Sharp estimates for metastable lifetimes in parabolic SPDEs: Kramers' law and beyond

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Abstract

We prove a Kramers-type law for metastable transition times for a class of one-dimensional parabolic stochastic partial differential equations (SPDEs) with bistable potential. The expected transition time between local minima of the potential energy depends exponentially on the energy barrier to overcome, with an explicit prefactor related to functional determinants. Our results cover situations where the functional determinants vanish owing to a bifurcation, thereby rigorously proving the results of formal computations announced in [BG09]. The proofs rely on a spectral Galerkin approximation of the SPDE by a finite-dimensional system, and on a potential-theoretic approach to the computation of transition times in finite dimension.

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1 Introduction

Metastability is a common physical phenomenon, in which a system quickly moved across a first-order phase transition line takes a long time to settle in its equilibrium state. This behaviour has been established rigorously in two main classes of mathematical models. The first class consists of lattice models with Markovian dynamics of Metropolis type, such as the Ising model with Glauber dynamics or the lattice gas with Kawasaki dynamics (see [dH04, OV05] for recent surveys).

The second class of models consists of stochastic differential equations driven by weak Gaussian white noise. For dissipative drift, sample paths of such equations tend to spend long time spans near attractors of the system without noise, with occasional transitions between attractors. In the particular case where the drift term is given by minus the gradient of a potential, the attractors are local minima of the potential, and the mean transition time between local minima is governed by Kramers' law [Eyr35, Kra40]: In the small-noise limit, the transition time is exponentially large in the potential barrier height between the minima, with a multiplicative prefactor depending on the curvature of the potential at the local minimum the process starts in and at the highest saddle crossed during the transition. While the exponential asymptotics was proved to hold by Freidlin and Wentzell using the theory of large deviations [VF69, FW98], the first rigorous proof of Kamers' law, including the prefactor, was obtained more recently by Bovier, Eckhoff, Gayrard and Klein [BEGK04, BGK05] via a potential-theoretic approach. See [Ber11] for a recent review.

The aim of the present work is to extend Kramers' law to a class of parabolic stochastic partial differential equations of the form

$$\mathrm{d}u_t(x) = \big[\Delta u_t(x) - U'(u_t(x))\big]\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W(t,x)\;, \tag{1.1}$$

where x belongs to an interval $[0, L]$, $u(x)$ is real-valued and $W(t, x)$ denotes space-time white noise. If the potential U has several local minima u_i, the deterministic limiting system admits several stable stationary solutions: these are simply the constant solutions, equal to u_i everywhere. It is natural to expect that the transition time between these stable solutions is also governed by a formula of Kramers type. In the case of the double-well potential $U(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$, the exponential asymptotics of the transition time was determined and proved to hold by Faris and Jona-Lasinio [FJL82]. The prefactor was computed formally, by analogy with the finite-dimensional case, by Maier and Stein [MS01, MS03, Ste05], except for particular interval lengths L at which Kamers' formula breaks down because of a bifurcation. The behaviour near bifurcation values has been derived formally in [BG09].

In the present work, we provide a full proof for Kramers' law for SPDEs of the form (1.1), for a general class of double-well potentials U. The results cover all finite positive values of the interval length, and thus include bifurcation values. One of the main ingredients of the proof is a result by Blömker and Jentzen on spectral Galerkin approximations [BJ09], which allows us to reduce the system to a finite-dimensional one. This reduction requires some a priori bounds on moments of transition times, which we obtain by large-deviation techniques (though it might be possible to obtain them by other methods). Transition times for the finite-dimensional equation can be accurately estimated by the potential-theoretic approach of [BEGK04, BGK05], provided one can control capacities uniformly in the dimension. Such a control has been achieved in [BBM10] in a particular case, the so-called synchronised regime of a chain of coupled bistable particles introduced in [BFG07a, BFG07b]. Part of the work of the present paper consists in establishing such a control for a general class of systems. We note that although we limit ourselves to the one-dimensional case, there seems to be no fundamental obstruction to extending the technique to higher dimensions. Very recently, Barret has independently obtained an alternative proof of Kramers' law for non-bifurcating one-dimensional SPDEs, using a different approach based on approximations by finite differences [Bar12].

The remainder of this paper is organised as follows. Section 2 contains the precise definition of the model, an overview of needed properties of the deterministic system, and the statement of all results. Section 3 outlines the essential steps of the proofs. Technical details of the proofs are deferred to subsequent sections. Section 4 contains the needed estimates on the deterministic partial differential equation, including an infinite-dimensional normal-form analysis of bifurcations. In Section 5 we derive the required a priori estimates for the stochastic system, mainly based on large-deviation principles. Section 6 contains the sharp estimates of capacities, while Section 7 combines the previous results to obtain precise estimates of expected transition times in finite dimension. Finally, Appendix A contains the proof of a sufficient condition on the potential U needed in the analysis.

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2 Results

2.1 Parabolic SPDEs with bistable potential

Let L be a positive constant, and let $E = \mathcal{C}([0,L],\mathbb{R})$ denote the Banach space of continuous functions $u : [0,L] \to \mathbb{R}$, equipped with the sup norm $\|\cdot\|_{L^\infty}$.

We consider the parabolic SPDE

$$\mathrm{d}u_t(x) = \big[\Delta u_t(x) - U'(u_t(x))\big]\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W(t,x)\;, \qquad t \in \mathbb{R}_+\;,\; x \in [0,L] \tag{2.1}$$

with

- either periodic boundary conditions (b.c.)

$$u(0) = u(L)\;, \tag{2.2}$$

- or zero-flux Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(L) = 0\;, \tag{2.3}$$

and initial condition $u_0 \in E$, satisfying the same boundary conditions.

In (2.1), Δ denotes the second derivative (the one-dimensional Laplacian), $\varepsilon > 0$ is a small parameter, and $W(t,x)$ denotes space–time white noise, defined as the cylindrical Wiener process compatible with the b.c. The local potential $U : \mathbb{R} \to \mathbb{R}$ will be assumed to satisfy a certain number of properties, which are detailed below. When considering a general class of local potentials, it is useful to keep in mind the example

$$U(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2\;. \tag{2.4}$$

Observe that U has two minima, located in $u = -1$ and $u = +1$, and a local maximum in $u = 0$. Furthermore, the quartic growth as $u \to \pm\infty$ makes U a confining potential. As a result, for small ε, solutions of (2.1) will be localised with high probability, with a preference for staying near $u = 1$ or $u = -1$.

The bistable and confining nature of U are two essential features that we want to keep for all considered local potentials. A first set of assumptions on U is the following:

Assumption 2.1 (Assumptions on the class of potentials U)**.**

U1: $U : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^3$.[1] In some cases (namely, when L is close to π for Neumann b.c. and close to 2π for periodic b.c.), our results require U to be of class $\mathcal{C}^5$.

U2: U has exactly two local minima and one local maximum, and U'' is nonzero at all three stationary points. Without loss of generality, we may assume that the local maximum is in $u = 0$ and that $U''(0) = -1$. The positions of the minima will be denoted by $u_- < 0 < u_+$.

U3: There exist constants $M_0 > 0$ and $p_0 \geqslant 2$ such that $U(u) \leqslant M_0(1 + |u|^{2p_0})$ for all $u \in \mathbb{R}$.

U4: There exist constants $\alpha \in \mathbb{R}, \beta > 0$ such that $U(u) \geqslant \beta u^2 - \alpha$ for all $u \in \mathbb{R}$.

U5: For any $\gamma > 0$, there exists an $M_1(\gamma)$ such that $U'(u)^2 \geqslant \gamma u^2 - M_1(\gamma)$ for all $u \in \mathbb{R}$.

U6: There exists a constant M_2 such that $U''(u) \geqslant -M_2$ for all $u \in \mathbb{R}$.

[1] Actually, for L bounded away from critical values, the results hold under the assumption $U \in \mathcal{C}^2$, with a weaker control on the error terms.

Let us recall the definition of a mild solution of (2.1). We denote by $\mathrm{e}^{\Delta t}$ the Markov semigroup of the heat equation $\partial_t u = \Delta u$, defined by the convolution

$$(\mathrm{e}^{\Delta t}\, u)(x) = \int_0^L G_t(x,y) u(y)\, \mathrm{d}y \;. \tag{2.5}$$

Here $G_t(x,y)$ denotes the Green function of the Laplacian compatible with the considered boundary conditions. It can be written as

$$G_t(x,y) = \sum_k \mathrm{e}^{-\nu_k t}\, e_k(x) \overline{e_k(y)} \;, \tag{2.6}$$

where the e_k form a complete orthonormal basis of eigenfunctions of the Laplacian, with eigenvalues $-\nu_k$. That is,

- for periodic b.c.,

$$e_k(x) = \frac{1}{\sqrt{L}}\, \mathrm{e}^{2\pi\,\mathrm{i}\,kx/L} \;, \quad k \in \mathbb{Z} \;, \qquad \text{and } \nu_k = \left(\frac{2\pi k}{L}\right)^2 ; \tag{2.7}$$

- for Neumann b.c.,

$$e_0(x) = \frac{1}{\sqrt{L}} \;,\; e_k(x) = \sqrt{\frac{2}{L}} \cos\!\left(\frac{\pi k x}{L}\right) , \quad k \in \mathbb{N} \;, \qquad \text{and } \nu_k = \left(\frac{\pi k}{L}\right)^2 . \tag{2.8}$$

A *mild solution* of the SPDE (2.1) is by definition a solution to the integral equation

$$u_t = \mathrm{e}^{\Delta t}\, u_0 - \int_0^t \mathrm{e}^{\Delta(t-s)}\, U'(u_s)\, \mathrm{d}s + \sqrt{2\varepsilon} \int_0^t \mathrm{e}^{\Delta(t-s)}\, \mathrm{d}W(s) \;. \tag{2.9}$$

Here the stochastic integral can be represented as a series of one-dimensional Itô integrals

$$\int_0^t \mathrm{e}^{\Delta(t-s)}\, \mathrm{d}W(s) = \sum_k \int_0^t \mathrm{e}^{-\nu_k(t-s)}\, \mathrm{d}W_s^{(k)} e_k \;, \tag{2.10}$$

where the $W_t^{(k)}$ are independent standard Wiener processes (see for instance [Jet86]). It is known that for a confining local potential U, (2.1) admits a pathwise unique mild solution in E [DPZ92].

2.2 The deterministic equation

Consider for a moment the deterministic partial differential equation

$$\partial_t u_t(x) = \Delta u_t(x) - U'(u_t(x)) \;. \tag{2.11}$$

Stationary solutions of (2.11) have to satisfy the second-order ordinary differential equation

$$u''(x) = U'(u(x)) \;, \tag{2.12}$$

together with the boundary conditions. Note that this equation describes the motion of a particle of unit mass in the inverted potential $-U$. There are exactly three stationary solutions which do not depend on x, given by

$$u_-^*(x) \equiv u_- \;, \qquad u_+^*(x) \equiv u_+ \;, \qquad u_0^*(x) \equiv 0 \;. \tag{2.13}$$

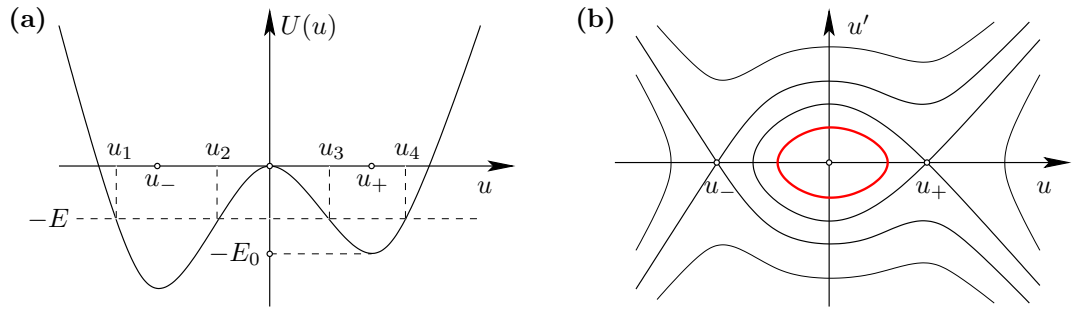

FIGURE 1. **(a)** Example of a local potential U. **(b)** Level sets of the first integral $H(u, u')$. Bounded stationary solutions compatible with the boundary conditions only exist in the inner region $\{H(u, u') < E_0\}$. The periodic orbit contained in $\{H(u, u') = E\}$ crosses the u-axis at points $u = u_2(E)$ and $u = u_3(E)$, defined in **(a)**.

