LAGRANGIAN FIBRATIONS ON HYPERKÄHLER MANIFOLDS: INTERSECTIONS OF LAGRANGIANS, *L*-REDUCTION, AND FOURFOLDS

DANIEL GREB, CHRISTIAN LEHN, AND SÖNKE ROLLENSKE

ABSTRACT. Let X be a projective hyperkähler manifold containing a Lagrangian subtorus L. We study intersections of deformations of L, defining a canonical almost holomorphic map called L-reduction, which is not birational if and only if X admits an almost holomorphic Lagrangian fibration with (strong) fibre L.

In dimension four we prove that in the above situation there is always a holomorphic Lagrangian fibration with fibre L, thus answering a question of Beauville in this particular case.

CONTENTS

1. INTRODUCTION

Let X be a hyperkähler manifold, that is, a compact, simply-connected Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a holomorphic symplectic form σ. By work of Matsushita it is well-known that the only possible non-trivial holomorphic maps from X to a lower-dimensional complex space are Lagrangian fibrations, see section 2. Moreover, a special version of the so-called Hyperkähler SYZ-conjecture asserts that any hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration.

Hence, it is an important problem to find geometric conditions on a given hyperkähler manifold that guarantee the existence of a Lagrangian fibration; here we address a question posed by Beauville [Bea11, Sect. 1.6]:

Question B — *Let X be a hyperkähler manifold and L a Lagrangian torus in X. Is L a fibre of a (meromorphic) Lagrangian fibration $f \colon X \to B$?*

We are building upon previous results that can be summarised as follows.

Theorem 1.1 (summarising [GLR11]) — *Let X be a hyperkähler manifold containing a smooth Lagrangian subtorus L.*

   (*i*) *If X is not projective, then X admits a holomorphic Lagrangian fibration with fibre L [GLR11, Thm. 4.1].*

  (*ii*) *If X is projective then X admits an almost holomorhic Lagrangian fibration wist strong fibre L if and only if there exists an effective divisor on X that restrict to zero on L [GLR11, Cor. 5.11].*

 (*iii*) *If X is projective, and if f : X --→ B is an almost holomorphic Lagrangian fibration, then there exists a holomorphic model for f on a birational hyperkähler manifold X', that is, there exists a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow & X' \\
f \downarrow & & \downarrow f' \\
B & \dashrightarrow & B'
\end{array}
$$

where f' is a Lagrangian fibration on X' and the horizontal maps are birational [GLR11, Thm. 6.2].

 (*iv*) *If X is projective, and f : X --→ B an almost holomorphic map with connected fibres onto a normal projective variety B such that $0 < \dim B < \dim X$, then $\dim B = \frac{1}{2} \dim X$, and f is an almost holomorphic Lagrangian fibration [GLR11, Thm. 6.7].*

Here, we consider a different approach based on a more detailed study of the deformation theory of L in X. For this, consider the component $\mathfrak{B}$ of the Barlet space that contains $[L]$ together with its universal family and the evaluation map to X:

$$
\begin{array}{ccc}
\mathfrak{U} & \xrightarrow{\ \varepsilon\ } & X \\
\pi \downarrow & & \\
\mathfrak{B}. & &
\end{array}
$$

It was shown in [GLR11, Lemma 3.1] that ε is surjective and generically finite, and that X admits an almost holomorphic Lagrangian fibration if and only if $\deg(\varepsilon) = 1$.

If the degree of ε is strictly bigger than one, some deformations of L intersect L in unexpected ways. In order to deal with this, we introduce the notion of *L-reduction*: for each projective hyperkähler manifold containing a Lagrangian torus there exists a projective variety $\mathfrak{T}$ and a rational map $\varphi_L \colon X \dashrightarrow \mathfrak{T}$, uniquely defined up to birational equivalence, whose fibre through a general point x coincides with the connected component of the intersection of all deformations of L through x. In this situation, we say that X is *L-separable* if φ_L is birational, and prove the following result:

Theorem 3.5 — *Let X be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then X admits an almost holomorphic fibration with strong fibre L if and only if X is not L-separable .*

If X is a hyperkähler fourfold, then we can exclude the case that X is L-separable by symplectic linear algebra. Moreover, based upon the rather explicit knowledge about the birational geometry of hyperkähler fourfolds we obtain a positive answer to the strongest form of Beauville's question:

Theorem 5.1 — *Let X be a four-dimensional hyperkähler manifold containing a Lagrangian torus L. Then X admits a holomorphic Lagrangian fibration with fibre L.*

At the Moscow conference "Geometric structures on complex manifolds" Ekaterina Amerik brought to our attention that she had independently shown a related result, based on an observation from [AC08], to the effect that in dimension four every hyperkähler manifold containing a Lagrangian subtorus *L* admits an almost holmorphic Lagrangian fibration with fibre *L* [A11].

