

One family of affinely homogeneous algebraic surfaces in \mathbb{C}^3

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Abstract.

In this article the 3-parameter family of real homogeneous hypersurfaces of 3-dimensional complex space is constructed. This family generalizes the series of examples that were published earlier. It contains both Levi nondegenerate surfaces (strictly pseudo convex and indefinite ones) and surfaces with degenerate Levi form.

Unlike the existing cumbersome descriptions of matrix algebras corresponding to surfaces under consideration, we suggest uppertriangular representation of these algebras with simple special bases. It is shown that all the affinely homogeneous surfaces of constructed family are algebraic ones of 1, 2, 3, 4 or 6 order.

In the article we consider also some properties of the obtained surfaces from the holomorphic homogeneity point of view.

Introduction

80 years ago E.Cartan published in [1] the classification of real hypersurfaces of 2-dimensional complex spaces, which are homogeneous according to holomorphic (pseudo-conformal) transformations. At the end of XX century the problems of the homogeneous manifolds description attracts the investigator's attention. But till nowadays the complete description of a class of homogeneous real hypersurfaces doesn't exist even in the case of 3-dimensional complex spaces.

Moreover there is a number of more simple problems, connected with homogeneity of embedded submanifolds, which don't have a complete decision yet.

Among such unsettled questions firstly we can mention the problem of the description of real hypersurfaces of complex space \mathbb{C}^3 which are homogeneous in a sence of affine transformations of this space. It should be noted that analogous problem in real space \mathbb{R}^3 has completely solved only in 1995-1996 (see [2]). The description of affinely homogeneous real hypersurfaces in space \mathbb{C}^2 was obtained in [3]-[4] in 2010-2011.

Because of the difference between affine and holomorphic geometries the solution of the last problem doesn't deduce from

Cartan's description and requires separate study of many cases. But any manifold which is homogeneous in a sence of affine transformations of some complex space is holomorphically homogeneous too. That's why the construction and study of different examples of affinely homogeneous submanifolds give us useful information for the solving of common problem of holomorphic homogeneity.

As the results of [5] -[8] show the considerable part of holomorphically homogeneous

objects can be reduced to affinely homogeneous ones by holomorphic transformation. It gives rise for more detailed study of the connection between different types of homogeneity. It should be mentioned in this connection that all the affinely homogeneous curves in real plane \mathbb{R}^2 (except exponential curve) turn out to be linearly homogeneous after appropriate affine coordinate change (complete list of such curves one can find in [9]-[10]).

Returning to complex space \mathbb{C}^3 let us define more exactly the main purpose of our article. The aim is to describe one big family of homogeneous manifolds from different points of view.

In theorem 1 we present the matrix Lie algebras related with the homogeneous surfaces under consideration in upper triangular form with simply constructed basis matrixes (note that previous descriptions of such algebras were tedious enough).

In theorems 2 and 3 the explicit equations are constructed for discussed surfaces and conclusion is made about their algebraic character.

Some results are also obtained in this article concerning the place of surfaces under consideration in holomorphic classification of homogeneous manifolds of 3-dimensional complex spaces.

§1. Extension of matrix Lie algebras

In our study of homogeneity of real surfaces in \mathbb{C}^3 we use two basic instruments: technique of matrix Lie algebras and canonical equations of surfaces under consideration.

We remind that manifold called to be homogeneous if some transformation group acts transitively on it. It's natural to discuss some Lie subgroups of the group $Aff(3, \mathbb{C})$ as a group acting on submanifold (real surface) of affine space \mathbb{C}^3 . Lie algebra corresponding to any such subgroup can be considered as an algebra of affine vector fields in \mathbb{C}^3 , having a form

$$\begin{aligned} Z = & (A_1 z_1 + A_2 z_2 + A_3 w + p) \frac{\partial}{\partial z_1} + \\ & + (B_1 z_1 + B_2 z_2 + B_3 w + s) \frac{\partial}{\partial z_2} + \\ & + (a z_1 + b z_2 + c w + q) \frac{\partial}{\partial w}, \end{aligned} \quad (1.1)$$

where z_1, z_2, w – complex coordinates in \mathbb{C}^3 , $A_k, B_k, a, b, c, p, s, q$ – complex constants.

Matrix representation of Lie algebra consisting of fields (1.1), allows to consider the algebra's elements as a 4×4 - matrixes of the form

$$\begin{pmatrix} A_1 & A_2 & A_3 & p \\ B_1 & B_2 & B_3 & s \\ a & b & c & q \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1.2)$$

Meanwhile, fields bracket transforms to matrix commutator (bracket)

$$[Z_1, Z_2] = Z_1 Z_2 - Z_2 Z_1.$$

We denote the Lie group of transitive affine transformations acting on a surface $M \in \mathbb{C}^3$ by $G(M)$, and Lie algebra corresponding to this group by $g(M)$. Vector fields from $g(M)$ are tangent to discussed affinely homogeneous real surface M . Analytically this fact can be written in such form

$$Re\{Z(\Phi)\}_{|M} = 0, \quad (1.3)$$

where Φ is a defining function of surface M . The real part symbol appears in (1.3) because we discuss a real surface M in complex space \mathbb{C}^3 .

The study of affinely homogeneous surfaces in \mathbb{C}^3 can be connected with description of algebras $g(M)$ corresponding to these surfaces. In that case we should remember that affine equivalence of two affinely homogeneous surfaces M_1 and M_2 means the similarity of corresponding algebras $g(M_1)$ and $g(M_2)$ (with some special similarity matrix).

One can take into account the possibility of such equivalence from the very beginning and consider the problem of homogeneity only for the surfaces with equations having already reduced to some simple form by affine transformation. In this case the matrixes constituting Lie algebras $g(M)$ will have some peculiarities in their structure.

Note that first of all we will discuss real analytic hypersurfaces in space \mathbb{C}^3 with non-degenerate Levi form (see [11]). Let us suppose, for example, that Levi form of such surface M is positively defined in some point ξ (i.e. M is strictly pseudo-convex (SPC) surface). Then according to [12] the equation of M can be reduced near ξ by affine transformation to the form

$$Imw = (|z_1|^2 + |z_2|^2) + \varepsilon_1(z_1^2 + \bar{z}_1^2) + \varepsilon_2(z_2^2 + \bar{z}_2^2) + \sum_{k \geq 3} F_k(z_1, z_2, \bar{z}_1, \bar{z}_2, Rew). \quad (1.4)$$

Here $k \geq 3$ is a weight of every polynomial term (weights of variables $z_1, z_2, \bar{z}_1, \bar{z}_2$ are equal to 1, weight of $u = Rew$ equals 2).

In this case a pair of non-negative coefficients $(\varepsilon_1, \varepsilon_2)$ is an affine invariant of the surface and the form (1.4) is called an affine canonical form for equation of M .

The equation of indefinite surface (having nondegenerate indefinite Levi form) can be reduced to the analogous canonical kind (having more complicated structure of quadratic part).

Below we discuss only *rigid* surfaces whose equations (1.4) are free of variable $u = Rew$. Coefficients of affine vector field (1.1) which is tangent to rigid surface (1.4) obey the following restrictions

$$\begin{aligned} q &\in \mathbb{R}, \\ \begin{cases} a = 2i(\bar{p} + 2\varepsilon_1 p), \\ b = 2i(\bar{s} + 2\varepsilon_2 s). \end{cases} & \end{aligned} \quad (1.5)$$

Another restriction on the elements of discussed algebras considerably simplifies the necessary calculations but it has an "artificial" character. All the algebras $g(M)$ from the articles [12] - [14] have the matrix representation (1.2) satisfying the following condition

$$A_3 = B_3 = 0. \quad (1.6)$$

Remark 1. If the coefficients $\varepsilon_1, \varepsilon_2$ of the surface equation (1.4) have "common position"

$$0 < \varepsilon_1, \varepsilon_2 \neq \frac{1}{2}, \quad (1.7)$$

then the equalities

$$ReA_3 = ReB_3 = 0, \quad (1.8)$$

can be deduced from (1.3). This remark justifies particularly the artificial conditions (1.6).

Remark 2. In the case of affinely homogeneous surfaces in \mathbb{C}^2 the unique coefficient of analogous canonical equation corresponds to the pair (A_3, B_3) . This coefficient can have nonzero values (see [3]).

Taking into account introduced restrictions we can write the matrix form (1.2) of vector fields in more simple manner

$$\begin{pmatrix} A_1 & A_2 & 0 & p \\ B_1 & B_2 & 0 & s \\ a & b & c & q \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (1.9)$$

There is a following geometric interpretation of such matrixes: their left-upper (2×2) -blocks correspond to infinitesimal actions of group $G(M)$ in complex tangent plane $T_0^{\mathbb{C}}M$ to discussed surface M , given near the origin by equation (1.4).

It should be noted that one can multiply left-upper (2×2) -blocks of matrixes (1.9) independently on any other elements of two multiplied matrixes (1.9). Because of this, the following simply checked statement for matrix algebras consisting of matrixes (1.9) is valid.

Proposition 1. *If g is a Lie subalgebra of algebra $GL(4, \mathbb{C})$, consisting of type (1.9) matrixes, then left-upper (2×2) -blocks of these matrixes constitute an algebra too.*

Denoting left-upper (2×2) -blocks of matrixes Z by e (with corresponding index) we can add the following formula to the proposition 1:

$$[Z_1, Z_2]_e = [e_1, e_2] = e_1 \cdot e_2 - e_2 \cdot e_1. \quad (1.10)$$

Proposition 1 is a base for introducing the following definition.

Definition 1. Let g be a Lie subalgebra of algebra $GL(4, \mathbb{C})$, whose elements have a form (1.9), h be a Lie subalgebra of algebra $GL(2, \mathbb{C})$. If algebra

$$g_e = \{Z_e | Z \in g\}$$

coincides with h , then g is called *an extension* of algebra h .

In connection with the problem of some classes description of affinely homogeneous surfaces in \mathbb{C}^3 the question of the extension property study in the sense of definition 1 arises for the Lie subalgebras of algebra $GL(2, \mathbb{C})$ to subalgebras of $GL(4, \mathbb{C})$.

In that case one can put a number of supplementary natural restrictions on extended algebras. For example, we can assume that extended algebras have a real dimension 5. Specifying this restrictions we will also require that the real linear hull of last columns of matrixes (1.9) constituting the discussed algebra g , has real dimension 5. The last requirement means

that the group $G(M)$ action allows to displace from the origin in any tangent direction to 5-dimensional surface M .

