - t_2)B^* \cdots S^\odot(t_{l-1} - t_l)B^*S^\odot(t_l)k_0, \quad (4.14)$$

where the arguments (t, t_1, \dots, t_l) belong to the set

$$\mathcal{T}_l := \{(t, t_1, \dots, t_l) : 0 \leq t_l \leq \cdots \leq t_1 \leq t\}. \quad (4.15)$$

In (4.14), we mean that the operators act in the following spaces, cf. (4.7),

$$S^\odot(t_l) : \mathcal{K}_{\alpha_0} \rightarrow \mathcal{K}_{\alpha_1}, \quad S^\odot(t_{l-s} - t_{l-s+1}) : \mathcal{K}_{\alpha_{2s}} \rightarrow \mathcal{K}_{\alpha_{2s+1}}, \quad s = 1, \dots, l,$$

and, cf. (4.12),

$$B^* : \mathcal{K}_{\alpha_{2s-1}} \rightarrow \mathcal{K}_{\alpha_{2s}}, \quad s = 1, \dots, l. \quad (4.16)$$

Here, for a positive $\delta < \alpha^* - \alpha$, we set

$$\begin{aligned} \alpha_{2s} &= \alpha^* - \frac{s}{l+1}\delta - s\epsilon, & \epsilon &= (\alpha^* - \alpha - \delta)/l, \\ \alpha_{2s+1} &= \alpha^* - \frac{s+1}{l+1}\delta - s\epsilon, & s &= 0, 1, \dots, l. \end{aligned} \quad (4.17)$$

Note that $\alpha_0 = \alpha^*$ and $\alpha_{2l+1} = \alpha$, and hence $K_l(t, t_1, \dots, t_l) \in \mathcal{K}_\alpha$. In view of Proposition 4.1 and (4.12), K_l is a continuous function of each of its variables on (4.15). Furthermore, it is differentiable in $t \in (0, +\infty)$ in every $\mathcal{K}_{\alpha'}$, $\alpha' \in (\alpha_*, \alpha)$, and the following holds, cf. (4.8) and (4.9),

$$\frac{d}{dt} K_l(t, t_1, \dots, t_l) = A^\odot K_l(t, t_1, \dots, t_l).$$

Now we set

$$k_t^{(n)} = S^\odot(t)k_0 + \sum_{l=1}^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} K_l(t, t_1, \dots, t_l) dt_1 \cdots dt_l. \quad (4.18)$$

For

$$T_\delta := \frac{\alpha^* - \alpha - \delta}{\alpha^* - \alpha_*} T_*, \quad (4.19)$$

the function $[0, T_\delta) \ni t \mapsto k_t^{(n)} \in \mathcal{K}_\alpha$ is continuous, whereas $(0, T_\delta) \ni t \mapsto k_t^{(n)} \in \mathcal{K}_{\alpha'}$ is differentiable, and the following holds, cf. (3.23),

$$\frac{d}{dt} k_t^{(n)} = A^\odot k_t^{(n)} + B^* k_t^{(n-1)}, \quad k_t^{(n)}|_{t=0} = k_0. \quad (4.20)$$

For $T < T_*$, let us show that there exists $\alpha \in (\alpha_*, \alpha^*)$ such that the sequence $\{k_t^{(n)}\}_{n \in \mathbb{N}}$ converges in \mathcal{K}_α uniformly on $[0, T]$. For this T , we pick $\alpha \in (\alpha_*, \alpha^*)$ and a positive $\delta < \alpha^* - \alpha$ such that also $T < T_\delta$, see (4.19). As in (3.26), for $t \in [0, T]$ we get

$$\begin{aligned} \|k_t^{(n)} - k_t^{(n-1)}\|_\alpha &\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|K_n(t, t_1, \dots, t_n)\|_\alpha dt_1 \cdots dt_n \\ &\leq \frac{T^n}{n!} \|k_0\|_{\alpha^*} \prod_{s=1}^n \|B^*\|_{\alpha_{2s-1} \alpha_{2s}}, \end{aligned}$$

where we have taken into account (4.7) and (4.16), (4.17) with $l = n$. Then by means of (4.12) we obtain

$$\begin{aligned} \|k_t^{(n)} - k_t^{(n-1)}\|_\alpha &\leq \frac{T^n}{n!} \left(\frac{n}{e}\right)^n \left(\frac{\alpha^* - \alpha_*}{(\alpha^* - \alpha - \delta)T_*}\right)^n \|k_0\|_{\alpha^*} \quad (4.21) \\ &= \frac{1}{n!} \left(\frac{n}{e}\right)^n \left(\frac{T}{T_\delta}\right)^n \|k_0\|_{\alpha^*}, \end{aligned}$$

which certainly yields the convergence to be proven. Now we take $\alpha' \in [\alpha_*, \alpha)$ and obtain the convergence of both sides of (4.20) in $\mathcal{K}_{\alpha'}$ where both operators are considered as bounded operators acting from \mathcal{K}_α to $\mathcal{K}_{\alpha'}$, see (4.7) and (4.12). This yields that the limit $k_t \in \mathcal{K}_{\alpha_*}$ of the sequence $\{k_t^{(n)}\}_{n \in \mathbb{N}}$ solves (2.26) with L^Δ given by (4.13). \square

Remark 4.3. From the proof given above one concludes that the evolution described by the problem (2.26) takes place in the scale of spaces $\{\mathcal{K}_\alpha\}_{\alpha \in [\alpha_*, \alpha^]}$ in the sense that, for every $t \in (0, T_*)$, there exists $\alpha_t \in (\alpha_*, \alpha^)$ such that the solution k_t lies in $\mathcal{K}_{\alpha_t} \subset \mathcal{K}_{\alpha_*}$.

4.3 The dual evolutions

Recall that the duality between correlation functions and quasi-observables is established by the relation (2.25).

Definition 4.4. Let α_* , α^* , T_* be as in Theorem 3.4, and for $G_0 \in \mathcal{G}_{\alpha_*}$, let G_t be the solution of the problem (2.19). For a given $k_0 \in \mathcal{K}_{\alpha^*}$, the dual evolution $k_0 \mapsto k_t^D$ is the weak*-continuous map $[0, T_*) \ni t \mapsto k_t^D \in \mathcal{K}_{\alpha_*}$ such that, for every $t \in [0, T_*)$, the following holds

$$\langle\langle G_t, k_0 \rangle\rangle = \langle\langle G_0, k_t^D \rangle\rangle. \quad (4.22)$$

Likewise, for $k_0 \in \mathcal{K}_{\alpha^*}$, let k_t be the solution of the problem (2.26), see Theorem 4.2. For a given $G_0 \in \mathcal{G}_{\alpha_*}$, the dual evolution $G_0 \mapsto G_t^D$ is the weak-continuous map $[0, T_*) \ni t \mapsto G_t^D \in \mathcal{G}_{\alpha^*}$ such that, for every $t \in [0, T_*)$, the following holds

$$\langle\langle G_0, k_t \rangle\rangle = \langle\langle G_t^D, k_0 \rangle\rangle. \quad (4.23)$$

Note that the solution of (2.26) need not coincide with k_t^D , and similarly, the solution of (2.19) need not be the same as G_t^D . It is even not obvious whether such dual evolutions exist since the topological dual to \mathcal{K}_α is not \mathcal{G}_α .

Theorem 4.5. *For any $G_0 \in \mathcal{G}_{\alpha_*}$ and any $k_0 \in \mathcal{K}_{\alpha^*}$, the dual evolutions $k_0 \mapsto k_t^D$ and $G_0 \mapsto G_t^D$ exist and are norm-continuous.*

Proof. First we prove the existence of k_t^D . For a given $k_0 \in \mathcal{K}_{\alpha^*}$ and a fixed $n \in \mathbb{N}$, let α , δ , and l be as in the proof of Theorem 4.2. Set

$$\begin{aligned} K_l^D(t, t_1, \dots, t_l) &:= S^\odot(t_l) B^* S^\odot(t_{l-1} - t_l) B^* \cdots S^\odot(t_1 - t_2) B^* \\ &\quad \times S^\odot(t - t_1) k_0, \end{aligned} \quad (4.24)$$

where the above operators act in the following spaces

$$\begin{aligned} S^\odot(t_s - t_{s+1}) &: \mathcal{K}_{\alpha_{2s}} \rightarrow \mathcal{K}_{\alpha_{2s+1}}, \quad s = 0, 1, \dots, l-1, \\ S^\odot(t - t_1) &: \mathcal{K}_{\alpha_0} \rightarrow \mathcal{K}_{\alpha_1}, \quad S^\odot(t_l) : \mathcal{K}_{\alpha_{2l}} \rightarrow \mathcal{K}_{\alpha_{2l+1}}, \end{aligned}$$

and B^* act as in (4.16). The numbers α_s are given by (4.17). Then we set, cf. (4.18),

$$k_t^{D,n} = S^\odot(t) k_0 + \sum_{l=1}^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} K_l^D(t, t_1, \dots, t_l) dt_1 \cdots dt_l. \quad (4.25)$$

Exactly as in the proof of Theorem 4.2 we obtain, cf. (4.21),

$$\|k_t^{D,n} - k_t^{D,n-1}\|_\alpha \leq \frac{1}{n!} \left(\frac{n}{e}\right)^n \left(\frac{T}{T_\delta}\right)^n \|k_0\|_{\alpha^*}$$

which yields that the sequence $\{k_t^{D,n}\}_{n \in \mathbb{N}}$ converges in \mathcal{K}_α uniformly on $[0, T]$. Hence its limit, which we denote by k_t^D , is a norm-continuous function from $[0, T_*)$ to \mathcal{K}_α , and $\mathcal{K}_\alpha \hookrightarrow \mathcal{K}_{\alpha^*}$. Note that $k_t^D \in \mathcal{K}_{\alpha_t}$ where $\alpha_t \in (\alpha_*, \alpha^*)$, cf. Remark 4.3.