2. PRELIMINARIES AND SETUP OF NOTATION

2.1. **Lagrangian fibrations.**

Definition 2.1 — Let *X* be a hyperkähler manifold. A *Lagrangian fibration* on *X* is a holomorphic map $f \colon X \to B$ with connected fibres onto a normal complex space *B* such that every irreducible component of the reduction of every fibre of *f* is a Lagrangian subvariety of *X*.

Due to fundamental results of Matsushita it is known that any fibration on a hyperkähler manifold is automatically Lagrangian:

Theorem 2.2 ([Mat99, Mat00, Mat01, Mat03]) — *Let X be a hyperkähler manifold of dimension 2n. If $f \colon X \to B$ is a morphism with connected fibres to a normal complex space B with $0 < \dim B < \dim X$, then f is a Lagrangian fibration. In particular, f is equidimensional and $\dim B = n$. Furthermore, every smooth fibre of f is a complex torus.*

2.2. **Meromorphic maps.** Let *X* be a normal complex space, *Y* a compact complex space, and $f \colon X \dashrightarrow Y$ a meromorphic map. Let

(1)
$$
\begin{array}{ccc}
 & \widetilde{X} & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle \widetilde{f}} \\
X \dashrightarrow & \xrightarrow{\ f\ } & Y
\end{array}
$$

be a resolution of the indeterminacies of *f*. The *fibre* F_y of *f* over a point $y \in Y$ is defined to be $F_y := p(\widetilde{f}^{-1}(y))$. This is independent of the chosen resolution.

Recall that a meromorphic map $f \colon X \dashrightarrow Y$ as above is called *almost holomorphic* if there is a Zariski-open subset $U \subset Y$ such that the restriction $f|_{f^{-1}(U)} \colon f^{-1}(U) \to U$

is holomorphic and proper. A *strong fibre* of an almost holomorphic map f is a fibre of $f|_{f^{-1}(U)}$.

Let X be a normal algebraic variety, B a complete algebraic variety, and $f \colon X \dashrightarrow B$ an almost holomorphic rational map. If A is a divisor on B, then its *pullback* via f is defined either geometrically as the closure of the pullback on the locus where f is holomorphic, or on the level of locally free sheaves as $f^*\mathcal{O}_B(A) := (p_*\widetilde{f}^*\mathcal{O}_B(A))^{\vee\vee}$, where $p \colon \widetilde{X} \to X$ is a resolution of indeterminacies as in diagram (1).

2.3. **Deformations of Lagrangian subtori.** The starting point for our approach to Beauville's question is the deformation theory of a Lagrangian subtorus L in a hyperkähler manifold X. We quickly recall the relevant results from [GLR11, Sects. 2 and 3].

The Barlet space $\mathfrak{B}(X)$ of X (or Chow scheme in the projective setting) parametrises compact cycles in X and it turns out (see (*i*) of Lemma 2.3 below) that there is a unique irreducible component $\mathfrak{B}$ of $\mathfrak{B}(X)$ containing the point $[L]$. Denoting by $\mathfrak{U}$ the graph of the universal family over $\mathfrak{B}$ and by Δ the *discriminant locus* of $\mathfrak{B}$, i.e., the set of points parametrising singular elements in the family $\mathfrak{B}$, we obtain the following diagram.

(2)
$$
\begin{array}{ccccc}
\mathfrak{U}_\Delta & \lhook\joinrel\longrightarrow & \mathfrak{U} & \xrightarrow{\;\varepsilon\;} & X \\
\downarrow & & {\scriptstyle \pi}\big\downarrow & & \\
\Delta & \lhook\joinrel\longrightarrow & \mathfrak{B}. & &
\end{array}
$$

A detailed analysis of the maps in diagram (2) shows that a small étale or analytic neighbourhood of L in X fibres over a neighbourhood of $[L]$ in $\mathfrak{B}$. More precisely, we have the following result.

