SEMI-INVARIANTS FOR CONCEALED CANONICAL ALGEBRAS. I. NON-PREHOMOGENEOUS CASE

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Throughout the paper $\Bbbk$ is a fixed algebraically closed field. By $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$ we denote the sets of the integers, the non-negative integers and the positive integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] := \{l \in \mathbb{Z} \mid i \leq k \leq j\}$ (in particular, $[i, j] = \varnothing$ if $i > j$).

Introduction

Concealed-canonical algebras have been introduced by Lenzing and Melzer [22] as a generalization of Ringel's canonical algebras [26]. An algebra is called concealed-canonical if it is isomorphism to the endomorphism ring of a tilting bundle over a weighted projective line. The concealed-canonical algebras can be characterized as the algebras which posses sincere separating exact subcategory [23] (see also [28]). Together with tilted algebras [7, 20], the concealed-canonical algebras form two most prominent classes of quasi-tilted algebras [19]. Moreover, according to a famous result of Happel [18], every quasi-tilted algebra is derived equivalent either to a tilted algebra or to a concealed-canonical algebra.

Despite investigations of a structure of the categories of modules over concealed-canonical algebras, geometric problems have been studied for this class of algebras (see for example [2, 3, 6, 14, 15, 17, 29]). Often these problem were studied for canonical algebras only and sometimes the authors restrict their attention to the concealed-canonical algebras of tame representation type.

In the paper we study a problem, which has been already completely solved in the case of canonical algebras. Namely, given a concealed-canonical algebra Λ and a module R, which is a direct sum of modules from of sincere separating exact subcategory of $\operatorname{mod} \Lambda$, we want to describe a structure of the ring of semi-invariants associated to Λ and the dimension vector of R. As it was mentioned above, this problem has been solved in the case of canonical algebras (the answers have been obtained independently by Skowroński and Weyman [29] and Domokos and Lenzing [14, 15]). This problem has also been solved for another class of concealed-canonical algebras, namely the path algebras of Euclidean quivers [30] (see also [12, 27]). The obtained results are very similar, although the applied methods are completely different. The aim my paper is to obtain a unified proof of the above results, which would generalize to arbitrary concealed-canonical algebras. This aim is achieved if the characteristic of $\Bbbk$ equals 0. If $\operatorname{char} \Bbbk > 0$, then we show that an analogous result is true if we study the semi-invariants which are the restrictions of the semi-invariants on the ambient affine space. The precise formulation of the obtained results can be found in Section 6.

The paper is organized as follows. In Section 1 we introduce a setup of quivers and their representations, which due to a result of Gabriel [16] is an equivalent way of thinking about algebras and modules. Next, in Section 2 we gather fact about concealed-canonical algebras (equivalently, quivers). In Section 3 we introduce semi-invariants and present their basic properties. Next, in Section 4 we study the semi-invariants in the case of concealed-canonical quivers more closely. Section 5 is devoted to presentation of necessary facts about the Kronecker quiver, which is the minimal concealed-canonical quiver. Finally, in Section 6 we present and proof the main result.

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1. Quivers and their representations

By a quiver Δ we mean a finite set Δ_0 (called the set of vertices of Δ) together with a finite set Δ_1 (called the set of arrows of Δ) and two maps $s, t : \Delta_1 \to \Delta_0$, which assign to each arrow α its starting vertex $s\alpha$ and its terminating vertex $t\alpha$, respectively. By a path of length $n \in \mathbb{N}_+$ in a quiver Δ we mean a sequence $\sigma = (\alpha_1, \ldots, \alpha_n)$ of arrows such that $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, n-1]$. In particular, we treat every arrow in Δ as a path of length 1. In the above situation we put $\ell\sigma := n$, $s\sigma := s\alpha_n$ and $t\sigma := t\alpha_1$. Moreover, for each vertex x we have a trivial path $\mathbf{1}_x$ at x such that $\ell\mathbf{1}_x := 0$ and $s\mathbf{1}_x := x =: t\mathbf{1}_x$. For the rest of the paper we assume that the considered quivers do not have oriented cycles, where by an oriented cycle we mean a path σ of positive length such that $s\sigma = t\sigma$.

Let Δ be a quiver. We define its path category $\Bbbk\Delta$ to be the category whose objects are the vertices of Δ and, for $x, y \in \Delta_0$, the morphisms from x to y are the formal $\Bbbk$-linear combinations of paths starting at x and terminating at y. If ω is a morphism from x to y, then we write $s\omega := x$ and $t\omega := y$. By a representation of Δ we mean a functor from $\Bbbk\Delta$ to the category $\operatorname{mod} \Bbbk$ of finite dimensional vector spaces. We denote the category of representations of Δ by $\operatorname{rep} \Delta$. Observe that every representation of Δ is uniquely determined by its values on the vertices and the arrows. Given a representation M of Δ we denote by $\mathbf{dim}\, M$ its dimension vector defined by the formula $(\mathbf{dim}\, M)(x) := \dim_{\Bbbk} M(x)$ for $x \in \Delta_0$. Observe that $\mathbf{dim}\, M \in \mathbb{N}^{\Delta_0}$ for each representation M of Δ. We call the elements of $\mathbb{N}^{\Delta_0}$ dimension vectors. A dimension vector $\mathbf{d}$ is called sincere if $\mathbf{d}(x) \neq 0$ for each $x \in \Delta_0$.

By a relation in a quiver Δ we mean a $\Bbbk$-linear combination of paths of lengths at least 2 having a common starting vertex and a common terminating vertex. Note that each relation in a quiver Δ is a morphism in $\Bbbk\Delta$. A set $\mathfrak{R}$ of relations in a quiver Δ is called minimal if $\langle \mathfrak{R}\setminus\{\rho\}\rangle \neq \langle \mathfrak{R}\rangle$ for each $\rho \in \mathfrak{R}$, where for a set $\mathfrak{X}$ of morphisms in Δ we denote by $\langle \mathfrak{X}\rangle$ the ideal in $\Bbbk\Delta$ generated by $\mathfrak{X}$. Observe that each minimal set of relations is finite. By a bound quiver $\mathbf{\Delta}$ we mean a quiver Δ together with a minimal set $\mathfrak{R}$ of relations. Given a bound quiver $\mathbf{\Delta}$ we denote by $\Bbbk\mathbf{\Delta}$ its path category, i.e. $\Bbbk\mathbf{\Delta} := \Bbbk\Delta/\langle\mathfrak{R}\rangle$. By a representation of a bound quiver $\mathbf{\Delta}$ we mean a functor from $\Bbbk\mathbf{\Delta}$ to $\operatorname{mod}\Bbbk$. In other words, a representation of $\mathbf{\Delta}$ is a representation M of Δ such that $M(\rho) = 0$ for each $\rho \in \mathfrak{R}$. We denote the category of representations of a bound quiver $\mathbf{\Delta}$ by $\operatorname{rep}\mathbf{\Delta}$. Moreover, we denote by $\operatorname{ind}\mathbf{\Delta}$ the full subcategory of $\operatorname{rep}\mathbf{\Delta}$ consisting of the indecomposable representations. It is known that $\operatorname{rep}\mathbf{\Delta}$ is an abelian Krull–Schmidt category.

An important role in the study of representations of quivers is played by the Auslander–Reiten translations τ and τ^- [1, Section IV.2], which assign to each representation of a bound quiver $\mathbf{\Delta}$ another representation of $\mathbf{\Delta}$. In particular, we will use the following consequences of the Auslander–Reiten formulas [1, Theorem IV.2.13]. Let M and N be representations of a bound quiver $\mathbf{\Delta}$. If $\operatorname{pdim}_{\mathbf{\Delta}} M \leq 1$, then

$$\dim_\Bbbk \operatorname{Ext}^1_{\mathbf{\Delta}}(M, N) = \dim_\Bbbk \operatorname{Hom}_{\mathbf{\Delta}}(N, \tau M). \tag{1.1}$$

Dually, if $\operatorname{idim}_{\mathbf{\Delta}} N \leq 1$, then

$$\dim_\Bbbk \operatorname{Ext}^1_{\mathbf{\Delta}}(M, N) = \dim_\Bbbk \operatorname{Hom}_{\mathbf{\Delta}}(\tau^- N, M). \tag{1.2}$$

Let $\mathbf{\Delta}$ be a bound quiver. We define the corresponding Tits form $\langle -, -\rangle_{\mathbf{\Delta}} : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ by the formula

$$\langle \mathbf{d}', \mathbf{d}''\rangle_{\mathbf{\Delta}} := \sum_{x\in\Delta_0} \mathbf{d}'(x)\cdot\mathbf{d}''(x) - \sum_{\alpha\in\Delta_1} \mathbf{d}'(s\alpha)\cdot\mathbf{d}''(t\alpha) + \sum_{\rho\in\mathfrak{R}} \mathbf{d}'(s\rho)\cdot\mathbf{d}''(t\rho)$$

for $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta_0}$. Bongartz [8, Proposition 2.2] has proved that

$$\begin{aligned}&\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_{\mathbf{\Delta}} \\ &\qquad = \dim_\Bbbk \operatorname{Hom}_{\mathbf{\Delta}}(M, N) - \dim_\Bbbk \operatorname{Ext}^1_{\mathbf{\Delta}}(M, N) + \dim_\Bbbk \operatorname{Ext}^2_{\mathbf{\Delta}}(M, N)\end{aligned}$$

for any $M, N \in \operatorname{rep}\mathbf{\Delta}$ provided $\operatorname{gldim}\mathbf{\Delta} \leq 2$.

2. Separating exact subcategories

In this section we present facts about sincere separating exact subcategories, which we use in our considerations. For the proofs we refer to [23, 26].

Let $\mathbf{\Delta}$ be a bound quiver and $\mathcal{X}$ a full subcategory of $\operatorname{ind}\mathbf{\Delta}$. We denote by $\operatorname{add}\mathcal{X}$ the full subcategory of $\operatorname{rep}\mathbf{\Delta}$ formed by the direct sums of representations from $\mathcal{X}$. We say that $\mathcal{X}$ is an exact subcategory of $\operatorname{ind}\mathbf{\Delta}$ if $\operatorname{add}\mathcal{X}$ is an exact subcategory of $\operatorname{rep}\mathbf{\Delta}$, where by an exact subcategory of $\operatorname{rep}\mathbf{\Delta}$ we mean a full subcategory $\mathcal{E}$ of $\operatorname{rep}\mathbf{\Delta}$ such that $\mathcal{E}$ is an abelian category and the inclusion functor $\mathcal{E} \hookrightarrow \operatorname{rep}\mathbf{\Delta}$ is exact. We put

$$\mathcal{X}_- := \{X \in \operatorname{ind}\mathbf{\Delta} : \operatorname{Hom}_{\mathbf{\Delta}}(\mathcal{X}, X) = 0\}$$

and

$$\mathcal{X}_+ := \{X \in \operatorname{ind}\mathbf{\Delta} : \operatorname{Hom}_{\mathbf{\Delta}}(X, \mathcal{X}) = 0\}.$$

Let $\mathbf{\Delta}$ be a bound quiver. Following [23] we say that $\mathcal{R}$ is a sincere separating exact subcategory of $\operatorname{ind}\mathbf{\Delta}$ provided the following conditions are satisfied:

(1) $\mathcal{R}$ is an exact subcategory of $\operatorname{ind}\mathbf{\Delta}$ stable under the actions of the Auslander–Reiten translations τ and τ^-.
(2) $\operatorname{ind}\mathbf{\Delta} = \mathcal{R}_- \cup \mathcal{R} \cup \mathcal{R}_+$.
(3) $\operatorname{Hom}_{\mathbf{\Delta}}(X, \mathcal{R}) \neq 0$ for each $X \in \mathcal{R}_-$ and $\operatorname{Hom}_{\mathbf{\Delta}}(\mathcal{R}, X) \neq 0$ for each $X \in \mathcal{R}_+$.
(4) $P \in \mathcal{R}_-$, for each indecomposable projective representation P of $\mathbf{\Delta}$, and $I \in \mathcal{R}_+$, for each indecomposable injective representation I of $\mathbf{\Delta}$.