For every $G \in \mathcal{G}_{\alpha^*}$, the map $\mathcal{K}_{\alpha^*} \ni k \mapsto \langle\langle G, k \rangle\rangle \in \mathbb{R}$ is continuous. Since each K_l^D in (4.25) is in \mathcal{K}_{α^*} , we have, cf. (4.24),

$$\begin{aligned} & \left\langle\left\langle G_0, \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} K_l^D(t, t_1, \dots, t_l) dt_1 \cdots dt_l \right\rangle\right\rangle & (4.26) \\ &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} \langle\langle G_0, K_l^D(t, t_1, \dots, t_l) \rangle\rangle dt_1 \cdots dt_l \\ &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} \langle\langle S(t-t_1)BS(t_1-t_2)B \\ & \quad \times \cdots \times S(t_{l-1}-t_l)BS(t_l)G_0, k_0 \rangle\rangle dt_1 \cdots dt_l. \end{aligned}$$

Thereafter, by (4.25) we obtain, cf. (3.25),

$$\langle\langle G_0, k_t^{D,n} \rangle\rangle = \langle\langle G_t^{(n)}, k_0 \rangle\rangle,$$

which holds for all $t \in [0, T_*)$ and $n \in \mathbb{N}$. Passing here to the limit $n \rightarrow \infty$ and taking into account the norm convergences $G_t^{(n)} \rightarrow G_t$, see Theorem 3.4, and $k_t^{D,n} \rightarrow k_t^D$ established above, we arrive at (4.22).

To prove (4.23), for $t \in [0, T_*)$ and $n \in \mathbb{N}$, we consider, cf. (4.24),

$$\begin{aligned} G_t^{D,n} &= S(t)G_0 + \sum_{l=1}^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} S(t_l)B \\ & \quad \times S(t_{l-1}-t_l)B \cdots S(t_1-t_2)BS(t-t_1)G_0 dt_1 \cdots dt_l. \end{aligned}$$

As in the proof of Theorem 3.4, we show that the sequence of $G_t^{D,n}$, $n \in \mathbb{N}$, converges in \mathcal{G}_{α^*} , uniformly on compact subsets of $[0, T_*)$. Let G_t^D be its limit. By the very construction, and due to the possibility of interchanging the integrations as in (4.26), we get

$$\langle\langle G_t^{D,n}, k_0 \rangle\rangle = \langle\langle G_0, k_t^{(n)} \rangle\rangle,$$

where $k_t^{(n)}$ is the same as in (4.18). Passing here to the limit $n \rightarrow \infty$ we arrive at (4.23). \square

Remark 4.6. As in Remark 4.3, from the above proof we conclude that, for each $t \in (0, T_*)$, there exists $\alpha_t \in (\alpha_*, \alpha^*)$ such that $G_t^D \in \mathcal{G}_{\alpha_t} \subset \mathcal{G}_{\alpha^*}$.

5 The Evolution of States

Theorem 4.2 does not ensure that the solutions k_t are correlation functions. Below we prove this holds under the condition (5.11). Recall that we also assume (3.12).

5.1 The evolution of local densities

Let a measure $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ be locally absolutely continuous with respect to the Poisson measure π . In that, for each $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, the projection μ^Λ is absolutely continuous with respect to π^Λ , and hence to λ^Λ , cf. (2.3). Consider

$$R^\Lambda(\eta) := \mathbb{I}_{\Gamma_\Lambda}(\eta) \frac{d\mu^\Lambda}{d\lambda^\Lambda}(\eta), \quad \eta \in \Gamma_0. \quad (5.1)$$

Clearly, R^Λ is a positive element of the Banach space $L^1(\Gamma_0, d\lambda)$ of unit norm. We call it a *local density*. The measure μ is characterized by the correlation measure (2.7), and thus by the correlation function (2.8) which can be written in the form (2.9), cf. Proposition 4.2 in [18],

$$k^\Lambda(\eta) = k(\eta) \mathbb{I}_{\Gamma_\Lambda}(\eta) = \int_{\Gamma_0} R^\Lambda(\eta \cup \xi) \lambda(d\xi), \quad \eta \in \Gamma_0. \quad (5.2)$$

Note that $k^\Lambda = k^\Lambda \mathbb{I}_{\Gamma_\Lambda}$.

As in [19], we say that a probability measure μ on $\mathcal{B}(\Gamma)$ obeys Dobrushin's exponential bound with a given $\alpha > 0$, if for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, there exists $C_\Lambda > 0$ such that

$$\int_{\Gamma_\Lambda} \exp(\alpha|\eta|) \mu^\Lambda(d\eta) \leq C_\Lambda. \quad (5.3)$$

For $\alpha > 0$, we set

$$b_\alpha(\eta) = \exp(\alpha|\eta|), \quad \eta \in \Gamma_0. \quad (5.4)$$

Clearly, if μ obeys (5.3) with a given $\alpha > 0$, then, for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$,

$$R^\Lambda \in \mathcal{R}_\alpha := L^1(\Gamma_0, b_\alpha d\lambda). \quad (5.5)$$

In this subsection, we study the evolution of local densities in the space \mathcal{R}_α .

As was mentioned above, we cannot define L as given in (1.5) on any space of functions $F : \Gamma \rightarrow \mathbb{R}$. However, it is possible to do in the case of bounded measurable functions $F : \Gamma_0 \rightarrow \mathbb{R}$, i.e., on the space $L^\infty(\Gamma_0, d\lambda)$. Set

$$\Xi(\eta) = E(\eta) + \langle a^+ \rangle |\eta|, \quad \eta \in \Gamma_0. \quad (5.6)$$

Then we rewrite (1.5) in the following form

$$\begin{aligned} (LF)(\eta) &= -\Xi(\eta)F(\eta) + \sum_{x \in \eta} (m + E^-(x, \eta \setminus x)) F(\eta \setminus x) \\ &\quad + \int_{\mathbb{R}^d} E^+(x, \eta) F(\eta \cup x) dx, \quad \eta \in \Gamma_0. \end{aligned} \quad (5.7)$$

Let $R \in L^1(\Gamma_0, d\lambda)$ be such that $\Xi R \in L^1(\Gamma_0, d\lambda)$. For such R and for any $F \in L^\infty(\Gamma_0, d\lambda)$, by (2.12) we get

$$\begin{aligned} \int_{\Gamma_0} (LF)(\eta) R(\eta) d\lambda(\eta) &= - \int_{\Gamma_0} \Xi(\eta) F(\eta) R(\eta) d\lambda(\eta) \\ &\quad + \int_{\Gamma_0} \int_{\mathbb{R}^d} (m + E^-(x, \eta)) F(\eta) R(\eta \cup x) dx d\lambda(\eta) \\ &\quad + \int_{\Gamma_0} \sum_{x \in \eta} E^+(x, \eta \setminus x) F(\eta) R(\eta \setminus x) d\lambda(\eta). \end{aligned}$$

Next, we define the following operator in $L^1(\Gamma_0, d\lambda)$

$$\begin{aligned} (L^\dagger R)(\eta) &= (A_0 R)(\eta) + (BR)(\eta) := -\Xi(\eta)R(\eta) \\ &\quad + \int_{\mathbb{R}^d} (m + E^-(y, \eta))R(\eta \cup y)dy + \sum_{x \in \eta} E^+(x, \eta \setminus x)R(\eta \setminus x), \end{aligned} \tag{5.8}$$

with

$$\text{Dom}(L^\dagger) = \{R \in L^1(\Gamma_0, d\lambda) : \Xi R \in L^1(\Gamma_0, d\lambda)\}. \tag{5.9}$$

Then, for any $F \in L^\infty(\Gamma_0, d\lambda)$, we have

$$\int_{\Gamma_0} LF \cdot R d\lambda = \int_{\Gamma_0} F \cdot L^\dagger R d\lambda. \tag{5.10}$$

Lemma 5.1. *Suppose that the following condition be satisfied*

$$m > \langle a^+ \rangle. \tag{5.11}$$

Then the closure of L^\dagger given in (5.8) and (5.9) is the generator of a stochastic C_0 -semigroup $\{S^\dagger(t)\}_{t \geq 0}$ of bounded linear operators in $L^1(\Gamma_0, d\lambda)$, which leave invariant each \mathcal{R}_α with $\alpha \leq \log m - \log \langle a^+ \rangle$. Moreover, the restrictions $S_\alpha^\dagger(t) := S^\dagger(t)|_{\mathcal{R}_\alpha}$, $t \geq 0$, constitute a positive C_0 -semigroup in \mathcal{R}_α , the generator L_α^\dagger of which is the restriction of $(L^\dagger, \text{Dom}(L^\dagger))$.