Lemma 2.3 ([GLR11, Lem. 3.1]) — *Let X be a hyperkähler manifold of dimension $2n$ and let L be a Lagrangian subtorus of X. Then, the following holds.*

    (*i*) *The Barlet space $\mathfrak{B}(X)$ is smooth of dimension n near $[L]$. In particular, $[L]$ is contained in a unique irreducible component $\mathfrak{B}$ of $\mathfrak{B}(X)$ and $\mathfrak{U}$ is smooth of dimension $2n$ near $\pi^{-1}([L])$.*

    (*ii*) *The morphism ε is finite étale along smooth fibres of π. In particular, a sufficiently small deformation of L is disjoint from L and there are no positive-dimensional families of smooth fibres through a general point $x \in X$.*

    (*iii*) *If $[L'] \in \mathfrak{B}$ with smooth L', then L' is a Lagrangian subtorus of X.*

Remark 2.4 — We remark two simple but useful consequences of Lemma 2.3.

    (*i*) The locus $X_\Delta := \varepsilon(\mathfrak{U}_\Delta)$ is the locus of points $x \in X$ such that there is a singular deformation of L passing through x. By dimension reasons it is a proper subset of X and by Lemma 2.3 (*ii*) the map ε is finite and étale on the preimage of $X \setminus X_\Delta$.

    (*ii*) Statement (*ii*) implies in particular that for any two points $[L], [M] \in \mathfrak{B}$ the intersection product $[L].[M]$ as cycles in X vanishes. It is therefore impossible for members of the family $\mathfrak{B}$ to intersect in a finite number of points.

2.4. **Almost holomorphic Lagrangian fibrations and Barlet spaces.** The following result relates the deformation theory of L in X discussed above to our question about globally defined almost holomorphic Lagrangian fibrations.

Lemma 2.5 ([GLR11, Lem. 3.2]) — *Let X be a hyperkähler manifold containing a Lagrangian subtorus L. Then X admits an almost holomorphic Lagrangian fibration with strong fibre L if and only if the evaluation map ε in diagram* (2) *is bimeromorphic.*

If ε is birational, then $\pi \circ \varepsilon^{-1}$ is the desired almost holomorphic fibration (up to normalisation of $\mathfrak{B}$). For the other direction one uses the Barlet space of a resolution of indeterminacies.

3. *L*-REDUCTION AND *L*-SEPARABLE MANIFOLDS

Let X be a hyperkähler manifold containing a Lagrangian subtorus L. In this section we start our analysis of the maps in the associated diagram (2). Recall from Lemma 2.5 above that in order to answer Beauville's question positively we have to show that the evaluation map ε is birational.

3.1. *L*-reduction.
Here, we construct a meromorphic map associated with the covering family $\{L_t\}_{t \in \mathfrak{B}}$. Generically, this map is a quotient map for the meromorphic equivalence relation defined by the family $\{L_t\}$, i.e., generically it identifies those points in X that cannot be separated by members of $\{L_t\}$.

3.1.1. *Construction of the L-reduction.*
We work in the setup summarised in diagram (2). We set $\mathfrak{U}_{reg} := \varepsilon^{-1}(X \setminus X_\Delta)$. Recall from Remark 2.4 that the map

$$\varepsilon|_{\mathfrak{U}_{reg}} \colon \mathfrak{U}_{reg} \to X \setminus X_\Delta$$

is finite étale; we denote its degree by d.

The map $\varepsilon|_{\mathfrak{U}_{reg}}$ induces a morphism $X \setminus X_\Delta \to \mathrm{Sym}^d(\mathfrak{U}_{reg})$. Composing this map with the natural morphism $\mathrm{Sym}^d(\mathfrak{U}_{reg}) \to \mathrm{Sym}^d(\mathfrak{B})$ induced by $\pi \colon \mathfrak{U} \to \mathfrak{B}$, we construct a morphism $X \setminus X_\Delta \to \mathrm{Sym}^d(\mathfrak{B})$. This morphism naturally extends to a rational map $\psi \colon X \dashrightarrow \mathrm{Sym}^d(\mathfrak{B})$. Let

