

Representations of quivers over the algebra of dual numbers.

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Abstract: The representations of a quiver Q over a field k (the kQ -modules, where kQ is the path algebra of Q over k) have been studied for a long time, and one knows quite well the structure of the module category $\text{mod } kQ$. It seems to be worthwhile to consider also representations of Q over arbitrary finite-dimensional k -algebras A . Here we draw the attention to the case when $A = k[\epsilon]$ is the algebra of dual numbers (the factor algebra of the polynomial ring $k[T]$ in one variable T modulo the ideal generated by T^2), thus to the Λ -modules, where $\Lambda = kQ[\epsilon] = kQ[T]/\langle T^2 \rangle$. The algebra Λ is a 1-Gorenstein algebra, thus the torsionless Λ -modules are known to be of special interest (as the Gorenstein-projective or maximal Cohen-Macaulay modules). They form a Frobenius category \mathcal{L} , thus the corresponding stable category $\underline{\mathcal{L}}$ is a triangulated category. As we will see, the category \mathcal{L} is the category of perfect differential kQ -modules and $\underline{\mathcal{L}}$ is the corresponding homotopy category. The category $\underline{\mathcal{L}}$ is triangle equivalent to the orbit category of the derived category $D^b(\text{mod } kQ)$ modulo the shift and the homology functor $H: \text{mod } \Lambda \rightarrow \text{mod } kQ$ yields a bijection between the indecomposables in $\underline{\mathcal{L}}$ and those in $\text{mod } kQ$. Our main interest lies in the inverse, it is given by the minimal \mathcal{L} -approximation. Also, we will determine the kernel of the restriction of the functor H to \mathcal{L} and describe the Auslander-Reiten quivers of \mathcal{L} and $\underline{\mathcal{L}}$.

Throughout the paper, k will be a field and Q will be a finite connected directed quiver. The starting point for the considerations of this paper is the following result which concerns the structure of the homotopy category of perfect differential kQ -modules. This assertion should be well-known, but we could not find a reference.

Let us recall that given a ring R , a differential R -module is by definition a pair (N, ϵ) where N is an R -module and ϵ an endomorphism of N such that $\epsilon^2 = 0$. If (N, ϵ) and (N', ϵ') are differential R -modules, a morphism $f: (N, \epsilon) \rightarrow (N', \epsilon')$ is given by an R -linear map $f: N \rightarrow N'$ such that $\epsilon'f = f\epsilon$. The morphism $f: (N, \epsilon) \rightarrow (N', \epsilon')$ is said to be *homotopic to zero* provided there exists an R -linear map $h: N \rightarrow N'$ such that $f = h\epsilon + \epsilon'h$. A differential R -module (N, ϵ) is said to be *perfect* provided N is a finitely generated projective R -module. We denote by $\text{diff}_{\text{perf}}(R)$ the category of perfect differential R -modules, and by $\underline{\text{diff}}_{\text{perf}}(R)$ the corresponding homotopy category. Let us denote by H the homology functor: it attaches to a differential R -module (N, ϵ) the R -module $H(N, \epsilon) = \text{Ker } \epsilon / \text{Im } \epsilon$. It is well-known that H vanishes on the maps which are homotopic to zero.

If R is noetherian, let us denote by $D^b(\text{mod } R)$ the bounded derived category of finitely generated R -modules. This is a triangulated category and its shift functor will be denoted by $[1]$.

Theorem 1. (a) *The category $\text{diff}_{\text{perf}}(kQ)$ of perfect differential kQ -module is a Frobenius category whose stable category $\underline{\text{diff}}_{\text{perf}}(R)$ is the orbit category $D^b(\text{mod } kQ)/[1]$.*

(b) *The homology functor $H: \text{diff}_{\text{perf}}(kQ) \rightarrow \text{mod } kQ$ is a full and dense functor which furnishes a bijection between the indecomposables in the homotopy category $\underline{\text{diff}}_{\text{perf}}(kQ)$ and those in $\text{mod } kQ$. It yields a quiver embedding ι of the Auslander-Reiten quiver of $\text{mod } kQ$ into the Auslander-Reiten quiver of the homotopy category $\underline{\text{diff}}_{\text{perf}}(kQ)$.*

We should remark that the study of differential modules themselves may have been neglected by the algebraists, however it is clear that the graded version, namely complexes, play an important role in many parts of mathematics. Theorem 1 is an immediate consequence of well-known results concerning perfect complexes over kQ : the category of perfect complexes is a Frobenius category, thus the corresponding stable category is $D^b(\text{mod } kQ)$; the homology functor H_0 from the category of perfect complexes to $\text{mod } kQ$ is full and dense and it furnishes a bijection between the shift orbits of the indecomposables in the homotopy category of perfect complexes and the indecomposables in $\text{mod } kQ$; also, it yields an embedding ι of the Auslander-Reiten quiver of $\text{mod } kQ$ into the Auslander-Reiten quiver of the homotopy category of the perfect complexes.

Theorem 1 follows from these assertions, using the covering theory as developed by Gabriel and his school (or the equivalent theory of group graded algebras by Gordon and Green); namely one just looks at the forgetful functor from the category of perfect complexes of kQ -modules to the category of perfect differential kQ -modules. The main property to be used is the local boundedness of the category of perfect complexes. For the fact that the orbit category $D^b(\text{mod } kQ)/[1]$ is triangulated, we may refer to the general criterion given recently by Keller [Ke], however one also may use directly the Frobenius category structure of the category of perfect differential kQ -modules. Some further comments on this proof will be given in section 3.

The proper framework for Theorem 1 seems to be the category of **all** differential kQ -modules, this category may be interpreted in several ways. It is the category of Λ -modules, where $\Lambda = kQ[\epsilon] = kQ[T]/\langle T^2 \rangle$ (with T a central variable). Note that $\Lambda = AQ$ may be considered as the path algebra of the quiver Q over the 2-dimensional local algebra $A = k[\epsilon] = k[T]/\langle T^2 \rangle$ of dual numbers, thus the Λ -modules are just the representations of Q over the ring A . Also, we may write Λ as the tensor product of kQ with A over k .

The aim of this paper to analyze Theorem 1 as dealing with two subcategories of the module category $\text{mod } \Lambda$, the subcategories of interest are on the one hand the category $\text{mod } kQ$ (these are the Λ -modules annihilated by ϵ), and the category of perfect differential kQ -modules on the other hand.

The decisive property which we will use is the fact that Λ is a 1-Gorenstein algebra. For any 1-Gorenstein algebra Λ , the category $\mathcal{L} = \mathcal{L}_\Lambda$ of the torsionless Λ -modules is of interest, these are the Λ -modules which are submodules of projective modules, but they also can be characterized differently: the torsionless Λ -modules are just the Gorenstein-projective or maximal Cohen-Macaulay modules as considered in [EJ,B]), and the modules of G -dimension 0 in the sense [AB]. Note that \mathcal{L} is a Frobenius category, thus the corresponding

stable category $\underline{\mathcal{L}}$ (obtained from \mathcal{L} by factoring out all the maps which factor through the subcategory \mathcal{P} of the projective Λ -modules) is a triangulated category. In our case $\Lambda = kQ[\epsilon]$, the category \mathcal{L} is precisely the category of perfect differential kQ -modules,

$$\mathcal{L} = \text{diff}_{\text{perf}}(kQ) \quad \text{and} \quad \underline{\mathcal{L}} = \underline{\text{diff}}_{\text{perf}}(R),$$

and every module in \mathcal{L} is even strongly Gorenstein-projective, see 4.10.

The basic functor to be considered is the homology functor $H: \text{mod } \Lambda \rightarrow \text{mod } kQ$, it sends a representation $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of the quiver Q over $k[\epsilon]$ to the homology with respect to the action of ϵ , thus, $H(M) = (H(M_i), H(M_\alpha))_{i, \alpha}$. Besides the functor $H: \text{mod } \Lambda \rightarrow \text{mod } kQ$ we will consider a reverse construction η which is not functorial (but of course stably functorial), the minimal right \mathcal{L} -approximation.

Theorem 2. *The algebra $\Lambda = kQ[\epsilon]$ is 1-Gorenstein. The functor $H: \mathcal{L} \rightarrow \text{mod } kQ$ is full and induces a bijection between the indecomposable Λ -modules in $\mathcal{L} \setminus \mathcal{P}$ and the indecomposable kQ -modules. The inverse bijection is given by taking the minimal right \mathcal{L} -approximation of an indecomposable kQ -module.*

We obtain an embedding ι of the Auslander-Reiten quiver $\Gamma(\text{mod } kQ)$ of $\text{mod } kQ$ into the Auslander-Reiten quiver $\Gamma(\underline{\mathcal{L}})$ of $\underline{\mathcal{L}}$ by sending an indecomposable kQ -module N to ηN . The only arrows which are not obtained in this way are the following “ghost maps”: For any vertex y of Q , let $P_0(y)$ and $I_0(y)$ be the corresponding indecomposable projective or injective kQ -module, respectively. For any y , we construct a homomorphism $c(y): \eta(I_0(y)/S(y)) \rightarrow P_0(y)$. These homomorphisms yield the arrows in $\Gamma(\underline{\mathcal{L}})$ which are not in the image of ι . In addition, we show that $\tau_{\mathcal{L}} P_0(y) = \eta I_0(y)$, where $\tau_{\mathcal{L}}$ is the Auslander-Reiten translation $\tau_{\mathcal{L}}$ of \mathcal{L} . In this way we get the required extension of the translation map of $\Gamma(\text{mod } kQ)$.

Of course, in order to obtain the Auslander-Reiten quiver of \mathcal{L} itself, we have to add the indecomposable projective Λ -modules $P(y) = P_0(y) \otimes_k A$. But this is easy: $\text{rad } P(y)$ belongs to \mathcal{L} and is indecomposable, actually $H(\text{rad } P(y))$ is just the simple module $S(y)$ corresponding to the vertex y .

Theorem 3. *The kernel of the functor $H: \mathcal{L} \rightarrow \text{mod } kQ$ is a finitely generated ideal of \mathcal{L} , it is generated by the identity morphisms of the indecomposable projective Λ -modules and the homomorphisms $\eta I(x) \rightarrow P(y)$, where x, y are vertices of Q .*

In fact, instead of using all the maps $\eta I(x) \rightarrow P(y)$, it is sufficient to take the maps $c(y)$ mentioned above.

Here is an outline of the paper. The first section describes the context of this investigation. In section 2, we show that the perfect differential kQ -modules are precisely the torsionless $kQ[\epsilon]$ -modules, thus the Gorenstein-projective $kQ[\epsilon]$ -modules. Section 3 provides some details for the covering approach. Section 4 is the central part, here we discuss in which way the homology functor H and the \mathcal{L} -approximation η are inverse to each other. Sections 5 and 6 deal with the ghost maps, section 7 with the position of the indecomposable projective Λ -modules in the Auslander-Reiten sequences of \mathcal{L} . Sections 8 to 10 are devoted to examples and further remarks.