As in Section 3, we employ the perturbation theory for positive semigroups developed in [29]. To proceed further, we need some facts in addition to those preceding Proposition 3.5. Recall that X stands for $L^1(E, d\nu)$.

Let $\rho \in L^1_{\text{loc}}(E, d\nu)$ be such that $p := \text{ess inf}_{x \in E} \rho(x) > 0$. We consider the Banach $X_\rho := L^1(E, \rho d\nu)$ with norm $\|\cdot\|_\rho$. Clearly, $X_\rho \hookrightarrow X$, where the embedding is dense and continuous. The latter follows from the fact that $\|f\| \leq p^{-1} \|f\|_\rho$ for all $f \in X_\rho$. Next, for $X_\rho^+ := X_\rho \cap X^+$ we have that X_ρ^+ is dense in X^+ and $X_\rho = X_\rho^+ - X_\rho^+$. Note that, $\|f + g\|_\rho = \|f\|_\rho + \|g\|_\rho$ for any $f, g \in X_\rho^+$.

Let $(A_0, D(A_0))$ be the generator of a positive C_0 -semigroup $\{S_0(t)\}_{t \geq 0}$ of contractions on X . Then we set $\tilde{S}_0(t) = S_0(t)|_{X_\rho}$, $t \geq 0$, and assume that the following holds:

- (a) The operators $S_0(t)$, $t \geq 0$, leave X_ρ invariant.
- (b) $\{\check{S}_0(t)\}_{t \geq 0}$ is a C_0 -semigroup on X_ρ .

By Proposition II.2.3 of [10], the generator \check{A}_0 of the semigroup $\{\check{S}_0(t)\}_{t \geq 0}$ is the part of A_0 . Namely, $\check{A}_0 f = A_0 f$ on the domain

$$D(\check{A}_0) = \{f \in D(A_0) \cap X_\rho : A_0 f \in X_\rho\}.$$

Set $D^+(\check{A}_0) = D(\check{A}_0) \cap X_\rho^+$. The next statement is an adaptation of Proposition 2.6 and Theorem 2.7 of [29].

Proposition 5.2. *Let conditions (a) and (b) above hold, and $-A_0$ be a positive linear operator in X . Suppose also that, cf. (3.17),*

$$\int_E ((A_0 + P)f)(x) \nu(dx) = 0,$$

where P is such that $P(D(\check{A}_0)) \subset X_\rho$. Finally, assume that there exist $c > 0, \varepsilon > 0$ such that, for all $f \in D^+(\check{A}_0)$, the following estimate holds

$$\int_E ((A_0 + P)f)(x) \rho(x) \nu(dx) \leq c \int_E f(x) \rho(x) \nu(dx) + \varepsilon \int_E (A_0 f(x)) \nu(dx).$$

Then the closure $(A, D(A))$ of the operator $(A_0 + P, D(A_0))$ is the generator of a stochastic semigroup $\{S(t)\}_{t \geq 0}$ on X . This semigroup leaves the space X_ρ invariant and induces a positive C_0 -semigroup, $\check{S}(t)$, on X_ρ with generator $(\check{A}, D(\check{A}))$, which is the restriction of $(A_0 + P, D(A_0))$ on X_ρ . Moreover, the operator $(A, D(A))$ is the closure of $(\check{A}, D(\check{A}))$ in X .

We shall use the version of Proposition 5.2 in which A_0 is a multiplication operator. Let $a : E \rightarrow \mathbb{R}_+$ be a measurable nonnegative function on E . Set

$$(A_0 f)(x) = -a(x) f(x), \quad x \in E, \quad D(A_0) := \{f \in X : af \in X\}.$$

Clearly, $-A_0$ is a positive operator in X . Then, by, e.g., Lemma II.2.9 in [10], $(A_0, D(A_0))$ is the generator of the C_0 -semigroup composed by the (positive) multiplication operators $S_0(t) = \exp\{-ta(x)\}$, $t \geq 0$. For any $f \in X_\rho$, we have $\|S_0(t)f\|_\rho \leq \|f\|_\rho$; hence, $S_0(t)$ leaves X_ρ invariant. By, e.g., Proposition I.4.12 and Lemma II.2.9 in [10], the restrictions $\check{S}_0(t) := S_0(t)|_{X_\rho}$, $t \geq 0$, constitute a C_0 -semigroup in X_ρ with generator \check{A}_0 which acts as $\check{A}_0 f = A_0 f$ on the domain

$$D(\check{A}_0) = \{f \in X_\rho : af \in X_\rho\} \subset D(A_0).$$

Lemma 5.3. *Let $P : D(A_0) \rightarrow X$ be a positive linear operator such that*

$$\int_E (Pf)(x) \nu(dx) = \int_E a(x) f(x) \nu(dx), \quad f \in D^+(A_0).$$

Suppose also that there exist $c > 0, \varepsilon > 0$ such that, for all $f \in D^+(\check{A}_0)$, the following holds

$$\begin{aligned} \int_E (Pf)(x) \rho(x) \nu(dx) &\leq \int_E (c + a(x)) f(x) \rho(x) \nu(dx) \\ &\quad - \varepsilon \int_E a(x) f(x) \nu(dx). \end{aligned} \quad (5.12)$$

Then the statements of Proposition 5.2 hold.

Proof. To apply Proposition 5.2 we should only show that $P(D(\check{A}_0)) \subset X_\rho$. Let us show that it follows from (5.12). Indeed, for $f \in D(\check{A}_0)$, we have that both f and af are in X_ρ . Set $f^+ = \max\{f; 0\}$, $f^- = -\min\{f; 0\}$. Then $f^\pm \in X_\rho^+$ and $|f^\pm| \leq |f|$, which yields $af^\pm \in X_\rho$. Hence, $f^\pm \in D^+(\check{A}_0)$, and therefore, by (5.12),

$$\int_E (Pf^\pm)(x) \rho(x) \nu(dx) < \infty. \quad (5.13)$$

Since $f = f^+ - f^-$ and P is positive, we have by (5.13)

$$\begin{aligned} \|Pf\|_{X_\rho} &= \int_E |(Pf^+)(x) - (Pf^-)(x)| \rho(x) \nu(dx) \\ &\leq \int_E (|(Pf^+)(x)| + |(Pf^-)(x)|) \rho(x) \nu(dx) \\ &= \int_E ((Pf^+)(x) + (Pf^-)(x)) \rho(x) \nu(dx) < \infty. \end{aligned}$$

□

Proof of Lemma 5.1. For any $R \in \text{Dom}(L^\dagger)$, by (2.12), we have

$$\begin{aligned} \int_{\Gamma_0} |(BR)(\eta)| \lambda(d\eta) &\leq \int_{\Gamma_0} \int_{\mathbb{R}^d} (m + E^-(x, \eta)) |R(\eta \cup x)| dx \lambda(d\eta) \\ &\quad + \int_{\Gamma_0} \sum_{x \in \eta} E^+(x, \eta \setminus x) |R(\eta \setminus x)| \lambda(d\eta) \\ &= \int_{\Gamma_0} \Xi(\eta) |R(\eta)| \lambda(d\eta) < \infty. \end{aligned}$$

Then $B : \text{Dom}(L^\dagger) \rightarrow L^1(\Gamma_0, d\lambda)$. Clearly, B is positive, and by (5.10) we have that, for any positive $R \in D(A_0)$,

$$\int_{\Gamma_0} (L^\dagger R)(\eta) \lambda(d\eta) = \int_{\Gamma_0} (L1)(\eta) R(\eta) \lambda(d\eta) = 0,$$

and hence,

$$\int_{\Gamma_0} (BR)(\eta) \lambda(d\eta) = \int_{\Gamma_0} \Xi(\eta) R(\eta) \lambda(d\eta).$$

Now we apply Lemma 5.3 with $P = B$ and $\rho = b_\alpha \geq 1$, cf. (5.4). Recall, that \check{A}_0 is given by $(\check{A}_0 R)(\eta) = -\Xi(\eta) R(\eta)$ on the domain

$$D(\check{A}_0) = \{R \in L^1(\Gamma_0, b_\alpha d\lambda) : \Xi R \in L^1(\Gamma_0, b_\alpha d\lambda)\}.$$

Then, for any $0 \leq R \in D(\check{A}_0)$, we have

$$\begin{aligned} & \int_{\Gamma_0} (BR)(\eta) b_\alpha(\eta) \lambda(d\eta) \\ &= \int_{\Gamma_0} (L^\dagger R)(\eta) b_\alpha(\eta) \lambda(d\eta) + \int_{\Gamma_0} \Xi(\eta) R(\eta) b_\alpha(\eta) \lambda(d\eta) \\ &= \int_{\Gamma_0} R(\eta) (Lb_\alpha)(\eta) \lambda(d\eta) + \int_{\Gamma_0} \Xi(\eta) R(\eta) b_\alpha(\eta) \lambda(d\eta), \end{aligned}$$

where we have used (5.10) both sides of which are finite, see (5.14) below. According to Lemma 5.3, we have to pick positive c and ε such that

$$\int_{\Gamma_0} (Lb_\alpha)(\eta) R(\eta) \lambda(d\eta) \leq \int_{\Gamma_0} [cb_\alpha(\eta) - \varepsilon\Xi(\eta)] R(\eta) \lambda(d\eta), \quad (5.14)$$

holding for any positive $R \in D(\check{A}_0)$. By (5.7) and (2.18), we get

$$(Lb_\alpha)(\eta) = -\Xi(\eta)e^{\alpha|\eta|} + e^{\alpha|\eta|}e^{-\alpha}E(\eta) + e^{\alpha|\eta|}e^\alpha\langle a^+ \rangle|\eta|.$$

