THE CHARACTERISTIC POLYNOMIAL OF A RANDOM PERMUTATION MATRIX AT DIFFERENT POINTS

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Abstract. We consider the logarithm of the characteristic polynomial of random permutation matrices, evaluated on a finite set of different points. The permutations are chosen with respect to the Ewens distribution on the symmetric group. We show that the behavior at different points is independent in the limit and are asymptotically normal. Our methods enables us to study more general matrices, closely related to permutation matrices, and multiplicative class functions.

Contents

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1. Introduction

The characteristic polynomial of a random matrix is a well studied object in Random Matrix Theory (RMT) (see for example [4], [5], [11], [9], [12], [10], [17], [18]). By a central limit theorem result of Keating and Snaith for $n \times n$ CUE matrices [12], the imaginary and the real part of the logarithm of the characteristic polynomial converge jointly in law to independent standard normal distributed random variables, after normalizing by $\sqrt{(1/2)\log n}$. Hughes, Keating and O'Connell refined this result in [11]: evaluating the logarithm of the characteristic polynomial, normalized by $\sqrt{(1/2)\log n}$, for a discrete set of points on the unit circle, this leads to a collection of i.i.d. standard (complex) normal random variables.

In [9], Hambly, Keevash, O'Connell and Stark give a Gaussian limit for the logarithm of the characteristic polynomial of random permutation matrices under uniform measure on the symmetric group. This result has been extended by Zeindler in [18] to the Ewens distribution on the symmetric group and to the logarithm of multiplicative class functions, introduced in [6].

In this paper, we will generalize the results in [9] and [18] in two ways. First, we follow the spirit of [11] by considering the behavior of the logarithm of the characteristic polynomial of a random permutation matrix at different points $x_1, \dots, x_d$. Second, we state CLT's for the logarithm of characteristic polynomials for matrix groups related to permutation matrices, such as some Weyl groups [6, section 7] and of the wreath product $\mathbb{T} \wr S_n$ [16] .
In particular, we consider $n \times n$-matrices $M = (M_{ij})_{1 \le i,j \le n}$ of the following form: For a permutation $\sigma \in S_n$ and a complex valued random variable z,

$$M_{ij}(\sigma, z) := z_i \delta_{i,\sigma(j)}, \tag{1.1}$$

where z_i is a family of i.i.d. random variables s.t. $z_i \stackrel{d}{=} z$, z_i independent of σ. Here, σ is chosen with respect to the Ewens distribution, i.e.

$$\mathbb{P}_\theta [\sigma] := \frac{\theta^{l_\sigma}}{\theta(\theta+1)\dots(\theta+n-1)}, \tag{1.2}$$

for fixed parameter $\theta > 0$ and l_σ being the total number of cycles of σ. The *Ewens measure* or *Ewens distribution* is a well-known measure on the the symmetric group S_n, appearing for example in population genetics [8]. It can be viewed as a generalization of the uniform distribution (i.e $\mathbb{P}[A] = \frac{|A|}{n!}$) and has an additional weight depending on the total number of cycles. The case $\theta = 1$ corresponds to the uniform measure. Matrices $M_{\sigma,z}$ of the form (1.1) can be viewed as generalized permutation matrices $M_\sigma = M_{\sigma,1}$, where the 1-entries are replaced by i.i.d. random variables. Also, it is easy to see that elements of the wreath product $\mathbb{T} \wr S_n$ for $z \in \mathbb{T}^n$ (see [16] and [6, section 4.2]) or elements of some Weyl groups (treated in [6, section 7]) are of the form (1.1). In this paper, we will not give any more details about wreath products and Weyl groups, since we do not use group structures.

We define the function $Z_{n,z}(x)$ by

$$Z_{n,z}(x) := \det(I - x^{-1} M_{\sigma,z}), \quad x \in \mathbb{C}. \tag{1.3}$$

Then, the characteristic polynomial of $M_{\sigma,z}$ has the same zeros as $Z_{n,z}(x)$. We will study the characteristic polynomial by identifying it with $Z_{n,z}(x)$, following the convention of [6], [17] or [18].
By using that the random variables z_i, $1 \leq i \leq n$ are i.i.d., a simple computation shows the following equality in law (see [6], Lemma 4.2):

$$Z_{n,z}(x) \stackrel{d}{=} \prod_{m=1}^{n} \prod_{k=1}^{C_m} (1 - x^{-m} T_{m,k}), \tag{1.4}$$

where C_m denotes the number of cycles of length m in σ and $(T_{m,k})_{1\leq m,k\leq\infty}$ is a family of independent random variables, independent of $\sigma \in S_n$, such that

$$T_{m,k} \stackrel{d}{=} \prod_{j=1}^{m} z_j. \tag{1.5}$$

Note that the characteristic polynomial $Z_{n,z}(x)$ of $M_{\sigma,z}$ depends strongly on the random variables C_m $(1 \leq m \leq n)$. The distribution of $(C_1, C_2, \cdots, C_n)$ with respect to the Ewens distribution with parameter θ was first derived by Ewens (1972), [8]. It can be computed, using the inclusion-exclusion formula, [2, chapter 4, (4.7)].

We are interested in the asymptotic behavior of the logarithm of (1.3) and therefore, we will study the characteristic polynomial of $M_{\sigma,z}$ in terms of (1.4), by choosing the branch of logarithm in a suitable way. In view of (1.4), it is natural to choose it as follows:

Definition 1.1. *Let $x = e^{2\pi i\varphi} \in \mathbb{T}$ be a fixed number and z a $\mathbb{T}$–valued random variable. Furthermore, let $(z_{m,k})_{m,k=1}^{\infty}$ and $(T_{m,k})_{m,k=1}^{\infty}$ be two sequences of independent random variables, independent of $\sigma \in S_n$ with*

$$z_{m,k} \stackrel{d}{=} z \quad \text{and} \quad T_{m,k} \stackrel{d}{=} \prod_{j=1}^{m} z_{j,k}. \tag{1.6}$$

We then set

$$\log\bigl(Z_{n,z}(x)\bigr) := \sum_{m=1}^{n} \sum_{k=1}^{C_m} \log(1 - x^{-m} T_{m,k}), \tag{1.7}$$

where we use for $\log(.)$ *the principal branch of logarithm. We will deal with negative values as follows:* $\log(-y) = \log y + i\pi$, $y \in \mathbb{R}_+$. *Note, that it is not necessary to specify the logarithm at* 0, *since we will deal only with cases where this occurs with probability* 0.

In this paper, we show that under various conditions, $\log Z_{n,z}(x)$ converges to a complex standard Gaussian distributed random variable after normalization and the behavior at different points is independent in the limit. Moreover, the normalization by $\sqrt{(\pi^2/12)\theta \log n}$ is independent of the random variable z. This covers the result in [9] for $\theta = 1$ and z being deterministic equal to 1. We state this more precisely:

Proposition 1.1. *Let S_n be endowed with the Ewens distribution with parameter θ, z a $\mathbb{T}$-valued random variable and $x \in \mathbb{T}$ be not a root of unity, i.e. $x^m \neq 1$ for all $m \in \mathbb{Z}$.*

Suppose that z is uniformly distributed. Then, as $n \to \infty$,

$$\frac{\mathrm{Re}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \xrightarrow{d} N_R \quad \text{and} \tag{1.8}$$

$$\frac{\mathrm{Im}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \xrightarrow{d} N_I, \tag{1.9}$$

with $N_R, N_I \sim \mathcal{N}(0,1)$.

In Proposition 1.1 $\mathrm{Re}\left(\log\left(Z_{n,z}(x)\right)\right)$ and $\mathrm{Im}\left(\log\left(Z_{n,z}(x)\right)\right)$ are converging to normal random variables without centering. This is due to that the expectation is $o(\sqrt{\log n})$. This will become more clear in the proof (see Section 4.1).
Furthermore, we state a CLT for $\log Z_{n,z}(x)$, evaluated on a finite set of different points $\{x_1, \dots, x_d\}$.

Proposition 1.2. *Let S_n be endowed with the Ewens distribution with parameter θ, $\overline{z} = (z_1, \dots, z_d)$ be a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \dots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*
Suppose that $z_1, \dots, z_d$ are uniformly distributed and independent. Then we have, as $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log\left(Z_{n,z_1}(x_1)\right) \\ \vdots \\ \log\left(Z_{n,z_d}(x_d)\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \dots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \dots, \mathrm{Im}(N_d)$ independent standard normal distributed random variables.

Note that $z_1, \dots, z_d$ are not equal to the family $(z_i)_{1 \le i \le n}$ of i.i.d. random variables in (1.1). In fact, we deal here with d different families of i.i.d. random variables, where the distributions are given by $z_1, \dots, z_d$. We will treat this more carefully in Section 4.2.

Proposition 1.2 shows that the characteristic polynomial of the random matrices $M_{\sigma,z}$ follows the tradition of matrices in the CUE, if evaluated at different points, due to the result by [11]. Moreover, the proof of Proposition 1.2 can also be used for regular random permutation matrices, i.e. $M_{\sigma,1}$, but requires further assumptions on the points $x_1, \dots, x_d$. We state this more precisely:

Proposition 1.3. *Let S_n be endowed with the Ewens distribution with parameter θ and $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be pairwise of finite type (see Definition 2.18). Then, as $n \to \infty$,*

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log\left(Z_{n,1}(x_1)\right) \\ \vdots \\ \log\left(Z_{n,1}(x_d)\right) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \dots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \dots, \mathrm{Im}(N_d)$ independent standard normal distributed random variables.

In fact, our methods allow us to prove much more. First, we are able to relax the conditions in the Propositions 1.1, 1.2 and 1.3 above. Also, these results on $\log Z_{n,z}(x)$ follow as corollaries of much more general statements (see Section 4). Indeed, the methods allow us to prove CLT's for multiplicative class functions. Multiplicative class functions have been studied by Dehaye and Dehaye-Zeindler, [6], [18].

Following [6], we present here two different types of multiplicative class functions.

Definition 1.2. *Let z be a complex valued random variable and $f : \mathbb{C} \to \mathbb{C}$ be given. Write for $\sigma \in S_n$*

$$W^1(f)(x) = W_z^{1,n}(f)(x)(\sigma) := \prod_{m=1}^{n} \prod_{k=1}^{C_m} f\left(z_m x^m\right), \tag{1.10}$$

where $z_m \overset{d}{=} z$, z_m i.i.d. and independent of σ. This defines the first multiplicative class function associated to f.

The second multiplicative class function is directly motivated by the expression (1.4) and is a slightly modified form of (1.10).

Definition 1.3. *Let z be a complex valued random variable and $f : \mathbb{C} \to \mathbb{C}$ be given. Write for $\sigma \in S_n$*

$$W^2(f)(x) = W_z^{2,n}(f)(x)(\sigma) := \prod_{m=1}^{n} \prod_{k=1}^{C_m} f\left(x^m T_{m,k}\right), \tag{1.11}$$

where $T_{m,k}$ is a family of independent random variables, $T_{m,k} \overset{d}{=} \prod_{j=1}^{m} z_j$ and $z_j \overset{d}{=} z$, for any $1 \leq j \leq n$. This defines the second multiplicative class function associated to f.

It is obvious from (1.4) and (1.11) that $Z_{n,z}(x)$ is the special case $f(x) = 1 - x^{-1}$ of $W^2(f)(x)$. This explains, why results on the second multiplicative class function cover in general results on $\log Z_{n,z}(x)$.

We postpone the statements of the more general theorems on multiplicative class functions to Section 4.

For the proofs we will make use of similar tools as in [9] and [18]. These tools include the Feller Coupling, uniformly distributed sequences and Diophantine approximations.

The structure of this paper is as follows: In Section 2, we will give some background of the Feller Coupling. Moreover, we recall some basic facts on uniformly distributed sequences and Diophantine approximations. In Section 3, we state some auxiliary CLT's on the symmetric group, which we will use in Section 4 to prove our main results for the characteristic polynomials and more generally, for multiplicative class functions.

2. Preliminaries

2.1. **The Feller coupling.** The reason why we expand the characteristic polynomial of $M_{\sigma,z}$ in terms of the cycle counts of σ as given in (1.4) is the fact that the asymptotic behavior of the numbers of cycles with length m in σ, denoted by $(C_m)_{1\leq m\leq n}$, has been well-studied, for example by [2] or [8]. In particular, they are in the limit independent Poisson random variables with mean θ/m, $m \geq 1$. To state this more precisely, we make use of the Feller coupling (see for instance [2], [8], [15]), which links the family of cycle lengths with the family of Poisson random variables.