1. The context.

1.1. An explicit description of the category of Gorenstein-projective modules is known only for very few algebras. In a recent paper Luo and Zhang [LZ] gave a characterization of the Gorenstein-projective AQ -modules, where Q is a finite directed quiver and A a any k -algebra, thus one may try to use this result in order to construct these modules explicitly. The present paper deals with the very special case of the algebra $A = k[\epsilon]$ of dual numbers, this may be considered as an interesting test case.

Let us stress that the class of 1-Gorenstein algebras is a class of algebras which includes both the hereditary and the self-injective algebras — two classes of algebras whose representations have been investigated very thoroughly and have been shown to be strongly related to Lie theory. Thus one might hope that all the 1-Gorenstein algebras have such a property. Now recently, Keller and Reiten [KR] identified another class of 1-Gorenstein algebras, namely the cluster tilted algebras, and this again is a class of algebras related to Lie theory. Of course, the result of the present paper also supports the hope.

As we have mentioned, an explicit description of the category of Gorenstein-projective modules is known only in few cases. Chen ([C], see also [RX]) recently has shown that for Λ of Loewy length at most 2, the stable category of Gorenstein-projective Λ -modules is a union of categories with Auslander-Reiten quiver of tree type \mathbb{A}_1 , thus not very exciting. Now, the next case of interest are the artin algebras of Loewy length 3, and for this case we provide a wealth of examples. Namely, if Q is a finite bipartite quiver (i.e., all vertices are sinks or sources), then the algebra $kQ[\epsilon]$ is of Loewy length at most 3.

1.2. The theorems allow to transfer a lot of results known for the module category $\text{mod } kQ$ to the category \mathcal{L} of Gorenstein-projective Λ -modules, where $\Lambda = kQ[\epsilon]$. For example, the Kac Theorem [K] yields:

The homology dimension vector $\mathbf{dim} H(-)$ maps any indecomposable object in $\mathcal{L} \setminus \mathcal{P}$ to a positive root of the corresponding Kac-Moody algebra \mathfrak{g} . For any positive real root \mathbf{r} , there is a unique isomorphism class of indecomposable modules M in \mathcal{L} with $\mathbf{dim} H(M) = \mathbf{r}$; if k is an infinite field, then for every positive imaginary root \mathbf{r} of \mathfrak{g} , there are infinitely many isomorphism classes of indecomposable modules M in \mathcal{L} with $\mathbf{dim} H(M) = \mathbf{r}$.

1.3. The relationship between abelian categories and triangulated categories has always been considered as fascinating, but also mysterious. It was clear from the beginning that starting with a suitable abelian category \mathcal{A} (namely the module category \mathcal{A} of a self-injective algebra) one may obtain a triangulated categories by factoring out some finitely generated ideal (namely the ideal of all maps which factor through a projective module). Only quite recently, it was observed that there are also examples in the reverse direction: if one starts with a cluster category (a triangulated category, according to [Ke]) and factors out the ideal of all maps which factor through the additive category generated by a fixed cluster-tilting object, then one obtains an abelian category, namely the module category of the corresponding cluster-tilted algebra (thus an abelian category), see Buan-Marsh-Reineke-Reiten-Todorov [BMRRT].

The results of the present paper should be seen in this context. As in the case of the cluster categories we start with a triangulated category \mathcal{T} and factor out a finitely generated ideal of \mathcal{T} in order to obtain an abelian category. But whereas in the cluster

category case the ideal is generated by some identity maps, here we deal with an ideal which lies inside the radical of \mathcal{T} .

It seems to be of interest that actually we deal with two related subcategories of a module category, one is a Frobenius category \mathcal{F} , the other an abelian category \mathcal{A} , such that there is a canonical bijection between the indecomposable objects in the stable category $\underline{\mathcal{F}}$ and the indecomposable objects in \mathcal{A} . Of course, if we consider for a self-injective algebra R the subcategories $\mathcal{F} = \text{mod } R$ and $\mathcal{A} = \text{mod } R/S(y)R$, then there is such a canonical bijection, namely the identity. The examples which we consider are more intricate: here, $\mathcal{F} = \mathcal{L}$ is the category of Gorenstein-projective $kQ[\epsilon]$ -modules, \mathcal{A} is the category of kQ -modules, and the bijection is given by the functor H and the \mathcal{L} -approximation η .

1.4. A long time ago, it has been shown by Buchweitz [B] that given a Gorenstein algebra Λ , the Verdier quotient of the bounded derived category $D^b(\text{mod } \Lambda)$ modulo the subcategory of perfect complexes can be identified with the stable category of Gorenstein-projective Λ -modules, and Orlov [O] proposed the name *triangulated category of singularities* for this Verdier quotient. In our case $\Lambda = kQ[\epsilon]$, we show that the triangulated category of singularities is just $D^b(\text{mod } kQ)/[1]$.

2. The basic observation.

Let R be an arbitrary ring. As above, we define $R[\epsilon] = R[T]/\langle T^2 \rangle$, where T is a variable which is supposed to commute with all the elements of R . The $R[\epsilon]$ -modules are just the *differential* R -modules, they may be written as (N, f) , where N is an R -module and $f: N \rightarrow N$ is an R -endomorphism with $f^2 = 0$ (namely, if such a pair (N, f) is given, then N can be considered as an $R[\epsilon]$ -module by defining the action of ϵ on N as being given by f); by abuse of notation, we sometimes will write ϵ instead of f . A differential R -module (N, f) will be said to be *perfect* provided N is a finitely generated projective R -module.

For an R -module N , let $N[\epsilon] = (N \oplus N; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$, this is an $R[\epsilon]$ -module (note that if we take $N = R$, then the module $R[\epsilon]$ is just the regular representation of the ring $R[\epsilon]$, thus there is no conflict of notation). If N is a finitely generated R -module, then $N[\epsilon]$ is a finitely generated $R[\epsilon]$ -module; if N is a projective R -module, then $N[\epsilon]$ is a projective $R[\epsilon]$ -module. In particular, if N is finitely generated and projective, then $N[\epsilon]$ is a perfect $R[\epsilon]$ -module. But a perfect $R[\epsilon]$ -module may not be of the form $N[\epsilon]$. Also note that a finitely generated projective $R[\epsilon]$ module is perfect, but the converse is not true.

Let us recall that an R -module is said to be *torsionless* provided it is a submodule of a projective R -module. Thus, a differential R -module is torsionless if it is a submodule of a projective $R[\epsilon]$ -module. Also recall that a ring R is *left hereditary* provided any torsionless left module is projective.

2.1. Lemma. *Let R be left noetherian and left hereditary. A differential R -module is finitely generated and torsionless if and only if it is perfect.*

Proof. Let (N, f) be a differential R -module.

First, assume that (N, f) is finitely generated and torsionless. Since (N, f) is torsionless, we know that (N, f) is a submodule of a free $R[\epsilon]$ -module, and since (N, f) is finitely

generated, it is even a submodule of a free $R[\epsilon]$ -module of finite rank. Thus (N, f) can be embedded into a module of the form $(R^t \oplus R^t, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$ for some natural number t . In particular, we see that N is a submodule of R^{2t} . Since R is left noetherian and hereditary, we conclude that N is a finitely generated projective R -module.

Conversely, assume that N is a finitely generated projective R -module. We denote by N' the kernel, by N'' the image of f ; let $u': N'' \rightarrow N'$ and $u: N' \rightarrow N$ be the inclusion maps. Note that f induces an epimorphism $f': N \rightarrow N''$ with $f = uu'f'$. Since R is left hereditary and N is a projective R -module, we see that N'' is also projective. It follows that there is a homomorphism $s: N'' \rightarrow N$ such that $f's = 1_{N''}$, and, as a consequence the map

$$[u \ s]: N' \oplus N'' \rightarrow N$$

is an isomorphism. But $uu' = uu'f's = fs$ shows that

$$[u \ s] \begin{bmatrix} 0 & u' \\ 0 & 0 \end{bmatrix} = f[u \ s],$$

therefore $[u \ s]$ is an isomorphism between $(N' \oplus N'', \begin{bmatrix} 0 & u' \\ 0 & 0 \end{bmatrix})$ and (N, f) . It remains to observe that there is the following embedding of differential R -modules:

$$\begin{bmatrix} 1 & 0 \\ 0 & u' \end{bmatrix} : (N' \oplus N'', \begin{bmatrix} 0 & u' \\ 0 & 0 \end{bmatrix}) \rightarrow (N' \oplus N', \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}),$$

and that $(N' \oplus N', \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$ is a projective Λ -module. This shows that $(N' \oplus N'', \begin{bmatrix} 0 & u' \\ 0 & 0 \end{bmatrix})$ and therefore (N, f) is torsionless. Since N is a finitely generated R -module, we see that (N, f) is a finitely generated Λ -module. \square

Let us consider now finite-dimensional k -algebras, where k is a field. Recall [AR1] that such an algebra R is called a *Gorenstein algebra of Gorenstein dimension 1* or just a *1-Gorenstein algebra* provided the injective dimension of ${}_R R$ as well as R_R is equal to 1. Since Λ is the tensor product of a hereditary and a self-injective algebra, one sees immediately that Λ is 1-Gorenstein.

Given any finite-dimensional k -algebra Λ , a Λ -module M is said to be *Gorenstein-projective* [EJ] provided there exists an exact (not necessarily bounded) complex $P_\bullet = (P_i, \delta_i)_i$ of finitely generated projective Λ -modules such that also $\text{Hom}_\Lambda(P_\bullet, \Lambda)$ is exact and such that M is the image of δ_0 . If Λ is Gorenstein, then a Λ -module M is Gorenstein-projective if and only if $\text{Ext}^i(M, \Lambda) = 0$ for all $i \geq 0$.

The following proposition is well-known.

2.2. Proposition. *Let R be a finite-dimensional k -algebra which is hereditary. Then $\Lambda = R[\epsilon]$ is a Gorenstein algebra of Gorenstein dimension 1. If M is any Λ -module. then the following conditions are equivalent:*

- (i) M is Gorenstein-projective,
- (ii) M is torsionless,
- (iii) $\text{Ext}_\Lambda^1(M, \Lambda) = 0$.

2.3. Given any Λ -module M , there is an exact sequence

$$0 \rightarrow P \rightarrow \eta M \xrightarrow{g} M \rightarrow 0,$$

such that P is projective, ηM belongs to \mathcal{L} and g is a right minimal map (see for example [EJ], Theorem 11.5.1).