Depending on the boundary conditions and the value of L, there may be additional, non-constant stationary solutions. They can be found by observing that (2.12) is a Hamiltonian system, with first integral

$$H(u, u') = \frac{1}{2}(u')^2 - U(u) \,. \tag{2.14}$$

Orbits of (2.12) belong to level sets of H (Figure 1b). Bounded orbits only exist for $H < E_0$, where [2]

$$E_0 = -(U(u_-) \vee U(u_+)) \,. \tag{2.15}$$

For any $E \in (0, E_0)$, there exist exactly four values $u_1(E) < u_2(E) < 0 < u_3(E) < u_4(E)$ of u for which $U(u) = -E$ (Figure 1a). The periodic solution corresponding to $H = E$ crosses the u-axis at $u = u_2(E)$ and $u = u_3(E)$, and has a period

$$T(E) = 2 \int_{u_2(E)}^{u_3(E)} \frac{\mathrm{d}u}{\sqrt{E + U(u)}} \,. \tag{2.16}$$

The fact that $U''(0) = -1$ implies that $\lim_{E \to 0} T(E) = 2\pi$ (in this limit, stationary solutions approach those of a harmonic oscillator with unit frequency). In addition, we have $\lim_{E \to E_0} T(E) = +\infty$, because the level set $H = E_0$ is composed of homoclinic orbits (or heteroclinic orbits if $U(u_-) = U(u_+)$).

We will make the following assumption, which imposes an additional condition on the local potential:

Assumption 2.2. The period $T(E)$ is strictly increasing on $[0, E_0)$.

Remark 2.3. A normal-form analysis (cf. Section 4.3) shows that if $U \in \mathcal{C}^5$, then $T(E)$ is increasing near $E = 0$ if and only if

$$U^{(4)}(0) > -\frac{5}{3} U'''(0)^2 \,. \tag{2.17}$$

Furthermore, a sufficient (but not necessary) condition for Assumption 2.2 to hold true is that

$$U'(u)^2 - 2U(u)U''(u) > 0 \qquad \text{for all } u \in (u_-, u_+) \setminus \{0\} \tag{2.18}$$

(see Appendix A). Note that this condition is satisfied for the particular potential (2.4).

[2] Here and below, we use the shorthands $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

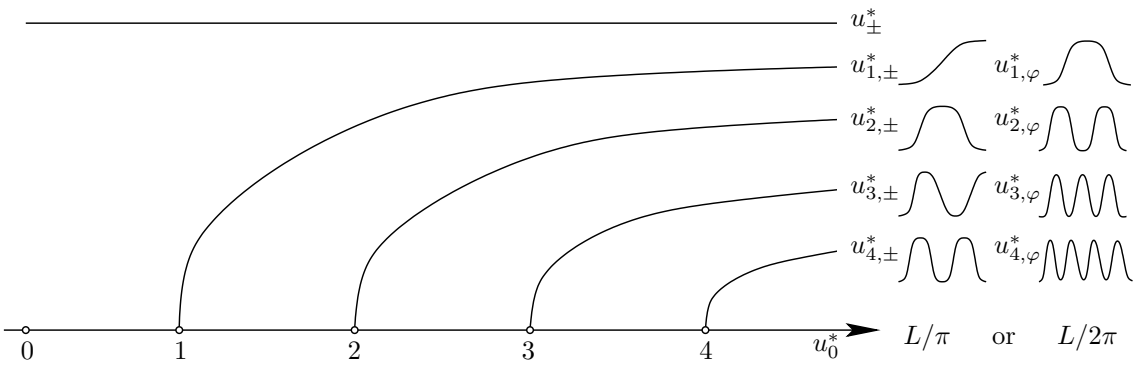

FIGURE 2. Schematic representation of the deterministic bifurcation diagram. Nonconstant stationary solutions appear whenever L is a multiple of π for Neumann b.c., and of 2π for periodic b.c. For Neumann b.c., the stationary solutions $u^*_{n,\pm}$ contain n kinks. For periodic b.c., all members of the family $\{u^*_{n,\varphi}, 0 \leqslant \varphi < L\}$ contain n kink–antikink pairs. The transition states ($n = 1$) are also called instantons [MS03].

Under Assumption 2.2, nonconstant stationary solutions satisfying periodic b.c. only exist for $L > 2\pi$, while stationary solutions satisfying Neumann b.c. only exist for $L > \pi$; they are obtained by taking the top or bottom half of a closed curve with constant H. Additional stationary solutions appear whenever L crosses a multiple of 2π or π. More precisely (Figure 2),

- for periodic b.c., there exist n families of nonconstant stationary solutions whenever $L \in (2n\pi, 2(n+1)\pi]$ for some $n \geqslant 1$, where members of a same family are of the form $u^*_{n,\varphi}(x) = u^*_{n,0}(x + \varphi)$, $0 \leqslant \varphi < L$;
- for Neumann b.c., there exist $2n$ nonconstant stationary solutions whenever $L \in (n\pi, (n+1)\pi]$ for some $n \geqslant 1$, where solutions occur in pairs $u^*_{n,\pm}(x)$ related by the symmetry $u^*_{n,-}(x) = u^*_{n,+}(L - x)$.

Next we examine the stability of these stationary solutions. Stability of a stationary solution u_0 is determined by the variational equation

$$\partial_t v_t(x) = \Delta v_t(x) - U''(u_0(x))v_t(x) =: Q[u_0]v_t(x) \,, \tag{2.19}$$

by way of the sign of the eigenvalues of the linear operator $Q[u_0] = \Delta - U''(u_0(\cdot))$. For the space-homogeneous stationary solutions (2.13), the eigenvalues of Q are simply shifted eigenvalues of the Laplacian. Thus

- For periodic b.c., the eigenvalues of $Q[u^*_0]$ are given by $-\lambda_k$, where

$$\lambda_k = \nu_k - 1 = \left(\frac{2\pi k}{L}\right)^2 - 1 \,, \quad k \in \mathbb{Z} \,. \tag{2.20}$$

  It follows that u^*_0 is always unstable: it has one positive eigenvalue for $L \leqslant 2\pi$, and the number of positive eigenvalues increases by 2 each time L crosses a multiple of 2π. The eigenvalues of $Q[u^*_\pm]$ are given by $-\nu_k - U''(u_\pm)$ and are always negative, implying that u^*_+ and u^*_- are stable.
- For Neumann b.c., the eigenvalues of $Q[u^*_0]$ are given by $-\lambda_k$, where

$$\lambda_k = \nu_k - 1 = \left(\frac{\pi k}{L}\right)^2 - 1 \,, \quad k \in \mathbb{N}_0 \,. \tag{2.21}$$

Again u_0^* is always unstable: it has one positive eigenvalue for $L \leqslant \pi$, and the number of positive eigenvalues increases by 1 each time L crosses a multiple of π. As before, u_+^* and u_-^* are always stable.

The problem of determining the stability of the nonconstant stationary solutions is equivalent to characterising the spectrum of a Schrödinger operator, and thus to solving a Sturm–Liouville problem. In general, there is no closed-form expression for the eigenvalues. However, a bifurcation analysis for L equal to multiples of 2π or π (cf. Section 4.3) shows that

- for periodic b.c., the stationary solutions $u_{n,\varphi}^*$ appearing at $L = 2n\pi$ have $2n-1$ positive eigenvalues and one eigenvalue equal to zero (associated with translation symmetry), the other eigenvalues being negative;
- for Neumann b.c., the stationary solutions $u_{n,\pm}^*$ appearing at $L = n\pi$ have n positive eigenvalues while the other eigenvalues are negative.

A last important object for the analysis is the potential energy

$$V[u] = \int_0^L \left[\frac{1}{2}u'(x)^2 + U(u(x))\right] \mathrm{d}x \; . \tag{2.22}$$

For $u+v$ satisfying the b.c., the Fréchet derivative of V at u in the direction v is given by

$$\begin{aligned} \nabla_v V[u] &:= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big(V[u+\varepsilon v] - V[u]\big) \\ &= \int_0^L \big[u'(x)v'(x) + U'(u(x))v(x)\big] \,\mathrm{d}x \\ &= \int_0^L \big[-u''(x) + U'(u(x))\big] v(x) \,\mathrm{d}x \; . \end{aligned} \tag{2.23}$$

Thus stationary solutions of the deterministic equation (2.11) are also stationary points of the potential energy. A similar computation shows that the second-order Fréchet derivative of V at u is the bilinear map

$$\nabla^2_{v_1,v_2} V[u] : (v_1, v_2) \mapsto -\int_0^L (Q[u]v_1)(x) v_2(x) \,\mathrm{d}x \; . \tag{2.24}$$

Hence the eigenvalues of the second derivative coincide, up to their sign, with those of the Sturm–Liouville problem for the variational equation (2.19). In particular, the stable stationary solutions u_+^* and u_-^* are local minima of the potential energy density.

We call *transition states* between u_+^* and u_-^* the stationary points of V at which $\nabla^2 V$ has one and only one negative eigenvalue. Thus

- for periodic b.c., u_0^* is the only transition state for $L \leqslant 2\pi$, while for $L > 2\pi$, all members of the family $u_{1,\varphi}^*$ are transition states;
- for Neumann b.c., u_0^* is the only transition state for $L \leqslant \pi$, while for $L > \pi$, the transition states are the two stationary solutions $u_{1,\pm}^*$.

Note that for given L and given b.c., V has the same value at all transition states. Transition states are characterised by the following property: Consider all continuous paths γ in E connecting u_-^* to u_+^*. For each of these paths, determine the maximal value of V along the path, and call *critical* those paths for which that value is the smallest possible. Then for any critical path, the maximal value of V is assumed on a transition state.

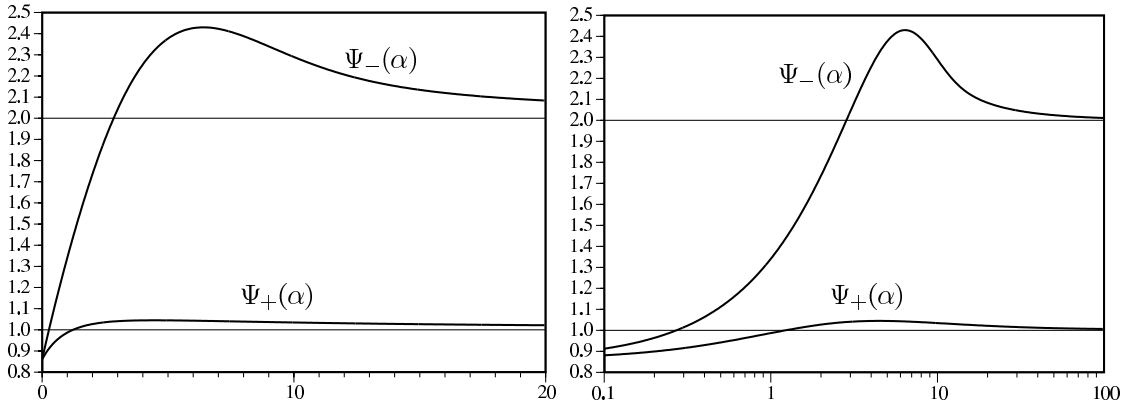

FIGURE 3. The functions $\Psi_\pm(\alpha)$ shown on a linear and on a logarithmic scale.