Let D_i, for $i \geq 1$, be independent random variables s.t. for $\theta > 1$

$$\mathbb{P}[D_i = 1] = \frac{\theta}{\theta + i - 1}, \qquad \mathbb{P}[D_i = j] = \frac{1}{\theta + i - 1}, \quad 2 \leq j \leq i.$$

$D_i = 1$ corresponds to starting a new cycle $(i\ldots$, whereas we put i after $j-1$ in the given cycle for $D_i = j$ $(2 \leq j \leq N)$, proceeding in order $i = 1$, $i = 2$ until $i = n$. Then the sequence

$$D_1 D_2 \ldots D_n$$

produces a permutation in $\mathcal{S}_n$ under the Ewens distribution P_θ, defined in (1.2).

There is even a one-to-one correspondence between S_n and sequences $(D_i)_{1\leq i\leq n}$. For given permutation σ in ordered cycle type, we set $D_i = 1$ if i is the first number in a cycle and $D_i = j$ if $j-1$ is the next number in front of i in the increasing subsequence of the cycle.

Definition 2.1. *Let $\xi_i = 1_{\{D_i=1\}}$ be independent Bernoulli random variables for $i \geq 1$ with*

$$\mathbb{P}\left[\xi_i = 1\right] = \frac{\theta}{\theta + i - 1} \quad \text{and} \quad \mathbb{P}\left[\xi_i = 0\right] = \frac{i-1}{\theta + i - 1}.$$

Define $C_m^{(n)}(\xi)$ to be the number of m-spacings in $1\xi_2\cdots\xi_n 1$ and $Y_m(\xi)$ to be the number of m-spacings in the limit sequence, i.e.

$$C_m^{(n)}(\xi) = \sum_{i=1}^{n-m} \xi_i(1-\xi_{i+1})\ldots(1-\xi_{i+m-1})\xi_{i+m} + \xi_{n-m+1}(1-\xi_{n-m+2})\ldots(1-\xi_n) \tag{2.1}$$

and

$$Y_m(\xi) = \sum_{i=1}^{\infty} \xi_i(1-\xi_{i+1})\ldots(1-\xi_{i+m-1})\xi_{i+m}. \tag{2.2}$$

Then the following theorem holds (see [2, Chapter 4, p. 87] and [1, Theorem 2]).

Theorem 2.2. *Under the Ewens distribution, we have that*

- *The above-constructed $C_m^{(n)}(\xi)$ has the same distribution as the variable $C_m^{(n)} = C_m$, the number of cycles of length m in σ.*
- *$Y_m(\xi)$ is a.s. finite and Poisson distributed with $\mathbb{E}\left[Y_m(\xi)\right] = \frac{\theta}{m}$.*
- *All $Y_m(\xi)$ are independent.*
- *For any fixed $b \in \mathbb{N}$,*

$$\mathbb{P}\left[(C_1^{(n)}(\xi), \cdots, C_b^{(n)}(\xi)) \neq (Y_1(\xi), \cdots, Y_b(\xi))\right] \to 0 \ (n \to \infty).$$

Furthermore, the distance between $C_m^{(n)}(\xi)$ and $Y_m(\xi)$ can be bounded from above (see for example [1], p. 525). We will give here the following bound (see [3], p. 15):

Lemma 2.3. *For any $\theta > 0$ there exists a constant $K(\theta)$ depending on θ, such that for every $1 \leq m \leq n$,*

$$\mathbb{E}_\theta\left[\left|C_m^{(n)}(\xi) - Y_m(\xi)\right|\right] \leq \frac{K(\theta)}{n} + \frac{\theta}{n}\Psi_n(m), \tag{2.3}$$

where

$$\Psi_n := \binom{n-m+\theta-1}{n-m}\binom{n+\theta-1}{n}^{-1}. \tag{2.4}$$

Note that Ψ_n satisfies the following equality:

Lemma 2.4. *For each $\theta > 0$, there exist some constants K_1 and K_2 such that*

$$\Psi_n(m) \leq \begin{cases} K_1(1-\frac{m}{n})^{\theta-1} & \text{for } m < n, \\ K_2 n^{1-\theta} & m = n. \end{cases} \tag{2.5}$$

Proof. Let $\gamma = \theta - 1$. It is well known (see [20], p.77) that

$$\lim_{k\to\infty} \frac{1}{k^\gamma}\binom{\gamma+k}{k} = \frac{1}{\Gamma(\theta)}. \tag{2.6}$$

Moreover,

$$\binom{\gamma+k}{k} = \frac{k^\gamma}{\Gamma(\theta)}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{2.7}$$

Then, the case where $m = n$ is clear. Consider the case $m < n$. By (2.6), there exist numbers $0 < a < A$ for an integer K_0 depending on θ such that, for $k \geq K_0$,

$$\frac{ak^\gamma}{\Gamma(\theta)} \leq \binom{\gamma+k}{k} \leq \frac{Ak^\gamma}{\Gamma(\theta)}. \tag{2.8}$$

Let $n \geq K_0$ and $\delta_n n \geq K_0$ for a sequence δ_n with $\delta_n n \to \infty$ as $n \to \infty$. Let $m \leq (1-\delta_n)n$, then by (2.8) we have

$$\binom{\gamma+n-m}{n-m}\binom{\gamma+n}{n}^{-1} \leq \frac{A}{a}\left(\frac{n-m}{n}\right)^\gamma. \tag{2.9}$$

This proves the claim. □

2.2. **Uniformly distributed sequences.** We introduce in this section uniformly distributed sequences and some of their properties. Most of this section is well-known. The only new result is Theorem 2.13, which is an extension of the Koksma-Hlawka inequality. For the other proofs (and statements), see the books by Drmota and Tichy [7] and by Kuipers and Niederreiter [13].

We begin by giving the definition of uniformly distributed sequences.

Definition 2.5. *Let $\boldsymbol{\varphi} = \left(\varphi^{(m)}\right)_{m=1}^\infty$ be a sequence in $[0,1]^d$. For $\overline{\alpha} = (\alpha_1, \ldots, \alpha_1) \in [0,1]^d$, we set*

$$A_n(\overline{\alpha}) = A_n(\overline{\alpha}, \boldsymbol{\varphi}) := \#\left\{1 \leq m \leq n; \varphi_m \in [0,\alpha_1] \times \cdots \times [0,\alpha_d]\right\}. \tag{2.10}$$

The sequence $\boldsymbol{\varphi}$ *is called uniformly distributed in* $[0,1]^d$ *if we have*

$$\lim_{n\to\infty}\left|\frac{A_n(\overline{\alpha})}{n}-\prod_{j=1}^{d}\alpha_j\right|=0 \text{ for any } \overline{\alpha}\in[0,1]^d. \tag{2.11}$$

The following theorem shows that the name uniformly distributed is well chosen.

Theorem 2.6. *Let* $h:[0,1]^d\to\mathbb{C}$ *be a proper Riemann integrable function and* $\boldsymbol{\varphi}=\left(\varphi^{(m)}\right)_{m\in\mathbb{N}}$ *be a uniformly distributed sequence in* $[0,1]^d$*. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}h(\varphi^{(m)})=\int\limits_{[0,1]^d}h(\overline{\phi})\;d\overline{\phi}, \tag{2.12}$$

where $d\overline{\phi}$ *is the* d*-dimensional Lebesgue measure.*

Proof. See [13, Theorem 6.1] □

Next, we introduce the discrepancy of a sequence $\boldsymbol{\varphi}$.

Definition 2.7. *Let* $\boldsymbol{\varphi}=\left(\varphi^{(m)}\right)_{m=1}^{\infty}$ *be a sequence in* $[0,1]^d$*. The* $*-$*discrepancy is defined as*

$$D_n^*=D_n^*(\boldsymbol{\varphi}):=\sup_{\overline{\alpha}\in[0,1]^d}\left|\frac{A_n(\overline{\alpha})}{n}-\prod_{j=1}^{d}\alpha_j\right|. \tag{2.13}$$

Remark There exists also a discrepancy D_n (without the $*$), which is equivalent to D_n^*, i.e. $D_n^*\le D_n\le 2^dD_n^*$. We need here only D_n^* and thus omit the definition of D_n.

By the following lemma, Theorem 2.6, the discrepancy and uniformly distributed sequences are closely related.

Lemma 2.8. *Let* $\boldsymbol{\varphi}=\left(\varphi^{(m)}\right)_{m=1}^{\infty}$ *be a sequence in* $[0,1]^d$*. The following statements are equivalent:*

(1) $\boldsymbol{\varphi}$ *is uniformly distributed in* $[0,1]^d$*.*
(2) $\lim_{n\to\infty}D_n^*(\boldsymbol{\varphi})=0$*.*
(3) *Let* $h:[0,1]^d\to\mathbb{C}$ *be a proper Riemann integrable function. Then*

$$\frac{1}{n}\sum_{m=1}^{n}h(\varphi^{(m)})\to\int\limits_{[0,1]^d}h(\overline{\phi})\;d\overline{\phi}\quad\text{for } n\to\infty.$$

The discrepancy allows us to estimate the rate of convergence in Theorem 2.6. To state this more precise, we introduce some notation.

Definition 2.9. *Let* $h:[0,1]^d\to\mathbb{C}$ *be a function. We call* h *of bounded variation in the sense of Hardy and Krause, if* h *is of bounded variation in the sense of Vitali and* h *restricted to each face* F *of dimension* $1,\dots,d-1$ *of* $[0,1]^d$ *is also of bounded variation in the sense of Vitali. We write* $V(h|F)$ *for the variation of* h *restricted to face* F*.*

Definition 2.10. *Let F be a face of $[0,1]^d$. We call a face F positive if there exists a sequence $j_1, \cdots, j_k$ in $\{1,\ldots,d\}$ s.t. $F = \bigcap_{m=1}^{k} \{s_{j_m} = 1\}$, with s_j, $1 \le j \le d$, being the canonical coordinates in $[0,1]^d$.*

Definition 2.11. *Let F be a face of $[0,1]^d$ and $\boldsymbol{\varphi}$ be sequence in $[0,1]^d$. Let $\pi_F(\boldsymbol{\varphi})$ be the projection of the sequence $\boldsymbol{\varphi}$ to the face F. We then write $D_n^*(F,\boldsymbol{\varphi})$ for the discrepancy of the projected sequence computed in the face F.*

We are now ready to state the following theorem:

Theorem 2.12 (Koksma-Hlawka inequality)**.** *Let $h : [0,1]^d \to \mathbb{C}$ be a function of bounded variation in the sense of Hardy and Krause. Let $\boldsymbol{\varphi} = \big(\varphi^{(m)}\big)_{m\in\mathbb{N}}$ be an arbitrary sequence in $[0,1]^d$. Then*

$$\left| \frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}) - \int\limits_{[0,1]^d} h(\overline{\phi})\ d\overline{\phi} \right| \le \sum_{k=1}^{d} \sum_{\substack{F\ positive \\ dim(F)=k}} D_n^*\big(F,\boldsymbol{\varphi}\big) V(h|F) \tag{2.14}$$

Proof. See [13, Theorem 5.5]. □

We will consider in this paper only functions of the form

$$h(\overline{\phi}) = h(\phi_1,\ldots,\phi_d) = \prod_{j=1}^{d} \log\big(f_j(e^{2\pi i \phi_j})\big), \tag{2.15}$$

with f_j being (piecewise) real analytic. In the context of the characteristic polynomial, we will choose $f_j(\phi_j) = |1 - e^{2\pi i\phi_j}|$. Unfortunately, we cannot apply Theorem 2.12 in this case, since $\log\big|1 - e^{2\pi i\phi_j}\big|$ is not of bounded variation. We thus reformulate Theorem 2.12. In order to do this, we follow the idea in [9] and [17] and replace $[0,1]^d$ by a slightly smaller set Q such that $\boldsymbol{\varphi} \subset Q$ and $h|Q$ is of bounded variation in the sense of Hardy and Krause.
We begin with the choice of Q. Considering (2.15), it is clear that the zeros of f_j cause problems. Thus, we choose Q such that f_j stays away from the zeros $(1 \le j \le d)$.
Let $a_{1,j} < \cdots < a_{k_j,j}$ be the zeros of f_j and define $a_{0,j} := 0$ and $a_{k_j+1,j} = 1$ (for $1 \le j \le d$). We then set for $\delta > 0$

$$Q := \bigcup_{\overline{q}\in\mathbb{N}^d} Q_{\overline{q}} \ \text{ with }\ Q_{\overline{q}} := \prod_{j=1}^{d} \big[a_{q_j,j}+\delta, a_{q_j+1,j}-\delta\big] \ \text{ and }\ \widetilde{Q}_{\overline{q}} := \prod_{j=1}^{d} \big[a_{q_j,j}, a_{q_j+1,j}\big]$$

Note that $\overline{q} = (q_1,\ldots,q_d) \in [0,\ldots,k_1+1] \times [0,\ldots,k_2+1] \times \cdots \times [0,\ldots,k_d+1]$ and we consider $Q_{\overline{q}}$ as empty if we have $q_j > k_j+1$ for any j. An illustration of possible Q is given in Figure 1.