Lenzing and de la Peña [23] have proved that there exists a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind}\mathbf{\Delta}$ if and only if $\mathbf{\Delta}$ is concealed-canonical, i.e. $\operatorname{rep}\mathbf{\Delta}$ is equivalent to the category of modules over a concealed-canonical algebra. In particular, if this is the case, then $\operatorname{gldim}\mathbf{\Delta} \leq 2$.

For the rest of the section we fix a concealed-canonical bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind}\mathbf{\Delta}$. Moreover, we put $\mathcal{P} := \mathcal{R}_-$ and $\mathcal{Q} := \mathcal{R}_+$. Finally, we denote by $\mathbf{P}$, $\mathbf{R}$ and $\mathbf{Q}$ the dimension vectors of the representations from $\operatorname{add}\mathcal{P}$, $\operatorname{add}\mathcal{R}$ and $\operatorname{add}\mathcal{Q}$, respectively.

It is known that $\operatorname{pdim}_{\mathbf{\Delta}} P \leq 1$ for each $P \in \mathcal{P}$ and $\operatorname{idim}_{\mathbf{\Delta}} Q \leq 1$ for each $Q \in \mathcal{Q}$. Next, $\operatorname{pdim}_{\mathbf{\Delta}} R = 1$ and $\operatorname{idim}_{\mathbf{\Delta}} R = 1$ for each $R \in \mathcal{R}$. The categories $\mathcal{P}$ and $\mathcal{Q}$ are closed under the actions of τ and τ^-, hence using the Auslander–Reiten formulas (1.1) and (1.2) we obtain that $\operatorname{Ext}^1_{\mathbf{\Delta}}(\mathcal{P}, \mathcal{R}) = 0 = \operatorname{Ext}^1_{\mathbf{\Delta}}(\mathcal{R}, \mathcal{Q})$. In particular,

$$\langle \mathbf{d}', \mathbf{d} \rangle_{\mathbf{\Delta}} \geq 0 \qquad \text{and} \qquad \langle \mathbf{d}, \mathbf{d}'' \rangle_{\mathbf{\Delta}} \geq 0 \tag{2.1}$$

for all $\mathbf{d}' \in \mathbf{P}$, $\mathbf{d} \in \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$.

We have $\mathcal{R} = \coprod_{\lambda \in \mathbb{P}^1_{\Bbbk}} \mathcal{R}_\lambda$ for connected uniserial categories $\mathcal{R}_\lambda$, $\lambda \in \mathbb{P}^1_{\Bbbk}$. For $\lambda \in \mathbb{P}^1_{\Bbbk}$ we denote by r_λ the number of the pairwise nonisomorphic simple objects in $\operatorname{add}\mathcal{R}_\lambda$. Then $r_\lambda < \infty$ for each $\lambda \in \mathbb{P}^1_{\Bbbk}$. Moreover, $\sum_{\lambda \in \mathbb{P}^1_{\Bbbk}} (r_\lambda - 1) = |\Delta_0| - 2$. In particular, if $\mathbb{X}_0 := \{\lambda \in \mathbb{P}^1_{\Bbbk} : r_\lambda > 1\}$, then $|\mathbb{X}_0| < \infty$.

Fix $\lambda \in \mathbb{P}^1_{\Bbbk}$. If $R_{\lambda,0}, \ldots, R_{\lambda,r_\lambda - 1}$ are chosen representatives of the isomorphisms classes of the simple objects in $\operatorname{add}\mathcal{R}_\lambda$, then we may assume that $\tau R_{\lambda,i} = R_{\lambda,i-1}$ for each $i \in [0, r_\lambda - 1]$, where we put $R_{\lambda,i} :=$

$R_{\lambda,i \bmod r_\lambda}$ for $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ there exists a unique (up to isomorphism) representation in $\mathcal{R}_\lambda$ whose socle and length in add $\mathcal{R}_\lambda$ are $R_{\lambda,i}$ and n, respectively. We fix such representation and denote it by $R_{\lambda,i}^{(n)}$ and its dimension vector by $\mathbf{e}_{\lambda,i}^n$. Then the composition factors of $R_{\lambda,i}^{(n)}$ are (starting from the socle) $R_{\lambda,i}, \ldots, R_{\lambda,i+n-1}$. Consequently, $\mathbf{e}_{\lambda,i}^n = \sum_{j\in[i,i+n-1]} \mathbf{e}_{\lambda,j}$, where $\mathbf{e}_{\lambda,j} := \mathbf{dim}\, R_{\lambda,j}$ for $j \in \mathbb{Z}$. Moreover, for all $i \in \mathbb{Z}$ and $n, m \in \mathbb{N}_+$ there exists an exact sequence

$$0 \to R_{\lambda,i}^{(n)} \to R_{\lambda,i}^{(n+m)} \to R_{\lambda,i+n}^{(m)} \to 0. \tag{2.2}$$

Obviously, for each $R \in \mathcal{R}_\lambda$ there exist $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ such that $R \simeq R_{\lambda,i}^{(n)}$. Moreover, it is known that the vectors $\mathbf{e}_{\lambda,0}, \ldots, \mathbf{e}_{\lambda,r_\lambda-1}$ are linearly independent. Consequently, if $R \in \operatorname{add}\mathcal{R}_\lambda$, then there exist uniquely determined $q_0^R, \ldots, q_{r_\lambda-1}^R \in \mathbb{N}$ such that $\mathbf{dim}\, R = \sum_{i\in[0,r_\lambda-1]} q_i^R \mathbf{e}_{\lambda,i}$. We put $q_i^R := q_{i \bmod r_\lambda}^R$ for $i \in \mathbb{Z}$. Observe that for each $i \in \mathbb{Z}$ the number $q_{\lambda,i}^R$ counts the multiplicity in which $R_{\lambda,i}$ appears as a composition factor in the Jordan–Hölder filtration of R in the category add $\mathcal{R}_\lambda$.

Let $R = \bigoplus_{\lambda\in\mathbb{P}^1_{\Bbbk}} R_\lambda$ for $R_\lambda \in \operatorname{add}\mathcal{R}_\lambda$, $\lambda \in \mathbb{P}^1_{\Bbbk}$. Then we put $q_{\lambda,i}^R := q_i^{R_\lambda}$ for $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in \mathbb{Z}$. Next, we put $p_\lambda^R := \min\{q_{\lambda,i}^R : i \in \mathbb{Z}\}$, for $\lambda \in \mathbb{P}^1_{\Bbbk}$, and $p_{\lambda,i}^R := q_{\lambda,i}^R - p_\lambda^R$, for $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in \mathbb{Z}$. Then

$$\mathbf{dim}\, R = \sum_{\lambda\in\mathbb{P}^1_{\Bbbk}} p_\lambda^R \cdot \mathbf{h}_\lambda + \sum_{\lambda\in\mathbb{P}^1_{\Bbbk}} \sum_{i\in[0,r_\lambda-1]} p_{\lambda,i}^R \cdot \mathbf{e}_{\lambda,i},$$

where $\mathbf{h}_\lambda := \sum_{i\in[0,r_\lambda-1]} \mathbf{e}_{\lambda,i}$ for $\lambda \in \mathbb{P}^1_{\Bbbk}$. It is known that $\mathbf{h}_\lambda = \mathbf{h}_\mu$ for any $\lambda, \mu \in \mathbb{P}^1_{\Bbbk}$. We denote this common value by $\mathbf{h}$. Then

$$\mathbf{dim}\, R = p^R \cdot \mathbf{h} + \sum_{\lambda\in\mathbb{P}^1_{\Bbbk}} \sum_{i\in[0,r_\lambda-1]} p_{\lambda,i}^R \cdot \mathbf{e}_{\lambda,i},$$

where $p^R := \sum_{\lambda\in\mathbb{P}^1_{\Bbbk}} p_\lambda^R$. It is known that if $R, R' \in \operatorname{add}\mathcal{R}$ and $\mathbf{dim}\, R = \mathbf{dim}\, R'$, then $p^R = p^{R'}$ and $p_{\lambda,i}^R = p_{\lambda,i}^{R'}$ for any $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in [0, r_\lambda - 1]$. Consequently, for each $\mathbf{d} \in \mathbf{R}$ there exist uniquely determined $p^{\mathbf{d}} \in \mathbb{N}$ and $p_{\lambda,i}^{\mathbf{d}} \in \mathbb{N}$ for $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in [0, r_\lambda - 1]$, such that

$$\mathbf{d} = p^{\mathbf{d}} \cdot \mathbf{h} + \sum_{\lambda\in\mathbb{P}^1_{\Bbbk}} \sum_{i\in[0,r_\lambda-1]} p_{\lambda,i}^{\mathbf{d}} \cdot \mathbf{e}_{\lambda,i}$$

and for each $\lambda \in \mathbb{P}^1_{\Bbbk}$ there exists $i \in [0, r_\lambda - 1]$ with $p_{\lambda,i}^{\mathbf{d}} = 0$. Again we put $p_{\lambda,i}^{\mathbf{d}} := p_{\lambda,i \bmod r_\lambda}^{\mathbf{d}}$ for $\mathbf{d} \in \mathbf{R}$, $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in \mathbb{Z}$.

