

# STABILITY PROBLEMS IN CRAMER-TYPE CHARACTERIZATION IN CASE OF I.I.D. SUMMANDS

S. G. BOBKOV<sup>1,4</sup>, G. P. CHISTYAKOV<sup>2,4</sup>, AND F. GÖTZE<sup>3,4</sup>

ABSTRACT. Stability property in Cramer's characterization of the normal law is considered in case of identically distributed summands. Opposite results are obtained in the problem with respect to the Kolmogorov metric, and the total variation norm and the entropic distance from the other side (addressing a question of M. Kac).

## 1. Introduction

A well-known theorem due to Cramer [Cr] states that, if the sum, say  $X + Y$ , of two independent random variables  $X$  and  $Y$  has a normal distribution, then both summands  $X$  and  $Y$  have to be normal. P. Lévy showed that this characterization property is stable in the topology of weak convergence of probability distributions on the real line. Using Lévy's metric,  $L(F, G)$ , to measure distances in weak convergence between any distribution functions  $F, G$ , there exists for any  $\varepsilon > 0$  a number  $\delta_\varepsilon$ , depending on  $\varepsilon$  only, such that  $\delta_\varepsilon \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ , and

$$L(F * G, \Phi) < \varepsilon \quad \text{implies} \quad L(F, \Phi_{a_1, \sigma_1}) < \delta_\varepsilon \quad \text{and} \quad L(G, \Phi_{a_2, \sigma_2}) < \delta_\varepsilon,$$

for some  $a_1, a_2 \in \mathbf{R}$  and  $\sigma_1, \sigma_2 > 0$ .

Here  $\Phi_{a, \sigma}$  denotes the distribution function of the normal law  $N(a, \sigma^2)$  with mean  $a$  and variance  $\sigma^2$ . In the standard case, where  $a = 0$  and  $\sigma = 1$ , we shall omit these indices. Furthermore, let  $F * G$  denote the convolution of the corresponding distributions.

There are similar results in terms of the Kolmogorov distance, including sharp quantitative estimates for the rate function  $\delta_\varepsilon$ . Here we refer to [Sa1-2], [Z1-3], [M1-2], [L-O], [Se], [C1-2], [Sh1-2], [C-G], [B-C-G2]. Other results in this context study characterizations and stability problems for distributions of more complicated functionals of independent summands, including linear and quadratic forms (cf. e.g. [K-L-R], [Ch-P-U]).

In the mid 1960's M. Kac, and then H.P. McKean raised the question of the stability in Cramer's theorem with respect to the entropic distance to normality; see [MK], pp. 365–366. Let us recall that, if a random variable  $X$  with finite second moment has density  $p(x)$ , the entropic distance of the distribution of  $X$  to a normal distribution, or to Gaussianity, is defined by

$$D(X) = h(Z) - h(X) = \int_{-\infty}^{+\infty} p(x) \log \frac{p(x)}{\varphi_{a, \sigma}(x)} dx,$$

---

1991 *Mathematics Subject Classification.* Primary 60E.

*Key words and phrases.* Cramer's theorem, normal characterization, stability problems.

1) School of Mathematics, University of Minnesota, USA; Email: bobkov@math.umn.edu.

2) Faculty of Mathematics, University of Bielefeld, Germany; Email: chistyak@math.uni-bielefeld.de.

3) Faculty of Mathematics, University of Bielefeld, Germany; Email: goetze@math.uni-bielefeld.de.

4) Research partially supported by NSF grant and SFB 701.

where  $\varphi_{a,\sigma}$  stands for the density of a normal random variable  $Z$  with parameters  $a = \mathbf{E}X$ ,  $\sigma^2 = \text{Var}(X)$ , and where

$$h(X) = - \int_{-\infty}^{+\infty} p(x) \log p(x) dx$$

denotes the classical entropy functional. The quantity  $D(X)$  represents the shortest distance from the distribution  $F$  of  $X$  to the family of all normal laws on the line with respect to the Kullback-Leibler distance.

In general  $0 \leq D(X) \leq +\infty$ . Similarly as in the case of the Kolmogorov (or uniform) distance  $\|F - \Phi_{a,\sigma}\| = \sup_x |F(x) - \Phi_{a,\sigma}(x)|$  and as in the case of the total variation distance  $\|F - \Phi_{a,\sigma}\|_{\text{TV}}$ , the value of  $D(X)$  does not depend on the mean or the variance of  $X$ . This distance is stronger than the total variation, in view of the Pinsker (or Pinsker-Csiszár-Kullback) inequality, which yields

$$D(X) \geq \frac{1}{2} \|F - \Phi_{a,\sigma}\|_{\text{TV}}^2.$$

Thus, Kac's question may be rephrased as to whether or not smallness of  $D(X + Y)$  for independent summands implies the smallness of  $D(X)$  and  $D(Y)$ . A negative answer to this question with non-identically distributed summands has recently been given in [B-C-G1]. Surprisingly the random variables  $X$  and  $Y$  in another counter-example to the Kac conjecture can be chosen to be identically distributed. More precisely, we prove in this note the following:

**Theorem 1.1.** *For any  $\varepsilon > 0$ , there exist independent random variables  $X$  and  $Y$  with a common absolutely continuous symmetric distribution  $F$  such that*

- a)  $D(X + Y) < \varepsilon$ ;
- b)  $\|F - \Phi_{a,\sigma}\|_{\text{TV}} > c$ , for all  $a \in \mathbf{R}$  and  $\sigma > 0$ ,

where  $c > 0$  is an absolute constant. In particular,  $D(X)$  is separated from zero.

As it is shown below, the distribution  $F$  in Theorem 1.1 may be chosen to be more or less well behaved. For example, in addition to the properties a) – b) (with constants  $c$  depending on  $\alpha$ ) it may have an almost standard normal density  $p$  in the sense that, for a prescribed  $\alpha \in (0, 1)$ ,

$$(1 - \alpha) \varphi(x) \leq p(x) \leq (1 + \alpha) \varphi(x), \quad \forall x \in \mathbf{R}.$$

In other words, a rather small oscillation of  $p$  around  $\varphi$  suffices to separate  $F$  and  $\Phi$ , while at the same time moving  $F * F$  arbitrarily close to  $\Phi * \Phi$  in a strong metric, like the total variation and even entropic distance.

Let us discuss this result in view of previous variants of stability in the Cramer theorem for the Kolmogorov distance, which is essentially due to Sapogov (cf. [Sa1-2], [B-C-G2]). If independent random variables  $X$  and  $Y$  have mean zero and (equal) variances  $\text{Var}(X) = \text{Var}(Y) = 1$ , but not necessarily equal distributions, say,  $F$  and  $G$ , then the condition

$$\|F * G - \Phi * \Phi\| \leq \varepsilon < 1$$

insures that

$$\|F - \Phi\| \leq \frac{C}{\sqrt{\log \frac{1}{\varepsilon}}} \quad \text{and} \quad \|G - \Phi\| \leq \frac{C}{\sqrt{\log \frac{1}{\varepsilon}}}$$

with some absolute constant  $C$ . This bound is asymptotically optimal for small values of  $\varepsilon$  (cf. [M1], [C1]). However, in case of identically distributed summands, it is rather far from being optimal, since the bound can be improved to a polynomial rate. It is observed that, if  $F$  has a median at zero (which is actually not important), the condition  $\|F * F - \Phi * \Phi\| \leq \varepsilon$  with  $\varepsilon$  small guarantees that

$$\|F - \Phi\| \leq C \varepsilon^{1/36} \log(1/\varepsilon).$$

We shall first sharpen the exponent in this estimate. In contrast to Sapogov's result, this can be done without using heavy tools from Complex Analysis, but rather using classical probabilistic arguments that involve Berry-Esseen's bound and Plancherel's formula.

**Theorem 1.2.** *If a distribution function  $F$  satisfies  $\|F * F - \Phi * \Phi\| \leq \varepsilon < 1/e$ , then we obtain with some absolute constant  $C$*

$$\|F - \Phi\| \leq C (\varepsilon \log(1/\varepsilon))^{2/5}.$$

Perhaps this bound is still not asymptotically attainable, and the exponent  $2/5$  might be replaced with  $1/2$  modulo a logarithmic factor. In order to see that the exponent  $1/2$  is unimprovable, we construct another family of probability distributions and prove:

**Theorem 1.3.** *For any  $\varepsilon \in (0, 1/e)$ , there exists a probability distribution  $F$  on the real line, such that  $\|F * F - \Phi * \Phi\| \leq \varepsilon$ , while*

$$\|F - \Phi\| \geq c \varepsilon^{1/2} \log^{-1/4}(1/\varepsilon),$$

where  $c > 0$  is an absolute constant.

In fact, even in the case of a stronger hypothesis, such as  $\|F * F - \Phi * \Phi\|_{\text{TV}} \leq \varepsilon$ , one can obtain a similar inequality  $\|F - \Phi\| \geq c \varepsilon^{1/2} \log^{-1/2}(1/\varepsilon)$ .

Finally, let us emphasize that the stability property itself for identically distributed independent random variables (similar as in Theorem 1.2) remains valid for a large class of probability distributions on the line, viewed as "points of attraction" (although, in general, the Lévy distance should replace the Kolmogorov distance for an improved control of weak convergence). In other words, there is nothing special about the normal law in Theorem 1.2 beyond the problem of optimal rates. This can be seen from the following more general result.