We will now adjust the Definitions 2.9, 2.10 and 2.11. The modification of Definition 2.9 is obvious. One simple takes h to be of bounded variation in the sense of Hardy and Krause in each $Q_{\overline{q}}$. The modification of Definition 2.10 is also straight forward. We call a face F of Q positive if there exists a $\overline{q} \in \mathbb{N}^d$ and a sequence $j_1, \cdots, j_k$ in $\{1,\ldots,d\}$ such that, for s_j $(1 \le j \le d)$ being the canonical coordinates in $[0,1]^d$,

$$F = \bigcap_{m=1}^{k} \left(\{ s_{j_m} = a_{q_{j_m}, j_m} - \delta \} \cap Q_{\overline{q}} \right).$$

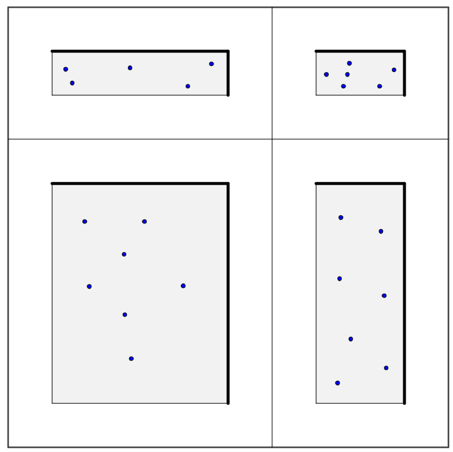

FIGURE 1. Illustration of Q, positive faces are bold

The modification of Definition 2.11 is slightly more tricky. Let F be a face of some $Q_{\overline{q}}$. Let $\boldsymbol{\varphi} \cap Q_{\overline{q}}$ be the subsequence of $\boldsymbol{\varphi}$ contained in $Q_{\overline{q}}$ and $\pi_F(\boldsymbol{\varphi} \cap Q_{\overline{q}})$ be the projection of $\boldsymbol{\varphi} \cap Q_{\overline{q}}$ to the face F. Unfortunately we cannot directly compute the discrepancy in the face F. We will see in the proof of Theorem 2.13 that we have to "extend F to the boundary of $\widetilde{Q}_{\overline{q}}$". More precisely this means to following: We set $\widetilde{F} := L \cap \widetilde{Q}_{\overline{q}}$, where L is the linear subspace containing F s.t. $\dim(L) = \dim(F)$ (see Figure 2 for an illustration). The discrepancy $D_n^*(F, \boldsymbol{\varphi})$ is then defined as the discrepancy of $\pi_F(\boldsymbol{\varphi} \cap Q_{\overline{q}})$ computed in $\widetilde{F}$.

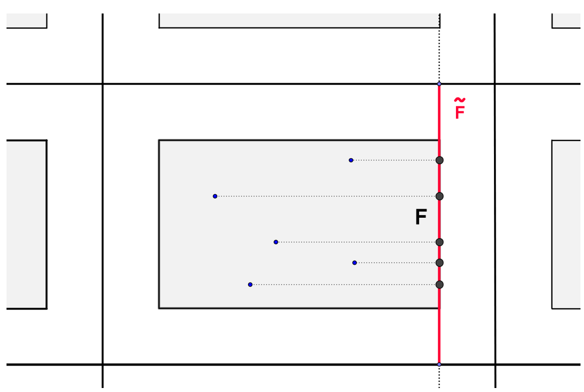

FIGURE 2. Illustration of $\widetilde{F}$

We are now ready to state an extended version of Theorem 2.12.

Theorem 2.13. *Let $\delta > 0$ be fixed and $\boldsymbol{\varphi} = (\varphi^{(m)})_{m=1}^{n}$ be a sequence in Q. Let $h : Q \to \mathbb{C}$ be a function of bounded variation in the sense of Hardy and Krause. We then have*

$$\left| \frac{1}{n} \sum_{m=1}^{n} h(\varphi^{(m)}) - \int_{Q} h(\overline{\phi})\ d\overline{\phi} \right| \leq \sum_{k=0}^{d-1} \delta^{d-k} \sum_{\substack{F \\ dim(F)=k}} \int_{F} h(\overline{\phi})\ dF(\overline{\phi}) \tag{2.16}$$

$$+ \sum_{k=1}^{d} \sum_{\substack{F\ positive \\ dim(F)=k}} D_n^*(F, \boldsymbol{\varphi}) V(h|F).$$

Proof for $d = 1$ and $d = 2$. We assume that $Q = [\delta, 1-\delta]^d$. The more general case can be proven in the same way.
The idea is to modify the proof of Theorem 2.12 in [13]. There are indeed only minor modifications necessary. We present here only the cases $d = 1$ and $d = 2$ since we only need these two cases.
$\underline{d = 1:}$ We consider the integral $I_1 = I_1(h) := \int_{\delta}^{1-\delta} \big(A_n(\phi) - \phi\big)\ dh(\phi)$ with $A_n(\phi)$ given as in Definition 2.5.
It is clear from the definition of $D_n^*(\boldsymbol{\varphi})$ that

$$\left| \int_{\delta}^{1-\delta} \big(A_n(\phi) - \phi\big)\ dh(\phi) \right| \leq D_n^*(\boldsymbol{\varphi}) \int_{\delta}^{1-\delta} |dh(\phi)| = D_n^*(\boldsymbol{\varphi}) V(h|[\delta, 1-\delta]). \tag{2.17}$$

On the other hand, one can use partial integration and partial summation to show that

$$I_1 = \big(\delta h(1-\delta) + \delta h(\delta)\big) + \int_{\delta}^{1-\delta} h(\phi)\ d\phi - \frac{1}{n} \sum_{m=1}^{n} h(\varphi^{(m)}). \tag{2.18}$$

This proves the theorem for $d = 1$.
$\underline{d = 2:}$ In this case we consider the integral $I_2 = \int_{[\delta,1-\delta]^2} \big(A_n(\phi_1, \phi_2) - \phi_1\phi_2\big)\ dh(\phi_1, \phi_2)$.
The argumentation is similar to the case $d = 1$. As above, it is immediate that I_2 is bounded by $D_n^*(\boldsymbol{\varphi}) V(h|[\delta, 1-\delta]^2)$. On the other hand, we get after consecutive partial integration

$$\int_{[\delta,1-\delta]^2} \phi_1\phi_2\ dh(\phi_1, \phi_2)$$
$$= \sum_{k=0}^{2} \sum_{\substack{F \\ \dim(F)=k}} \delta^{2-k} \int_{F} h\ dF - \sum_{\substack{F\ \text{positive} \\ \dim(F)=1}} \int_{F} h\ dF$$
$$+ h(1-\delta, 1-\delta) - 2\delta h(1-\delta, 1-\delta) - \delta h(\delta, 1-\delta) - \delta h(1-\delta, \delta) \tag{2.19}$$

and with two times partial summation

$$\int\limits_{[\delta,1-\delta]^2} A_n(\phi_1,\phi_2)\ dh(\phi_1,\phi_2)$$
$$= h(1-\delta,1-\delta) - \sum_{\substack{F\\ \dim(F)=1}} \frac{1}{n}\sum_{m=1}^{n} h\big(\pi_F(\varphi^{(m)})\big) + \frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}). \tag{2.20}$$

We now subtract (2.19) from (2.20) and expand the sum over the positive faces (with $\varphi^{(m)} = (\varphi_1^{(m)}, \varphi_2^{(m)})$). We get

$$I_2 = \left(\frac{1}{n}\sum_{m=1}^{n} h(\varphi^{(m)}) - \int\limits_{Q} h(\overline{\phi})\ d\overline{\phi}\right) - \sum_{k=1}^{2}\sum_{\substack{F\\ \dim(F)=k}} \delta^{2-k}\int\limits_{F} h\ dF \tag{2.21}$$

$$+ \left(\int\limits_{\delta}^{1-\delta} h(u,1-\delta)\ du - \frac{1}{n}\sum_{m=1}^{n} h(\varphi_1^{(m)},1-\delta) + \delta h(\delta,1-\delta) + \delta h(1-\delta,1-\delta)\right) \tag{2.22}$$

$$+ \left(\int\limits_{\delta}^{1-\delta} h(1-\delta,v)\ dv - \frac{1}{n}\sum_{m=1}^{n} h(1-\delta,\varphi_2^{(m)}) + \delta h(1-\delta,\delta) + \delta h(1-\delta,1-\delta)\right). \tag{2.23}$$

The brackets (2.22) and (2.23) agree with (2.18) if we set "$h(s) = h(1-\delta,s)$" in (2.22), respectively "$h(s) = h(s,1-\delta)$" in (2.23). We thus can interpret the brackets (2.22) and (2.23) as integrals over the positive faces of Q and apply the induction hypothesis ($d = 1$). A simple application of the triangle inequality proves the theorem for $d = 2$.
It is important to point out that the discrepancy of $(\varphi_1^{(m)})_{m=1}^n$ and $(\varphi_2^{(m)})_{m=1}^n$ is computed in $[0,1]$ and not in $[\delta,1-\delta]$. This observation is the origin for the definition of $D_n^*(F,\boldsymbol{\varphi})$ before Theorem 2.13.

□

In Section 4.2, we will consider sums of the form

$$\frac{1}{n}\sum_{m=1}^{n} \log\left(f_j\left(e^{2\pi i m\varphi_j}\right)\right)\log\left(f_\ell\left(e^{2\pi i m\varphi_\ell}\right)\right). \tag{2.24}$$

We are thus primary interested in (d-dimensional) sequences $\boldsymbol{\varphi}_{\mathrm{Kro}} = (\varphi_{\mathrm{Kro}}^{(m)})_{m=1}^\infty$, for given $\overline{\varphi} = (\varphi_1,\cdots,\varphi_d) \in \mathbb{R}^d$, defined as follows:

$$\varphi_{\mathrm{Kro}}^{(m)} = (\{m\varphi_1\},\ldots,\{m\varphi_d\}), \tag{2.25}$$

where $\{s\} := s - [s]$ and $[s] := \max\{n \in \mathbb{Z}, n \le s\}$. The sequence $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\mathrm{Kro}}$ is called *Kronecker-sequence of* $\overline{\varphi}$. The next lemma shows that the Kronecker-sequence is for almost all $\overline{\varphi} \in \mathbb{R}^d$ uniformly distributed.

Lemma 2.14. *Let* $\overline{\varphi} = (\varphi_1,\ldots,\varphi_d) \in \mathbb{R}^d$ *be given. The Kronecker-sequence of* $\boldsymbol{\varphi}$ *is uniformly distributed in* $[0,1]^d$ *if and only if* $1,\varphi_1,\ldots,\varphi_d$ *are linearly independent over* $\mathbb{Z}$.

Proof. See [7, Theorem 1.76] □

Our aim is to apply Theorem 2.12 and Theorem 2.13 for Kronecker sequences. We thus have to estimate the discrepancy in this case and find a suitable $\delta > 0$. We start by giving an upper bound for the discrepancy.

Lemma 2.15. *Let $\overline{\varphi} = (\varphi_1, \dots, \varphi_d) \in [0,1]^d$ be given with $1, \varphi_1, \dots, \varphi_d$ linearly independent over $\mathbb{Z}$. Let $\boldsymbol{\varphi}$ be the Kronecker sequence of $\overline{\varphi}$. We then have for each $H \in \mathbb{N}$*

$$D_n^*(\boldsymbol{\varphi}) \leq 3^d \left(\frac{2}{H+1} + \frac{1}{n} \sum_{0 < \|\overline{q}\|_\infty \leq H} \frac{1}{r(\overline{q}) \|\overline{q} \cdot \overline{\varphi}\|} \right) \tag{2.26}$$

with $\|.\|_\infty$ being the maximum norm, $\|a\| := \inf_{n \in \mathbb{Z}} |a - n|$ and $r(\overline{q}) = \prod_{i=1}^d \max\{1, q_i\}$ for $\overline{q} = (q_1, \cdots, q_d) \in \mathbb{N}^d$.