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\widetilde{\psi}} & \mathrm{Sym}^d(\mathfrak{B}) \\
{\scriptstyle p}\downarrow & & \\
X & &
\end{array}
$$

be a resolution of singularities of the indeterminacies of ψ with $\widetilde{X}$ nonsingular. The Stein factorisation of $\widetilde{\psi}$ then yields the following diagram.

$$
\begin{array}{ccc}
X & \xleftarrow{\ p\ } & \widetilde{X} \\
 & {\scriptstyle \varphi_L}\diagdown & \ \downarrow{\scriptstyle \widetilde{\varphi}}\ \diagdown{\scriptstyle \widetilde{\psi}} \\
 & & \mathfrak{T} \longrightarrow \mathrm{Sym}^d(\mathfrak{B}).
\end{array}
$$

Here, $\varphi_L = \widetilde{\varphi} \circ p^{-1} \colon X \dashrightarrow \mathfrak{T}$ is the rational map induced by $\widetilde{\varphi}$. Noting that $\varphi_L \colon X \dashrightarrow \mathfrak{T}$ is unique up to birational equivalence, and hence canonically associated with the pair (X, L), we call it *the L-reduction* of X.

Remark 3.1 — For every point $x \in X \setminus X_\Delta$ there are exactly d pairwise distinct smooth tori $L_1, \ldots, L_d$ in the family $\{L_t\}_{t \in \mathfrak{B}}$ containg x. By construction, φ_L is defined at x and maps it to the class of $([L_1], \ldots, [L_d])$ in $\mathrm{Sym}^d(\mathfrak{B})$.

3.1.2. *First properties of the L-reduction.* The following set-theoretical assertion is an immediate consequence of the construction of φ_L.

Lemma 3.2 — *The fibre of φ_L through a point $x \in X \setminus X_\Delta$ coincides with the connected component of*

$$\bigcap_{[M] \in \mathfrak{B},\, x \in M} M$$

containing x.

Proof. If $x \in X \setminus X_\Delta$, then ε is étale in every point of the preimage $\varepsilon^{-1}(x)$. Thus the image $\pi(\varepsilon^{-1}(x)) = \{[L_1], \ldots, [L_d]\}$ consists of the points in $\mathfrak{B}$ that parametrise the d pairwise distinct subtori in X through x. In particular, the meromorphic map $\psi \colon X \dashrightarrow \mathrm{Sym}^d(\mathfrak{B})$ is defined at x and its fibre is

(3) $$\psi^{-1}(\psi(x)) = \bigcap_i L_i.$$

After taking the Stein factorisation, the fibre of φ_L is the component of (3) through x, as claimed. □

Lemma 3.3 — *Let X be a projective hyperkähler manifold containing a Lagrangian subtorus L. Then the L-reduction $\varphi_L \colon X \dashrightarrow \mathfrak{T}$ is almost holomorphic.*

Proof. Let $X' \subset X \times \mathfrak{T}$ be the graph of φ_L with projections $p \colon X' \to X$ and $\varphi'_L \colon X' \to \mathfrak{T}$. If $\mathrm{dom}(\varphi_L)$ is the domain of definition of φ_L, and $Z = X \setminus \mathrm{dom}(\varphi_L)$ is the locus where φ_L is not defined, then

(4) $$Z = \{x \in X \mid \dim p^{-1}(x) > 0\},$$

cf. [Deb01, Sect. 1.39]. We have to show that the general fibre of φ_L does not intersect Z.

Aiming for a contradiction, suppose that there exists $x_0 \in X \setminus X_\Delta$ such that φ_L has maximal rank at x_0 and such that the fibre F_{x_0} over x_0 intersects Z nontrivially. Take a point $z \in F_{x_0} \cap Z$. Since φ_L has maximal rank at x_0, there exists a connected open analytic neighbourhood U of x_0 in $X \setminus X_\Delta$ such that $\varphi_L(U) =: V$ is open in $\mathfrak{T}$, such that $\varphi'_L(p^{-1}(z)) \cap V$ is connected, and such that the restriction $\varphi_L|_U \colon U \to V$ is a trivialisable holomorphic fibre bundle with connected fibres over V. Let $s \colon V \to U$ be a holomorphic section of $\varphi_L|_U$ with $x_0 \in \sigma(V)$ and let $C := s(\varphi'_L(p^{-1}(z)))$. Let $L_1, \ldots, L_d$ be the d pairwise distinct tori in the family $\{L_t\}_{t \in \mathfrak{B}}$ containing x_0. Since C is connected and $C \not\subset F_{x_0}$ by construction, Lemma 3.2 implies that there exists $k \in \{1, \ldots, d\}$ such that $C \not\subset L_k$. As $\varepsilon|_{\varepsilon^{-1}(U)}$ is finite étale, arbitrarily close to x_0 there exist points $y \in C \setminus L_k$ with the following property: there exists a small deformation $\hat{L}_k$ of L_k with $y \in \hat{L}_k$ and $\hat{L}_k \cap L_k = \emptyset$.

