c-Sections, solvability and large subgroups of finite groups

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Abstract

c-Sections of maximal subgroups in a finite group and their relation to solvability were extensively researched in recent years (see [SW], [W] and [LS]). A fundamental result [W] is that a finite group is solvable if and only if the c-sections of all its maximal subgroups are trivial. In this paper we prove (Theorem 1.2), that if for each maximal subgroup of a finite group G, the corresponding c-section order is smaller than the index of the maximal subgroup, then each composition factor of G is either cyclic or isomorphic to the O'Nan sporadic group (the opposite direction does not hold). Furthermore, by a certain "refining" of the latter theorem we obtain an equivalent condition for solvability. Finally, we provide an existence result for large subgroups in the sense of [L].

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1 Introduction

All groups in this paper are finite. Most of our notation is standard. For $A \leq G$ we denote the class of all the subgroups conjugate to A in G by $Con_G(A)$. If $A \leq G$ and $|A| \geq |G|^{1/2}$ then A is called a *large subgroup* of G.

Let M be a maximal subgroup of a group G and K/L be a chief factor of G such that $L \leq M$ while $K \not\leq M$. Following Shirong and Wang in [SW], we call the group $M \cap K/L$ a *c-section of* M. It was proved [SW, 1.1] that for a fixed maximal subgroup M of G all the c-sections of M are isomorphic. We denote the abstract group isomorphic to a c-section (and so to all c-sections) of M by $Sec(M)$.

In [W] it was proved (although not using this terminology) that a group is solvable if and only if the c-sections of all its maximal subgroups are trivial. Further solvability conditions were proved in [SW]. In particular, a group is solvable if and only if the c-sections of all its maximal subgroups are 2-closed ([SW], Theorem 2.1), and if and only if the c-sections of all its maximal subgroups are nilpotent ([SW], Theorem 2.2). The case when all the c-sections are supersolvable was discussed in [LS].

In this paper we study further the notion of c-sections and its connection to solvability. In particular, for a maximal subgroup M we consider the relation between the order of the c-section $|Sec(M)|$ and the index $|G : M|$. By the above, if G is solvable then obviously $|Sec(M)| < |G : M|$ for each maximal subgroup M of G. It turns out that the opposite direction is not true.

Example 1.1 Let $T = O'Nan$, the O'Nan simple sporadic group, and let $G = Aut(T) = T : 2$. We show that $|Sec(M)| < |G : M|$ for all maximal subgroups M of G. If $M = T$ then $|Sec(M)| = 1 < |G : M| = 2$. Let M be maximal in G, $M \neq T$. Since $T/1$ is a chief factor of G and $M \not\geq T$, $M > 1$, we have $S := Sec(M) = M \cap T$. By $G = MT$ it follows that for each $g \in G$ there exists $t \in T$ such that $S^g = S^t$. Thus $Con_T(S) = Con_G(S)$, and so $Con_T(N_T(S)) = Con_G(N_T(S))$. Assume now that $|Sec(M)| \geq |G : M|$. Then $|S| \geq |G : M|$, implying $|S| \geq |T : S|$ and $|S| \geq |T|^{1/2}$, that is, S is a large subgroup of T. By checking the list of maximal subgroups of $T = O'Nan$ in [At], we deduce that S is contained in a maximal subgroup of T isomorphic to $L_3(7) : 2$. Considering the maximal subgroups of $L_3(7) : 2$, it follows that the only possibilites are either $S \cong L_3(7) : 2$ or $S \cong L_3(7)$, and in any case $N_T(S) \cong L_3(7) : 2$. By the information in [At] we deduce $Con_T(N_T(S)) \neq Con_G(N_T(S))$, contradicting our previous observation. Thus $|Sec(M)| < |G : M|$ for all maximal subgroups M of G.

The involvement of $O'Nan$ in Example 1.1 is not a coincidence. We have the following result.

Theorem 1.2 *Let G be a group such that $|Sec(M)| < |G : M|$ for all maximal subgroups M of G. Then every composition factor of G is either cyclic or isomorphic to $O'Nan$.*

The opposite direction of Theorem 1.2 is not true. Indeed for $G = O'Nan$ there exists a large maximal subgroup M, hence $|Sec(M)| = |M| \geq |G : M|$. Actually, it was proved in [L] that each simple non-abelian group has a proper large subgroup (and hence a large maximal subgroup). A key step in proving Theorem 1.2 is the following.