Proof. The proof is a direct application of the Erdös-Turán-Koksma inequality (see [7, Theorem 1.21]). □

It is clear that we can use Lemma 2.15 to give an upper bound for the discrepancy, if we can find a lower bound for $\|\overline{q} \cdot \overline{\varphi}\|$. The most natural is thus to assume that $\overline{\varphi}$ fulfills some diophantine equation. In order to state this more precise, we give the following definition:

Definition 2.16. *Let $\overline{\varphi} \in [0,1]^d$ be given. We call $\overline{\varphi}$ of finite type if there exist constants $K > 0$ and $\gamma \geq 1$ such that*

$$\|\overline{q} \cdot \overline{\varphi}\| \geq \frac{K}{(\|\overline{q}\|_\infty)^\gamma} \quad \text{for all } \overline{q} \in \mathbb{Z}^d \setminus \{0\}. \tag{2.27}$$

If $\overline{\varphi} = (\varphi_1, \dots, \varphi_d)$ is of finite type, then it follows immediately from the definition that each φ_j is also of finite type and the sequence $1, \varphi_1, \dots, \varphi_d$ is linearly independent over $\mathbb{Z}$.

One can now show the following:

Theorem 2.17. *Let $\overline{\varphi} \in [0,1]^d$ be of finite type and $\boldsymbol{\varphi}$ be the Kronecker sequence of $\overline{\varphi}$. Then*

$$D_n^*(\boldsymbol{\varphi}) = O(n^{-\alpha}) \quad \text{for some } \alpha > 0. \tag{2.28}$$

Proof. This theorem is a direct consequence of Lemma 2.15 and a simple computation. Further details can be found in [7, Theorem 1.80] or in [19]. □

As already mentioned above, we will consider in Section 4.2 sums of the form

$$\frac{1}{n} \sum_{m=1}^n \log\left(f_j\left(e^{2\pi i m \varphi_j}\right)\right) \log\left(f_\ell\left(e^{2\pi i m \varphi_\ell}\right)\right). \tag{2.29}$$

Surprisingly, it is not necessary to consider summands with more than two factors, even when we study the joint behavior at more than two points. We thus give the following definition:

Definition 2.18. *Let $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d}$ be given. We call both sequences $(x_j)_{j=1}^d$ and $(\varphi_j)_{j=1}^d$ pairwise of finite type, if we have for all $j \neq \ell$ that $(\varphi_j, \varphi_\ell) \in [0,1]^2$ is of finite type in the sense of Definition 2.16.*

3. Central Limit Theorems for the Symmetric Group

In this section, we state general Central Limit Theorems (CLT's) on the symmetric group. These theorems will allow us to prove CLT's for the logarithm of the characteristic polynomial and for multiplicative class functions.

3.1. **One dimensional CLT.** For a permutation $\sigma \in S_n$, chosen with respect to the Ewens distribution with parameter θ, let C_m be the random variable corresponding to the number of cycles of length m of σ. In order to state the CLT's on the symmetric group, we introduce random variables

$$A_n := \sum_{m=1}^{n} \sum_{k=1}^{C_m} X_{m,k}, \tag{3.1}$$

where we consider $X_{m,k}$ to be independent real valued random variables with $X_{m,k} \overset{d}{=} X_{m,1}$, for all $1 \leq m \leq n$ and $k \geq 1$. Furthermore, all $X_{m,k}$ are independent of σ. Of course, if $X_{m,k} = \mathrm{Re}(\log(1 - x^{-m} T_{m,k}))$ (or $\mathrm{Im}(\log(1 - x^{-m} T_{m,k}))$), then A_n is equal in law to the real (or imaginary) part of $\log Z_{n,z}(x)$, which is the logarithm of the characteristic polynomial of $M_{\sigma,z}$. This will be treated in Section 4.

We state the first result:

Theorem 3.1. *Let $\theta > 0$ be fixed. Assume that the sequence $X_{m,1}$ fulfills the following conditions*

(i) $\frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1}|\right] = O(1)$,
(ii) $\frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[X_{m,1}^2\right] \to V$, *as* $n \to \infty$
(iii) $\frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1}|^3\right] = o\left((\log n)^{1/2}\right)$,
(iv) $\mathbb{E}\left[|X_{m,1}|\right] = O(\log m)$, *and*
(v) There exists a $p > 1/\theta$ *such that* $\frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1}|^p\right] = O(1)$.

Then

$$\frac{A_n - \mathbb{E}\left[A_n\right]}{\sqrt{\log n}} \tag{3.2}$$

converges in law to the normal distribution $\mathcal{N}(0, \theta V)$.

Proof. For the proof, we will make use of the Feller coupling (see Section 2.1). This ensures that the random variables C_m and Y_m are defined on the same space and we can compare them with Lemma 2.3. The strategy of the proof is the following: We define

$$B_n = \sum_{m=1}^{n} \sum_{k=1}^{Y_m} X_{m,k}, \tag{3.3}$$

and show that A_n and B_n have the same asymptotic behavior after normalization. In particular, we will show the following lemma:

Lemma 3.2. *Suppose that the conditions of Theorem 3.1 hold. Then*

$$\mathbb{E}\left[|A_n - B_n|\right] = O(1). \tag{3.4}$$

In particular, it follows immediately that

$$\frac{A_n - B_n}{\sqrt{\log n}} \xrightarrow{d} 0 \quad \text{as } n \to \infty \tag{3.5}$$

and that the random variables $A_n - \mathbb{E}[A_n]$ *and* $B_n - \mathbb{E}[B_n]$ *have the same asymptotic behavior after normalization with* $\sqrt{\log n}$.

Proof of Lemma 3.2. We only have to prove (3.4). The other statements follow directly with Markov's inequality and Slutsky's theorem.
We use that $X_{m,k}$ is independent of C_m and Y_m and that $\mathbb{E}[X_{m,k}] = \mathbb{E}[X_{m,1}]$. We get

$$\begin{aligned}
&\mathbb{E}[|A_n - B_n|] \\
&\quad = \mathbb{E}\left[\left|\sum_{m=1}^{n}\left(\sum_{k=1}^{C_m} X_{m,k} - \sum_{k=1}^{Y_m} X_{m,k}\right)\right|\right] \leq \sum_{m=1}^{n} \mathbb{E}\left[\left|\sum_{k=(C_m \wedge Y_m)+1}^{C_m \vee Y_m} X_{m,k}\right|\right] \\
&\quad \leq \sum_{m=1}^{n} \mathbb{E}\left[\sum_{k=(C_m \wedge Y_m)+1}^{C_m \vee Y_m} \mathbb{E}[|X_{m,k}|]\right] \leq \sum_{m=1}^{n} \mathbb{E}[|X_{m,1}|]\,\mathbb{E}[|C_m - Y_m|]
\end{aligned} \tag{3.6}$$

By Lemma 2.3, there exists for any $\theta > 0$ a constant $K(\theta)$, such that

$$\sum_{m=1}^{n} \mathbb{E}[|X_{m,k}|]\,\mathbb{E}[|C_m - Y_m|] \leq \frac{K(\theta)}{n} \sum_{m=1}^{n} \mathbb{E}[|X_{m,1}|] + \frac{\theta}{n} \sum_{m=1}^{n} \Psi_n(m)\mathbb{E}[|X_{m,1}|]\,. \tag{3.7}$$

By assumption (i), it is clear that the first term on the RHS in (3.7) is always $O(1)$. For $\theta \geq 1$, the second term on the RHS is also bounded, since $\Psi_n(m) \leq 1$. We will use condition (v) for $\theta < 1$ and Lemma 2.4. Then, by the Hölder inequality, the second term on the RHS is bounded for $\theta < 1$, since

$$\begin{aligned}
&\frac{1}{n} \sum_{m=1}^{n} \Psi_n(m)\mathbb{E}[|X_{m,1}|] \\
&\quad \leq K_2 \frac{\mathbb{E}[|X_{n,1}|]}{n} n^{1-\theta} + \frac{K_1}{n} \sum_{m=1}^{n-1} \mathbb{E}[|X_{m,1}|]\left(1 - \frac{m}{n}\right)^{\theta-1} \\
&\quad \leq \frac{O(\log n)}{n^{\theta}} + \left(\frac{K_1}{n} \sum_{m=1}^{n-1} \mathbb{E}[|X_{m,1}|^p]\right)^{1/p} \cdot \left(\sum_{m=1}^{n-1}\left(1 - \frac{m}{n}\right)^{q(\theta-1)}\right)^{1/q},
\end{aligned} \tag{3.8}$$

where the constants K_1 and K_2 are chosen as in Lemma 2.4 and p is such that condition (v) is satisfied. By a simple change of variable $m \to n - m$, the second factor in (3.8) is bounded. This proves the lemma. □

Lemma 3.2 shows that it is enough to consider B_n. We now complete the proof of Theorem 3.1 by computing the characteristic function $\chi_n(t)$ of $B_n - \mathbb{E}[B_n]$, normalized by $\sqrt{\log n}$. To simplify the notation, we define the constant

$$K := \prod_{m=1}^{n} \exp\left(-\frac{it}{\sqrt{\log n}} \frac{\theta}{m} \mathbb{E}[X_{m,1}]\right). \tag{3.9}$$

Then,

$$\begin{aligned}\chi_n(t) &:= \mathbb{E}\left[\exp\left(it\frac{B_n - \mathbb{E}[B_n]}{\sqrt{\log n}}\right)\right] \\ &= \mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}}\sum_{m=1}^{n}\left(\sum_{k=1}^{Y_m} X_{m,k} - \frac{\theta}{m}\mathbb{E}[X_{m,k}]\right)\right)\right] \\ &= K\mathbb{E}\left[\prod_{m=1}^{n}\prod_{k=1}^{Y_m}\mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}}X_{m,k}\right)\right]\right] \\ &= K\prod_{m=1}^{n}\mathbb{E}\left[\left(\mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}}X_{m,1}\right)\right]\right)^{Y_m}\right] \\ &= K\prod_{m=1}^{n}\exp\left(\frac{\theta}{m}\left(\mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}}X_{m,1}\right)\right] - 1\right)\right) \\ &= K\prod_{m=1}^{n}\exp\left(\frac{\theta}{m}\mathbb{E}\left[\exp\left(\frac{it}{\sqrt{\log n}}X_{m,1}\right) - 1\right]\right)\end{aligned} \tag{3.10}$$

We now use the fact that $|(e^{is}-1)-(is-s^2)| \le |s^3|$ and get

$$\exp\left(\frac{it}{\sqrt{\log n}}X_{m,1}\right) - 1 = \frac{it}{\sqrt{\log n}}X_{m,1} - \frac{t^2}{2\log n}X_{m,1}^2 + O\left(\frac{t^3 X_{m,1}^3}{(\log n)^{3/2}}\right). \tag{3.11}$$

We combine (3.10), (3.11) and the definition of K to get

$$\chi_n(t) = \exp\left(-\theta\frac{t^2}{2\log n}\sum_{m=1}^{n}\frac{\mathbb{E}\left[X_{m,1}^2\right]}{m} + O\left(\frac{t^3}{(\log n)^{3/2}}\sum_{m=1}^{n}\frac{\mathbb{E}\left[X_{m,1}^3\right]}{m}\right)\right), \tag{3.12}$$

which goes to $\exp\left(-\theta\frac{t^2}{2}\right)$ by the assumptions (ii), (iii) and the following well-known lemma:

Lemma 3.3. *Let $(a_k)_{k\ge1}$ be a real sequence. Suppose that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}a_k = L. \tag{3.13}$$

Then

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{a_k}{k} = L. \tag{3.14}$$

This concludes the proof of Theorem 3.1 □

Remark: From the proof of Theorem 3.1 it is clear, that the normalization is not restricted to $\sqrt{\log n}$. In fact, we could normalize by any term which goes to infinity with n. Also, it is worth to notice that condition iv is optimal when it comes to regular permutation matrices. Then, the real part of the logarithm of the characteristic polynomial of $M_{\sigma,1}$ is determined by $X_{m,1} = \log|1-x^{-m}| = O(\log m)$. Condition v is only necessary for $\theta < 1$ and can be omitted for $\theta \ge 1$.

3.2. **Multi dimensional central limit theorems.** In this section, we replace the random variables $X_{m,k}$ in Theorem 3.1 by $\mathbb{R}^d$-valued random variables $\overline{X}_{m,k} = (X_{m,k,1}, \ldots, X_{m,k,d})$ and prove a CLT for

$$\overline{A}_{n,d} := \sum_{m=1}^{n} \sum_{k=1}^{C_m} \overline{X}_{m,k}. \tag{3.15}$$

As before, we assume that $\overline{X}_{m,k}$ is a sequence of independent random variables such that $\overline{X}_{m,k} \stackrel{d}{=} \overline{X}_{m,1}$ and all $\overline{X}_{m,k}$ and $\sigma \in S_n$ are independent. We will prove the following theorem:

Theorem 3.4. *Let $\theta > 0$ and $d \in \mathbb{N}$ be fixed. Furthermore, assume that for any $1 \leq j \leq d$ the following conditions are satisfied:*

(i) $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1,j}|\right] = O(1)$,
(ii) $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[X_{m,1,j}X_{m,1,\ell}\right] \to \sigma_{j,\ell}$, *as* $n \to \infty$,
(iii) $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1,j}^3|\right] = o\left(\log^{1/2}(n)\right)$,
(iv) $\mathbb{E}\left[X_{m,1.j}\right] = O(\log m)$,
(v) there exists a $p > 1/\theta$ *s.t.* $\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1,j}|^p\right] = O(1)$.

