On the rate of convergence to the semi-circular law

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Abstract

Let $\mathbf{X} = (X_{jk})$ denote a Hermitian random matrix with entries X_{jk}, which are independent for $1 \le j \le k$. We consider the rate of convergence of the empirical spectral distribution function of the matrix $\mathbf{X}$ to the semi-circular law assuming that $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$ and that the distributions of the matrix elements X_{jk} have a uniform sub exponential decay in the sense that there exists a constant $\varkappa > 0$ such that for any $1 \le j \le k \le n$ and any $t \ge 1$ we have

$$\Pr\{|X_{jk}| > t\} \le \varkappa^{-1} \exp\{-t^{\varkappa}\}.$$

By means of a short recursion argument it is shown that the Kolmogorov distance between the empirical spectral distribution of the Wigner matrix $\mathbf{W} = \frac{1}{\sqrt{n}}\mathbf{X}$ and the semicircular law is of order $O(n^{-1}\log^b n)$ with some positive constant $b > 0$.

1 Introduction

Consider a family $\mathbf{X} = \{X_{jk}\}$, $1 \le j \le k \le n$, of independent random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$. Assume that $X_{jk} = X_{kj}$, for $1 \le k <$

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$j \le n$, and introduce the symmetric matrices

$$\mathbf{W} = \frac{1}{\sqrt{n}}\begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}.$$

The matrix $\mathbf{W}$ has a random spectrum $\{\lambda_1, \dots, \lambda_n\}$ and an associated spectral distribution function

$$\mathcal{F}_n(x) = \frac{1}{n}\,\mathrm{card}\,\{j \le n : \lambda_j \le x\}, \quad x \in \mathbb{R}.$$

Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions

$$F_n(x) = \mathbf{E}\,\mathcal{F}_n(x).$$

Let $G(x)$ denote the semi-circular distribution function with density $g(x) = G'(x) = \frac{1}{2\pi}\sqrt{4-x^2}I_{[-2,2]}(x)$, where $I_{[a,b]}(x)$ denotes an indicator–function of interval $[a, b]$. We shall study the rate of convergence $\mathcal{F}_n(x)$ to the semi-circular law under the condition $\Pr\{|X_{jk}| > t\} \le \varkappa^{-1}\exp\{-t^{\varkappa}\}$ for some $\varkappa > 0$. This problem has been studied by several authors. The authors proved in [7] that the Kolmogorov distance between $\mathcal{F}_n(x)$ and the distribution function $G(x)$, $\Delta_n^* := \sup_x |\mathcal{F}_n(x) - G(x)|$ is of order $O_P(n^{-\frac12})$. Bai, [1], and Girko, [4], showed that $\Delta_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac12})$. Bobkov, Götze and Tikhomirov [3] proved that Δ_n and $\mathbf{E}\Delta_n^*$ have order $O(n^{-\frac23})$ assuming a Poincaré inequality for the distribution of the matrix elements. For the Gaussian Unitary Ensemble in [6] and for the Gaussian Orthogonal Ensemble in [11] it has been shown that $\Delta_n = O(n^{-1})$. Denote by $\gamma_{n1} \le \ldots \le \gamma_{nn}$, the quantiles of G, i.e. $G(\gamma_{nj}) = \frac{j}{n}$. We introduce the notation

$$\mathrm{llog}_n := \log\log n \tag{1.1}$$

In Erdös, Yau and Yin [9] showed that for matrix elements X_{jk} which have a uniformly sub exponential decay in the sense that there exists a constant $\varkappa > 0$ such that for any $1 \le j \le k \le n$ and any $t \ge 1$

$$\Pr\{|X_{jk}| \ge t\} \le \varkappa^{-1}\exp\{-t^{\varkappa}\},$$

the following result holds

$$\Pr\Big\{\,\exists\, j : |\lambda_j - \gamma_j| \ge (\log n)^{C\,\mathrm{llog}_n}\Big[\min\{(j, N-j+1)\Big]^{-\frac13} n^{-\frac23}\Big\} \le C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\}$$

for large n enough. It is straightforward to check that this bound implies that with high probability

$$\Pr\Big\{\sup_x |\mathcal{F}_n(x) - G(x)| \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}\Big\} \ge 1 - C\exp\{-(\log n)^{c\,\mathrm{llog}_n}\}. \tag{1.2}$$

From the last inequality it is follows that

$$\mathbf{E}\Delta_n^* \le Cn^{-1}(\log n)^{C\,\mathrm{llog}_n}.$$

In this paper we derive some improvement of the result (1.2) (reducing the power of logarithm) using arguments similar to [9] and provide a selfcontained proof based on recursion methods developed in the papers of Götze and Tikhomirov [7], [5], [12].

For any positive constants $\alpha > 0$ and $\varkappa > 0$ define the quantities

$$l_{n,\alpha} := \log n(\log\log n)^{\alpha} \quad \text{and} \quad \beta_n := (l_{n,\alpha})^{\frac{1}{\varkappa}+\frac{1}{2}}. \tag{1.3}$$

The main result of this paper is the following

Theorem 1.1. *Let* $\mathbf{E}X_{jk} = 0$, $\mathbf{E}X_{jk}^2 = 1$. *Assume that there exists a constant* $\varkappa > 0$ *such that for any* $1 \le j \le k \le n$ *and any* $t \ge 1$,

$$\Pr\{|X_{jk}| \ge t\} \le \varkappa^{-1}\exp\{-t^{\varkappa}\}. \tag{1.4}$$

Then, for any positive $\alpha > 0$ *there exist a positive constants* C *and* c *depending on* $\varkappa$ *and* α *only such that*

$$\Pr\left\{\sup_x |\mathcal{F}_n(x) - G(x)| > n^{-1}\beta_n^2\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{1.5}$$

We apply the result of Theorem 1.1 to the investigation of the eigenvectors of the matrix $\mathbf{W}$. Let $\mathbf{u}_j = (u_{j1}, \ldots, u_{jn})^T$ be eigenvectors of the matrix $\mathbf{W}$ corresponding to the eigenvalues λ_j, $j = 1, \ldots, n$. We prove the following result.

Theorem 1.2. *Under the conditions of Theorem 1.1 for any positive* $\alpha > 0$ *there exist positive constants* C *and* c, *depending on* $\varkappa$ *and* α *only such that*

$$\Pr\Big\{\max_{1\le j,k\le n}|u_{jk}|^2 > \frac{\beta_n^2}{n}\Big\} \le C\exp\{-cl_{n,\alpha}\}, \tag{1.6}$$

and

$$\Pr\Big\{\max_{1\le k\le n}\Big|\sum_{\nu=1}^{k}|u_{j\nu}|^2 - \frac{k}{n}\Big| > \frac{\beta_n}{\sqrt{n}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{1.7}$$

2 Proof of the main Theorem

To bound error Δ_n^* we shall use an approach developed in our paper [7]. We shall apply a bound of the Kolmogorov distance between distribution functions via the distance between their Stieltjes transforms. We denote the Stieltjes transform of $\mathcal{F}_n(x)$ by $m_n(z)$ and the Stieltjes transform of a semi-circular law by $s(z)$. Let $\mathbf{R} = \mathbf{R}(z)$ be the resolvent matrix of $\mathbf{W}$ given by

$$\mathbf{R} = (\mathbf{W} - z\mathbf{I}_n)^{-1},$$

for all $z = u + iv$ with $v \neq 0$. Here and in what follows $\mathbf{I}_n$ denotes the identity matrix of dimension n. Sometimes we shall omit the sub index in the notation of an identity matrix. It is well-known that the Stieltjes transform of a semi-circular distribution satisfies the equation

$$s^2(z) + zs(z) + 1 = 0$$

(see, for example, equality (4.20) in [7]). Furthermore, the Stieltjes transform of empirical spectral distribution function $\mathcal{F}_n(x)$, say $m_n(z)$, is given by

$$m_n(z) = \frac{1}{n}\sum_{j=1}^{n} R_{jj} = \frac{1}{n}\mathbf{E}\mathrm{Tr}\,\mathbf{R}.$$