Proposition 1.3 *Let G be a simple non-abelian group. Then the following are equivalent:*
(1) G has a proper large subgroup H such that $Con_G(H) = Con_{Aut(G)}(H)$.
(2) $G \ncong O'Nan$.

By a certain "refinement" of Theorem 1.2, we get an equivalent condition for solvability in Theorem 1.4 below. Throughout this paper, we denote $\beta := log(175560)/log(2624832) \simeq 0,817$ (this number is connected to the largest proper subgroup H of $G = O'Nan$ satisfying $Con_G(H) = Con_{Aut(G)}(H)$).

Theorem 1.4 *Let G be a group. Then G is solvable if and only if $|Sec(M)| < |G : M|^\beta$ for all maximal subgroups M of G.*

We show in Proposition 2.8 that the (non-solvable) group $G = Aut(O'Nan)$ satisfies $|Sec(M)| \leq |G : M|^\beta$ (with equality in some cases) for all maximal subgroups M of G. Thus β can not be replaced by a larger constant in Theorem 1.4.

Next, we include the following result, which, unlike the other results of this paper, is "classification-free".

Theorem 1.5 *Let G be a group. Then the following are equivalent.*
(1) $|Sec(M)| < |G : M|$ for all maximal subgroups M of G.
(2) For each non-abelian chief factor K/L of G, and for each $L < B < K$ such that B/L is large in K/L, we have $Con_K(B) \neq Con_G(B)$.

Let G be a group satisfying the conditions of Theorem 1.5. We note that, by our Theorem 1.2, it follows that each non-cyclic composition factor of G (if exists) is isomorphic to $O'Nan$.

The main result of [L] is that each group of composite order has a proper large subgroup. By applying Proposition 1.3 we prove the following.

Theorem 1.6 *Let G be a group such that $|G|$ is divisible by two primes at least. Assume that G does not have composition factors isomorphic to $O'Nan$. Then G has a proper large subgroup H such that $Con_G(H) = Con_{Aut(G)}(H)$.*

The restriction on the composition factors of G in Theorem 1.6 can not be removed. This is clearly demonstrated by considering $G = O'Nan$. Furthermore, the statement of this theorem does not hold in general for p-groups (where p is a prime), as can be shown by the example of any elementary abelian p-group.

The proof of Proposition 1.3 is given in Section 2. The proofs of Theorems 1.2, 1.4, 1.5 and 1.6 are given in Section 3.

2 Proof of Proposition 1.3

Notice first that in Example 1.1 we showed that if $T = O'Nan$ and S is a proper large subgroup of T, then $Con_T(S) \neq Con_{Aut(T)}(S)$. Thus the implication (1) $\Rightarrow$ (2) of Proposition 1.3 is proved. It remains to prove that each simple non-abelian group G, except $O'Nan$, has a proper large subgroup H satisfying $Con_G(H) = Con_{Aut(G)}(H)$. We prove this separately for the sporadic simple groups, the simple groups of Lie type and the alternating groups; see Proposition 2.1, Corollary 2.4 and Proposition 2.5, respectively.

Proposition 2.1 *Let G be a sporadic simple group which is not isomorphic to O'Nan. Then G has a proper large subgroup H such that $Con_G(H) = Con_{Aut(G)}(H)$.*

Proof. As mentioned above, it was proved in [L] that each simple non-abelian group G has a large maximal subgroup. When $Out(G) = 1$ this large subgroup H certainly satisfies our extra condition. In Table 1 we give for each sporadic group G with $Out(G) > 1$, except $O'Nan$, a corresponding large maximal subgroup H such that $Con_G(H) = Con_{Aut(G)}(H)$. The information is based on [At]. This information completes the proof. $\square$

Table 1: Large subgroups H such that $Con_G(H) = Con_{Aut(G)}(H)$

G	H	$\|H\|$	$\|G:H\|$
M_{12}	$L_2(11)$	660	144
M_{22}	$L_3(4)$	20160	22
Suz	$G_2(4)$	251596800	1782
HS	M_{22}	443520	100
M^CL	$U_4(3)$	3265920	275
He	$S_4(4):2$	1958400	2058
HN	A_{12}	239500800	1140000
J_2	$U_3(3)$	6048	100
J_3	$L_2(16):2$	8160	6156
Fi_{22}	$2 \cdot U_6(2)$	18393661440	3510
Fi'_{24}	Fi_{23}	4089470473293004800	306936