It is known that $\mathbf{h}$ is sincere. Moreover, $\mathbf{h}$ can be used in order to distinguish between representations from $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. Namely, if X is an indecomposable representation of $\mathbf{\Delta}$, then

$$X \in \mathcal{P} \quad \text{if and only if} \quad \langle \mathbf{dim}\, X, \mathbf{h} \rangle_{\mathbf{\Delta}} > 0. \tag{2.3}$$

Dually, if X is an indecomposable representation of $\mathbf{\Delta}$, then

$$X \in \mathcal{Q} \qquad \text{if and only if} \qquad \langle \mathbf{h}, \mathbf{dim}\, Q, \rangle_{\mathbf{\Delta}} > 0. \tag{2.4}$$

Let $\lambda, \mu \in \mathbb{P}^1_{\Bbbk}$, $i, j \in \mathbb{Z}$ and $m, n \in \mathbb{N}_+$. Then

$$\dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(R^{(n)}_{\lambda,i}, R^{(m)}_{\mu,j}) = \min\{q^{R^{(m)}_{\mu,j}}_{\lambda,i+n-1}, q^{R^{(n)}_{\lambda,i}}_{\mu,j}\}$$

(in particular, $\operatorname{Hom}_{\mathbf{\Delta}}(R^{(n)}_{\lambda,i}, R^{(m)}_{\mu,j}) = 0$ if $\lambda \neq \mu$). The above formula together with the Auslander–Reiten formula (1.1) imply that

$$\langle \mathbf{e}^n_{\lambda,i}, \mathbf{d} \rangle_{\mathbf{\Delta}} = p^{\mathbf{d}}_{\lambda,i+n-1} - p^{\mathbf{d}}_{\lambda,i-1} \tag{2.5}$$

for any $\lambda \in \mathbb{P}^1_{\Bbbk}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}_+$ and $\mathbf{d} \in \mathbf{R}$. In particular,

$$\langle \mathbf{h}, \mathbf{d} \rangle_{\mathbf{\Delta}} = 0 = \langle \mathbf{d}, \mathbf{h} \rangle_{\mathbf{\Delta}} \tag{2.6}$$

for each $\mathbf{d} \in \mathbf{R}$.

An important role in the proofs will be played by ext-minimal representations. We call a representation V ext-minimal if there is no decomposition $V = V_1 \oplus V_2$ with $\operatorname{Ext}^1_{\mathbf{\Delta}}(V_1, V_2) \neq 0$. We recall facts on ext-minimal representations belonging to $\operatorname{add} \mathcal{R}$.

We start with the case $\mathbf{d} \in \mathbf{R}$ such that $p^{\mathbf{d}} = 0$. In this case there is a unique (up to isomorphism) ext-minimal representation $W \in \operatorname{add} \mathcal{R}$ with dimension vector $\mathbf{d}$, which is constructed inductively in the following way. For $\lambda \in \mathbb{P}^1_{\Bbbk}$ let $I_\lambda := \{i \in [0, r_\lambda - 1] : p^{\mathbf{d}}_{\lambda,i} \neq 0 \text{ and } p^{\mathbf{d}}_{\lambda,i-1} = 0\}$. For $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in I_\lambda$ we denote by $m_{\lambda,i}$ the minimal $m \in \mathbb{N}_+$ such that $p^{\mathbf{d}}_{\lambda,i+m} = 0$. By induction there exists (unique up to isomorphism) ext-minimal representation $W' \in \operatorname{add} \mathcal{R}$ with dimension vector $\mathbf{d} - \sum_{\lambda \in \mathbb{P}^1_{\Bbbk}} \sum_{i \in I_\lambda} \mathbf{e}^{m_{\lambda,i}}_{\lambda,i}$. Then $W := W' \oplus \bigoplus_{\lambda \in \mathbb{P}^1_{\Bbbk}} \bigoplus_{i \in I_\lambda} R^{(m_{\lambda,i})}_{\lambda,i}$ is ext-minimal.

We will use the following property of the above module.

Lemma 2.1. *Assume $\mathbf{d} \in \mathbf{R}$ and $p^{\mathbf{d}} = 0$. Let $W \in \operatorname{add} \mathcal{R}$ be an ext-minimal representation with dimension vector $\mathbf{d}$. If $\lambda \in \mathbb{P}^1_{\Bbbk}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}_+$, $p^{\mathbf{d}}_{\lambda,i} = p^{\mathbf{d}}_{\lambda,i+n}$ and $p^{\mathbf{d}}_{\lambda,j} \geq p^{\mathbf{d}}_{\lambda,i}$ for each $j \in [i, i+n]$, then $\operatorname{Hom}_{\mathbf{\Delta}}(R^{(n)}_{\lambda,i+1}, W) = 0$.*

Proof. Observe that $\operatorname{Hom}_{\mathbf{\Delta}}(R^{(n)}_{\lambda,i+1}, R^{(m_{\lambda,k})}_{\lambda,k}) = 0$ for each $k \in I_\lambda$, since one easily checks that either $q^{R^{(m_{\lambda,k})}_{\lambda,k}}_{\lambda,i+n} = 0$ (if $p^{\mathbf{d}}_{\lambda,i} = 0$) or $q^{R^{(n)}_{\lambda,i+1}}_{\lambda,k} = 0$ (if $p^{\mathbf{d}}_{\lambda,i} > 0$). Now the claim follows by induction. □

Now let $\mathbf{d} \in \mathbf{R}$ be arbitrary. The description of the ext-minimal representations with dimension vector $\mathbf{d}$, which belong to $\operatorname{add} \mathcal{R}$, has been given in [25, Theorem 3.5] (this theorem has been formulated in the case $\mathbf{\Delta} = (\Delta, \varnothing)$ for a Euclidean quiver Δ, but its proof translates to an arbitrary concealed-canonical bound quiver). We will not repeat the formulation here, but only mention some consequences. First, if

$W \in \operatorname{add}\mathcal{R}$ and $\mathbf{dim}\, W = \mathbf{d}$, then W is ext-minimal if and only if $\dim_{\Bbbk} \operatorname{End}_{\mathbf{\Delta}}(W) = p^{\mathbf{d}} + \langle \mathbf{d}, \mathbf{d} \rangle_{\mathbf{\Delta}}$. In particular,

$$(2.7) \quad p^{\mathbf{d}} + \langle \mathbf{d}, \mathbf{d} \rangle_{\mathbf{\Delta}} = \min\{\dim_{\Bbbk} \operatorname{End}_{\mathbf{\Delta}}(W) : W \in \operatorname{add}\mathcal{R} \text{ such that } \mathbf{dim}\, W = \mathbf{d}\}$$

(here we use also [25, Lemma 2.1]). Next, if $W \in \operatorname{add}\mathcal{R}$ is an ext-minimal representation with dimension vector $\mathbf{d}$ and $W' \in \operatorname{add}\mathcal{R}$ is an ext-minimal representation with dimension vector $\mathbf{d} - p^{\mathbf{d}} \cdot \mathbf{h}$, then there exists an exact sequence $0 \to \bigoplus_{\lambda \in \mathbb{P}^1_{\Bbbk}} R_\lambda \to W \to W' \to 0$ with $R_\lambda \in \mathcal{R}_\lambda$ (in particular, indecomposable) for each $\lambda \in \mathbb{P}^1_{\Bbbk}$ (obviously, $\mathbf{dim}\, R_\lambda$ is a multiplicity of $\mathbf{h}$ for each $\lambda \in \mathbb{P}^1_{\Bbbk}$).

3. Semi-invariants

Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. By $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ we denote the set of the representations M of $\mathbf{\Delta}$ such that $M(x) = \Bbbk^{\mathbf{d}(x)}$ for each $x \in \Delta_0$. We may identify $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ with a Zariski-closed subset of the affine space $\operatorname{rep}_{\Delta}(\mathbf{d}) := \prod_{\alpha \in \Delta_1} \mathbb{M}_{\mathbf{d}(t\alpha) \times \mathbf{d}(s\alpha)}(\Bbbk)$, hence it has a structure of an affine variety. The group $\operatorname{GL}(\mathbf{d}) := \prod_{x \in \Delta_0} \operatorname{GL}(\mathbf{d}(x))$ acts on $\operatorname{rep}_{\Delta}(\mathbf{d})$ by conjugation: $(g * M)(\alpha) := g(t\alpha) \cdot M(\alpha) \cdot g(s\alpha)^{-1}$ for $g \in \operatorname{GL}(\mathbf{d})$, $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$ and $\alpha \in \Delta_1$. The set $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ is a $\operatorname{GL}(\mathbf{d})$-invariant subset of $\operatorname{rep}_{\Delta}(\mathbf{d})$ and the $\operatorname{GL}(\mathbf{d})$-orbits in $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ correspond to the isomorphism classes of the representations of $\mathbf{\Delta}$ with dimension vector $\mathbf{d}$. If $\mathcal{X}$ is a full subcategory of $\operatorname{ind} \mathbf{\Delta}$, then we denote by $\mathcal{X}(\mathbf{d})$ the set of $V \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ such that $V \in \operatorname{add} \mathcal{X}$.

Let Δ be a quiver and $\theta \in \mathbb{Z}^{\Delta_0}$. We treat θ as a $\mathbb{Z}$-linear function $\mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ in a usual way. If $\mathbf{d}$ is a dimension vector, then by a semi-invariant of weight θ we mean every function $f \in \Bbbk[\operatorname{rep}_{\Delta}(\mathbf{d})]$ such that $f(g^{-1} * M) = \chi^\theta(g) \cdot f(M)$ for any $g \in \operatorname{GL}(\mathbf{d})$ and $M \in \operatorname{rep}_{\Delta}(\mathbf{d})$, where $\chi^\theta(g) := \prod_{x \in \Delta_0} (\det g(x))^{\theta(x)}$ for $g \in \operatorname{GL}(\mathbf{d})$.

Now let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If $\theta \in \mathbb{Z}^{\Delta_0}$, then a function $f \in \Bbbk[\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})]$ is called a semi-invariant of weight θ if f is the restriction of a semi-invariant of weight θ from $\Bbbk[\operatorname{rep}_{\Delta}(\mathbf{d})]$. This definition differs from the definition used in other papers on the subject (see for example [5, 11, 13, 15]), however these are the semi-invariants which one needs to understand in order to study King's moduli spaces for representations of bound quivers [21]. Moreover, the two approaches coincide if the characteristic of $\Bbbk$ equals 0. We denote the space of the semi-invariants of weight θ by $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta$. If $\mathbf{d}$ is sincere, then we put $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}] := \bigoplus_{\theta \in \mathbb{Z}^{\Delta_0}} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta$ and call it the algebra of semi-invariants for $\mathbf{\Delta}$ and $\mathbf{d}$.

We recall a construction from [13]. Let $\mathbf{\Delta}$ be a bound quiver. Fix a representation V of $\mathbf{\Delta}$ and define $\theta^V : \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ by the condition:

$$\theta^V(\mathbf{dim}\, M) = \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(V, M) - \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(M, \tau V)$$

for each representation M of $\mathbf{\Delta}$. The formula (1.1) implies that $\theta^V = \langle \mathbf{dim}\, V, -\rangle_{\mathbf{\Delta}}$ if $\operatorname{pdim}_{\mathbf{\Delta}} V \leq 1$. Dually, if V has no indecomposable projective direct summands (i.e. $\tau^-\tau V \simeq V$ [1, Theorem IV.2.10]) and $\operatorname{idim}_{\mathbf{\Delta}} \tau V \leq 1$, then $\theta^V = -\langle -, \mathbf{dim}\, \tau V\rangle_{\mathbf{\Delta}}$ by the formula (1.2).