Only few Lie subalgebras of algebra $GL(2, \mathbb{C})$ are shown in articles [14]-[15] to allow such extension. For example, there is only one (up to matrix similarity) extended real 3-dimensional subalgebra $GL(2, \mathbb{C})$. This is the algebra \hat{g} with basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}. \quad (1.11)$$

Proposition 2 ([16], Theorem 5). *There exists a 3-parameter family of affinely different affinely homogeneous strictly pseudo convex (SPC) real hypersurfaces in space \mathbb{C}^3 whose algebras of linear vector fields are the extensions of algebra \hat{g} . Bases of these algebras are*

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & -i(1 - 4\varepsilon_2^2)\omega \\ 0 & 1/2 & 0 & 2i(\zeta\omega + 2\varepsilon_2\bar{\zeta}\bar{\omega}) \\ -4(1 - 4\varepsilon_2^2)\bar{\omega} & 2(1 - 4\varepsilon_2^2)(2\varepsilon_2\zeta\omega + \bar{\zeta}\bar{\omega}) & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 1 & 0 & 4i(\zeta\omega + 2\varepsilon_2\bar{\zeta}\bar{\omega}) \\ 0 & 0 & 0 & 4i((1 - \zeta^2)\omega + 2\varepsilon_2(1 - |\zeta|^2)\bar{\omega}) \\ a_2 & 8(1 - 4\varepsilon_2^2)\bar{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (1.12)$$

where $a_2 = 16(\varepsilon_2 - \varepsilon_1)\zeta\omega - 8(4\varepsilon_1\varepsilon_2 - 1)\bar{\zeta}\bar{\omega} - 8(4\varepsilon_2^2 - 1)\zeta\bar{\omega}$,

$$E_3 = \begin{pmatrix} 0 & i & 0 & -4(\zeta\omega + 2\varepsilon_2\bar{\zeta}\bar{\omega}) \\ 0 & 0 & 0 & 4((1 + \zeta^2)\omega - 2\varepsilon_2(1 - |\zeta|^2)\bar{\omega}) \\ a_3 & 8i(1 - 4\varepsilon_2^2)\bar{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_3 = -8i(2(\varepsilon_1 + \varepsilon_2)\zeta\omega + (4\varepsilon_2(\varepsilon_1\bar{\zeta} + \varepsilon_2\zeta) + (\bar{\zeta} - \zeta))\bar{\omega})$,

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 \\ 0 & 2i(2\varepsilon_2\zeta\omega + \bar{\zeta}\bar{\omega}) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and parameters entering these matrixes are connected by the relations:

$$|\omega| = 1, \quad 2(\varepsilon_1 + \varepsilon_2\zeta^2)\omega + (1 + |\zeta|^2)\bar{\omega} = 0.$$

Simplified 2-parameter version of this family that corresponds to the case $\omega = 1$ is given in [14].

Algebra \hat{g} as it is shown in [17], [13] can be extended to algebras that relate to indefinite affinely homogeneous hypersurfaces. The formulas describing such extensions in indefinite case turn out still more complicated than in SPC situation.

Integration of all the algebras mentioned above (and construction of concrete homogeneous surfaces) is extremely difficult in realization. The cases in which such integration was realized in [13]-[14] reduce to algebraic surfaces of 3, 4 and 6 orders.

The main purpose of this article is to simplify a visual representation of algebras and surfaces obtained in [13]-[14]. Such simplification allows to put the discussed examples in expected full classifications of affinely homogeneous and holomorphically homogeneous real surfaces of space \mathbb{C}^3 .

§2. Simplified representation of extended matrix algebras

Let us consider 5-dimensional Lie subalgebra g of matrix algebra $GL(2, \mathbb{C})$. We will require further that all the matrixes g have a form (1.2), and the algebra g itself is an extension of the 3-dimensional algebra (1.11). Moreover let us suppose that g satisfies to some conditions that are necessary in order to this algebra be a representation of matrix algebra $g(M)$ for some rigid affinely homogeneous surface $M \in \mathbb{C}^3$ (given by canonical equation (1.4)).

We enumerate these conditions:

- 1) elements $q_k (k = 1, \dots, 5)$ of all the basis matrixes $E_1 - E_5$ of algebra g are real;
- 2) basis matrix E_5 has a form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(that corresponds to rigidity of integral surface M of an algebra $g(M)$);

- 3) real dimension of linear hull $V = \langle \begin{pmatrix} p_k \\ s_k \\ q_k \end{pmatrix}, k = 1, \dots, 5 \rangle$ of the last columns of the basis matrixes is equal to 5.

Let's come to an agreement without loss of discussing generality that basis matrixes E_1, E_2, E_3 of algebra g have matrixes e_1, e_2, e_3 from (1.11) as a e -blocks exactly; e -blocks of matrixes E_4, E_5 are considered to be equal to zero.

Finally, besides necessary conditions already formulated, we require (only for simplifying of calculations) that following relations for the elements of ordered basis matrixes of discussed algebra are satisfied:

$$4) p_2 = 2s_1, p_3 = 2is_1, b_2 = -2a_1, b_3 = -2ia_1. \quad (2.1)$$

Remark. In articles [13]-[14] an "artificial" conditions (2.1) are used also. In that case as it is already noticed above, it is possible to construct the 3-dimensional family of different affinely homogeneous hypersurfaces in space \mathbb{C}^3 . This remark emphasizes once more the difficulty of complete solution of description problem for such surfaces.

THEOREM 1. *If Lie algebra g , consisting of matrixes (1.2) and satisfying the conditions 1)-4) enumerated above, is an extension of algebra (1.11), then it is conjugate to*

upper triangular algebra. Basis of such triangular algebra has one of two following types:

$$1) E_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & A & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & (iA - s) & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & (iA + s) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.2)$$

$$E_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A = m + in, m, n, s \in \mathbb{R},$$

or

$$2) E_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & (iA + s) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.3)$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A = m + in, m, n, s \in \mathbb{R}.$$

Remark. The conjugation of algebra g from theorem 1 and upper triangular algebra g^* (of the first or second types) is realized by formula $g^* = CgC^{-1}$ with some nondegenerate

$$\text{matrix } C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the proof of Theorem 1 we need a sequence of subsidiary statements (Propositions 3-7 below) about properties of discussed algebras and their basis matrixes. Let's write these matrixes taking into account noticed requirements to them. Then

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & p_1 \\ 0 & 1/2 & 0 & s_1 \\ a_1 & b_1 & c_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 & 0 & p_2 \\ 0 & 0 & 0 & s_2 \\ a_2 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & i & 0 & p_3 \\ 0 & 0 & 0 & s_3 \\ a_3 & b_3 & c_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.4)$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & s_4 \\ a_4 & b_4 & c_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here (3,4)-elements of basis matrixes E_1, E_2, E_3, E_4 are supposed to be equal to zero because we can consider corrected matrixes

$$E_k^* = E_k - q_k E_5 \quad (k = 1, \dots, 4).$$

instead of the initial ones.

Proposition 3. Elements $c_k (k = 1, \dots, 4)$ of basis matrixes E_1, E_2, E_3, E_4 of algebra (2.4) are real.

Proof. The closedness of an algebra (2.4) in relation to matrix bracket means that the bracket of any two its basis matrixes expands by this basis with some real coefficients. In that case $[E_k, E_5]_{(3,4)} = c_k$ for all $k = 1, \dots, 4$ and these values must be a real E_5 -coefficients of the expansion of bracket under consideration via the basis E_1, E_2, E_3, E_4, E_5 . ■

Closedness property in relation to bracket allows us to get a lot of other important information about basis matrixes. For example, the following stronger version of proposition 3 is valid for an algebra \hat{g} extensions (see the proof in [14]).

Proposition 4. Following conditions are satisfied for elements of basis matrixes E_1, E_2, E_3, E_4 of algebra (2.4):

$$c_1 = 3/2, c_2 = c_3 = c_4 = 0. \quad (2.5)$$

Here we won't repeat the proof of this fact. It will be considered to be realized for discussed algebras together with equalities (2.1). Exactly such values of coefficients c_k are considered in all the examples of algebras from articles [12]-[14].

Taking into account proposition 4 one can simplify the basis of discussed algebra (2.4) to the form

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & p_1 \\ 0 & 1/2 & 0 & s_1 \\ a_1 & b_1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 & p_2 \\ 0 & 0 & 0 & s_2 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & i & 0 & p_3 \\ 0 & 0 & 0 & s_3 \\ a_3 & b_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.6)$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & s_4 \\ a_4 & b_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Another simple fact is also a consequence of algebra (2.6) closedness property.

Proposition 5. The following equalities are valid for the matrix Lie algebra with basis (2.6):

$$a_4 = 0, \quad s_4 = 0. \quad (2.7)$$

Proof. Brackets

$$[E_2, E_4] = \begin{pmatrix} 0 & 0 & 0 & s_4 \\ 0 & 0 & 0 & 0 \\ 0 & -a_4 & 0 & -a_4 p_2 - b_4 s_2 + a_2 p_4 + b_2 s_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[E_3, E_4] = \begin{pmatrix} 0 & 0 & 0 & i s_4 \\ 0 & 0 & 0 & 0 \\ 0 & -i a_4 & 0 & a_3 p_4 + b_3 s_4 - a_4 p_3 - b_4 s_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

have to expand as linear combinations with real coefficients via the basis E_1, E_2, E_3, E_4, E_5 . Since e -blocks of these brackets are equal to zero, then only E_4 and E_5 take part in expansion of each bracket. In that case (1,4)- and (3,2)- elements of these brackets, distinguishing by factor i , have to be equal to zero. ■

Corollary. If an algebra (2.6) satisfies the conditions 1) - 4) written above, then inequality for element p_4 of basis matrix E_4 is valid

$$p_4 \neq 0. \quad (2.8)$$

Matrix E_1 from constructed basis (2.6) has a simple spectrum ($\mu_1 = 1, \mu_2 = 1/2, \mu_3 = 3/2, \mu_4 = 0$), therefore this matrix can be reduced to diagonal form. Let us go to new algebra by means of the conjugation $E_k \rightarrow S^{-1}E_kS$ with matrix S , consisting of the eigenvectors of E_1 .