(see, for instance, equality (4.3) in [7]). Introduce the matrices $\mathbf{W}^{(j)}$, which are obtained from $\mathbf{W}$ by deleting the j-th row and the j-th column, and the corresponding resolvent matrix $\mathbf{R}^{(j)}$ defined by $\mathbf{R}^{(j)} := (\mathbf{W}^{(j)} - zI_{n-1})^{-1}$ and let $m_n^{(j)}(z) := \frac{1}{n}\mathrm{Tr}\,\mathbf{R}^{(j)}$. Consider the index sets $\mathbb{T}_j := \{1, \dots, n\} \setminus \{j\}$. We shall use the representation

$$R_{jj} = \frac{1}{-z + \frac{1}{\sqrt{n}}X_{jj} - \frac{1}{n}\sum_{k,l\in\mathbb{T}_j} X_{jk}X_{jl}R_{kl}^{(j)}}, \tag{2.1}$$

(see, for example, equality (4.6) in [7]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)}\varepsilon_j R_{jj}, \tag{2.2}$$

where $\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}$ with

$$\varepsilon_{j1} := \frac{1}{\sqrt{n}}X_{jj}, \quad \varepsilon_{j2} := \frac{1}{n}\sum_{k\in\mathbb{T}_j}(X_{jk}^2 - 1)R_{kk}^{(j)},$$
$$\varepsilon_{j3} := \frac{1}{n}\sum_{k\neq l\in\mathbb{T}_j} X_{jk}X_{jl}R_{kl}^{(j)}, \quad \varepsilon_{j4} := \frac{1}{n}(\mathrm{Tr}\,R^{(j)} - \mathrm{Tr}\,R).$$

This relation immediately implies the following two equations

$$R_{jj} = -\frac{1}{z + m_n(z)} - \sum_{\nu=1}^{3}\frac{\varepsilon_{j\nu}}{(z + m_n(z))^2} + \sum_{\nu=1}^{3}\frac{1}{(z + m_n(z))^2}\varepsilon_{j\nu}\varepsilon_j R_{jj} + \frac{1}{z + m_n(z)}\varepsilon_{j4}R_{jj}, \tag{2.3}$$

and

$$m_n(z) = -\frac{1}{z+m_n(z)} - \frac{1}{(z+m_n(z))}\frac{1}{n}\sum_{j=1}^n \varepsilon_j R_{jj}$$
$$= -\frac{1}{z+m_n(z)} - \frac{1}{(z+m_n(z))^2}\frac{1}{n}\sum_{\nu=1}^3\sum_{j=1}^n \varepsilon_{j\nu}$$
$$+ \frac{1}{(z+m_n(z))^2}\frac{1}{n}\sum_{\nu=1}^3\sum_{j=1}^n \varepsilon_{j\nu}\varepsilon_j R_{jj} + \frac{1}{z+m_n(z)}\frac{1}{n}\sum_{j=1}^n \varepsilon_{j4}R_{jj}. \tag{2.4}$$

2.1 Large deviations I

In the following Lemmas we shall bound $\varepsilon_{j\nu}$, for $\nu = 1, \dots, 4$ and $j = 1, \dots, n$.

Lemma 2.1. *Assuming the conditions of Theorem 1.1 there exists positive constants C and c depending on $\varkappa$ and α such that, for any $j = 1, \dots, n$*

$$\Pr\{|\varepsilon_{j1}| \geq 2 l_{n,\alpha}^{\frac{1}{\varkappa}} n^{-\frac{1}{2}}\} \leq C\exp\{-c l_{n,\alpha}\}$$

Proof. The result follows immediately from the hypothesis (1.4). □

Lemma 2.2. *Assuming the conditions of Theorem 1.1 we have, for any $z = u + iv$ with $v > 0$ and any $j = 1, \dots, n$, we have*

$$|\varepsilon_{j4}| \leq \frac{1}{nv}.$$

Proof. The conclusion of Lemma 2.2 follows immediately from the obvious inequality $|\operatorname{Tr}\mathbf{R} - \operatorname{Tr}\mathbf{R}^{(j)}| \leq v^{-1}$ (see Lemma 4.1 in [7]). □

Lemma 2.3. *Assuming the conditions of Theorem 1.1, for all $v > 0$, the following inequality holds*

$$\Pr\Big\{|\varepsilon_{j2}| > 2 l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}} n^{-\frac{1}{2}} (n^{-1}\sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2)^{\frac{1}{2}}\Big\} \leq C\exp\{-c l_{n,\alpha}\}$$

for some positive constants $c > 0$ and C, depending on $\varkappa$ and α only.

Proof. We use the following inequality for sums of independent random variables. Let $\xi_1, \dots, \xi_n$ be independent random variables such that $\mathbf{E}\xi_j = 0$ and $|\xi_j| \leq \sigma_j$. Then

$$\Pr\{|\sum_{j=1}^n \xi_j| > x\} \leq c(1 - \Phi(x/\sigma) \leq \frac{\sigma}{x}\exp\{-\frac{x^2}{2\sigma^2}\}, \tag{2.5}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \exp\{-\frac{y^2}{2}\}dy$ and $\sigma^2 = \sigma_1^2 + \dots + \sigma_n^2$. We put $\eta_l = X_{jl}^2 - 1$, and define,

$$\xi_l = \big(\eta_l \mathbb{I}\{|X_{jl}| \leq l_{n,\alpha}^{\frac{1}{\varkappa}}\} - \mathbf{E}\eta_l\mathbb{I}\{|X_{jl}| \leq l_{n,\alpha}^{\frac{1}{\varkappa}}\}\big) R_{ll}^{(j)}.$$

Note that $\mathbf{E}\xi_l = 0$ and $|\xi_l| \le 2 l_{n,\alpha}^{\frac{2}{\varkappa}} |R_{ll}^{(j)}|$. Introduce the σ-algebra $\mathfrak{M}^{(j)}$ generated by the random variables X_{kl} with $k,l \in \mathbb{T}_j$. Let $\mathbf{E}_j$ and $\Pr_j$ denote the conditional expectation and the conditional probability with respect to $\mathfrak{M}^{(j)}$. Note that the random variables X_{jl} and the σ-algebra $\mathfrak{M}^{(j)}$ are independent. Applying inequality (2.5) with $x := 2n^{\frac12} l_{n,\alpha}^{\frac{2}{\varkappa}+\frac12} \big(n^{-1}\sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2\big)^{1/2}$, we get

$$\Pr\Big\{|\sum_{l\in\mathbb{T}_j} \xi_j| > x\Big\} = \mathbf{E}\Pr_j\Big\{|\sum_{l\in\mathbb{T}_j} \xi_j| \ge x\Big\} \le \mathbf{E}\exp\Big\{-\frac{x^2}{\sigma^2}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \quad (2.6)$$

Furthermore, note that

$$|\mathbf{E}_j \eta_l \mathbb{I}\{|\xi_l| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}| \le \mathbf{E}_j^{\frac12} |\eta_l|^2 \Pr_j^{\frac12}\{|\xi_l| > l_{n,\alpha}\} \le \mathbf{E}^{\frac12}|\eta_l|^2 \exp\{-\frac{c}{2} l_{n,\alpha}\} \quad (2.7)$$

The last inequality implies that

$$\Big|\sum_{l\in\mathbb{T}_j} \mathbf{E}_j \eta_l \mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}})\} R_{ll}^{(j)}\Big| \le C\exp\{-\frac{c}{2} l_{n,\alpha}\}(\frac1n \sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2)^{\frac12}. \quad (2.8)$$

The inequalities (2.6) and (2.8) together conclude the proof of Lemma 2.3. Thus the Lemma is proved.