Recall that a *Borel subgroup* B of a group of Lie type G in characteristic p is the normalizer of a Sylow p-subgroup of G. Since the Sylow p-subgroups of G are conjugate in G, it follows that $Con_G(B) = Con_{Aut(G)}(B)$. The following proposition states that in most cases B is large in G. We did not find a reference for this property, which may have an independent interest. In order to shorten notation, we say that the twisted group of Lie type ${}^\sigma\mathcal{L}_l(q^\sigma)$ is defined over the field $GF(q)$.

Proposition 2.2 *Let G be a simple group of Lie type ${}^\sigma\mathcal{L}_l(q^\sigma)$ of rank l defined over the field with q elements, where $q > 2$. Then a Borel subgroup B of G is a large subgroup of G.*

Proof. We deal separately with the cases when G is twisted or not.

Case 1. G is a non-twisted group of Lie type.
Then according to [Ca, 9.4.10]

$$|G| = \frac{1}{d}q^N(q^{d_1}-1)\cdots(q^{d_l}-1), |B| = \frac{1}{d}q^N(q-1)^l \text{ and } |G:B| = (q^{d_1}-1)\cdots(q^{d_l}-1)/(q-1)^l,$$

where d is as in 9.4.10 of [Ca], $N = |\Phi^+|$ is the number of positive roots of the root system related to G and $d_1 + \cdots + d_l = N + l$ [Ca, 9.3.4].

By assumption $q \geq 3$. Assume $l = 1$. Then even $q \geq 4$, $N = 1$ and $d_1 = N + l = 2$. Hence

$$|G:B| = (q^2-1)/(q-1) = q+1 \text{ and } |B| = q(q-1)/(q-1,2).$$

As $q(q-1) \geq 3q$ and $3q > 2(q+1)$, the assertion follows.

Now let $l \geq 2$. If $l = 2$ and $q = 3$, then either $d = 1$ and $G \cong L_3(3)$ or $G_2(3)$, or $d = 2$ and $G \cong PSp_4(3)$. In the first case $|B| = 2^2 \cdot 3^3$ or $2^2 \cdot 3^6$ and $|G:B| = 2^2 \cdot 13$ or $2^4 \cdot 7 \cdot 13$, respectively. Thus B is a large subgroup of G. If $G \cong PSp_4(3)$ then $|B| = 2 \cdot 3^4 = 162$ and $|G:B| = 2^5 \cdot 5 = 160$ and the assertion holds again.

From now on we assume $l \geq 3$ if $q = 3$ and $l \geq 2$ otherwise. We aim to show

$$(q^{d_1}-1)\cdots(q^{d_l}-1) < \frac{1}{d}q^N(q-1)^{2l}.$$

We have $(q^{d_1}-1)\cdots(q^{d_l}-1) < q^{\sum_{i=1}^l d_i} = q^{N+l}$ and claim that $(q-1)^{2l-1} > q^l$, which then yields the assertion. First let $q = 3$. Then $l \geq 3$, $(\frac{4}{3})^l > 2$ and so $2^{2l-1} > 3^l$ as required. Now suppose $q \geq 4$. Then $(q-1)^{2l} > (q^2-2q)^l = q^l(q-2)^l$. Thus it remains to show that $(q-2)^l \geq q-1$. This holds, as $(q-2)^l \geq (q-2)^2 = q^2-4q+4$ and $q^2 \geq 5(q-1)$.

Case 2. G is a twisted group of Lie type.
We choose the notation as it is given in [Ca]. So G is isomorphic to one of the following groups:

$${}^2A_l(q^2), {}^2B_2(q^2), {}^2D_l(q^2), {}^3D_4(q^3), {}^2E_6(q^2), {}^2F_4(q^2), {}^2G_2(q^2),$$

where $q^2 = 2^{2m+1}$ (resp. $q^2 = 3^{2m+1}$) if $\mathcal{L}$ is of type B_2 or F_4 (resp. of type G_2).