Note firstly that diagonalization of one matrix from the algebra "improves" in some sense other basis matrixes due to algebraic structure of the linear hull

$$g = \langle E_1, E_2, E_3, E_4, E_5 \rangle.$$

Second remark concerns the proposed ordering of eigenvalues of matrix E_1 . Their monotonous decreasing is most convenient as it guarantees the upper triangular form of new matrix. And though upper triangular form is rather ordinary for matrix algebras it wasn't easy to get it in our case from initial geometric discussions of the problem under consideration.

So, let's consider the matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & -p_1 \\ 0 & 0 & 1 & -2s_1 \\ 1 & -2a_1 & -b_1 & (2a_1p_1 + 4b_1s_1)/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.9)$$

consisting of ordered eigenvectors of E_1 .

Proposition 6. If algebra (2.6) satisfies the conditions (2.1) and (2.7), then it transforms under conjugation with similarity matrix S to the algebra with basis

$$E_1^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2^* = \begin{pmatrix} 0 & a_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3^* = \begin{pmatrix} 0 & a_3 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.10)$$

$$E_4^* = \begin{pmatrix} 0 & 0 & b_4 & 0 \\ 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Proof of proposition 6 is purely technical. However, it should be noted that direct application of S -similarity to basis matrix gives the required matrixes only for E_1 and E_5 .

For example, matrix E_4 transforms by this conjugation not to E_4^* but to

$$\begin{pmatrix} 0 & 0 & b_4 & (-2b_4s_1 + 2a_1p_4) \\ 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Last matrix can be reduced to form E_4^* due to reality condition of the expression

$$(-2b_4s_1 + 2a_1p_4).$$

This fact is a consequence of closedness property for initial algebra. The condition that bracket of initial basis matrixes

$$[E_1, E_4] = \begin{pmatrix} 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & 0 \\ 0 & b_4 & 0 & (-b_4s_1 + a_1p_4) \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

belongs to the algebra (2.6) provides the required reality.

Analogously but by using the conditions (2.1) the upper triangular matrixes $S^{-1}E_2S$ and $S^{-1}E_3S$ can be transformed to more simple "one-diagonal" matrixes E_2^* and E_3^* .

Proposition 6 is considered to be proved. ■

Triangular 5-dimensional algebras with bases (2.10) allow the further simplification owing to conjugations. The similarity

$$Z \rightarrow C^{-1}ZC$$

with diagonal matrix

$$C = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.11)$$

transforms each element z_{kl} of matrix Z in $\lambda_k^{-1}z_{kl}\lambda_l$.

For example, it means that in order to save E_5^* it is sufficient to put

$$\lambda_1 = 1 \quad (2.12)$$

in matrix (2.11). Matrix E_4^* transforms by discussed similarity to

$$\begin{pmatrix} 0 & 0 & b_4\lambda_3 & 0 \\ 0 & 0 & 0 & p_4\lambda_2^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further discussion decomposes on subcases. For example, when both the elements p_4, b_4 in matrix E_4 are not equal to zero we can put in addition to (2.12)

$$\lambda_2 = 1/p_4, \quad \lambda_3 = b_4. \quad (2.13)$$

Then both elements p_4, b_4 are transformed into unities, and (2,3)-elements 1 and i of matrixes E_2^* and E_3^* are multiplied on the same non-zero number $\lambda_2^{-1}\lambda_3$.

Instead of the exact transformed matrixes E_2^* and E_3^* we consider their real linear combinations, so we can restore the normed (2,3)-elements 1 and i .

Therefore, there are two versions of matrix E_4^* from basis of algebra (2.10) after using of diagonal conjugation (with saving of the form of the rest four basis matrixes):

$$E_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ or } E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.14)$$

Proposition 7. *In first case from (2.14) the initial algebra (2.6) transforms by conjugation into algebra (2.2); in second case - into algebra (2.3).*

Proof. For the proof we use the closedness of algebra and bracket's restrictions following from this property. These restrictions are rather bulky in initial form of algebra (2.6). Now after diagonal correction they become rather elegant and constructive. For example, for matrix elements of the form (2.10), written in generalized notations as

$$E_1 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & A_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & B_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & A_3 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & B_3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & 0 & A_4 & 0 \\ 0 & 0 & 0 & B_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.15)$$

the following conditions hold:

$$1) A_2B_4 - A_4B_2 \in \mathbb{R}, \quad A_3B_4 - A_4B_3 \in \mathbb{R} \quad (2.16)$$

and

$$2) iA_2 - A_3 = rA_4, \quad B_3 - iB_2 = rB_4 \quad (2.17)$$

for some real r .

Substituting equalities $A_4 = B_4 = 1$ in (2.16) and (2.17), we obtain:

$$A_3 = iA_2 - r, \quad B_3 = iB_2 + r, \quad B_2 = A_2.$$

In that case investigated algebra depends on 3 real parameteres ($A \in \mathbb{C}, r \in \mathbb{R}$) and coincides with (2.2). Analogously for the case $A_4 = 0$ the reduction of the algebra under consideration to form (2.3) follows from (2.16) and (2.17). ■

Now using the Propositions 3 - 7 easily allows us to get the proof of Theorem 1.

Proof of Theorem 1. Initial form (2.4) of basis matrixes of extended algebra is simplified firstly to the condition (2.6) with corrected values of parameters $c_1 - c_4$ (Propositions

3 and 4). Then, after using the Propositions 5 and 6, we reduce a basis (2.6) to the form (2.10) that contains only one-diagonal basis matrixes. Finally the transformations connected with the Proposition 7 turn the algebra under consideration into one of two kinds: either (2.2) or (2.3). ■

Remark 1. Second formal case in (2.14) is impossible in the above-mentioned SPC-hypersurfaces situation of common position owing to formulas (1.5). We emphasize that investigations of our article have more wide and summarized essence in comparison with already discussed articles.

Remark 2. Below when we will get the affinely homogeneous hypersurfaces from constructed algebras we will use not triangular but slightly changed form of the matrixes. In the first place it is connected with a desire to have a "standard" field $\partial/\partial w$ in algebra. The presence of such field means the rigidity (independence of variable u) of homogeneous surface defining function. The property of rigidity is traditionally considered (see. [18]-[20]) to be interesting in discussed questions.

In our case the field E_5 takes the required form owing to simple change of variables $z_1 \leftrightarrow w$. In that case the bases of algebras (2.2) and (2.3) transform to

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & A \\ 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & (iA + s) \\ i & 0 & 0 & 0 \\ 0 & (iA - s) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.19)$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = m + in, \quad m, n, s \in \mathbb{R},$$

and

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & (iA + s) \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.20)$$

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = m + in, \quad m, n, s \in \mathbb{R}.$$

Below we will operate only with such bases.

§3. Integration of matrix Lie algebras

The main purpose of this paragraph is the construction of affinely homogeneous real hypersurfaces relating to algebras of families (2.2) and (2.3). We use the Frobenius theorem as a basic idea of such construction.

3.1 Frobenius theorem and rank of the algebra

Let 5-dimensional algebra of vector fields in space \mathbb{C}^3 has a rank 5 in a fixed point of this space. Then according to Frobenius theorem there is a unique integral manifold of discussed algebra of the same dimension 5 passing through the discussed point. We will integrate our algebras in such "right" points of space \mathbb{C}^3 (with respect to each concrete algebra) and get the required homogeneous surfaces as their integral manifolds.

We don't deal with the points of space \mathbb{C}^3 in which the rank of discussed algebra is not maximal one. Thereby it is necessary to consider the system of five transition vectors relating to the basis matrixes of algebra and analyze the rank of this system for any discussed algebra (2.2) or (2.3) at each point of \mathbb{C}^3 .

For example, we should examine the real rank of the system of shift components of complex vector fields for algebra of family (2.20) at any point $Q(z_1, z_2, w) \in \mathbb{C}^3$

$$\begin{pmatrix} z_1 & (m + in) & (s - n) + im & 0 & 0 \\ 2z_2 & z_1 & iz_1 & 1 & 0 \\ 3w & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.1)$$

Going from complex notations to real ones, we can discuss 6×5 -matrix

$$\begin{pmatrix} x_1 & m & (s - n) & 0 & 0 \\ y_1 & n & m & 0 & 0 \\ 2x_2 & x_1 & -y_1 & 1 & 0 \\ 2y_2 & y_1 & x_1 & 0 & 0 \\ 3u & 0 & 0 & 0 & 1 \\ 3v & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.2)$$

In order to have a maximal rank in the case of matrix (3.2) (according to standard definition of rank) it suffices that at least one of its six minors of 5-th order be different from zero. These minors can be easily calculated:

$$\begin{aligned} D_1 &= 3v(-nx_1 + my_1), & D_2 &= 3v(-mx_1 + (s - n)y_1), \\ D_3 &= 0, & D_4 &= 3v(m^2 + n^2 - ns), \\ D_5 &= 0, & D_6 &= (n - s)y_1^2 - nx_1^2 + 2mx_1y_1 - 2(m^2 + n^2 - ns)y_2. \end{aligned} \quad (3.3)$$

For the family (2.19) it is necessary to calculate six defining minors of analogous matrix

$$\begin{pmatrix} x_1 & m & (s - n) & 0 & 0 \\ y_1 & n & m & 0 & 0 \\ 2x_2 & x_1 & -y_1 & 1 & 0 \\ 2y_2 & y_1 & x_1 & 0 & 0 \\ 3u & (mx_2 - ny_2) & -(s + n)x_2 - my_2 & x_1 & 1 \\ 3v & (nx_2 + my_2) & mx_2 - (s + n)y_2 & y_1 & 0 \end{pmatrix}. \quad (3.4)$$

Minor

$$D_6 = (n - s)y_1^2 - nx_1^2 + 2mx_1y_1 - 2(m^2 + n^2 - sn)y_2. \quad (3.5)$$

having the same form for both matrixes (3.2) and (3.4), is most important for us because of the following reason. Many operations with discussed fields mean the representation of required homogeneous surfaces by equation

$$v = F(z, \bar{z}, u),$$

that is solved according the v -coordinate (with unknown analytic function F).