Hence, (5.14) holds if, for (λ -almost) all $\eta \in \Gamma_0$, we have that

$$e^{\alpha|\eta|}e^{-\alpha}E(\eta) + e^{\alpha|\eta|}e^\alpha\langle a^+ \rangle|\eta| \leq (c + \Xi(\eta))e^{\alpha|\eta|} - \varepsilon\Xi(\eta),$$

which is equivalent to

$$\varepsilon\Xi(\eta)e^{-\alpha|\eta|} + (e^\alpha - 1)(\langle a^+ \rangle|\eta| - e^{-\alpha}E(\eta)) \leq c. \quad (5.15)$$

For a given $\alpha > 0$ and any $c > 0$, by (5.6), (2.18), (2.17), and (3.7), it follows that

$$\varepsilon\Xi(\eta)e^{-\alpha|\eta|} \leq c, \quad \eta \in \Gamma_0,$$

for some $\varepsilon > 0$. Next, by (2.18) the second term in the left-hand side of (5.15) is non-positive whenever $\langle a^+ \rangle \leq e^{-\alpha}m$, which holds for sufficiently small $\alpha > 0$ in view of (5.11). \square

5.2 Dual local evolution

Our aim now is to construct the evolution dual to that of $R^\Lambda \mapsto S_\alpha^\dagger(t)R^\Lambda$ obtained in Lemma 5.1. Let \mathcal{F}_α be the dual space to \mathcal{R}_α as in (5.5). It is a weighted L^∞ space on Γ_0 with measure λ and norm

$$\|F\|_\alpha = \operatorname{ess\,sup}_{\eta \in \Gamma_0} |F(\eta)| \exp(-\alpha|\eta|). \quad (5.16)$$

Let \tilde{L}_α^\dagger be the operator dual to $L_\alpha^\dagger = L^\dagger|_{\mathcal{R}_\alpha}$ as in Lemma 5.1. Then the action of \tilde{L}_α^\dagger is described in (1.5). Let us show that, for any $\alpha' < \alpha$,

$$\mathcal{F}_{\alpha'} \subset \text{Dom}(\tilde{L}_\alpha^\dagger). \quad (5.17)$$

By (5.16) we have that $|F(\eta)| \leq \|F\|_{\alpha'} \exp(\alpha'|\eta|)$. Then

$$\begin{aligned} \|\tilde{L}_\alpha^\dagger F\|_\alpha &\leq \|F\|_{\alpha'} \text{ess sup}_{\eta \in \Gamma_0} (E(\eta) + \langle a^+ \rangle |\eta|) \exp(-(\alpha - \alpha')|\eta|) \\ &\quad + e^{-\alpha'} \|F\|_{\alpha'} \text{ess sup}_{\eta \in \Gamma_0} E(\eta) \exp(-(\alpha - \alpha')|\eta|) \\ &\quad + e^{\alpha'} \|F\|_{\alpha'} \text{ess sup}_{\eta \in \Gamma_0} \langle a^+ \rangle |\eta| \exp(-(\alpha - \alpha')|\eta|), \end{aligned}$$

which can be rewritten in the form

$$\|\tilde{L}_\alpha^\dagger F\|_\alpha \leq \|F\|_{\alpha'} (1 + e^{\alpha'}) \Delta_+(\alpha - \alpha') + \|F\|_{\alpha'} (1 + e^{-\alpha'}) \Delta_-(\alpha - \alpha'), \quad (5.18)$$

where, for $\beta > 0$,

$$\begin{aligned} \Delta_+(\beta) &:= \text{ess sup}_{\eta \in \Gamma_0} \langle a_+ \rangle |\eta| e^{-\beta|\eta|}, \\ \Delta_-(\beta) &:= \text{ess sup}_{\eta \in \Gamma_0} E(\eta) e^{-\beta|\eta|}. \end{aligned}$$

Let \mathcal{L}_α stand for the closure of $\text{Dom}(\tilde{L}_\alpha^\dagger)$ in \mathcal{F}_α . Note that \mathcal{L}_α is a proper subspace of \mathcal{F}_α . Set

$$\mathcal{L}_\alpha^\circ = \{F \in \text{Dom}(\tilde{L}_\alpha^\dagger) : \tilde{L}_\alpha^\dagger F \in \mathcal{L}_\alpha\}.$$

For $t \geq 0$, let $\tilde{S}_\alpha^\circ(t)$ be the restriction, to \mathcal{L}_α , of the operator dual to $S_\alpha^\dagger(t)$. By Theorem 10.4 in [26], the operators $\tilde{S}_\alpha^\circ(t)$, $t \geq 0$, constitute a C_0 -semigroup on \mathcal{L}_α , generated by $\tilde{L}_\alpha^\dagger|_{\mathcal{L}_\alpha^\circ}$. The latter operator, which is the part of \tilde{L}_α^\dagger in \mathcal{L}_α , will be denoted by \tilde{L}_α° . Note that, in view of (5.17) and (5.18), for any $\alpha' < \alpha$, the action of \tilde{L}_α° on $F \in \mathcal{F}_{\alpha'}$ is given by (1.5). Moreover, for any $\alpha'' < \alpha' < \alpha$, \tilde{L}_α° acts from $\mathcal{F}_{\alpha''}$ to $\mathcal{F}_{\alpha'}$, both considered as subsets of \mathcal{L}_α° .

For $\alpha' < \alpha$ and $F_0 \in \mathcal{F}_{\alpha'}$, we set

$$F_t = \tilde{S}_\alpha^\circ(t) F_0, \quad t > 0. \quad (5.19)$$

Then, see, e.g., page 5 in [26],

$$F_t = F_0 + \int_0^t \tilde{L}_\alpha^\circ F_s ds. \quad (5.20)$$

5.3 The main statement

We recall that any $k \in \mathcal{K}_\alpha$ is in fact a sequence of $k^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, $n \in \mathbb{N}_0$, such that

$$\sup_{n \in \mathbb{N}} e^{\alpha n} \|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} < \infty, \quad \alpha \in \mathbb{R},$$

see (4.1) and (4.2). By Proposition 2.1, such $k \in \mathcal{K}_\alpha$ is a correlation function of a unique $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ whenever $k^{(0)} = 1$ and

$$\langle\langle G, k \rangle\rangle \geq 0, \quad \text{for all } G \in B_{\text{bs}}^+(\Gamma_0), \quad (5.21)$$

see (2.6) and (2.10). For $\alpha \in \mathbb{R}$, we set

$$\mathcal{M}_\alpha(\Gamma) = \{\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma) : k_\mu \in \mathcal{K}_\alpha\},$$

where μ and k_μ are as in (2.8).

Theorem 5.4. *Let θ , α_* , α^* , and T_* be as in Theorem 3.4. Let also (5.11) hold, and let $k_0 \in \mathcal{K}_{\alpha^*}$ be the correlation function of μ_0 , and k_t be the solution of (2.26) with $k_t|_{t=0} = k_0$, as in Theorem 4.2. Then, there exists $\mu_t \in \mathcal{M}_{\alpha^*}(\Gamma)$ such that $k_{\mu_t} = k_t$. In other words, the evolution $k_0 \mapsto k_t$ uniquely determines the evolution of the corresponding states*

$$\mathcal{M}_{\alpha^*}(\Gamma) \ni \mu_0 \mapsto \mu_t \in \mathcal{M}_{\alpha^*}(\Gamma), \quad t > 0.$$

Proof. The main idea of the proof is to show that k_t can be approximated in a certain sense by a sequence of ‘correlation functions’, for which (5.21) holds. To this end we use two sequences $\{\Lambda_n\}_{n \in \mathbb{N}} \subset \mathcal{B}_b(\mathbb{R}^d)$ and $\{N_l\}_{l \in \mathbb{N}} \subset \mathbb{N}$. Both are increasing, and $\{\Lambda_n\}_{n \in \mathbb{N}}$ is exhausting, which means that each $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is contained in Λ_n with big enough n .