Let B be a Borel subgroup of T. Then by [Ca, 14.1.2]

$$|B| = \frac{1}{d}q^N(q-\eta_1)(q-\eta_2)\cdots(q-\eta_l),$$

where N is the number of positive roots in the root system related to $\mathcal{L}_l(q)$, d will be indicated in each case and $\eta_1, \ldots, \eta_l$ are the eigenvalues of the isometry τ of the vector space spanned by the roots which is related to the symmetry of the diagram for $\mathcal{L}_l(q)$. By [Ca, 14.3.2] we know $|G|$ and can calculate the index $|G:B|$ in all cases. Now we discuss all the possibilities.

Let $G \cong {}^2A_l(q^2)$ be a unitary group. We distinguish between the cases l even and l odd.

l even. Then $d = (q+1, l+1), N = l(l+1)/2, \eta_1 = \ldots = \eta_{l/2} = 1$ and $\eta_{l/2+1} = \ldots = \eta_l = -1$. So,

$$|B| = \frac{1}{d} q^{l(l+1)/2}(q-1)^{l/2}(q+1)^{l/2} \text{ and } |G:B| = \prod_{i=1}^{l}(q^{i+1} - (-1)^{i+1})/(q-1)^{l/2}(q+1)^{l/2}.$$

Notice that $(q^m - 1)(q^{m+1}+1) < q^{m+m+1}$. Thus $|G:B| < q^{2+3+\cdots+(l+1)}/(q-1)^{l/2}(q+1)^{l/2} = q^{(l(l+1)/2)+l}/(q-1)^{l/2}(q+1)^{l/2}$. So it is enough to show that $q^l \leq \frac{1}{d}(q-1)^l(q+1)^l$, or $q \leq \frac{1}{d^{1/l}}(q-1)(q+1)$. Since the "worst" case is $d = q+1$, it suffices to show $q \leq (q-1)(q+1)^{1-1/l}$. Since even $q \leq (q-1)(q+1)^{1/2}$ holds for every $q > 2$, we are done.

l odd, $l \geq 3$. Then $d = (q+1, l+1), N = l(l+1)/2, \eta_1 = \ldots = \eta_{(l+1)/2} = 1$ and $\eta_{((l+1)/2)+1} = \ldots = \eta_l = -1$. So,

$$|B| = \frac{1}{d} q^{l(l+1)/2}(q-1)^{(l+1)/2}(q+1)^{(l-1)/2} \text{ and}$$

$$|G:B| = \prod_{i=1}^{l}(q^{i+1} - (-1)^{i+1})/(q-1)^{(l+1)/2}(q+1)^{(l-1)/2}.$$

Similarly to the previous case we obtain $|G:B| < q^{(l(l+1)/2)+l}/(q-1)^{(l+1)/2}(q+1)^{(l-1)/2}$. Thus it is enough to show $q^l \leq \frac{1}{d}(q-1)^{l+1}(q+1)^{l-1}$. Again we take the worst case $d = q+1$, so it suffices to show $q^l \leq (q-1)^{l+1}(q+1)^{l-2}$, or $q^l \leq (q^2-1)^{l-2}(q-1)^3$. As $q < q^2 - 1$, it suffices to show $q^3 < (q^2-1)(q-1)^3$. Since the latter holds for every $q > 2$, this case is completed as well.

Let $G \cong {}^2B_2(q^2)$ be a Suzuki group. Then $d = 1, N = 4, \eta_1 = 1$ and $\eta_2 = -1$. Thus

$$|B| = q^4(q^2-1),\ |G:B| = q^4 + 1$$

and the assertion holds for every q.

Let $G \cong {}^2D_l(q^2)$ be an orthogonal group of minus type. Then $d = (4, q^l+1), N = l(l-1), \eta_1 = \ldots = \eta_{l-1} = 1$ and $\eta_l = -1$. Thus

$$|B| = \frac{1}{d} q^{l(l-1)}(q-1)^{l-1}(q+1) \text{ and } |G:B| = (q^l+1)\Big(\prod_{i=1}^{l-1}(q^{2i}-1)\Big)/(q-1)^{l-1}(q+1).$$

Then $|G:B| < q^{l-1}2^{l-1}\prod_{i=1}^{l-1} q^{2i-1} = 2^{l-1}\prod_{i=1}^{l-1} q^{2i} = 2^{l-1}q^{l(l-1)} \leq q^{l(l-1)}(q-1)^{l-1}$ as $q > 2$. Hence B is a large subgroup in that case.