The map g (or also the module ηM) is called the minimal right \mathcal{L} -approximation, since any map $h: L \rightarrow M$ with $L \in \mathcal{L}$, can be factorized as $h = h'g$ with $h': L \rightarrow \eta M$. Of course, this factorization property and the minimality of g implies that ηM is uniquely determined by M , up to an isomorphism, but there there is not necessarily a canonical isomorphism. Also, one should note that ηM is the universal extension of M from below, using projective modules.

3. Proof of theorem 1.

Given any ring R , we denote by $\mathcal{P} = \mathcal{P}_R$ the category of finitely generated projective R -modules.

We are interested in complexes of R -modules, such complexes may be considered as differential graded R -modules. In particular, we will consider perfect complexes (or perfect differential graded R -modules), these are the bounded complexes which use only finitely generated projective R -modules. We denote by $C^b(\mathcal{P}_R)$ the category of perfect complexes, and by $K^b(\mathcal{P}_R)$ the corresponding homotopy category. Let us stress that $C^b(\mathcal{P}_R)$ is a Frobenius category and $K^b(\mathcal{P}_R)$ is just the corresponding stable category, say with stabilization functor $\pi: C^b(\mathcal{P}_R) \rightarrow K^b(\mathcal{P}_R)$, it sends a map to its homotopy class.

Note that in case $R = kQ$, where Q is a finite directed quiver, the ring R has finite global dimension, and $K^b(\mathcal{P}_R)$ can be identified with the bounded derived category $D^b(\text{mod } R)$. In dealing with categories of complexes, the shift functor will be denoted by $[1]$.

There is the following commutative diagram

$$\begin{array}{ccc} C^b(\mathcal{P}_R) & \xrightarrow{\pi} & K^b(\mathcal{P}_R) \\ \gamma \downarrow & & \downarrow \gamma \\ \text{diff}_{\text{perf}}(R) & \xrightarrow{\pi} & \underline{\text{diff}}_{\text{perf}}(R) \end{array}$$

here, the horizontal functors π are just the stabilization functors, whereas the vertical functors γ are obtained by forgetting the grading (such a forgetful functor is sometimes called a compression functor or, in the covering theory of the Gabriel school, the corresponding pushdown functor).

What is important here, is the fact that for $R = kQ$ the category $C^b(\mathcal{P}_R)$ is locally bounded, thus the functors γ are dense and provide a bijection between the shift-orbits of the indecomposable objects in $C^b(\mathcal{P}_R)$ and the indecomposable objects in $\text{diff}_{\text{perf}}(R)$ and similarly, between the shift-orbits of the indecomposable objects in $K^b(\mathcal{P}_R)$ and the indecomposable objects in $\underline{\text{diff}}_{\text{perf}}(R)$.

Let us denote by $H_0: C^b(\mathcal{P}_R) \rightarrow \text{mod } R$ the functor which attaches to a complex $P_\bullet = (P_i, \delta_i)$ the homology at the position 0. Then it is well-known [H1] that H_0 provides a bijection between the shift orbits of indecomposable objects in $K^b(\mathcal{P}_{kQ})$ and the indecomposable kQ -modules. This completes the proof. \square

For the case $R = kQ$, let us exhibit the diagram above again, but now using the notation $\mathcal{L} = \text{diff}_{\text{perf}}(kQ)$ and $\underline{\mathcal{L}} = \underline{\text{diff}}_{\text{perf}}(kQ)$ which is used in the further parts of the paper:

$$\begin{array}{ccc} C^b(\mathcal{P}_{kQ}) & \xrightarrow{\pi} & K^b(\mathcal{P}_{kQ}) \\ \gamma \downarrow & & \downarrow \gamma \\ \mathcal{L} & \xrightarrow{\pi} & \underline{\mathcal{L}} \end{array}$$

4. Explicit bijections.

Let Q be a finite directed quiver, kQ its path algebra and $\Lambda = kQ[\epsilon]$. Recall that we denote by \mathcal{L} the category of (finitely generated) torsionless modules. Given any Λ -module M , let $\eta M \rightarrow M$ be a minimal right \mathcal{L} -approximation.

We consider $\text{mod } kQ$ as the full subcategory of $\text{mod } \Lambda$ given by all the Λ -modules N which are annihilated by ϵ .

4.1. *The Λ -modules in $\mathcal{L} \cap \text{mod } kQ$ are just the projective kQ -modules.*

Proof. Let M be a module in $\mathcal{L} \cap \text{mod } kQ$. Since M is in \mathcal{L} , we may consider M as a submodule of $(\Lambda)^\epsilon$ for some ϵ . Since M is annihilated by ϵ , it must lie inside the submodule $(\epsilon\Lambda)^\epsilon$ (since $\{x \in \Lambda \mid \epsilon x = 0\} = \epsilon\Lambda$). But $\epsilon\Lambda \simeq kQ$ as Λ -modules and thus also as kQ -modules. Now M is a submodule of kQ , thus it is a projective kQ -module. Of course, conversely, a projective kQ -module belongs to \mathcal{L} . \square

4.2. *Let M be in \mathcal{L} , let $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$. Then $M'' \subseteq M'$, $H(M) = M'/M''$ and the exact sequence*

$$0 \rightarrow M'' \rightarrow M' \rightarrow H(M) \rightarrow 0$$

is a projective kQ -resolution of $H(M)$.

Proof. Since M' is a submodule of M , it belongs to \mathcal{L} . Since M' is annihilated by ϵ , it belongs to $\text{mod } kQ$, thus M' is a projective kQ -module by 4.1. \square

4.3. *Let M be a Λ -module with $H(M) = 0$. If $N = \epsilon M$ is a projective kQ -module, then M is isomorphic to $N[\epsilon]$, thus M is a projective Λ -module.*

Proof. Since $H(M) = 0$, the multiplication with ϵ yields an isomorphism from $N = M/\epsilon M$ onto ϵM , in particular, these modules have the same k -dimension. Let $P = N[\epsilon]$ and consider the canonical map $p: P \rightarrow P/\epsilon P = N$. We can lift this map to a map

$p': P \rightarrow M$ which again has to be surjective (since ϵM lies in the radical of M). Now $\dim P = 2 \dim N = \dim M/\epsilon M + \dim \epsilon M = \dim M$. This shows that p' is an isomorphism. \square

4.4. *Any M in \mathcal{L} has a projective submodule U such that $\epsilon M \subseteq U$ and such that M/U can be identified with $H(M)$.*

Proof. Let $M \in \mathcal{L}$. Let $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$. Then $M'' \subseteq M'$, since $\epsilon^2 = 0$ and the multiplication with ϵ yields an isomorphism $M/M' \rightarrow M''$. Obviously, M/M' is annihilated by ϵ , thus a kQ -module. On the other hand, M'' as a submodule of M is torsionless. Thus the module $M/M' \simeq M''$ belongs to $\mathcal{L} \cap kQ\text{-mod}$, and therefore is a projective kQ -module. But this means that the inclusion map $M'/M'' \rightarrow M/M''$ is a split monomorphism (of Λ -modules), since these are kQ -modules and the cokernel is a projective kQ -module. In this way, we obtain a Λ -submodule U of M such that $U \cap M' = M''$ and $U + M' = M$. Note that $M/U \simeq M'/M'' = H(M)$.

It remains to see that U is a projective Λ -module. In order to use 4.3, we have to show that $H(U) = 0$ and that ϵU is a projective kQ -module. Let $U' = \{x \in U \mid \epsilon x = 0\}$, then $U' = U \cap M'$, thus $U' = M''$. Let us show that $\epsilon U = M''$. On the one hand $\epsilon U \subseteq \epsilon M = M''$. On the other hand, any element in $M'' = \epsilon M$ has the form ϵx with $x \in M = U + M'$, thus $x = u + x'$ with $u \in U$ and $x' \in M'$, thus $\epsilon x = \epsilon(u + x') = \epsilon u$ belongs to ϵU . It follows that $H(U) = U'/\epsilon U = M''/M'' = 0$. Also, we know that $\epsilon U = M''$ is a projective kQ -module. \square

We see: If M belongs to \mathcal{L} , then there is an exact sequence

$$0 \rightarrow U \rightarrow M \xrightarrow{g} H(M) \rightarrow 0$$

with U a projective Λ -module. This means that we have obtained a right \mathcal{L} -approximation g of $H(M)$. But note that we do cannot obtain a canonical such map, since the submodule U is not uniquely determined. Also observe that this is a minimal right \mathcal{L} -approximation if and only if M has no non-zero projective direct summand.

4.5. *If M is in \mathcal{L} , indecomposable and not projective, then $H(M)$ is indecomposable and $\eta H(M)$ is isomorphic to M .*

Proof. If there is a direct decomposition $H(M) = N_1 \oplus N_2$, let $M_i \rightarrow N_i$ be a minimal right \mathcal{L} -approximation of N_i , for $i = 1, 2$. Then $M_1 \oplus M_2 \rightarrow N_1 \oplus N_2$ is a minimal right \mathcal{L} -approximation. The uniqueness of minimal right \mathcal{L} -approximations yields that M is isomorphic to $M_1 \oplus M_2$, thus one of the M_i is zero, say $M_2 = 0$. But then also $N_2 = 0$.

Since $M \rightarrow H(M)$ is a minimal right \mathcal{L} -approximation, we see that $\eta H(M)$ and M are isomorphic. \square

In 4.4 we have seen that for any module $M \in \mathcal{L}$ there is a projective submodule U of M such that $\epsilon M \subseteq U$ and such that M/U can be identified with $H(M)$. Let us show that for any projective submodule U' with $\epsilon M \subseteq U'$, the projection map $p: M \rightarrow M/U'$ induces an isomorphism $H(p)$.

4.6. *Let M be a module in \mathcal{L} . Let U be a projective submodule of M such that $\epsilon M \subseteq U$. Then the map $H(M) \rightarrow H(M/U) = M/U$ induced by the projection is an isomorphism.*

Proof. First, let us show that $\epsilon M = \epsilon U$. In order to prove this, we may consider M as an A -module and U as an A -submodule of M . Since U is projective as a Λ -module, U is also projective as an A -module (since $kQ[\epsilon]$ is projective as an A -module). But A is self-injective, thus any projective A -module is also injective as an A -module. This shows that the embedding $U \rightarrow M$ splits as an embedding of A -modules. Thus there is an A -submodule U' of M such that $M = U \oplus U'$. But U' is isomorphic to M/U as an A -module, thus annihilated by ϵ . This shows that $\epsilon M = \epsilon U \oplus \epsilon U' = \epsilon U$.

Let $M' = \{x \in M \mid \epsilon x = 0\}$. Then $M' \cap U = \{x \in U \mid \epsilon x = 0\} = \epsilon U = \epsilon M$.