**Theorem 1.4.** *Assume a distribution function  $G$  has a real positive characteristic function. There exists a positive, increasing function  $\alpha = \alpha_G(\varepsilon)$ , such that  $\alpha(\varepsilon) \rightarrow 0$ , as  $\varepsilon \rightarrow 0$ , and the following property holds. For any distribution function  $F$  satisfying*

$$L(F * F, G * G) \leq \varepsilon \leq 1,$$

we have  $L(F, G) \leq \alpha(\varepsilon)$ .

Here, the assumption on the positivity of the characteristic function of  $G$  cannot be removed. For example, it is well-known (cf. [F], Chapter XV) that there exist two different distributions  $F$  and  $G$  on the real line, such that  $F * F = G * G$ . Hence,  $G$  contradicts to the conclusion of Theorem 1.4. One of these distributions, say  $G$ , has a non-negative 2-periodic characteristic

function  $g$  defined by  $g(t) = 1 - |t|$  for  $|t| \leq 1$ , while  $F$  has the characteristic function  $f(t) = 2g(t/2) - 1$ .

The organization of the paper is as follows. In Section 2 we derive an elementary, but useful bound on the deviations of characteristic functions in terms of the Lévy metric. In Section 3 we consider some properties of squares of characteristic functions. Theorem 1.4 is proved in Section 4. Some additional technical bounds needed for the Gaussian case are collected in Section 5. The proof of Theorem 1.2 is completed in Section 6. Sections 7-9 are devoted to the proof of Theorem 1.1. Finally, Theorem 1.3 is proved in Section 4.

### Contents

1. Introduction
2. A bound on deviations of characteristic functions
3. Squares of characteristic functions
4. Stability theorem (proof of Theorem 1.4)
5. The Gaussian case
6. Proof of Theorem 1.2
7. Oscillating densities
8. Computations and bounds for convolutions
9. Proof of Theorem 1.1
10. Proof of Theorem 1.3

## 2. A Bound on Deviations of Characteristic Functions

Throughout the paper we deal with the following classical metrics in the space of all probability distributions on the real line:

- 1) The Kolmogorov distance  $\|F - G\| = \sup_x |F(x) - G(x)|$ ;
- 2) The Lévy distance

$$L(F, G) = \min \{h \geq 0 : G(x - h) - h \leq F(x) \leq G(x + h) + h, \quad \forall x \in \mathbf{R}\};$$

- 3) The total variation distance

$$\|F - G\|_{\text{TV}} = \sup \sum |(F(x_k) - G(x_k)) - (F(y_k) - G(y_k))|,$$

where the sup is taken over all finite collections of points  $y_1 < x_1 < \dots < y_n < x_n$ .

Here,  $F$  and  $G$  denote arbitrary distribution functions.

Introduce the characteristic functions

$$f(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x), \quad g(t) = \int_{-\infty}^{+\infty} e^{itx} dG(x) \quad (t \in \mathbf{R}).$$

**Lemma 2.1.** *For all  $t \in \mathbf{R}$  and  $N > 0$ ,*

$$|f(t) - g(t)| \leq 2L(F, G)(2 + (N + 2)|t|) + 3G\{|x| > N\}. \quad (2.1)$$

Here  $G\{|x| > N\} = \int_{\{|x| > N\}} dG(x)$ , and we use similar notations below.

**Proof.** The argument is rather standard, and we include it for completeness.

Put  $h = L(F, G)$ . Without loss of generality, one may assume that  $N$  and  $N \pm h$  are points of continuity of both  $F$  and  $G$ . Given  $t \in \mathbf{R}$ , write

$$f(t) - g(t) = \int_{-N-h}^{N+h} e^{itx} d(F(x) - G(x)) + \int_{\{|x| > N+h\}} e^{itx} d(F(x) - G(x)). \quad (2.2)$$

The last integral is bounded from above in absolute value by

$$F\{|x| > N + h\} + G\{|x| > N + h\}.$$

Also, by the definition of the Lévy distance,

$$\begin{aligned} F\{|x| > N + h\} &= F(-N - h) + (1 - F(N + h)) \\ &\leq (G(-N) + h) + (1 - G(N)) + h = G\{|x| \geq N\} + 2h. \end{aligned}$$

Hence, the last integral in (2.2) is bounded by

$$2G\{|x| \geq N\} + 2h. \quad (2.3)$$

Next, integrating by parts, we have

$$\int_{-N-h}^{N+h} e^{itx} d(F(x) - G(x)) = e^{itx} (F(x) - G(x)) \Big|_{x=-N-h}^{x=N+h} \quad (2.4)$$

$$-it \int_{-N-h}^{N+h} e^{itx} (F(x) - G(x)) dx. \quad (2.5)$$

Since

$$F(-N - h) - G(-N - h) \leq F(-N - h) \leq G(-N) + h$$

and  $G(-N - h) - F(-N - h) \leq G(-N)$ , we have that

$$|F(-N - h) - G(-N - h)| \leq G(-N) + h.$$

Similarly,  $|F(N + h) - G(N + h)| \leq (1 - G(N)) + h$ , so the increment term in (2.4) is bounded in absolute value by

$$G\{|x| \geq N\} + 2h. \quad (2.6)$$

Now, let's turn to the integral in (2.5) over the interval  $(-N - h, N + h)$ . Again, by the definition of the Lévy distance,

$$F(x) - G(x) = (F(x) - G(x + h)) + (G(x + h) - G(x)) \leq h + (G(x + h) - G(x)),$$

and similarly,

$$G(x) - F(x) \leq h + (F(x + h) - F(x)).$$

Hence,

$$|F(x) - G(x)| \leq h + (G(x + h) - G(x)) + (F(x + h) - F(x)),$$

which implies that

$$\begin{aligned} \int_{-N-h}^{N+h} |F(x) - G(x)| dx &\leq 2(N + h)h \\ &\quad + \int_{-\infty}^{+\infty} (G(x + h) - G(x)) dx + \int_{-\infty}^{+\infty} (F(x + h) - F(x)) dx. \end{aligned}$$

Each of the last two integrals does not exceed  $h$ , so

$$\left| \int_{-N-h}^{N+h} e^{itx} (F(x) - G(x)) dx \right| \leq 2(N + 1 + h)h \leq 2(N + 2)h.$$

Together with the bounds (2.3) and (2.6), we obtain that

$$|f(t) - g(t)| \leq 3G\{|x| \geq N\} + 4h + 2|t|(N + 2)h,$$

which is exactly (2.1).

**Remark.** With the same argument one may derive a similar inequality in terms of the classical  $\beta$ -metric defined in the space of all probability distributions via

$$\beta(F, G) = \sup_{\|u\|_{\text{BL}} \leq 1} \left| \int u(x) dF(x) - \int u(x) dG(x) \right|,$$

where  $\|u\|_{\text{BL}} = \sup_x |u(x)| + \|u\|_{\text{Lip}}$ , cf. [D], Ch.11. Replacing the function  $e^{itx}$  in the proof of Lemma 2.1 with  $u(x)$  and maximizing over all admissible  $u$ , one arrives at

$$\beta(F, G) \leq 2L(F, G)(N + 4) + 3G\{|x| > N\}.$$

There is also an elementary reverse bound as well  $L(F, G) \leq 2\sqrt{\beta(F, G)}$ .

Note that there is a slight refinement in Lemma 2.1 when switching to the Kolmogorov distance.

**Lemma 2.2.** *For all  $t \in \mathbf{R}$  and  $N > 0$ ,*

$$|f(t) - g(t)| \leq 2\|F - G\|(2 + N|t|) + 2G\{|x| > N\}. \quad (2.7)$$

Indeed, write

$$f(t) - g(t) = \int_{-N}^N e^{itx} d(F(x) - G(x)) + \int_{\{|x| > N\}} e^{itx} d(F(x) - G(x)).$$

The last integral is bounded in absolute value from above by

$$F\{|x| > N\} + G\{|x| > N\} \leq 2G\{|x| > N\} + 2\|F - G\|.$$

In addition, integrating by parts, we have

$$\int_{-N}^N e^{itx} d(F(x) - G(x)) = e^{itx} (F(x) - G(x)) \Big|_{x=-N}^{x=N} - it \int_{-N}^N e^{itx} (F(x) - G(x)) dx,$$

which is bounded in absolute value by  $2\|F - G\|(1 + N|t|)$ . The two bounds lead to (2.7).

### 3. Squares of Characteristic Functions

Let  $F$  and  $G$  be two distribution functions with characteristic functions  $f$  and  $g$ , respectively. We want to estimate the difference  $f(t) - g(t)$  on large intervals in terms of the Lévy distance  $L(F * F, G * G)$  between these convolutions. When  $g$  is real and positive, we shall use the following special quantity, which we denote by  $T_g$ .

**Notation.** Define

$$T_g(\varepsilon) = \min\{t \geq 0 : |g(t)| \leq \varepsilon\}, \quad 0 \leq \varepsilon \leq 1,$$

where we put  $T_g(\varepsilon) = +\infty$  in case  $|g(t)| > \varepsilon$ , for all  $t > 0$ .