Then the distribution of

$$\frac{\overline{A}_{n,d} - \mathbb{E}\left[\overline{A}_{n,d}\right]}{\sqrt{\log n}} \tag{3.16}$$

converges in law to the normal distribution $\mathcal{N}(0, \theta\Sigma)$, where Σ is the covariance matrix $(\sigma_{ij})_{1 \leq i,j \leq d}$.

Proof. The theorem follows from the Cramer-Wold theorem if we can show for each $\overline{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$

$$\overline{t} \cdot \frac{\overline{A}_n - \mathbb{E}\left[\overline{A}_n\right]}{\sqrt{\log n}} \xrightarrow{d} N(0, \overline{t}\Sigma\overline{t}^T). \tag{3.17}$$

A simple computation shows that

$$\overline{t} \cdot \overline{A}_n = \sum_{m=1}^{n} \sum_{k=1}^{C_m} V_{m,k}^{(d)} \tag{3.18}$$

with

$$V_{m,k}^{(d)} := V_{m,k} = \sum_{j=1}^{d} t_j X_{m,k,j}. \tag{3.19}$$

We now show that $V_{m,k}$ fulfills the conditions of Theorem 3.1. Clearly, $V_{m,k}$ is a sequence of independent random variables, $V_{m,k} \stackrel{d}{=} V_{m,1}$ and $V_{m,k}$ is independent of C_b for all m, k, b. The Conditions (i), (iv) and (v) of Theorem 3.1 are straight forward. We thus proceed by verifying Conditions (ii) and (iii). We have

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[V_{m,1}^2\right] = \frac{1}{n}\sum_{m=1}^{n}\sum_{j,\ell=1}^{d} t_j t_\ell \mathbb{E}\left[X_{m,1,j}X_{m,1,\ell}\right]$$
$$\to \sum_{j,\ell=1}^{d} t_j t_\ell \sigma_{j,\ell} = \bar{t}\Sigma\bar{t}^T \tag{3.20}$$

with $\Sigma = (\sigma_{j,\ell})_{1\leq j,\ell\leq d}$. This shows that (ii) is fulfilled. We now look at (iii). We use the generalized Hölder inequality and get

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[V_{m,1}^3\right] \leq \frac{1}{n}\sum_{j_1,j_2,j_3=1}^{d}|t_{j_1}t_{j_2}t_{j_3}|\left(\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j_1}X_{m,1,j_2}X_{m,1,j_3}|\right]\right)$$
$$\leq \sum_{j_1,j_2,j_3=1}^{d}\frac{|t_{j_1}t_{j_2}t_{j_3}|}{n}\left(\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j_1}|^3\right]^{1/3}\mathbb{E}\left[|X_{m,1,j_2}|^3\right]^{1/3}\mathbb{E}\left[|X_{m,1,j_3}|^3\right]^{1/3}\right)$$
$$\leq \sum_{j_1,j_2,j_3=1}^{d}\frac{|t_{j_1}t_{j_2}t_{j_3}|}{n}\prod_{a=1}^{3}\left(\sum_{m=1}^{n}\mathbb{E}\left[|X_{m,1,j_a}|^3\right]\right)^{1/3} = o\left(\log^{1/2}(n)\right).$$

This concludes the proof of Theorem 3.4. □

Remark: It is clear that Theorem 3.4 can be used for complex random variables, by identifying $\mathbb{C}$ by $\mathbb{R}^2$.

4. Results on the Characteristic Polynomial and Multiplicative Class Functions

In this section we apply the theorems in Section 3 to the logarithm of the characteristic polynomial and the logarithm of multiplicative class functions. We study in Section 4.1 the of behavior of the real and imaginary part separately and then consider in Section 4.2 the joint behavior and the behavior at different points.

We first recall the branch of logarithm we use for $Z_{n,z}(x)$ and specify the branch of logarithm for $W_z^{1,n}(f)$ and $W_z^{2,n}(f)$, given by Definitions 1.2 and 1.3. As in Definition 1.1, it is natural to choose the branch of logarithm as follows:

Definition 4.1. *Let $x = e^{2\pi i\varphi} \in \mathbb{T}$ be a fixed number, z a $\mathbb{T}$–valued random variable and $f : \mathbb{T} \to \mathbb{C}$ a real analytic function. Furthermore, let $(z_{m,k})_{m,k=1}^{\infty}$ and $(T_{m,k})_{m,k=1}^{\infty}$ be two sequences of independent random variables, independent of $\sigma \in S_n$ with*

$$z_{m,k} \stackrel{d}{=} z \quad \text{and} \quad T_{m,k} \stackrel{d}{=} \prod_{j=1}^{m} z_{j,k}. \tag{4.1}$$

We then set

$$\log\big(Z_{n,z}(x)\big) := \sum_{m=1}^{n}\sum_{k=1}^{C_m} \log(1 - x^{-m} T_{m,k}), \tag{4.2}$$

$$w^{1,n}(f)(x) := \log\left(W_z^{1,n}(f)(x)\right) := \sum_{m=1}^{n}\sum_{k=1}^{C_m} \log\big(f(x^m z_{m,k})\big), \tag{4.3}$$

$$w^{2,n}(f) := \log\left(W_z^{2,n}(f)(x)\right) := \sum_{m=1}^{n}\sum_{k=1}^{C_m} \log\big(f(x^m T_{m,k})\big), \tag{4.4}$$

where we have used for $\log(.)$ *the principal branch of logarithm. We will deal with negative values as follows:* $\log(-y) = \log y + i\pi$, $y \in \mathbb{R}_+$. *Note, that it is not necessary to specify the logarithm at* 0, *since we will deal only with cases where this occurs with probability* 0.

4.1. **Limit behavior at** 1 **point.** We will discuss some important cases for which the conditions in Theorem 3.1 are fulfilled. The results in this section are

Theorem 4.2. *Let* S_n *be endowed with the Ewens distribution with parameter* θ, f *be a non zero real analytic function,* z *a* $\mathbb{T}$*-valued random variable and* $x = e^{2\pi i\varphi} \in \mathbb{T}$ *be not a root of unity, i.e.* $x^m \neq 1$ *for all* $m \in \mathbb{Z}$.
Suppose that one of the following conditions is fulfilled,

- z *is uniformly distributed,*
- z *is absolutely continuous with bounded, Riemann integrable density,*
- z *is discrete, there exists a* $\rho > 0$ *with* $z^\rho \equiv 1$, *all zeros of* f *are roots of unity and* x *is of finite type (see Definition 2.16).*

We then have

$$\frac{\operatorname{Re}\left(w^{1,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_R(f)\sqrt{\log n} \xrightarrow{d} N_R, \tag{4.5}$$

$$\frac{\operatorname{Im}\left(w^{1,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_I(f)\sqrt{\log n} \xrightarrow{d} N_I \tag{4.6}$$

with $N_R \sim \mathcal{N}\left(0, \theta V_R(f)\right), N_I \sim \mathcal{N}\left(0, \theta V_I(f)\right)$ *and*

$$m_R(f) = \operatorname{Re}\left(\int_0^1 \log\big(f(e^{2\pi i\phi})\big)\, d\phi\right), \quad V_R(f) = \int_0^1 \log^2\big|f(e^{2\pi i\phi})\big|\, d\phi, \tag{4.7}$$

$$m_I(f) = \operatorname{Im}\left(\int_0^1 \log\big(f(e^{2\pi i\phi})\big)\, d\phi\right), \quad V_I(f) = \int_0^1 \arg^2\big(f(e^{2\pi i\phi})\big)\, d\phi. \tag{4.8}$$

Theorem 4.3. *Let* S_n *be endowed with the Ewens distribution with parameter* θ, f *be a non zero real analytic function,* z *a* $\mathbb{T}$*-valued random variable and* $x \in \mathbb{T}$ *be not a root of unity.*
Suppose that one of the following conditions is fulfilled,

- z *is uniformly distributed,*

- *z is absolutely continuous with density $g : [0,1] \to \mathbb{R}_+$, such that*

$$g(\phi) = \sum_{j\in\mathbb{Z}} c_j e^{2\pi i j\phi} \quad \text{with} \quad \sum_{j\in\mathbb{Z}} |c_j| < \infty. \tag{4.9}$$

- *z is discrete, there exists a $\rho > 0$ with $z^\rho \equiv 1$, all zeros of f are roots of unity, x is of finite type (see Definition 2.16) and for each $1 \le k \le \rho$,*

$$\mathbb{P}\left[z = e^{2\pi i k/\rho}\right] = \frac{1}{\rho}\sum_{j=0}^{\rho-1} c_j e^{2\pi i jk} \quad \text{with } |c_j| < 1 \text{ for } j \neq 0. \tag{4.10}$$

We then have

$$\frac{\operatorname{Re}\left(w^{2,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_R(f)\sqrt{\log n} \xrightarrow{d} N_R, \tag{4.11}$$

$$\frac{\operatorname{Im}\left(w^{2,n}(f)\right)}{\sqrt{\log n}} - \theta \cdot m_I(f)\sqrt{\log n} \xrightarrow{d} N_I, \tag{4.12}$$

with $m_R(f), m_I(f), N_R$ and N_I as in Theorem 4.2.

Since $Z_{n,z}(x)$ is the special case $f(x) = 1 - x^{-1}$ of W^2, we get immediately with a short computation the following corollary, which covers Proposition 1.1:

Corollary 4.4. *Let S_n be endowed with the Ewens distribution with parameter θ, z a $\mathbb{T}$-valued random variable and $x \in \mathbb{T}$ be not a root of unity, i.e. $x^m \neq 1$ for all $m \in \mathbb{Z}$.*
Suppose that one of the conditions in Theorem 4.3 holds, then

$$\frac{\operatorname{Re}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\log n}} \xrightarrow{d} N_R \quad \text{and} \tag{4.13}$$

$$\frac{\operatorname{Im}\left(\log\left(Z_{n,z}(x)\right)\right)}{\sqrt{\log n}} \xrightarrow{d} N_I, \tag{4.14}$$

with $N_R, N_I \sim \mathcal{N}\left(0, \theta\frac{\pi^2}{12}\right)$.

In Corollary 4.4 $\operatorname{Re}\left(\log\left(Z_{n,z}(x)\right)\right)$ and $\operatorname{Im}\left(\log\left(Z_{n,z}(x)\right)\right)$ are converging to normal random variables without centering. This is due to that the expectation is $o(\sqrt{\log n})$. This will become more clear below.

Remark: The case x a root of unity can be treated similarly. The computations are indeed much simpler, see for instance [18] for $z \equiv 1$.
The rest of this section is devoted to the proofs of Theorem 4.2 and 4.3. We have to distinguish the cases where z is absolutely continuous and z is discrete. Thus, we divide the proof into subsections. We will verify the assumptions of Theorem 3.1 only for the real part, since the computations for the imaginary part are much simpler and easy to show. To illustrate the computations, we begin with the simplest case.

Proofs of Theorem 4.2 and 4.3:

4.1.1. *Uniform measure on the unit circle.* We consider z to be uniformly distributed on the unit circle $\mathbb{T}$. Under this condition, we prove that Theorem 4.2, Theorem 4.3 and Corollary 4.4 hold. We start with the proof of Corollary 4.4.