Also, $M' + U = M$. For the proof, consider an element $x \in M$. Since $\epsilon M = \epsilon U$, there is $u \in U$ such that $\epsilon x = \epsilon u$, thus $\epsilon(x - u) = 0$. This shows that $x - u \in M'$ and therefore $x = (x - u) + u \in M' + U$.

There is the canonical (Noether-) isomorphism

$$M'/(M' \cap U) \rightarrow (M' + U)/U.$$

It yields the required isomorphism

$$H(M) = M'/\epsilon M = M'/(M' \cap U) \simeq (M' + U)/U = M/U = H(M/U).$$

□

If we start with a module $N \in \text{mod } kQ$ and form ηN , then there is given an exact sequence

$$0 \rightarrow U \xrightarrow{u} \eta N \xrightarrow{g} N \rightarrow 0,$$

such that U is projective and ηN belongs to \mathcal{L} . thus we can apply 4.6 in order to see that the map $H(\eta N) \rightarrow H(N) = N$ induced by g is an isomorphism. This shows:

4.7. *If $N \in \text{mod } kQ$, then $H(\eta N)$ is canonically isomorphic to N .*

□

It remains to study the minimal right \mathcal{L} -approximation for $N \in \text{mod } kQ$. Start with a minimal projective kQ -resolution of N , say

$$0 \rightarrow \Omega_0 N \rightarrow P_0 N \rightarrow N \rightarrow 0$$

and embed $\Omega_0 N$ into $(\Omega_0 N)[\epsilon]$. Since $\Omega_0 N$ is a projective kQ -module, we know that $(\Omega_0 N)[\epsilon]$ is a projective Λ -module. Forming the pushout of the given embeddings of $\Omega_0 N$ into $P_0 N$ and $(\Omega_0 N)[\epsilon]$, we obtain the following commutative diagram of Λ -modules with

exact rows and columns

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_0 N & \longrightarrow & P_0 N & \longrightarrow & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & (\Omega_0 N)[\epsilon] & \longrightarrow & X & \xrightarrow{g} & N \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& \longrightarrow & \Omega_0 N & \xlongequal{\quad} & \Omega_0 N & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

4.8. *The module X belongs to \mathcal{L} and has no non-zero projective direct summand. As a consequence, the map $g: X \rightarrow N$ is a minimal right \mathcal{L} -approximation of N .*

Thus, we can identify X with ηN .

Proof. The vertical sequence in the middle shows that X is an extension of $P_0 N$ and $\Omega_0 N$. Both $P_0 N$ and $\Omega_0 N$ belong to \mathcal{L} and \mathcal{L} is closed under extensions, thus X belongs to \mathcal{L} . On the other hand, we have already mentioned that $(\Omega_0 N)[\epsilon]$ is a projective Λ -module, thus g is a right \mathcal{L} -approximation.

Let us show that any projective direct summand P of X is zero. According to 4.6, we know that $H(X) = N$. Now assume that there is a direct decomposition of Λ -modules $X = M \oplus P$, with P projective. Then $H(M) = H(X)$. Let $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$. According to 4.2 the following exact sequence

$$0 \rightarrow M'' \rightarrow M' \rightarrow H(M) \rightarrow 0$$

is a projective kQ -resolution of $H(M)$. Since we are using a minimal projective kQ -resolution of $N = H(X) = H(M)$, we see that there is a projective kQ -module C with $C \oplus \Omega_0 N = M''$ and $C \oplus P_0 N = M'$. It remains to compare the dimensions: The equality

$$\begin{aligned}
\dim M &= 2 \dim M'' + \dim N \\
&= 2 \dim C + 2 \dim \Omega_0 N + \dim N \\
&= 2 \dim C + \dim X \\
&= 2 \dim C + \dim M + \dim P
\end{aligned}$$

shows that P (and C) have to be zero (the first line uses that $H(M) = N$).

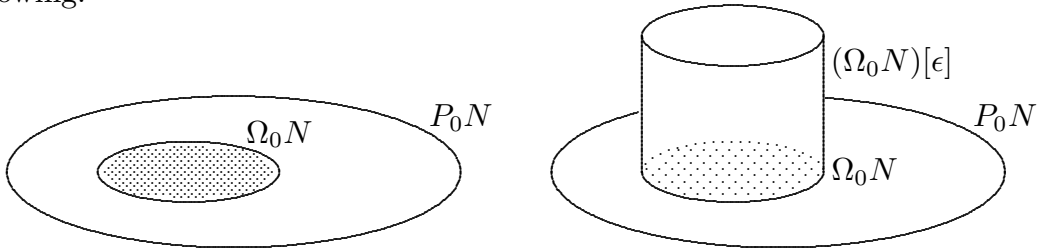
This implies that g is minimal, since otherwise there is a non-trivial direct decomposition $X = X' \oplus X''$ with say X'' contained in the kernel of g . But then X'' is isomorphic to a direct summand of the kernel of g , thus projective. \square

4.9. Proof of Theorem 2. Let M be an indecomposable module in $\mathcal{L} \setminus \mathcal{P}$. We attach to it $H(M)$, this is a kQ -module. According to 4.5, we know that $H(M)$ is indecomposable and also that $\eta H(M) = M$.

Conversely, let us start with an indecomposable kQ -module N . By construction, ηN belongs to \mathcal{L} . We know by 4.8 that ηN has no non-zero projective direct summands. And we know by 4.7 that $H(\eta N) = N$. We still have to show that ηN is indecomposable. Thus assume that there is a non-trivial direct decomposition $\eta N = M_1 \oplus M_2$ of Λ -modules. Both modules M_1, M_2 belong to $\mathcal{L} \setminus \mathcal{P}$, thus according to the first part of the proof, $H(M_1)$ and $H(M_2)$ are non-zero. As a consequence, $N = H(\eta N) = H(M_1) \oplus H(M_2)$ is a non-trivial direct decomposition. This contradicts the assumption that N is indecomposable.

It remains to be shown that $H: \mathcal{L} \rightarrow \text{mod } kQ$ is full. Let M_1, M_2 be modules in \mathcal{L} . We have to show that H yields a surjection $\text{Hom}_\Lambda(M_1, M_2) \rightarrow \text{Hom}_{kQ}(H(M_1), H(M_2))$. Of course, we can assume that M_1 and M_2 both are indecomposable. If one of the modules M_i is projective, then $H(M_i) = 0$ and nothing has to be shown. Thus we can assume that both modules belong to $\mathcal{L} \setminus \mathcal{P}$. It follows that there are \mathcal{L} -approximations $g_i: M_i \rightarrow H(M_i)$ for $i = 1, 2$. But then any homomorphism $f: H(M_1) \rightarrow H(M_2)$ can be lifted to a map $\tilde{f}: M_1 \rightarrow M_2$ with $g_2 \tilde{f} = f g_1$ and $H(\tilde{f}) = f$. \square \square

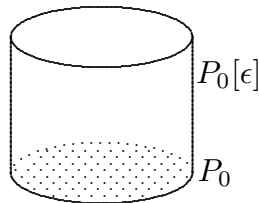
Illustration. The picture to have in mind when dealing with ηN for $N \in \text{mod } kQ$ is the following:



On the left, we show $\Omega_0 N$ as a submodule of $P_0 N$, note that N is obtained from $P_0 N$ by factoring out $\Omega_0 N$. The right pictures depicts ηN as an amalgamation of $P_0 N$ and $(\Omega_0 N)[\epsilon]$ along $\Omega_0 N$. Note: if we write $M = \eta N$, and set $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$, then $M' = P_0 N$ and $M'' = \Omega_0 N$.

The right picture looks similar to the usual depiction of a mapping cylinder in topology, but better it should be compared with a mapping cone. After all, what here looks like a cylinder is a projective Λ -module, thus something that one should consider as a contractible object.

Of course, the same kind of pictures can be used also to depict the projective Λ -modules, stressing that they are of the form $P_0[\epsilon]$, with P_0 a projective kQ -module.



Remark. If one starts with an indecomposable non-projective kQ -module N and compares the Λ -modules N and ηN , the decisive difference is that ηN has $\Omega_0 N$ as a factor

module, and this is a non-zero projective kQ -module. Thus $\text{Hom}_\Lambda(\eta N, kQ) \neq 0$, whereas, of course, $\text{Hom}_\Lambda(N, kQ) = 0$.

Let us insert an interesting property of the torsionless Λ -modules. Recall that a module M is said to be *strongly Gorenstein-projective* [BM] provided there exists an exact sequence

$$0 \rightarrow M \xrightarrow{u} P \xrightarrow{p} M \rightarrow 0$$

such that u is a left Λ -approximation. This means, that M is the kernel of a map $f: P \rightarrow P$, where

$$\dots \rightarrow P \xrightarrow{f} P \xrightarrow{f} P \rightarrow \dots$$

is a complete projective resolution (namely, $f = up$).

4.10. Proposition. *Any torsionless $kQ[\epsilon]$ -module is strongly Gorenstein-projective.*

Proof: Let M be a torsionless Λ -module, where $\Lambda = kQ[\epsilon]$. It is sufficient to construct an exact sequence of the form

$$0 \rightarrow M \xrightarrow{u} P \xrightarrow{p} M \rightarrow 0$$

with P projective. Namely, since Λ is 1-Gorenstein, any injective map $M \rightarrow P$ is a left Λ -approximation. Alternatively, we construct an endomorphism $f: P \rightarrow P$ of a projective Λ -module P with $\text{Im}(f) = \text{Ker}(f) = M$. Of course, we can assume that M is indecomposable, but also that M is not projective. Thus $M = \eta N$ for some kQ -module N .

The construction of ηN starts with a projective presentation

$$0 \rightarrow \Omega_0 N \xrightarrow{u} P_0 N \rightarrow N \rightarrow 0,$$

and the map u gives rise to a map $u[\epsilon]: (\Omega_0 N)[\epsilon] \rightarrow (P_0 N)[\epsilon]$, thus we may consider the map

$$f = \begin{bmatrix} -\epsilon & 0 \\ u[\epsilon] & \epsilon \end{bmatrix} : P \longrightarrow P, \quad \text{where } P = (\Omega_0 N)[\epsilon] \oplus (P_0 N)[\epsilon].$$

First of all, one easily verifies that $f^2 = 0$ (since $\epsilon^2 = 0$ and ϵ commutes with all maps), thus $\text{Im}(f) \subseteq \text{Ker}(f)$. We claim that the cokernel $\text{Cok}(f)$ maps onto ηN .