□

Corollary 2.4. *Assuming the conditions of Theorem 1.1 for any $\alpha > 0$ there exists positive constants c and C, depending on $\varkappa$ and α such that for any $z = u + iv$ with $v > 0$*

$$\Pr\{|\varepsilon_{j2}| > \beta_n^2 (nv)^{-\frac12} (\operatorname{Im} m_n^{(j)}(z))^{\frac12}\} \le C\exp\{-cl_{n,\alpha}\}. \quad (2.9)$$

Proof. Note that

$$n^{-1}\sum_{l\in\mathbb{T}_j} |R_{ll}^{(j)}|^2 \le n^{-1}\operatorname{Tr}|\mathbf{R}^{(j)}|^2 = \frac1v \operatorname{Im} m_n^{(j)}(z), \quad (2.10)$$

where $|\mathbf{R}^{(j)}|^2 = \mathbf{R}^{(j)}\mathbf{R}^{(j)*}$. The result follows now from Lemma 2.5. □

Lemma 2.5. *Assuming the conditions of Theorem 1.1, for any $j = 1, \ldots, n$ and for any $v > 0$, the following inequality holds*

$$\Pr\Big\{|\varepsilon_{j3}| > \beta_n^2 n^{-\frac12} (\frac1n \sum_{k\ne l\in\mathbb{T}_j} |R_{kl}^{(j)}|^2)^{\frac12}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \quad (2.11)$$

Proof. We shall use a large deviation bound for quadratic forms which follows from results by Ledoux (see [10]).

Proposition 2.1. *Let $\xi_1,\dots,\xi_n$ be independent random variables such that $|\xi_j|\le 1$. Let a_{ij} denote complex numbers such that $a_{ij}=a_{ji}$ and $a_{jj}=0$. Let $Z=\sum_{l,k=1}^n\xi_l\xi_k a_{lk}$. Let $\sigma^2=\sum_{l,k=1}^n|a_{lk}|^2$. Then for every $t>0$ there exists some positive constant $c>0$ such that the following inequality holds*

$$\Pr\{|Z|\ge\frac{3}{2}\mathbf{E}^{\frac12}|Z|^2+t\}\le\exp\{-\frac{ct}{\sigma}\}. \tag{2.12}$$

Proof. Proposition 2.1 follows from Theorem 3.1 in [10]. □

In order to bound ε_{j3} we use Proposition 2.1 with

$$\xi_l=(X_{jl}\mathbb{I}\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}\}-\mathbf{E}X_{jl}\mathbb{I}\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}\})/2\,l_{n,\alpha}^{\frac{1}{\varkappa}}. \tag{2.13}$$

Note that the random variables X_{jl}, $l\in\mathbb{T}_j$ and the matrix $\mathbf{R}^{(j)}$ are mutually independent for any fixed $j=1,\dots,n$. Moreover, we have $|\xi_l|\le 1$. Put $Z:=\sum_{k\ne l\in\mathbb{T}_j}\xi_l\xi_k R_{kl}^{(j)}$. Applying Proposition 2.1 with $t=l_{n,\alpha}(\sum_{l\ne k\in\mathbb{T}_j}|R_{lk}^{(j)}|^2)^{\frac12}$, we get

$$\mathbf{E}\Pr_j\Big\{|Z|\ge l_{n,\alpha}(\sum_{l\ne k\in\mathbb{T}_j}|R_{lk}^{(j)}|^2)^{\frac12}\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.14}$$

Furthermore,

$$\Pr\{\exists j,l\in[1,\dots,n]:\,|X_{jl}|>l_{n,\alpha}^{\frac{1}{\varkappa}}\}\le C\exp\{-cl_{n,\alpha}\} \tag{2.15}$$

and

$$|\mathbf{E}X_{jl}\mathbb{I}\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}|\le\Pr^{\frac12}\{\exists j,l,k\in[1,\dots,n]:\,|X_{jl}|>l_{n,\alpha}^{\frac{1}{\varkappa}}\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.16}$$

Introduce the random variables $\widehat{\xi}_l=\xi_{jl}\mathbb{I}\{|X_{jl}|\le l_{n,\alpha}^{\frac{1}{\varkappa}}\}/2l_{n,\alpha}^{\frac{1}{\varkappa}}$ and $\widehat{Z}=\sum_{l,k\in\mathbb{T}_j}\xi_l\xi_k R_{lk}^{(j)}$. Note that

$$\Pr\Big\{\sum_{l,k\in\mathbb{T}_j}X_{jk}X_{jl}R_{kl}^{(j)}\ne\sum_{l,k\in\mathbb{T}_j}\widehat{\xi}_k\widehat{\xi}_l R_{kl}^{(j)}\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.17}$$

Inequalities (2.14)–(2.17) together imply

$$\Pr\Big\{|\varepsilon_{j3}|>\beta_n^2 n^{-\frac12}(\frac{1}{n}\sum_{k\ne l\in\mathbb{T}_j}|R_{kl}^{(j)}|^2)^{\frac12}\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.18}$$

Thus Lemma 2.5 is proved. □

Corollary 2.6. *Under the conditions of Theorem 1.1 there exist positive constants c and C depending on $\varkappa$ and α such that for any $z=u+iv$ with $v>0$*

$$\Pr\{|\varepsilon_{j3}|>\beta_n^2(nv)^{-\frac12}(\operatorname{Im}m_n^{(j)}(z))^{\frac12}\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.19}$$

Proof. Note that

$$n^{-1}\sum_{k\neq l\in\mathbb{T}_j}|R^{(j)}_{kl}|^2\le n^{-1}\mathrm{Tr}\,|\mathbf{R}^{(j)}|^2=\frac{1}{v}\mathrm{Im}\,m_n^{(j)}(z). \tag{2.20}$$

The result now follows from Lemma 2.5.

□

To summarize these results we recall

$$\beta_n=(l_{n,\alpha})^{\frac{1}{\varkappa}+\frac12}, \tag{2.21}$$

defined previously in (1.3). Then we may write that, for $\nu=1,2,3$

$$\Pr\Big\{|\varepsilon_{j\nu}|>\frac{\beta_n}{\sqrt{n}}\Big(1+\frac{\mathrm{Im}^{\frac12}m_n(z)}{\sqrt{v}}+\frac{1}{\sqrt{v}\sqrt{nv}}\Big)\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{2.22}$$

Denote by

$$\Omega_n(z)=\Big\{\omega\in\Omega:\,|\varepsilon_j|\le\frac{\beta_n}{\sqrt{n}}\Big(1+\frac{\mathrm{Im}^{\frac12}m_n(z)}{\sqrt{v}}+\frac{1}{\sqrt{nv}}\Big)\Big\}. \tag{2.23}$$

Let $v_0=\frac{a\beta_n}{n}$ with a sufficiently small positive constant $a>0$. We introduce the region $\mathcal{D}=\{z=u+iv\in\mathbb{C}:|u|\le 2,\,v_0<v\le 2\}$. Furthermore, we introduce the sequence $z_l=u_l+v_l$ in $\mathcal{D}$, recursively defined via $u_{l+1}-u_l=\frac{4}{n^8}$ and $v_{l+1}-v_l=\frac{2}{n^8}$. Using a union bound, we have

$$\Pr\{\cap_{z_l\in\mathcal{D}}\Omega_n(z_l)\}\ge 1-C\exp\{-cl_{n,\alpha}\}. \tag{2.24}$$

It is straightforward to check that

$$|\varepsilon_j(z)-\varepsilon_j(z')|\le\frac{|z-z'|}{v_0^2}\frac{\beta_n}{\sqrt{n}}\big(1+\frac{\mathrm{Im}^{\frac12}m_n(z)}{\sqrt{v}}+\frac{1}{\sqrt{n}v}\big). \tag{2.25}$$

This immediately implies that

$$\Pr\{\cap_{z\in\mathcal{D}}\Omega_n(z)\}\ge 1-C\exp\{-cl_{n,\alpha}\}, \tag{2.26}$$

for appropriately some chosen constant in the definition (2.23) of the event $\Omega_n(z)$.