Let $G \cong {}^3D_4(q^3)$. Then $d = 1, N = 12$ and $\eta_i = \alpha^{i-1}$ with $\alpha \neq 1$ a third root of unity, for $1 \leq i \leq 3$. Hence $|B| = q^{12}(q-1)(q-\alpha)(q-\alpha^2) = q^{12}(q^3-1)$ and

$$|G:B| = (q^8+q^4+1)(q^3+1)(q^2-1) < 2q^{13} < q^{12}(q^3-1),$$

and the assertion holds for every q (including $q = 2$).

Let $G \cong {}^2E_6(q^2)$. Then $d = (3, q+1), N = 36$, $\eta_1 = \ldots = \eta_4 = 1$, $\eta_5 = \eta_6 = -1$,

$$|B| = \frac{1}{d}q^{36}(q-1)^4(q+1)^2 \text{ and } |G : B| = (q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)/(q-1)^4(q+1)^2.$$

Here $|G : B| < 2q^{11}q^8 2q^7 2q^5(q^5+1) = 2^3q^{31}(q^5+1)$ and $q^5(q-1)^4(q+1) > 2^3(q^5+1)$, which shows the assertion.

Let $G \cong {}^2F_4(q^2)$. Then $d = 1$, $N = 24$, $\eta_1 = \eta_2 = 1$ and $\eta_3 = \eta_4 = -1$. So

$$|B| = q^{24}(q-1)^2(q+1)^2 = q^{24}(q^2-1)^2 \text{ and } |G : B| = (q^{12}+1)(q^8-1)(q^6+1)(q^2-1)/(q^2-1)^2.$$

Now let $r := q^2 = 2^{2m+1} > 2$. Then $|G : B| = (r^6+1)(r^3+r^2+r+1)(r^3+1) \leq (r^6+1)2r^3(r^3+1) < r^{12}(r-1)^2 = |B|$ and B is a large subgroup of G.

Let $G \cong {}^2G_2(q^2)$. Then $d = 1$, $N = 6, \eta_1 = 1$ and $\eta_2 = -1$. Then

$$|B| = q^6(q^2-1) \text{ and } |G : B| = (q^6+1)$$

and the assertion holds in all cases. □

We note that Proposition 2.2 can not be extended to the case $q = 2$, but a Borel subgroup is a large subgroup of G if $G \cong {}^3D_4(2)$ (this was shown in the proof of Proposition 2.2).

Next we consider the linear groups defined over $GF(2)$. We have the following general result.

Proposition 2.3 *Let G be a special linear group of rank $l \geq 2$ defined over the field with q elements. Let V be the natural module for T and (V_1, V_l) be two subspaces of dimension 1 and l, respectively, such that $V_1 \subseteq V_l$. Let P_i be the stabilizer of V_i in T, for $i = 1, l$. If $(l, q) \neq (2, 2)$, then $R := P_1 \cap P_l$ is a large subgroup of G, and $Con_G(R) = Con_{Aut(G)}(R)$.*

Proof. Recall that the field and diagonal automorphisms of G act on the set of maximal parabolic subgroups of type i, for $1 \leq i \leq l$ [Ca] and that the graph automorphisms interchange the sets of maximal parabolics of type 1 and l. Since P_l acts transitively on the 1-dimensional subspaces of V_l, it follows that $Con_G(R) = Con_{Aut(G)}(R)$.

Then $n := |G : R|$ is the number of flags (W_1, W_l), where W_i an i-dimensional subspace of V and $W_1 \subseteq W_l$. We have $n = (q^{l+1}-1)(q^l-1)/(q-1)^2$. As

$$|G| = \frac{1}{d}q^{l(l+1)/2}(q^{l+1}-1)\cdots(q^2-1),$$

where $d = (q-1, l+1)$, we get $|R| = \frac{1}{d}q^{l(l+1)/2}(q^{l-1}-1)\cdots(q^2-1)(q-1)^2$.