On the one hand, Lemma 3.2 then implies that $F_y \cap F_{x_0} = \emptyset$. On the other hand, since $y \in C = s(\varphi'_L(p^{-1}(z)))$ we have $z \in F_y \cap F_{x_0}$, a contradiction. □

Definition 3.4 — A projective hyperkähler manifold X containing a Lagrangian subtorus L is called *L-separable* if its L-reduction $\varphi_L \colon X \dashrightarrow \mathfrak{T}$ is birational.

3.2. **Lagrangian fibrations on non-L-separable manifolds.**

Theorem 3.5 — *Let X be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then X admits an almost holomorphic fibration with strong fibre L if and only if X is not L-separable.*

As a consequence of this result we can reformulate Beauville's question in the following way.

Question B" — *Does there exist a projective hyperkähler manifold X together with a Lagrangian subtorus L such that X is L-separable?*

Proof of Theorem 3.5. If X is not L-separable, the L-reduction $\varphi_L \colon X \dashrightarrow \mathfrak{T}$ is an almost holomorphic map (Lemma 3.3) such that $0 < \dim \mathfrak{T} < \dim X$. Thus by part (*iv*) of Theorem 1.1, the map φ_L is an almost holomorphic Lagrangian fibration on X. By the description of the general fibre of the L-reduction (Lemma 3.2), the torus L is a strong fibre of φ_L.

If conversely $f \colon X \dashrightarrow B$ is an almost holomorphic Lagrangian fibration with strong fibre L, then through the general point there is a unique Lagrangian subtorus in $\mathfrak{B}$ and the L-reduction coincides with the rational map $\pi \circ \varepsilon^{-1} \colon X \dashrightarrow \mathfrak{B}$. In particular, X is not L-separable. $\square$

4. INTERSECTIONS OF LAGRANGIAN SUBTORI

As before, let X be a projective hyperkähler manifold containing a Lagrangian subtorus L. In this section we study a neighbourhood of L in X more closely, which leads to several results about the geometry of intersections of different members in the family $\mathfrak{B}$ of deformations of L. We are going to use the notation and the results of Section 2.3 throughout.

By Lemma 2.3, $\mathfrak{B}$ is smooth at $[L]$ and we can find a neighbourhood V of $[L]$ such that the restriction $\varepsilon|_{\mathfrak{U}_V} \colon \mathfrak{U}_V \to X$ of the evaluation map to the preimage $\mathfrak{U}_V := \pi^{-1}(V)$ embeds $\mathfrak{U}_V$ into X. We may thus consider $\mathfrak{U}_V$ as an open subset of X. The intersection of $\mathfrak{U}_V$ with a submanifold $M \subset X$ is depicted in Figure 1.

Lemma 4.1 — *Let $M \subseteq X$ be a smooth and proper submanifold, and $L \subset X$ a smooth Lagrangian torus that intersects M nontrivially. Then a generic small deformation of L has smooth intersection with M.*

Proof. We continue to use the notation introduced above. Since $\mathfrak{U}_V$ is open in X, the intersection $M \cap \mathfrak{U}_V$ is smooth. Furthermore, the map $\pi|_{M \cap \mathfrak{U}_V} \colon M \cap \mathfrak{U}_V \to V$ is proper, because π is proper and M is compact. We can therefore apply the theorem on generic smoothness to $\pi|_{M \cap \mathfrak{U}_V}$ which proves the result. $\square$