2.3 Main results

We can now state the main results of this work. We start with the case of Neumann b.c. We fix parameters $r, \rho > 0$ and an initial condition u_0 such that $\|u_0 - u_-^*\|_{L^\infty} \leqslant r$. Let

$$\tau_+ = \inf\bigl\{t > 0 \colon \|u_t - u_+^*\|_{L^\infty} < \rho\bigr\} \,. \tag{2.25}$$

We are interested in sharp estimates of the expected first-hitting time $\mathbb{E}^{u_0}\{\tau_+\}$ for small values of ε.

Recall from (2.21) that the eigenvalues of the variational equation at $u_0^* \equiv 0$ are given by $-\lambda_k$ where $\lambda_k = (k\pi/L)^2 - 1$. Those at u_-^* are given by $-\nu_k^-$ where

$$\nu_k^- = \left(\frac{k\pi}{L}\right)^2 + U''(u_-) \,. \tag{2.26}$$

When $L > \pi$, we denote the eigenvalues at the transition states $u_{1,\pm}^*$ by $-\mu_k$ where

$$\mu_0 < 0 < \mu_1 < \mu_2 < \dots \tag{2.27}$$

We further introduce two functions $\Psi_\pm : \mathbb{R}_+ \to \mathbb{R}_+$, which play a rôle for the behaviour of $\mathbb{E}^{u_0}\{\tau_+\}$ when L is close to π. They are given by

$$\Psi_+(\alpha) = \sqrt{\frac{\alpha(1+\alpha)}{8\pi}}\, \mathrm{e}^{\alpha^2/16}\, K_{1/4}\left(\frac{\alpha^2}{16}\right) , \tag{2.28}$$

$$\Psi_-(\alpha) = \sqrt{\frac{\pi\alpha(1+\alpha)}{32}}\, \mathrm{e}^{-\alpha^2/64}\left[I_{-1/4}\left(\frac{\alpha^2}{64}\right) + I_{1/4}\left(\frac{\alpha^2}{64}\right)\right] , \tag{2.29}$$

where $I_{\pm 1/4}$ and $K_{1/4}$ denote modified Bessel functions of first and second kind. The functions $\Psi_\pm$ are bounded below and above by positive constants, and satisfy

$$\lim_{\alpha\to+\infty} \Psi_+(\alpha) = 1 \,, \qquad \lim_{\alpha\to-\infty} \Psi_-(\alpha) = 2 \,, \tag{2.30}$$

and

$$\lim_{\alpha\to 0} \Psi_+(\alpha) = \lim_{\alpha\to 0} \Psi_-(\alpha) = \frac{\Gamma(1/4)}{2^{5/4}\sqrt{\pi}} \,. \tag{2.31}$$

See Figure 3 for plots of these functions.

Theorem 2.4 (Neumann boundary conditions)**.** *For Neumann b.c. and sufficiently small r, ρ and ε, the following holds true.*

1. *If $L < \pi$ and L is bounded away from π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = 2\pi\left(\frac{1}{|\lambda_0|\nu_0^-}\prod_{k\geqslant 1}\frac{\lambda_k}{\nu_k^-}\right)^{1/2} \mathrm{e}^{(V[u_0^*]-V[u_-^*])/\varepsilon}\left[1+\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\right] . \tag{2.32}$$

2. *If $L > \pi$ and L is bounded away from π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = \pi\left(\frac{1}{|\mu_0|\nu_0^-}\prod_{k\geqslant 1}\frac{\mu_k}{\nu_k^-}\right)^{1/2} \mathrm{e}^{(V[u_{1,\pm}^*]-V[u_-^*])/\varepsilon}\left[1+\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\right] . \tag{2.33}$$

3. *If $L \leqslant \pi$ and L is in a neighbourhood of π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = 2\pi\left(\frac{\lambda_1+\sqrt{C\varepsilon}}{|\lambda_0|\nu_0^-\nu_1^-}\prod_{k\geqslant 2}\frac{\lambda_k}{\nu_k^-}\right)^{1/2} \frac{\mathrm{e}^{(V[u_0^*]-V[u_-^*])/\varepsilon}}{\Psi_+(\lambda_1/\sqrt{C\varepsilon})}\left[1+R_+(\varepsilon,\lambda_1)\right] , \tag{2.34}$$

   where

$$C = \frac{1}{4L}\left[U^{(4)}(0) + \frac{8\pi^2-3L^2}{4\pi^2-L^2}U'''(0)^2\right] , \tag{2.35}$$

   and the remainder R_+ satisfies

$$R_+(\varepsilon,\lambda) = \mathcal{O}\left(\left[\frac{\varepsilon|\log\varepsilon|^3}{\max\{|\lambda|,\sqrt{\varepsilon|\log\varepsilon|}\}}\right]^{1/2}\right) . \tag{2.36}$$

4. *If $L \geqslant \pi$ and L is in a neighbourhood of π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = 2\pi\left(\frac{\mu_1+\sqrt{C\varepsilon}}{|\mu_0|\nu_0^-\nu_1^-}\prod_{k\geqslant 2}\frac{\mu_k}{\nu_k^-}\right)^{1/2} \frac{\mathrm{e}^{(V[u_{1,\pm}^*]-V[u_-^*])/\varepsilon}}{\Psi_-(\mu_1/\sqrt{C\varepsilon})}\left[1+R_-(\varepsilon,\mu_1)\right] , \tag{2.37}$$

   where C is given by (2.35)*, and the remainder R_- is of the same order as R_+.*

Note that (2.31) (together with the fact that $\mu_k(L) \to \lambda_k(\pi)$ as $L \to \pi_+$) shows that $\mathbb{E}^{u_0}\{\tau_+\}$ is indeed continuous at $L = \pi$. In a neighbourhood of order $\sqrt{\varepsilon}$ of $L = \pi$, the prefactor of the transition time is of order $\varepsilon^{1/4}$, while it is constant to leading order when L is bounded away from π.

We have written here the different expressions for the expected transition time in a generic way, in terms of eigenvalues and potential-energy differences. Note however that several quantities appearing in the theorem admit more explicit expressions:

- We have $V[u_0^*] = 0$ and $V[u_-^*] = U(u_-)$, while $V[u_{1,\pm}^*]$ is determined by solving (2.12) with the help of the first integral (2.14). For the symmetric double-well potential (2.4), it can be expressed explicitly in terms of elliptic integrals.
- The two identities

$$\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right) = \frac{\sin(\pi x)}{\pi x} , \qquad \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right) = \frac{\sinh(\pi x)}{\pi x} \tag{2.38}$$

  imply that the prefactor in (2.32) is given by

$$2\pi\left(\frac{1}{|\lambda_0|\nu_0^-}\prod_{k\geqslant 1}\frac{\lambda_k}{\nu_k^-}\right)^{1/2} = 2\pi\left(\frac{\sin L}{\sqrt{U''(u_-)}\,\sinh\bigl(L\sqrt{U''(u_-)}\bigr)}\right)^{1/2} . \tag{2.39}$$

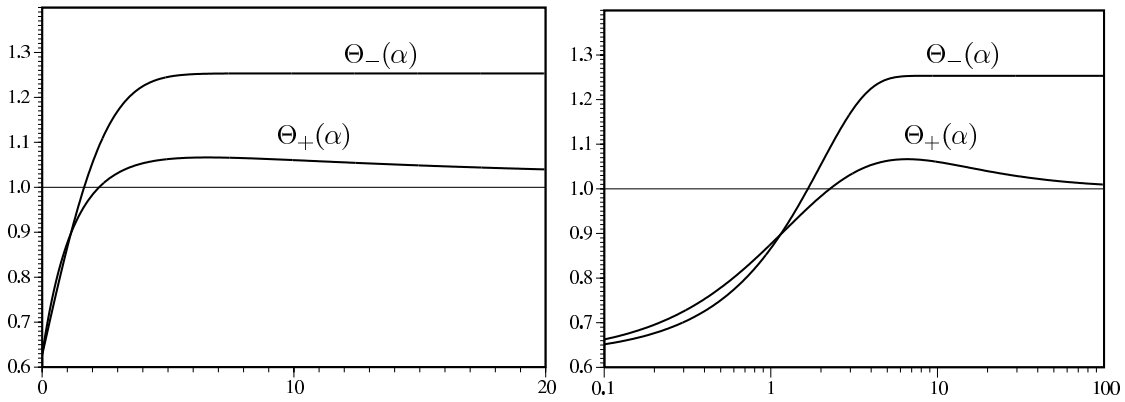

FIGURE 4. The functions $\Theta_\pm(\alpha)$ shown on a linear and on a logarithmic scale.

- Since there is no closed-form expression for the eigenvalues μ_k, it might seem impossible to compute the prefactor appearing in (2.33). In fact, techniques developed for the computation of Feynman integrals allow to compute the product of such ratios of eigenvalues, also called a ratio of functional determinants, see [For87, MT95, CdV99, MS01, MS03] .

We now turn to the case of periodic b.c. In that case, the eigenvalues of the variational equation at $u_0^* \equiv 0$ are given by $-\lambda_k$ where $\lambda_k = (2k\pi/L)^2 - 1$. Those at u_-^* are given by $-\nu_k^-$ where

$$\nu_k^- = \left(\frac{2k\pi}{L}\right)^2 + U''(u_-) \; . \tag{2.40}$$

When $L > 2\pi$, we denote the eigenvalues at the family of transition states $u_{1,\varphi}^*$ by $-\mu_k$ where

$$\mu_0 < \mu_{-1} = 0 < \mu_1 < \mu_2, \mu_{-2} < \dots \tag{2.41}$$

We further introduce two functions $\Theta_\pm : \mathbb{R}_+ \to \mathbb{R}_+$, which play a rôle for the behaviour of $\mathbb{E}^{u_0}\{\tau_+\}$ when L is close to 2π. They are given by

$$\Theta_+(\alpha) = \sqrt{\frac{\pi}{2}}(1+\alpha)\,\mathrm{e}^{\alpha^2/8}\,\Phi\left(-\frac{\alpha}{2}\right) , \tag{2.42}$$

$$\Theta_-(\alpha) = \sqrt{\frac{\pi}{2}}\Phi\left(\frac{\alpha}{2}\right) , \tag{2.43}$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^x \mathrm{e}^{-t^2/2}\,\mathrm{d}t$ denotes the distribution function of a standard Gaussian random variable. The functions $\Theta_\pm$ are bounded below and above by positive constants, and satisfy

$$\lim_{\alpha\to+\infty}\Theta_+(\alpha) = 1 \; , \qquad \lim_{\alpha\to-\infty}\Theta_-(\alpha) = \sqrt{\frac{\pi}{2}} \; , \tag{2.44}$$

and

$$\lim_{\alpha\to 0}\Theta_+(\alpha) = \lim_{\alpha\to 0}\Theta_-(\alpha) = \sqrt{\frac{\pi}{8}} \; . \tag{2.45}$$

See Figure 4 for plots of these functions.