Proposition 8. *If minor D_6 of matrix (3.2) (or (3.4)) is not equal to zero at some point $Q(z_1, z_2, w)$ of space \mathbb{C}^3 then required hypersurface allows the representation in a form $v = F(z, \bar{z}, u)$ at this point.*

Proof. Really, the inequality $D_6 \neq 0$ means that tangent vectors to surface under study (at discussed point) cover all the directions in 5-dimensional space, which is formed by coordinates x_1, y_1, x_2, y_2, u . And v -variable is expressed in terms of other 5 vector coordinates in this tangent space. Thereby we can express variable v in terms of other variables not only in tangent plane to surface, but on the surface itself too. ■

Proposition 8 suggests a consideration of the sets

$$\Gamma_6^{(m,n,s)} = \{(n - s)y_1^2 - nx_1^2 + 2mx_1y_1 - 2(m^2 + n^2 - sn)y_2 = 0\}, \quad (3.6)$$

that are defined by vanishing of minor D_6 under fixed values of parameters (m, n, s) .

For the non-trivial triples of parameters (m, n, s) all such sets can be divided into two types depending on the value of expression

$$N = (m^2 + n^2 - ns). \quad (3.7)$$

Proposition 9. *Let $(m, n, s) \neq (0, 0, 0)$ be a nontrivial triple of parameters. If combination (3.7) of these parameters is equal to zero, then the set $\Gamma_6^{(m,n,s)}$ is a real hyperplane of \mathbb{C}^3 .*

If $N = (m^2 + n^2 - ns) \neq 0$, then the set

$$\Gamma_6^{(m,n,s)} = \mathbb{C}_w \times \gamma_6^{(m,n,s)} \quad (3.8)$$

is a direct product of the plane \mathbb{C}_w onto paraboloid $\gamma_6^{(m,n,s)}$ with the same equation (3.6).

Proof. The validity of the first part of Proposition 9 follows from the fact that expression of $N = (m^2 + n^2 - ns)$ is (up to nonzero factor) the coefficient in front of the variable y_2 in equation (3.6) and simultaneously it is the determinant of a quadratic form in the variables x_1, y_1 from the same equation. Thus, the second-order equation (3.6) describes a twice-covered real hyperplane in this case.

The second part of proposition 9 is obvious. ■

First of all we are interested in ranks of algebras (2.19) and (2.20) at the points of surfaces $\Gamma_6^{(m,n,s)}$ corresponding to these algebras. The main interest is related here to the points where the rank is 5.

According to the Frobenius theorem, homogeneous hypersurfaces pass through these points, which are the integral manifolds of the discussed algebras. The purpose of this section, we recall, is a description of all such hypersurfaces.

Note that the ranks of algebras under consideration can differ from 5 at the points of surfaces $\Gamma_6^{(m,n,s)}$. For example, in the case of non-zero parameters m, n, s , satisfying the equality

$$N = (m^2 + n^2 - ns) = 0,$$

all six minors (3.3) vanish at all points of the hypersurface $\Gamma_6^{(m,n,s)}$.

At the same time the following statement is true.

Proposition 10. *Let $(m, n, s) \neq (0, 0, 0)$ be a non-trivial set of parameters, $g = g^{(m,n,s)}$ be the Lie algebra from the family (2.19) or (2.20) and $\Gamma_6 = \Gamma_6^{(m,n,s)}$ be the surface (3.6). Let, moreover, algebra g has a full rank at some point Q in Γ_6 . Then the affinely homogeneous integral hypersurface of an algebra g passing through the point Q , is the surface Γ_6 itself.*

Proof. The proof of this assertion follows from two observations.

Note at first, that affine homogeneity is a well-known property of real hyperplanes as well as of paraboloids of the form

$$Imz_n = \sum_{k=1}^{n-1} (\alpha_k(Rez_k)^2 + \beta_k(Imz_k)^2), \quad \alpha_k, \beta_k \in \mathbb{R}$$

in the space \mathbb{C}^n of any dimension. Then any surface $\Gamma_6^{(m,n,s)}$ of the space \mathbb{C}^3 is affinely homogeneous.

Secondly, we note that all the vector fields of each algebra (2.19) or (2.20) are tangent to the corresponding surface $\Gamma_6^{(m,n,s)}$. This fact is easily verified for all basic vector fields of discussed algebras.

Then at the points of full rank the surface $\Gamma_6^{(m,n,s)}$ is an integral manifold of any algebra $g^{(m,n,s)}$. Proposition 10 is proved. ■

Therefore, besides the main case, connected with conditions

$$(m, n, s) \neq (0, 0, 0), \quad D_6 \neq 0, \tag{3.9}$$

it is necessary to consider only algebras relating to trivial set of parameters

$$m = 0, \quad n = 0, \quad s = 0. \tag{3.10}$$

Proposition 11. *If the condition (3.10) is satisfied then rank of algebra (2.20) is incomplete at all the points of space \mathbb{C}^3 ;*

rank of algebra (2.19) is maximal one at all the points satisfying the inequality $y_1 \neq 0$.

Proof. The proof of proposition 11 is purely calculating. All six minors of 5-th order for algebra (2.20) are equal to zero in this case. In case of algebra (2.19) we have formulas

$$D_1 = -y_1^2(x_1^2 + y_1^2), \quad D_2 = -x_1y_1(x_1^2 + y_1^2), \quad D_3 = D_4 = D_5 = D_6 = 0,$$

proving second part of the statement. ■

At the end of this chapter we discuss the integration of algebra (2.19) at points of half-spaces $y_1 < 0$ and $y_1 > 0$ under the condition (3.10).

Proposition 12. *Let point $Q(\mu, \nu, \eta) \in \mathbb{C}^3$ satisfies the condition $Im \mu \neq 0$ and let M be an integral surface of algebra (2.19) with trivial set of parameters (m, n, s) , passing through Q . Then M is affinely equivalent to the real hypersurface $v = 0$.*

Proof. For the proof we consider the basis fields

$$E_1 = z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + 3w \frac{\partial}{\partial w}, \quad (3.14)$$

$$E_2 = z_1 \frac{\partial}{\partial z_2}, \quad E_3 = iE_2, \quad E_4 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial w}, \quad E_5 = \frac{\partial}{\partial w}$$

of discussed algebra.

Tangent plane to integral hypersurface at a given point, generated by vectors (3.11), can be defined by the equation

$$Im(\bar{\mu}z_1) = 0.$$

Generally speaking, such position of the surface does not allow us to solve the equation of the required integral surface according to the variable v . That's why we change the coordinates in space \mathbb{C}^3 by the formulas

$$z_1^* = w, \quad z_2^* = z_2, \quad w^* = -iz_1.$$

In this case

$$z_1 = iw^*, \quad w = z_1^*, \quad \frac{\partial}{\partial z_1} = -i \frac{\partial}{\partial w^*}, \quad \frac{\partial}{\partial w} = \frac{\partial}{\partial z_1^*},$$

and the differentiation with respect to coordinate z_2 remains unchangeable.

Then basis fields of our algebra have the form

$$E_1 = 3z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + w \frac{\partial}{\partial w},$$

$$E_2 = iw \frac{\partial}{\partial z_2}, \quad E_3 = -w \frac{\partial}{\partial z_2}, \quad E_4 = iw \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}, \quad E_5 = \frac{\partial}{\partial z_1}.$$

in new coordinates (we omit asterisks for the simplicity).

Now the minor $D_6 = -u^2(u^2 + v^2)$. of matrix (3.4) won't be equal to zero.

It allows us to give the required surface by equation $\Phi = -v + F(z, \bar{z}, u) = 0$ in desired form. The condition (1.3) for the fields E_2 and E_3 tangent to the surface means that F doesn't depend on variable z_2 .

The existence of field $E_5 = \partial/\partial z_1$ in algebra under consideration leads to independence the function F on another variable $x_1 = Re z_1$. Then analogous tangency condition (1.3) for the field E_4 can be written in the form

$$Re \left(-(u + iF) \frac{\partial F}{\partial y_1} \right) = -u \frac{\partial F}{\partial y_1} = 0. \quad (3.12)$$

This equation means the independence of F on the variable y_1 therefore condition (1.3) for the field E_1 get the form

$$u \frac{\partial F}{\partial u} - F = 0.$$

Here the formula for the common solution is $F = Cu$ where C is arbitrary constant. It means that required integral manifolds are hyperplanes

$$v = Cu \tag{3.13}$$

with arbitrary values of constant C . The meaning of this parameter at a fixed point $Q \in \mathbb{C}^3$ is defined by real and imaginary parts of third coordinate of the point Q . It's clear that for any C the surface (3.13) is affinely equivalent to a real hyperplane $v = 0$. Proposition 12 is proved. ■

3.2. Integration of the family (2.19)

* The results of this section connected with computer integration of the system of partial differential equations are obtained with active participation of Sergeev V.G.

The main result of this section can be formulated in common form as follows.

THEOREM 2. *Each affinely homogeneous hypersurface which is an integral manifold of algebra of the family (2.19), becomes an algebraic surface of order $k \in \{1, 2, 3, 4, 6\}$.*

It will be obtained as a collection of partial results (Propositions 13 - 17) connected with the integration of the family (2.19) in some particular cases.

The well-known method of integration for matrix Lie algebras is connected with the use of exponential maps [22]. It means the transition from matrix algebras to the corresponding Lie groups.

One can also consistently solve a system of partial differential equations corresponding to the basis fields of algebra. Any individual partial differential equation reduces under this approach to the ODE system [23].

Each of two mentioned approaches to solving the problem of integrating has technically complex realization. For this reason, below we describe in detail the procedure of integration.

In notations of §1 we can write the basis vector fields of any algebra (2.19) in the following form:

$$\begin{aligned} E_1 &= z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + 3w \frac{\partial}{\partial w}, & E_2 &= A \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2} + Az_2 \frac{\partial}{\partial w}, \\ E_3 &= (iA + s) \frac{\partial}{\partial z_1} + iz_1 \frac{\partial}{\partial z_2} + (iA - s) \frac{\partial}{\partial w}, & E_4 &= \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial w}, & E_5 &= \frac{\partial}{\partial w}. \end{aligned} \tag{3.14}$$

As we noted in previous section it is sufficient to integrate algebras (2.19) with

$$(m, n, s) \neq (0, 0, 0)$$

beyond the points of hypersurfaces $\Gamma_6^{(m,n,s)}$. Under this condition defining function of any desired surface is guaranteed to have a "convenient" form $v = F(z, \bar{z}, u)$ at all such points, and even $v = F(z, \bar{z})$ (by the presence of the field E_5 in algebra).