Proof Corollary 4.4 for uniform z. We start with the characteristic polynomial. We set

$$X_{m,k} := \operatorname{Re}\big(\log(1 - x^{-m} T_{m,k})\big) = \log|1 - x^{-m} T_{m,k}|. \tag{4.15}$$

It is easy to check that in this case $T_{m,k}$ is also uniformly distributed. Thus,

$$x^{-m} T_{m,k} \overset{d}{=} T_{m,k} \overset{d}{=} z \overset{d}{=} z_{m,k} \quad \text{and} \quad X_{m,k} \overset{d}{=} \log|1 - z_{m,k}|. \tag{4.16}$$

We have

$$\mathbb{E}\left(|X_{m,k}|\right) \;=\; \int_0^1 \left|\log|1 - e^{-2i\pi\phi}|\right| \, d\phi = \int_{-1/2}^{1/2} \left|\log|1 - e^{-2i\pi\phi}|\right| \, d\phi. \tag{4.17}$$

This integral exists, since $|1 - e^{-2i\pi\phi}| \sim 2\pi\phi$ for $\phi \to 0$, in particular,

$$\mathbb{E}\left[|X_{m,1}|\right] \le \int_{-1/2}^{1/2} \left|\log\left|\frac{1 - e^{-2i\pi\phi}}{\phi}\right|\right| \, d\phi + \int_{-1/2}^{1/2} |\log|\phi|| \, dt < \infty. \tag{4.18}$$

This shows that the first moment exists and so naturally, condition (i) and (iv) of Theorem 3.1 are fulfilled by the independency of m and the upper bound in (4.18). We proceed by showing that the conditions (ii), (iii) and (v) hold for uniformly chosen z.
One can use partial integration and induction to see that $\mathbb{E}\left[|X_{m,1}|^p\right]$ exists for all $p \ge 1$. Moreover, as for $p = 1$, $\mathbb{E}\left[|X_{m,1}|^p\right]$ is independent of m. In particular,

$$\frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[\log^2|1 - z_{m,1}|\right] = \int_0^1 \log^2|1 - e^{-2i\pi\phi}| \, d\phi = \frac{\pi^2}{12}. \tag{4.19}$$

We thus have

$$\frac{\operatorname{Re}\left(\log\big(Z_{n,z}\big)\right) - \mathbb{E}\left[\operatorname{Re}\left(\log\big(Z_{n,z}\big)\right)\right]}{\sqrt{\log n}} \xrightarrow{d} N_R \tag{4.20}$$

with $N_R \sim \mathcal{N}\left(0, \theta\frac{\pi^2}{12}\right)$.
It remains to show that $\mathbb{E}\left[\operatorname{Re}\left(\log\big(Z_{n,z}(x)\big)\right)\right]$ is $o(\sqrt{\log n})$. We have

$$\begin{aligned}\mathbb{E}\left[\operatorname{Re}\left(\log\big(Z_{n,z}(x)\big)\right)\right] &= \sum_{m=1}^{n} \mathbb{E}\left[C_m\right] \mathbb{E}\left[\log|1 - z_{m,1}|\right] \\ &= \left(\int_0^1 \log|1 - e^{-2i\pi\phi}| \, d\phi\right)\left(\sum_{m=1}^{n} \mathbb{E}\left[C_m\right]\right).\end{aligned} \tag{4.21}$$

By Jensen's formula, $\int_0^1 \log|1 - e^{-2i\pi\phi}| \, d\phi = 0$ []. This completes the proof for the real part of $Z_{n,z}(x)$.

For the imaginary part we use that $\operatorname{Im}\big(\log(1-e^{-2\pi i\phi})\big) = \frac{\pi}{2} - \phi\pi$ for $\phi \in [0,1[$. By similar computations as for the real part, it is easy to see that all the conditions of Theorem 3.1 are fulfilled and thus, Corollary 4.4 holds for uniform z. □

We now proceed with multiplicative class functions, for uniform z. We start by giving the proof of Theorem 4.2.

Proof of Theorem 4.2 and Theorem 4.3 for uniform z. Since $T_{m,k}$ is uniformly distributed, we have $T_{m,k} \stackrel{d}{=} z_{m,k}$ and thus $w^{1,n}(f) \stackrel{d}{=} w^{2,n}(f)$. We therefore do not have to distinguish these cases. Furthermore, if $x_0 = e^{2\pi i\phi_0}$ is a zero of f, then the real part behaves as follows:

$$\log\big|f(e^{2\pi i\phi})\big| \sim K \log|\phi - \phi_0|, \tag{4.22}$$

for $\phi \to \phi_0$ and a $K > 0$. Also, the imaginary part $\arg\big(f(e^{2\pi i\phi})\big)$ is bounded and piecewise real analytic with at most finitely many discontinuity points. This shows that all expectations exist and we can use the same argumentation as for $\log\big(Z_{n,z}(x)\big)$ in the proof of Corollary 4.4, for z being uniformly distributed.
The verification of the assumptions of Theorem 3.1 are then straight forward and we thus omit the details.
The only point that needs a little bit more explanation is the behavior of $\mathbb{E}\left[\operatorname{Re}\left(w^{j,n}(f)\right)\right]$ and $\mathbb{E}\left[\operatorname{Im}\left(w^{j,n}(f)\right)\right]$. We can use Lemma 3.2 and get

$$\begin{aligned}
\mathbb{E}\left[\operatorname{Re}\left(\log\big(Z_{n,z}(x)\big)\right)\right] &= \sum_{m=1}^{n} \mathbb{E}\left[C_m\right]\mathbb{E}\left[\log\big|f(z_{m,1})\big|\right] \\
&= \left(\sum_{m=1}^{n} \mathbb{E}\left[Y_m\right]\mathbb{E}\left[\log\big|f(z_{m,1})\big|\right]\right) + O(1) \\
&= \left(\int_0^1 \log\big|f(e^{2\pi i\phi})\big|\, d\phi\right)\left(\sum_{m=1}^{n}\frac{\theta}{m}\right) + O(1) \\
&= \theta \cdot m_R(f)\log n + O(1).
\end{aligned} \tag{4.23}$$

This works similar for the imaginary part. So, this completes the proof of the case where z is uniformly distributed. □

4.1.2. *Absolute continuous on the unit circle.* We consider here z absolutely continuous. We assume that the density g of z is bounded and Riemann integrable, i.e

$$\mathbb{P}\left[z \in [e^{2\pi i\alpha}, e^{2\pi i\beta}]\right] = \int_\alpha^\beta g(\phi)\, d\phi \quad \text{for } 0 \le \alpha \le \beta \le 1. \tag{4.24}$$

In this situation, $w^{1,n}(f) \neq w^{2,n}(f)$ and we thus have to distinguish this cases. In the following, we only give the proofs for Theorem 4.2 and 4.3, since Corollary 4.4 follows immediately from Theorem 4.3.
We begin with $w^{1,n}(f)$ since the computations are simpler.

Proof of Theorem 4.2 for absolutely continuous z. We set

$$X_{m,k} := \log\big|f(z_{m,k}x^m)\big| \tag{4.25}$$

and $x = e^{2\pi i\varphi}$. For simplicity, we write $h(\phi) := \log\left|f(e^{2\pi i\phi})\right|$. We now show that the assumptions of Theorem 3.1 are fulfilled. It is easy to see that

$$\mathbb{E}\left[|X_{m,k}|\right] = \int_0^1 \left|\log\left|f(x^m e^{2\pi i\phi})\right|\right| g(\phi)\ d\phi = \int_0^1 |h(\phi + m\varphi)| g(\phi)\ d\phi$$
$$\leq \sup_{\alpha\in[0,1]} |g(\alpha)| \int_0^1 |h(\phi)|\ d\phi < \infty. \tag{4.26}$$

This shows that the first moment can be bounded independently of m and so assumptions (i) and (iv) are fulfilled.

We now verify the other assumptions of Theorem 3.1. For simplicity, we will consider g as a periodic function with periodicity 1. We then have for any $p \geq 1$

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,1}|^p\right] = \frac{1}{n}\sum_{m=1}^{n} \int_0^1 |h(m\varphi + \phi)|^p g(\phi)\ d\phi = \frac{1}{n}\sum_{m=1}^{n} \int_0^1 |h(\phi)|^p g(\phi - m\varphi)\ d\phi$$
$$= \int_0^1 |h|(\phi)|^p \left(\frac{1}{n}\sum_{m=1}^{n} g(\phi - m\varphi)\right)\ d\phi. \tag{4.27}$$

All integrals in (4.27) exist, since g is bounded. We now take a closer look at $\frac{1}{n}\sum_{m=1}^{n} g(\phi - m\varphi)$. By assumption, $x = e^{2i\pi\varphi}$ is not a root of unity and φ is thus irrational. Therefore, $(\{m\varphi\})_{m=1}^{\infty}$ is uniformly distributed in $[0,1]$ and we can apply Theorem 2.6 for fixed ϕ, since g is Riemann integrable. We get as $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n} g(\phi - m\varphi) = \frac{1}{n}\sum_{m=1}^{n} g(\phi - \{m\varphi\}) \longrightarrow \int_0^1 g(\phi)\ d\phi. \tag{4.28}$$

We now can use dominated convergence in (4.27), since g is bounded and h^p is integrable. We get

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,k}|^p\right] \to \int_0^1 |h(\phi)|^p\ d\phi. \tag{4.29}$$

As above, the arguments can be also applied for $X_{m,k} = \arg(f(z_{m,k}x^m))$. So, all assumptions of Theorem 3.1 are satisfied for both, the real and the imaginary part of $w^{1,n}(f)$.
It remains to show that the real part of $\mathbb{E}\left[w^{1,n}(f)\right]$ can be replaced by $\theta \cdot m_R(f)\log n$ and the imaginary part by $\theta \cdot m_I(f)\log n$. But this is clear by (4.29). So, this concludes the proof for absolutely continuous z. □

We now come to $w^{2,n}$.

Proof of Theorem 4.3 for absolutely continuous z. We set in this case

$$X_{m,k} := \log\left|f(x^m T_{m,k})\right|. \tag{4.30}$$

The density of $T_{m,k}$ is g^{*m}, where g^{*m} is the $m-$times convolution of g with itself. We have by assumption

$$g(\phi) = \sum_{j\in\mathbb{Z}} c_j e^{2\pi i j\phi} \quad \text{with} \quad |c_j| \leq 1 \text{ for } j \neq 0 \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |c_j| < \infty. \tag{4.31}$$

We use that $\widehat{g^{*m}}(j) = (c_j)^m$ and write as above $h(\phi) := \log\left|f(e^{2\pi i\phi})\right|$. We get

$$\begin{aligned}\mathbb{E}\left[|X_{m,k}|\right] &= \int_0^1 |h(\phi + m\varphi)| g^{*m}(\phi)\ d\phi \leq \sum_{j\in\mathbb{Z}} |c_j|^m \int_0^1 |h(\phi)|\ d\phi \\ &\leq \sum_{j\in\mathbb{Z}} |c_j| \int_0^1 |h(\phi)|\ d\phi < \infty \end{aligned}\tag{4.32}$$

This shows that again, the first moment can be bounded independently of m and so, assumptions (i) and (iv) of Theorem 3.1 are satisfied. We now come to assumptions (ii), (iii) and (v).

Let $p \geq 1$ be given, then

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,k}|^p\right] &= \frac{1}{n}\sum_{m=1}^{n} \int_0^1 |h(\phi + m\varphi)|^p g^{*m}(\phi)\ d\phi \\ &= \int_0^1 |h(\phi)|^p \left(\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi)\right)\ d\phi. \end{aligned}\tag{4.33}$$

Consider now $\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi)$. By assumption,

$$g(\phi) = \sum_{j\in\mathbb{Z}} c_j e^{2\pi i j\phi} \quad \text{with} \quad |c_j| \leq 1 \text{ for } j \neq 0 \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |c_j| < \infty. \tag{4.34}$$

We use that $\widehat{g^{*m}}(j) = (c_j)^m$ and get

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi) &= \frac{1}{n}\sum_{m=1}^{n}\sum_{j\in\mathbb{Z}} c_j^m e^{2\pi i j(\phi - m\varphi)} \\ &= \sum_{j\in\mathbb{Z}} e^{2\pi i j\phi}\left(\frac{1}{n}\sum_{m=1}^{n} c_j^m e^{-2\pi i j m\varphi}\right). \end{aligned}\tag{4.35}$$

We now compute the behavior of $\frac{1}{n}\sum_{m=1}^{n} c_j^m e^{-2\pi i j m\varphi}$. For $j = 0$, this expression is just 1, since $c_0 = \int_0^1 g(\phi)\ d\phi = 1$. For $j \neq 0$, we use that $|c_j| \leq 1$ and get for some constant K and almost all ϕ

$$\frac{1}{n}\left|\sum_{m=1}^{n} c_j^m e^{-2\pi i j m\varphi}\right| \leq \frac{1}{n}\frac{1 - c_j^{n+1} e^{2i\pi j\phi(n+1)}}{1 - c_j e^{2i\pi j\phi}} \leq \frac{K}{n}. \tag{4.36}$$

Also, we have $\frac{1}{n}\sum_{m=1}^{n} |c_j^m| \leq |c_j|$ and thus

$$\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi) \leq \left|\sum_{j\in\mathbb{Z}} e^{2\pi i j\phi}\left(\frac{1}{n}\sum_{m=1}^{n} c_j^m e^{-2\pi i j m\varphi}\right)\right| \leq \sum_{j\in\mathbb{Z}} |c_j| < \infty \tag{4.37}$$

So we can apply dominated convergence in (4.35). Therefore, as $n \to \infty$,

$$\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi) \longrightarrow 1 \quad (n \to \infty). \tag{4.38}$$

Furthermore, $\sum_j |c_j|$ is also an upper bound for $\frac{1}{n}\sum_{m=1}^{n} g^{*m}(\phi - m\varphi)$. So again, we can use in (4.33) dominated convergence to get

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[|X_{m,k}|^p\right] \to \int_0^1 |h|^2(\phi)\, d\phi. \tag{4.39}$$

This completes the proof for absolutely continuous z.