We have to show that $|G:R| \leq |R|$. If $l=2$ and $q \geq 3$ then

$$|G:R| = (q^3-1)(q^2-1)/(q-1)^2 = (q^2+q+1)(q+1) < \frac{1}{q-1}q^3(q-1)^2 \leq |R|.$$

If $l=3$ then $|G:R| = (q^4-1)(q^3-1)/(q-1)^2 < \frac{1}{q-1}q^6(q^2-1)(q-1)^2 \leq |R|$ and if $l \geq 4$ then

$$|G:R| = (q^{l+1}-1)(q^l-1)/(q-1)^2 < q^{2l+1} < q^{l(l+1)/2} < |R|,$$

completing the proof. □

Notice that the assertion of Proposition 2.3 is false for $G \cong L_3(2)$.

Corollary 2.4 *Let G be a simple group of Lie type. Then G has a proper large subgroup H such that $Con_G(H) = Con_{Aut(G)}(H)$.*

Proof. If $T \cong {}^3D_4(2)$ or if G is not defined over $GF(2)$, then the assertion follows by Proposition 2.2 and the remark after it. Therefore we may assume that G is defined over $GF(2)$.

If G is of type A_l, $l > 2$, then the statement is a consequence of Proposition 2.3. If $G \cong B_2(2)' \cong A_6 \cong L_2(9)$, $G \cong A_2(2) \cong L_3(2) \cong L_2(7)$ or $G \cong G_2(2)' \cong U_3(3)$, then we obtain the assertion by Proposition 2.2. If G is as listed in Table 2, then H is a large subgroup of G such that $Con_G(H) = Con_{Aut(G)}(H)$ (the details are taken from [At]).

Table 2: Large subgroups H such that $Con_G(H) = Con_{Aut(G)}(H)$

G	H	$\lvert H\rvert$	$\lvert G:H\rvert$
$D_4(2)$	$3^4:2^3.S_4$	15552	11200
$F_4(2)$	$[2^{20}]A_6.2$	754974720	4385745
${}^2F_4(2^2)'$	$2.[2^8]:5:4$	10240	1755

If G is one of the remaining groups of Lie type with $q=2$, i.e. G is isomorphic to one of the following groups

$$B_l(2), D_l(2) (l \geq 5), E_6(2), E_7(2), E_8(2), {}^2A_l(2^2), {}^2D_l(2^2), {}^2E_6(2^2),$$

then it is easily verified that the large subgroup H of G given by Table II of [L] satisfies $Con_G(H) = Con_{Aut(G)}(H)$. This completes the proof. □

It remains to consider the alternating groups.

Proposition 2.5 *Let $G \cong A_n$, $n \geq 5$. Then G has a proper large subgroup H such that $Con_G(H) = Con_{Aut(G)}(H)$.*

Proof. The case $G = A_6 \cong L_2(9)$ has already been handled in Proposition 2.2. Thus we may assume $n \neq 6$, in which case $Aut(G) = S_n$. Let H be a point stabilizer in $G = A_n$, then $Con_{A_n}(H) = Con_{S_n}(H)$, and clearly H is large in $G = A_n$. This completes the proof.

□

Now Proposition 1.3 follows by Proposition 2.1, Corollary 2.4 and Proposition 2.5.

The following will be used in the proof of Theorem 1.4

Proposition 2.6 *Let G be a simple non-abelian group. Then G has a proper subgroup H such that $|H| \geq |G : H|^\beta$ and $Con_G(H) = Con_{Aut(G)}(H)$.*

Proof. In view of Proposition 1.3, it is left to consider the case $G = O'Nan$. By [At] G has a (maximal) subgroup $H \cong J_1$, $|H| = 175560$, $|G : H| = 2624832$, such that $Con_G(H) = Con_{Aut(G)}(H)$. Since $|H| = |G : H|^\beta$, the proof is completed. □

Remark 2.7 The number β can not be replaced by a larger constant in Proposition 2.6. Indeed, let $T := O'Nan$ and let $A < T$ be such that $Con_T(A) = Con_{Aut(T)}(A)$. We show that $|A| \leq |T : A|^\beta$. Set $G = Aut(T)$. By Frattini's argument $G = TN_G(A)$ and so $|T : A| \geq |T : T \cap N_G(A)| = |G : N_G(A)|$. The list of maximal subgroups of $G = Aut(T) \cong O'Nan : 2$ is determined in [Wi]. By this list $S := J_1 \times 2$ is the largest maximal subgroup of $Aut(T)$ distinct from T. Thus $|T : A| \geq |G : S| = 2624832$, which implies $|A| \leq |T : A|^\beta$ as required.