In this case the system of four equations corresponding to family of algebras (2.19), has in real coordinates a form

$$\begin{aligned}
x_1 \frac{\partial F}{\partial x_1} + y_1 \frac{\partial F}{\partial y_1} + 2x_2 \frac{\partial F}{\partial x_2} + 2y_2 \frac{\partial F}{\partial y_2} &= 3F, \\
m \frac{\partial F}{\partial x_1} + n \frac{\partial F}{\partial y_1} + x_1 \frac{\partial F}{\partial x_2} + y_1 \frac{\partial F}{\partial y_2} &= (nx_2 + my_2), \\
(s-n) \frac{\partial F}{\partial x_1} + m \frac{\partial F}{\partial y_1} - y_1 \frac{\partial F}{\partial x_2} + x_1 \frac{\partial F}{\partial y_2} &= mx_1 - (n+s)y_2, \\
\frac{\partial F}{\partial x_2} &= y_1.
\end{aligned} \tag{3.15}$$

We use the step-by-step solving of individual equations of (3.15) and start from the most simple 4-th equation of the system. It's clear that

$$F = F(x_1, y_1, x_2, y_2) = x_2 y_1 + G(x_1, y_1, y_2), \tag{3.16}$$

where $G(x_1, y_1, y_2)$ is unknown function of three variables. Substitution (3.16) into other equations of the system (3.15) gives the new system of three equations

$$\begin{aligned}
x_1 \frac{\partial G}{\partial x_1} + y_1 \frac{\partial G}{\partial y_1} + 2y_2 \frac{\partial G}{\partial y_2} &= 3G, \\
m \frac{\partial G}{\partial x_1} + n \frac{\partial G}{\partial y_1} + y_1 \frac{\partial G}{\partial y_2} &= my_2 - x_1 y_1, \\
(s-n) \frac{\partial G}{\partial x_1} + m \frac{\partial G}{\partial y_1} - x_1 \frac{\partial G}{\partial y_2} &= y_1^2 - (n+s)y_2.
\end{aligned} \tag{3.17}$$

It is the system (3.17) that will be discussed in this section below.

Algorithm of its solving certainly depends on being zero of the coefficients in front of derivatives of the desired function, i.e. on the parameters $m, n, (n-s)$. Let us indicate once more important combination $N = (m^2 + n^2 - ns)$ of written parameters, which has already met us above and will be needed below. We have to investigate a lot of cases, connected with four named coefficients. It should be emphasized that not every combinations of these coefficients influence considerably on the solution of the system (3.17).

The following cases are interesting in a whole:

- 1) $m = 0, n = 0, s = 0,$
- 2) $m = 0, n = 0, s \neq 0,$
- 3) $n \neq 0, N = 0,$
- 4) $n \neq 0, N \neq 0,$
- 5) $m \neq 0, n = 0.$

We remind that the first of these cases has been already discussed in previous section in connection with the both families (2.19) and (2.20). We write the equations of the surfaces $\Gamma_6^{(m,n,s)}$ for each item from the list 2) - 5) in convenient form before we begin to discuss the rest of combinations (m, n, s) .

Proposition 13. *The equations of the surfaces $\Gamma_6^{(m,n,s)}$ in the cases 2) - 5) have the form:*

$$2) y_1^2 = 0,$$

$$3) (nx_1 - my_1)^2 = 0,$$

$$4) (nx_1 - my_1)^2 - (m^2 + n^2 - ns)(y_1^2 - 2ny_2) = 0,$$

$$5) 2m(x_1y_1 - my_2) - sy_1^2 = 0.$$

Proof. The validity of this statement in the cases 2) and 5) is obvious. In the case 3) it is necessary to remove the summand with zero-coefficient $N = m^2 + n^2 - ns$ from the general formula for $\Gamma_6^{(m,n,s)}$ and substitute the expression $s = (m^2 + n^2)/n$ instead of s in this formula.

Finally, in general case 4) the left part of the suggested expression can be written (after removing the first bracket) in the form:

$$\begin{aligned} n^2x_1^2 - 2mnx_1y_1 + m^2y_1^2 - (m^2 + n^2 - ns)y_1^2 + 2n(m^2 + n^2 - ns)y_2 = \\ = n(nx_1^2 - 2mx_1y_1 - (n - s)y_1^2 + 2(m^2 + n^2 - ns)y_2). \end{aligned}$$

This expression coincides with the formula for the minor D_6 up to the factor $(-n)$. Proposition 13 is proved. ■

Now we will integrate the family of algebras (2.19) in rather simple case 2).

Proposition 14. *If $m = 0$, $n = 0$, $s \neq 0$ then all the solutions of the system (3.15) present an algebraic surfaces of 3 degree with equations*

$$v = x_2y_1 - x_1y_2 + \frac{1}{3s}x_1^3 + \frac{1}{s}x_1y_1^2 + Cy_1^3, \quad (3.18)$$

where C is arbitrary constant.

Remark. In this case the surfaces $\Gamma_6^{(0,0,s)}$ have the equation $sy_1^2 = 0$. According to the agreements defined above, we can consider the equations (3.18) in two half-spaces $y_1 < 0$ and $y_1 > 0$.

Proof. In order to prove proposition 14 we notice that in this case the system (3.17) has a very simple form

$$\begin{aligned} x_1 \frac{\partial G}{\partial x_1} + y_1 \frac{\partial G}{\partial y_1} + 2y_2 \frac{\partial G}{\partial y_2} &= 3G, \\ y_1 \frac{\partial G}{\partial y_2} &= -x_1y_1, \\ s \frac{\partial G}{\partial x_1} - x_1 \frac{\partial G}{\partial y_2} &= y_1^2 - sy_2. \end{aligned} \quad (3.19)$$

As $y_1 \neq 0$, then the second equation of the system (3.19) can be divided by y_1 . In the next step it is necessary to solve this equation.

Remark. After the solving of each such equation as well as in the previous integration a new unknown analytic function of reduced number of variables appears. We will always designate such functions by G, H, φ without additional comments. It should be noted that the last function in this collection depends on unique real variable. This function will be defined from ODE, that appears in the last step of the solving each system.

Returning to the solution of the second equation of the system (3.19)

$$G = -x_1y_2 + H(x_1, y_1),$$

we substitute it into the two remaining equations of the system. So we get a new system

$$x_1 \frac{\partial H}{\partial x_1} + y_1 \frac{\partial H}{\partial y_1} = 3H, \quad s \frac{\partial H}{\partial x_1} = (x_1^2 + y_1^2). \quad (3.20)$$

We substitute the solution of its second equation

$$H = \frac{1}{s} \left(\frac{1}{3}x_1^3 + x_1y_1^2 \right) + \varphi(y_1) \quad (3.21)$$

into the first one. It reduces to ODE

$$y_1 \varphi'(y_1) = 3\varphi(y_1).$$

Combining the solution $\varphi(y_1) = Cy_1^3$ of this equation with the solutions of all the equations, which have been got in previous steps, we get the formula (3.18). ■

Remark. The family of affinely homogeneous surfaces (3.18) depends formally on two real parameters s, C . At the same time the coherent dilation of variables

$$z_1 \rightarrow (3s)^{1/3}z_1, \quad z_2 \rightarrow (3s)^{-1/3}z_2$$

transforms (3.18) into equation

$$v = x_2y_1 - x_1y_2 + x_1^3 + 3x_1y_1^2 + (3sC)y_1^3,$$

depending on the one generalized parameter $3sC$.

Going to the discussion of more difficult cases, let us suppose firstly that

$$n \neq 0.$$

Then general solution of the second equation of the system (3.17) will be described by the formula

$$G(x_1, y_1, y_2) = \frac{m}{n}y_1y_2 - \frac{1}{2n}x_1y_1^2 - \frac{m}{6n^2}y_1^3 + H \left(\frac{y_1^2}{2} - ny_2, nx_1 - my_1 \right) \quad (3.22)$$

independently of vanishing or nonvanishing parameter m .

It should be noted that formula (3.22) is obtained with the help of the finding of the integrals of ODEs system

$$\frac{dx_1}{m} = \frac{dy_1}{n} = \frac{dy_2}{y_1} = \frac{dG}{my_2 - x_1y_1}, \quad (3.23)$$

corresponding to the discussed partial differential equation.

Substitution of this solution into two remaining equations of the system (3.17), we get a new system

$$\begin{aligned} 2t_1 \frac{\partial H}{\partial t_1} + t_2 \frac{\partial H}{\partial t_2} &= 0, \\ t_2 \frac{\partial H}{\partial t_1} + (m^2 + n^2 - ns) \frac{\partial H}{\partial t_2} + \frac{(m^2 + n^2 + ns)}{n^2} t_1 &= 0, \end{aligned} \quad (3.24)$$

where

$$t_1 = \frac{y_1^2}{2} - ny_2, \quad t_2 = nx_1 - my_1.$$

In this system the combination of parameters $N = (m^2 + n^2 - ns)$ plays an important role. At first let

$$(m^2 + n^2 - ns) = 0, \quad (3.25)$$

in accordance with the case 3.

Proposition 15. *If $n \neq 0$, $(m^2 + n^2 - ns) = 0$ then all the solutions of the system (3.15) present an algebraic surfaces of 4-th order with equations*

$$v = x_2y_1 + \left(\frac{m}{n}y_1y_2 - \frac{1}{2n}x_1y_1^2 - \frac{m}{6n^2}y_1^3 \right) - \frac{s(y_1^2 - 2ny_2)^2}{4n(nx_1 - my_1)} + C(nx_1 - my_1)^3, \quad (3.26)$$

where C is arbitrary constant.

Proof. In order to prove this statement we solve the second equation of the system (3.24) under assumption (3.25):

$$H(t_1, t_2) = -\frac{st_1^2}{nt_2} + \varphi(t_2). \quad (3.27)$$

Function $\varphi(t_2)$ is determined from the first equation of the system.

When we substitute the formula (3.27) into this equation it takes the form

$$t_2\varphi'(t_2) = 3\varphi(t_2).$$

Combination of all the intermediate formulas reduces to the proof of the proposition 15. ■

Now let

$$n \neq 0, \quad N = (m^2 + n^2 - ns) \neq 0.$$

Proposition 16. *If $n \neq 0$, $N = (m^2 + n^2 - ns) \neq 0$ then all the solutions of the system (3.15) present an algebraic surfaces of 3-rd or 6-th order with equations*

$$v = x_2y_1 + \left(\frac{m}{n}y_1y_2 - \frac{1}{2n}x_1y_1^2 - \frac{m}{6n}y_1^3 \right) + \\ + \frac{(N + 2ns)(nx_1 - my_1)}{6n^2N^2} (2(nx_1 - my_1)^2 - 3N(y_1^2 - 2ny_2)) + \\ + C |(nx_1 - my_1)^2 - N(y_1^2 - 2ny_2)|^{3/2}, \quad (3.28)$$

where C is arbitrary constant.