□

4.1.3. *Discrete measure on the unit circle.* In this section, we will prove Theorem 4.2 and 4.3 for discrete z with $z^\rho \equiv 1$.

The special case $z \equiv 1$ has been considered by Hambly, Keevash, O'Connell and Stark [9] for the (regular) characteristic polynomial and the uniform distribution, and by Zeindler [18] for multiplicative class functions and the Ewens distribution on S_n.

We need here the following result proven in [18, p. 14–15].

Lemma 4.5. *Let $f : \mathbb{T} \to \mathbb{C}$ be real analytic with only roots of unity as zeros and let $x = e^{2\pi i\varphi}$ be of finite type (see Definition 2.16). We then have $\log\left|f(x^n)\right| = O\big(\log n\big)$ and for any $p \geq 1$*

$$\frac{1}{n}\sum_{m=1}^{n} \log^p\left|f(x^m)\right| \longrightarrow \int_0^1 \log^p\left|f(e^{2\pi i\phi})\right|\, d\phi. \tag{4.40}$$

$$\frac{1}{n}\sum_{m=1}^{n} \left|\log^p\left|f(x^m)\right|\right| \longrightarrow \int_0^1 \left|\log^p\left|f(e^{2\pi i\phi})\right|\right|\, d\phi. \tag{4.41}$$

The idea of the proof of Lemma 4.5 is to use Theorem 2.13 for $d = 1$. The assumption x of finite type allows to choose a suitable δ and to estimate the discrepancy D_n^*.

Remark: The upper bound for $\log\left|f(x^n)\right|$ in Lemma 4.5 is optimal, i.e. if f has a zero, then there exists a infinite sequence $(n_k)_{k=1}^\infty$ and a constant $K > 0$ such that $n_k \to \infty$ and

$$\left|\log\left|f(x^{n_k})\right|\right| \geq K \log(n_k). \tag{4.42}$$

We begin as in Section 4.1.2 with the proof for $w^{1,n}(f)$.

Proof of Theorem 4.2 for discrete z. By $z^\rho \equiv 1$, we have for any $p \geq 1$

$$\frac{1}{n}\sum_{m=1}^{n} \mathbb{E}\left[\log^p\left|f(z_{m,1}x^m)\right|\right] = \sum_{k=1}^{\rho} \mathbb{P}\left[z = e^{2\pi i k/\rho}\right]\left(\frac{1}{n}\sum_{m=1}^{n} \log^p\left|f(e^{2\pi i k/\rho}x^m)\right|\right). \tag{4.43}$$

We set $f_k(y) := f(e^{2\pi ik/\rho}y)$ for any $y \in \mathbb{T}$. By assumption, the zeros of f are only roots of unity and thus, the zeros of f_k are also only roots of unity. This shows that we can apply Lemma 4.5 in (4.43) for each k and we immediately get as $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[\log^p\left|f(z_{m,1}x^m)\right|\right] \to \int_0^1 \log^p\left|f(e^{2\pi i\phi})\right|\,d\phi. \tag{4.44}$$

So, the moments of $w^{1,n}(f)$ follow the same behavior as in the uniform case. By the proof of Theorem 4.2 for uniformly distributed z, we conclude the proof for discrete z.

□

We now give the proof for $w^{2,n}(f)$, if z is discrete.

Proof of Theorem 4.3 for discrete z. We start the proof by recalling, that for discrete z with $z^\rho \equiv 1$, there exist always a sequence $(c_j)_{0\le j\le \rho-1}$ such that

$$\mathbb{P}\left[z = e^{2\pi ik/\rho}\right] = \frac{1}{\rho}\sum_{j=0}^{\rho-1} c_j e^{2\pi ijk}. \tag{4.45}$$

(See for more details [14], chapter 7.)
It follows immediately

$$\mathbb{P}\left[T_{m,1} = e^{2\pi ik/\rho}\right] = \frac{1}{\rho}\sum_{j=0}^{\rho-1} c_j^m e^{2\pi ijk}. \tag{4.46}$$

For any $p \ge 1$, we have

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[\log^p\left|f(x^mT_{m,1})\right|\right] &= \frac{1}{n}\sum_{m=1}^{n}\sum_{k=0}^{\rho-1}\log^p\left|f(x^me^{2\pi ik/\rho})\right|\mathbb{P}\left[T_{m,1} = e^{2\pi ijk/\rho}\right]\\ &= \frac{1}{n\rho}\sum_{m=1}^{n}\sum_{k=0}^{\rho-1}\sum_{j=0}^{\rho-1} c_j^m e^{2\pi ijk}\log^p\left|f(x^me^{2\pi ik/\rho})\right|\\ &= \frac{1}{\rho}\sum_{k=0}^{\rho-1}\sum_{j=0}^{\rho-1} e^{2\pi ijk}\left(\frac{1}{n}\sum_{m=1}^{n} c_j^m\log^p\left|f(x^me^{2\pi ik/\rho})\right|\right).\end{aligned} \tag{4.47}$$

First, we consider only summands with $j \ne 0$ and we show that they vanish in the limit. Let k be fixed and $\epsilon > 0$ be arbitrary. Since by assumption $|c_j| < 1$, we can find a m_0 such that $|c_j|^m < \epsilon$ for $m \ge m_0$. We get

$$\begin{aligned}\left|\frac{1}{n}\sum_{m=1}^{n} c_j^m\log^p\left|f(x^me^{2\pi ik/\rho})\right|\right| &\le \frac{1}{n}\sum_{m=1}^{n}|c_j|^m\left|\log^p\left|f(x^me^{2\pi ik/\rho})\right|\right|\\ &\le o(1) + \frac{\epsilon}{n}\sum_{m=m_0}^{n}\left|\log^p\left|f(x^me^{2\pi ik/\rho})\right|\right|\end{aligned} \tag{4.48}$$

Since x and f satisfy the assumptions of Lemma 4.5, we can use (4.41) to see that the last sum converges to $\epsilon\int_0^1\log^p\left|f(e^{2\pi i\phi})\right|\,d\phi$. This proves that the terms are vanishing in the limit, since ϵ was arbitrary. The remaining terms in (4.47), i.e. the terms with $j = 0$, are thus

$$\frac{1}{\rho}\sum_{k=0}^{\rho-1}\left(\frac{1}{n}\sum_{m=1}^{n}\log^p\left|f(x^m e^{2\pi ik/\rho})\right|\right). \tag{4.49}$$

Again, by Lemma 4.5 we get as $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n}\mathbb{E}\left[\log^p\left|f(x^m T_{m,k})\right|\right] \longrightarrow \int_0^1 \log^p\left|f(e^{2\pi i\phi})\right|\, d\phi. \tag{4.50}$$

So, the moments of $w^{2,n}(f)$ follow the same behavior as in the uniform case. By the proof of Theorem 4.3 for uniformly distributed z, we conclude the proof for discrete z.

□

This completes the proofs of Theorem 4.2 and 4.3.

4.2. **Behavior at different points.** In this section, we study the joint behavior of the real and the imaginary parts of the characteristic polynomial of $M_{\sigma,z}$ and of multiplicative class functions. Furthermore, we consider the behavior at a finite set of different points $x_1 = e^{2\pi i\varphi_1}, \ldots, x_d = e^{2\pi i\varphi_d}$, $d \in \mathbb{N}$ fixed. We will follow the structure of Section 4.1.

Before we state the results of this section, it is important to emphasize that we will allow different random variables $z_1, \ldots, z_d$ at the different points $x_1, \ldots, x_d$. Of course, we need to specify the joint behavior at the different points. For the multiplicative class function $w^{1,n}(f_j)(x_j)$, we define the following joint behavior. Let $\overline{z} = (z_1, \ldots, z_d)$ be a random variable with values in $\mathbb{T}^d$. Let further $\overline{z}^{(m,k)} = (z_1^{(m,k)}, \ldots, z_d^{(m,k)})$ be a sequence of i.i.d. random variables with $\overline{z}^{(m,k)} \overset{d}{=} \overline{z}$ (in m and k, for $1 \le m \le n$ and $1 \le k \le C_m$, where C_m denotes the number of cycles of m in σ). Then, for functions $f_1, \ldots, f_d$ and for any fixed $1 \le j \le d$,

$$w^{1,n}(f_j)(x_j) = w^{1,n}_{z_j}(f_j)(x_j) := \sum_{m=1}^{n}\sum_{k=1}^{C_m}\log\left(f_j\left(z_j^{(m,k)}x_j^m\right)\right). \tag{4.51}$$

This means that the behavior in different cycles of σ is independent. But the behavior in a given cycle at different points is determined by $\overline{z}$.

For the logarithm of the characteristic polynomial $\log\big(Z_{n,z}(x_j)\big)$ and for the multiplicative class function $w^{2,n}(f_j)(x_j)$, we do something similar. Intuitively, we construct for each point x_j a matrix M_{σ,z_j} as in (1.1), where we choose for M_{σ,z_1} n i.i.d. random variables, which are equal in distribution to z_1. At point x_2, we choose again n i.i.d random variables, which are equal in distribution to z_2 and so on. Formally, we define for (the same sequence as above) $\overline{z}^{(m,k)} = (z_1^{(m,k)}, \ldots, z_d^{(m,k)})$ another sequence (in m and in k) $\overline{T}^{(m,k)} = (T_1^{(m,k)}, \ldots, T_d^{(m,k)})$ of independent random variables, so that for any fixed $1 \leq j \leq d$ and fixed $1 \leq m \leq n$,

$$T_j^{(m,k)} \stackrel{d}{=} \prod_{\ell=1}^{m} z_j^{(m,\ell)}$$

and

$$(T_1^{(m,k)}, \dots, T_d^{(m,k)}) \stackrel{d}{=} \left(\prod_{\ell=1}^{m} z_1^{(m,\ell)}, \dots, \prod_{\ell=1}^{m} z_d^{(m,\ell)} \right). \tag{4.52}$$

This gives for fixed j's and function f_j:

$$w^{2,n}(f_j)(x_j) = w_{z_j}^{2,n}(f_j)(x_j) := \sum_{m=1}^{n} \sum_{k=1}^{C_m} \log \left(f_j \left(T_j^{(m,k)} x_j^m \right) \right). \tag{4.53}$$

We now state the results of this section:

Theorem 4.6. *Let S_n be endowed with the Ewens distribution with parameter θ, $f_1, \dots, f_d$ be non zero real analytic functions, $\overline{z} = (z_1, \dots, z_d)$ a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \dots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*
Suppose that one of the following conditions is satisfied:

- *$z_1, \dots, z_d$ are uniformly distributed and independent.*
- *For all $1 \leq j \leq d$, z_j is absolutely continuous. The common density of z_j and z_ℓ is bounded and Riemann integrable for all $j \neq \ell$.*
- *For all $1 \leq j \leq d$, z_j is trivial, i.e. $z_j \equiv 1$, and all zeros of f_j are roots of unity. Furthermore, $x_1, \dots, x_d$ are pairwise of finite type (see Definition 2.18).*
- *For all $1 \leq j \leq d$, there exists a $\rho_j > 0$ with $(z_j)^{\rho_j} \equiv 1$, all zeros of f_j are roots of unity and $x_1, \dots, x_d$ are pairwise of finite type.*

We then have, as $n \to \infty$,

$$\frac{1}{\sqrt{\log n}} \begin{pmatrix} w^{1,n}(f_1)(x_1) \\ \vdots \\ w^{1,n}(f_d)(x_d) \end{pmatrix} - \theta \sqrt{\log n} \begin{pmatrix} m(f_1) \\ \vdots \\ m(f_d) \end{pmatrix} \stackrel{d}{\longrightarrow} N = \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix},$$

where N is a $d-$variate complex normal distributed random variable with, for $j \neq \ell$,

$$\mathrm{Cov}\big(\mathrm{Re}(N_j), \mathrm{Re}(N_\ell)\big) = \theta \int_{[0,1]^2} \log\big|f_j(e^{2\pi i u})\big| \log\big|f_\ell(e^{2\pi i v})\big| \, dudv, \tag{4.54}$$

$$\mathrm{Cov}\big(\mathrm{Re}(N_j), \mathrm{Im}(N_\ell)\big) = \theta \int_{[0,1]^2} \log\big|f_j(e^{2\pi i u})\big| \arg\big(f_\ell(e^{2\pi i v})\big) \, dudv, \tag{4.55}$$

$$\mathrm{Cov}\big(\mathrm{Im}(N_j), \mathrm{Im}(N_\ell)\big) = \theta \int_{[0,1]^2} \arg\big(f_j(e^{2\pi i u})\big) \arg\big(f_\ell(e^{2\pi i v})\big) \, dudv. \tag{4.56}$$