Now let $\mathbf{d}$ be a dimension vector. If $\theta^V(\mathbf{d}) = 0$, then we define a function $c^V_{\mathbf{d}} \in \Bbbk[\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})]$ in the following way. Let $P_1 \xrightarrow{f} P_0 \to V \to 0$ be the minimal projective presentation of V. One shows that

$$\dim_{\Bbbk} \operatorname{Ker} \operatorname{Hom}_{\mathbf{\Delta}}(f, M) = \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(V, M)$$

and

$$\dim_{\Bbbk} \operatorname{Coker} \operatorname{Hom}_{\mathbf{\Delta}}(f, M) = \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(M, \tau V),$$

hence

$$\begin{aligned}(3.1)\quad & \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(P_0, M) - \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(P_1, M) \\ & \qquad = \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(V, M) - \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(M, \tau V) = \theta^V(\mathbf{d}) = 0,\end{aligned}$$

for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. Thus, we may define $c^V_{\mathbf{d}} \in \Bbbk[\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})]$ by the formula $c^V_{\mathbf{d}}(M) := \det \operatorname{Hom}_{\mathbf{\Delta}}(f, M)$ for $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. Note that $c^V_{\mathbf{d}}$ is defined only up to a non-zero scalar. If $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then $c^V_{\mathbf{d}}(M) = 0$ if and only if $\operatorname{Hom}_{\mathbf{\Delta}}(V, M) \neq 0$. Moreover, if $\operatorname{pdim}_{\mathbf{\Delta}} V \leq 1$ and $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then $c^V_{\mathbf{d}}(M) = 0$ if and only if $\operatorname{Ext}^1_{\mathbf{\Delta}}(V, M) \neq 0$. It is known that $c^V_{\mathbf{d}} \in \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\theta^V}$. This function depends on the choice of f, but the functions obtained for different f's differ only by non-zero scalars.

In fact, we could start with an arbitrary $\mathbf{d}$-admissible projective presentation, where for a representation V of a bound quiver $\mathbf{\Delta}$ and a dimension vector $\mathbf{d}$ we call a projective representation $P_1 \xrightarrow{f} P_0 \to V \to 0$ of V $\mathbf{d}$-admissible if $\dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(P_0, M) = \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(P_1, M)$ for any (equivalently, some) $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$.

Lemma 3.1. *Let $\mathbf{\Delta}$ be a bound quiver, $\mathbf{d}$ a dimension vector and $P_1 \xrightarrow{f} P_0 \to V \to 0$ a $\mathbf{d}$-admissible projective presentation of a representation V of $\mathbf{\Delta}$.*

1. *If $\theta^V(\mathbf{d}) = 0$, then there exists $\xi \in \Bbbk$ such $\xi \neq 0$ and $c^V_{\mathbf{d}}(M) = \xi \cdot \det \operatorname{Hom}_{\mathbf{\Delta}}(f, M)$ for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$.*
2. *If there exists $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ such that $\det \operatorname{Hom}_{\mathbf{\Delta}}(f, M) \neq 0$, then $\theta^V(\mathbf{d}) = 0$.*

Proof. Let $P_1' \xrightarrow{f'} P_0' \to V \to 0$ be the minimal projective presentation of V. There exists projective representations P and Q of $\mathbf{\Delta}$ and isomorphisms $g_1 : P_1 \to P_1' \oplus P \oplus Q$ and $g_0 : P_0 \to P_0' \oplus P$ such that

$$f = g_0^{-1} \circ \begin{bmatrix} f' & 0 & 0 \\ 0 & \operatorname{Id}_P & 0 \end{bmatrix} \circ g_1.$$

Consequently,

$$\text{(3.2)}\quad \operatorname{Hom}_{\mathbf{\Delta}}(f,M) = \operatorname{Hom}_{\mathbf{\Delta}}(g_1,M) \circ \begin{bmatrix} \operatorname{Hom}_{\mathbf{\Delta}}(f',M) & 0 \\ 0 & \operatorname{Hom}_{\mathbf{\Delta}}(\mathrm{Id}_P,M) \\ 0 & 0 \end{bmatrix} \circ \operatorname{Hom}_{\mathbf{\Delta}}(g_0^{-1},M)$$

for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. Since the presentation $P_1 \xrightarrow{f} P_0 \to V \to 0$ is $\mathbf{d}$-admissible, (3.1) implies that the condition $\theta^V(\mathbf{d}) = 0$ is equivalent to the condition $\dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(Q,M) = 0$ for each for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. Together with (3.2) this implies our claims. □

As an immediate consequence we obtain the following.

Corollary 3.2. *Let $\mathbf{\Delta}$ be a bound quiver, $\mathbf{d}$ a dimension vector and $0 \to V_1 \to V \to V_2 \to 0$ an exact sequence such that $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$.*

1. *If $\theta^V(\mathbf{d}) = 0$, then (up to a non-zero scalar) $c_{\mathbf{d}}^V = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2}$.*
2. *If $c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2} \neq 0$, then $\theta^V(\mathbf{d}) = 0$ and (up to a non-zero scalar) $c_{\mathbf{d}}^V = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2}$.*

Proof. Let $P_1' \xrightarrow{f'} P_0' \to V_1 \to 0$ and $P_1'' \xrightarrow{f''} P_0' \to V_2 \to 0$ be the minimal projective presentations of V_1 and V_2, respectively. Then there exists a projective presentation of V of the form

$$P_1' \oplus P_1'' \xrightarrow{f} P_0' \oplus P_0'' \to V \to 0,$$

where $f = \begin{bmatrix} f' & g \\ 0 & f'' \end{bmatrix}$ for some $g \in \operatorname{Hom}_{\mathbf{\Delta}}(P_1'', P_0')$. One easily sees that $\det \operatorname{Hom}_{\mathbf{\Delta}}(f,M) = c_{\mathbf{d}}^{V_1}(M) \cdot c_{\mathbf{d}}^{V_2}(M)$ for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, hence the claims follows from Lemma 3.1. □

The following fact is an extension of [10, Lemma 1(a)] to the setup of bound quivers.

Lemma 3.3. *Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence, $\theta^V(\mathbf{d}) = 0$ and $c_{\mathbf{d}}^V \neq 0$, then $\theta^{V_2}(\mathbf{d}) \leq 0$.*

Proof. If $\theta^{V_2}(\mathbf{d}) > 0$, then

$$\dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(V_2, M) \geq \theta^{V_2}(\mathbf{d}) > 0$$

for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$. This immediately implies that $\operatorname{Hom}_{\mathbf{\Delta}}(V,M) \neq 0$ for each $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, hence $c_{\mathbf{d}}^V = 0$, contradiction. □

We have the following multiplicative property.

Lemma 3.4. *Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If V_1 and V_2 are representations of $\mathbf{\Delta}$, $V := V_1 \oplus V_2$, $\theta^V(\mathbf{d}) = 0$ and $c_{\mathbf{d}}^V \neq 0$, then $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$ and $c_{\mathbf{d}}^V = c_{\mathbf{d}}^{V_1} \cdot c_{\mathbf{d}}^{V_2}$ (up to a non-zero scalar).*

Proof. See [13, Lemma 3.3]. □

We will also use another multiplicative property.

Lemma 3.5. *Let $\mathbf{\Delta}$ be a bound quiver and V a representation of $\mathbf{\Delta}$. If $\mathbf{d}'$ and $\mathbf{d}''$ are dimension vectors and $\theta^V(\mathbf{d}') = 0 = \theta^V(\mathbf{d}'')$, then*

$$c^V_{\mathbf{d}'+\mathbf{d}''}(W' \oplus W'') = c^V_{\mathbf{d}'}(W') \cdot c^V_{\mathbf{d}''}(W'')$$

for all $(W', W'') \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}') \times \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d}'')$.

Proof. Let $P_1 \xrightarrow{f} P_0 \to V \to 0$ be the minimal projective presentation of V. Then

$$\operatorname{Hom}_{\mathbf{\Delta}}(f, W' \oplus W'') = \begin{bmatrix} \operatorname{Hom}_{\mathbf{\Delta}}(f, W') & \\ 0 & \operatorname{Hom}_{\mathbf{\Delta}}(f, W'') \end{bmatrix}$$

and both $\operatorname{Hom}_{\mathbf{\Delta}}(f, W')$ and $\operatorname{Hom}_{\mathbf{\Delta}}(f, W'')$ are square matrices, hence the claim follows. □

The following result follows from the proof of [13, Theorem 3.2] (note that the assumption about the characteristic of $\Bbbk$ made in [13, Theorem 3.2] is only necessary to prove surjectivity of the restriction morphism, which we have for free with our definition of semi-invariants).

Proposition 3.6. *Let $\mathbf{\Delta}$ be a bound quiver, $\mathbf{d}$ a dimension vector and $\theta \in \mathbb{Z}^{\Delta_0}$.*

(1) *If $\theta(\mathbf{d}) \neq 0$, then $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta = 0$.*
(2) *If $\theta(\mathbf{d}) = 0$, then the space $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta$ is spanned by the functions $c^V_{\mathbf{d}}$ for $V \in \operatorname{rep} \mathbf{\Delta}$ such that $\theta^V = \theta$ and $c^V_{\mathbf{d}} \neq 0$.* □

In fact we may take a smaller spanning set.

Corollary 3.7. *Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a dimension vector. If $\theta \in \mathbb{Z}^{\Delta_0}$ and $\theta(\mathbf{d}) = 0$, then the space $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta$ is spanned by the functions $c^V_{\mathbf{d}}$ for ext-minimal $V \in \operatorname{rep} \mathbf{\Delta}$ such that $\theta^V = \theta$ and $c^V_{\mathbf{d}} \neq 0$.*

Proof. Assume that V is a representation of $\mathbf{\Delta}$ such that $\theta^V = \theta$, $c^V_{\mathbf{d}} \neq 0$ and there is a decomposition $V = V_1 \oplus V_2$ with $\operatorname{Ext}^1_{\mathbf{\Delta}}(V_1, V_2) \neq 0$. Lemma 3.4 implies that $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$ and $c^{V_1}_{\mathbf{d}} \cdot c^{V_2}_{\mathbf{d}} \neq 0$. If $0 \to V_2 \to W \to V_1 \to 0$ is a non-split exact sequence, then Corollary 3.2(2) and Lemma 3.4 imply that (up to a non-zero scalar) $c^W_{\mathbf{d}} = c^{V_1}_{\mathbf{d}} \cdot c^{V_2}_{\mathbf{d}} = c^V_{\mathbf{d}}$. Since $\dim_{\Bbbk} \operatorname{End}_{\mathbf{\Delta}}(W) < \dim_{\Bbbk} \operatorname{End}_{\mathbf{\Delta}}(V)$ (see for example [25, Lemma 2.1]), the claim follows by induction. □

We may even take a smaller set, if we are only interested in generators of $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$. Namely, we have the following.

Corollary 3.8. *Let $\mathbf{\Delta}$ be a bound quiver and $\mathbf{d}$ a sincere dimension vector. Then the algebra $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$ is generated by the semi-invariants $c^V_{\mathbf{d}}$ for $V \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ such that $\theta^V(\mathbf{d}) = 0$, $c^V_{\mathbf{d}} \neq 0$ and V is indecomposable.*

Proof. This follows from Proposition 3.6 and Lemma 3.4 (this is also the content of [13, Corollary 3.4]). □

4. Preliminary results

Throughout this section we fix a concealed-canonical bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind}\mathbf{\Delta}$. We will use notation introduced in Section 2. We also fix $\mathbf{d} \in \mathbf{R}$ such that $p := p^{\mathbf{d}} > 0$. Notice that this implies that $\mathbf{d}$ is sincere.