3 Large deviations II

In this Section we obtain bounds for large deviation probabilities of the sum of ε_j. We start with

$$\delta_{n1}=\frac{1}{n}\sum_{j=1}^n\varepsilon_{j1}.$$

Lemma 3.1. *There exist constants c and C such that*

$$\Pr\{|\delta_{n1}| > n^{-1}\beta_n\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.1}$$

Proof. We repeat the proof of Lemma 2.1. Consider the truncated random variables

$$\widehat{X}_{jj} = X_{jj}\mathbb{I}\{|X_{jj}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}\}. \tag{3.2}$$

By assumption (1.4),

$$\Pr\{|X_{jj}| > l_{n,\alpha}^{\frac{1}{\varkappa}}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.3}$$

Moreover,

$$|\mathbf{E}\widehat{X}_{jj}| \le C\exp\{-cl_{n,\alpha}\}. \tag{3.4}$$

We define

$$\widetilde{X}_{jj} = \widehat{X}_{jj} - \mathbf{E}\widehat{X}_{jj} \tag{3.5}$$

and consider the sum

$$\widetilde{\delta}_{n1} := \frac{1}{n\sqrt{n}}\sum_{j=1}^{n}\widetilde{X}_{jj}. \tag{3.6}$$

Since

$$|\widetilde{X}_{jj}| \le 2l_{n,\alpha}^{\frac{1}{\varkappa}},$$

we have

$$\Pr\{|\widetilde{\delta}_{n1}| > n^{-1}l_{n,\alpha}^{\frac{1}{\varkappa}+\frac{1}{2}}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.7}$$

Note that

$$|\widetilde{\delta}_{n1} - \delta_{n1}| \le \frac{1}{n}\sum_{j=1}^{n}|\mathbf{E}\widehat{X}_{jj}| \le \exp\{-cl_{n,\alpha}\}. \tag{3.8}$$

This inequality and inequality ((3.7)) together imply

$$\Pr\Big\{|\delta_{n1}| > n^{-1}l_{n,\alpha}^{\frac{1}{\varkappa}+\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.9}$$

Thus, Lemma 3.1 is proved. □

Consider now the quantity

$$\delta_{n2} := \frac{1}{n^2}\sum_{j=1}^{n}\sum_{l\in\mathbb{T}_j}(X_{jl}^2 - 1)R_{ll}^{(j)}. \tag{3.10}$$

We prove the following Lemma

Lemma 3.2. *Assume that there exists a constant C such that for any $j = 1, \ldots, n$ and any $l \in \mathbb{T}_j$*

$$|R_{ll}^{(j)}| \le C. \tag{3.11}$$

Then there exist constants c and C, depending on $\varkappa$ and α such that

$$\Pr\{|\delta_{n2}| \le n^{-1}\beta_n^2\} \le C\exp\{-cl_{n,\alpha}\} \tag{3.12}$$

Proof. Introduce the truncated random variables

$$\xi_{jl} = \widehat{X}_{jl}^2 - \mathbf{E}\widehat{X}_{jl}^2, \tag{3.13}$$

where $\widehat{X}_{jl} = X_{jl}\mathbb{I}\{|X_{jl}| \le l_{n,\alpha}^{\frac{1}{\varkappa}}$. It is straightforward to check that

$$0 \le 1 - E\widehat{X}_{jl}^2 \le C\exp\{-cl_{n,\alpha}\}. \tag{3.14}$$

We shall need the following quantities as well

$$\widehat{\delta}_{n2} = \frac{1}{n^2}\sum_{j=1}^n\sum_{l\in\mathbb{T}_j}(\widehat{X}_{jl}^2-1)R_{ll}^{(j)} \quad \text{and} \quad \widetilde{\delta}_{n2} = \frac{1}{n^2}\sum_{j=1}^n\sum_{l\in\mathbb{T}_j}\xi_{lj}R_{ll}^{(j)}. \tag{3.15}$$

By assumption (1.4),

$$\Pr\{\delta_{n2} \ne \widehat{\delta}_{n2}\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.16}$$

Let

$$\zeta_j := \frac{1}{\sqrt{n}}\sum_{l\in\mathbb{T}_j}\xi_{jl}R_{ll}^{(j)}. \tag{3.17}$$

Then

$$\widehat{\delta}_{n2} = \frac{1}{n^2}\sum_{j=1}^n\zeta_j. \tag{3.18}$$

Note that the sequence $\widehat{\delta}_{n2}$ is a martingale with respect to the σ–algebras $\mathfrak{M}_j$. In fact,

$$\mathbf{E}\{\zeta_j|\mathfrak{M}_{j-1}\} = \mathbf{E}\{\mathbf{E}\{\zeta_j|\mathfrak{M}^{(j)}\}|\mathfrak{M}_{j-1}\} = 0. \tag{3.19}$$

In order to use large deviation bounds for $\widehat{\delta}_{n2}$ we replace the differences ζ_j by truncated random variables. We put

$$\widehat{\zeta}_j = \zeta_j\mathbb{I}\{|\zeta_j| \le l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2)^{\frac{1}{2}}\}. \tag{3.20}$$

Since ζ_j is a sum of independent bounded random variables with mean zero, we have

$$\Pr\Big\{|\zeta_j| > l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}(\frac{1}{n}\sum_{l\in\mathbb{T}_j}|R_{ll}^{(j)}|^2)^{\frac{1}{2}}\Big\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.21}$$

This implies that

$$\Pr\{\sum_{j=1}^n\zeta_j \ne \sum_{j=1}^n\widehat{\zeta}_j\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.22}$$

Furthermore, introduce the random variables

$$\widetilde{\zeta}_j = \widehat{\zeta}_j - \mathbf{E}\{\widehat{\zeta}_j|\mathfrak{M}_{j-1}\}. \tag{3.23}$$

Using the Cauchy-Schwartz inequality and the boundedness of the random variables $\xi_{jl} R_{ll}^{(j)}$ we may show that

$$|\mathbf{E}\{\widehat{\zeta}_j|\mathfrak{M}_{j-1}\}| \le C \exp\{-c l_{n,\alpha}\}. \tag{3.24}$$