Theorem 2.5 (Periodic boundary conditions)**.** *For periodic b.c. and sufficiently small r, ρ and ε, the following holds true.*

1. *If $L < 2\pi$ and L is bounded away from 2π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0^-}}\Big(\prod_{k\geqslant 1}\frac{\lambda_k}{\nu_k^-}\Big)\,\mathrm{e}^{(V[u_0^*]-V[u_-^*])/\varepsilon}\big[1+\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\big]\,. \tag{2.46}$$

2. *If $L \leqslant 2\pi$ and L is in a neighbourhood of 2π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = \frac{2\pi}{\sqrt{|\lambda_0|\nu_0^-}}\frac{\lambda_1+\sqrt{2C\varepsilon}}{\nu_1^-}\Big(\prod_{k\geqslant 2}\frac{\lambda_k}{\nu_k^-}\Big)\frac{\mathrm{e}^{(V[u_0^*]-V[u_-^*])/\varepsilon}}{\Theta_+(\lambda_1/\sqrt{2C\varepsilon}\,)}\big[1+R_+(\varepsilon,\lambda_1)\big]\,, \tag{2.47}$$

   where

$$C = \frac{1}{4L}\Big[U^{(4)}(0)+\frac{32\pi^2-3L^2}{16\pi^2-L^2}U'''(0)^2\Big]\,, \tag{2.48}$$

   and the remainder R_+ satisfies (2.36)*.*

3. *If $L \geqslant 2\pi$ and L is in a neighbourhood of 2π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = \frac{2\pi}{\sqrt{|\mu_0|\nu_0^-}}\frac{\sqrt{2C\varepsilon}}{\nu_1^-}\Big(\prod_{k\geqslant 2}\frac{\sqrt{\mu_k\mu_{-k}}}{\nu_k^-}\Big)\frac{\mathrm{e}^{(V[u_{1,\varphi}^*]-V[u_-^*])/\varepsilon}}{\Theta_-(\mu_1/\sqrt{8C\varepsilon}\,)}\big[1+R_-(\varepsilon,\mu_1)\big]\,, \tag{2.49}$$

   where C is given by (2.48)*, and the remainder R_- is of the same order as R_+.*

4. *If $L > 2\pi$ and L is bounded away from 2π, then*

$$\mathbb{E}^{u_0}\{\tau_+\} = \frac{2\pi}{\sqrt{|\mu_0|\nu_0^-}}\frac{\sqrt{2\pi\varepsilon\mu_1}}{\nu_1^-}\Big(\prod_{k\geqslant 2}\frac{\sqrt{\mu_k\mu_{-k}}}{\nu_k^-}\Big)\frac{\mathrm{e}^{(V[u_{1,0}^*]-V[u_-^*])/\varepsilon}}{L\|(u_{1,0}^*)'\|_{L^2}}\big[1+\mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^{3/2})\big]\,. \tag{2.50}$$

Note that for $L \geqslant 2\pi - \mathcal{O}(\sqrt{\varepsilon})$, the prefactor of $\mathbb{E}^{u_0}\{\tau_+\}$ is proportional to $\sqrt{\varepsilon}/L$. This is due to the existence of the continuous family of transition states

$$u_{1,\varphi}^*(x) = u_{1,0}^*(x+\varphi)\,, \qquad 0 \leqslant \varphi < L \tag{2.51}$$

owing to translation symmetry. The quantity

$$L\|(u_{1,0}^*)'\|_{L^2} = L\left[\int_0^L\Big(\frac{\mathrm{d}}{\mathrm{d}x}u_{1,0}^*(x)\Big)^2\,\mathrm{d}x\right]^{1/2} \tag{2.52}$$

plays the rôle of the "length of the saddle". One shows (cf. Section 6.2) that for L close to 2π, μ_1 is close to $-2\lambda_1$ and

$$L\|(u_{1,0}^*)'\|_{L^2} = 2\pi\sqrt{\frac{\mu_1}{8C}}+\mathcal{O}(\mu_1)\,, \tag{2.53}$$

which implies that $\mathbb{E}^{u_0}\{\tau_+\}$ is indeed continuous at $L = 2\pi$.

As in the case of Neumann b.c., several of the above quantities admit more explicit expressions. For instance, the identities (2.38) imply that the prefactor in (2.46) is given by

$$\frac{2\pi}{\sqrt{|\lambda_0|\nu_0^-}}\Big(\prod_{k\geqslant 1}\frac{\lambda_k}{\nu_k^-}\Big) = \frac{2\pi\sin(L/2)}{\sinh\big(\sqrt{U''(u_-)}\,L/2\big)}\,. \tag{2.54}$$

See [Ste04] for an explicit expression of the prefactor for $L > 2\pi$, for a particular class of double-well potentials.

3 Outline of the proof

3.1 Potential theory

A first key ingredient of the proof is the potential-theoretic approach to metastability of finite-dimensional SDEs developed in [BEGK04, BGK05]. Given a confining potential $V : \mathbb{R}^d \to \mathbb{R}$, consider the diffusion defined by

$$\mathrm{d}x_t = -\nabla V(x_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_t\;, \tag{3.1}$$

where W_t denotes d-dimensional Brownian motion. The diffusion is reversible with respect to the invariant measure

$$\mu(\mathrm{d}x) = \frac{1}{Z}\,\mathrm{e}^{-V(x)/\varepsilon}\,\mathrm{d}x\;, \tag{3.2}$$

where Z is the normalisation. This follows from the fact that its infinitesimal generator

$$\mathcal{L} = \varepsilon\Delta - \nabla V(x)\cdot\nabla = \varepsilon\,\mathrm{e}^{V/\varepsilon}\,\nabla\cdot\mathrm{e}^{-V/\varepsilon}\,\nabla \tag{3.3}$$

is self-adjoint in $L^2(\mathbb{R}^d, \mu(\mathrm{d}x))$.

Let $A, B, C \subset \mathbb{R}^d$ be measurable sets which are regular (that is, their complement is a region with continuously differentiable boundary). We are interested in the expected first-hitting time

$$w_A(x) = \mathbb{E}^x\big\{\tau_A\big\}\;. \tag{3.4}$$

Dynkin's formula shows that $w_A(x)$ solves the Poisson problem

$$\begin{aligned} \mathcal{L}w_A(x) &= -1 && x \in A^c\;,\\ w_A(x) &= 0 && x \in A\;. \end{aligned} \tag{3.5}$$

The solution of (3.5) can be expressed in terms of the Green function $G_{A^c}(x,y)$ as

$$w_A(x) = -\int_{A^c} G_{A^c}(x,y)\,\mathrm{d}y\;. \tag{3.6}$$

Reversibility implies that the Green function satisfies the symmetry

$$\mathrm{e}^{-V(x)/\varepsilon}\,G_{A^c}(x,y) = \mathrm{e}^{-V(y)/\varepsilon}\,G_{A^c}(y,x)\;. \tag{3.7}$$

Another important quantity is the *equilibrium potential*

$$h_{A,B}(x) = \mathbb{P}^x\big\{\tau_A < \tau_B\big\}\;. \tag{3.8}$$

It satisfies the Dirichlet problem

$$\begin{aligned} \mathcal{L}h_{A,B}(x) &= 0 && x \in (A\cup B)^c\;,\\ h_{A,B}(x) &= 1 && x \in A\;,\\ h_{A,B}(x) &= 0 && x \in B\;, \end{aligned} \tag{3.9}$$

whose solution can be expressed in terms of the Green function and an *equilibrium measure* $e_{A,B}(\mathrm{d}y)$ on ∂A defined by

$$h_{A,B}(x) = \int_{\partial A} G_{B^c}(x,y)e_{A,B}(\mathrm{d}y)\;. \tag{3.10}$$

Finally, the *capacity* between A and B is defined as

$$\mathrm{cap}_A(B) = -\int_{\partial A} \mathrm{e}^{-V(y)/\varepsilon}\, e_{A,B}(\mathrm{d}y)\ . \tag{3.11}$$

The key observation is that the relations (3.10), (3.7) and (3.6) can be combined to yield

$$\begin{aligned}\int_{A^c} h_{C,A}(y)\, \mathrm{e}^{-V(y)/\varepsilon}\, \mathrm{d}y &= \int_{A^c}\int_{\partial C} G_{A^c}(y,z) e_{C,A}(\mathrm{d}z)\, \mathrm{e}^{-V(y)/\varepsilon}\, \mathrm{d}y \\ &= -\int_{\partial C} w_A(z)\, \mathrm{e}^{-V(z)/\varepsilon}\, e_{C,A}(\mathrm{d}z)\ . \end{aligned}\tag{3.12}$$

The approach used in [BEGK04] is to take C to be a ball of radius ε centred in x, and to use Harnack inequalities to show that $w_A(z) \simeq w_A(x)$ on C. It then follows from (3.11) that

$$\int_{A^c} h_{C,A}(y)\, \mathrm{e}^{-V(y)/\varepsilon}\, \mathrm{d}y \simeq w_A(x)\, \mathrm{cap}_C(A)\ . \tag{3.13}$$

The left-hand side can be estimated using a priori bounds on the equilibrium potential. Thus a sufficiently precise estimate of the capacity $\mathrm{cap}_C(A)$ will yield a good estimate for $\mathbb{E}^x\{\tau_A\} = w_A(x)$. Now it follows from Green's identities that the capacity can also be expressed as a Dirichlet form evaluated at the equilibrium potential:

$$\mathrm{cap}_A(B) = \varepsilon \int_{(A\cup B)^c} \|\nabla h_{A,B}(x)\|^2\, \mathrm{e}^{-V(x)/\varepsilon}\, \mathrm{d}x\ . \tag{3.14}$$

Even more useful is the variational representation

$$\mathrm{cap}_A(B) = \varepsilon \inf_{h\in\mathcal{H}_{A,B}} \int_{(A\cup B)^c} \|\nabla h(x)\|^2\, \mathrm{e}^{-V(x)/\varepsilon}\, \mathrm{d}x\ , \tag{3.15}$$

where $\mathcal{H}_{A,B}$ denotes the set of twice weakly differentiable functions satisfying the boundary conditions in (3.9). Indeed, inserting a sufficiently good guess for the equilibrium potential on the right-hand side immediately yields a good upper bound. A matching lower bound can be obtained by a slightly more involved argument.

Several difficulties prevent us from applying the same strategy directly to the infinite-dimensional equation (2.1). It is possible, however, to approximate (2.1) by a finite-dimensional system, using a spectral Galerkin method, to estimate first-hitting times for the finite-dimensional system using the above ideas, and then to pass to the limit.