The module ηN was constructed as the following pushout

$$\begin{array}{ccc} \Omega_0 N & \xrightarrow{u} & P_0 N \\ \mu \downarrow & & \downarrow \mu' \\ (\Omega_0 N)[\epsilon] & \xrightarrow{u'} & \eta N \end{array}$$

Let us denote by $\pi: (\Omega_0 N)[\epsilon] \rightarrow \Omega_0$ and $\pi': (P_0 N)[\epsilon] \rightarrow P_0 N$ the canonical projection maps (thus $\mu\pi$ is the multiplication by ϵ on $(\Omega_0 N)[\epsilon]$ and $\pi' \cdot u[\epsilon] = u\pi$). Let us consider the map

$$g = [u' \quad \mu'\pi] : (\Omega_0 N)[\epsilon] \oplus (P_0 N)[\epsilon] \longrightarrow \eta N.$$

The composition of f and g is

$$gf = \begin{bmatrix} u' & \mu'\pi \end{bmatrix} \begin{bmatrix} -\epsilon & 0 \\ u[\epsilon] & \epsilon \end{bmatrix} = \begin{bmatrix} -u'\epsilon + \mu'\pi u[\epsilon] & \mu'\pi\epsilon \end{bmatrix},$$

but this is the zero map since $u'\epsilon = u'\mu\pi = \mu'u\pi = \mu'\pi'u[\epsilon]$ and $\pi'\epsilon = 0$. Therefore g factors through the cokernel of f , thus g is of the form $P \rightarrow \text{Cok}(f) \xrightarrow{\zeta} \eta N$ for some map $\zeta: \text{Cok}(f) \rightarrow \eta N$. Since $[u', \mu']$ is surjective, also g is surjective, and therefore ζ is surjective.

It remains to look at the length of ηN . For any Λ -module M , let $|M|$ denote its length (and note that the simple Λ -modules are the simple kQ -modules). We have obtained a surjective map $\zeta: \text{Cok}(f) \rightarrow \eta N$, thus, in particular, $|\eta N| \leq |\text{Cok}(f)|$. Since ηN is an extension of $P_0 N$ by $\Omega_0 N$, we see that $|\eta N| = |P_0 N| + |\Omega_0 N|$ and therefore

$$|P| = |(P_0 N)[\epsilon]| + |(\Omega_0 N)[\epsilon]| = 2|\eta N|.$$

Finally, we use that $f^2 = 0$, therefore $2|\text{Cok}(f)| \leq |P|$. Combining the inequalities, we see that

$$|P| = 2|\eta N| \leq 2|\text{Cok}(f)| \leq |P|.$$

Consequently, we must have both $|\eta N| = |\text{Cok}(f)|$ and $2|\text{Cok}(f)| = |P|$. The first equality means that ζ is an isomorphism, thus the cokernel of f is isomorphic to ηN . The second equality means that $\text{Im}(f) = \text{Ker}(f)$. Altogether, we conclude that $f: P \rightarrow P$ is an endomorphism of the projective Λ -module P with $\text{Im}(f) = \text{Ker}(f) = M$. \square

5. The ghost sequences.

Let us construct the Auslander-Reiten sequences in \mathcal{L} which end in an indecomposable projective kQ -module $P_0(y)$. Note that there is the corresponding indecomposable projective Λ -module $P(y) = (P_0(y))[\epsilon] = P_0(y) \otimes_k A$. We may identify its submodule $P_0(y) \otimes_k k\epsilon$ with $P_0(y)$ and we have $P(y)/P_0(y) \simeq P_0(y)$.

We will need the Auslander-Reiten translation τ in $\text{mod } \Lambda$. We denote by $\tau_{\mathcal{L}}$ the Auslander-Reiten translation in \mathcal{L} .

5.1. Lemma. *For any vertex y of Q , we have $\tau P_0(y) = I_0(y)$ and $\tau_{\mathcal{L}} P_0(y) = \eta I_0(y)$.*

Proof. In order to see that $\tau P_0(y) = I_0(y)$, one notes that $P_0(y)$ is the cokernel of the multiplication map $\epsilon: P(y) \rightarrow P(y)$, thus $\tau P_0(y)$ has to be the kernel of the multiplication map $\epsilon: I(y) \rightarrow I(y)$, and this is $I_0(y)$. But for any indecomposable module M in $\mathcal{L} \setminus \mathcal{P}$ the Auslander-Reiten translate $\tau_{\mathcal{L}} M$ of M in \mathcal{L} is a non-zero direct summand of $\eta \tau M$. Since $\eta I_0(y)$ is indecomposable, we conclude that $\tau_{\mathcal{L}} P_0(y) = \eta I_0(y)$. \square

We will distinguish whether the vertex y of Q is a source or not. First, we consider the case where y is a source.

5.2. Lemma. *Let y be a source of Q . Then the Auslander-Reiten sequence in \mathcal{L} ending in $P_0(y)$ is of the form*

$$0 \rightarrow \eta I_0(y) \rightarrow P(y) \oplus \text{rad } P_0(y) \rightarrow P_0(y) \rightarrow 0.$$

Proof. We denote by $\iota: \text{rad } P_0(y) \rightarrow P_0(y)$ the inclusion map and by $\epsilon': P(y) \rightarrow P_0(y)$ the surjective map induced by the multiplication with ϵ . We claim that the map $[\epsilon' \ \iota]: P(y) \oplus \text{rad } P_0(y) \rightarrow P_0(y)$ is minimal right almost split.

Let L be an indecomposable module in \mathcal{L} and consider a map $f: L \rightarrow P_0(y)$. Now either f maps into the radical $\text{rad } P_0(y)$ of $P_0(y)$, then we have a lifting. Or else it maps onto $P_0(y)$. But then there is a map $f': P(y) \rightarrow L$ such that $ff' = \epsilon'$. If the image of f is annihilated by ϵ , then f' factors through ϵ' , say $f' = f''\epsilon'$ and then $ff''\epsilon' = ff' = \epsilon'$ and therefore $ff'' = 1$, thus f is a split epimorphism. Otherwise we use an embedding of L into a projective Λ -module in order to see that f' has to be an isomorphism (here we use that y is a source). But then $f = \epsilon'(f')^{-1}$ is the required factorization.

It remains to show that the map is minimal. Now $P(y)$ cannot be deleted, since the map ϵ' is surjective, whereas the map ι is not surjective. Also, if N is an indecomposable direct summand of $\text{rad } P_0(y)$, say $\text{rad } P_0(y) = N \oplus N'$, then the inclusion map $N \rightarrow P_0(y)$ cannot be factor through the restriction of $[\epsilon' \ \iota]$ to $P(y) \oplus N'$, since the composition of any map $N \rightarrow P(y)$ with ϵ' is zero. \square

5.3. Lemma. *Let y be not a source of Q . Then the Auslander-Reiten sequence in \mathcal{L} ending in $P_0(y)$ is of the form*

$$0 \rightarrow \eta I_0(y) \rightarrow \eta(I_0(y)/S(y)) \oplus \text{rad } P_0(y) \rightarrow P_0(y) \rightarrow 0.$$

Proof. First, let us construct an Auslander-Reiten sequence ending in $P_0(y)$ in $\text{mod } \Lambda$. Since the radical of the endomorphism ring of $P_0(y)$ is zero, it is sufficient to construct an arbitrary non-split exact sequence with end terms $I_0(y)$ and $P_0(y)$. We start with the non-split exact sequence

$$0 \rightarrow S(y) \rightarrow S(y)[\epsilon] \rightarrow S(y) \rightarrow 0,$$

and form the induced exact sequences first with respect to the inclusion $S(y) \rightarrow I_0(y)$ and then with respect to the projection $P_0(y) \rightarrow S(y)$ in order to obtain a sequence

$$0 \rightarrow I_0(y) \rightarrow M \rightarrow P_0(y) \rightarrow 0$$

(actually, this is an Auslander-Reiten sequences with indecomposable middle term, as considered in [BR]). Note that M has a filtration $M_2 \subset M_1 \subset M$, with

$$M_2 = \text{rad } P_0(y), \quad M_1/M_2 = S(y)[\epsilon], \quad M/M_1 = I_0(y)/\text{soc}.$$

Next, the corresponding Auslander-Reiten sequence in \mathcal{L} ending in $P_0(y)$ is of the form

$$0 \rightarrow \eta I_0(y) \rightarrow \overline{M} \rightarrow P_0(y) \rightarrow 0,$$

where \overline{M} is an \mathcal{L} -approximation of M . Our aim is to determine \overline{M} . The exact sequence $0 \rightarrow M_2 \rightarrow M \rightarrow M/M_2 \rightarrow 0$ yields an exact sequence

$$(*) \quad 0 \rightarrow M_2 \rightarrow \overline{M} \rightarrow \eta(M/M_2) \rightarrow 0,$$

here we use that M_2 is a projective kQ -module, so that $\eta M_2 = M_2$.

Let us assume now that y is not a source.

Claim. *We have $\eta(M/M_2) = \eta(M/M_1)$ and the sequence $(*)$ splits.*

The exact sequence

$$0 \rightarrow S(y)[\epsilon] \rightarrow M/M_2 \xrightarrow{q} M/M_1 \rightarrow 0$$

yields an exact sequence

$$0 \rightarrow \eta(S(y)[\epsilon]) \rightarrow L \rightarrow \eta(M/M_1) \rightarrow 0$$

where L is a (not necessarily minimal) \mathcal{L} -approximation of M/M_2 . Clearly, $\eta(S(y)[\epsilon]) = P(y)$, but this implies that the sequence splits (since $\eta(M/M_1) \in \mathcal{L}$ and $\text{Ext}^1(\mathcal{L}, \mathcal{P}) = 0$). Thus $L = P(y) \oplus \eta(M/M_1)$ is an \mathcal{L} -approximation of M/M_2 . Now $M/M_1 = I_0(y)/S(y)$ is the direct sum of modules of the form $I_0(z)$ with $z \in Q_0$ (actually, the direct summands are just all the modules $I_0(s(\alpha))$, where α is an arrow with $t(\alpha) = y$), and, as we know, the minimal \mathcal{L} -approximations of these indecomposable modules have no non-zero projective direct summands. As a consequence, also a minimal \mathcal{L} -approximation of M/M_1 has no non-zero projective direct summand. Thus we may delete $P(y)$ and see that $\eta(M/M_1)$ is an \mathcal{L} -approximation of M/M_2 . But actually, this has to be a minimal \mathcal{L} -approximation. (Here is the proof. Let $g: \eta(M/M_1) \rightarrow M/M_1$ be the minimal right \mathcal{L} -approximation and let $g': \eta(M/M_1) \rightarrow M/M_2$ (with $g = qg'$) be a lifting which is a \mathcal{L} -approximation. Assume that we have a direct decomposition $\eta(M/M_1) = L \oplus L'$ such that $g'u$ is an \mathcal{L} -approximation of M/M_2 , where $u: L \rightarrow \eta(M/M_1)$ is the inclusion map. Then we claim that gu is an \mathcal{L} -approximation of M/M_1 . Assume that there is given a map $h: L'' \rightarrow M/M_1$ with $L'' \in \mathcal{L}$. Then there is a lifting $h': L'' \rightarrow \eta(M/M_1)$ such that $gh' = h$. In this way, we get a map $g'h': L'' \rightarrow M/M_2$. We use now that $g'u$ is an \mathcal{L} -approximation of M/M_2 and obtain $h'': L'' \rightarrow L$ with $g'h' = g'u h''$. But then $h = gh' = qg'h' = qg'u h'' = guh''$ shows that h can be factorized through g .)