Here, we may use a martingale bound due to Bentkus, [2], Theorem 1.1. It provides the following result. Let $\mathfrak{M}_0 = \{\emptyset, \Omega\} \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_n \subset \mathfrak{M}$ be a family of σ-algebras of a measurable space $\{\Omega, \mathfrak{M}\}$. Let $M_n = \xi_1 + \ldots + \xi_n$ be a martingale with bounded differences $\xi_j = M_j - M_{j-1}$ such that

$$\Pr\{|\xi_j| \le b_j\} = 1 \quad \text{for} \quad j = 1, \ldots, n.$$

Then, for $x > \sqrt{8}$

$$\Pr\{|M_n| \ge x\} \le c(1 - \Phi(\frac{x}{\sigma})) = \int_{\frac{x}{\sigma}}^{\infty} \varphi(t) dt, \quad \varphi(t) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{t^2}{2}\}. \tag{3.25}$$

with some numerical constant $c > 0$ and $\sigma^2 = b_1^2 + \cdots + b_n^2$. Note that for $t > C$

$$1 - \Phi(t) \le \frac{1}{C} \varphi(t).$$

Thus, this leads to the inequality

$$\Pr\{|M_n| \ge x\} \le \exp\{-\frac{x^2}{2\sigma^2}\}, \tag{3.26}$$

which we shall use to bound $\widetilde{\delta}_{n2}$. By assumption (3.11) and definition of $\widetilde{\zeta}_j$, we may take $\beta_j = l_{n,\alpha}^{\frac{2}{\varkappa}+\frac{1}{2}}$. We get

$$\Pr\{|\widetilde{\delta}_{n2}| > n^{-1} \beta_n^2\} \le C \exp\{-c l_{n,\alpha}\} \tag{3.27}$$

Inequalities (3.22)–(3.27) together conclude the proof of Lemma 3.2. □

Let

$$\delta_{n3} := \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \ne k \in \mathbb{T}_j} X_{jl} X_{jk} R_{lk}^{(j)}. \tag{3.28}$$

Lemma 3.3. *Assume that there exists a constant $B > 0$ such that for any $j = 1, \ldots n$*

$$\operatorname{Im} m_n^{(j)}(z) \le B. \tag{3.29}$$

Then there exist constants c and C, depending on $\varkappa$ and α such that

$$\Pr\{|\delta_{n3}| > \frac{\beta_n^2}{n\sqrt{v}}\} \le C \exp\{-c l_{n,\alpha}\} \tag{3.30}$$

Proof. The proof of this Lemma is similar to the proof of Lemma 3.2. We introduce the random variables

$$\eta_j = \frac{1}{\sqrt{n}} \sum_{l \neq k \in \mathbb{T}_j} X_{jk} X_{jl} R^{(j)}_{lk} \tag{3.31}$$

and note that the sequence

$$M_n = \sum_{j=1}^{n} \eta_j \tag{3.32}$$

is martingale with respect to the σ–algebras $\mathfrak{M}_j$. Then we apply the martingale bound of Bentkus twice replacing η_j by truncated random variables. Thus the Lemma is proved. □

Finally, we shall bound

$$\delta_{n4} := \frac{1}{n^2} \sum_{j=1}^{n} (\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}) R_{jj}. \tag{3.33}$$

Lemma 3.4. *For any $z = u + iv$ with $v > 0$ the following inequality*

$$|\delta_{n4}| \leq \frac{1}{nv} \operatorname{Im} m_n(z) \tag{3.34}$$

holds.

Proof. By formula (5.4) in [7], we have

$$(\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}) R_{jj} = (1 + \frac{1}{n} \sum_{l,k \in T_j} X_{jl} X_{jk} ((R^{(j)})^2)_{lk}) R^2_{jj} = \frac{d}{dz} R_{jj}. \tag{3.35}$$

From here it follows that

$$\frac{1}{n^2} \sum_{j=1}^{n} (\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}) R_{jj} = \frac{1}{n^2} \frac{d}{dz} \operatorname{Tr} \mathbf{R} = \frac{1}{n^2} \operatorname{Tr} \mathbf{R}^2. \tag{3.36}$$

Finally, we note that

$$|\frac{1}{n^2} \operatorname{Tr} \mathbf{R}^2| \leq \frac{1}{nv} \operatorname{Im} m_n(z). \tag{3.37}$$

The last inequality concludes the proof. Thus, Lemma 3.4 is proved. □

3.1 Stieltjes transforms

We shall derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $\mathbf{X}$ and the Stieltjes transform $s(z)$ of the semi-circular law. Introduce the additional notations

$$\delta_n := \delta_{n1} + \delta_{n2} + \delta_{n3}, \quad \widehat{\delta}_n := \delta_{n4}, \quad \overline{\delta}_n := \frac{1}{n} \sum_{\nu=1}^{3} \sum_{j=1}^{n} \varepsilon_{j\nu} \varepsilon_j R_{jj}. \tag{3.38}$$

Recall that $s(z)$ satisfies the equation

$$s(z) = -\frac{1}{z+s(z)}. \tag{3.39}$$

Introduce $g_n(z) := m_n(z) - s(z)$. The representation (2.4) and equality (3.39) together imply

$$g_n(z) = \frac{g_n(z)}{(z+s(z))(z+m_n(z))} - \frac{\delta_n}{(z+m_n(z))^2} + \frac{\widehat{\delta}_n}{z+m_n(z)} + \frac{\overline{\delta}_n}{(z+m_n(z))^2}. \tag{3.40}$$

This equality yields

$$|g_n(z)| \le \frac{|\delta_n| + |\overline{\delta}_n|}{|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{|\widehat{\delta}_n|}{|z+s(z)+m_n(z)|}. \tag{3.41}$$

For any $z \in \mathcal{D}$ introduce the events

$$\widehat{\Omega}_n(z) := \{\omega \in \Omega : |\delta_n| \le \frac{\beta_n}{n\sqrt{v}}\}, \quad \widetilde{\Omega}_n(z) := \{\omega \in \Omega : |\widehat{\delta}_n| \le \frac{C\,\mathrm{Im}\, m_n(z)}{nv}\}, \tag{3.42}$$

$$\overline{\Omega}_n(z) := \{\omega \in \Omega : |\overline{\delta}_n| \le \Big(\frac{\beta_n^2 \mathrm{Im}\, m_n(z)}{nv} + \frac{\beta_n^2}{(nv)^2}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|^2\}. \tag{3.43}$$

Put $\Omega_n^*(z) := \widehat{\Omega}_n(z) \cup \widetilde{\Omega}_n(z) \cup \overline{\Omega}_n(z)$. By Lemmas 3.1–3.4, we have

$$\Pr\{\widehat{\Omega}_n(z)\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{3.44}$$

By Lemma 3.4,

$$\Pr\{\widetilde{\Omega}_n(z)\} = 1. \tag{3.45}$$

Note that

$$|\varepsilon_{j\nu}\varepsilon_{j4}| \le \frac{1}{2}(|\varepsilon_{j\nu}|^2 + |\varepsilon_{j4}|^2). \tag{3.46}$$

By inequality (2.22), we have, for $\nu = 1, 2, 3$,

$$\Pr\Big\{|\varepsilon_{j\nu}|^2 > \frac{\beta_n^2}{n}(1 + \frac{\mathrm{Im}\, m_n(z)}{v} + \frac{1}{nv^2})\Big\} \le C\exp\{-cl_{n,\alpha}\} \tag{3.47}$$

and

$$\Pr\{|\varepsilon_{j4}|^2 \le \frac{1}{n^2v^2}\} = 1. \tag{3.48}$$

Similarly as in (2.26) we may show that

$$\Pr\{\cap_{z\in\mathcal{D}}\Omega_n^*(z) \cap \Omega_n\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{3.49}$$

Let

$$\Omega_n^* := \cap_{z\in\mathcal{D}}\Omega_n^*(z) \cap \Omega_n. \tag{3.50}$$

We now prove now some auxiliary Lemmas.