The variance is given by

$$\operatorname{Var}\left(\operatorname{Re}(N_j)\right) = \theta \int\limits_{[0,1]} \log^2\left|f_j(e^{2\pi i u})\right| \, du \tag{4.57}$$

and

$$\operatorname{Var}\left(\operatorname{Im}(N_j)\right) = \theta \int\limits_{[0,1]} \arg^2\left(f_j(e^{2\pi i v})\right) \, dv. \tag{4.58}$$

Theorem 4.7. *Let S_n be endowed with the Ewens distribution with parameter θ, $f_1, \dots, f_d$ be non zero real analytic functions, $\overline{z} = (z_1, \dots, z_d)$ a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \dots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*
Suppose that one of the following conditions is satisfied:

- *$z_1, \dots, z_d$ are uniformly distributed and independent.*
- *For all $1 \le j \le d$, z_j is absolutely continuous. For each $j \neq \ell$, the joint density $g_{j,\ell}$ of z_j and z_ℓ satisfies*
$$g_{j,\ell}(\phi_j, \phi_\ell) = \sum_{a,b\in\mathbb{Z}} c_{a,b} e^{2\pi i (a\phi_j + b\phi_\ell)} \quad \textit{and} \quad \sum_{a,b\in\mathbb{Z}} |c_{a,b}| < \infty. \tag{4.59}$$
- *For all $1 \le j \le d$, z_j is trivial, i.e. $z_j \equiv 1$, and all zeros of f_j are roots of unity. Furthermore, $x_1, \dots, x_d$ are pairwise of finite type (see Definition 2.18),*
- *For all $1 \le j \le d$, z_j is discrete, there exists a $\rho_j > 0$ with $(z_j)^{\rho_j} \equiv 1$, all zeros of f_j are roots of unity. Furthermore, assume that $x_1, \dots, x_d$ are pairwise of finite type (see Definition 2.18) and that for $j \neq \ell$*

$$\mathbb{P}\left[z_j = e^{2\pi i k_1/\rho_j}, z_\ell = e^{2\pi i k_2/\rho_\ell}\right] = \frac{1}{\rho_j \rho_\ell} \sum_{a=0}^{\rho_j - 1} \sum_{b=0}^{\rho_\ell - 1} c_{a,b} e^{2\pi i (a k_1 + b k_2)} \tag{4.60}$$

$$\textit{with} \sum_{a=0}^{\rho_j - 1} |c_{a,b}| < 1, \ \textit{for } b \neq 0 \quad \textit{and} \sum_{b=0}^{\rho_\ell - 1} |c_{a,b}| < 1, \ \textit{for } a \neq 0. \tag{4.61}$$

We then have, as $n \to \infty$,

$$\frac{1}{\sqrt{\log n}} \begin{pmatrix} w^{2,n}(f_1)(x_1) \\ \vdots \\ w^{2,n}(f_d)(x_d) \end{pmatrix} - \theta \sqrt{\log n} \begin{pmatrix} m(f_1) \\ \vdots \\ m(f_d) \end{pmatrix} \xrightarrow{d} N = \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $m(f_j)$ and N as in Theorem 4.6.

As before, we get as simple corollary, which covers Proposition 1.2:

Corollary 4.8. *Let S_n be endowed with the Ewens distribution with parameter θ, $\overline{z} = (z_1, \dots, z_d)$ be a $\mathbb{T}^d$-valued random variable and $x_1 = e^{2\pi i \varphi_1}, \dots, x_d = e^{2\pi i \varphi_d} \in \mathbb{T}$ be such that $1, \varphi_1, \dots, \varphi_d$ are linearly independent over $\mathbb{Z}$.*

Suppose that one of the conditions in Theorem 4.7 is satisfied: We then have, as $n \to \infty$,

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta \log n}} \begin{pmatrix} \log\big(Z_{n,z_1}(x_1)\big) \\ \vdots \\ \log\big(Z_{n,z_d}(x_d)\big) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N_1 \\ \vdots \\ N_d \end{pmatrix}$$

with $\mathrm{Re}(N_1), \dots, \mathrm{Re}(N_d), \mathrm{Im}(N_1), \dots, \mathrm{Im}(N_d)$ *independent standard normal distributed random variables.*

Proof of Theorem 4.6 and 4.7. We consider $w^{1,n}(f)$ and $w^{2,n}(f)$ as $\mathbb{R}^2$-valued random variables and argue with Theorem 3.4. The assumptions of Theorem 3.1 and Theorem 3.4 are almost the same. The only difference lies in condition (ii) in Theorem 3.4:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbb{E}\left[X_{m,1,j} X_{m,1,\ell}\right] = \sigma_{j,\ell}. \tag{4.62}$$

The computations for uniformly distributed and for absolute continuous $z_1, \dots, z_d$ are for both, $w^{1,n}$ and $w^{2,n}$ the same as in Section 4.1, namely in Section 4.1.1 and Section 4.1.2. Thus, we have a closer look at the third and the forth condition in Theorem 4.6 and 4.7, the trivial and the discrete case. The behavior in one point, where $z \equiv 1$ has been treated by [9]. For the behavior at different points, we have to extend Lemma 4.5:

Lemma 4.9. *Let $f_1, f_2 : \mathbb{T} \to \mathbb{C}$ be real analytic with only roots of unity as zeros and let $x_1 = e^{2\pi i \varphi_1}$ and $x_2 = e^{2\pi i \varphi_2}$ be such that $(x_1, x_2) \in \mathbb{T}^2$ be of finite type (see Definition 2.16). We then have $\log\big|f_j(x_j^n)\big| = O\big(\log n\big)$ for $j \in \{1, 2\}$. Moreover, as $n \to \infty$*

$$\frac{1}{n} \sum_{m=1}^{n} \log\left|f_1\left(x_1^m\right)\right| \log\left|f_2\left(x_2^m\right)\right| \longrightarrow \int_{[0,1]^2} \log\big|f_1(e^{2\pi i u})\big| \log\big|f_2(e^{2\pi i v})\big| \, du dv, \tag{4.63}$$

$$\frac{1}{n} \sum_{m=1}^{n} \arg\left(f_1\left(x_1^m\right)\right) \log\left|f_2\left(x_2^m\right)\right| \longrightarrow \int_{[0,1]^2} \arg\big(f_1(e^{2\pi i u})\big) \log\big|f_2(e^{2\pi i v})\big| \, du dv \tag{4.64}$$

and

$$\frac{1}{n} \sum_{m=1}^{n} \arg\left(f_1\left(x_1^m\right)\right) \arg\left(f_2\left(x_2^m\right)\right) \longrightarrow \int_{[0,1]^2} \arg\big(f_1(e^{2\pi i u})\big) \arg\big(f_2(e^{2\pi i v})\big) \, du dv. \tag{4.65}$$

By using Lemma 4.9, the proof for $z_1, \dots, z_d$ being discrete is the same as in Section 4.1.3. Thus, in order to conclude the proofs for Theorem 4.6 and 4.7, we will proceed by giving the proof of Lemma 4.9:

Proof. We start by considering (4.63). Since x_1 and x_2 are not roots of unity, we expect for $n \to \infty$

$$\frac{1}{n}\sum_{m=1}^{n} \log|f_1(x_1^m)| \log|f_2(x_2^m)| \longrightarrow \int\limits_{[0,1]^2} \log\left|f_1(e^{2\pi i u})\right| \log\left|f_2(e^{2\pi i v})\right| \, dudv. \quad (4.66)$$

Unfortunately this is not automatically true since $\log(f_j)$ is not Riemann integrable and we thus cannot apply Theorem 2.6. We show here that (4.66) is true by using the assumption that (x_1, x_2) is of finite type.

We use the notations:

$$\begin{aligned} h_1(\phi) &:= \log|f_1(e^{2\pi i\phi})|,\ \varphi_1^{(m)} := \{m\varphi_1\},\ \boldsymbol{\varphi}_1 := (\varphi_1^{(m)})_{m=1}^{\infty}, \\ h_2(\phi) &:= \log|f_2(e^{2\pi i\phi})|,\ \varphi_2^{(m)} := \{m\varphi_2\},\ \boldsymbol{\varphi}_2 := (\varphi_2^{(m)})_{m=1}^{\infty}, \\ \overline{\varphi} &:= (\varphi_1, \varphi_2),\ \varphi^{(m)} := (\varphi_1^{(m)}, \varphi_2^{(m)})\ \boldsymbol{\varphi} := \left(\varphi^{(m)}\right)_{m=1}^{\infty}. \end{aligned} \quad (4.67)$$

We thus can reformulate the LHS of (4.63) as

$$\frac{1}{n}\sum_{m=1}^{n} h_1(\varphi_1^{(m)}) h_2(\varphi_2^{(m)}). \quad (4.68)$$

If f_1 and f_2 are zero free, then h_1 and h_2 are Riemann integrable. Furthermore, $1, \varphi_1, \varphi_2$ are by assumption linearly independent over $\mathbb{Z}$, and thus $\boldsymbol{\varphi}$ is a uniformly distributed sequence by Lemma 2.14. Equation (4.66) now follows immediately with Theorem 2.6.

If f_1 and f_2 are not zero free, we have to be more careful. We use in this case Theorem 2.13 for $d = 2$. We assume for simplicity that 0 and 1 are to the only singularities of h_1 and h_2. The more general case with roots of unity as zeros is completely similar.

We first have to choose a suitable $\delta = \delta(n)$ such that $\varphi^{(m)} \in [\delta, 1-\delta]^2$ for $1 \le m \le n$. Since by assumption $\overline{\varphi}$ is of finite type, there exists $K > 0, \gamma > 1$ such that

$$\|\overline{q} \cdot \overline{\varphi}\| \ge \frac{K}{(\|\overline{q}\|_\infty)^\gamma} \quad \text{for all } \overline{q} \in \mathbb{Z}^2 \setminus \{0\} \quad (4.69)$$

with $\|a\| := \inf_{m\in\mathbb{Z}} |a - m|$. We thus can chose $\delta = \frac{K}{n^\gamma}$.

Next, we have to estimate the discrepancies of the sequences $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2$ and $\boldsymbol{\varphi}$. Since $\overline{\varphi}, \varphi_1, \varphi_2$ are of finite type, we can use Theorem 2.17 and get

$$D_n^*(\boldsymbol{\varphi}_1) = O(n^{-\alpha_1}),\ D_n^*(\boldsymbol{\varphi}_2) = O(n^{-\alpha_2}) \ \text{ and } \ D_n^*(\boldsymbol{\varphi}) = O(n^{-\alpha}) \quad (4.70)$$

for some $\alpha_1, \alpha_2, \alpha > 0$.

We can show now with Theorem 2.13 that the error made by the approximation in (4.66) goes to 0 by showing that all summands on the RHS of (2.16) go to 0. This computation is straight forward and very similar for each summand. We restrict ourselves to illustrate the computations only on the summands corresponding to the face F of $[\delta, 1-\delta]^2$ with $\phi_1 = 1 - \delta$. We get with $h(\phi_1, \phi_2) := h_1(\phi_1)h_2(\phi_2)$,

$$\delta|h_1(1-\delta)| \int\limits_{\delta}^{1-\delta} |h_2(u)| \, du + D_n^*(\boldsymbol{\varphi}_2)|h_1(1-\delta)| V(h_2|[\delta, 1-\delta]), \quad (4.71)$$

where $V(h_2|[\delta, 1-\delta])$ is the variation of $h_2|[\delta, 1-\delta]$. It is easy to see that, for $\phi \to 0$ and some $K_1 > 0$, $h_1(\phi) \sim K_1 \log(\phi) \sim h_1(1-\phi)$. Thus, the first summand in (4.71) goes to 0 for $n \to \infty$. On the other hand we have

$$D_n^*(\boldsymbol{\varphi}_2)|h_1(1-\delta)|V(h_2|[\delta, 1-\delta]) \sim K_2 D_n^*(\boldsymbol{\varphi}_2) \log^2 \delta \leq K_3 n^{-\alpha_2} \log^2 n. \tag{4.72}$$

for constants $K_2, K_3 > 0$. This shows that also the second term in (4.71) goes to 0. So, we proved (4.63). Equations (4.64) and (4.65) are straightforward, with the given computations above and we conclude Lemma 4.9. □

This completes the proofs of Theorem 4.6 and 4.7. □

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