3.2 Spectral Galerkin approximation

Let $P_d : E \to E$ be the projection operator defined by

$$(P_d u)(x) = \sum_{|k|\leqslant d} y_k e_k(x)\ , \qquad y_k = y_k[u] = \int_0^L \overline{e_k(y)} u(y)\, \mathrm{d}y\ , \tag{3.16}$$

where the e_k are the basis vectors compatible with the boundary conditions, given by (2.7) or (2.8). We denote by E_d the finite-dimensional image of E under P_d. Let $u_t(x)$ be the mild solution of the SPDE (2.1) and let $u_t^{(d)}(x)$ be the solution of the projected equation

$$\mathrm{d}u_t^{(d)}(x) = P_d\big[\Delta u_t^{(d)}(x) - U'(u_t^{(d)}(x))\big]\, \mathrm{d}t + \sqrt{2\varepsilon} P_d\, \mathrm{d}W(t,x)\ , \tag{3.17}$$

called Galerkin approximation of order d. It is known (see, for instance, [Jet86]) that (3.17) is equivalent to the finite-dimensional system of SDEs

$$\mathrm{d}y_k(t) = -\frac{\partial}{\partial y_k}\widehat{V}(y(t))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}W_k(t)\;, \qquad |k| \leqslant d\;, \tag{3.18}$$

where the $W_k(t)$ are independent standard Brownian motions, and the potential is given by

$$\widehat{V}(y) = \widehat{V}\left[\sum_{|k|\leqslant d} y_k e_k\right]\;. \tag{3.19}$$

We will need an estimate of the deviation of the Galerkin approximation $u_t^{(d)}$ from u_t. Such estimates are available in the numerical analysis literature. For instance, [Liu03] provides an estimate for a weighted norm of the form $\|u\|_{H_0^r}^2 = \sum_k (1+k^2)^r |y_k|^2$, with $r < 1/2$. We shall use the more precise results in [BJ09], which allow for a control in the (stronger) sup norm. Namely, we have the following result:

Theorem 3.1. *Fix a $T > 0$. Let U' be locally Lipschitz, and assume*

$$\sup_{d\in\mathbb{N}} \sup_{0\leqslant t\leqslant T} \|u_t^{(d)}(\omega)\|_{L^\infty} < \infty \tag{3.20}$$

for all $\omega \in \Omega$. Then, for any $\gamma \in (0, 1/2)$, there exists an almost surely finite random variable $Z : \Omega \to \mathbb{R}_+$ such that

$$\sup_{0\leqslant t\leqslant T} \|u_t(\omega) - u_t^{(d)}(\omega)\|_{L^\infty} \leqslant Z(\omega) d^{-\gamma} \tag{3.21}$$

for all $\omega \in \Omega$.

Proof: The result follows directly from [BJ09, Theorem 3.1], provided we verify the validity of four assumptions given in [BJ09, Section 2].

- Assumption 1 concerns the regularity of the semigroup $\mathrm{e}^{\Delta t}$ associated with the heat kernel, and is satisfied as shown in [BJ09, Lemma 4.1].
- Assumption 2 is the local Lipschitz condition on U'.
- Assumption 3 concerns the deviation of $P_d W(t,x)$ from $W(t,x)$ and is satisfied according to [BJ09, Proposition 4.2].
- Assumption 4 is (3.20). □

3.3 Proof of the main result

For $r, \rho > 0$ sufficiently small constants we define the balls

$$A = A(r) = \{u \in E\colon \|u - u_-^*\|_{L^\infty} \leqslant r\}\;, \tag{3.22}$$

$$B = B(\rho) = \{u \in E\colon \|u - u_+^*\|_{L^\infty} \leqslant \rho\}\;. \tag{3.23}$$

If u_{ts}^* stands for a transition state between u_-^* and u_+^*, we denote by

$$H_0 = V[u_{\mathrm{ts}}^*] - V[u_-^*] \tag{3.24}$$

the communication height from u_-^* to u_+^*. We fix an initial condition $u_0 \in A$, and write $u_0^{(d)} = P_d u_0$ for its projection on the finite-dimensional space E_d. Finally we set $A_d = A \cap E_d$. Consider the first-hitting times

$$\begin{aligned} \tau_B^{(d)} &= \inf\{t > 0 \colon u_t^{(d)} \in B\} \,, \\ \tau_B &= \inf\{t > 0 \colon u_t \in B\} \,. \end{aligned} \tag{3.25}$$

We first need some a priori bounds on moments of these hitting times. They are stated in the following result, which is proved in Section 5.

Proposition 3.2 (A priori bound on moments of hitting times)**.** *For any $\eta > 0$, there exist constants $\varepsilon_0 = \varepsilon_0(\eta), T_1 = T_1(\eta), H_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a $d_0(\varepsilon) > 0$ such that*

$$\sup_{v_0 \in A} \mathbb{E}^{v_0}\left\{\tau_B^2\right\} \leqslant T_1^2 \, \mathrm{e}^{2(H_0+\eta)/\varepsilon} \qquad \text{and} \qquad \sup_{d \geqslant d_0} \sup_{v_0 \in A} \mathbb{E}^{v_0^{(d)}}\left\{(\tau_B^{(d)})^2\right\} \leqslant T_1^2 \, \mathrm{e}^{2H_1/\varepsilon} \,. \tag{3.26}$$

The next result applies to all finite-dimensional Galerkin approximations, and is based on the potential-theoretic approach. The detailed proof is given in Sections 6 and 7.

Proposition 3.3 (Bounds on expected hitting times in finite dimension)**.** *There exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists a $d_0 = d_0(\varepsilon) < \infty$ such that for all $d \geqslant d_0$, there exists a probability measure $\nu_{d,B}$ supported on ∂A_d such that*

$$C(d,\varepsilon) \, \mathrm{e}^{H(d)/\varepsilon}\big[1 - R_{d,B}^-(\varepsilon)\big] \leqslant \int_{\partial A_d} \mathbb{E}^{v_0}\big\{\tau_B^{(d)}\big\} \nu_{d,B}(\mathrm{d}v_0) \leqslant C(d,\varepsilon) \, \mathrm{e}^{H(d)/\varepsilon}\big[1 + R_{d,B}^+(\varepsilon)\big] \,, \tag{3.27}$$

where the quantities $C(d,\varepsilon)$, $H(d)$ and $R_{d,B}^\pm(\varepsilon)$ are explicitly known. They satisfy

- $\lim_{d\to\infty} C(d,\varepsilon) =: C(\infty,\varepsilon)$ *exists and is finite;*
- $\lim_{d\to\infty} H(d) = H_0$ *is given by the communication height;*
- *the remainders $R_{d,B}^\pm(\varepsilon)$ are uniformly bounded in d and $R_B^\pm(\varepsilon) = \sup_d R_{d,B}^\pm(\varepsilon)$ satisfies $\lim_{\varepsilon\to 0} R_B^\pm(\varepsilon) = 0$.*

Then we have the following result.

Proposition 3.4 (Averaged bounds on the expected first-hitting time in infinite dimension)**.** *Pick a $\delta \in (0, \rho)$. There exist $\varepsilon_0 > 0$ and probability measures ν_+ and ν_- on ∂A such that for $0 < \varepsilon < \varepsilon_0$,*

$$\begin{aligned} \int_{\partial A} \mathbb{E}^{v_0}\big\{\tau_{B(\rho)}\big\} \nu_+(\mathrm{d}v_0) &\leqslant C(\infty,\varepsilon) \, \mathrm{e}^{H_0/\varepsilon}\big[1 + 2R_{B(\rho-\delta)}^+(\varepsilon)\big] \,, \\ \int_{\partial A} \mathbb{E}^{v_0}\big\{\tau_{B(\rho)}\big\} \nu_-(\mathrm{d}v_0) &\geqslant C(\infty,\varepsilon) \, \mathrm{e}^{H_0/\varepsilon}\big[1 - 2R_{B(\rho+\delta)}^-(\varepsilon)\big] \,. \end{aligned} \tag{3.28}$$

Proof: To ease notation, we write $B = B(\rho)$, $B_\pm = B(\rho \pm \delta)$ and $T_{\mathrm{Kr}} = C(\infty,\varepsilon) \, \mathrm{e}^{H_0/\varepsilon}$. For given $v_0 \in \partial A$ and $K > 0$, define the event

$$\Omega_{K,d} := \left\{ \sup_{t \in [0, KT_{\mathrm{Kr}}]} \|v_t - v_t^{(d)}\|_{L^\infty} \leqslant \delta, \, \tau_{B_-}^{(d)} \leqslant KT_{\mathrm{Kr}} \right\} , \tag{3.29}$$

where v_t and $v_t^{(d)}$ denote the solutions of the original and the projected equation with respective initial conditions v_0 and $P_d v_0$. Theorem 3.1 and Markov's inequality imply

$$\mathbb{P}(\Omega_{K,d}^c) \leqslant \mathbb{P}\{Z > \delta d^\gamma\} + \frac{\mathbb{E}^{v_0^{(d)}}\{\tau_{B_-}^{(d)}\}}{KT_{\mathrm{Kr}}} \,. \tag{3.30}$$

Choosing ε_0 such that $R_{B_-}^+(\varepsilon) < 1$ and (3.27) holds for all $\varepsilon < \varepsilon_0$ and sufficiently large d, the last summand can be made smaller than $2/K$. Thus for all $\varepsilon < \varepsilon_0$, we have

$$\limsup_{d\to\infty} \mathbb{P}(\Omega_{K,d}^c) \leqslant \frac{2}{K} \,. \tag{3.31}$$

We decompose

$$\mathbb{E}^{v_0}\{\tau_B\} = \mathbb{E}^{v_0}\{\tau_B 1_{\Omega_{K,d}}\} + \mathbb{E}^{v_0}\{\tau_B 1_{\Omega_{K,d}^c}\} \,. \tag{3.32}$$

In order to estimate the first summand, we note that by definition of B, B_+ and B_-,

$$\tau_{B_+}^{(d)} \leqslant \tau_B \leqslant \tau_{B_-}^{(d)} \qquad \text{on } \Omega_{K,d} \,. \tag{3.33}$$

It follows that

$$\begin{aligned} \mathbb{E}^{v_0^{(d)}}\{\tau_{B_+}^{(d)}\} - \mathbb{E}^{v_0^{(d)}}\{\tau_{B_+}^{(d)} 1_{\Omega_{K,d}^c}\} &= \mathbb{E}^{v_0^{(d)}}\{\tau_{B_+}^{(d)} 1_{\Omega_{K,d}}\} \\ &\leqslant \mathbb{E}^{v_0}\{\tau_B 1_{\Omega_{K,d}}\} \\ &\leqslant \mathbb{E}^{v_0^{(d)}}\{\tau_{B_-}^{(d)} 1_{\Omega_{K,d}}\} \leqslant \mathbb{E}^{v_0^{(d)}}\{\tau_{B_-}^{(d)}\} \,. \end{aligned} \tag{3.34}$$

The second summand in (3.32) can be bounded by Cauchy–Schwartz:

$$0 \leqslant \mathbb{E}^{v_0}\{\tau_B 1_{\Omega_{K,d}^c}\} \leqslant \sqrt{\mathbb{E}^{v_0}\{\tau_B^2\}}\sqrt{\mathbb{P}(\Omega_{K,d}^c)} \,. \tag{3.35}$$

This shows that

$$\mathbb{E}^{v_0^{(d)}}\{\tau_{B_+}^{(d)}\} - \sqrt{\mathbb{E}^{v_0}\{(\tau_{B_+}^{(d)})^2\}\,\mathbb{P}(\Omega_{K,d}^c)} \leqslant \mathbb{E}^{v_0}\{\tau_B\} \leqslant \mathbb{E}^{v_0^{(d)}}\{\tau_{B_-}^{(d)}\} + \sqrt{\mathbb{E}^{v_0}\{\tau_B^2\}\,\mathbb{P}(\Omega_{K,d}^c)} \,. \tag{3.36}$$

Proposition 3.3 shows that

$$\limsup_{d\to\infty} \int_{\partial A_d} \mathbb{E}^{v_0}\{\tau_{B_-}^{(d)}\}\nu_{d,B_-}(\mathrm{d}v_0) \leqslant T_{\mathrm{Kr}}\big[1 + R_{B_-}^+(\varepsilon)\big] \,, \tag{3.37}$$

while Lebesgues's dominated convergence theorem and (3.31) yield

$$\limsup_{d\to\infty} \int_{\partial A_d} \sqrt{\mathbb{E}^{v_0}\{\tau_B^2\}\,\mathbb{P}(\Omega_{K,d}^c)}\,\nu_{d,B_-}(\mathrm{d}v_0) \leqslant \sqrt{\sup_{v_0\in A} \mathbb{E}^{v_0}\{\tau_B^2\}\,\frac{2}{K}} \,. \tag{3.38}$$