Lemma 3.5. *Let $z = u + iv \in \mathcal{D}$ and $\omega \in \Omega_n^*$. Assume that*

$$|g_n(z)| \le \frac{1}{2}. \tag{3.51}$$

Then the following bound holds

$$|g_n(z)| \le \frac{C\beta_n^2}{nv} + \frac{C\beta_n^2}{n^2v^2\sqrt{\gamma+v}}.$$

Proof. First we note that the inequality $|g_n(z)| \le \frac{1}{2}$ implies

$$|z + m_n(z)| \ge |z + s(z)| - |g_n(z)| \ge \frac{1}{2}. \tag{3.52}$$

Moreover,

$$\operatorname{Im} m_n^{(j)}(z) \le |m_n^{(j)}(z)| \le |m_n(z)| + \frac{1}{nv} \le |s(z)| + |g_n(z)| + \frac{1}{nv} \le C. \tag{3.53}$$

Furthermore, we obviously obtain

$$|z+s_n(z)+s(z)| \ge \operatorname{Im} z+\operatorname{Im} m_n(z)+\operatorname{Im} s(z) \ge \operatorname{Im}(z+s(z)) \ge \operatorname{Im}\{\sqrt{z^2-4}\}. \tag{3.54}$$

For $z \in \mathcal{D}$ we get $\operatorname{Re}(z^2-4) \le 0$ and $\frac{\pi}{2} \le \arg(z^2-4) \le \frac{3\pi}{2}$. Therefore,

$$\operatorname{Im}\{\sqrt{z^2-4}\} \ge \frac{1}{\sqrt{2}}|z^2-4|^{\frac{1}{2}} \ge \frac{1}{4}\sqrt{\gamma+v}, \tag{3.55}$$

where $\gamma = 2 - |u|$. Inequality (3.41) implies that for $\omega \in \Omega_n^*$

$$\begin{aligned} |g_n(z)| \le{}& \frac{\beta_n}{n\sqrt{v}|z+m_n(z)||z+s(z)+m_n(z)|} + \frac{C\operatorname{Im} m_n(z)}{nv|z+s(z)+m_n(z)|} \\ &+ \frac{\beta_n^2}{nv|z+m_n(z)||z+s(z)+m_n(z)|}\Big(\operatorname{Im} m_n(z) + \frac{1}{nv}\Big)\frac{1}{n}\sum_{j=1}^n |R_{jj}|^2. \end{aligned} \tag{3.56}$$

Furthermore, equation (2.2), inequality (3.52) and the definition of Ω_n^* in (2.23) together imply that, for $\omega \in \Omega_n^*$ and $z \in \mathcal{D}$

$$|R_{jj}| \le \frac{2}{|z+s_n(z)|}. \tag{3.57}$$

Equation (3.56) and inequality (3.57) together imply

$$|g_n(z)| \le \frac{C\beta_n^2}{nv}\Big(1 + \frac{1}{nv\sqrt{\gamma+v}}\Big). \tag{3.58}$$

This inequality completes the proof of lemma. □

Put now $v_0' := v_0'(z) = \frac{v_0}{\sqrt{\gamma}}$, where $\gamma := 2 - |u|$ and $z = u + iv$. Denote $\widehat{\mathcal{D}} := \{z \in \mathcal{D} : v \ge v_0'\}$.

Corollary 3.6. *Assume that for* $\omega \in \Omega_n^*$ *and* $z = u + iv \in \widehat{\mathcal{D}}$

$$|g_n(z)| \le \frac{1}{2}.$$

Then

$$|g_n(z)| \le \frac{1}{100}.$$

Proof. Note that for $v \ge v_0'$

$$\frac{C\beta_n^2}{nv} + \frac{C\beta_n^2}{n^2v^2\sqrt{\gamma+v}} \le \frac{1}{100} \tag{3.59}$$

Thus, the Corollary is proved. □

Corollary 3.7. *Let* $z = u + iv \in \mathcal{D}$. *Assume that*

$$|z + g_n(z)| > \frac{1}{2}. \tag{3.60}$$

Then for any $\omega \in \Omega_n$, *the following bound holds*

$$|g_n(z)| \le \frac{C\beta_n^2}{nv} + \frac{C\beta_n^2}{n^2v^2\sqrt{\gamma+v}}.$$

Proof. In the proof of Lemma 3.5 we have only used condition (3.51) of Lemma 3.5 to prove inequality (3.60). This proves the Corollary. □

Assume that N_0 is sufficiently large number such that for any $n \ge N_0$ and for any $v \in \mathcal{D}$ the right hand side of inequality (3.58) is smaller then $\frac{1}{100}$. In the what follows we shall assume that $n \ge N_0$ is fixed. The following lemma plays a crucial role in our proof. It is similar to Lemma 3.4 in [8].

Lemma 3.8. *Assume that for some* $z = u + iv \in \mathcal{D}$ *with* $v \ge v_0$ *condition* (3.51) *holds. Then it holds for* $z' = u + iv' \in \mathcal{D}$ *with* $v \ge v' \ge v - n^{-8}$.

Proof. First of all note that

$$|m_n(z) - m_n(z')| = \frac{1}{n}(v - v')|\operatorname{Tr}\mathbf{R}(z)\mathbf{R}(z')| \le \frac{v - v'}{vv'} \le \frac{C}{n^4} \le \frac{1}{100} \tag{3.61}$$

and

$$|s(z) - s(z')| \le \frac{1}{100} \tag{3.62}$$

By Corollary 3.7, we have

$$|g_n(z)| \le \frac{1}{100}. \tag{3.63}$$

All these inequalities together imply

$$|g_n(z')| \le \frac{3}{100} < \frac{1}{2}. \tag{3.64}$$

Thus, the Lemma is proved.

□

Proposition 3.1. *There exists positive constants C, c, depending on α and $\varkappa$ only such that*

$$\Pr\left\{|g_n(z)| > \frac{\beta_n^2(\operatorname{Im} m_n + \frac{1}{nv})^{\frac{1}{2}}}{n\sqrt{v}\sqrt{\gamma+v}}\right\} \le C\exp\{-cl_{n,\alpha}\}. \tag{3.65}$$

for all $z \in \mathcal{D}$

Proof. Note that for $v = 2$ we have, for any $\omega \in \Omega_n^*$,

$$|z + m_n(z)| \ge \operatorname{Im}(z + m_n(z)) \ge 2 \ge \frac{1}{2}. \tag{3.66}$$

By Lemma 3.5, we obtain inequality (3.65) for $v = 2$. By Lemma 3.8, this inequality holds for any v with $v_0 \le v \le 2$ as well. Thus Proposition 3.1 is proved.