Given $\mu_0 \in \mathcal{M}_{\alpha^*}$, let $k_0 \in \mathcal{K}_{\alpha^*}$ be such that $k_{\mu_0} = k_0$. Recall that this means that the projections μ^Λ are absolutely continuous with respect to λ , see (2.8) and (5.1). For this μ_0 , and for Λ_n and N_l as above, we set

$$R_0^{\Lambda_n, N_l}(\eta) = R_0^{\Lambda_n}(\eta) I_{N_l}(\eta), \quad (5.22)$$

where $R_0^{\Lambda_n}$ is the local density as in (5.1), and

$$I_N(\eta) := \begin{cases} 1, & \text{if } |\eta| \leq N; \\ 0, & \text{otherwise.} \end{cases} \quad (5.23)$$

Noteworthy, $R_0^{\Lambda_n, N_l}$ is a positive element of $L^1(\Gamma_0, d\lambda)$ with $\|R_0^{\Lambda_n, N_l}\|_{L^1(\Gamma_0, d\lambda)} \leq 1$, and $R_0^{\Lambda_n, N_l} \in \mathcal{R}_\alpha$ for any $\alpha > 0$. Indeed, cf. (5.5),

$$\|R_0^{\Lambda_n, N_l}\|_{\mathcal{R}_\alpha} = \sum_{m=0}^{N_l} \frac{r_m}{m!} e^{\alpha m}, \quad r_m := \left\| (R_0^{\Lambda_n, N_l})^{(m)} \right\|_{L^1(\mathbb{R}^{md})}.$$

Then, for any $\alpha > 0$ and any $t \geq 0$, we can apply $S_\alpha^\dagger(t)$, as in Theorem 5.1, and obtain

$$R_t^{\Lambda_n, N_l} = S_\alpha^\dagger(t) R_0^{\Lambda_n, N_l} \in \mathcal{R}_\alpha^+ := \{R \in \mathcal{R}_\alpha : R \geq 0\}, \quad (5.24)$$

which yields, cf. (5.20),

$$R_t^{\Lambda_n, N_l} = R_0^{\Lambda_n, N_l} + \int_0^t L_\alpha^\dagger R_s^{\Lambda_n, N_l} ds.$$

For $G_0 \in B_{\text{bs}}^+(\Gamma_0)$, see (2.6), let us consider

$$F_0(\eta) = \sum_{\xi \subset \eta} G_0(\xi). \quad (5.25)$$

Since $G_0(\xi) = 0$ for all ξ such that $|\xi|$ exceeds some $N(G_0)$, see (2.4), we have that

$$|F_0(\eta)| \leq (1 + |\eta|)^{N(G_0)} C(G_0), \quad (5.26)$$

for some $C(G_0) > 0$, and hence $F_0 \in \mathcal{F}_\alpha$ for any $\alpha > 0$. Therefore, the map $\mathcal{R}_\alpha \ni R \mapsto \langle\langle F_0, R \rangle\rangle$ is continuous, and thus we can write, see (5.18),

$$\begin{aligned} \langle\langle F_0, R_t^{\Lambda_n} \rangle\rangle &= \langle\langle F_0, R_0^{\Lambda_n} \rangle\rangle + \int_0^t \langle\langle F_0, L_\alpha^\dagger R_s^{\Lambda_n} \rangle\rangle ds \\ &= \langle\langle F_0, R_0^{\Lambda_n} \rangle\rangle + \int_0^t \langle\langle \tilde{L}_\alpha^\dagger F_0, R_s^{\Lambda_n} \rangle\rangle ds. \end{aligned} \quad (5.27)$$

Now we set, cf. (5.2),

$$q_t^{\Lambda_n, N_l}(\eta) = \int_{\Gamma_0} R_t^{\Lambda_n, N_l}(\eta \cup \xi) \lambda(d\xi), \quad t \geq 0. \quad (5.28)$$

For $G_0 \in B_{\text{bs}}^+(\Gamma_0)$ and any $t \geq 0$, by (2.12) and (5.25) we have

$$\begin{aligned} \langle\langle G_0, q_t^{\Lambda_n, N_l} \rangle\rangle &= \int_{\Gamma_0} \int_{\Gamma_0} G_0(\eta) R_t^{\Lambda_n, N_l}(\eta \cup \xi) \lambda(d\xi) \lambda(d\eta) \\ &= \int_{\Gamma_0} \left(\sum_{\xi \subset \eta} G_0(\xi) \right) R_t^{\Lambda_n, N_l}(\eta) \lambda(d\eta) \\ &= \langle\langle F_0, R_t^{\Lambda_n, N_l} \rangle\rangle, \end{aligned} \quad (5.29)$$

which in view of (2.6) and (5.24) yields

$$\langle\langle G_0, q_t^{\Lambda_n, N_l} \rangle\rangle \geq 0. \quad (5.30)$$

Applying again (2.12), for $\alpha > 0$ we obtain, cf. (5.29),

$$\begin{aligned} \int_{\Gamma_0} e^{\alpha|\eta|} q_t^{\Lambda_n, N_l}(\eta) \lambda(d\eta) &= \int_{\Gamma_0} \left(\sum_{\xi \subset \eta} e^{\alpha|\xi|} \right) R_t^{\Lambda_n, N_l}(\eta) \lambda(d\eta) \\ &= \int_{\Gamma_0} (1 + e^\alpha)^{|\eta|} R_t^{\Lambda_n, N_l}(\eta) \lambda(d\eta). \end{aligned}$$

Since both $q_t^{\Lambda_n, N_t}$ and $R_t^{\Lambda_n, N_t}$ are positive and $R_t^{\Lambda_n, N_t}$ is in $\mathcal{R}_{\alpha'}$ for any $\alpha' > 0$, the latter yields that, for any $\alpha > 0$ and $t \geq 0$,

$$q_t^{\Lambda_n, N_t} \in \mathcal{R}_\alpha.$$

As was already mentioned, our aim is to show that, in a weak sense, $q_t^{\Lambda_n, N_t}$ converges to k_t as in Theorems 4.2. Note that k_t belongs to \mathcal{K}_{α^*} , which is a completely different space than \mathcal{R}_{α^*} , see (4.1).

To proceed further, we need to define the action of powers of \widehat{L} as in (2.21) on suitable sets of functions, which include $B_{\text{bs}}(\Gamma_0)$. Recall that any function $h : \Gamma_0 \rightarrow \mathbb{R}$ is a sequence of symmetric functions $h^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$, where $h^{(0)}$ is a constant function. Let \mathcal{H}_{fin} be the set of measurable $h : \Gamma_0 \rightarrow \mathbb{R}$, for each of which there exists $N(h) \in \mathbb{N}_0$ such that $h^{(n)} = 0$ whenever $n > N(h)$. Then we set

$$\begin{aligned} \mathcal{H}_{\text{fin}}^1 &= \{h \in \mathcal{H}_{\text{fin}} : h^{(n)} \in L^1((\mathbb{R}^d)^n), \text{ for } n \leq N(h)\}, \\ \mathcal{H}_{\text{fin}}^\infty &= \{h \in \mathcal{H}_{\text{fin}} : h^{(n)} \in L^\infty((\mathbb{R}^d)^n), \text{ for } n \leq N(h)\}. \end{aligned} \quad (5.31)$$

Note that

$$B_{\text{bs}}(\Gamma_0) \subset \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty, \quad (5.32)$$

and, for any $\alpha > 0$ and $\alpha' \in \mathbb{R}$,

$$\mathcal{H}_{\text{fin}}^1 \subset \mathcal{R}_\alpha, \quad \mathcal{H}_{\text{fin}}^\infty \subset \mathcal{K}_{\alpha'}. \quad (5.33)$$

Furthermore, cf. (2.5) and (5.26), for any $\alpha > 0$,

$$K : \mathcal{H}_{\text{fin}}^\infty \rightarrow \mathcal{F}_\alpha. \quad (5.34)$$

Let A and B be as in (2.21) and (2.22), (2.24). Then, for $G \in \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty$ and $n \in \mathbb{N}_0$, we have, see (2.15),

$$\begin{aligned} \|(AG)^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} &\leq (nm + n^2 \|a^-\|) \|G^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \\ &\quad + n \langle a^+ \rangle \|G^{(n+1)}\|_{L^\infty((\mathbb{R}^d)^n)}, \\ \|(BG)^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} &\leq n(n-1) \|a^-\| \|G^{(n+1)}\|_{L^\infty((\mathbb{R}^d)^n)} \\ &\quad + n \langle a^+ \rangle \|G^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}, \\ \|(AG)^{(n)}\|_{L^1((\mathbb{R}^d)^n)} &\leq (nm + n^2 \|a^-\|) \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} \\ &\quad + n \|a^+\| \|G^{(n+1)}\|_{L^1((\mathbb{R}^d)^n)}, \\ \|(BG)^{(n)}\|_{L^1((\mathbb{R}^d)^n)} &\leq n \langle a^- \rangle \|G^{(n-1)}\|_{L^1((\mathbb{R}^d)^n)} \\ &\quad + n \|a^+\| \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}. \end{aligned}$$

Thus, \widehat{L} given in (2.21) can be defined on both sets (5.31) and

$$N(\widehat{L}G) = N(G) + 1, \quad \widehat{L} : \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty \rightarrow \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty. \quad (5.35)$$

Now we fix Λ_n and N_l , and let μ_0 be in $\mathcal{M}_{\alpha^*}(\Gamma)$. Then for $\eta \in \Gamma_{\Lambda_n}$, $k_0(\eta)$ is given by (2.8), and hence, see (5.28), (5.22), and (5.2),

$$q_0^{\Lambda_n, N_l}(\eta) \leq \int_{\Gamma_0} R_0^{\Lambda_n}(\eta \cup \xi) \lambda(d\xi) = k_0(\eta), \quad \eta \in \Gamma_{\Lambda_n}, \quad (5.36)$$

which can readily be extended to all $\eta \in \Gamma_0$. Thus, $q_0^{\Lambda_n, N_l} \in \mathcal{K}_{\alpha^*}$. For $t \in [0, T_*)$, let $k_t^{\Lambda_n, N_l}$ be the solution of the problem (2.26) with $k_t^{\Lambda_n, N_l}|_{t=0} = q_0^{\Lambda_n, N_l}$, as in Theorem 4.2. Then, for $t \in (0, T_*)$,

$$k_t^{\Lambda_n, N_l} = q_0^{\Lambda_n, N_l} + \int_0^t L^\Delta k_s^{\Lambda_n, N_l} ds,$$

where L^Δ is defined in (4.13). In view of (5.33), for any $G \in \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty$, we then have

$$\langle\langle G, k_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G, q_0^{\Lambda_n, N_l} \rangle\rangle + \int_0^t \langle\langle \widehat{L}G, k_s^{\Lambda_n, N_l} \rangle\rangle ds. \quad (5.37)$$