Inserting (3.37) and (3.38) in (3.36) shows that

$$\limsup_{d\to\infty} \frac{1}{T_{\mathrm{Kr}}} \int_{\partial A_d} \mathbb{E}^{v_0}\{\tau_B\}\,\nu_{d,B_-}(\mathrm{d}v_0) \leqslant 1 + R_{B_-}^+(\varepsilon) + \sqrt{\frac{\sup_{v_0\in A}\mathbb{E}^{v_0}\{\tau_B^2\}}{T_{\mathrm{Kr}}^2}\frac{2}{K}} \,. \tag{3.39}$$

Taking K sufficiently large, the third summand can be made smaller than the second one. The upper bound in (3.28) then follows with $\nu_+ = \nu_{d,B_-}$ for d sufficiently large. The proof of the lower bound is analogous. □

To finish the proof of the main result, we need

Theorem 3.5. *There exist constants $\varepsilon_0, \kappa, t_0, m, c > 0$ such that such that for all $\varepsilon < \varepsilon_0$,*

$$\mathbb{P}\biggl\{ \sup_{u_0,v_0\in A} \frac{\|u_t - v_t\|_{L^\infty}}{\|u_0 - v_0\|_{L^\infty}} < c\,\mathrm{e}^{-mt} \ \forall t > t_0 \biggr\} \geqslant 1 - \mathrm{e}^{-\kappa/\varepsilon} \ . \tag{3.40}$$

Here u_t and v_t denote the mild solutions of the SPDE (2.1) *with respective initial conditions u_0 and v_0.*

This result has been proved in [MOS89, Corollary 3.1] in the case of Dirichlet b.c., under the condition $L \neq \pi$. The reason for this restriction is that for $L = \pi$, the Hessian at the potential minimum has a zero eigenvalue. For Neumann and periodic b.c., this difficulty does not occur, because the potential minima always have only strictly negative eigenvalues.

Proposition 3.6 (Main result)**.** *Pick $\delta \in (0, \rho/2)$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$C(\infty,\varepsilon)\,\mathrm{e}^{H_0/\varepsilon}\bigl[1 - 3R^-_{B(\rho+2\delta)}(\varepsilon)\bigr] \leqslant \mathbb{E}^{u_0}\bigl\{\tau_B(\rho)\bigr\} \leqslant C(\infty,\varepsilon)\,\mathrm{e}^{H_0/\varepsilon}\bigl[1 + 3R^+_{B(\rho-2\delta)}(\varepsilon)\bigr] \ . \tag{3.41}$$

Proof: As before we write $B = B(\rho)$, $B_\pm = B(\rho \pm \delta)$. Given a constant $T \geqslant t_0$, consider the event

$$\Omega_T = \biggl\{ \tau_{B_+} \geqslant T, \sup_{v_0\in A} \frac{\|v_t - u_t\|_{L^\infty}}{\|v_0 - u_0\|_{L^\infty}} < c\,\mathrm{e}^{-mt} \ \forall t \geqslant T \biggr\} \ . \tag{3.42}$$

Then Theorem 3.5 and a standard large-deviation estimate show that for any $T > 0$, there exist constants $\varepsilon_0, \kappa_1 > 0$ such that for $\varepsilon < \varepsilon_0$,

$$\mathbb{P}(\Omega_T^c) \leqslant \mathrm{e}^{-\kappa_1/\varepsilon} \ . \tag{3.43}$$

Note that for all $v_0 \in A$,

$$\tau^{v_0}_{B_+} \leqslant \tau^{u_0}_B \leqslant \tau^{v_0}_{B_-} \qquad \text{on } \Omega_T \ , \tag{3.44}$$

provided T is large enough that $rc\,\mathrm{e}^{-mT} \leqslant \delta/2$. In order to prove the upper bound, we start by observing that

$$\mathbb{E}^{u_0}\bigl\{\tau_B 1_{\Omega_T}\bigr\} = \mathbb{E}^{u_0}\bigl\{\tau_B 1_{\Omega_T}\bigr\} \int_{\partial A} \nu_+(\mathrm{d}v_0) \leqslant \int_{\partial A} \mathbb{E}^{v_0}\bigl\{\tau_{B_-}\bigr\}\nu_+(\mathrm{d}v_0) \ , \tag{3.45}$$

which can be bounded above with Proposition 3.4. Furthermore, by Cauchy–Schwartz, we have

$$\mathbb{E}^{u_0}\bigl\{\tau_B 1_{\Omega_T^c}\bigr\} \leqslant \sqrt{\mathbb{E}^{u_0}\bigl\{\tau_B^2\bigr\}}\sqrt{\mathbb{P}(\Omega_1^c)} \leqslant T_1\,\mathrm{e}^{(H_0+\eta-\kappa_1/2)/\varepsilon} \ . \tag{3.46}$$

For the lower bound, we use the decomposition

$$\begin{aligned}\mathbb{E}^{u_0}\bigl\{\tau_B\bigr\} &\geqslant \mathbb{E}^{u_0}\bigl\{\tau_B 1_{\Omega_T}\bigr\} \int_{\partial A} \nu_-(\mathrm{d}v_0) \\ &\geqslant \int_{\partial A} \mathbb{E}^{v_0}\bigl\{\tau_{B_+}\bigr\}\nu_-(\mathrm{d}v_0) - \int_{\partial A} \mathbb{E}^{v_0}\bigl\{\tau_{B_+} 1_{\Omega_T^c}\bigr\}\nu_-(\mathrm{d}v_0) \ .\end{aligned} \tag{3.47}$$

The first term on the right-hand side can be bounded below with Proposition 3.4, while the second one is bounded above by

$$\sqrt{\sup_{v_0\in\partial A} \mathbb{E}^{v_0}\bigl\{\tau_{B_+}^2\bigr\}}\sqrt{\mathbb{P}(\Omega_1^c)} \leqslant T_1\,\mathrm{e}^{(H_0+\eta-\kappa_1/2)/\varepsilon} \ . \tag{3.48}$$

This concludes the proof, provided we choose $\eta < \kappa_1/2$ when applying Proposition 3.4. □

4 Deterministic system

This section gathers a number of needed results on the deterministic partial differential equation (2.11). Some general properties of the equation are discussed e.g. in [CI75, Jol89].

In Section 4.1 we introduce various function spaces and inequalities required in the analysis. In Section 4.2, we establish some general bounds on the potential energy V and its derivative. Section 4.3 analyses the behaviour of the potential energy at bifurcation points, and Section 4.4 contains a result on the relation between V and its restrictions to finite-dimensional subspaces.

4.1 Function spaces

We introduce two scales of function spaces that will play a rôle in the sequel. Let I denote either a compact interval $[0, L] \subset \mathbb{R}$ or the circle $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

We denote by $\mathcal{C}^0 = \mathcal{C}^0(I)$ the space of all continuous functions $u : I \to \mathbb{R}$. Note that I is compact so that the functions from $\mathcal{C}^0$ are bounded. When equipped with the sup norm $\|u\|_{\mathcal{C}^0} = \sup_{x\in I}|u(x)|$, $\mathcal{C}^0$ is a Banach space. For $\alpha > 0$, we define the Hölder seminorm

$$[u]_\alpha = \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} , \tag{4.1}$$

the Hölder norm $\|u\|_{\mathcal{C}^\alpha} = \|u\|_{\mathcal{C}^0} + [u]_\alpha$, and write $\mathcal{C}^\alpha = \{u \in \mathcal{C}^0 \colon \|u\|_{\mathcal{C}^\alpha} < \infty\}$ for the associated Banach space.

For $1 \leqslant p \leqslant \infty$, $L^p = L^p(I)$ denotes the space of all $u : I \to \mathbb{R}$ with bounded L^p-norm. Note that for $u \in \mathcal{C}^0$, $\|u\|_{L^\infty} = \|u\|_{\mathcal{C}^0}$. When $u \in L^2(\mathbb{T}^1)$, we write its Fourier series as

$$u(x) = \sum_{k\in\mathbb{Z}} y_k e_k(x) , \qquad e_k(x) = \frac{\mathrm{e}^{\mathrm{i}\,kx}}{\sqrt{2\pi}} . \tag{4.2}$$

For $s \geqslant 0$, we define the Sobolev norm

$$\|u\|_{H^s}^2 \equiv \|y\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} (1 + k^2)^s |y_k|^2 , \tag{4.3}$$

and denote by $H^s = H^s(\mathbb{T}^1) = \{u \in L^2(\mathbb{T}^1) \colon \|u\|_{H^s} < \infty\}$ the fractional Sobolev space (also called Bessel potential space). Note that H^s is a Hilbert space, and $H^0 = L^2$. The norm $\|u\|_{H^1}$ can be equivalently defined by $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2$, where u' is the weak derivative of u.

Lemma 4.1. *For any $\alpha \geqslant 0$ and $s > \alpha + 1/2$, there exists a constant $C = C(\alpha, s)$ such that*

$$\|u\|_{\mathcal{C}^\alpha} \leqslant C\|u\|_{H^s} \qquad \forall u \in H^s(\mathbb{T}^1) . \tag{4.4}$$

As a consequence, we have $H^s(\mathbb{T}^1) \subset \mathcal{C}^\alpha(\mathbb{T}^1)$.

Remark 4.2. In the particular case $s = 1$, (4.4) can be strengthened to Morrey's inequality

$$\|u\|_{\mathcal{C}^{1/2}} \leqslant C\|u\|_{H^1} \qquad \forall u \in H^1(\mathbb{T}^1) . \tag{4.5}$$

Let p, q satisfy $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Hausdorff–Young inequalities [DS88] state that there exist constants $C_1(p)$ and $C_2(p)$ such that

$$\|u\|_{L^q} \leqslant C_1 \|y\|_{\ell^p} \qquad \text{and} \qquad \|y\|_{\ell^q} \leqslant C_2 \|u\|_{L^p} \; . \tag{4.6}$$

We consider now some properties of convolutions $y * z$ defined by

$$(y * z)_k = \sum_{l \in \mathbb{Z}} y_l z_{k-l} \; . \tag{4.7}$$

Young's inequality states that for $1 \leqslant p, q, r \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$,

$$\|y * z\|_{\ell^r} \leqslant \|y\|_{\ell^p} \|z\|_{\ell^q} \; . \tag{4.8}$$

Lemma 4.3. *Let $r, s, t \in (0, 1/2)$ be such that $t < r + s - 1/2$. Then there exists a constant $C = C(r, s, t)$ such that*

$$\|y * z\|_{H^t} \leqslant C \|y\|_{H^r} \|z\|_{H^s} \; . \tag{4.9}$$

Proof: Define w_k by

$$\frac{1}{w_k} = \sum_{l \in Z} \frac{1}{(1 + l^2)^r} \frac{1}{(1 + (k-l)^2)^s} \; . \tag{4.10}$$

Splitting the sum at $-|k|, 0, |k|/2, |k|$ and $2|k|$, and bounding each sum by an integral, one easily shows that

$$\frac{1}{w_k} \leqslant \frac{C}{(1 + k^2)^t} \; . \tag{4.11}$$

Let $\tilde{y}_k = (1 + k^2)^{r/2} |y_k|$ and $\tilde{z}_k = (1 + k^2)^{s/2} |z_k|$. Adapting a computation in [Sai00], we write

$$\big|(y * z)_k\big| \leqslant \sum_{l \in \mathbb{Z}} |y_l| |z_{k-l}| = \sum_{l \in \mathbb{Z}} \frac{1}{(1 + l^2)^{r/2} (1 + (k-l)^2)^{s/2}} \tilde{y}_l \tilde{z}_{k-l} \; . \tag{4.12}$$

By the Cauchy–Schwartz inequality,

$$\big((y * z)_k\big)^2 \leqslant \frac{1}{w_k} \sum_{l \in \mathbb{Z}} \tilde{y}_l^2 \tilde{z}_{k-l}^2 \; . \tag{4.13}$$

If $\tilde{y}^2, \tilde{z}^2$ denote the vectors with components $\tilde{y}_l^2$ and $\tilde{z}_l^2$, it follows from (4.11) that

$$\|y * z\|_{H^t}^2 \leqslant \sum_{k \in \mathbb{Z}} C \sum_{l \in \mathbb{Z}} \tilde{y}_l^2 \tilde{z}_{k-l}^2 = C \sum_{k \in \mathbb{Z}} (\tilde{y}^2 * \tilde{z}^2)_k = C \|\tilde{y}^2 * \tilde{z}^2\|_{\ell^1} \leqslant C \|\tilde{y}^2\|_{\ell^1} \|\tilde{z}^2\|_{\ell^1} \tag{4.14}$$

by Young's inequality, and the results follows since $\|\tilde{y}^2\|_{\ell^1} = \|y\|_{H^r}^2$, $\|\tilde{z}^2\|_{\ell^1} = \|z\|_{H^s}^2$. □

Finally, the following estimate allows to bound the usual ℓ^r-norm in terms of Sobolev norms.