We have shown that $\eta(M/M_2) = \eta(M/M_1)$. In order to see that the sequence

$$0 \rightarrow M_2 \rightarrow \overline{M} \rightarrow \eta(M/M_1) \rightarrow 0$$

splits, we use the fact that there is an exact sequence of the form

$$0 \rightarrow P \rightarrow \eta(M/M_2) \rightarrow M/M_2 \rightarrow 0.$$

Of course, $\text{Ext}_\Lambda^1(P, M_2) = 0$, since P is projective. But also $\text{Ext}_\Lambda^1(M/M_1, M_2) = 0$, since the support of $M/M_1 = I_0(y)/S(y)$ are the proper predecessors of y , whereas the support of $M_2 = \text{rad } P_0(y)$ are the proper successors of y , thus these supports are separated by the vertex y .

Altogether, we see that $\overline{M} = M_2 \oplus \eta(M/M_1) = \text{rad } P_0(y) \oplus \eta I_0(y)$, and this is the middle term of the Auslander-Reiten sequence in \mathcal{L} ending in $P_0(y)$, as we want to show. \square

6. The ghost maps.

A *ghost map* is by definition a homomorphism f such that $H(f) = 0$. Let us start with a characterization of the ghost maps between indecomposable Λ -modules in \mathcal{L} .

6.1. Proposition. *Let X, Y be indecomposable Λ -modules in \mathcal{L} and let $f: X \rightarrow Y$ be a homomorphism. Let X' be the kernel of the multiplication map $\epsilon: X \rightarrow X$, let Y_P be a projective submodule of Y such that $\epsilon Y \subseteq Y_P$. Then $H(f) = 0$ if and only if f can be written as the sum of two homomorphisms $f_0, f_1: X \rightarrow Y$ such that f_1 vanishes on X' , whereas the image of f_0 is contained in Y_P .*

Proof. First let us show that maps of the form f_0 and f_1 as mentioned in the assertion belong to the kernel of H . Now $H(f_1)$ is induced by the restriction of f_1 to X' , thus if this restriction is zero, then $H(f_1) = 0$. And, if the image of f_0 is contained in Y_P , then f_0 factors through Y_P , but Y_P is a projective Λ -module, thus $H(Y_P) = 0$ and therefore $H(f_0) = 0$.

Conversely, let $f: X \rightarrow Y$ be a homomorphism such that $H(f) = 0$. Let $X'' = \epsilon X$, $Y'' = \epsilon Y$, and let Y' be the kernel of the multiplication map $\epsilon: Y \rightarrow Y$. We can write X and Y as pushouts according to the following diagrams (where all the maps are the canonical inclusion maps).

$$\begin{array}{ccc} X'' & \xrightarrow{u'} & X' \\ u'' \downarrow & & \downarrow u \\ X''[\epsilon] & \xrightarrow{u'''} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} Y'' & \xrightarrow{v'} & Y' \\ v'' \downarrow & & \downarrow v \\ Y''[\epsilon] & \xrightarrow{v'''} & Y \end{array}$$

Note that we may identify $Y''[\epsilon]$ with the submodule Y_P of Y .

Now $H(f) = 0$ means that $f(X') \subseteq \epsilon Y$. We denote by $f': X' \rightarrow Y''$ the restriction of f to X' , it satisfies $vv'f' = fu$. Also, let $f'': X'' \rightarrow Y''$ be the restriction of f to X'' , thus $f'' = f'u'$. We can extend the map $f'': X'' \rightarrow Y''$ to a map $f''[\epsilon]: X''[\epsilon] \rightarrow Y''[\epsilon]$ such that $v''f'' = f''[\epsilon]u''$. Since X is the pushout of the maps u' and u'' , and $v''f'u' = v''f'' = f''[\epsilon]u''$, we obtain a map $h: X \rightarrow Y''[\epsilon]$ such that $hu = v''f'$ and $hu''' = f''[\epsilon]$. Let $f_0 = v'''h$. Then $(f - f_0)u = fu - v'''hu = vv'f' - v'''v''f' = (vv' - v'''v'')f' = 0$. This shows that $f_1 = f - f_0$ vanishes on X' . Altogether we see that $f = f_0 + f_1$, that f_1 vanishes on X_1 and that the image of f_0 is contained in the image of h , thus in Y_P . \square

If we look at the two maps f_0 and f_1 ,

$$\begin{array}{ccc} & & f_0 \\ & \curvearrowright & \\ & & Y''[\epsilon] \\ & \nearrow & \searrow \\ X & & Y \\ & \searrow & \nearrow \\ & & X/X' \\ & \curvearrowleft & \\ & & f_1 \end{array}$$

we see that they are of completely different nature: Namely, f_0 factors through a projective Λ -module (namely $Y_P = Y''[\epsilon]$), thus through an object which vanishes under H , whereas f_1 is a factorization inside the stable category $\underline{\mathcal{L}}$. The following special case is of interest:

6.2. Corollary. *Let X be an indecomposable Λ -modules in \mathcal{L} and denote by X' the kernel of the multiplication map $\epsilon: X \rightarrow X$. If Y is a projective kQ -module, and $f: X \rightarrow Y$ a homomorphisms, then $H(f) = 0$ if and only if $f(X') = 0$.*

Proof. Since Y is a projective kQ -module, the only projective Λ -module contained in Y is the zero module. \square

6.3. Lemma. *Let M be indecomposable in $\mathcal{L} \setminus \mathcal{P}$ and let M' be the kernel of the multiplication map ϵ . Let Y be a projective kQ -module. Then, any homomorphism $f: M \rightarrow Y$ vanishes on M' , thus is a ghost map.*

Proof. Let $M'' = \epsilon M$. Since Y is a kQ -module, we see that $M'' \subseteq \text{Ker}(f)$. Using the Noether isomorphism, we obtain an embedding

$$M'/(M' \cap \text{Ker}(f)) \rightarrow (M' + \text{Ker}(f))/\text{Ker}(f) \subseteq Y.$$

Now Y is a projective kQ -module, thus also $M'/(M' \cap \text{Ker}(f))$ is a projective kQ -module. It is a factor module of $M'/M'' = H(M)$, thus a direct summand of $H(M)$. But M is indecomposable and in $\mathcal{L} \setminus \mathcal{P}$, thus we know that $H(M)$ has no non-zero projective direct summands. This shows that $M'/(M' \cap \text{Ker}(f)) = 0$, thus $M' \subseteq \text{Ker}(f)$. \square

6.4. Given a vertex y of Q which is not a source, we have exhibited in 5.3 an Auslander-Reiten sequence of \mathcal{L} ending in $P(y)$. Let us denote by $u(y): \text{rad } P_0(y) \rightarrow P_0(y)$ the inclusion map and choose some map $c(y): \eta(I_0(y)/S(y)) \rightarrow P_0(y)$ such that

$$[c(y), u(y)]: \eta(I_0(y)/S(y)) \oplus \text{rad } P_0(y) \rightarrow P_0(y)$$

is a minimal right almost split map. If the vertex y is a source, then we may denote by $c(y)$ the zero map $0 = \eta(I_0(y)/S(y)) \oplus \text{rad } P_0(y) \rightarrow P_0(y)$.

Then we have:

Theorem 3. *The ideal of ghost maps in \mathcal{L} is generated by the identity maps of the indecomposable projective Λ -modules as well as the maps $c(y)$ for the vertices y of Q .*

Proof. Let \mathcal{I} be the ideal in \mathcal{L} generated by the the identity maps of the indecomposable projective Λ -modules as well as the maps $c(y)$ with $y \in Q_0$. Of course, all the maps in \mathcal{I} are ghost maps.

(a) First, let us show that *all the maps $M \rightarrow Y$, where M is indecomposable in $\mathcal{L} \setminus \mathcal{P}$ and not a projective kQ -module, whereas Y is a projective kQ -module, belong to \mathcal{I} .* For the proof, we can assume that Y is also indecomposable, say $Y = P_0(y)$ for some vertex y . We use induction on $l(y)$, where $l(y)$ is the maximal length of a path in Q ending in y . If $l(y) = 0$, then y is a sink, and therefore $\text{rad } P_0(y) = 0$. According to 5.3, the Auslander-Reiten sequence in \mathcal{L} ending in y shows that the minimal right almost split map for $P_0(y)$ is of the form

$$c(y): \eta(I_0(y)/\text{soc}) \rightarrow P_0(y).$$

Since we assume that M is not projective, we can factor f through $c(y)$, but $c(y)$ belongs to \mathcal{I} , therefore f belongs to \mathcal{I} .

Next, assume that $l(y) > 0$. If y is not a source, then the right almost split map ending in $P_0(y)$ is of the form

$$g = [c(y), u(y)]: \eta(I_0(y)/\text{soc}) \oplus \text{rad } P_0(y) \rightarrow P_0(y),$$

again according to 5.3. We factor f through g , thus we can write f as a sum of maps, where one factors through the map $c(y)$, thus belongs to \mathcal{I} , whereas the other maps factor through an indecomposable direct summand of $\text{rad } P_0(y)$. But all the indecomposable direct summands of $\text{rad } P_0(y)$ are of the form $P_0(x)$ with $l(x) < l(y)$, thus by induction we know already that the maps $M \rightarrow P_0(x)$ belong to \mathcal{I} , and therefore $f \in \mathcal{I}$.

It remains to look at the case where y is a source, so that the right almost split map ending in $P_0(y)$ is of the form

$$g: P(y) \oplus \text{rad } P_0(y) \rightarrow P_0(y),$$

now using 5.2. Again, we factor f through g , thus we can write f as a sum of maps, where one map factors through the projective module $P(y)$, and therefore belongs to \mathcal{I} , whereas the other maps factor through a direct summand $P_0(x)$ of $\text{rad } P_0(y)$. Again, we must have $l(x) < l(y)$, thus by induction all the maps $M \rightarrow P_0(x)$ belong to \mathcal{I} . This shows again that f belongs to \mathcal{I} .