At the same time, for such G , we have that KG is in each \mathcal{F}_α , $\alpha > 0$, cf. (5.34), and hence, see (5.27) and (5.29),

$$\langle\langle G, q_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G, q_0^{\Lambda_n, N_l} \rangle\rangle + \int_0^t \langle\langle \widetilde{L}_\alpha^\dagger KG, R_s^{\Lambda_n, N_l} \rangle\rangle ds.$$

As was mentioned at the beginning of Subsection 5.2, the action of $\widetilde{L}_\alpha^\dagger$ is described in (1.5). Thus, by (2.20) we obtain from the latter

$$\langle\langle G, q_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G, q_0^{\Lambda_n, N_l} \rangle\rangle + \int_0^t \langle\langle \widehat{L}G, q_s^{\Lambda_n, N_l} \rangle\rangle ds. \quad (5.38)$$

For G as in (5.37) and (5.38), we set

$$\phi(t, G) = \langle\langle G, k_t^{\Lambda_n, N_l} \rangle\rangle - \langle\langle G, q_t^{\Lambda_n, N_l} \rangle\rangle. \quad (5.39)$$

Then

$$\phi(t, G) = \int_0^t \phi(s, \widehat{L}G) ds, \quad \phi(0, G) = 0.$$

For any $n \in \mathbb{N}$, the latter yields

$$\frac{d^n}{dt^n} \phi(t, G) = \phi(t, \widehat{L}^n G). \quad (5.40)$$

In view of (5.35), $\phi(t, G)$ is infinitely differentiable on $(0, T_*)$, and

$$\frac{d^n}{dt^n} \phi(0, G) = 0, \quad \text{for all } n \in \mathbb{N}_0.$$

Thus, $\phi(t, G) \equiv 0$, and hence, for all $G_0 \in B_{\text{bs}}^+(\Gamma_0)$, we have, see (5.30) and (5.32),

$$\langle\langle G_0, q_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_0, k_t^{\Lambda_n, N_l} \rangle\rangle \geq 0. \quad (5.41)$$

Now let k_t be the solution of (2.26) with $k_t|_{t=0} = k_0$. In Appendix, we prove that, for any $G \in B_{\text{bs}}(\Gamma_0)$,

$$\langle\langle G, k_t \rangle\rangle = \lim_{n \rightarrow \infty} \lim_{l \rightarrow \infty} \langle\langle G, k_t^{\Lambda_n, N_l} \rangle\rangle, \quad (5.42)$$

point-wise on $[0, T_*)$. Then by (5.41) we get that, for each $t \in (0, T_*)$ and any $G \in B_{\text{bs}}^+(\Gamma_0)$,

$$\langle\langle G, k_t \rangle\rangle \geq 0,$$

which together with the fact that $k_t \in \mathcal{K}_{\alpha^*}$ by Proposition 2.1 yields that k_t is the correlation function for a certain unique $\mu_t \in \mathcal{M}_{\alpha^*}(\Gamma)$. \square

Remark 5.5. Theorem 5.4 holds true for $m = 0$ and $a^+ \equiv 0$, which can be seen from (5.15).

Proposition 5.6. *Let the conditions of Theorem 5.4 hold. Then k_t , as in Theorem 4.2, and k_t^D , as in Theorem 4.5, coincide for all $t \in [0, T_*)$, whenever $k_0^D = k_0$.*

Proof. As in the proof of Theorem 5.4, we are going to show that k_t^D can be approximated by the same sequence of ‘correlation functions’ (5.28). For $G_0 \in B_{\text{bs}}(\Gamma_0)$, let F_0 be as in (5.25). Since F_0 is polynomially bounded, see (5.26), we have that $F_0 \in \mathcal{F}_{\alpha'}$ for any $\alpha' < \alpha$, where α is as in Theorem 5.1. Then we can apply (5.19) and obtain (5.20). For fixed Λ_n and N_l , $R_0^{\Lambda_n, N_l}$ is in any $\mathcal{R}_{\alpha'}$, and hence the map

$$\mathcal{F}_\alpha \ni F \mapsto \langle\langle F, R_0^{\Lambda_n, N_l} \rangle\rangle \in \mathbb{R}$$

is continuous. Since the Bochner integral in (5.20) is in \mathcal{F}_α , we have

$$\langle\langle F_t, R_0^{\Lambda_n, N_l} \rangle\rangle = \langle\langle F_0, R_0^{\Lambda_n, N_l} \rangle\rangle + \int_0^t \langle\langle F_s, \tilde{L}_\alpha^\dagger R_0^{\Lambda_n, N_l} \rangle\rangle ds. \quad (5.43)$$

On the other hand,

$$\tilde{G}_t(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F_t(\xi)$$

is in \mathcal{F}_β , $\beta = \log(1 + e^\alpha)$. Thus, we can rewrite (5.43) in the form

$$\langle\langle \tilde{G}_t, q_0^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_0, q_0^{\Lambda_n, N_l} \rangle\rangle + \int_0^t \langle\langle \tilde{G}_s, L^\Delta q_0^{\Lambda_n, N_l} \rangle\rangle ds. \quad (5.44)$$

For the evolution $G_0 \mapsto G_t \in \mathcal{G}_{\alpha^*}$ described by Theorem 3.4, in a similar way we have

$$\langle\langle G_t, q_0^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_0, q_0^{\Lambda_n, N_l} \rangle\rangle + \int_0^t \langle\langle G_s, L^\Delta q_0^{\Lambda_n, N_l} \rangle\rangle ds. \quad (5.45)$$

It is easy to see that $N(q_0^{\Lambda_n, N_l}) = N_l$ and, for any $m \leq N_l$, cf. (5.36),

$$\begin{aligned} \left\| (q_0^{\Lambda_n, N_l})^{(m)} \right\|_{L^\infty(\mathbb{R}^{md})} &\leq \left\| k_0^{(m)} \right\|_{L^\infty(\mathbb{R}^{md})}, \\ \left\| (q_0^{\Lambda_n, N_l})^{(m)} \right\|_{L^1(\mathbb{R}^{md})} &\leq \sum_{s=0}^{N_l-m} \frac{1}{s!} \left\| (R_0^{\Lambda_n})^{(m+s)} \right\|_{L^1(\Lambda^m)}. \end{aligned}$$

Hence,

$$q_0^{\Lambda_n, N_l} \in \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty.$$

Similarly as in (5.35), one can show that

$$L^\Delta : \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty \rightarrow \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty.$$

Now, for $h \in \mathcal{H}_{\text{fin}}^1 \cap \mathcal{H}_{\text{fin}}^\infty$, we introduce, cf. (5.39),

$$\phi(t, h) = \langle\langle \tilde{G}_t, h \rangle\rangle - \langle\langle G_t, h \rangle\rangle,$$

for which by (5.45) and (5.44) we get

$$\phi(t, h) = \int_0^t \phi(s, L^\Delta h) ds, \quad \phi(0, h) = 0.$$

Employing the same arguments as in (5.40), (5.41) we then obtain

$$\langle\langle \tilde{G}_t, q_0^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_t, q_0^{\Lambda_n, N_l} \rangle\rangle. \quad (5.46)$$

On the other hand, by (4.22) we have

$$\langle\langle G_t, q_0^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_0, \tilde{k}_t^{\Lambda_n, N_l} \rangle\rangle,$$

where the evolution $q_0^{\Lambda_n, N_l} \mapsto \tilde{k}_t^{\Lambda_n, N_l}$ is described by Theorem 4.5. At the same time, see (5.29),

$$\begin{aligned} \langle\langle \tilde{G}_t, q_0^{\Lambda_n, N_l} \rangle\rangle &= \langle\langle F_t, R_0^{\Lambda_n, N_l} \rangle\rangle = \langle\langle F_0, R_t^{\Lambda_n, N_l} \rangle\rangle \\ &= \langle\langle G_0, q_t^{\Lambda_n, N_l} \rangle\rangle, \end{aligned}$$

where $q_t^{\Lambda_n, N_l}$ is the same as in (5.28) and (5.41). Then (5.46) can be rewritten

$$\langle\langle G_0, \tilde{k}_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_0, q_t^{\Lambda_n, N_l} \rangle\rangle,$$

which holds for all $G_0 \in B_{\text{bs}}(\Gamma_0)$. Then, by (5.41) we have that, for all $G_0 \in B_{\text{bs}}(\Gamma_0)$,

$$\langle\langle G_0, \tilde{k}_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_0, k_t^{\Lambda_n, N_l} \rangle\rangle,$$

and, for $G_0 \in B_{\text{bs}}^+(\Gamma_0)$,

$$\langle\langle G_0, \tilde{k}_t^{\Lambda_n, N_l} \rangle\rangle \geq 0.$$

At the same time, by (4.22) we have

$$\begin{aligned} \langle\langle G_0, k_t^D \rangle\rangle - \langle\langle G_0, \tilde{k}_t^{\Lambda_n, N_t} \rangle\rangle &= \langle\langle G_t, k_0 \rangle\rangle - \langle\langle G_t, q_0^{\Lambda_n, N_t} \rangle\rangle \\ &= \int_{\Gamma_0} G_t(\eta) k_0(\eta) (1 - \mathbb{I}_{\Gamma_{\Lambda_n}}(\eta)) \lambda(d\eta) \\ &\quad + \int_{\Gamma_0} G_t(\eta) \left[k_0(\eta) \mathbb{I}_{\Gamma_{\Lambda_n}}(\eta) - q_0^{\Lambda_n, N_t}(\eta) \right] \lambda(d\eta). \end{aligned}$$