First we prove that the algebra $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]$ is controlled by the representations from $\operatorname{add}\mathcal{R}$.

Lemma 4.1. *Let V be a representation of $\mathbf{\Delta}$ such that $\theta^V(\mathbf{d}) = 0$. If $c_{\mathbf{d}}^V \neq 0$, then $V \in \operatorname{add}\mathcal{R}$ and $\theta^V = \langle \mathbf{dim}\, V, - \rangle_{\mathbf{\Delta}}$.*

Proof. Assume that $P \in \mathcal{P}$ is a direct summand of V. Since $\operatorname{pdim}_{\mathbf{\Delta}} P \leq 1$, (2.1) and (2.3) imply

$$\theta^P(\mathbf{d}) = \langle \mathbf{dim}\, P, \mathbf{d} \rangle_{\mathbf{\Delta}} \geq \langle \mathbf{dim}\, P, \mathbf{h} \rangle_{\mathbf{\Delta}} > 0.$$

Consequently, $c_{\mathbf{d}}^V = 0$ by Lemma 3.4, contradiction. Dually, V cannot have a direct summand from $\mathcal{Q}$. Finally, since $\operatorname{pdim}_{\mathbf{\Delta}} V = 1$, $\theta^V = \langle \mathbf{dim}\, V, - \rangle_{\mathbf{\Delta}}$. □

Together with Corollary 3.7 this lemma immediately implies the following.

Corollary 4.2. *Let $\theta \in \mathbb{Z}^{\mathbf{\Delta}_0}$ be such that $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_\theta \neq 0$. Then there exists $\mathbf{r} \in \mathbf{R}$ such that $\theta = \langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}$ and $\langle \mathbf{r}, \mathbf{d} \rangle_{\mathbf{\Delta}} = 0$.* □

Taking into account Corollary 3.8 we need to identify $V \in \operatorname{ind}\mathbf{\Delta}$ such that $\theta^V(\mathbf{d}) = 0$ and $c_{\mathbf{d}}^V \neq 0$. The first step in this direction is the following.

Lemma 4.3. *Let V be an indecomposable representation of $\mathbf{\Delta}$. If $\theta^V(\mathbf{d}) = 0$ and $c_{\mathbf{d}}^V \neq 0$, then $V = R_{\lambda,i+1}^{(n)}$ for some $\lambda \in \mathbb{P}_{\Bbbk}^1$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ such that $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$ and $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i}^{\mathbf{d}}$ for each $j \in [i+1, i+n-1]$.*

Proof. We know from Lemma 4.1 that $V \in \mathcal{R}$, hence there exists $\lambda \in \mathbb{P}_{\Bbbk}^1$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ such that $V = R_{\lambda,i+1}^{(n)}$. Then $\theta^V(\mathbf{d}) = p_{\lambda,i+n}^{\mathbf{d}} - p_{\lambda,i}^{\mathbf{d}}$ by (2.5), thus the condition $\theta^V(\mathbf{d}) = 0$ means that $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$. Finally, the condition $c_{\mathbf{d}}^V \neq 0$ and Lemma 3.3 imply that $\theta^{V'}(\mathbf{d}) \leq 0$ for each factor representation V' of V. The sequence (2.2) implies that $R_{\lambda,j+1}^{(n+i-j)}$ is a factor representation of V for each $j \in [i+1, i+n-1]$, hence the claim follows. □

Now we show that the representations described in the above lemma give rise to non-zero semi-invariants.

Lemma 4.4. *Let $\lambda \in \mathbb{P}_{\Bbbk}^1$, $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ be such that $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$ and $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i+n}^{\mathbf{d}}$ for each $j \in [i+1, i+n-1]$. If $V := R_{\lambda,i+1}^{(n)}$, then $\theta^V(\mathbf{d}) = 0$ and there exists $R \in \mathcal{R}(\mathbf{d})$ such that $c_{\mathbf{d}}^V(R) \neq 0$.*

Proof. We only need to show that there exists $R \in \mathcal{R}(\mathbf{d})$ such that $c_{\mathbf{d}}^V(R) \neq 0$. Let $W \in \operatorname{add} \mathcal{R}$ be an ext-minimal representation for $\mathbf{d} - p \cdot \mathbf{h}$ and fix $\mu \in \mathbb{P}^1_{\Bbbk}$ different from λ such that $r_\mu = 1$. If $R := W \oplus R_{\mu,0}^{(p)}$, then $R \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ and $\operatorname{Hom}_{\mathbf{\Delta}}(V, R) = \operatorname{Hom}_{\mathbf{\Delta}}(V, W) = 0$ by Lemma 2.1, hence the claim follows. □

As a consequence we present a smaller generating set of $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$. First we introduce some notation. For $\lambda \in \mathbb{P}^1_{\Bbbk}$ we denote by $\mathcal{I}_\lambda$ the set of $i \in [0, r_\lambda - 1]$ such that there exists $n \in \mathbb{N}_+$ with $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$ and $p_{\lambda,j}^{\mathbf{d}} > p_{\lambda,i}^{\mathbf{d}}$ for each $j \in [i+1, i+n-1]$ (such n, if exists, is uniquely determined by λ and i, and we denote it by $n_{\lambda,i}$). Observe that $\mathcal{I}_\lambda = \{0\}$ and $n_{\lambda,0} = 1$ if $r_\lambda = 1$.

Corollary 4.5. *The algebra* $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$ *is generated by the semi-invariants* $c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n_{\lambda,i})}}$ *for* $\lambda \in \mathbb{P}^1_{\Bbbk}$ *and* $i \in \mathcal{I}_\lambda$.

Proof. For $\lambda \in \mathbb{P}^1_{\Bbbk}$ we denote by $\mathcal{I}_\lambda'$ the set of all pairs $(i, n) \in [0, r_\lambda - 1] \times \mathbb{N}_+$ such that $p_{\lambda,i}^{\mathbf{d}} = p_{\lambda,i+n}^{\mathbf{d}}$ and $p_{\lambda,j}^{\mathbf{d}} \geq p_{\lambda,i}^{\mathbf{d}}$ for each $j \in [i+1, i+n-1]$. Observe that if $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $(i, n) \in \mathcal{I}_\lambda'$, then $i \in \mathcal{I}_\lambda$. Corollary 3.8 and Lemma 4.3 imply that the algebra $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$ is generated by the semi-invariants $c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n)}}$ for $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $(i, n) \in \mathcal{I}_\lambda'$. Now, let $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $(i, n) \in \mathcal{I}_\lambda'$. Obviously, $n \geq n_{\lambda,i_\lambda}$. If $n > n_{\lambda,i_\lambda}$, then $c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n)}} = c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n_{\lambda,i})}} \cdot c_{\mathbf{d}}^{R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})}}$ by Corollary 3.2(1), as according to (2.2) we have an exact sequence

$$0 \to R_{\lambda,i+1}^{(n_{\lambda,i})} \to R_{\lambda,i+1}^{(n)} \to R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})} \to 0.$$

Since $R_{\lambda,i+n_{\lambda,i}+1}^{(n-n_{\lambda,i})} = R_{\lambda,(i+n_{\lambda,i}+1) \bmod r_\lambda}^{(n-n_{\lambda,i})}$ and $((i + n_{\lambda,i}) \bmod r_\lambda, n - n_{\lambda,i}) \in \mathcal{I}_\lambda'$, the claim follows by induction. □

At the later stage we will prove that for each non-zero semi-invariant f there exists $R \in \mathcal{R}(\mathbf{d})$ such that $f(R) \neq 0$. At the moment we formulate the following versions of this fact.

Lemma 4.6. *Let* V *be a representation of* $\mathbf{\Delta}$ *such that* $\theta^V(\mathbf{d}) = 0$ *and* $c_{\mathbf{d}}^V \neq 0$. *Then there exists* $R \in \mathcal{R}(\mathbf{d})$ *such that* $c_{\mathbf{d}}^V(R) \neq 0$.

Proof. Let X be an indecomposable direct summand of V. Lemma 3.4 implies that $c_{\mathbf{d}}^X \neq 0$. Consequently, Lemmas 4.3 and 4.4 imply that there exists $R_X \in \mathcal{R}(\mathbf{d})$ such that $c_{\mathbf{d}}^X(R_X) \neq 0$. Since $\mathcal{R}(\mathbf{d})$ is an irreducible and open subset of $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ [15, Section 4], there exists $R \in \mathcal{R}(\mathbf{d})$ such that $c_{\mathbf{d}}^X(R) \neq 0$ for each indecomposable direct summand X of V. Using once more Lemma 3.4 we obtain that $c_{\mathbf{d}}^V(R) \neq 0$. □

Lemma 4.7. *If* $q \in \mathbb{N}$ *and* $f \in \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\mathbf{\Delta}}}$ *is non-zero, then there exists* $R \in \mathcal{R}(\mathbf{d})$ *such that* $f(R) \neq 0$.

Proof. If $q = 0$, then the claim is obvious, since $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_0 = \Bbbk$. Thus assume $q > 0$. We know that $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}$ is spanned by the functions $c_{\mathbf{d}}^V$ for $V \in \operatorname{add}\mathcal{R}$ with dimension vector $q \cdot \mathbf{h}$. It is enough to prove that $c_{\mathbf{d}}^V(M) = 0$ for all $V \in \operatorname{add}\mathcal{R}$ and $M \in \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ such that $\mathbf{dim}\, V = q \cdot \mathbf{h}$ and $M \notin \mathcal{R}(\mathbf{d})$. Every such M has an indecomposable direct summand Q from $\mathcal{Q}$. Indeed, since $M \notin \mathcal{R}(\mathbf{d})$, it has an indecomposable direct summand X which belongs to $\mathcal{P} \cup \mathcal{Q}$. If $X \in \mathcal{Q}$, then we take $Q := X$. If $X \in \mathcal{P}$, then $\langle \mathbf{dim}\, M - \mathbf{dim}\, X, \mathbf{h}\rangle_{\mathbf{\Delta}} < 0$ by (2.3) and (2.6). Consequently, M has an indecomposable direct summand Q with $\langle \mathbf{dim}\, Q, \mathbf{h}\rangle_{\mathbf{\Delta}} < 0$. Using again (2.3) and (2.6) we get $Q \in \mathcal{Q}$. Then

$$\dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(V, M) \geq \dim_{\Bbbk} \operatorname{Hom}_{\mathbf{\Delta}}(V, Q) = \langle q \cdot \mathbf{h}, \mathbf{dim}\, Q\rangle_{\mathbf{\Delta}} > 0$$

by (2.4) and the claim follows. □

Recall from Corollary 4.2 that the possible weights are of the form $\langle \mathbf{r}, -\rangle_{\mathbf{\Delta}}$ for $\mathbf{r} \in \mathbf{R}$ such that $\langle \mathbf{r}, \mathbf{d}\rangle_{\mathbf{\Delta}} = 0$. Our next aim is to show that it is enough to understand those which are for the form $\langle q \cdot \mathbf{h}, -\rangle_{\mathbf{\Delta}}$ for $q \in \mathbb{N}$.

We start with the following easy lemma.