By Lemma 2.1, if  $L(F * F, G * G) \leq \varepsilon \leq 1$ , then

$$|f(t)^2 - g(t)^2| \leq 2\varepsilon(2 + (N+2)|t|) + 3G * G\{|x| > N\}. \quad (3.1)$$

To simplify the right-hand side, let  $X, X'$  be independent random variables with distribution function  $G$ . Then

$$\begin{aligned} G * G\{|x| > N\} &= \mathbf{P}\{|X + X'| > N\} \\ &\leq \mathbf{P}\{|X| > N/2\} + \mathbf{P}\{|X'| > N/2\} = 2G\{X > N/2\}. \end{aligned}$$

Replacing  $N$  with  $2N$ , we thus get from (3.1) that

$$|f(t)^2 - g(t)^2| \leq \varepsilon' \equiv 4\varepsilon(1 + (N+1)|t|) + 6G\{|x| > N\}.$$

Now, assume that  $g(t)$  is real and positive.

Since  $\operatorname{Re}[f(t)^2] = (\operatorname{Re}f(t))^2 - (\operatorname{Im}f(t))^2$ , we then get that

$$|(\operatorname{Re}f(t))^2 - (\operatorname{Im}f(t))^2 - g(t)^2| \leq \varepsilon'$$

This insures that on a certain large interval  $[-a, a]$ ,  $\operatorname{Re}f(t) \geq 0$ . Indeed, otherwise, for some  $t_0 \in [-a, a]$ , we would have that  $\operatorname{Re}f(t_0) = 0$ , which implies that

$$(\operatorname{Im}f(t_0))^2 + g(t_0)^2 \leq \varepsilon',$$

and therefore  $|g(t_0)| \leq \sqrt{\varepsilon'}$ . But, if  $\varepsilon'$  is small enough, the inequality  $|g(t_0)| \leq \sqrt{\varepsilon'}$  is possible for sufficiently large  $t_0$ , only.

Thus, we conclude that

$$\begin{aligned} \operatorname{Re}f(t) < 0 &\Rightarrow \exists t_0 \text{ such that } 0 < t_0 < |t| \text{ and } \operatorname{Re}f(t_0) = 0 \\ &\Rightarrow |g(t_0)| \leq \sqrt{\varepsilon'} \\ &\Rightarrow |t| > t_0 \geq T_g(\sqrt{\varepsilon'}), \end{aligned}$$

provided that  $\varepsilon' \leq 1$ . Hence,

$$|f(t)^2 - g(t)^2| \leq \varepsilon' \leq 1 \Rightarrow \operatorname{Re}f(t) \geq 0 \text{ on the interval } |t| \leq T_g(\sqrt{\varepsilon'}).$$

On the other hand, the inequality  $|f(t)^2 - g(t)^2| \leq \varepsilon'$  requires that

$$|f(t) - g(t)| \leq \sqrt{\varepsilon'} \quad \text{or} \quad |f(t) + g(t)| \leq \sqrt{\varepsilon'}.$$

But for  $|t| \leq T_g(\sqrt{\varepsilon'})$  we have  $\operatorname{Re}f(t) \geq 0$ , so the second assertion would imply that  $g(t) \leq \sqrt{\varepsilon'}$ , which is only possible for  $|t| \geq T_g(\sqrt{\varepsilon'})$ . Hence,

$$|f(t)^2 - g(t)^2| \leq \varepsilon' \leq 1 \Rightarrow |f(t) - g(t)| \leq \sqrt{\varepsilon'} \text{ on the interval } |t| \leq T_g(\sqrt{\varepsilon'}).$$

One can summarize:

**Lemma 3.1.** *Assume that  $G$  has a real positive characteristic function  $g$ . Suppose that  $L(F * F, G * G) \leq \varepsilon$ , and for  $N > 0$  and  $t \in \mathbf{R}$ , let*

$$\varepsilon' = 4\varepsilon(1 + (N+1)|t|) + 6G\{|x| > N\} \leq 1. \quad (3.2)$$

If  $|t| \leq T_g(\sqrt{\varepsilon'})$ , then

$$|f(t) - g(t)| \leq \sqrt{\varepsilon'}. \quad (3.3)$$

**Remark 3.2.** Using similar arguments, and taking into account Lemma 2.2 and the inequality (2.7), the stronger assumption  $\|F * F - G * G\| \leq \varepsilon$  implies (3.3) for a smaller number

$$\varepsilon' = 4\varepsilon(1 + N|t|) + 4G\{|x| > N\}, \quad (3.4)$$

provided that  $\varepsilon' \leq 1$  and  $|t| \leq T_g(\sqrt{\varepsilon'})$ .

#### 4. Stability Theorem

We are now ready to establish the stability property of convolutions for equal factors with respect to the Lévy distance, as described in Theorem 1.4.

It is enough to show that, for any  $\delta > 0$ , one can find  $\varepsilon > 0$ , depending on  $\varepsilon$  and  $G$ , only, such that  $L(F, G) \leq \delta$ , as long as  $L(F * F, G * G) \leq \varepsilon$ .

We may additionally assume that the characteristic function  $g$  of  $G$  is vanishing at infinity (otherwise, a slight modification of the argument is needed). Under this assumption, the function  $\varepsilon \rightarrow T_g(\varepsilon)$  is finite for  $0 < \varepsilon < 1$ . In addition, it is non-increasing and satisfies  $T_g(\varepsilon) \rightarrow +\infty$ , as  $\varepsilon \rightarrow 0$ .

We shall use the so-called  $\lambda$ -metric, introduced by Senatov and Zolotarev [S-Z]:

$$\lambda(F, G) = \min_{T>0} \max_{|t| \leq T} \left\{ \frac{1}{2} |f(t) - g(t)|, \frac{1}{T} \right\}.$$

Thus,

$$\forall |t| \leq T \quad |f(t) - g(t)| \leq a \Rightarrow \lambda(F, G) \leq \max \left\{ \frac{a}{2}, \frac{1}{T} \right\}.$$

Assume  $L(F * F, G * G) \leq \varepsilon$  and apply Lemma 3.1 with, say,  $N = \varepsilon^{-1/3}$ . If  $|t| \leq \varepsilon^{-1/3}$ , we may bound  $\varepsilon'$  in (3.2) by

$$\varepsilon' \leq \varepsilon'_G(\varepsilon) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0,$$

with some positive function  $\varepsilon'_G$ , depending on  $G$ , only. Put

$$T(\varepsilon) = \min \left\{ \varepsilon^{-1/3}, T_g(\sqrt{\varepsilon'_G(\varepsilon)}) \right\}.$$

Then the conclusion (3.3) in Lemma 3.1 takes the form

$$\max_{|t| \leq T(\varepsilon)} |f(t) - g(t)| \leq \sqrt{\varepsilon'_G(\varepsilon)},$$

so

$$\lambda(F, G) \leq \lambda_G(\varepsilon) \equiv \max \left\{ \frac{1}{2} \sqrt{\varepsilon'_G(\varepsilon)}, \frac{1}{T(\varepsilon)} \right\} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

But, according to Theorem 2 in [Se-Z], the Lévy distance  $L = L(F, G)$  is estimated in terms of  $\lambda = \lambda_G(\varepsilon)$  by virtue of the inequality

$$L \leq 8\lambda \log(Y/\lambda),$$

where  $Y \geq e$  may be any number, satisfying  $G\{|x| > Y/e\} \leq \lambda \log(Y/\lambda)$ . This insures that  $L \leq \psi_G(\lambda)$ , for some non-negative increasing function  $\psi_G$  such that  $\psi_G(\lambda) \rightarrow 0$ , as  $\lambda \rightarrow 0$ . Therefore,

$$L(F, G) \leq \psi_G(\lambda_G(\varepsilon)) \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0.$$

Thus, Theorem 1.4 is proved.

## 5. The Gaussian Case

In order to find an explicit rate function  $\alpha(\varepsilon)$  in the Gaussian case, one could use similar arguments as in the proof of the more general Theorem 1.4. However, this would lead us to unappropriate rates. So, we have to return to Lemma 3.1 (or Remark 3.2 concerning the Kolmogorov distance) and apply it in a more sophisticated way.

Thus, let  $G = \Phi$  denote the standard normal distribution, and assume that, for a given distribution function  $F$ , we have

$$\|F * F - \Phi * \Phi\| \leq \varepsilon \quad (5.1)$$

with  $\varepsilon \in (0, 1/e]$ . Note that, since  $\Phi$  has a bounded density, it does not matter whether the Lévy or Kolmogorov distances are used to measure distance.

We need to find appropriate bounds for the difference of the characteristic functions,  $f(t) - g(t)$ , where now  $g(t) = e^{-t^2/2}$ , on different intervals of  $t$ -axis.