□

4 Proof of Theorem 1.1

To conclude the proof of Theorem 1.1 we modify the bound of the Kolmogorov distance of the spectral distribution functions via Stieltjes transforms obtained in [7] Lemma 2.1. Given $\varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon = [-2+\varepsilon, 2-\varepsilon]$ and $\mathbb{J}'_\varepsilon = [-2+\frac{1}{2}\varepsilon, 2-\frac{1}{2}\varepsilon]$ For any $x \in \mathbb{J}_\varepsilon$ define $\gamma = \gamma(x) := 2 - |x|$. For any distribution function F denote by $S_F(z)$ its Stieltjes transform,

$$S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF(x).$$

Proposition 4.1. *Let $v > 0$ and a and $\varepsilon > 0$ be positive numbers such that*

$$\alpha = \frac{1}{\pi}\int_{|u|\le a} \frac{1}{u^2+1} du = \frac{3}{4}, \tag{4.1}$$

and

$$2va \le \varepsilon\sqrt{\gamma}. \tag{4.2}$$

If G denotes the distribution function of the standard semi-circular law, and F is any distribution function, there exists some absolute constants C_1, C_2 and C_3 such that

$$\begin{aligned}\Delta(F,G) &:= \sup_x |F(x) - G(x)| \\ &\le \sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+i\frac{v}{\sqrt{\gamma}}) - S_G(u+i\frac{v}{\sqrt{\gamma}}))du\right| + C_2 v + C_3\varepsilon^{\frac{3}{2}}.\end{aligned} \tag{4.3}$$

Proof. The proof of Proposition 4.1 is straightforward adaptation of the proof of Lemma 2.1 from [7]. We include it here for the sake of completeness. Note that

$$\sup_x |F(x)-G(x)| \le \sup_{x\in\mathbb{J}_\varepsilon} |F(x)-G(x)| + G(-2+\varepsilon), \tag{4.4}$$

and

$$G(-2+\varepsilon) \le C\varepsilon^{\frac{3}{2}}. \tag{4.5}$$

Let $x\in\mathbb{J}_\varepsilon$. Then according to condition (4.2) $x+\frac{va}{\sqrt{\gamma}}\in\mathbb{J}'_\varepsilon$. Denote by $v'=\frac{v}{\sqrt{\gamma}}$. For any $x\in\mathbb{J}'_\varepsilon$

$$\begin{aligned}
&\Big|\frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\Big)\Big| \\
&\quad\ge \frac{1}{\pi}\mathrm{Im}\Big(\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\Big) \\
&\quad= \frac{1}{\pi}\Big[\int_{-\infty}^{\infty}\frac{v'd(F(y)-G(y))}{(y-u)^2+v'^2}\Big]du \\
&\quad= \frac{1}{\pi}\int_{-\infty}^{x}\Big[\int_{-\infty}^{\infty}\frac{2v'(y-u)(F(y)-G(y))dy}{((y-u)^2+v'^2)^2}\Big] \\
&\quad= \frac{1}{\pi}\int_{-\infty}^{\infty}(F(y)-G(y))\Big[\int_{-\infty}^{x}\frac{2v'(y-u)du}{((y-u)^2+v'^2)^2}dy\Big] \\
&\quad= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(F(x-v'y)-G(x-v'y))dy}{y^2+1}.
\end{aligned} \tag{4.6}$$

Since F is non decreasing, we have

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(F(x-v'y)-G(x-v'y))dy}{y^2+1} \\
&\qquad\ge \alpha(F(x-v'a)-G(x-v'a)) - \frac{1}{\pi}\int_{|y|\le a}|G(x-v'y)-G(x-v'a))|dy \\
&\qquad\ge \alpha(F(x-v'a)-G(x-v'a)) - \frac{1}{v'\pi}\int_{|y|\le v'a}|G(x-y)-G(x-v'a))|dy.
\end{aligned} \tag{4.7}$$

Denote by $\Delta_\varepsilon(F,G)=\sup_{x\in\mathbb{J}_\varepsilon}|F(x)-G(x)|$. Let $x_n\in\mathbb{J}_\varepsilon$ such that $F(x_n)-G(x_n)\to\Delta_\varepsilon(F,G)$. Then $x_n=x_n+v'a\in\mathbb{J}'_\varepsilon$. We have

$$\begin{aligned}
\sup_{x\in\mathbb{J}'_\varepsilon}\Big|\mathrm{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\Big| &\ge \alpha(F(x_n)-G(x_n)) \\
-\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\frac{1}{\sqrt{\gamma}}\int_{|y|<2v'a}&|G(x+y)-G(x)|dy-(1-\alpha)\Delta(F,G).
\end{aligned} \tag{4.8}$$

Note that

$$\frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\frac{1}{\sqrt{\gamma}}\int_{|y|<2v'a}|G(x+y)-G(x)|dy \\ \leq \frac{1}{\pi v}\sup_{x\in\mathbb{J}'_\varepsilon}\frac{1}{\sqrt{\gamma}}\sqrt{4-x^2}\leq Cv. \tag{4.9}$$

Inequalities (4.4), (4.8) and (4.9) together imply

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\operatorname{Im}\int_{-\infty}^{x}(S_F(u+iv')-S_G(u+iv'))du\right|\geq(2\alpha-1)\Delta_\varepsilon(F,G)-Cv-C\varepsilon^{\frac{3}{2}}. \tag{4.10}$$

Similar arguments may be used for the sequence $x_n\in\mathbb{J}_\varepsilon$ such $F(x_n)-G(x_n)\to -\Delta_\varepsilon(F,G)$. This completes the proof. □

Corollary 4.1. *Under the conditions of Proposition 4.1, for any $V>v$, the following inequality holds*

$$\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{-\infty}^{x}(\operatorname{Im}(S_F(u+iv')-S_G(u+iv'))du\right| \\ \leq\int_{-\infty}^{\infty}|S_F(u+iV)-S_G(u+iV)|du \\ +\sup_{x\in\mathbb{J}'_\varepsilon}\left|\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))\,du\right|. \tag{4.11}$$

Proof. Put $z=u+iv'$. $v'\leq 2$. Since the functions of $S_F(z)$ and $S_G(z)$ are analytic in the upper half-plane, it is enough to use Cauchy's theorem. We can write

$$\int_{-\infty}^{\infty}\operatorname{Im}(S_F(z)-S_G(z))du=\lim_{L\to\infty}\int_{-L}^{x}(S_F(u+iv')-S_G(u+iv'))du. \tag{4.12}$$

Since $v'=\frac{v}{\sqrt{\gamma}}\leq\frac{\varepsilon}{2a}$, we may assume without loss of generality that $v'\leq 2$. By Cauchy's integral formula, we have

$$\int_{-L}^{x}(S_F(z)-S_G(z))du=\int_{-L}^{x}(S_F(u+iV)-S_G(u+iV))du \\ +\int_{v'}^{V}(S_F(-L+iu)-S_G(-L+iu))du \\ -\int_{v'}^{V}(S_F(x+iu)-S_G(x+iu))du. \tag{4.13}$$

Denote by $\xi(\eta)$ a random variable with distribution function $F(x)\,(G(x))$. Then we have

$$|S_F(-L+iv')|=\left|\mathbf{E}\frac{1}{\xi+L-iv'}\right|\leq v'^{-1}\Pr\{|\xi|>L/2\}+\frac{2}{L}. \tag{4.14}$$

Similarly,

$$|S_G(-L+iv')| \le {v'}^{-1}\Pr\{|\eta|>L/2\} + \frac{2}{L}. \tag{4.15}$$

These inequalities imply that

$$\left|\int_{v'}^{V} (S_F(-L+iu) - S_G(-L+iu))du\right| \to 0 \quad \text{as} \quad L\to\infty, \tag{4.16}$$

which completes the proof. □

Combining the results of Proposition 4.1 and Corollary 4.1, we get

Corollary 4.2. *Under the conditions of Proposition 4.1 the following inequality holds*

$$\begin{aligned}\Delta(F,G) \le 7C_1\int_{-\infty}^{\infty} |S_F(u+iV) - S_G(u+iV)|du + C_2 v + C_3\varepsilon^{\frac{3}{2}}\\ + C_1\sup_{x\in\mathbb{J}'_\varepsilon}\int_{v'}^{V}|S_F(x+iu)-S_G(x+iu)|du,\end{aligned} \tag{4.17}$$

where $v' = \frac{v}{\sqrt{\gamma}}$ *with* $\gamma = 2-|x|$.