(b) Now let us consider arbitrary modules X, Y in \mathcal{L} and let $f: X \rightarrow Y$ be a ghost map. We want to show that f belongs to \mathcal{I} . We can assume that both modules X and Y are indecomposable. Also we can assume that none of the modules X, Y belongs to \mathcal{P} .

Let us exclude the case that X is a projective kQ -module. In that case $H(f) = 0$ means that $f(X) \subseteq \epsilon Y$, but then we write f as the composition of the following three maps

$$X \rightarrow X[\epsilon] \xrightarrow{f[\epsilon]} (\epsilon Y)[\epsilon] \rightarrow Y$$

where the last map is some embedding (we know that such an embedding exists). This shows that f factors through a projective Λ -module.

(c) Thus it remains to consider the following setting: There is given a ghost map $f: X \rightarrow Y$, where X, Y are indecomposable modules in $\mathcal{L} \setminus \mathcal{P}$, and X is not a projective kQ -module. Let X' be the kernel of the multiplication map $\epsilon: X \rightarrow X$. According to 6.1, we can write $f = f_0 + f_1$ where f_0 factors through a projective Λ -module and where f_1 vanishes on X' . Now f_0 belongs to \mathcal{I} , thus it remains to be seen that f_1 is in \mathcal{I} . Since f_1 vanishes on X' , we can factor f_1 as

$$X \rightarrow X/X' \rightarrow Y.$$

Now X is indecomposable and not a projective kQ -module, whereas X/X' is a projective kQ -module, thus we have seen in (a) that the map $X \rightarrow X/X'$ belongs to \mathcal{I} . This shows that also f_1 belongs to \mathcal{I} and thus f is in \mathcal{I} . \square

6.5. Finally, let us describe the maps $c(y)$ in terms of the arrows of the quiver.

Given a vertex y , we may decompose $\eta(I_0(y)/S(y))$ as the direct sum of the modules $I_0(s(\alpha))$, where α runs through the arrows with $t(\alpha) = y$. Thus the map $c(y)$ considered in 6.4 is given by maps $I_0(s(\alpha)) \rightarrow P_0(y)$, for the various arrows $\alpha: s(\alpha) \rightarrow y$. Let us construct such maps explicitly.

Thus, for every arrow $\alpha: i \rightarrow j$ in Q , we want to construct a map

$$c(\alpha): \eta I_0(i) \rightarrow P_0(j)$$

which is irreducible in \mathcal{L} .

In order to define $\eta I_0(i)$, we start with a minimal projective kQ -presentation

$$0 \rightarrow \Omega_0 I_0(i) \rightarrow P_0 I_0(i) \rightarrow I_0(i) \rightarrow 0,$$

this yields a canonical map $\eta I_0(i) \rightarrow \Omega_0 I_0(i)$. Now consider any arrow $\alpha: i \rightarrow j$ in Q . There is up to isomorphism a unique indecomposable kQ -module $N(\alpha)$ with a non-split exact sequence

$$0 \rightarrow P_0(j) \rightarrow N(\alpha) \rightarrow I_0(i) \rightarrow 0.$$

(Since there is an arrow $i \rightarrow j$ and Q is directed, the supports of $P_0(j)$ and $I_0(j)$ do not intersect and $P_0(j)_j = k$, $I_0(i)_i = k$. Thus we define $N(\alpha)$ by using for $N(\alpha)_\alpha$ the identity map $k \rightarrow k$.)

Since $P_0 I_0(i)$ is projective, and $N(\alpha) \rightarrow I_0(i)$ is an epimorphism, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_0 I_0(i) & \longrightarrow & P_0 I_0(i) & \longrightarrow & I_0(i) & \longrightarrow & 0 \\ & & \downarrow c'(\alpha) & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & P_0(j) & \longrightarrow & N(\alpha) & \longrightarrow & I_0(i) & \longrightarrow & 0 \end{array}$$

Note that the map $c'(\alpha)$ has to be surjective (namely, consider the induced exact sequence using the projection map $\pi: P_0(j) \rightarrow S(j)$; by the construction of $N(\alpha)$, this sequence does not split, thus the composition of $c(\alpha)$ and π cannot be the zero map).

The required map $c(\alpha): \eta I_0(i) \rightarrow P_0(j)$ is the composition of the canonical map $\eta I_0(i) \rightarrow \Omega_0 I_0(i)$ and $c'(\alpha)$:

$$\eta I_0(i) \rightarrow \Omega_0 I_0(i) \xrightarrow{c'(\alpha)} P_0(j).$$

One may also use a combined construction in order to deal with the various arrows β starting in y at the same time, or also to deal with the various arrows α ending in y . Let $N(y)$ be obtained by identifying the modules $I_0(y)$ and $P_0(y)$ at the vertex y , thus there are the following two exact sequences

$$0 \longrightarrow P_0(y) \longrightarrow N(y) \longrightarrow I_0(y)/S(y) \longrightarrow 0$$

$$0 \longrightarrow \text{rad } P_0(y) \longrightarrow N(y) \longrightarrow I_0(y) \longrightarrow 0$$

Of course, $I_0(y)/S(y)$ is the direct sum of the modules $I_0(s(\alpha))$ where α is an arrow with $t(\alpha) = y$, whereas $\text{rad } P_0(y)$ is the direct sum of the modules $P_0(t(\beta))$ where β is an arrow with $s(\beta) = y$.

As above, we take a minimal projective resolution of $I_0(y)/S(y)$ or of $I_0(y)$ and obtain maps $c'(y): \Omega_0(I_0(y)/S(y)) \rightarrow P_0(y)$ and $d'(y): \Omega_0 I_0(y) \rightarrow \text{rad } P_0(y)$ as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_0(I_0(y)/S(y)) & \longrightarrow & P_0(I_0(y)/S(y)) & \longrightarrow & I_0(y)/S(y) & \longrightarrow & 0 \\
& & \downarrow c'(y) & & \downarrow & & \parallel & & \\
0 & \longrightarrow & P_0(y) & \longrightarrow & N(y) & \longrightarrow & I_0(y)/S(y) & \longrightarrow & 0 \\
& & & & & & & & \\
0 & \longrightarrow & \Omega_0 I_0(y) & \longrightarrow & P_0 I_0(y) & \longrightarrow & I_0(y) & \longrightarrow & 0 \\
& & \downarrow d'(y) & & \downarrow & & \parallel & & \\
0 & \longrightarrow & \text{rad } P_0(y) & \longrightarrow & N(y) & \longrightarrow & I_0(y) & \longrightarrow & 0
\end{array}$$

The composition of $c'(y)$ with the canonical map $\eta(I_0(y)/S(y)) \rightarrow \Omega_0(I_0(y)/S(y))$ yields a map $c(y): \eta(I_0(y)/S(y)) \rightarrow P_0(y)$. Similarly, we compose $d'(y)$ with the canonical map $\eta I_0(y) \rightarrow \Omega_0 I_0(y)$ and obtain $d(y): \eta I_0(y) \rightarrow \text{rad } P_0(y)$. These maps $c(y)$ and $d(y)$ can be used in the Auslander-Reiten sequences exhibited in 5.2 and 5.3.

7. The position of the indecomposable projective modules.

7.1. Lemma. *Let $P(x)$ be the indecomposable projective Λ -module corresponding to the vertex x . Then $H(\text{rad } P(x)) = S(x)$.*

Proof. We write $P(x) = P_0(x)[\epsilon]$, thus there is an exact sequence of the following form

$$0 \rightarrow (\text{rad } P_0(x))[\epsilon] \rightarrow P_0(x)[\epsilon] \rightarrow S(x)[\epsilon] \rightarrow 0.$$

This implies that we obtain for the radical of $P(x)$ an exact sequence of the form

$$0 \rightarrow (\text{rad } P_0(x))[\epsilon] \rightarrow \text{rad } P(x) \rightarrow S(x) \rightarrow 0,$$

therefore $H(\text{rad } P(x)) = S(x)$. □

Thus, the Auslander-Reiten sequence with $P(x)$ as a direct summand of the middle term starts with $\eta S(x)$. We distinguish whether x is a source or not.

7.2 If x is a source, then $I_0(x) = S(x)$, thus we deal with the Auslander-Reiten sequence exhibited in Lemma 5.2:

$$\begin{array}{ccccc}
& & P(x) & & \\
& \nearrow & & \searrow & \\
\eta S(x) & & & & P_0(x) \\
& \searrow & & \nearrow & \\
& & \text{rad } P_0(x) & &
\end{array}$$

What is of interest and should be remembered is the fact that in this case the irreducible map starting in $P(x)$ is surjective with target $P_0(x)$.

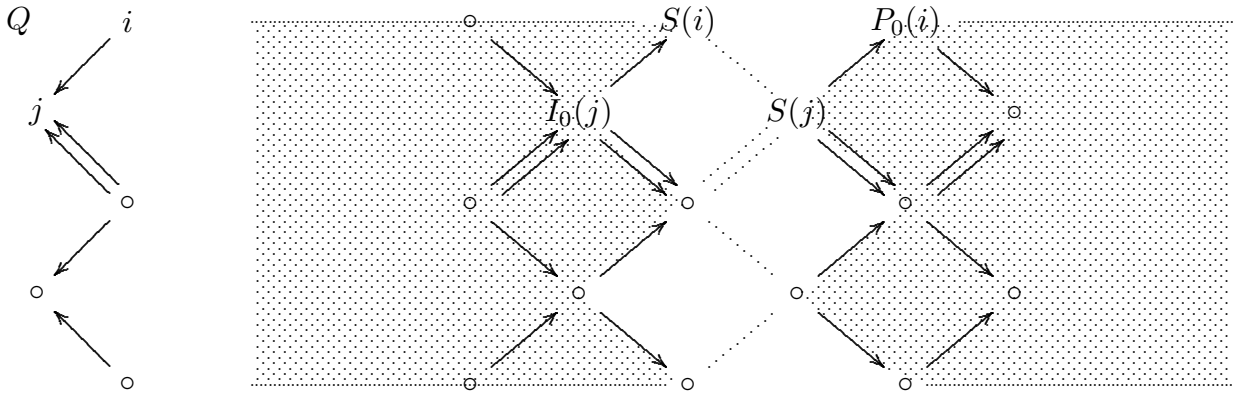
It remains to consider the case that x is not a source. Let us denote by τ_0 the Auslander-Reiten translation for $\text{mod } kQ$. Since x is not a source, $S(x)$ is not an injective kQ -module, thus $\tau_0^{-1}S(x)$ is defined.