Lemma 4.8. *Let $W \in \operatorname{add}\mathcal{R}$ be such that $\theta^W(\mathbf{d}) = 0$ and $c_{\mathbf{d}}^W \neq 0$. If $q \in \mathbb{N}$ and $f \in \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}$ is non-zero, then there exists $R \in \mathcal{R}(\mathbf{d})$ such that $c_{\mathbf{d}}^W(R) \cdot f(R) \neq 0$.*

Proof. Since $\mathcal{R}(\mathbf{d})$ is an open irreducible subset of $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$, the claim follows from Lemmas 4.6 and 4.7. □

Proposition 4.9. *Let $\mathbf{r} \in \mathbf{R}$, $\langle \mathbf{r}, \mathbf{d}\rangle_{\mathbf{\Delta}} = 0$ and $W \in \operatorname{add}\mathcal{R}$ be an ext-minimal representation for $\mathbf{r} - p^{\mathbf{r}} \cdot \mathbf{h}$.*

1. *If $c_{\mathbf{d}}^W = 0$, then $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r},-\rangle_{\mathbf{\Delta}}} = 0$.*
2. *If $c_{\mathbf{d}}^W \neq 0$, then the map*

$$\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle p^{\mathbf{r}}\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}} \to \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r},-\rangle_{\mathbf{\Delta}}}, f \mapsto c_{\mathbf{d}}^W \cdot f,$$

is an isomorphism of vector spaces.

Proof. Let $\Phi : \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle p^{\mathbf{r}}\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}} \to \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r},-\rangle_{\mathbf{\Delta}}}$ be the map given by $\Phi(f) := c_{\mathbf{d}}^W \cdot f$ for $f \in \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle p^{\mathbf{r}}\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}$.

It follows from Corollary 3.7 and Lemma 4.1 that $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r},-\rangle_{\mathbf{\Delta}}}$ is spanned by the functions $c_{\mathbf{d}}^V$ for ext-minimal $V \in \operatorname{add}\mathcal{R}$ such that $\mathbf{dim}\, V = \mathbf{r}$. If $V \in \operatorname{add}\mathcal{R}$ is ext-minimal and $\mathbf{dim}\, V = \mathbf{r}$, then there exists an exact sequence $0 \to R \to V \to W \to 0$, where $R \in \operatorname{add}\mathcal{R}$ and $\mathbf{dim}\, R = p^{\mathbf{r}} \cdot \mathbf{h}$. Thus Corollary 3.2(1) implies that $c_{\mathbf{d}}^V = c_{\mathbf{d}}^W \cdot c_{\mathbf{d}}^R = \Phi(c_{\mathbf{d}}^R)$. This shows that Φ is an epimorphism. In particular, $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r},-\rangle_{\mathbf{\Delta}}} = 0$ if $c_{\mathbf{d}}^W = 0$. On the other hand, if $c_{\mathbf{d}}^W \neq 0$, then Φ is a monomorphism (hence an isomorphism) by Lemma 4.8. □

In the previous papers on the subject the authors have studied either the semi-invariants on the whole variety $\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ [14, 15] or on the closure of $\mathcal{R}(\mathbf{d})$ only [29]. However, the answers they have obtained did not differ. We have the following explanation of this phenomena.

Proposition 4.10. *If $f \in \Bbbk[\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})]$ is a non-zero semi-invariant, then there exists $R \in \mathcal{R}(\mathbf{d})$ such that $f(R) \neq 0$.*

Proof. Fix $\mathbf{r} \in \mathbf{R}$ such that $f \in \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}}$. The previous lemma implies that $f = c_{\mathbf{d}}^W \cdot f'$, where $W \in \operatorname{add} \mathcal{R}$ is an ext-minimal representation with dimension vector $\mathbf{r} - p^{\mathbf{r}} \cdot \mathbf{h}$ and $f' \in \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\mathbf{\Delta}}}$. Consequently, the claim follows from Lemma 4.8. □

Observe that this proposition means in particular, that $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$ is a domain, hence the product of two non-zero semi-invariants is non-zero again.

5. The Kronecker quiver

Our aim in this section is to collect necessary facts about representations and semi-invariants for the Kronecker quiver K_2, i.e. the quiver $\underset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \underset{2}{\bullet}$ with the empty set of relations. In this case a sincere separating exact subcategory is uniquely determined. Let $\mathcal{T} = \coprod_{\lambda \in \mathbb{P}^1_{\Bbbk}} \mathcal{T}_\lambda$ by the sincere separating exact subcategory of $\operatorname{ind} K_2$.

For $\zeta, \xi \in \Bbbk$ let $N_{\zeta,\xi}$ be the representation $\Bbbk \underset{\xi}{\overset{\zeta}{\leftleftarrows}} \Bbbk$. Then the simple objects in $\operatorname{add} \mathcal{T}$ are precisely the representations $N_{\zeta,\xi}$ for $(\zeta : \xi) \in \mathbb{P}^1_k$. Moreover, if $(\zeta : \xi), (\zeta' : \xi') \in \mathbb{P}^1_{\Bbbk}$, then $N_{\zeta,\xi} \simeq N_{\zeta',\xi'}$ if and only if $(\zeta : \xi) = (\zeta' : \xi')$. Consequently, by abuse of notation, we will denote $N_{\zeta,\xi}$ by $N_{(\zeta:\xi)}$ for $(\zeta : \xi) \in \mathbb{P}^1_k$. By choosing our parameterization appropriately we may assume that $N_\lambda \in \mathcal{T}_\lambda$ for each $\lambda \in \mathbb{P}^1_k$. In particular, $\tau N_\lambda = N_\lambda$ for each $\lambda \in \mathbb{P}^1_{\Bbbk}$.

The Kronecker quiver can be viewed as the minimal concealed-canonical bound quiver. Namely, we can embed the category $\operatorname{rep} K_2$ into the category of representations of an arbitrary concealed-canonical quiver. We describe a construction of such embeddings more precisely.

Let $\mathbf{\Delta}$ be a bound quiver with a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind} \mathbf{\Delta}$. Let $R := \bigoplus_{\lambda \in \mathbb{P}^1_{\Bbbk}} \bigoplus_{i \in I_\lambda} R_{\lambda,i}$ for subsets $I_\lambda \subseteq [0, r_\lambda - 1]$ such that $|I_\lambda| = r_\lambda - 1$ (in particular, $I_\lambda = \varnothing$ if $r_\lambda = 1$), where we use notation introduced in Section 2. Let $R^\perp$ denote the full subcategory of $\operatorname{rep} \mathbf{\Delta}$, whose objects are $M \in \operatorname{rep} \mathbf{\Delta}$ such that $\operatorname{Hom}_{\mathbf{\Delta}}(R, M) = 0 = \operatorname{Ext}^1_{\mathbf{\Delta}}(R, M)$. Lenzing and de la Peña [23, Proposition 4.2] have proved that there exists a full faithfully exact functor $F : \operatorname{rep} K_2 \to \operatorname{rep} \mathbf{\Delta}$ which induces an equivalence between $\operatorname{rep} K_2$ and $R^\perp$. Moreover, F induces an equivalence between $\mathcal{T}$ and $R^\perp \cap \mathcal{R}$. The simple objects in $R^{\perp} \cap (\operatorname{add} \mathcal{R})$, which are the images of the simple objects in $\operatorname{add} \mathcal{T}$, are of the form $R_{\lambda,i_\lambda}^{(r_\lambda)}$ for $\lambda \in \mathbb{P}^1_{\Bbbk}$, where for $\lambda \in \mathbb{P}^1_{\Bbbk}$ we denote by i_λ the unique element of $[0, r_\lambda - 1] \setminus I_\lambda$. Consequently, (if we choose appropriate parameterization) $F(N_\lambda) \simeq R_{\lambda,i_\lambda}^{(r_\lambda)}$ for each $\lambda \in \mathbb{P}^1_{\Bbbk}$.

Let $p \in \mathbb{N}$. We define the functions $f_{(p,p)}^{(0)}, \ldots, f_{(p,p)}^{(p)} \in \Bbbk[\operatorname{rep}_{K_2}(p,p)]$ by the condition: if $V \in \operatorname{rep}_{K_2}(p,p)$, then

$$\det(S \cdot V_\alpha - T \cdot V_\beta) = \sum_{i \in [0,p]} S^i \cdot T^{p-i} \cdot f_{(p,p)}^{(i)}(V).$$

Note that $f_{(p,p)}^{(0)}, \ldots, f_{(p,p)}^{(p)}$ are semi-invariants of weight $(-1,1)$. If $(\zeta : \xi) \in \mathbb{P}^1_{\Bbbk}$, then (by choosing a projective presentation of $N_{\zeta,\xi}$ in an appropriate way) we get

$$c_{(p,p)}^{N_{\zeta,\xi}}(V) = \det(\xi \cdot V_\alpha - \zeta \cdot V_\beta) = \sum_{i \in [0,p]} \xi^i \cdot \zeta^{p-i} \cdot f_{(p,p)}^{(i)}(V). \tag{5.1}$$

It is well known (see for example [30]) that $\operatorname{SI}[K_2,(p,p)]$ is the polynomial algebra in $f_{(p,p)}^{(0)}, \ldots, f_{(p,p)}^{(p)}$. In particular,

$$\dim_{\Bbbk} \operatorname{SI}[K_2,(p,p)]_{(-q,q)} = \binom{q+p}{q} \tag{5.2}$$

for each $q \in \mathbb{N}$.

We will need the following lemma.

Lemma 5.1. *If* $f_1, f_2 \in \operatorname{SI}[K_2,(p,p)]_{(-1,1)}$ *and*

$$\{V \in \operatorname{rep}_{K_2}(p,p) : f_1(V) = 0\} = \{V \in \operatorname{rep}_{K_2}(p,p) : f_2(V) = 0\},$$

then (*up to a non-zero scalar*) $f_1 = f_2$.

Proof. From the description of $\operatorname{SI}[K_2,(p,p)]$ it follows that f_1 and f_2 are irreducible, hence the claim follows. □

6. The main result

Throughout this section we fix a concealed-canonical bound quiver $\mathbf{\Delta}$ and a sincere separating exact subcategory $\mathcal{R}$ of $\operatorname{ind} \mathbf{\Delta}$. We use freely notation introduced in Section 2. We also fix $\mathbf{d} \in \mathbf{R}$ such that $p := p^{\mathbf{d}} > 0$.

First we investigate the algebra $\bigoplus_{q \in \mathbb{N}} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle q \cdot \mathbf{h}, - \rangle_{\mathbf{\Delta}}}$. We introduce some notation. For $\lambda \in \mathbb{P}^1_{\Bbbk}$ we denote by $\mathcal{I}_\lambda^0$ the set of $i \in [0, r_\lambda - 1]$ such that $p_{\lambda,i}^{\mathbf{d}} = 0$. Observe that $\mathcal{I}_\lambda^0 \subseteq \mathcal{I}_\lambda$ for each $\lambda \in \mathbb{P}^1_{\Bbbk}$. Recall that, for $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in \mathcal{I}_\lambda$, $n_{\lambda,i}$ denotes the minimal $n \in \mathbb{N}_+$ such that $p_{\lambda,i+n}^{\mathbf{d}} = 0$. We put $c_{\mathbf{d}}^\lambda := \prod_{i \in \mathcal{I}_\lambda^0} c_{\mathbf{d}}^{R_{\lambda,i}^{(n_{\lambda,i})}}$. An iterated application of Corollary 3.2(1) to exact sequences of the form (2.2) implies that $c_{\mathbf{d}}^\lambda = c_{\mathbf{d}}^{R_{\lambda,i}^{(r_\lambda)}}$ for each $i \in \mathcal{I}_\lambda^0$.