Then exactly as in (5.42) we obtain

$$\langle\langle G_0, k_t^D \rangle\rangle = \lim_{n \rightarrow \infty} \lim_{l \rightarrow \infty} \langle\langle G_0, q_t^{\Lambda_n, N_t} \rangle\rangle = \lim_{n \rightarrow \infty} \lim_{l \rightarrow \infty} \langle\langle G_0, \tilde{k}_t^{\Lambda_n, N_t} \rangle\rangle,$$

which holds for any $G_0 \in B_{\text{bs}}(\Gamma_0)$. Thus, for all $G_0 \in B_{\text{bs}}^+(\Gamma_0)$,

- (a) $\forall G_0 \in B_{\text{bs}}(\Gamma_0) \quad \langle\langle G_0, k_t^D \rangle\rangle = \langle\langle G_0, k_t \rangle\rangle,$
- (b) $\forall G_0 \in B_{\text{bs}}^+(\Gamma_0) \quad \langle\langle G_0, k_t^D \rangle\rangle \geq 0.$

The latter property yields that k_t^D is a correlation function. To complete the proof we have to show that (a) implies $k_t = k_t^D$. In the topology induced from Γ , each Γ_Λ , $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, is a Polish space. Let \mathcal{C}_Λ be the set of all bounded continuous $G : \Gamma_\Lambda \rightarrow \mathbb{R}$. Since (a) holds for all $G \in \mathcal{C}_\Lambda \cap B_{\text{bs}}(\Gamma_0)$, the projections of the correlation measures (2.7) corresponding to k_t and k_t^D on each $\mathcal{B}(\Gamma_\Lambda)$ coincide, see e.g. Proposition 1.3.27 in [1]. This yields $k_t = k_t^D$, and hence completes the proof. \square

6 Concluding Remarks

In Theorem 3.4, we have shown that the evolution of quasi-observables exists with the only condition that (3.12) holds. However, this evolution is restricted in time and takes place in the scale of Banach spaces (3.1), (3.2). In [15], the analytic semigroup that defines the evolution of quasi-observables was constructed. Thus, the evolution $G_0 \mapsto G_t$ defined by this semigroup takes place in one space \mathcal{G}_α for all $t > 0$. However, this result was obtained under the additional condition that in our notations takes the form

$$m > 4(\langle a^- \rangle e^{-\alpha} + \langle a^+ \rangle), \tag{6.1}$$

by which the constant mortality should dominate not only the dispersion but also the competition. In our Theorem 3.4 the value of m can be arbitrary and even equal to zero.

As was shown in [13], under conditions similar to (3.12) and (6.1) the corresponding semigroup evolution $k_0 \mapsto k_t$ exists in a proper Banach subspace of \mathcal{K}_α , cf. (4.1). In our Theorem 4.2, we construct the evolution $k_0 \mapsto k_t$ in the scale of spaces (4.3), restricted in time, but under the condition of (3.12)

only. Hence, it holds for any $m \geq 0$. Note that the problem of whether k_t for $t > 0$ are correlation functions of probability measures on $\mathcal{B}(\Gamma)$, provided k_0 is, has not been studied yet in the literature. In our Theorem 5.4, we prove that this evolution corresponds to the evolution of probability measures if $m > \langle a^+ \rangle$ holds in addition to (3.12), cf. Remark 5.5.

A Proof of (5.42)

For fixed $t \in (0, T_*)$ and $G_0 \in B_{\text{bs}}(\Gamma_0)$, by (4.23) we have

$$\langle\langle G_0, k_t \rangle\rangle - \langle\langle G_0, k_t^{\Lambda_n, N_l} \rangle\rangle = \langle\langle G_t^D, k_0 \rangle\rangle - \langle\langle G_t^D, q_0^{\Lambda_n, N_l} \rangle\rangle = \mathcal{I}_n^{(1)} + \mathcal{I}_{n,l}^{(2)}, \quad (\text{A.1})$$

where we set

$$\begin{aligned} \mathcal{I}_n^{(1)} &= \int_{\Gamma_0} G_t^D(\eta) k_0(\eta) (1 - \mathbb{I}_{\Gamma_{\Lambda_n}}(\eta)) \lambda(d\eta), \\ \mathcal{I}_{n,l}^{(2)} &= \int_{\Gamma_0} G_t^D(\eta) \left[k_0(\eta) \mathbb{I}_{\Gamma_{\Lambda_n}}(\eta) - q_0^{\Lambda_n, N_l}(\eta) \right] \lambda(d\eta). \end{aligned}$$

Let us prove that, for an arbitrary $\varepsilon > 0$,

$$|\mathcal{I}_n^{(1)}| < \varepsilon/2, \quad (\text{A.2})$$

for sufficiently big Λ_n . Recall that k_0 is a correlation function, and hence is positive. Taking into account that

$$\mathbb{I}_{\Gamma_{\Lambda}}(\eta) = \prod_{x \in \eta} \mathbb{I}_{\Lambda}(x),$$

we have

$$\begin{aligned} |\mathcal{I}_n^{(1)}| &\leq \int_{\Gamma_0} |G_t^D(\eta)| k_0(\eta) (1 - \mathbb{I}_{\Gamma_{\Lambda_n}}(\eta)) \lambda(d\eta) \\ &= \sum_{p=1}^{\infty} \frac{1}{p!} \int_{(\mathbb{R}^d)^p} \left| (G_t^D)^{(p)}(x_1, \dots, x_p) \right| k_0^{(p)}(x_1, \dots, x_p) \\ &\quad \times J_{\Lambda_n}(x_1, \dots, x_p) dx_1 \cdots dx_p, \end{aligned} \quad (\text{A.3})$$

where

$$\begin{aligned} J_{\Lambda}(x_1, \dots, x_p) &:= 1 - \mathbb{I}_{\Lambda}(x_1) \cdots \mathbb{I}_{\Lambda}(x_p) \\ &= \mathbb{I}_{\Lambda^c}(x_1) \mathbb{I}_{\Lambda}(x_2) \cdots \mathbb{I}_{\Lambda}(x_p) + \mathbb{I}_{\Lambda^c}(x_2) \mathbb{I}_{\Lambda}(x_3) \cdots \mathbb{I}_{\Lambda}(x_p) \\ &\quad + \cdots + \mathbb{I}_{\Lambda^c}(x_{p-1}) \mathbb{I}_{\Lambda}(x_p) + \mathbb{I}_{\Lambda^c}(x_p), \\ &\leq \sum_{s=1}^p \mathbb{I}_{\Lambda^c}(x_s), \end{aligned} \quad (\text{A.4})$$

and $\Lambda^c := \mathbb{R}^d \setminus \Lambda$. Taking into account that $k_0 \in \mathcal{K}_{\alpha^*}$, cf. (4.1) and (4.2), by (A.4) we obtain in (A.3)

$$|\mathcal{I}_n^{(1)}| \leq \|k_0\|_{\alpha^*} \sum_{p=1}^{\infty} \frac{p}{p!} e^{-\alpha^* p} \int_{\Lambda_n^c} \int_{(\mathbb{R}^d)^{p-1}} |(G_t^D)^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p. \quad (\text{A.5})$$

For t as in (A.1), one finds $\alpha < \alpha^*$ such that $G_t^D \in \mathcal{G}_\alpha$, see Remark 4.6. For this α and ε as in (A.2), we pick $\bar{p} \in \mathbb{N}$ such that

$$\sum_{p=\bar{p}+1}^{\infty} \frac{e^{-\alpha p}}{p!} \int_{(\mathbb{R}^d)^p} |(G_t^D)^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p < \frac{\varepsilon e(\alpha^* - \alpha)}{4\|k_0\|_{\alpha^*}}. \quad (\text{A.6})$$

Then we apply (A.6) and the following evident estimate

$$pe^{-\alpha^* p} \leq e^{-\alpha p} / e(\alpha^* - \alpha),$$

and obtain in (A.5) the following

$$|\mathcal{I}_n^{(1)}| < \frac{\|k_0\|_{\alpha^*}}{e(\alpha^* - \alpha)} \sum_{p=1}^{\bar{p}} \frac{p}{p!} e^{-\alpha^* p} \int_{\Lambda_n^c} \int_{(\mathbb{R}^d)^{p-1}} |(G_t^D)^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p + \frac{\varepsilon}{4}.$$

Here the first term contains a finite number of summands, in each of which $(G_t^D)^{(p)}$ is in $L^1((\mathbb{R}^d)^p)$. Hence, it can be made strictly smaller than $\varepsilon/4$ by picking big enough Λ_n , which yields (A.2).