7.3. *If x is not a source, then the Auslander-Reiten sequence with $P(x)$ as a direct summand of the middle term starts with $\eta S(x)$ and ends in $\eta\tau_0^{-1}S(x)$.* \square

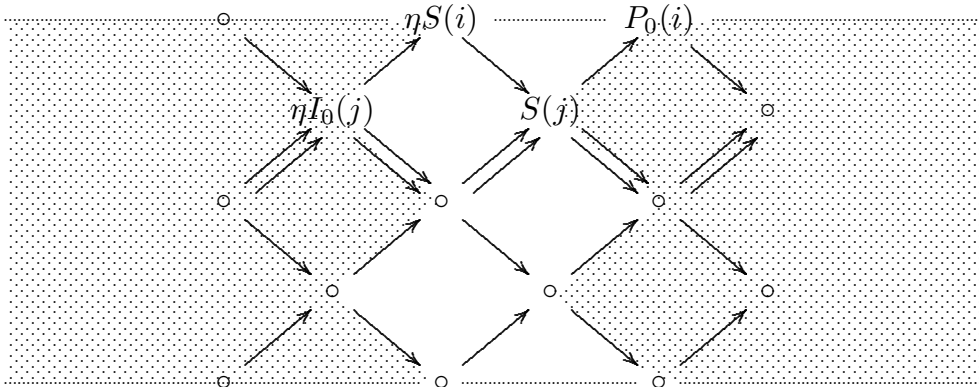
8. Examples

8.1. First example. As a first example, we take a bipartite quiver Q (this means that all the vertices are sinks or sources). If $\alpha: i \rightarrow j$ is an arrow in Q , then $I_0(i) = S(i)$ and $P(j) = S(j)$ are simple kQ -modules.

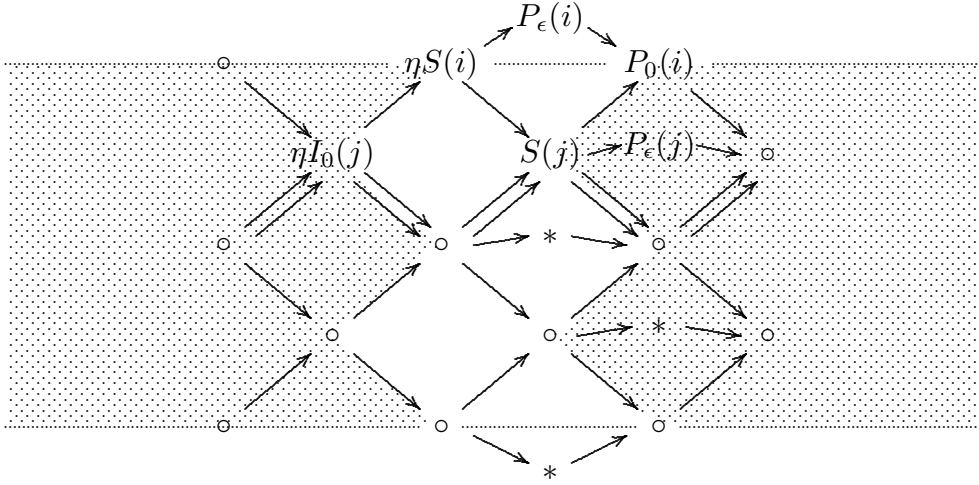
We consider the following quiver Q exhibited below on the left. On the right side, we present the decisive parts of the preinjective component and the preprojective component of $\Gamma(kQ)$, already as subquivers of the translation quiver $\mathbb{Z}(Q^{\text{op}})$ separated only by some arrows (they are marked by dotted diagonal lines):



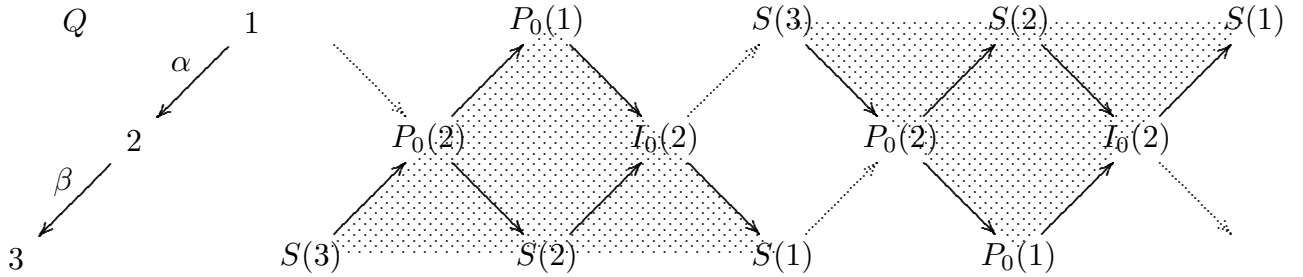
If we consider now $\underline{\mathcal{L}}$, we have to replace any kQ -module N by ηN , and we have to add an arrow $\eta S(i) \rightarrow S(j)$ for any arrow $i \rightarrow j$ in the quiver Q . These new arrows represent a k -basis of $\text{Hom}(\eta S(i), S(j))$. Note that the new arrows represent ghost maps. Here is this part of $\Gamma(\underline{\mathcal{L}})$.



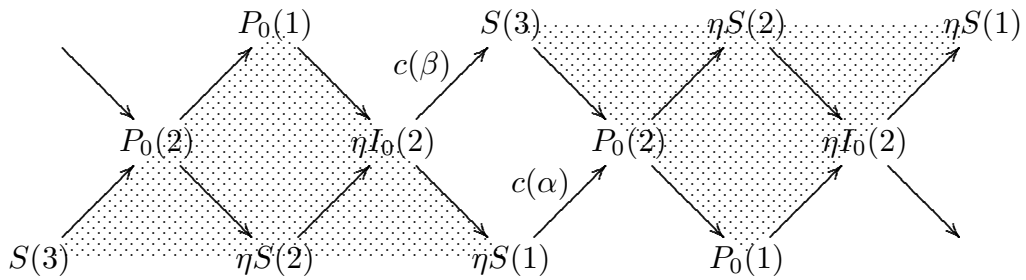
Finally, let us show the corresponding part of $\Gamma(\mathcal{L})$. Here, the indecomposable projective Λ -modules are added.



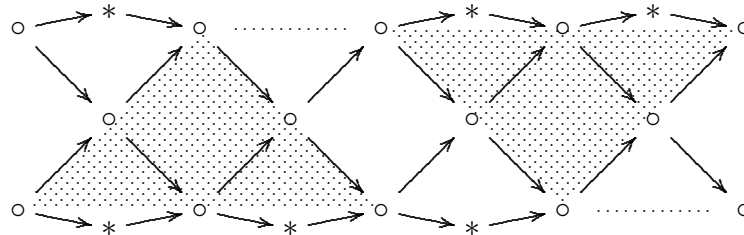
8.2. Second example. As a second example we take the quiver Q of type \mathbb{A}_3 with linear orientation. First, let us show two copies of $\Gamma(\text{mod } kQ)$ appropriately embedded into the translation quiver $\mathbb{Z}\mathbb{A}_3$.



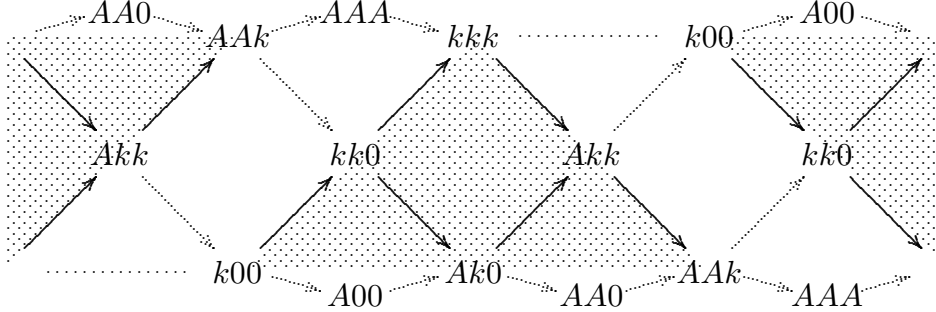
Now we present the corresponding torsionless Λ -modules and insert the ghost arrows $c(\alpha): \eta I_0(1) \rightarrow P(2)$ and $c(\beta): \eta I_0(2) \rightarrow S(3)$.



The Auslander-Reiten quiver of \mathcal{L} (or better its universal covering) looks as follows:



It may be helpful for the reader to use this example in order to write down all the Λ -modules as representations of Q over A (for example, the module $M = \eta Q(1) = \eta S(1)$ is written as AAk , since $M_3 = M_2 = A$ and $M_1 = k$). As in some illustrations before, we use dotted arrows in order to mark the ghost maps.



9. Remarks on the behavior of the homology functor H on $\text{mod } \Lambda$.

We have seen that the homology functor has nice properties when restricted to the subcategory $\mathcal{L} \setminus \mathcal{P}$. For example, it maps indecomposable modules to indecomposables, and non-isomorphic indecomposable modules to non-isomorphic ones. Also, for a fixed homology dimension vector \mathbf{r} , there is either one indecomposable object in $\mathcal{L} \setminus \mathcal{P}$ with $\dim H(M) = \mathbf{r}$ or at least a 1-parameter family. But all these assertions are only true for the restriction of H to $\mathcal{L} \setminus \mathcal{P}$. In general, one cannot expect such a pleasant behavior.

9.1. Example. Consider the quiver Q of type \mathbb{A}_3 with two sources and a sink. Then the AQ module $M = k \rightarrow A \leftarrow k$ is indecomposable, but $K(M) = k \rightarrow 0 \leftarrow k$ is decomposable.

9.2. Example. Let Q be the Kronecker quiver with two arrows from 2 to 1. Then there is a \mathbb{P}_1 -family of indecomposable AQ modules M with $M_2 = k$ and $M_1 = A$. For all of them $H(M)$ is the simple module $S(2)$. Thus here we have many non-isomorphic AG -modules with isomorphic homology modules.

9.3. Example. Let Q be the quiver of type \mathbb{D}_4 with subspace orientation, let 1, 2, 3 be the sources of Q and 0 the sink. Then there are precisely 2 isomorphism classes of indecomposable AQ -modules M with $H(M) = S(1) \oplus S(2) \oplus S(3)$, for one of them $M_0 = A$, for the other one, $M_0 = A^2$. (For the proof, one has to observe that in this case Λ is a tubular algebra, and one has to study the corresponding root system in detail.)

10. Generalization.

Instead of looking at the path algebra kQ of a quiver, one may start with an arbitrary finite-dimensional k -algebra H which is hereditary and take instead of $\Lambda = kQ[\epsilon]$ the corresponding algebra $H[\epsilon]$. Observe that the proofs of the theorems given here work in general. What is special in the quiver case is just the possibility to describe the maps $c(y)$ in terms of the arrows of the quiver, see 6.5.

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