First, in order to control the relatively small values of  $t$ , we apply Remark 3.2 with the value  $N = \sqrt{2 \log(1/\varepsilon)}$ . In particular,  $\Phi\{|x| > N\} \leq e^{-N^2/2} = \varepsilon$ , so the quantity  $\varepsilon'$  in (3.4) for  $\varepsilon \leq \frac{1}{e}$  is bounded by

$$4\varepsilon \left(1 + \sqrt{2 \log(1/\varepsilon)} |t|\right) + 4\varepsilon = 8\varepsilon + 4\varepsilon \sqrt{2 \log(1/\varepsilon)} |t|.$$

For  $|t| \leq N$ , the last expression may be bounded from above by

$$8\varepsilon + 8\varepsilon \log(1/\varepsilon) \leq 16\varepsilon \log(1/\varepsilon).$$

Thus, in case  $\varepsilon \leq 1/e$ , the concluding inequality (3.3),  $|f(t) - g(t)| \leq \sqrt{\varepsilon'}$ , holds with

$$\varepsilon' = 16\varepsilon \log(1/\varepsilon),$$

provided that  $\varepsilon' \leq 1$  and  $|t| \leq \min\{N, T_g(\sqrt{\varepsilon'})\}$ . But

$$N \geq T_g(\sqrt{\varepsilon'}) = \sqrt{\log \frac{1}{\varepsilon'}} = \sqrt{\log \frac{1}{\varepsilon} - \log \log(\varepsilon^{-16})},$$

since the inequality on the left is equivalent to  $e^{-N^2/2} \leq \sqrt{\varepsilon'}$ , that is,  $16\varepsilon \log(1/\varepsilon) \geq \varepsilon^2$ , which is true for  $\varepsilon \leq 1/e$ . By Lemma 3.1/Remark 3.2, we obtain:

**Lemma 5.1.** *Assuming (5.1), with  $\varepsilon' = 16\varepsilon \log \frac{1}{\varepsilon} \leq 1$  and  $|t| \leq \sqrt{\log(1/\varepsilon')}$ , we have*

$$|f(t) - e^{-t^2/2}| \leq 4\sqrt{\varepsilon \log(1/\varepsilon)}.$$

Furthermore, in order to control the deviations of  $f(t)$  around  $g(t)$  for very small values of  $t$ , or for relatively large  $t$ , a truncation procedure will be needed.

**Definition.** Given a random variable  $X$  with distribution function  $F$ , put  $X^* = X$  in case  $|X| \leq M$  and  $X^* = 0$  in case  $|X| > M$ , where

$$M = M(\varepsilon) = 1 + 2\sqrt{\log(1/\varepsilon)}.$$

Denote by  $F^*$  the distribution function and by  $f^*$  the characteristic function of  $X^*$ .

Since  $F^*$  is supported on the interval  $[-M, M]$ , its characteristic function satisfies, for all  $t$ ,

$$|1 - f^*(t)| \leq \int_{-M}^M |1 - e^{itx}| dF^*(x) \leq |t| \int_{-M}^M |x| dF^*(x) \leq M|t|.$$

Using a similar inequality for the normal distribution, we have that

$$|f^*(t) - e^{-t^2/2}| \leq 2M(\varepsilon) |t|. \quad (5.2)$$

Before using this bound, one needs to quantify the distance between  $F^*$  and  $F$  in the Kolmogorov metric. More generally, the next lemma allows to reduce Theorem 1.2 to the case of compactly supported distributions.

**Lemma 5.2.** *Under (5.1),*

$$\|F^* - F\| \leq 13\varepsilon \quad \text{and} \quad \|F^* * F^* - \Phi * \Phi\| \leq 13\varepsilon. \quad (5.3)$$

**Proof.** The argument is rather standard. With a slight modification we borrow it from [S1], [Se], where the non-i.i.d. case is treated.

Assume that  $\varepsilon$  is small enough, say,  $\varepsilon \leq \varepsilon_0 = \frac{1}{4} - \Phi(-1) = 0.091345\dots$ . This assumption allows one to bound a median  $m = m(X)$ . Indeed, taking an independent copy  $Y$  of  $X$  and applying (5.1), we get

$$\frac{1}{4} \leq \mathbf{P}\{X \leq m, Y \leq m\} \leq \mathbf{P}\{X + Y \leq 2m\} \leq \Phi * \Phi(2m) + \varepsilon = \Phi(\sqrt{2}m) + \varepsilon.$$

Hence,  $\Phi(\sqrt{2}m) \geq \frac{1}{4} - \varepsilon_0 = \Phi(-1)$ . So,  $m \geq -1/\sqrt{2}$ . With a similar bound for  $-m$ , we obtain that

$$|m(X)| < 1. \quad (5.4)$$

To simplify matters, we may assume that  $M$  is a point of continuity of  $F$ . By definition,

$$\Phi * \Phi(-(M-1)) = 1 - \Phi * \Phi(M-1) \leq \frac{1}{2} e^{-(M-1)^2/4} = \frac{1}{2} \varepsilon.$$

Hence, by (5.1) once more,

$$\begin{aligned} \mathbf{P}\{X \leq 1, Y \leq -M\} &\leq \mathbf{P}\{X + Y \leq -(M-1)\} \\ &\leq \Phi * \Phi(-(M-1)) + \varepsilon \leq \frac{3}{2} \varepsilon, \end{aligned}$$

and, by (5.4),  $F(-M) \leq 3\varepsilon$ . Analogously,  $1 - F(M) \leq 3\varepsilon$ . Thus,  $F\{|x| \geq M\} \leq 6\varepsilon$ , where  $F$  is treated as a measure.

Now, for  $x < -M$ , we have  $|F^*(x) - F(x)| = F(x) \leq 6\varepsilon$ , and similarly for  $x > M$ .

If  $-M < x < 0$ , then  $F^*(x) = F(x) - F(-M)$ , and if  $0 < x < M$ , we have  $F^*(x) = F(x) + (1 - F(M))$ . In both cases,  $|F^*(x) - F(x)| \leq 6\varepsilon$ . Therefore,

$$\|F^* - F\| \leq 6\varepsilon.$$

From this, by the triangle inequality,

$$\begin{aligned} \|F^* * F^* - F * F\| &\leq \|F^* * F^* - F^* * F\| + \|F^* * F - F * F\| \\ &\leq \|F^* - F\| + \|F^* - F\| \leq 12\varepsilon. \end{aligned}$$

Finally, by (5.1) and the triangle inequality again, we get

$$\|F^* * F^* - \Phi * \Phi\| \leq \|F^* * F^* - F * F\| + \|F * F - \Phi * \Phi\| \leq 13\varepsilon.$$

Thus, the bounds (5.3) hold for the values  $\varepsilon \leq \varepsilon_0$ . For the range  $\varepsilon > \varepsilon_0$ , they hold automatically, since  $13\varepsilon_0 > 1$ . Thus, Lemma 5.2 is proved.

**Lemma 5.3.** *Let  $\varepsilon' = 16\varepsilon \log \frac{1}{\varepsilon} \leq 1/e$  and  $T_1 = \sqrt{\log(1/\varepsilon')} < T$ . If  $F$  is supported on  $[-M(\varepsilon), M(\varepsilon)]$ , then, assuming (5.1), we have*

$$\int_{T_1}^T \frac{|f(t) - e^{-t^2/2}|}{t} dt \leq CT^{1/4} \sqrt{\varepsilon \log(1/\varepsilon)},$$

where  $C$  is an absolute constant.

**Proof.** Recall that  $g(t) = e^{-t^2/2}$ . Since

$$\frac{f(t)^2 - g(t)^2}{-it} = \int_{-\infty}^{+\infty} e^{itx} ((F * F)(x) - (\Phi * \Phi)(x)) dx,$$

one may apply Plancherel's formula

$$\frac{1}{\pi} \int_0^{+\infty} \frac{|f(t)^2 - g(t)^2|^2}{t^2} dt = \int_{-\infty}^{+\infty} |(F * F)(x) - (\Phi * \Phi)(x)|^2 dx. \quad (5.5)$$

By the assumption (5.1), the integrand on the right-hand side is bounded by  $\varepsilon^2$ , so

$$\int_{-2M}^{2M} |(F * F)(x) - (\Phi * \Phi)(x)|^2 dx \leq 4M\varepsilon^2.$$

On the other hand, since  $F$  is supported on  $[-M, M]$ , for the region  $x > 2M$  we have  $(F * F)(x) = 1$ , while  $(\Phi * \Phi)(x) = 0$  in  $x < -2M$ . Hence, in terms of a standard normal random variable  $Z$ ,

$$\begin{aligned} \int_{2M}^{+\infty} |(F * F)(x) - (\Phi * \Phi)(x)|^2 dx &= \int_{2M}^{+\infty} |1 - (\Phi * \Phi)(x)|^2 dx \\ &= \int_{2M}^{+\infty} \mathbf{P}\{Z > x/\sqrt{2}\}^2 dx \leq \frac{1}{4} \int_{2M}^{+\infty} e^{-x^2/2} dx \\ &= \frac{\sqrt{2\pi}}{4} \mathbf{P}\{Z > 2M\} < \frac{1}{2} e^{-2M^2} < \frac{1}{2} \varepsilon^2. \end{aligned}$$

With a similar bound for the region  $x < -2M$ , we can estimate the right integral in (5.5) by  $(1 + 4M)\varepsilon^2 < 5M\varepsilon^2$ , which results in

$$\int_0^{+\infty} \frac{|f(t)^2 - g(t)^2|^2}{t^2} dt \leq 5M\pi\varepsilon^2. \quad (5.6)$$

Now, just write

$$|f(t)|^4 = |(f(t)^2 - g(t)^2) + g(t)^2|^2 \leq 2|f(t)^2 - g(t)^2|^2 + 2g(t)^4,$$

so that (5.6) implies

$$\int_{T_1}^{+\infty} \frac{|f(t)|^4}{t^2} dt \leq 10M\pi\varepsilon^2 + 2 \int_{T_1}^{+\infty} \frac{g(t)^4}{t^2} dt.$$