Proposition 4.2 — *Let $M \subseteq X$ be a compact submanifold and $L \subseteq X$ be a general Lagrangian subtorus, such that $L \cap M \neq \varnothing$. Then $N_{L \cap M / M}$ is trivial. If M is a complex torus, then $L \cap M$ is a disjoint union of tori.*
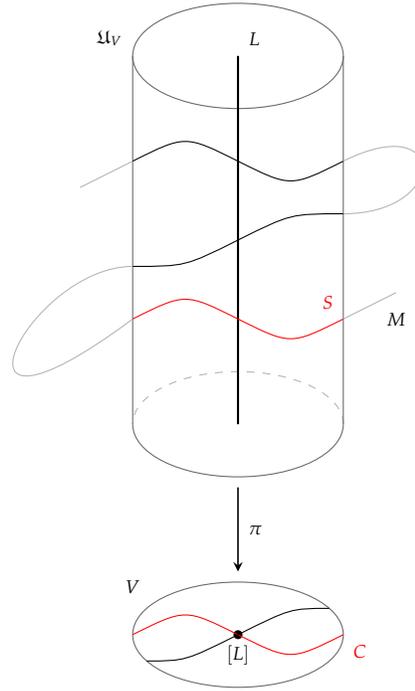
FIGURE 1. The neighbourhood $\mathfrak{U}_V$ of L and its projection to $V \subset \mathfrak{B}$.

Proof. As L is general, the intersection $L \cap M$ is smooth by Lemma 4.1. Moreover, both statements can be verified by looking at one connected component of $L \cap M$ at a time. We go back to the local situation depicted in Figure 1, invoking the notation introduced in the beginning of this section, and let T be a connected component of $L \cap M$. If V is sufficiently small, then the inclusion $L \cap M \hookrightarrow \mathfrak{U}_V \cap M$ induces a one-to-one correspondence of their respective connected components. Let S be the unique component of $\mathfrak{U}_V \cap M$ corresponding to T. By generality of L we may assume that $\pi|_S$ is a smooth map, thus $C := \pi(S) \subset V$ is smooth of dimension $n - \dim T$ near $[L]$. Moreover, C parametrizes those small deformations of L that induce a flat deformation of T inside M. Corresponding to the family $S \to C$ we thus obtain a classyfying map $\chi \colon C \to \mathscr{D}(M)$ from C to the Douady-space of M.

On the level of tangent spaces we have $T_C([L]) \subset T_{\mathfrak{B}}([L]) = H^0(L, N_{L/X})$, where the last equality comes from the Hilbert-Chow morphism, compare [GLR11, Lem. 3.1]. The morphism χ induces a map $\chi_* \colon T_C([L]) \to H^0(T, N_{T/M})$. But small deformations of T inside M induced by deformations of L are disjoint from T by Lemma 2.3 (*ii*). Thus the map χ_* is injective, and the image of $T_C([L])$ consists of nowhere vanishing sections. For dimension reasons these sections generate the normal bundle of T in M, and consequently $N_{T/M}$ is trivial, as claimed.

If M is a torus as well, then $T_M|_T$ is likewise trivial. So, by the normal bundle sequence

$$0 \longrightarrow T_T \longrightarrow T_M|_T \longrightarrow N_{T/M} \longrightarrow 0$$

the tangent bundle T_T is trivial, and thus T is a complex torus.   □

Based on the preceeding result we can now refine the observation in Remark 2.4(ii):

Lemma 4.3 — *Let X be a four-dimensional hyperkähler manifold. Let L and M be two Lagrangian tori intersecting smoothly, and set $I = L \cap M$. Then, I is a finite disjoint union of elliptic curves.*

Proof. It remains to exclude the existence of zero-dimensional connected components of I. By general Lagrangian intersection theory, see for example [BF09, Introduction], we have

$$[L].[M] = \chi(I).$$

However, this already implies the claim, since by Proposition 4.2 above, any positive dimensional component of I is a smooth elliptic curve, contributing zero to the Euler characteristic $\chi(I)$. □

Corollary 4.4 — *Let X be a four-dimensional hyperkähler manifold, L a Lagrangian subtorus. Assume that X is L-separable. Then, the evaluation map ε in Diagram (2) has degree at least three.*

The following result is not necessary for our arguments in dimension four, however, it might be of independent interest for dealing with higher-dimensional cases.