Lemma 4.4. *Fix $1 \leqslant r < 2$. For any $s > 1/r - 1/2$, there exists a finite $C(s)$ such that*

$$\|y\|_{\ell^r} \leqslant C(s) \|y\|_{H^s} \; . \tag{4.15}$$

Proof: Apply Hölder's inequality, with $p = 2/(2 - r)$ and $q = 2/r$, to the decomposition $|y_k|^r = (1 + k^2)^{-rs/2} \cdot (1 + k^2)^{rs/2} |y_k|^r$. □

By the Hausdorff–Young inequality (4.6), this implies the Sobolev embedding theorem

$$\|u\|_{L^p} \leqslant C(s, p) \|y\|_{H^s} \tag{4.16}$$

whenever $p \geqslant 2$ and $s > 1/2 - 1/p$.

4.2 Bounds on the potential energy

In this subsection, we derive some bounds involving the potential energy

$$V[u] = \int_0^L \left[\frac{1}{2}u'(x)^2 + U(u(x))\right] \mathrm{d}x \tag{4.17}$$

and its gradient. Periodic and Neumann boundary conditions can be treated in a unified way by writing the Fourier series as

$$u(x) = \sum_{k\in\mathbb{Z}} z_k \frac{\mathrm{e}^{\mathrm{i}\, bk\pi x/L}}{\sqrt{L}} , \qquad z_k = \overline{z_{-k}} , \tag{4.18}$$

where $b = 1$ and $z_k = z_{-k}$ for Neumann b.c., and $b = 2$ for periodic b.c. The value of the potential expressed in Fourier variables becomes

$$\widehat{V}(z) = V[u(\cdot)] = \frac{1}{2}\sum_{k\in\mathbb{Z}} \nu_k |z_k|^2 + \int_0^L U\left(\sum_{k\in\mathbb{Z}} z_k \frac{\mathrm{e}^{\mathrm{i}\, bk\pi x/L}}{\sqrt{L}}\right) \mathrm{d}x , \tag{4.19}$$

where $\nu_k = (b\pi k)^2/L^2$.

Lemma 4.5 (Bounds on $\widehat{V}$)**.** *There exist constants $\alpha', \beta', M_0' > 0$ such that*

$$\beta'\|z\|_{H^1}^2 - \alpha' \leqslant \widehat{V}(z) \leqslant M_0'(1 + \|z\|_{H^1}^2)^{p_0} . \tag{4.20}$$

Proof: By Assumption 2.1 (U3), we have

$$V[u] \leqslant \frac{1}{2}\|u'\|_{L^2}^2 + M_0(1 + \|u\|_{L^{2p_0}}^{2p_0}) , \tag{4.21}$$

where

$$\|u'\|_{L^2}^2 = \frac{b^2\pi^2}{L^2}\sum_{k\in\mathbb{Z}} k^2 z_k^2 \leqslant \frac{b^2\pi^2}{L^2}\|z\|_{H^1}^2 . \tag{4.22}$$

By (4.16) we have

$$\|u\|_{L^{2p_0}} \leqslant C(1, 2p_0)\|z\|_{H^1} , \tag{4.23}$$

which implies the upper bound. The lower bound is obtained in a similar way, using Assumption 2.1 (U4). □

The gradient of $\widehat{V}(z)$ and the Fréchet derivative of $V[u]$ are related by

$$\frac{\partial \widehat{V}}{\partial z_k}(z) = \nabla_{e_k} V[u] , \tag{4.24}$$

where e_k is defined in (2.7) or (2.8), respectively. Thus, by (2.23) and Parseval's identity,

$$\|\nabla\widehat{V}(z)\|_{\ell^2}^2 = \sum_{k\in\mathbb{Z}} \frac{\partial \widehat{V}}{\partial z_k}\overline{\frac{\partial \widehat{V}}{\partial z_k}}(z) = \int_0^L \left[-u''(x) + U'(u(x))\right]^2 \mathrm{d}x . \tag{4.25}$$

Lemma 4.6 (Lower bound on $\|\nabla\widehat{V}\|_{\ell^2}^2$)**.** *For any $\rho > 0$ there exists $M_1'(\rho)$ such that*

$$\|\nabla\widehat{V}(z)\|_{\ell^2}^2 \geqslant \rho\|z\|_{H^1}^2 - M_1'(\rho) . \tag{4.26}$$

PROOF: We expand the square in (4.25) and evaluate the terms separately. Using Assumptions 2.1 (U5) und (U6) and integration by parts, we have for any $\gamma > 0$

$$\begin{aligned}
\int_0^L u''(x)^2 \,\mathrm{d}x &= \sum_{k\in\mathbb{Z}} \frac{b^4\pi^4}{L^4} k^4 |z_k|^2 \;, \\
\int_0^L U'(u(x))^2 \,\mathrm{d}x &\geqslant \gamma \sum_{k\in\mathbb{Z}} |z_k|^2 - LM_1(\gamma) \;, \\
-2\int_0^L u''(x)U'(u(x)) \,\mathrm{d}x = 2\int_0^L u'(x)^2 U''(u(x)) \,\mathrm{d}x &\geqslant -2M_2 \sum_{k\in\mathbb{Z}} \frac{b^2\pi^2}{L^2} k^2 |z_k|^2 \;,
\end{aligned} \tag{4.27}$$

so that

$$\|\nabla \widehat{V}(z)\|_{\ell^2}^2 \geqslant \sum_{k\in\mathbb{Z}} \biggl[\frac{b^4\pi^4}{L^4} k^4 - 2M_2 \frac{b^2\pi^2}{L^2} k^2 + \gamma \biggr] z_k^2 - M_1(\gamma) \;. \tag{4.28}$$

For any $\rho > 0$, we can find a γ such that the term in brackets is bounded below by $2\rho(1+k^2)$, uniformly in k. This proves the result. □

Corollary 4.7. *For any $\delta > 0$, there exists $H = H(\delta)$ such that $\|\nabla \widehat{V}(z)\|_{\ell^2}^2 \geqslant \delta^2$ whenever $\widehat{V}(z) \geqslant H$. As a consequence, all stationary points of $\widehat{V}$ belong to $\{z\colon \widehat{V}(z) \leqslant H(0)\}$.*

PROOF: This immediately follows from $\|\nabla \widehat{V}(z)\|_{\ell^2}^2 \geqslant \alpha'(\sqrt{\widehat{V}(z)/M_0'} - 1) - M_1'(\alpha')$. □

4.3 Normal forms

We will rely on normal forms when analysing the system for L near a critical value. In this situation we will always assume that the local potential U is in $\mathcal{C}^5$, so that we can write its Taylor expansion as

$$U(u) = -\frac{1}{2}u^2 + \frac{a_3\sqrt{L}}{3}u^3 + \frac{a_4 L}{4}u^4 + \mathcal{O}(u^5) \;. \tag{4.29}$$

Then for small u, the potential energy admits the expansion

$$V[u] = \frac{1}{2}\|u'\|_{L^2}^2 - \frac{1}{2}\|u\|_{L^2}^2 + \frac{a_3\sqrt{L}}{3}\int_0^L u(x)^3 \,\mathrm{d}x + \frac{a_4 L}{4}\|u\|_{L^4}^4 + \mathcal{O}(\|u\|_{L^5}^5) \;. \tag{4.30}$$

Equation (4.19) shows that the potential energy in Fourier variables can be decomposed as

$$\widehat{V}(z) = \widehat{V}_2(z) + \widehat{V}_3(z) + \widehat{V}_4(z) + R(z) \;, \tag{4.31}$$

where the $\widehat{V}_n(z)$ are given by the convolutions

$$\begin{aligned}
\widehat{V}_2(z) &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k z_k z_{-k} \;, \\
\widehat{V}_3(z) &= \frac{a_3}{3} \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z} \\ k_1+k_2+k_3=0}} z_{k_1} z_{k_2} z_{k_3} \;, \\
\widehat{V}_4(z) &= \frac{a_4}{4} \sum_{\substack{k_1,k_2,k_3,k_4\in\mathbb{Z} \\ k_1+k_2+k_3+k_4=0}} z_{k_1} z_{k_2} z_{k_3} z_{k_4} \;,
\end{aligned} \tag{4.32}$$

where $\lambda_k = \nu_k - 1$ and the remainder satisfies $R(z) = \mathcal{O}(\|z\|_{H^s}^5)$ for all $s > 3/10$.

Proposition 4.8. *Let L be such that λ_k is bounded away from 0 for all $k \neq \pm 1$. Then there exists a map $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ such that*

$$\widehat{V}(z + g(z)) = \widehat{V}_2(z) + C_4 z_1^2 z_{-1}^2 + R_1(z) \,, \tag{4.33}$$

where

$$C_4 = \frac{3}{2} a_4 + 2a_3^2 \left(\frac{1}{|\lambda_0|} - \frac{1}{2\lambda_2} \right) , \tag{4.34}$$

and the remainder satisfies

$$R_1(z) = \mathcal{O}(\|z\|_{H^s}^5) \tag{4.35}$$

for all $5/12 < s < 1/2$. Furthermore, $\|g(z)\|_{H^t} = \mathcal{O}(\|z\|_{H^s}^2)$ for $t < 2s - 1/2$ and the Jacobian of the transformation $z \mapsto z + g(z)$ satisfies

$$\det(\mathbb{1} + \partial_z g(z)) = 1 + \mathcal{O}(a_3 \|z\|_{H^s}) + \mathcal{O}(a_4 \|z\|_{H^s}^2) \tag{4.36}$$

on the set $\{z_k = z_{-k}\}$.

Proof: In the course of the proof, we will need Sobolev norms with indices q, r, t, satisfying the relations

$$0 < q, r < t < s < 1/2 \,, \quad t < 2s - 1/2 \,, \quad r < 2t - 1/2 \quad \text{and} \quad q < 3t - 1 \,.$$

This is always possible for $5/12 < s < 1/2$.