We shall now apply the result of Corollary 4.2 to the empirical spectral distribution function $\mathcal{F}_n(x)$ of the random matrix $\mathbf{X}$. At first we bound the integral over the line $V=2$. Note that in this case we have $|z+m_n(z)|\ge 1$. Moreover, $\operatorname{Im} m_n^{(j)}(z) \le \frac{1}{V} \le \frac{1}{2}$. We may now apply the results of the previous Lemmas regarding large deviation probabilities. This implies the following bound for $g_n(z)$ for all $z=u+iV$ with $u\in\mathbb{R}$.

$$\begin{aligned}|g_n(z)| \le \frac{\beta_n}{n\sqrt{V}|z+m_n(z)||z+s(z)+m_n(z)|}\Big(1+\frac{\beta_n \operatorname{Im} m_n(z)}{\sqrt{V}} + \frac{\beta_n}{nV\sqrt{V}}\Big)\\ + \frac{C\operatorname{Im} m_n(z)}{nV|z+s(z)+m_n(z)|}.\end{aligned} \tag{4.18}$$

Note that for $V=2$

$$|z+m_n(z)||z+m_n(z)+s(z)| \ge \begin{cases} 4 & \text{for} \quad |u|\le 2,\\ \frac{1}{4}|z|^2 & \text{for} |u|>2.\end{cases} \tag{4.19}$$

We may rewrite the bound (4.18) as follows

$$|g_n(z)| \le \frac{C\beta_n^2}{n(|z|^2+1)} + \frac{C\operatorname{Im} m_n(z)}{nV}. \tag{4.20}$$

Note that for any distribution function $F(x)$ we have

$$\int_{-\infty}^{\infty} \operatorname{Im} s_F(u+iv)du \le \pi \tag{4.21}$$

From here it follows that, for $V = 2$

$$\int_{|u|\geq n} |m_n(z) - s(z)|du \leq \frac{C}{n} \tag{4.22}$$

Denote $\overline{\mathcal{D}}_n := \{z = u + 2i : |u| \leq n\}$ and

$$\overline{\Omega}_n := \Big\{\cap_{z\in\overline{\mathcal{D}}_n} \{\omega \in \Omega : \ |g_n(z)| \leq \frac{C\beta_n^2}{n(|z|^2+1)}\}\Big\} \cap \Omega_n^*$$

Using a union bound, we may show that

$$\Pr\{\overline{\Omega}_n\} \geq 1 - C\exp\{-cl_{n,\alpha}\}. \tag{4.23}$$

It is straightforward to check that for $\omega \in \overline{\Omega}_n$

$$\int_{-\infty}^{\infty} |m_n(z) - s(z)|du \leq \frac{C}{n} \tag{4.24}$$

We put $\varepsilon = n^{-\frac{2}{3}}$ and $v_0 = \frac{\beta_n^2}{n}$. To conclude the proof we need to consider the "vertical" integrals for $z = x + iv'$ with $x \in \mathbb{J}'_\varepsilon$, $v' = \frac{v_0}{\sqrt{\gamma}}$ and $\gamma = 2 - |x|$. It is enough to consider one of these integrals only. For example

$$\int_{v'}^{2} \frac{1}{n^2v^2\sqrt{\gamma+v}}dv \leq \frac{1}{n^2v'\sqrt{\gamma}} \leq \frac{1}{n^2v_0} \leq \frac{\beta_n^2}{n}. \tag{4.25}$$

Finally, we get for any $\omega \in \overline{\Omega}_n$

$$\Delta(F_n, G) = \sup_x |F_n(x) - G(x)| \leq \frac{\beta_n^2}{n}. \tag{4.26}$$

Thus Theorem 1.1 is proved.

5 Proof of Theorem 1.2

We may express the diagonal entries of the resolvent matrix $\mathbf{R}$ as follows

$$R_{jj} = \sum_{k=1}^{n} \frac{1}{\lambda_k - z}|u_{jk}|^2. \tag{5.1}$$

Consider the distribution function, say $F_{nj}(x)$, of the probability distribution of the eigenvalues λ_k

$$F_{nj}(x) = \sum_{k=1}^{n} |u_{jk}|^2 \mathbb{I}\{\lambda_k \leq x\}. \tag{5.2}$$

Then we have

$$R_{jj}=R_{jj}(z)=\int_{-\infty}^{\infty}\frac{1}{x-z}dF_{nj}(x). \tag{5.3}$$

which means that R_{jj} is the Stieltjes transform of the distribution $F_{nj}(x)$. Note that, for any $\lambda>0$

$$\max_{1\le k\le n}|u_{jk}|^2\le\sup_x(F_{nj}(x+\lambda)-F_{nj}(x))=:Q_{nj}(\lambda). \tag{5.4}$$

On the other hand side, it is easy to check that

$$Q_{nj}(\lambda)\le 2\sup_u\lambda\operatorname{Im}R_{jj}(u+i\lambda). \tag{5.5}$$

By relations (2.23) and (2.26), we obtain for any $v\ge v_0$ with $v_0=\frac{c\beta_n}{n}$ with a sufficiently small constant c,

$$\Pr\Big\{\frac{|\varepsilon_j|}{|z+m_n(z)|}\le\frac{1}{2}\Big\}\le\exp\{-cl_{n,\alpha}\}. \tag{5.6}$$

Furthermore, the representation (2.2) and inequality (5.6) together imply, for $v\ge v_0$,

$$\operatorname{Im}R_{jj}\le|R_{jj}|\le C_1 \tag{5.7}$$

with some positive constant $C_1>0$ depending on $\varkappa$, α. This implies that

$$\Pr\Big\{\max_{1\le k\le n}|u_{jk}|^2\le\frac{\beta_n}{n}\Big\}\le C\exp\{-cl_{n,\alpha}\}. \tag{5.8}$$

By a union bound we arrive at the inequality (1.6). To prove inequality (1.7), we consider the quantity

$$r_j:=R_{jj}-s(z). \tag{5.9}$$

Using equalities (2.2) and (3.39), we get

$$r_j=-\frac{s(z)g_n(z)}{z+m_n(z)}+\frac{\varepsilon_j}{z+m_n(z)}R_{jj}. \tag{5.10}$$

By inequalities (3.65) and (2.26), we have

$$|r_j|\le\frac{c\beta_n}{\sqrt{nv}}. \tag{5.11}$$

From here it follows that

$$\sup_{x\in\mathbb{J}_\varepsilon}\int_{v'}^{V}|r_j(x+iv)|dv\le\frac{C}{\sqrt{n}}. \tag{5.12}$$

Similar to (4.24) we get

$$\int_{-\infty}^{\infty}|r_j(x+iV)|dx\le\frac{C}{\sqrt{n}}. \tag{5.13}$$

Applying Corollary 4.2, we get

$$\Pr\{\sup_x |F_{nj}(x) - G(x)| \le \frac{\beta_n}{\sqrt{n}}\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.14}$$

Using now that

$$\Pr\Big\{\sup_x |F_n(x) - G(x)| \le \frac{\beta_n^2}{n}\Big\} \ge 1 - C\exp\{-cl_{n,\alpha}\}, \tag{5.15}$$

we get

$$\Pr\Big\{\sup_x |F_{nj}(x) - G(x)| \le \frac{\beta_n}{\sqrt{n}}\Big\} \ge 1 - C\exp\{-cl_{n,\alpha}\}. \tag{5.16}$$

Thus, Theorem 1.2 is proved.

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