Since  $T_1 \geq 1$ , the last integral does not exceed

$$\int_{T_1}^{+\infty} e^{-2t^2} dt = \frac{\sqrt{2\pi}}{2} \mathbf{P}\{Z > 2T_1\} \leq \frac{\sqrt{2\pi}}{2} e^{-2T_1^2} = \frac{\sqrt{2\pi}}{2} \varepsilon'^2 \leq C\varepsilon^2 \log^2(1/\varepsilon)$$

with some absolute constant  $C$ . Hence we get (using a different constant), for any  $T > T_1$ ,

$$\int_{T_1}^T \frac{|f(t)|^4}{t^2} dt \leq C\varepsilon^2 \log^2(1/\varepsilon). \quad (5.7)$$

Applying Hölder's inequality with exponents  $p = 4$ ,  $q = 4/3$  with respect to the measure  $d\mu(t) = dt/t^2$  on the interval  $[T_1, T]$ , and using (5.7), we get that

$$\begin{aligned} \int_{T_1}^T \frac{|f(t)|}{t} dt &= \int_{T_1}^T |f(t)| t d\mu(t) \\ &\leq \left( \int_{T_1}^T |f(t)|^4 d\mu(t) \right)^{1/4} \left( \int_{T_1}^T t^{4/3} d\mu(t) \right)^{3/4} \\ &\leq C\sqrt{\varepsilon \log(1/\varepsilon)} (T^{1/3} - T_1^{1/3})^{3/4}. \end{aligned}$$

Thus, with some absolute constant

$$\int_{T_1}^T \frac{|f(t)|}{t} dt \leq CT^{1/4} \sqrt{\varepsilon \log(1/\varepsilon)}.$$

It remains to note that

$$\frac{1}{\sqrt{2\pi}} \int_{T_1}^{+\infty} \frac{g(t)}{t} dt \leq \mathbf{P}\{Z > T_1\} \leq e^{-T_1^2/2} = \sqrt{\varepsilon'} = 4\sqrt{\varepsilon \log(1/\varepsilon)}.$$

Thus, Lemma 5.3 is proved.

## 6. Proof of Theorem 1.2

According to Lemma 5.2, we may assume that the distribution  $F$  is supported on the interval  $[-M, M]$ , where

$$M = M(\varepsilon) = 1 + 2\sqrt{\log(1/\varepsilon)}$$

with  $\varepsilon > 0$ , such that  $\varepsilon' = 16\varepsilon \log \frac{1}{\varepsilon} \leq \frac{1}{e}$ .

In particular, the characteristic function  $f$  of  $F$  satisfies the bound (5.2).

At this point we shall employ the Berry-Esseen bound (cf. e.g. [P]): For any  $T > 0$ ,

$$\|F - \Phi\| \leq c_1 \int_0^T \frac{|f(t) - g(t)|}{t} dt + c_2 \frac{1}{T}, \quad (6.1)$$

where  $c_1, c_2 > 0$  are universal constants and  $g(t) = e^{-t^2/2}$ .

We split the integral into the three parts. Put

$$T_0 = \sqrt{\varepsilon}, \quad T_1 = \sqrt{\log(1/\varepsilon')}.$$

Clearly,  $T_0 < T_1$ . The value  $T > T_1$  will be of order  $\varepsilon^{-2/5}$  modulo a logarithmic term, and will be specified somewhat later. Anyway,  $\log T \leq C \log \frac{1}{\varepsilon}$  with some absolute constant.

By (5.2), the part of the integral (6.1) over the interval  $0 < t < T_0$  does not exceed

$$2M(\varepsilon) \sqrt{\varepsilon} \leq 8\sqrt{\varepsilon \log(1/\varepsilon)}.$$

By Lemma 5.1, the integral along the interval  $T_0 < t < T_1$  does not exceed

$$4\sqrt{\varepsilon \log(1/\varepsilon)} (\log T_1 - \log T_0).$$

Since  $-\log T_0 = \frac{1}{2} \log \frac{1}{\varepsilon}$  and  $\log T_1$  is of order  $\log \log \frac{1}{\varepsilon}$ , we conclude that

$$\int_{T_0}^{T_1} \frac{|f(t) - g(t)|}{t} dt \leq C\sqrt{\varepsilon} \log^{3/2}(1/\varepsilon)$$

with some absolute constant  $C$ .

To bound the integral along the interval  $T_1 < t < T$ , we apply Lemma 5.3, which yields the bound  $CT^{1/4}\sqrt{\varepsilon \log(1/\varepsilon)}$ .

Collecting all these bounds and applying (6.1), we arrive at

$$\|F - \Phi\| \leq C \left[ \sqrt{\varepsilon} \log^{3/2}(1/\varepsilon) + T^{1/4} \sqrt{\varepsilon \log(1/\varepsilon)} + \frac{1}{T} \right].$$

with some absolute constant  $C$ . It remains to choose the value  $T = (\varepsilon \log(1/\varepsilon))^{-2/5}$ , which is almost optimal. Thus, Theorem 1.2 is proved.

## 7. Oscillating Densities

Let us now focus on Theorem 1.1. At first we shall describe the class of probability distributions on the real line, which provide a counter-example to Kac's conjecture. Given parameters  $T > 0$  and  $\alpha \in (0, 1]$ , put

$$p_T(x) = \frac{1}{c} (1 - \alpha \cos(Tx^2)) \varphi(x), \quad x \in \mathbf{R}, \quad (7.1)$$

where  $\varphi$  denotes the density of the standard normal law, and  $c = c_T > 0$  is a normalizing constant (meaning that  $c$  depends both on  $T$  and  $\alpha$ ). Denote by  $F_T$  the corresponding distribution function,

$$F_T(x) = \int_{-\infty}^x p_T(y) dy = \frac{1}{c} \left[ \Phi(x) - \alpha \int_{-\infty}^x \cos(Ty^2) d\Phi(y) \right]. \quad (7.2)$$

By construction, the density  $p_T$  has oscillations around  $\varphi$ , bounded by a multiple of  $\alpha\varphi$ . When  $T$  is large, this type of oscillation on one hand provides closeness of  $p_T$  to  $\varphi$  in the weak sense, but on the other hand also separates  $p_T$  from the normal in the strong sense.

To carefully develop the participating Fourier transforms, we start with a well-known elementary identity.

**Lemma 7.1.** *For all complex numbers  $a$  and  $z$ , such that  $\operatorname{Re}(z) > 0$ ,*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ax - zx^2/2} dx = \frac{1}{\sqrt{z}} e^{\frac{a^2}{2z}}, \quad (7.3)$$

where  $\sqrt{z}$  is the principal branch of the square root function in the half-plane  $\operatorname{Re}(z) > 0$ .

For example, applying (7.3) with  $a = 0$ ,  $z = 1 - 2iT$ , we find that  $\int e^{iT x^2} \varphi(x) dx = \frac{1}{\sqrt{1-2iT}}$ .

Write  $\cos(Tx^2) = \frac{1}{2} e^{iT x^2} + \frac{1}{2} e^{-iT x^2}$ , so that to express the normalizing constant  $c = c_T$  in (7.1) explicitly in terms of  $T$  and  $\alpha$ :

$$c_T = 1 - \frac{\alpha}{2} \left( \frac{1}{\sqrt{1-2iT}} + \frac{1}{\sqrt{1+2iT}} \right).$$

As an immediate consequence, we obtain:

**Lemma 7.2.** *We have*

$$|1 - c_T| \leq \frac{\alpha}{(1 + 4T^2)^{1/4}} < \frac{\alpha}{\sqrt{T}}. \quad (7.4)$$

Thus,  $c_T \rightarrow 1$ , as  $T \rightarrow +\infty$ , uniformly over all allowed values of  $\alpha$ . From (7.4), whenever  $T \geq 4$ , one easily has

$$\frac{\alpha}{4\sqrt{T}} \leq \left| \frac{1}{c_T} - 1 \right| \leq \frac{2\alpha}{\sqrt{T}}. \quad (7.5)$$

For  $T \geq 4$ , one also has a uniform bound

$$|p_T(x) - \varphi(x)| \leq 3\alpha \varphi(x), \quad x \in \mathbf{R}. \quad (7.6)$$

Moreover, one can easily compute the characteristic function  $f_T$  of  $F_T$ , by applying (7.3) with  $a = it$  and  $z = 1 \pm 2iT$ . Namely, for all  $t \in \mathbf{R}$ , we get

$$c_T f_T(t) = e^{-t^2/2} - \frac{\alpha}{2} \left( \frac{e^{-t^2/2(1-2iT)}}{\sqrt{1-2iT}} + \frac{e^{-t^2/2(1+2iT)}}{\sqrt{1+2iT}} \right).$$

From this

$$|c_T f_T(t) - e^{-t^2/2}| \leq \frac{\alpha}{(1 + 4T^2)^{1/4}} < \frac{1}{\sqrt{T}},$$

and using Lemma 7.2, we arrive at the following conclusion.

**Proposition 7.3.** *Uniformly for all  $\alpha \in (0, 1]$ , we have  $F_T \rightarrow \Phi$  weakly, as  $T \rightarrow +\infty$ .*

Equivalently,  $L(F_T, \Phi) \rightarrow 0$  for the Lévy distance.