Proposition 4.5 — *Let L, M be general tori in the family $\mathfrak{B}$ such that $L \cap M \neq \varnothing$. Then there is a subtorus $T \subset L$ such that $L \cap M$ is a union of translates of T in L.*

Proof. We use the same setup as in Proposition 4.2 and work in a small neighbourhood $\mathfrak{U}_V$ of L where every component T_i of $L \cap M$ corresponds to a unique component S_i of $M \cap \mathfrak{U}_V$. Since L and M are general, every component of the intersection is a smooth torus, and we may assume that

(5) $\qquad$ each component S_i has the same image $C = \pi(S_i) \subset V$.

Now let $x \in T_i$. Using the identification $T_{\mathfrak{B},[L]} = H^0(L, N_{L/X}) = N_{L/X,x} = \Omega_{L,x}$ we have the sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T_{L,x} & \longrightarrow & T_{X,x} & \xrightarrow{\ p\ } & \Omega_{L,x} & \longrightarrow & 0 \\
& & \big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & T_{T_i,x} & \longrightarrow & T_{M,x} & \longrightarrow & T_{C,[L]} & \longrightarrow & 0.
\end{array}
$$

Now observe that for $\xi \in T_{L,x}$ and $\xi' \in T_{X,x}$ we have by definition of our identification of the normal bundle and the cotangent bundle $\sigma(\xi, \xi') = p(\xi')(\xi)$, where the latter is evaluation of 1-forms on tangent vectors. As $T_{M,x}$ is a Lagrangian subspace of $T_{X,x}$ the subspace $T_{T_i,x}$ is exactly the annihilator of $T_{C,[L]} \hookrightarrow \Omega_{L,x}$ in $T_{L,x}$ with respect to the natural pairing. The latter does not depend on i by (5) and by triviality of Ω_L. Consequently, all components T_i are translates of a fixed subtorus T, since a subtorus is up to translation determined by its tangent space in one point. □

5. HYPERKÄHLER FOURFOLDS

Using the results from the last section we can now prove our main result which gives the strongest possible positve answer to Beauville's question:

Theorem 5.1 — *Let X be a four-dimensional hyperkähler manifold containing a Lagrangian torus L. Then X admits a holomorphic Lagrangian fibration with fibre L.*

Remark 5.2 — We are grateful to E. Amerik for communicating the following linear algebra observation to us which serves to exclude L-separable manifolds $X \supset L$ in dimension four. This greatly simplified a previous deformation-theoretic argument.

Lemma 5.3 — *Let V be a four-dimensional symplectic vector space with symplectic form σ, and let $W_1, W_2, W_3 \subset V$ be three Lagrangian subspaces satisfying $\dim W_i \cap W_j = 1$ for all $i \neq j$. Then $W_1 \cap W_2 \cap W_3 \neq \{0\}$.*

Proof. Suppose on the contrary that $W_1 \cap W_2 \cap W_3 = \{0\}$ and consider the span $\langle W_1, W_2 \rangle$. It is of dimension 3 as $\dim W_1 \cap W_2 = 1$. Moreover, we claim that

$$(6) \qquad\qquad W_3 \subset \langle W_1, W_2 \rangle.$$

Indeed, otherwise we would have $\dim W_3 \cap \langle W_1, W_2 \rangle = 1$, implying that the intersections $W_3 \cap \langle W_1, W_2 \rangle = W_3 \cap W_1 = W_3 \cap W_2$ are all one-dimensional, in contradiction to our assumption that $W_1 \cap W_2 \cap W_3 = \{0\}$.

Now, again using $W_1 \cap W_2 \cap W_3 = \{0\}$ we write $\langle W_1, W_2 \rangle = W_3 \oplus (W_1 \cap W_2)$. As V is symplectic and W_3 is Lagrangian, there is $v \in W_1 \cap W_2$ and $w \in W_3$ such that $\sigma(v, w) \neq 0$. According to the inclusion (6) we can write $w = w_1 + w_2$ with $w_i \in W_i$, so that

$$0 \neq \sigma(v, w) = \sigma(v, w_1) + \sigma(v, w_2) = 0 + 0 = 0,$$

as W_1 and W_2 are Lagrangian. Contradiction.   □

Proposition 5.4 — *Let X be a four-dimensional hyperkähler manifold containing a Lagrangian torus L. Then X is not L-separable.*