As noted in the introduction, the following shows that Theorem 1.4 can not be improved by replacing β by a larger constant.

Proposition 2.8 *Let $T = O'Nan$ and $G = Aut(T)$. Then $|Sec(M)| \leq |G : M|^\beta$ for each maximal subgroup M of G.*

Proof. Let M be a maximal subgroup of G. If $M = T$ then $Sec(M) = 1$, so we may assume that $G = MT$ and $M \cap T < T$. For $g \in G$ there exist $u \in M, t \in T$ such that $g = ut$ and so $(M \cap T)^g = M^g \cap T = M^t \cap T = (M \cap T)^t$. This shows that $Con_T(M \cap T) = Con_G(M \cap T)$, and thus by Remark 2.7 $|M \cap T| \leq |T : M \cap T|^\beta = |MT : M|^\beta = |G : M|^\beta$. Since $T/1$ is a chief factor of G and $T \not\leq M$, $1 < M$, we have $Sec(M) = M \cap T$, so by the above $|Sec(M)| \leq |G : M|^\beta$ as required.

□

3 Proofs of Theorems 1.1, 1.2, 1.5 and 1.6

We start with the proof of our classification-free result.

Proof of Theorem 1.5. Suppose that (2) does not hold. Then there exist a non-abelian chief factor K/L of G, and a large proper subgroup B/L of K/L such that $Con_K(B) = Con_G(B)$. We shall show that G/L has a maximal subgroup M/L such that $|Sec(M)| \geq |G : M|$, hence (1) fails. It is no loss here to assume that $L = 1$. By Frattini's argument $G = KN_G(B)$. Since B is not normal in G we can choose M, a maximal subgroup of G containing $N_G(B)$. Then $M \not\geq K$ and K is minimal normal, hence $Sec(M) = M \cap K$. But $M \cap K \geq B$ and B is a large subgroup of K. Thus $|M \cap K| \geq |K : M \cap K| = |G : M|$, which implies $|Sec(M)| \geq |G : M|$.

In the other direction, suppose that (1) does not hold and let M be a maximal subgroup of G with $|Sec(M)| \geq |G : M|$. Let K/L be a chief factor of G satisfying $L \leq M$ and $K \not\leq M$. Then $G = KM$ implying $|G : M| = |K : M \cap K|$ and so $|(M \cap K)/L| \geq |K : M \cap K|$. Thus $(M \cap K)/L$ is a large proper subgroup of K/L (notice that K/L is non-abelian, since otherwise $M \cap K/L \lhd G/L$, contradicting the fact that K/L is a chief factor). In order to see that (2) fails, it is left to show that $Con_K(M \cap K) = Con_G(M \cap K)$. Let $g \in G$, then $g = mk$, where $m \in M$, $k \in K$. Thus $(M \cap K)^g = (M \cap K)^k$, and the proof is completed. □

We proceed with a useful lemma.

Lemma 3.1 *Let G be a group, $N \trianglelefteq G$, $N = T^m$, where T is a simple non-abelian group. Suppose $B \leq T$ and $Con_T(B) = Con_{Aut(T)}(B)$. Let $A := B^m$ be a subgroup of N. Then $Con_N(A) = Con_G(A)$.*

Proof. By construction $Aut(N) = Aut(T)\ wr\ S_m = N_{Aut(N)}(A)Aut(T)^m$. Since each $g \in G$ acts on N (by conjugation) like an element of $Aut(N)$, the assertion follows now from the assumption $Con_T(B) = Con_{Aut(T)}(B)$. □

Theorems 1.2, 1.4 and 1.6 can now be proved.