To test the closeness with respect to the total variation norm, write

$$\begin{aligned} \int_{-\infty}^{+\infty} |c_T p_T(x) - \varphi(x)| dx &= \frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |\cos(Tx^2)| e^{-x^2/2} dx \\ &= \frac{\alpha}{\sqrt{2\pi} T} \int_0^{+\infty} \frac{|\cos(y)|}{\sqrt{y}} e^{-y/(2T)} dy. \end{aligned}$$

Restricting the integration to  $0 < y < T$  and assuming  $T \geq 1$ , we obtain that

$$\int_{-\infty}^{+\infty} |c_T p_T(x) - \varphi(x)| dx > \frac{\alpha}{\sqrt{2\pi e} T} \int_0^T \frac{|\cos(y)|}{\sqrt{y}} dy > c_T \kappa \alpha,$$

for some absolute constant  $\kappa > 0$ . Hence, using the upper bound in (7.5),

$$\begin{aligned} \|F_T - \Phi\|_{\text{TV}} &= \int_{-\infty}^{+\infty} |p_T(x) - \varphi(x)| dx \\ &\geq \int_{-\infty}^{+\infty} \left| p_T(x) - \frac{1}{c_T} \varphi(x) \right| dx - \left| \frac{1}{c_T} - 1 \right| \geq \alpha \left( \kappa - \frac{2}{\sqrt{T}} \right). \end{aligned}$$

One may now combine this bound with (7.6) and summarize.

**Proposition 7.4.** *For some absolute constants  $T_0 > 0$  and  $\kappa > 0$ ,*

$$\kappa\alpha \leq \|F_T - \Phi\|_{\text{TV}} \leq 3\alpha \quad (T \geq T_0).$$

## 8. Computations and Bounds for Convolutions

In the following we shall study the behaviour of the convolutions  $p_T * p_T$  for the growing parameter  $T$ . Again applying Lemma 7.1 as in the previous section, one can obtain explicit formulas for the densities of convolutions (so, the formula for the characteristic function  $f_T$  is not needed).

In analogy with notations for distribution functions, for integrable functions  $p(x)$  and  $q(x)$  we write  $(p * q)(x) = p(x) * q(x) = \int p(x - y) q(y) dy$ .

For  $T$  real, define

$$u_T(x) = e^{iTx^2} \varphi(x), \quad x \in \mathbf{R}.$$

In particular,  $u_0 = \varphi$ , so

$$u_0(x) * u_0(x) = \varphi(x) * \varphi(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}.$$

**Step 1.** More generally, in the integral

$$u_T(x) * \varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iTy^2} e^{-\frac{1}{2}((x-y)^2+y^2)} dy \quad (8.1)$$

we change variable via  $y = \frac{x}{2} + \frac{s}{\sqrt{2}}$ . Then the integral (8.1) becomes

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\frac{iTx^2}{4}} e^{\frac{iTx}{\sqrt{2}}s + \frac{iTs^2}{2}} e^{-\frac{x^2}{4} - \frac{s^2}{2}} \frac{ds}{\sqrt{2}} &= \frac{e^{-\frac{x^2}{4}}}{2\pi\sqrt{2}} e^{\frac{iTx^2}{4}} \int_{-\infty}^{+\infty} e^{\frac{iTx}{\sqrt{2}}s - \frac{1-iT}{2}s^2} ds \\ &= \varphi(x) * \varphi(x) e^{\frac{iTx^2}{4}} \frac{1}{\sqrt{1-iT}} e^{-\frac{T^2x^2}{4(1-iT)}}, \end{aligned}$$

where we used (7.3) with  $a = \frac{iTx}{\sqrt{2}}$  and  $z = 1 - iT$ . Thus,

$$\frac{u_T(x) * \varphi(x)}{\varphi(x) * \varphi(x)} = e^{\frac{iTx^2}{4}} \frac{1}{\sqrt{1-iT}} e^{-\frac{T^2x^2}{4(1-iT)}}.$$

Since  $\frac{1}{1-iT} = \frac{1+iT}{1+T^2}$  has a positive real part, we obtain that

$$\frac{|u_T(x) * \varphi(x)|}{\varphi(x) * \varphi(x)} \leq \frac{1}{(1+T^2)^{1/4}} < \frac{1}{\sqrt{|T|}}.$$

Applying a similar representation to  $-T$  and using  $\cos(Tx^2) \varphi(x) = \frac{1}{2} u_T(x) + \frac{1}{2} u_{-T}(x)$ , we conclude that

$$\frac{|[\cos(Tx^2) \varphi(x)] * \varphi(x)|}{\varphi(x) * \varphi(x)} < \frac{1}{\sqrt{|T|}}. \quad (8.2)$$

**Step 2.** Here, similarly as before, consider the convolution

$$\begin{aligned} u_T(x) * u_T(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iT((x-y)^2+y^2)} e^{-\frac{1}{2}((x-y)^2+y^2)} dy \\ &= \varphi(x) * \varphi(x) e^{\frac{iTx^2}{2}} \frac{1}{\sqrt{1-2iT}}, \end{aligned}$$

which gives  $|u_T(x) * u_T(x)| = \varphi(x) * \varphi(x) \frac{1}{(1+4T^2)^{1/4}}$ , and again

$$\frac{|u_T(x) * u_T(x)|}{\varphi(x) * \varphi(x)} < \frac{1}{\sqrt{|T|}}. \quad (8.3)$$

**Step 3.** Using  $(x-y)^2 - y^2 = -\sqrt{2}xs$ , we get

$$\begin{aligned} u_T(x) * u_{-T}(x) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iT((x-y)^2-y^2)} e^{-\frac{1}{2}((x-y)^2+y^2)} dy \\ &= \frac{1}{2\pi\sqrt{2}} e^{-\frac{x^4}{4}} \int_{-\infty}^{+\infty} e^{-iT\sqrt{2}xs - \frac{1}{2}s^2} ds \\ &= \varphi(x) * \varphi(x) e^{-T^2x^2}, \end{aligned}$$

where now we applied (7.3) with  $a = -iT\sqrt{2}$  and  $z = 1$ . Hence,

$$\frac{u_T(x) * u_{-T}(x)}{\varphi(x) * \varphi(x)} = e^{-T^2x^2}. \quad (8.4)$$

**Step 4.** Furthermore, for  $T > 0$ , consider the convolution

$$\psi_T(x) = [\cos(Tx^2) \varphi(x)] * [\cos(Tx^2) \varphi(x)].$$

Representing

$$\psi_T = \frac{u_T + u_{-T}}{2} * \frac{u_T + u_{-T}}{2} = \frac{u_T * u_T + 2u_T * u_{-T} + u_{-T} * u_{-T}}{4}$$

and applying the Steps 2-3 (i.e., relations (8.3) and (8.4)), we obtain that, for all  $x \in \mathbf{R}$ ,

$$\frac{|\psi_T(x)|}{\varphi(x) * \varphi(x)} < \frac{1}{2} \left( \frac{1}{\sqrt{T}} + e^{-T^2x^2} \right). \quad (8.5)$$

**Step 5.** Given  $T > 0$  and  $\alpha \in [0, 1]$ , at last consider

$$\begin{aligned} \zeta_T(x) &= [(1 - \alpha \cos(Tx^2)) \varphi(x)] * [(1 - \alpha \cos(Tx^2)) \varphi(x)] \\ &= \varphi(x) * \varphi(x) - 2\alpha [\cos(Tx^2) \varphi(x)] * \varphi(x) + \alpha^2 \psi_T(x). \end{aligned}$$

By Step 1 (the inequality (8.2)) and Step 4 (the inequality (8.5)), we get

$$\left| \frac{\zeta_T(x)}{\varphi(x) * \varphi(x)} - 1 \right| < \frac{5}{2\sqrt{T}} + \frac{1}{2} e^{-T^2x^2}. \quad (8.6)$$

Finally, for the density  $p_T$ , introduced in (7.1), we have  $\zeta_T = c_T^2 p_T * p_T$ . Using (8.6), we obtain that

$$\left| \frac{p_T(x) * p_T(x)}{\varphi(x) * \varphi(x)} - 1 \right| < \left| \frac{1}{c_T^2} - 1 \right| + \frac{1}{c_T^2} \left( \frac{5}{2\sqrt{T}} + \frac{1}{2} e^{-T^2x^2} \right). \quad (8.7)$$

By (7.4)–(7.5), if  $T \geq 4$ , we have  $|\frac{1}{c_T} - 1| \leq \frac{2}{\sqrt{T}}$  and  $c_T \geq \frac{1}{2}$ . Hence, the right-hand side of (8.7) does not exceed

$$\frac{2}{\sqrt{T}} \cdot \frac{3}{2} + 4 \left( \frac{5}{2\sqrt{T}} + \frac{1}{2} e^{-T^2 x^2} \right) = \frac{13}{\sqrt{T}} + 2 e^{-T^2 x^2}.$$

Thus, we have arrived at:

**Lemma 8.1.** *Given  $T \geq 4$  and  $\alpha \in [0, 1]$ , for all  $x \in \mathbf{R}$ ,*

$$\left| \frac{p_T(x) * p_T(x)}{\varphi(x) * \varphi(x)} - 1 \right| < \frac{13}{\sqrt{T}} + 2 e^{-T^2 x^2}. \quad (8.8)$$

## 9. Proof of Theorem 1.1

Having established Proposition 7.4 and Lemma 8.1, we may proceed with the last step of the proof of Theorem 1.1. More precisely, for large values of  $T$ , we need to verify statement *a*) in Theorem 1.1 for the distributions  $F = F_T$  described in (7.1)–(7.2). Note that statement *b*) is provided by Proposition 7.4, when  $\alpha$  is separated from zero (take  $\alpha = 1$ , for example).