We have the following fact.

Lemma 6.1. *The algebra* $\bigoplus_{q\in\mathbb{N}} \mathrm{SI}[\mathbf{\Delta},\mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}$ *is generated by the semi-invariants* $c_{\mathbf{d}}^{\lambda}$ *for* $\lambda\in\mathbb{X}$.

Proof. This fact has been proved in [4], but for completeness we include its (shorter) proof here.

Fix $q\in\mathbb{N}$. Proposition 3.7 and Lemma 4.1 imply that $\mathrm{SI}[\mathbf{\Delta},\mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}$ is spanned by the semi-invariants $c_{\mathbf{d}}^{V}$ for ext-minimal $V\in\operatorname{add}\mathcal{R}$ with dimension vector $q\cdot\mathbf{h}$. Fix such V. Since V is ext-minimal with dimension vector $q\cdot\mathbf{h}$, $V=\bigoplus_{\lambda\in\mathbb{X}} R_{\lambda,i_\lambda}^{(k_\lambda\cdot r_\lambda)}$, where $\mathbb{X}\subseteq\mathbb{P}^1_{\Bbbk}$ and, for each $\lambda\in\mathbb{X}$, $i_\lambda\in[0,r_\lambda-1]$ and $k_\lambda\in\mathbb{N}_+$. Moreover, Lemma 4.3 implies that $i_\lambda\in\mathcal{I}_\lambda^0$ for each $\lambda\in\mathbb{X}$. An iterated application of Corollary 3.2(1) to exact sequences of the form (2.2) implies that $c_{\mathbf{d}}^{R_{\lambda,i_\lambda}^{(k_\lambda\cdot r_\lambda)}}=(c_{\mathbf{d}}^{\lambda})^{k_\lambda}$ for each $\lambda\in\mathbb{X}$. Consequently, $c_{\mathbf{d}}^{V}=\prod_{\lambda\in\mathbb{X}}(c_{\mathbf{d}}^{\lambda})^{k_\lambda}$ by Lemma 3.4, hence the claim follows. □

The following fact is crucial.

Proposition 6.2. *There exists a regular map*

$$\Phi:\operatorname{rep}_{K_2}(p,p)\to\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$$

such that Φ^* *induces an isomorphism*

$$\bigoplus_{q\in\mathbb{N}}\mathrm{SI}[\mathbf{\Delta},\mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}\to\bigoplus_{q\in\mathbb{N}}\mathrm{SI}[K_2,(p,p)]_{(-q,q)}$$

of $\mathbb{N}$*-graded rings and (up to a non-zero scalar)* $\Phi^*(c_{\mathbf{d}}^{\lambda})=c_{(p,p)}^{N_\lambda}$ *for each* $\lambda\in\mathbb{P}^1_{\Bbbk}$.

Proof. For each $\lambda\in\mathbb{P}^1_{\Bbbk}$ we fix $i_\lambda\in\mathcal{I}_\lambda^0$. From Section 5 we know that there exists a fully faithful exact functor $F:\operatorname{rep}K_2\to\operatorname{rep}\mathbf{\Delta}$ such that $F(N_\lambda)\simeq R_{\lambda,i_\lambda}^{(r_\lambda)}$ for each $\lambda\in\mathbb{P}^1_{\Bbbk}$. Observe that for each $R\in\operatorname{add}(\coprod_{\lambda\in\mathbb{P}^1_{\Bbbk}\setminus\mathbb{X}_0}\mathcal{R}_\lambda)$ (recall that $\mathbb{X}_0$ is the set of $\lambda\in\mathbb{P}^1_{\Bbbk}$ such that $r_\lambda>1$) there exists $N\in\mathcal{T}$ such that $F(N)\simeq R$.

Put $E_1:=F(S_1)$ and $E_2:=F(S_2)$, where S_i is the simple representation of K_2 at i for $i\in\{1,2\}$, i.e.

$$S_1:=\ \Bbbk \leftleftarrows 0 \qquad\text{and}\qquad S_2:=\ 0\leftleftarrows \Bbbk\ .$$

Then [24, Proposition 2.3] (see also [9, Proposition 5.2]) implies that there exists a regular map $\Phi':\operatorname{rep}_{K_2}(p,p)\to\operatorname{rep}_{\mathbf{\Delta}}(p\cdot\mathbf{h})$ such that $\Phi'(N)\simeq F(N)$ for each $N\in\operatorname{rep}_{K_2}(p,p)$. Moreover, there exists a morphism $\varphi:\mathrm{GL}(p,p)\to\mathrm{GL}(p\cdot\mathbf{h})$ of algebraic groups such that $\Phi'(g*N)=\varphi(g)*\Phi'(N)$, for all $g\in\mathrm{GL}(p,p)$ and $N\in\operatorname{rep}_{\mathbf{\Delta}}(p\cdot\mathbf{h})$, and

$$\chi^{\theta}(\varphi(g))=(\det(g(1))^{\theta(\mathbf{dim}\,E_1)}\cdot(\det g(2))^{\theta(\mathbf{dim}\,E_2)}, \tag{6.1}$$

for all $g\in\mathrm{GL}(p,p)$ and $\theta\in\mathbb{Z}^{\Delta_0}$.

Let $W\in\operatorname{add}\mathcal{R}$ be an ext-minimal representation for $\mathbf{d}':=\mathbf{d}-p\cdot\mathbf{h}$. We define $\Phi:\operatorname{rep}_{K_2}(p,p)\to\operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ by $\Phi(N):=\Phi'(N)\oplus W$ for $N\in\operatorname{rep}_{K_2}(p,p)$.

Let $q \in \mathbb{N}$. We show that $\Phi^*(f)$ is a semi-invariant of weight $(-q, q)$ for each $f \in \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}$. Using Proposition 3.6 and Lemma 4.1 it suffices to show that $\Phi^*(c^V)$ is a semi-invariant of weight $(-q, q)$ for each representation V of $\mathbf{\Delta}$ with dimension vector $q \cdot \mathbf{h}$. Now, if $g \in \mathrm{GL}(p, p)$ and $N \in \mathrm{rep}_{K_2}(p, p)$, then

$$\begin{aligned}(\Phi^*(c^V_{\mathbf{d}}))(g^{-1} * N) &= c^V_{\mathbf{d}}(W \oplus \Phi'(g^{-1} * N)) \\ &= c^V_{\mathbf{d}'}(W) \cdot c^V_{p\cdot\mathbf{h}}(\varphi(g^{-1}) * \Phi'(N)) \\ &= c^V_{\mathbf{d}'}(W) \cdot \chi^{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}(\varphi(g)) \cdot c^V_{p\cdot\mathbf{h}}(\Phi(N)) \\ &= \chi^{\langle \mathbf{h},-\rangle_{\mathbf{\Delta}}}(\varphi(g)) \cdot (\Phi^*(c^V_{\mathbf{d}}))(N),\end{aligned}$$

where the second and the last equalities follow from Lemma 3.5. Using (6.1) we get

$$\chi^{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}}(\varphi(g)) = (\det(g(1))^{-q} \cdot (\det(g(2)))^q,$$

since

$$\langle \mathbf{h}, \mathbf{dim}\, E_i\rangle_{\mathbf{\Delta}} = \langle (1, 1), \mathbf{dim}\, S_i\rangle_{K_2} = (-1)^i$$

for each $i \in \{1, 2\}$ (we use here that F is exact).

The above implies that Φ^* induces a homomorphism

$$\bigoplus_{q\in\mathbb{N}} \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle q\cdot\mathbf{h},-\rangle_{\mathbf{\Delta}}} \to \bigoplus_{q\in\mathbb{N}} \mathrm{SI}[K_2, (p, p)]_{(-q,q)} \tag{6.2}$$

of $\mathbb{N}$-graded rings. We need to show that this is an isomorphism.

First we show $\Phi^*(f) \neq 0$ for each non-zero semi-invariant f (in particular, this will imply that (6.2) is a monomorphism). Let

$$\mathcal{Z} := \{M \in \mathrm{rep}_{\mathbf{\Delta}}(\mathbf{d}) : \text{there exists } N \in \mathrm{rep}_{K_2}(p, p) \text{ such that } M \simeq W \oplus \Phi(N)\}.$$

In other words, $\mathcal{Z}$ in the closure of the image of Φ under the action of $\mathrm{GL}(\mathbf{d})$. Using Proposition 4.10 it suffices to show that $\mathcal{Z}$ contains a non-empty open subset of $\mathcal{R}(\mathbf{d})$. Let

$$\mathcal{U} := \{M \in \mathcal{R}(\mathbf{d}) : c^\lambda_{\mathbf{d}}(M) \neq 0 \text{ for each } \lambda \in \mathbb{X}_0 \text{ and } \dim_{\Bbbk} \mathrm{End}_{\mathbf{\Delta}}(M) = p + \langle \mathbf{d}, \mathbf{d}\rangle_{\mathbf{\Delta}}\}.$$

Since the function

$$\mathrm{rep}_{\mathbf{\Delta}}(\mathbf{d}) \ni M \mapsto \dim_{\Bbbk} \mathrm{End}_{\mathbf{\Delta}}(M) \in \mathbb{Z}$$

is upper semi-continuous, (2.7) implies that $\mathcal{U}$ is a non-empty open subset $\mathcal{R}(\mathbf{d})$, which consists of ext-minimal representations. In particular, if $M \in \mathcal{U}$, then there exists an exact sequence of the form $0 \to R \to M \to W \to 0$ with $R \in \mathrm{add}\,\mathcal{R}$ such that $\mathbf{dim}\, R = p \cdot \mathbf{h}$. If $p^R_\lambda \neq 0$ for some $\lambda \in \mathbb{X}_0$, then $\mathrm{Hom}_{\mathbf{\Delta}}(R^{(r_\lambda)}_{\lambda,i_\lambda}, R) \neq 0$. Consequently, $\mathrm{Hom}_{\mathbf{\Delta}}(R^{(r_\lambda)}_{\lambda,i_\lambda}, M) \neq 0$, hence $c^\lambda_{\mathbf{d}}(M) = 0$, contradiction. Thus $p^R_\lambda = 0$

for each $\lambda \in \mathbb{X}_0$, hence $M \simeq W \oplus R$ and $R \in \operatorname{add}(\coprod_{\lambda \in \mathbb{P}^1_{\Bbbk} \setminus \mathbb{X}_0} \mathcal{R}_\lambda)$. In particular, there exists $N \in \operatorname{rep} \mathcal{T}$ such that $F(N) \simeq R$, hence $M \in \mathcal{Z}$.