Proof. Let introduce in this case the designation

$$\xi = \frac{t_2^2}{2} - Nt_1.$$

Then the solution of the second equation of the system (3.24) can be written in the form

$$H = \frac{N + 2sn}{N^2n^2} \left(\frac{t_2^3}{3} - Nt_1t_2 \right) + \varphi(\xi). \quad (3.29)$$

Then ODE, which we get on the last step of the solving of the system (3.24), has the form

$$2\xi\varphi' = 3\varphi. \quad (3.30)$$

So the formula (3.28) appears with fractional index of degree. After squaring the equation this formula transforms to algebraic equation of 6-th order. ■

Now we should analyse the last announced case, connected with conditions

$$n = 0, \quad m \neq 0.$$

We note that here $N = (m^2 + n^2 - ns) \neq 0$.

Proposition 17. *If $n = 0$, $m \neq 0$, then all the solutions of the system (3.15) present an algebraic surfaces of 3-rd or 6-th order with the equations*

$$v = x_2y_1 + x_1y_2 - \frac{x_1^2y_1}{m} + \frac{(m^2 - 2s^2)y_1^3}{3m^3} + \frac{2sy_1(x_1y_1 - my_2)}{m^2} + C |sy_1^2 - 2m(x_1y_1 - my_2)|^{3/2}, \quad (3.31)$$

where C is arbitrary constant.

Proof. In order to prove this statement we note, that here the system (3.17) can be simplified to the form

$$x_1 \frac{\partial G}{\partial x_1} + y_1 \frac{\partial G}{\partial y_1} + 2y_2 \frac{\partial G}{\partial y_2} = 3G, \\ m \frac{\partial G}{\partial x_1} + y_1 \frac{\partial G}{\partial y_2} = my_2 - x_1y_1, \\ s \frac{\partial G}{\partial x_1} + m \frac{\partial G}{\partial y_1} - x_1 \frac{\partial G}{\partial y_2} = y_1^2 - sy_2. \quad (3.32)$$

The second equation has the solution

$$G = x_1 y_2 - \frac{1}{m} y_1 x_1^2 + H(y_1, x_1 y_1 - m y_2).$$

Appeared system of two equations can be written in the form ($t_1 = y_1$, $t_2 = x_1 y_1 - m y_2$):

$$t_1 \frac{\partial H}{\partial t_1} + 2t_2 \frac{\partial H}{\partial t_2} = 3H, \quad m \frac{\partial H}{\partial t_1} + st_1 \frac{\partial H}{\partial t_2} = t_1^2 + \frac{2st_2}{m}.$$

We substitute the solution of the second equation

$$H = \frac{m^2 - 2s^2}{3m^3} t_1^3 + \frac{2s}{m^2} t_1 t_2 + \varphi(st_1^2 - 2mt_2)$$

into the first one, noting here $\xi = st_1^2 - 2mt_2$. So we get ODE (3.30) again. It reduces to the formula (3.31), which is more simple than analogous one of proposition 16, but in a whole is similar to it. ■

Summarize this section by taking into account information previously obtained we get the main result of this section i.e. the Theorem 2.

3.3. Integration of the family (2.20)

In this case in order to get a list of homogeneous manifolds we can also start with hypersurfaces Γ_6 , given by equations (3.6). But the main part of the description is given in the next theorem.

THEOREM 3. *Affinely homogeneous surface of space \mathbb{C}^3 , that differs from (3.6) and be an integral manifold of some algebra of the family (2.20) with $(m, n, s) \neq (0, 0, 0)$ is affinely equivalent near any its point either to real hyperplane*

$$v = 0, \tag{3.33}$$

or to surface

$$v = x_1^3 (x_1 \neq 0), \tag{3.34}$$

or to one of the surfaces

$$v = |y_2 + (x_1^2 + \gamma y_1^2)|^{3/2} \tag{3.35}$$

with some $\gamma \in \mathbb{R}$.

Remark 1. The equation (3.35) presents a locally homogeneous surface near any its point whose coordinates are satisfied the restriction

$$y_2 + (x_1^2 + \gamma y_1^2) \neq 0.$$

Proof of Theorem 3. Proof of the theorem 3 can be obtained by the scheme that was used in the proof of the theorem 2. Below we consider the case which is formally more simple than previous one. But in this case there is also a lot of rather delicate points. For this reason, we reduce the detailed proof of theorem 3.

To begin we show that besides the manifolds with simplest equations (3.33) and (3.34) there are homogeneous surfaces with equations

$$v = |y_2 + (\alpha x_1^2 + \beta x_1 y_1 + \gamma y_1^2)|^{3/2}, \quad \alpha, \beta, \gamma \in \mathbb{R}. \quad (3.36)$$

in the case under consideration.

Then we reduce any such equation to more simple form (3.35) by affine transformation.

Turning immediately to the proof we note that any of discussed algebras contains the fields

$$E_4 = \frac{\partial}{\partial z_2}, \quad E_5 = \frac{\partial}{\partial w}.$$

It means that the integral manifold of such algebra, passing through the point outside the surface Γ_6 , can be given by equation

$$v = F(x_1, y_1, y_2),$$

that is free from the variables x_2, u .

Then the system of equations, relating to other basis fields of algebra (2.20), has a form

$$\begin{aligned} x_1 \frac{\partial F}{\partial x_1} + y_1 \frac{\partial F}{\partial y_1} + 2y_2 \frac{\partial F}{\partial y_2} - 3F &= 0, \\ m \frac{\partial F}{\partial x_1} + n \frac{\partial F}{\partial y_1} + y_1 \frac{\partial F}{\partial y_2} &= 0, \\ (s - n) \frac{\partial F}{\partial x_1} + m \frac{\partial F}{\partial y_1} + x_1 \frac{\partial F}{\partial y_2} &= 0. \end{aligned} \quad (3.37)$$

We consider the solution of this system under supplementary restriction

$$x_1 \neq 0, \quad (3.38)$$

which isn't formally following from the equations of the system .

So, when x_1 is not equal to zero, the general solution of the first equation of the system (3.37) has a form

$$F = x_1^3 G(t_1, t_2), \quad t_1 = \frac{y_1}{x_1}, \quad t_2 = \frac{y_2}{x_1^2} \quad (3.39)$$

with arbitrary analytic function $G(t_1, t_2)$.

Taking into account this formula we can transform two remaining equations of the system (3.37) to

$$\begin{aligned} (n - mt_1) \frac{\partial G}{\partial t_1} + (t_1 - 2mt_2) \frac{\partial G}{\partial t_2} &= -3mG, \\ (m - (s - n)t_1) \frac{\partial G}{\partial t_1} + (1 - 2(s - n)t_2) \frac{\partial G}{\partial t_2} &= -3(s - n)G. \end{aligned} \quad (3.40)$$

Let us consider the coefficients in front of derivative $\partial G/\partial t_1$ in these equations. Recall that here we don't discuss the situation when both linear functions $n - mt_1$ and $m - (s - n)t_1$ are equal to zero, i.e. $m = n = s = 0$.

If the parameter collection (m, n, s) is non-trivial then two cases are interesting that correspond to the proportional or non-proportional functions $n - mt_1$ and $m - (s - n)t_1$.

At first we regard the case with the determinant

$$\begin{vmatrix} n & -m \\ m & -(s - n) \end{vmatrix} = N = m^2 + n^2 - ns = 0. \quad (3.41)$$

For definiteness, we will assume that the function $n - mt_1$ of the first equation (3.40) differs from zero; the case of non-vanishing function $m - (s - n)t_1$ can be considered analogously.

Then for some real λ two equalities

$$m = \lambda n, \quad (s - n) = \lambda m.$$

hold.

Because of these equalities the combination of the equations of the system (3.40) with coefficients $-\lambda$ and 1 respectively transforms to the equation

$$(1 - \lambda t_1) \frac{\partial G}{\partial t_2} = 0.$$

It means that under the condition $1 - \lambda t_1 \neq 0$ the solution $G(t_1, t_2)$ of the system (3.40) is in fact a function of unique variable t_1 . Then the first equation (3.40) has a form

$$(n - mt_1) \frac{\partial G}{\partial t_1} = -3mG.$$

Its general solution is a function $G(t_1) = C(n - mt_1)^3$ with arbitrary constant C . Returning to the systems (3.40) and (3.37), in this case we obtain the formulas

$$v = 0 \text{ (if } C = 0 \text{ or if } m = n = 0 \text{) or}$$

$$\frac{v}{C} = (nx_1 - my_1)^3. \quad (3.42)$$

Up to affine equivalence formula (3.42) changes to

$$v = x_1^3. \quad (3.43)$$

The last formula and equation of the plane (3.33) describe (up to affine equivalence) all the integral manifolds of algebras (2.20), that are satisfied to supplementary conditions (3.41) and (3.38). It should be noted that the local affine homogeneity of plane curve

$$y = x^3$$

hold at all the points except the origin of the coordinates, which is a twist point of the curve. For the same reason the obtained surface (3.43) can be locally affinely homogeneous according to the (complex) affine transformations only at points which satisfy the condition (3.38). Therefore, the using of such restriction is natural despite the right hand side function

$F = (nx_1 - my_1)^3$ of the equation (3.42) satisfies all the equations of the system (3.37) at all their points.

In the second case when

$$\begin{vmatrix} n & -m \\ m & -(s-n) \end{vmatrix} = N = m^2 + n^2 - ns \neq 0, \quad (3.44)$$

we introduce the notation

$$r = (s - n).$$

Then the second equation of the system (3.40) takes the form

$$(m - rt_1) \frac{\partial G}{\partial t_1} + (1 - 2rt_1) \frac{\partial G}{\partial t_2} = -3rG. \quad (3.45)$$

In the first subcase when $r = 0$ the general solution of this equation has a form

$$G = H(t_1 - mt_2)$$

with arbitrary analytic function $H(\xi)$ of the argument $\xi = t_1 - mt_2$.