Let us recall that in the setting of an arbitrary probability space  $(\Omega, \mu)$ , the Kullback-Leibler distance (also called informational divergence) is well-defined for any probability measure  $\nu$ , which is absolutely continuous with respect to  $\mu$ . Namely, if  $u$  is density of  $\nu$  with respect to  $\mu$ , the distance is given by

$$D(\nu||\mu) = \int_{\Omega} u \log u \, d\mu.$$

**Lemma 9.1.** *If  $|u - 1| \leq A$  with some constant  $A \geq 0$ , then*

$$D(\nu||\mu) \leq \log((A + 1)e) \int_{\Omega} |u - 1| \, d\mu. \quad (9.1)$$

Indeed, if  $u \geq 1/e$ , we have  $|u \log u| \leq \log((A + 1)e) |u - 1|$ , by the mean value theorem. For  $0 \leq u \leq 1/e$  the latter inequality holds automatically, since  $u \log u$  is decreasing on that interval.

**Proof of Theorem 1.1.** On the real line  $\Omega = \mathbf{R}$ , taking  $\mu$  to be the normal law  $N(0, 2)$  with mean zero and variance 2, and taking  $\nu$  to be the distribution  $F_T * F_T$ , the corresponding density  $u$  in Lemma 9.1 is given by

$$u = \frac{d\nu}{d\mu} = \frac{p_T * p_T}{\varphi * \varphi}.$$

By Lemma 8.1 with  $T \geq 4$ , we have  $|u - 1| \leq 8.5$ . Hence, the assumption of Lemma 9.1 is fulfilled with  $A = 8.5$ . Using  $\log((A + 1)e) < 4$ , we obtain by (9.1) and (8.8) that

$$D(F_T * F_T||N(0, 2)) \leq 4 \int_{-\infty}^{+\infty} \left( \frac{13}{\sqrt{T}} + 2 e^{-T^2 x^2} \right) d\mu(x) \leq \frac{C}{\sqrt{T}}, \quad (9.2)$$

where  $C$  is an absolute constant.

Finally, let us apply the well-known fact that, if a random variable  $\xi$  has a distribution  $\nu$ , which is absolutely continuous with respect to Lebesgue measure and has a finite second

moment, the entropic distance  $D(\xi)$  to normality represents the shortest Kullback-Leibler distance from  $\nu$  to the class of all normal laws. That is,

$$\min_{a, \sigma} D(\nu || N(a, \sigma^2)) = D(\xi) \quad (9.3)$$

with minimum being attained for  $a = \mathbf{E}\xi$ ,  $\sigma^2 = \text{Var}(\xi)$ . Hence, if  $X_T$  and  $Y_T$  are independent random variables distributed according to  $F_T$ , by (9.2)–(9.3), applied with  $\xi = X_T + Y_T$ , we obtain that

$$D(X_T + Y_T) \leq \frac{C}{\sqrt{T}} \rightarrow 0, \quad \text{as } T \rightarrow +\infty.$$

Note there is no need to compute the parameters  $a$  and  $\sigma^2$  for  $\xi$ .

Hence, Theorem 1.1 is now proved.

### 10. Proof of Theorem 1.3

To prove Theorem 1.3, we introduce another family of probability distributions on the real line, which oscillate around the standard normal law and can be described by explicit formulas for their distribution functions. Using distribution functions seems more convenient for estimating the Kolmogorov distances compared to an "explicit density approach". This family has a much simpler structure than the family described in the proof of Theorem 1.1, and it can also be used to construct a counter example to Kac's question – however, not in the i.i.d. case.

Given  $T$  large enough (say,  $T \geq 6$ ), consider the distribution function

$$F_T(x) = \Phi(x) + \frac{1}{2T} \cos(T_0 x) \varphi(x) 1_{\{|x| < T_0\}},$$

where  $T_0 = 4\sqrt{\log T}$  and, as before,  $\Phi$  and  $\varphi$  stand for the density and the distribution function of the standard normal law. If  $\cos(T_0^2) = 0$ , which we may assume from now on,  $F_T$  has density

$$p_T(x) = \varphi(x) - \frac{1}{2} \left( \frac{T_0}{T} \sin(T_0 x) + \frac{x}{T} \cos(T_0 x) \right) \varphi(x) 1_{\{|x| < T_0\}}.$$

Since  $T_0 < T$ , the function  $p_T$  is positive and continuous everywhere (except for  $|x| = T_0$ ). Since in addition  $F_T(-\infty) = 0$  and  $F_T(+\infty) = 1$ ,  $F_T$  is indeed a distribution function.

It is easy to check that, if  $X_T$  is a random variable with distribution  $F_T$ , we have  $\mathbf{E}X_T \rightarrow 0$  and  $\text{Var}(X_T) \rightarrow 1$ , as  $T \rightarrow +\infty$ .

It follows immediately from the definition that  $|F_T(x) - \Phi(x)|$  attains its maximum at the origin, which we formulate below in terms of the Kolmogorov distance.

**Lemma 10.1.** *For all  $T > 0$ , we have  $\|F_T - \Phi\| = \frac{1}{2T\sqrt{2\pi}}$ .*

Hence, to prove Theorem 1.3, it will be sufficient to show that for the corresponding convolutions the distance

$$\|F_T * F_T - \Phi * \Phi\|$$

is of order  $1/T^2$  modulo a logarithmic factor. Consequently, one may take  $\varepsilon$  to be of order  $1/T^2$ .

The remaining part of this section is devoted to the confirmation of this conjecture. Here and below, the convolution operator is applied to functions of bounded variation that are regarded as positive or signed measures. For example,

$$F_T(x) * F_T(x) = \int_{-\infty}^{+\infty} F_T(x-y) dF_T(y).$$

Rewrite  $F_T$  in the form  $F_T = \Phi + G_T + H_T$ , where

$$G_T(x) = \frac{1}{2T} \cos(T_0 x) \varphi(x), \quad H_T(x) = \frac{1}{2T} \cos(T_0 x) \varphi(x) 1_{\{|x| \geq T_0\}}. \quad (10.1)$$

Hence,

$$F_T * F_T = (\Phi + G_T) * (\Phi + G_T) - 2(\Phi + G_T) * H_T + H_T * H_T. \quad (10.2)$$

Obviously, we have the bounds  $\|G_T\|_{TV} < 1$  and  $\|H_T\|_{TV} < e^{-T_0^2/2}$ , hence the sum of the last two terms in (10.2) has total variation less than  $5 e^{-T_0^2/2} = 5/T^8$ .

The remaining convolutions in (10.2) are readily computable. Changing the variable  $y = \frac{x}{2} + \frac{s}{\sqrt{2}}$ , we have

$$\begin{aligned} 2T G_T(x) * \Phi(x) &= 2T \int_{-\infty}^{+\infty} G_T(x-y) d\Phi(y) \\ &= \int_{-\infty}^{+\infty} \cos(T_0(x-y)) \varphi(x-y) \varphi(y) dy \\ &= \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{+\infty} e^{iT_0(x-y)} e^{-\frac{1}{2}((x-y)^2+y^2)} dy \\ &= \frac{1}{2\pi} \operatorname{Re} \left[ e^{\frac{1}{2}iT_0x - \frac{1}{4}x^2} \int_{-\infty}^{+\infty} e^{-iT_0s/\sqrt{2}} e^{-\frac{1}{2}s^2} \frac{ds}{\sqrt{2}} \right] \\ &= \frac{1}{2\sqrt{\pi}} \cos(T_0x/2) e^{-x^2/4} e^{-T_0^2/4}. \end{aligned}$$

By simple computations, it follows that

$$\|G_T * \Phi\|_{TV} = \frac{e^{-T_0^2/4}}{4T\sqrt{\pi}} \int_{-\infty}^{+\infty} \left| \frac{d}{dx} \cos(T_0x/2) e^{-x^2/4} \right| dx < e^{-T_0^2/4} = \frac{1}{T^4}.$$

Taking into account the previous bounds, we get the following preliminary conclusion.

**Lemma 10.2.** *If  $T \geq 6$ , we have*

$$F_T * F_T = \Phi * \Phi + G_T * G_T + U_T,$$

where the function  $G_T$  is defined in (10.1) and  $U_T$  has a total variation less than  $7/T^4$ .

It remains to proceed to the last step. The following simple identity is verified directly.