Now we fix $\lambda \in \mathbb{P}^1_k$. We show that (up to a non-zero scalar) $\Phi^*(c^\lambda_{\mathbf{d}}) = c^{N_\lambda}_{(p,p)}$ for each $\lambda \in \mathbb{P}^1_{\Bbbk}$. According to Lemma 6.1 this will imply that (6.2) is an epimorphism, hence finish the proof. Fix $N \in \operatorname{rep}_{K_2}(p,p)$. Then

$$(\Phi^*(c^\lambda_{\mathbf{d}}))(N) = 0 \text{ if and only if } \operatorname{Hom}_{\mathbf{\Delta}}(R^{(r_\lambda)}_{\lambda,i_\lambda}, F(N)) \neq 0.$$

Since $R^{(r_\lambda)}_{\lambda,i_\lambda} \simeq F(N_\lambda)$ and F is fully faithful, we get

$$(\Phi^*(c^\lambda_{\mathbf{d}}))(N) = 0 \text{ if and only if } \operatorname{Hom}_{K_2}(N_\lambda, N) \neq 0.$$

Similarly, if $N \in \operatorname{rep}_{K_2}(p,p)$, then

$$c^{N_\lambda}_{(p,p)}(N) = 0 \text{ if and only if } \operatorname{Hom}_{K_2}(N_\lambda, N) \neq 0.$$

Consequently, the claim follows from Lemma 5.1. □

Corollary 6.3. *If* $\mathbf{r} \in \mathbf{R}$ *and* $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}} \neq 0$, *then*

$$\dim_{\Bbbk} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}} = \binom{p^{\mathbf{r}} + p}{p^{\mathbf{r}}}.$$

Proof. Proposition 4.9(2) implies that

$$\dim_{\Bbbk} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}} = \dim_{\Bbbk} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\mathbf{\Delta}}}.$$

Next,

$$\dim_{\Bbbk} \operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle p^{\mathbf{r}} \cdot \mathbf{h}, - \rangle_{\mathbf{\Delta}}} = \dim_{\Bbbk} \operatorname{SI}[K_2, (p,p)]_{(-p^{\mathbf{r}}, p^{\mathbf{r}})}$$

by Proposition 6.2, hence the claim follows from (5.2). □

Let $\Phi : \operatorname{rep}_{K_2}(p,p) \to \operatorname{rep}_{\mathbf{\Delta}}(\mathbf{d})$ be a regular map constructed in Proposition 6.2. For $j \in [0,p]$ we denote by $f^{(j)}_{\mathbf{d}}$ the inverse image of $f^{(j)}_{(p,p)}$ under Φ^*. Then (5.1) implies that (up to a non-zero scalar)

$$c^{(\zeta:\xi)}_{\mathbf{d}} = \sum_{i \in [0,p]} \xi^i \cdot \zeta^{p-i} \cdot f^{(i)}_{\mathbf{d}} \tag{6.3}$$

for each $(\zeta : \xi) \in \mathbb{P}^1_{\Bbbk}$. As the first application we get the following (smaller) set of generators of $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$.

Proposition 6.4. *The algebra* $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$ *is generated by the semi-invariants* $f^{(0)}_{\mathbf{d}}, \ldots, f^{(p)}_{\mathbf{d}}$ *and* $c^{R^{(n_{\lambda,i})}_{\lambda,i+1}}_{\mathbf{d}}$ *for* $\lambda \in \mathbb{X}_0$ *and* $i \in \mathcal{I}_\lambda$.

Proof. Recall from Corollary 4.5 that the algebra $\operatorname{SI}[\mathbf{\Delta}, \mathbf{d}]$ is generated by the semi-invariants $c^{R^{(n_{\lambda,i})}_{\lambda,i+1}}_{\mathbf{d}}$ for $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $i \in \mathcal{I}_\lambda$. Thus we only need to express, for each $\lambda \in \mathbb{P}^1_{\Bbbk} \setminus \mathbb{X}_0$ and $i \in \mathcal{I}_\lambda$, $c^{R^{(n_{\lambda,i})}_{\lambda,i+1}}_{\mathbf{d}}$ as the polynomial in the semi-invariants listed in the proposition. However, if $\lambda \in \mathbb{P}^1_{\Bbbk} \setminus \mathbb{X}_0$ and $i \in \mathcal{I}_\lambda$, then $c^{R^{(n_{\lambda,i})}_{\lambda,i+1}}_{\mathbf{d}} = c^\lambda_{\mathbf{d}}$, hence the claim follows from (6.3). □

We give another formulation of Proposition 6.4. Let $\mathcal{A}$ be the polynomial algebra in the indeterminates $S_0, \dots, S_p$ and $T_{\lambda,i}$ for $\lambda \in \mathbb{X}_0$ and $i \in \mathcal{I}_\lambda$. Proposition 6.4 says that the homomorphism $\Psi : \mathcal{A} \to \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]$ given by the formulas: $\Psi(S_i) := f_{\mathbf{d}}^{(i)}$ for $i \in [0,p]$ and $\Psi(T_{\lambda,i}) := c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n_{\lambda,i})}}$ for $\lambda \in \mathbb{X}_0$ and $i \in \mathcal{I}_\lambda$, is an epimorphism. Our last aim is to describe its kernel.

First, we introduce a grading by $\mathbf{R}$ in $\mathcal{A}$ by specifying the degrees of the indeterminates as follows: $\deg(S_i) := \mathbf{h}$ for $i \in [0,p]$ and $\deg(T_{\lambda,i}) := \mathbf{e}_{\lambda,i}^{n_{\lambda,i}}$ for $\lambda \in \mathbb{X}_0$ and $i \in \mathcal{I}_\lambda$. Note that Ψ is a homogenenous map, i.e. $\Psi(\mathcal{A}_{\mathbf{r}}) = \mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}}$ for each $\mathbf{r} \in \mathbf{R}$.

Let $\mathbf{R}_0$ be the submonoid of $\mathbf{R}$ generated by the elements $\mathbf{h}$ and $\mathbf{e}_{\lambda,i}^{n_{\lambda,i}}$ for $\lambda \in \mathbb{X}_0$ and $i \in \mathcal{I}_\lambda$. Obviously, if $\mathbf{r} \in \mathbf{R}$, then $\mathcal{A}_{\mathbf{r}} \neq 0$ if and only if $\mathbf{r} \in \mathbf{R}_0$. Similarly, Corollary 4.5 implies that $\mathrm{SI}[\mathbf{\Delta}, \mathbf{d}]_{\langle \mathbf{r}, - \rangle_{\mathbf{\Delta}}} \neq 0$ if and only if $\mathbf{r} \in \mathbf{R}_0$.

Lemma 6.5. *If $\mathbf{r} \in \mathbf{R}_0$, then*

$$\dim_{\Bbbk} \mathcal{A}_{\mathbf{r}} = \binom{p^{\mathbf{r}} + p + |\mathbb{X}_0|}{p^{\mathbf{r}}}.$$

Proof. One easily observes that there is an isomorphism $\mathcal{A}_{p^{\mathbf{r}} \cdot \mathbf{h}} \to \mathcal{A}_{\mathbf{r}}$ of vector spaces (induced by multiplying by the unique monomial of degree $\mathbf{r} - p^{\mathbf{r}} \cdot \mathbf{h}$). Moreover, $\bigoplus_{q \in \mathbb{N}} \mathcal{A}_{q \cdot \mathbf{h}}$ is the polynomial algebra generated by $S_0, \dots, S_p$ and $\prod_{i \in \mathcal{I}_\lambda^0} T_{\lambda,i}$ for $\lambda \in \mathbb{X}_0$. Now the claim follows. $\square$

The formula (6.3) implies that for each $\lambda \in \mathbb{X}_0$ there exist $\zeta_\lambda, \xi_\lambda \in \Bbbk$ such that

$$\prod_{i \in \mathcal{I}_\lambda^0} c_{\mathbf{d}}^{R_{\lambda,i+1}^{(n_{\lambda,i})}} = \sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot f_{\mathbf{d}}^{(j)}$$

Obviously, $(\zeta_\lambda, \xi_\lambda) \neq (0,0)$ and $(\zeta_\lambda : \xi_\lambda) = \lambda$.

Proposition 6.6. *We have*

$$\operatorname{Ker} \Psi = \Big(\sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in \mathcal{I}_\lambda^0} T_{i,\lambda} : \lambda \in \mathbb{X}_0 \Big).$$

Proof. Let

$$\mathcal{J} := \Big(\sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in \mathcal{I}_\lambda^0} T_{i,\lambda} : \lambda \in \mathbb{X}_0 \Big).$$

Obviously, $\operatorname{Ker} \Psi \subseteq I$. Observe that both $\operatorname{Ker} \Psi$ and $\mathcal{J}$ are graded ideals (with respect to the grading introduced above). Consequently, in order to prove our claim it suffices to show that $\dim_{\Bbbk} \mathcal{J}_{\mathbf{r}} = \dim_{\Bbbk} \operatorname{Ker} \Psi_{\mathbf{r}}$ for each $\mathbf{r} \in \mathbf{R}_0$.

We already know from Lemma 6.5 and Corollary 6.3 that

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Ker} \Psi_{\mathbf{r}} &= \dim_{\Bbbk} A_{\mathbf{r}} - \dim_{\Bbbk} \mathrm{SI}[\boldsymbol{\Delta}, \mathbf{r}]_{\langle \mathbf{r}, - \rangle_{\boldsymbol{\Delta}}} \\ &= \binom{p^{\mathbf{r}} + p + |\mathbb{X}_0|}{p^{\mathbf{r}}} - \binom{p^{\mathbf{r}} + p}{p^{\mathbf{r}}}\end{aligned}$$

for each $\mathbf{r} \in \mathbf{R}_0$. On the other hand, similarly as in the proof of Lemma 6.5, we show that $\dim_{\Bbbk} \mathcal{J}_{\mathbf{r}} = \dim_{\Bbbk} \mathcal{J}_{p^{\mathbf{r}} \cdot \mathbf{h}}$ for each $\mathbf{r} \in \mathbf{R}$. Moreover, the algebra $\bigoplus_{q \in \mathbb{N}} (\mathcal{A}/\mathcal{J})_{q \cdot \mathbf{h}}$ is obviously the polynomial algebra in $p^{\mathbf{r}} + p$ indeterminates. This, together with Lemma 6.5, immediately implies our claim. □

We may summarize our considerations in the following theorem.

Theorem 6.7. *We have the isomorphism*

$$\mathrm{SI}[\boldsymbol{\Delta}, \mathbf{d}] \simeq \mathcal{A}/\Big(\sum_{j \in [0,p]} \xi_\lambda^j \cdot \zeta_\lambda^{p-j} \cdot S_j - \prod_{i \in \mathcal{I}_\lambda^0} T_{i,\lambda} : \lambda \in \mathbb{X}_0 \Big).$$

If

$$i(\mathbf{d}) := \{\lambda \in \mathbb{X}_0 : |\mathcal{I}_\lambda| > 1\},$$

then $\mathrm{SI}[\boldsymbol{\Delta}, \mathbf{d}]$ *is a complete intersection given by* $\max(0, i(\mathbf{d}) - p - 1)$ *equations. In particular,* $\mathrm{SI}[\boldsymbol{\Delta}, \mathbf{d}]$ *is polynomial algebra if and only if* $i(\mathbf{d}) \leq p + 1$.

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