Meanwhile the first equation of the system (3.40) transforms to ODE

$$(n - 2m\xi)H' = -3mH. \quad (3.46)$$

We note that for $r = 0$ the inequality $m \neq 0$ follows from the condition (3.44). Then the general solution of the equation (3.45) which can be rewritten in the form

$$\left(\xi - \frac{n}{2m}\right) H' = \frac{3}{2}H,$$

is a function

$$H = C \left| \xi - \frac{n}{2m} \right|^{3/2}$$

with arbitrary constant C .

Returning to the original variables and functions we obtain the equations of discussed homogeneous surfaces in this subcase in the form

$$v = 0 \text{ (if } C = 0) \quad \text{or} \quad \frac{v}{C} = \left| y_1 x_1 - m y_2 - \frac{n}{2m} x_1^2 \right|^{3/2}. \quad (3.47)$$

Affine images of these surfaces contain in formulas (3.33) and (3.36).

Now we consider the equation (3.45) in the second subcase, supposing $r \neq 0$. Under this condition the equation (3.45) can be rewritten in the form

$$(t_1 - \alpha) \frac{\partial G}{\partial t_1} + (2t_2 - \beta) \frac{\partial G}{\partial t_2} = 3G, \quad \alpha = \frac{m}{r}, \quad \beta = \frac{1}{2r}. \quad (3.48)$$

For $t_1 \neq \alpha$ the solution of the equation (3.48) can be expressed as follows

$$G = (t_1 - \alpha)^3 H(\tau), \quad \tau = \frac{t_2 - \beta}{(t_1 - \alpha)^2}. \quad (3.49)$$

Instead of the first equation of the system (3.40) it is convenient to consider the linear combination of two equations of this system with coefficients r and $-m$ respectively. The proposed combination has a form

$$(rn - m^2) \frac{\partial G}{\partial t_1} + (rt_1 - m) \frac{\partial G}{\partial t_2} = 0. \quad (3.50)$$

In this case the coefficient in front of the derivative $\partial G/\partial t_1$ in this equation is equal to

$$(s - n)n - m^2 = -(m^2 + n^2 - ns) = -N \neq 0.$$

We substitute the derivatives of the function (3.49) on the variables t_1, t_2 and obtain the equation

$$(2N(t_2 - \beta) + r(t_1 - \alpha)^2) H' - 3N(t_1 - \alpha)^2 H = 0$$

or

$$\left(\tau + \frac{r}{2N}\right) H' = \frac{3}{2} H.$$

When

$$\tau \neq -r/2N \quad (3.51)$$

the general solution of this ODE has the form

$$H = C \left| \tau + \frac{r}{2N} \right|^{3/2}, \quad C \in \mathbb{R}.$$

Then in this subcase the equations of the homogeneous surfaces (under the condition (3.51)) have the form

$$v = 0 \text{ (if } C = 0) \text{ or}$$

$$\frac{v}{C} = x_1^3 \cdot \left(\frac{y_1}{x_1} - \alpha\right)^3 \cdot \left| \left(\frac{y_2}{x_1^2} - \beta\right) + \frac{r}{2N} \left(\frac{y_1}{x_1} - \alpha\right)^2 \right|^{3/2} \cdot \left| \frac{y_1}{x_1} - \alpha \right|^{-3}.$$

The simplified form of the last equation is

$$v = \left| (y_2 - \beta x_1^2) + \frac{r}{2N} (y_1 - \alpha x_1)^2 \right|^{3/2}, \quad y_1 - \alpha x_1 \neq 0. \quad (3.52)$$

Formula (3.52) coincides with (3.36) up to the redescription of the constants.

To conclude the proof of the theorem 3, we note that any surface (3.36) with non-zero combination of coefficients (α, β, γ) is affinely equivalent near its any point to one of the surfaces (3.35).

Really, mixed product from the expression

$$\alpha x_1^2 + \beta x_1 y_1 + \gamma y_1^2$$

can be eliminated by rotation in z_1 -plane.

By the following dilation $z_1 \rightarrow tz_1$ ($t \in \mathbb{R}$) one of two remained non-zero coefficients (we can assume that it be α) can be transformed to ± 1 .

Returning to the equation (3.36) in the space of three complex variables, we can change the sign in front of coordinate y_2 by the mapping $z_2 \rightarrow -z_2$ and obtain the required equation (3.35).

The theorem 3 is completely proved. ■

§4. Holomorphic properties of affinely homogeneous surfaces

In [4] the "gluing" property is illustrated for large families of affinely different affinely homogeneous real surfaces of the space \mathbb{C}^2 under holomorphic transformations. Any manifold from 3-parameter family of affinely different affinely homogeneous Levi degenerate real surfaces in space \mathbb{C}_{z_1, z_2}^2 is shown in [4] to be rectified to the same real hyperplane $Imz_2 = 0$ by (local) holomorphic transformations of this space.

Generally speaking this property has long been known. For example, affinely different tubes in space \mathbb{C}^2 over the curves $y = x^2$ and $y = e^x$ are spherical surfaces (holomorphically equivalent to ordinary sphere) at each point. Consequently these tubes are holomorphically equivalent to each other. There are the similar effects in spaces of larger dimension (see [24], [25]).

Such effect takes place in discussed family too. At the beginning we consider affinely homogeneous real surfaces relating to the family of algebras (2.20).

Proposition 18. *Affinely homogeneous surface corresponding to any algebra (2.20) is holomorphically equivalent to real hyperplane $v = 0$ or to one of 4 algebraic tubes:*

- 1) $v = x_1^3$;
- 2) $v = x_1^{2/3}$ (or up to affine transformation $v = x_1^{3/2}$);
- 3) $v = x_1^2 - x_2^{2/3}$;
- 4) $v = x_1^2 + x_2^{2/3}$.

Remark. First and second surfaces given above are Levi degenerate; third and fourth surfaces are strictly pseudo convex and indefinite nondegenerate ones respectively.

Proof. In order to prove proposition 18 let's consider equation (3.36)

$$v = (y_2 + \alpha x_1^2 + \beta x_1 y_1 + \gamma y_1^2)^{3/2}.$$

We rewrite it in the form

$$v^{2/3} = y_2 + \alpha x_1^2 + \beta x_1 y_1 + \gamma y_1^2,$$

or up to change of variables,

$$v = x_2^{2/3} - (\alpha x_1^2 + \beta x_1 y_1 + \gamma y_1^2).$$

Using of the complex notation in variable $z_1 = x_1 + iy_1$ transforms last equation to

$$v = x_2^{2/3} + \varepsilon |z_1|^2 + (\mu z_1^2 + \bar{\mu} \bar{z}_1^2)$$

with some real ε and complex in general μ .

We notice that quadratic change

$$w^* = w - 2i\mu z_1^2$$

simplifies it to the form

$$v = x_2^{2/3} + \varepsilon|z_1|^2.$$

Depending of the sign of ε we obtain (owing to possible dilation of the variable z_1) equations from items 2), 3 or 4) of the proved proposition 18. ■

It's clear that the family of affinely homogeneous real surfaces relating to algebras (2.19) has more complicated structure. That's why we discuss only holomorphic properties of homogeneous surfaces of 3 and 4 degrees given above as subtotal cases in the integration of the family of algebras (2.19). Note that cubic surfaces have some applications in modern complex analysis investigations (see, for instance, [26]).

Mainly, as well in proposition 18, we will be interesting by holomorphic distinction of surfaces relating to different sets of parameters of family (2.19).

Surfaces of 3-rd order appear in 3 cases in this family. The question is about the formulas (3.31) and (3.28) with zero value of parameter C and general formula (3.18) (with arbitrary value of C). According to these formulas we have the surfaces with equations:

$$v = x_2 y_1 - x_1 y_2 + \frac{1}{3s} x_1^3 + \frac{1}{s} x_1 y_1^2 + C y_1^3, \quad (4.1)$$

$$v = x_2 y_1 + x_1 y_2 - \frac{x_1^2 y_1}{m} + \frac{(m^2 - 2s^2) y_1^3}{3m^3} + \frac{2s y_1 (x_1 y_1 - m y_2)}{m^2} \quad (4.2)$$

$$v = x_2 y_1 + \left(\frac{m}{n} y_1 y_2 - \frac{1}{2n} x_1 y_1^2 - \frac{m}{6n} y_1^3 \right) + \frac{(N + 2ns)(n x_1 - m y_1)}{6n^2 N^2} (2(n x_1 - m y_1)^2 - 3N(y_1^2 - 2n y_2)). \quad (4.3)$$

It should be noted that cubic part of all three equations depends on unique complex variable. In this connection the following simple fact should be used:

Proposition 19. *If a real hypersurface be given in complex space \mathbb{C}^3 by equation*

$$v = z_1 \bar{z}_2 + z_2 \bar{z}_1 + T_3(z_1, \bar{z}_1) \quad (4.4)$$

with arbitrary cubic polynomial T_3 , then it is holomorphically equivalent to (sign-indefinite) quadric

$$v = z_1 \bar{z}_2 + z_2 \bar{z}_1. \quad (4.5)$$

Proof. In fact, the real polynomial $T_3(z_1, \bar{z}_1)$ can be written in the form

$$T_3(z_1, \bar{z}_1) = A_{30} z_1^3 + A_{21} z_1^2 \bar{z}_1 + A_{12} z_1 \bar{z}_1^2 + A_{03} \bar{z}_1^3$$

with pairwise conjugate complex coefficients $A_{30} = \bar{A}_{03}$, $A_{21} = \bar{A}_{12}$.

Then cubic change of variables

$$z_1^* = z_1, \quad z_2^* = z_2, \quad w^* = w - 2iA_{30}z_1^3 \quad (4.6)$$

simplifies equation (4.4), deleting the terms of 3 degree $A_{30}z_1^3 + A_{03}\bar{z}_1^3$.

And the next quadratic coordinate change

$$z_1^* = z_1, \quad z_2^* = z_2 + A_{21}z_1^2, \quad w^* = w$$

transforms already simplified equation to (4.5). ■

It remains to note that after simple quadratic change (4.6) (and by multiplication the variable z_2 by i in the case (4.2)) each of the equations (4.1) and (4.2) reduces to the form (4.4).