**Lemma 10.3.** *For all  $x$ ,*

$$4T^2 G_T(x) * G_T(x) = -\frac{T_0}{4\sqrt{\pi}} \sin(T_0x) e^{-x^2/4} - \frac{x e^{-x^2/4}}{8\sqrt{\pi}} \left( \cos(T_0x) + e^{-T_0^2} \right).$$

**Proof.** Write

$$\begin{aligned} 4T^2 G_T(x) * G_T(x) &= T^2 \int_{-\infty}^{+\infty} G_T(x-y) dG_T(y) \\ &= \int_{-\infty}^{+\infty} \cos(T_0(x-y)) \varphi(x-y) d(\cos(T_0 y) \varphi(y)) \\ &= -T_0 I_1 - I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(T_0(x-y)) \sin(T_0 y) e^{-\frac{1}{2}((x-y)^2+y^2)} dy, \\ I_2 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(T_0(x-y)) \cos(T_0 y) y e^{-\frac{1}{2}((x-y)^2+y^2)} dy. \end{aligned}$$

Changing  $y = \frac{x}{2} + \frac{s}{\sqrt{2}}$ , we get  $T_0 y + T_0(x-y) = T_0 x$  and  $T_0 y - T_0(x-y) = T_0 \sqrt{2} s$ , so

$$I_1 = \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\sin(T_0 x) + \sin(T_0(2y-x))) e^{-\frac{1}{2}((x-y)^2+y^2)} dy = \frac{1}{4\sqrt{\pi}} \sin(T_0 x) e^{-x^2/4}$$

and

$$I_2 = \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\cos(T_0 x) + \cos(T_0(2y-x))) y e^{-\frac{1}{2}((x-y)^2+y^2)} dy = I'_2 + I''_2.$$

Here

$$I'_2 = \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(T_0 x) y e^{-\frac{1}{2}((x-y)^2+y^2)} dy = \frac{1}{8\sqrt{\pi}} x \cos(T_0 x) e^{-x^2/4}$$

and

$$I''_2 = \frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cos(T_0(2y-x)) y e^{-\frac{1}{2}((x-y)^2+y^2)} dy = \frac{1}{8\sqrt{\pi}} x e^{-x^2/4} e^{-T_0^2}.$$

It remains to collect these formulas, and the proof is complete.

**Proof of Theorem 1.3.** In view of the equality of Lemma 10.3, we first note that  $x e^{-x^2/4} < 1$  and obtain that  $4T^2 |G_T(x) * G_T(x)| < T_0$ , whenever  $T \geq 6$ . By Lemma 10.2,

$$\|F_T * F_T - \Phi * \Phi\| < \frac{T_0}{4T^2} = \frac{\sqrt{\log T}}{T^2}.$$

Hence, choosing  $T = \varepsilon^{-1/2} \log^{1/4}(1/\varepsilon)$  with  $\varepsilon > 0$  small enough, we get  $\|F_T * F_T - \Phi * \Phi\| < \varepsilon$ , while, by Lemma 10.1,

$$\|F - \Phi\| = \frac{1}{2T\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}} \varepsilon^{1/2} \log^{-1/4}(1/\varepsilon).$$

Thus, Theorem 1.3 is proved.

**Remark.** Inspecting the last step of the proof of Lemma 10.3, one recognizes that there is a stronger property, namely,  $T^2 \|G_T * G_T\|_{\text{TV}} < CT_0^2$ , which, by Lemma 10.2, leads to

$$\|F_T * F_T - \Phi * \Phi\|_{\text{TV}} < C \frac{\log T}{T^2}$$

with some absolute constant  $C$ . Hence, modulo a logarithmic factor, the statement of Proposition 1.3 may be sharpened: For  $\varepsilon > 0$  small enough and  $T$  of the same order  $\varepsilon^{-1/2} \log^{1/4}(1/\varepsilon)$ , we have

$$\|F_T * F_T - \Phi * \Phi\|_{\text{TV}} < \varepsilon, \quad \text{while} \quad \|F - \Phi\| > c\varepsilon^{1/2} \log^{-1/2}(1/\varepsilon),$$

where  $c > 0$  is an absolute constant.

#### REFERENCES

- [B-C-G1] Bobkov, S. G., Chistyakov, G. P., Götze, F. Entropic instability of Cramer's characterization of the normal law. Preprint (2009). To appear in: The Selected Works of Willem van Zwet. A separate volume by the IMS, Springer.
- [B-C-G2] Bobkov, S. G., Chistyakov, G. P., Götze, F. Regularized distributions and entropic stability of Cramer's characterization of the normal law. Bielefeld, Preprint, 2010.
- [C1] Chistyakov, G. P. The sharpness of the estimates in theorems on the stability of decompositions of a normal distribution and a Poisson distribution. (Russian) *Teor. Funkcii Funkcional. Anal. i Prilozhen.* Vyp. 26 (1976), 119-128, iii.
- [C2] Chistyakov, G. P. A remark on a theorem of N. A. Sapogov on the stability of decompositions of a normal distribution. (Russian) *Operators in function spaces and problems in function theory* (Russian), 108-116, 147, "Naukova Dumka", Kiev, 1987.
- [C-G] Chistyakov, G. P., Golinskii, L. B. Order-sharp estimates for the stability of decompositions of the normal distribution in the Levy metric. (Russian) Translated in *J. Math. Sci.* 72 (1994), no. 1, 2848–2871. *Stability problems for stochastic models* (Russian) (Moscow, 1991), 16–40, Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1991.
- [Ch-P-U] Christoph, G., Prohorov, Yu. V., Ulyanov, V. V. Characterization and stability problems for finite quadratic forms. *Asymptotic methods in probability and statistics with applications* (St. Petersburg, 1998), 39-50, *Stat. Ind. Technol.*, Birkhauser Boston, Boston, MA, 2001.
- [Cr] Cramér, H. Ueber eine Eigenschaft der Normalen Verteilungsfunktion. *Math. Zeitschrift*, Bd. 41 (1936), 405–414.
- [D] Dudley, R. M.. *Real Analysis and Probability*. Chapman & Hall Math. Series, 1999.
- [F] Feller, W. *An introduction to Probability Theory and its Applications*. Vol.2. Second edition, John Wiley & Sons, Inc., 1971.
- [I] Il'in, V. V. On the quantitative stability of the Cramér theorem in a uniform metric in the case of identically distributed random variables. (Russian) *Dokl. Akad. Nauk SSSR*, 260 (1981), no. 3, 525–526.
- [K-L-R] Kagan, A. M., Linnik, Yu. V., Rao, C. R. *Characterization problems in mathematical statistics*. Translated from the Russian by B. Ramachandran. *Wiley Series in Probability and Mathematical Statistics*. John Wiley & Sons, New York-London-Sydney, 1973, xii+499 pp.
- [L-O] Linnik, Yu. V., Ostrovskii, I. V. *Decompositions of random variables and random vectors*. (Russian) Izdat. "Nauka", Moscow, 1972, 479 pp. Translated from the Russian: *Translations of Mathematical Monographs*, Vol. 48. American Mathematical Society, Providence, R. I., 1977, ix+380 pp.
- [M1] Maloshevskii, S. G. Unimprovability of N. A. Sapogov's result in the stability problem of H. Cramer's theorem. (Russian) *Teor. Veroyatnost. i Primenen.* 13 (1968), 522–525.

- [M2] Maloshevskii, S. G. Estimates connected with the stability problem of H. Cramer's theorem. (Russian) *Mat. Zametki* 7 (1970), 281-288.
- [MK] McKean, H. P., Jr. Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas. *Arch. Rational Mech. Anal.* 21 (1966), 343-367.
- [P] Petrov, V. V. Sums of independent random variables. Springer-Verlag, 1975.
- [Sa1] Sapogov, N. A. The stability problem for a theorem of Cramér. (Russian) *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 15 (1951), 205-218.
- [Sa2] Sapogov, N. A. The problem of stability for a theorem of Cramér. (Russian) *Vestnik Leningrad. Univ.* 10 (1955), no. 11, 61-64.
- [Se] Senatov, V. V. Refinement of estimates of stability for a theorem of H. Cramér. (Russian) *Continuity and stability in problems of probability theory and mathematical statistics. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 61 (1976), 125-134.
- [Se-Z] Senatov, V. V., Zolotarev, V. M. Two-sided estimates of the Levy metric. (Russian) *Teor. Veroyatnost. i Primenen.* 20 (1975), no. 2, 239-250.
- [Sh1] Shiganov, I. S. Some estimates connected with the stability of H. Cramer's theorem. (Russian) *Studies in mathematical statistics, 3. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 87 (1979), 187-195.
- [Sh2] Shiganov, I. S. On stability estimates of Cramer's theorem. *Stability problems for stochastic models (Varna, 1985)*, 178-181, *Lecture Notes in Math.*, 1233, Springer, Berlin, 1987.
- [Z1] Zolotarev, V. M. On the problem of the stability of the decomposition of the normal law into components. (Russian) *Teor. Veroyatnost. i Primenen.*, 13 (1968), 738-742.
- [Z2] Zolotarev, V. M. Several new probabilistic inequalities connected with the Levy metric. (Russian) *Dokl. Akad. Nauk SSSR*, 190 (1970), 1019-1021.
- [Z3] Zolotarev, V. M. Estimates of the difference between distributions in the Levy metric. (Russian) *Collection of articles dedicated to Academician Ivan Matveyevich Vinogradov on his eightieth birthday, I. Trudy Mat. Inst. Steklov.* 112 (1971), 224-231.

SERGEY G. BOBKOV  
 SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA  
 127 VINCENT HALL, 206 CHURCH ST. S.E., MINNEAPOLIS, MN 55455 USA  
*E-mail address:* bobkov@math.umn.edu

GENNADIY P. CHISTYAKOV  
 FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD  
 POSTFACH 100131, 33501 BIELEFELD, GERMANY  
*E-mail address:* chistyak@math.uni-bielefeld.de

FRIEDRICH GÖTZE  
 FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD  
 POSTFACH 100131, 33501 BIELEFELD, GERMANY  
*E-mail address:* goetze@mathematik.uni-bielefeld.de