Let us show the same for the second integral in (A.1). Write, see (5.2), (5.28), (5.22), and (5.23),

$$\begin{aligned} \mathcal{I}_{n,l}^{(2)} &= \int_{\Gamma_0} G_t^D(\eta) \int_{\Gamma_0} R_0^{\Lambda_n}(\eta \cup \xi) [1 - I_{N_l}(\eta \cup \xi)] \lambda(d\eta) \lambda(d\xi) \\ &= \int_{\Gamma_0} F_t(\eta) R_0^{\Lambda_n}(\eta) [1 - I_{N_l}(\eta)] \lambda(d\eta) \\ &= \sum_{m=N_l+1}^{\infty} \frac{1}{m!} \int_{\Lambda_n^m} (R_0^{\Lambda_n})^{(m)}(x_1, \dots, x_m) F_t^{(m)}(x_1, \dots, x_m) dx_1 \cdots dx_m, \end{aligned}$$

where

$$F_t(\eta) := \sum_{\xi \subset \eta} G_t^D(\xi),$$

and hence

$$F_t^{(m)}(x_1, \dots, x_m) = \sum_{s=0}^m \sum_{\{i_1, \dots, i_s\} \subset \{1, \dots, m\}} (G_t^D)^{(s)}(x_{i_1}, \dots, x_{i_s}). \quad (\text{A.7})$$

By (5.2), for $x_i \in \Lambda_n$, $i = 1, \dots, m$, we have

$$k_0^{(m)}(x_1, \dots, x_m) = \sum_{s=0}^{\infty} \int_{\Lambda_n^s} (R_0^{\Lambda_n})^{(m+s)}(x_1, \dots, x_m, y_1, \dots, y_s) dy_1 \cdots dy_s,$$

from which we immediately get that

$$(R_0^{\Lambda_n})^{(m)}(x_1, \dots, x_m) \leq k_0^{(m)}(x_1, \dots, x_m) \leq e^{-\alpha^* m} \|k_0\|_{\alpha^*},$$

since $k_0 \in \mathcal{K}_{\alpha^*}$. Now let Λ_n be such that (A.2) holds. Then we can have

$$|\mathcal{I}_{n,t}^{(2)}| < \varepsilon/2, \tag{A.8}$$

holding for big enough N_l if $e^{-\alpha^* |\cdot|} F_t$ is in $L^1(\Lambda_n, d\lambda)$. By (A.7),

$$\begin{aligned} & \sum_{p=0}^{\infty} \frac{1}{p!} e^{-\alpha^* p} \int_{\Lambda_n^p} |F^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p \\ & \leq \sum_{p=0}^{\infty} \sum_{s=0}^p \frac{1}{s!(p-s)!} e^{-\alpha^* s} \left\| (G_t^D)^{(s)} \right\|_{L^1(\Gamma_0, d\lambda)} e^{-\alpha^*(p-s)} [\ell(\Lambda_n)]^{p-s} \\ & = \|G_t^D\|_{\alpha^*} \exp\left(e^{-\alpha^*} \ell(\Lambda_n)\right), \end{aligned}$$

where $\ell(\Lambda_n)$ is the Lebesgue measure of Λ_n . This yields (A.8) and thereby also (5.42).

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References

- [1] S. Albeverio, Y. Kondratiev, Y. Kozitsky, and M. Röckner, *The statistical mechanics of quantum lattice systems. A path integral approach*. EMS Tracts in Mathematics, 8. European Mathematical Society (EMS), Zürich, 2009.
- [2] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces, *J. Func. Anal.*, **154** (1998) 444–500.
- [3] V. P. Belavkin, V. P. Maslov, and S. E. Tariverdiev, The asymptotic dynamics of a system with a large number of particles described by Kolmogorov–Feller equations. (Russian) *Teoret. Mat. Fiz.*, **49** (1981) 298–306.
- [4] D. A. Birch and W. R. Young, A master equation for a spatial population model with pair interactions, *Theoret. Population Biol.* **70** (2006) 26–42.

- [5] B. M. Bolker and S. W. Pacala, Using moment equations to understand stochastically driven spatial pattern formation in ecological systems, *Theoret. Population Biol.* **52** (1997) 179–197.
- [6] B. M. Bolker and S. W. Pacala, Spatial moment equations for plant competitions: Understanding spatial strategies and the advantages of short dispersal, *The American Naturalist* **153** (1999) 575–602.
- [7] B. M. Bolker, S. W. Pacala, and C. Neuhauser, Spatial dynamics in model plant communities: What do we really know? *The American Naturalist* **162** (2003) 135–148.
- [8] U. Dieckmann and R. Law, Relaxation projections and the method of moments, *The Geometry of Ecological Interactions*, (Cambridge University Press, Cambridge, UK, 2000), pp. 412–455.
- [9] R. L. Dobrushin, Y. G. Sinai, and Y. M. Sukhov, Dynamical systems of statistical mechanics, *Itogi Nauki*, (VINITI, 1985), pp. 235–284; eng. transl. *Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics*, II ed. Ya. G. Sinai, (Encyclopaedia Math. Sci., Springer, Berlin Heidelberg, 1989).
- [10] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt, Graduate Texts in Mathematics, 194. (Springer-Verlag, New York, 2000).
- [11] R. C. Fetecau, Collective behavior of biological aggregations in two dimensions: a nonlocal kinetic model, *Math. Models Methods Appl. Sci.* **21** (2011) 1539–1569.
- [12] D. L. Finkelshtein, Yu. G. Kondratiev, and O. Kutovyi, Operator approach to Vlasov scaling for some models of spatial ecology, arXiv preprint: 1104.2418.
- [13] D. L. Finkelshtein, Yu. G. Kondratiev, and O. Kutovyi, Semigroup approach to birth-and-death stochastic dynamics in continuum, *J. Funct. Anal.* (2012) doi:10.1016/j.jfa.2011.11.005.
- [14] D. L. Finkelshtein, Yu. G. Kondratiev, and O. Kutovyi, Vlasov scaling for stochastic dynamics of continuous systems, *J. Statist. Phys.* **141** (2010) 158–178.
- [15] D. L. Finkelshtein, Yu. G. Kondratiev, and O. Kutovyi, Individual based model with competition in spatial ecology, *SIAM J. Math. Anal.* **41** (2009) 297–317.

- [16] D. L. Finkelshtein, Yu. G. Kondratiev, and M. J. Oliveira, Markov evolution and hierarchical equations in the continuum. I: One-component systems, *J. Evol. Equ.* **9** (2009) 197–233.
- [17] N. Fournier and S. Méléard, A microscopic probabilistic description of a locally regulated population and macroscopic approximations, *Ann. Appl. Probab.* **14** (2004) 1880–1919.
- [18] Yu. Kondratiev and T. Kuna, Harmonic analysis on configuration space. I. General theory, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002) 201–233.
- [19] Yu. Kondratiev, T. Kuna, and O. Kutoviy, On relations between *a priori* bounds for measures on configuration spaces, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **7** (2004) 195–213.
- [20] Yu. Kondratiev and O. Kutovyi, On the metrical properties of the configuration space, *Math. Nachr.* **279** (2006) 774–783.
- [21] Yu. Kondratiev, O. Kutovyi, and R. Minlos, On non-equilibrium stochastic dynamics for interacting particle systems in continuum, *J. Func. Anal.* **255** (2008) 200–227.
- [22] Yu. Kondratiev, O. Kutovyi, and S. Pirogov, Correlation functions and invariant measures in continuous contact model, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **11** (2008) 231–258.
- [23] Yu. G. Kondratiev and A. V. Skorokhod, On contact models in continuum, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **9** (2006) 187–198.
- [24] D. J. Murrell, U. Dieckmann, and R. Law, On moment closures for population dynamics in continuous space, *J. Theoret. Biol.* **229** (2004) 421–432.
- [25] N. Obata, Configuration space and unitary representations of the group of diffeomorphisms, *RIMS Kôkyûroku* **615** (1987) 129–153.
- [26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44. (Springer-Verlag, New York, 1983).
- [27] D. Ruelle, *Statistical Mechanics: Rigorous Results*, (World Scientific, Singapore. 1999).
- [28] H. Spohn, Kinetic equations from Hamiltonian dynamics: Markovian limits, *Rev. Modern Phys.*, **52** (1980) 569–615.
- [29] H. R. Thieme and J. Voigt, Stochastic semigroups: their construction by perturbation and approximation, *Positivity IV—Theory and Applications*, eds. M. R. Weber and J. Voigt (Tech. Univ. Dresden, Dresden, 2006), pp. 135–146.

- [30] F. Trèves, *Ovcyannikov Theorem and Hyperdifferential Operators*, Notas de Matemática, No. 46 (Instituto de Matemática Pura e Aplicada, Conselho Nacional de Pesquisas, Rio de Janeiro, 1968).
- [31] Xiao-Qiang Zhao, *Dynamical Systems in Population Biology*, CMS Books in Mathematics, Springer-Verlag, New York Inc., 2003.