There is a more subtle situation with equation (4.3). Going to the complex coordinates we consider the hermitian, quadratic and cubic terms in right hand side of this equation. Hermitian form looks here as follows:

$$H(z, \bar{z}) = \left(1 - \frac{N + 2ns}{N}\right) \left(\left(\frac{m}{n} - i\right) z_1 \bar{z}_2 + \left(\frac{m}{n} + i\right) z_2 \bar{z}_1 \right). \quad (4.7)$$

This means that for $s \neq 0$ the equation (4.3) can be reduced by polynomial holomorphic change of variables to the form (4.5).

If $s = 0$ then hermitian form (4.7) identically vanishes. In both cases under consideration we can delete the quadratic terms in right hand side of (4.3). But if $s \neq 0$ we need in special consideration of the cubic terms in (4.3). The sum of all such terms can be written as

$$\begin{aligned} T_3 &= \frac{1}{3n^2N}(nx_1 - my_1)^3 - \frac{1}{6n^2}((3nx_1 + my_1) + 3(nx_1 - my_1)) = \\ &= \frac{1}{3N}(nx_1^3 - 3mx_1^2y_1 - 3nx_1y_1^2 + my_1^3) \end{aligned} \quad (4.8)$$

It's easy to see that T_3 equals to the real part of the expression

$$\frac{n + im}{3N}z_1^3.$$

Hence this term can be removed too from the equation (4.3) by cubic change of coordinates similar to (4.6). After this we get the equation

$$v = 0.$$

Thus, the following classifying proposition is valid:

Proposition 20. *Any surface of 3 order from constructed families of affinely homogeneous surfaces (4.1)-(4.3) is holomorphically equivalent either to indefinite quadric (4.5) or to Levi-degenerate real hyperplane $v = 0$.*

Let's discuss now from the holomorphic point of view affinely homogeneous surfaces of 4-th order.

We remind that they appear in basic family (2.19) under restrictions

$$m^2 + n^2 - ns = 0, \quad n \neq 0. \quad (4.9)$$

It means that subfamily of algebras (2.19), connected with surfaces of 4 order, depends only on two real parameters, for example $n \in \mathbb{R} \setminus \{0\}$ and $t = m/n \in \mathbb{R}$ (in that case $s = (m^2 + n^2)/n = n(1 + t^2)$).

When we integrate corresponding algebras (see §3) another real parameter C appears; so the surfaces interesting for us, are given by explicit form

$$v = (y_1x_2 + ty_1y_2) - \frac{1}{6n}(ty_1^3 + 3x_1y_1^2) - \frac{1+t^2}{4n} \cdot \frac{(y_1^2 - 3ny_2)^2}{x_1 - ty_1} + Cn^3(x_1 - ty_1)^3, \quad (4.10)$$

depending on three real parameters (n, t, C) .

Method that we use for surfaces (4.10) distinction is connected with holomorphic normal form of real hypersurfaces equations in complex spaces (see [19]). Modification of Moser's normal form that were developed for 3-dimensional case in [5]-[6] allows to decide raised problem in many cases. However, a lot of routine calculations is required to define Taylor's coefficients of normal equation, which are holomorphic invariants of surface under consideration.

We remind that the most known holomorphic invariant of smooth real hypersurface in multidimensional complex space is its Levi form ([11]) and Levi-degeneration (or non-degeneration) of surface connected with this form. This property is checked by using 2-order Taylor's coefficients of studied surface.

In spaces \mathbb{C}^n with $n \geq 3$ there are more subtle holomorphically invariant characteristics of non-degenerate surface (also connected with its Levi form). We mean the strictly pseudoconvexity (SPC) property of a surface or its indefinite type.

All the properties mentioned above depend in general on the point of the surface (4.10). We will analyze the situation at the point $Q(1, 0, iCn^3)$, lying on the surface with arbitrary fixed values of parameters. Shifting this point to origin of the space \mathbb{C}^3 by the change

$$z_1^* = z_1 - 1, \quad z_2^* = z_2, \quad w^* = w - iCn^3 \quad (4.11)$$

and then using rotation of axes, we write equation (4.10) in power expansion

$$v = F_2 + F_3 + F_4 + F_5 + F_6 + \dots \quad (4.12)$$

Explicit expressions for low-order terms from (4.12) have the form

$$F_2 = (y_1x_2 + ty_1y_2) - \frac{1}{2n}y_1^2 - sy_2^2 + 3Cn^3(x_1 - ty_1)^2, \quad (4.13)$$

$$F_3 = -\frac{1}{6n}(ty_1^3 + 3x_1y_1^2) - \frac{1+t^2}{4n}(-4ny_1^2y_2 + 4n^2y_2^2(ty_1 - x_1)) + Cn^3(x_1 - ty_1)^3, \quad (4.14)$$

$$F_4 = -\frac{1+t^2}{4n}(y_1^4 - 4ny_1^2y_2(ty_1 - x_1) + 4n^2y_2^2(ty_1 - x_1)^2), \quad (4.15)$$

and for $k \geq 4$

$$F_k = F_4 \cdot (ty_1 - x_1)^{k-4}. \quad (4.16)$$

Moser normalization of equation (4.12) can be realized for a few standard steps (see [19], [5]).

At first step Hermitian form $H(z, \bar{z})$ being Levi form of discussed surface M is picked from term F_2 out and then and then this form is reduced to the canonical type. In our case after transition to complex coordinates (multiplied by 2 for convenience)

$$x_k = z_k + \bar{z}_k, \quad y_k = i(\bar{z}_k - z_k), \quad k = 1, 2. \quad (4.17)$$

Levi form of surface (4.10) has the form

$$H(z, \bar{z}) = \left(-\frac{1}{n} + 6CsN^2\right)|z_1|^2 + (t - i)z_1\bar{z}_2 + (t + i)z_2\bar{z}_1 - 2s|z_2|^2. \quad (4.18)$$

Calculating the determinant of this Hermitian form matrix

$$\begin{pmatrix} \left(-\frac{1}{n} + 6CsN^2\right) & (t - i) \\ (t + i) & -2s \end{pmatrix} \quad (4.19)$$

we obtain the value

$$\Delta = \det(H) = \frac{s}{n}(1 - 12Csn^3). \quad (4.20)$$

Proposition 21. *The surfaces (4.10) are strictly pseudo convex if*

$$C < \frac{1}{12n^4(1 + t^2)},$$

are indefinite if

$$C > \frac{1}{12n^4(1 + t^2)},$$

are Levi-degenerate if

$$C = \frac{1}{12n^4(1 + t^2)}.$$

Proof. In order to prove this statement we notice that discussed cases are connected with the sign of determinant (4.20). Because of the restriction (4.9) this sign coincides with the sign of expression

$$1 - 12Csn^3 = 1 - 12Cn^3(n \cdot (1 + t^2)) = 1 - 12Cn^4(1 + t^2).$$

■

We can conclude from this simplest description of the surfaces (4.10) that any surface of this family with "small" value of parameter $C < (12n^4(1 + t^2))^{-1}$ holomorphically differs from any other one with "large" value of parameter $C > (12n^4(1 + t^2))^{-1}$.

More detailed investigations of family (4.10) were fulfilled only for zero value of parameter C . All such surfaces are SPC-surfaces according to proposition 21. In that case after Levi form reduction to canonical type any of them is described by equation

$$v = (|z_1|^2 + |z_2|^2) + F_{20} + F_{02} + \sum_{k+l \geq 3} F_{kl}(z, \bar{z}), \quad (4.21)$$

where F_{kl} - homogeneous polynomial of degree k in variable z and degree l in \bar{z} .

The coordinate transformation guaranteeing the Levi form reduction from (4.18) to $H = |z_1|^2 + |z_2|^2$ can be given by formulas

$$z_1^* = z_1 - (m + in)z_2, \quad z_2^* = \sqrt{m^2 + n^2}z_2, \quad w^* = -nw. \quad (4.22)$$

Taking into account these formulas the low-order polynomials F_{kl} from equation (4.21) can be simply calculated.

Two following steps of equation (4.21) normalization are connected with changes of variables

$$z^* = z, \quad w^* = w - 2i(F_{20} + F_{30} + \dots) \quad (4.23)$$

and

$$z^* = z + f_2 + f_3 + f_4 + \dots, \quad w^* = w, \quad \langle f_k, z \rangle = F_{k1} \quad (k = 2, 3, \dots). \quad (4.24)$$

They release the equation (4.21) from terms of $(k, 0)$ and $(0, k)$ types (change (4.23)) and from $(k, 1)$ and $(1, k)$ types with $k > 1$ (change (4.24)) respectively.

After these changes the equation of surface (4.21) will take a form

$$v = (|z_1|^2 + |z_2|^2) + \sum_{k,l \geq 2} H_{kl}(z, \bar{z}), \quad (4.25)$$

with new components in right hand side.

Final normalization transformation which is close to identical mapping corrects first of all the polynomials

$$H_{22}(z, \bar{z}), \quad H_{32}(z, \bar{z}), \quad H_{33}(z, \bar{z})$$

from equation (4.25).

It is important to notice that the initial equation of normalized surface can lose the property of rigidity, i.e. freedom of variable $u = \text{Re } w$, at this stage.

Rather sizeable computer calculations reduce to following result.

Proposition 22. *Normalization of the surfaces (4.10) by above-described scheme reduce them to equations*

$$v = (|z_1|^2 + |z_2|^2) + \sum_{k+l+2m \geq 4} N_{klm}(z, \bar{z})u^m, \quad (4.26)$$

in which

$$N_{220} = E_0 = |z_1|^4 - 4|z_1|^2|z_2|^2 + |z_2|^4, \quad (4.27)$$

$$N_{320} = z_1(|z_1|^4 - 6|z_1|^2|z_2|^2 + 3|z_2|^4) + z_2(3|z_1|^4 - 6|z_1|^2|z_2|^2 + |z_2|^4)$$

with arbitrary values of parameters, defining family (4.10).

Thus, low-order terms of Moser normal equations show no distinction in sets of parameters of discussed family. One need to calculate another terms of equation (4.26) for complete solution of this problem.

Recall in this connection that according to [27] any holomorphically homogeneous SPC-hypersurface is completely determined by coefficients of its normal equation of 6 order at

most. For the present we do not succeeded in overcoming all the technical difficulties in calculation of necessary coefficients.

Preliminary calculations show that in discussed family (4.10) of 4-order surfaces there is at least one-parameter subfamily of manifolds with holomorphically different polynomials N_{420} from Moser normal equation.

But we emphasize once more preliminary character of this statement.

R E F E R E N C E S

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