Proof of Theorem 1.2 Let G be a group such that $|Sec(M)| < |G : M|$ for all maximal subgroups M of G. Suppose to the contrary that G has a chief factor $K/L = T^m$, T is a simple non-abelian group and $T \not\cong O'Nan$. By Proposition 1.3 there exists a proper large subgroup B of T such that $Con_T(B) = Con_{Aut(T)}(B)$. Let $A = B^m$, a subgroup of K/L. Then it is easily verified that A is a proper large subgroup of K/L, and by Lemma 3.1 $Con_{K/L}(A) = Con_{G/L}(A)$. Let H be the preimage of A in G, then clearly $Con_K(H) = Con_G(H)$, so condition (2) of Theorem 1.5 is not satisfied by G. Since condition (1) of the same theorem is satisfied, we reached the desired contradiction. □

Proof of Theorem 1.4 The *only if* part is known, as mentioned in the introduction. We prove the *if* part. Let G be a minimal counterexample. Since the condition on the c-sections of G is inherited by quotients of G, we have that G/N is solvable for each $1 < N \trianglelefteq G$. Hence G has a unique minimal normal subgroup N, and $N = T^m$, where T is a simple non-abelian group. Furthermore $N = T^m \leq G \leq Aut(T)\ wr\ S_m = Aut(N)$. By Proposition 2.6 there exists a proper subgroup H of T such that $|H| \geq |G : H|^\beta$ and $Con_T(H) = Con_{Aut(T)}(H)$. Define $A = H^m$, a subgroup of N. Then it is easily verified that $|A| \geq |N : A|^\beta$, and by Lemma 3.1 $Con_N(A) = Con_G(A)$. By Frattini's argument we get $G = NN_G(A)$. Notice that $A < N$ forces that A is not normal in G. Let M be a maximal subgroup of G containing $N_G(A)$. Then $N \not\leq M$ and since N is minimal normal we have $Sec(M) \cong M \cap N$. Now $M \cap N \geq A$, implying that $|M \cap N| \geq |N : A|^\beta \geq |N : M \cap N|^\beta$. But since $G = MN$ we have $|N : M \cap N| = |G : M|$, hence $|Sec(M)| \geq |G : M|^\beta$, the desired contradiction. □

Proof of Theorem 1.6 Assume that the theorem is false and let G be a minimal counterexample. Suppose first that G does not have proper non-trivial characteristic subgroups. Then, since G is not a p-group, $G = T^m$, where T is a simple non-abelian group. Moreover, by assumption $T \not\cong O'Nan$. By Proposition 1.3 there exists $S < T$ such that $Con_T(S) = Con_{Aut(T)}(S)$ and S is large in T. Set $H = S^m$, a subgroup of $G = T^m$. Then H is a proper large subgroup of G, and $Con_G(H) = Con_{Aut(G)}(H)$ by Lemma 3.1. Therefore, G is not a counterexample in this case. Hence we may assume from now on that G has proper non-trivial characteristic subgroups.

Let K be a minimal characteristic subgroup of G. Then $K = T^m$, where T is a simple group (T may be of prime order). Suppose $|G/K|$ is divisible by two primes at least. Then, since G/K is not a counterexample, there exists $K < H < G$ such that H/K is large in G/K and $Con_{G/K}(H/K) = Con_{Aut(G/K)}(H/K)$. Let $\alpha \in Aut(G)$, then α induces an automorphism $\overline{\alpha}$ of G/K and $(H/K)^{\overline{\alpha}} = H^g/K$ for some $g \in G$. Thus $H^\alpha = H^g$, and it follows that $Con_G(H) = Con_{Aut(G)}(H)$. Since H is large in G, we deduce that G is not a counterexample.

It is left, therefore, to consider the case where G/K is a non-trivial p-group for a prime p. If K is elementary abelian then K is a q-group for a prime q distinct from p. Now, either a Sylow p-subgroup or a Sylow q-subgroup of G is large in G. Since this Sylow subgroup, say H, satisfies $Con_G(H) = Con_{Aut(G)}(H)$ by Sylow theorem, we deduce again that G is not a counterexample. Finally, suppose that K is non-solvable. Let R be a non-normal Sylow subgroup of K. By Frattini's argument we have $G = KN_G(R)$ and so either K or $N_G(R)$ is a proper large subgroup of G. Denote this large subgroup by H. If $H = K$ then $Con_G(H) = Con_{Aut(G)}(H) = \{H\}$. If $H = N_G(R)$, notice that for $\alpha \in Aut(G)$ there exists $u \in K$ such that $R^\alpha = R^u$. Thus $H^\alpha = H^u$, and it follows that $Con_G(H) = Con_{Aut(G)}(H)$. This shows that G is not a counterexample in this case, as well. The proof is now completed. □

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