Proof. Suppose on the contrary that X is L-separable. Given a general point $x \in X$, it follows from Lemma 3.2 and Corollary 4.4 that there exists a natural number $d \geq 3$, and d smooth Lagrangian subtori locally cutting out x. For dimension reasons, three of these Lagrangian subtori, say L_1, L_2, L_3, suffice to cut out x. The point x being general, Lemma 4.3 implies that the intersection of every subcollection of these tori is a smooth elliptic curve at x. Consequently, the three Lagrangian subspaces $W_i := T_{L_i, x} \subset T_{X, x}$ satisfy the assumptions of Lemma 5.3. It follows that $W_1 \cap W_2 \cap W_3 \neq \{0\}$, contradicting our choice of L_1, L_2, L_3. Therefore, X cannot be L-separable.   □

Proof of Theorem 5.1. If X is not projective, we are done by Theorem 1.1 (i), so we may assume X to be projective. By Proposition 5.4, X is not L-separable and hence admits an almost holomorphic Lagrangian fibration $f : X \dashrightarrow B$ by Theorem 3.5. It remains to show that the existence of an almost holomorphic Lagrangian fibration implies the existence of a holomorphic one, which will be done in Lemma 5.5.   □

Lemma 5.5 — *Let $f: X \dashrightarrow B$ be an almost holomorphic Lagrangian fibration on a projective hyperkähler fourfold. Then there exists a birational modification $\psi: B \dashrightarrow B'$ such that $\psi \circ f: X \to B'$ is a holomorphic Lagrangian fibration.*

The proof of Lemma 5.5 rests on the explicit knowledge of the birational geometry of hyperkähler fourfolds. For this we recall the notion of *Mukai flop*: Assume that a hyperkähler fourfold X contains a smooth subvariety $P \cong \mathbb{P}^2$. If we blow up P, the exceptional divisor is isomorphic to the projective bundle $\mathbb{P}(\Omega^1_{\mathbb{P}^2})$, and it is well known that it can be blown down in the other direction to yield another hyperkähler manifold X'. The resulting birational transformation $X \dashrightarrow X'$ is called the *Mukai flop* at P.

Proof of Lemma 5.5. By Theorem 1.1 (*iii*) there exists a holomorphic model for f, that is, a Lagrangian fibration $f': X' \to B'$ on a possibly different hyperkähler manifold X' and a diagram

$$
\begin{array}{ccc}
X & \overset{\varphi}{\dashrightarrow} & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
B & \overset{\psi}{\dashrightarrow} & B'
\end{array}
$$

with birational horizontal arrows such that φ is an isomorphism near the general fibre of f.

We claim that the composition $f' \circ \varphi = \psi \circ f$ is holomorphic and thus a Lagrangian fibration on X. To see this first note that by [WW03, Thm. 1.2] the map φ factors as a finite composition of Mukai flops, so by induction we may assume that φ^{-1} is the simultaneous Mukai flop of a disjoint union of embedded projective planes $\mathbb{P}^2 \cong P_i \subset X'$.

As φ is holomorphic near a general fibre of f', none of the P_i's can intersect the general fibre. Thus $f'(P_i)$ is a proper subset of B' and hence of dimension at most 1. Since there is no non-constant map from $\mathbb{P}^2$ to a curve, $f'(P_i)$ is a single point. In other words, the locus of indeterminacy of φ^{-1} is contained in the fibres of f', and thus the composition $f' \circ \varphi$ remains holomorphic. $\square$

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DANIEL GREB, INSTITUT FÜR MATHEMATIK, ABTEILUNG FÜR REINE MATHEMATIK, ALBERT-LUDWIGS-
UNIVERSITÄT FREIBURG, ECKERSTR. 1, 79104 FREIBURG IM BREISGAU, GERMANY
    E-mail address: `daniel.greb@math.uni-freiburg.de`

CHRISTIAN LEHN, INSTITUT FOURIER, 100 RUE DES MATHS, BP 74, 38402 ST MARTIN D'HERES,
FRANCE
    E-mail address: `christian.lehn@ujf-grenoble.fr`

SÖNKE ROLLENSKE, FAKULTÄT FÜR MATHEMATIK, UNIVERSTÄT BIELEFELD, UNIVERSITÄTSSTR. 25,
33615 BIELEFELD, GERMANY
    E-mail address: `rollenske@math.uni-bielefeld.de`