1. Let $g^{(2)} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ be homogeneous of degree 2, and satisfy $g^{(2)}_{-k}(z) = g^{(2)}_k(z)$. Then, expanding and grouping terms of equal order we get

$$\begin{aligned} \widehat{V}(z + g^{(2)}(z)) = \widehat{V}_2(z) + \sum_k \lambda_k z_k g^{(2)}_{-k}(z) + \frac{1}{2} \sum_k \lambda_k g^{(2)}_k(z) g^{(2)}_{-k}(z) & \\ + \widehat{V}_3(z) \qquad + \sum_k \frac{\partial \widehat{V}_3}{\partial z_k}(z) g^{(2)}_k(z) \qquad + r_1(z) & \\ + \widehat{V}_4(z) \qquad + r_2(z) & \\ + R(z + g^{(2)}(z)) \,, & \end{aligned} \tag{4.37}$$

with remainders that can be written as

$$\begin{aligned} r_1(z) &= \frac{1}{2} \sum_{k,l} \frac{\partial^2 \widehat{V}_3}{\partial z_k \partial z_l} \big(z + \theta_1 g^{(2)}(z)\big) g^{(2)}_k(z) g^{(2)}_l(z) \\ r_2(z) &= \sum_k \frac{\partial \widehat{V}_4}{\partial z_k} \big(z + \theta_2 g^{(2)}(z)\big) g^{(2)}_k(z) \end{aligned} \tag{4.38}$$

for some $\theta_1, \theta_2 \in [0, 1]$.
We want to choose $g^{(2)}(z)$ in such a way that the terms of order 3 in $\widehat{V}(z + g^{(2)}(z))$ cancel. This can be achieved by taking

$$\lambda_k g^{(2)}_k(z) =: \tilde{g}^{(2)}_k(z) = \begin{cases} 0 & \text{if } |k| = 1 \,, \\ -\dfrac{a_3}{3} \displaystyle\sum_{k_1 + k_2 = k} b_{k_1, k_2} z_{k_1} z_{k_2} & \text{otherwise} \,, \end{cases} \tag{4.39}$$

for appropriate coefficients b_{k_1,k_2} satisfying $b_{k_1,k_2} = b_{-k_1,-k_2}$. The choice of these coefficients is not unique, but we can make it unique by imposing the symmetry conditions

$$b_{k,l} = b_{l,k} = b_{l,-k-l} = b_{-k-l,l} = b_{k,-k-l} = b_{-k-l,k} \; . \tag{4.40}$$

Indeed, these are all the terms contributing to the monomial $z_k z_l z_{-k-l}$ in the first sum in (4.37). Then simple combinatorics show that all $b_{k,l}$ belong to the interval $[1/6, 6]$. This choice has the further advantage that on the set $\{z_k = z_{-k}\}$,

$$\frac{\partial \tilde{g}_k^{(2)}}{\partial z_l}(z) = -\frac{2a_3}{3} b_{l,k-l} z_{k-l} = -\frac{2a_3}{3} b_{k,l-k} z_{l-k} = \frac{\partial \tilde{g}_l^{(2)}}{\partial z_k}(z) \tag{4.41}$$

whenever $|k|, |l| \neq 1$.
The term of order 4 of $\widehat{V}(z + g^{(2)}(z))$ is given by

$$\widetilde{V}_4(z) = \widehat{V}_4(z) + \frac{1}{2} \sum_k \lambda_k g_k^{(2)}(z) g_{-k}^{(2)}(z) + \sum_k \frac{\partial \widehat{V}_3}{\partial z_k}(z) g_k^{(2)}(z) \; . \tag{4.42}$$

Note that the convolution structure is preserved. In order to show that the sums indeed converge, we first note that since $t < 2s - 1/2$, we have

$$\|\tilde{g}^{(2)}(z)\|_{H^t} \leqslant 2|a_3| \|z * z\|_{H^t} \leqslant 2|a_3| \|z\|_{H^s}^2 \tag{4.43}$$

by Lemma 4.3. Since $|\lambda_k|^{-1} \leqslant 1 \leqslant (1 + k^2)^t \; \forall k \neq \pm 1$, the first sum in (4.42) can be bounded by

$$\sum_k \frac{\tilde{g}_k^{(2)}(z)^2}{|\lambda_k|} \leqslant \|\tilde{g}^{(2)}(z)\|_{H^t}^2 \leqslant 4a_3^2 \|z\|_{H^s}^4 \; . \tag{4.44}$$

The second sum in (4.42) can be bounded as follows:

$$\begin{aligned}
\left| \sum_k \frac{\partial \widehat{V}_3}{\partial z_k}(z) g_k^{(2)}(z) \right| &\leqslant \sum_k \left[\frac{|a_3|}{3} \sum_{k_1 + k_2 = -k} 3 |z_{k_1}| |z_{k_2}| \right] \left| g_k^{(2)}(z) \right| \\
&= |a_3| \Big[|z| * |z| * \left| g^{(2)}(z) \right| \Big]_0 \\
&\leqslant |a_3| \Big\| |z| * |z| * \left| g^{(2)}(z) \right| \Big\|_{H^r} \\
&\leqslant |a_3| \|z\|_{H^s}^2 \left\| g^{(2)}(z) \right\|_{H^t} \\
&\leqslant 4a_3^2 \|z\|_{H^s}^4 \; .
\end{aligned} \tag{4.45}$$

This shows that $\widetilde{V}_4(z)$ indeed exists, and satisfies

$$\left| \widetilde{V}_4(z) \right| \leqslant \left(\frac{a_4}{4} + 6a_3^2 \right) \|z\|_{H^s}^4 \; . \tag{4.46}$$

Next we estimate the remainders. The remainder $r_1(z)$ can be bounded as follows:

$$\begin{aligned}
|r_1(z)| &\leqslant \frac{1}{2}\sum_{k,l} 6\frac{|a_3|}{3}\big|z_{-k-l}+\theta_1 g^{(2)}_{-k-l}(z)\big|\big|g^{(2)}_k(z)\big|\big|g^{(2)}_l(z)\big| \\
&\leqslant |a_3|\Big[\big|z+\theta_1 g^{(2)}(z)\big| * \big|g^{(2)}(z)\big| * \big|g^{(2)}(z)\big|\Big]_0 \\
&\leqslant |a_3|\Big\|\big|z+\theta_1 g^{(2)}(z)\big| * \big|g^{(2)}(z)\big| * \big|g^{(2)}(z)\big|\Big\|_{H^q} \\
&\leqslant |a_3|\big\|z+\theta_1 g^{(2)}(z)\big\|_{H^t}\big\|g^{(2)}(z)\big\|^2_{H^t} \\
&\leqslant 4|a_3|^3\|z\|^5_{H^s}\Big[1+2|a_3|\|z\|_{H^s}\Big] \,. \qquad (4.47)
\end{aligned}$$

A similar computation yields

$$|r_2(z)| \leqslant \frac{3}{2}|a_3 a_4|\|z\|^5_{H^s}\Big[1+8|a_3|^3\|z\|^3_{H^s}\Big] \,. \qquad (4.48)$$

Finally, clearly

$$\big|R(z+g^{(2)}(z))\big| = \mathcal{O}(\|z\|^5_{H^s}) \,. \qquad (4.49)$$

We have thus obtained

$$\widehat{V}(z+g^{(2)}(z)) =: \widetilde{V}(z) = V_2(z) + \widetilde{V}_4(z) + \mathcal{O}(\|z\|^5_{H^s}) \,. \qquad (4.50)$$

2. The Jacobian matrix of $z \mapsto z+g^{(2)}(z)$ is given by $\mathbb{1}+A(z)$, where the elements of $A(z) = \partial_z g^{(2)}(z)$ can be deduced from (4.41). By construction, $A(z)$ is self-adjoint with respect to the scalar product weighted by the λ_k, and thus has real eigenvalues. Its ℓ^1-operator norm satisfies

$$\|A(z)\|_{\ell^1} = \max_l \sum_k \left|\frac{\partial g^{(2)}_k(y)}{\partial y_l}\right| \leqslant 4|a_3| \max_l \sum_k \frac{|z_{k-l}|}{|\lambda_k|} \leqslant const\,\|z\|_{\ell^\infty} \leqslant const\,\|z\|_{H^s} \,. \qquad (4.51)$$

Hence the spectral radius $\rho(z)$ of $A(z)$ has order $\|z\|_{H^s}$. Now if $\rho(z) < 1$ and we denote the eigenvalues of $A(z)$ by $a_k(z)$,

$$\log\det(\mathbb{1}+A(z)) = \sum_k \log(1+a_k(z)) \leqslant \sum_k a_k(z) = \operatorname{Tr} A(z) \,. \qquad (4.52)$$

It follows that $|\det(\mathbb{1}+A(z))| \leqslant \mathrm{e}^{\operatorname{Tr}(A(z))}$, and one easily shows that $\operatorname{Tr}(A(z)) = \mathcal{O}(a_3|z_0|) \leqslant \mathcal{O}(a_3\|z\|_{H^s})$. A matching lower bound can be obtained in a similar way. This proves that the Jacobian of the transformation $z \mapsto z+g^{(2)}(z)$ is $1+\mathcal{O}(a_3\|z\|_{H^s})$.

3. Let $g^{(3)} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ be homogeneous of degree 3, and satisfy $g^{(3)}_{-k}(z) = g^{(3)}_k(z)$. We choose it of the form

$$\lambda_k g^{(3)}_k(z) =: \tilde{g}^{(3)}_k(z) = \begin{cases} 0 & \text{if } |k| = 1 \,, \\ \displaystyle\sum_{k_1+k_2+k_3=k} b_{k_1,k_2,k_3} z_{k_1} z_{k_2} z_{k_3} & \text{otherwise} \,, \end{cases} \qquad (4.53)$$

where the coefficients b_{k_1,k_2,k_3} are invariant under permutations of k_1, k_2 and k_3. In addition, we require invariance under sign change and

$$b_{k_1,k_2,k_3} = b_{k_1,k_2,-k_1-k_2-k_3} \,. \qquad (4.54)$$

This guarantees in particular that

$$\frac{\partial \tilde{g}_k^{(3)}}{\partial z_l}(z) = \frac{\partial \tilde{g}_l^{(3)}}{\partial z_k}(z) \tag{4.55}$$

holds on the set $\{z_k = z_{-k}\}$. The coefficients b_{k_1,k_2,k_3} can now be chosen in such a way that

$$\sum_k z_k \tilde{g}_k^{(3)}(z) + \tilde{V}_4(z) \tag{4.56}$$

contains only one term, proportional to $z_{-1}^2 z_1^2$, which cannot be eliminated because $\tilde{g}_1^{(3)}(z) = \tilde{g}_{-1}^{(3)}(z) = 0$. It follows that

$$\widetilde{V}(z + g^{(3)}(z)) = \widehat{V}_2(z) + C_4 z_{-1}^2 z_1^2 + R_1(z) \tag{4.57}$$

for some constant C_4. Along the lines of the above calculations, one checks that $R_1(z) = \mathcal{O}(\|z\|_{H^s}^5)$, and that the Jacobian of the transformation $z \mapsto z + g^{(3)}(z)$ is

$$\det\bigl(\mathbb{1} + \partial_z g^{(3)}(z)\bigr) = 1 + \mathcal{O}\bigl((a_4 + a_3^2)\|z\|_{H^s}^2\bigr) \,. \tag{4.58}$$

This proves (4.35) and (4.36).

4. It remains to compute the coefficient C_4 of the resonant term. To do this, it is sufficient to compute the terms containing $z_{\pm 1}$ of $\tilde{g}_0^{(2)}(z)$ and $\tilde{g}_{\pm 2}^{(2)}(z)$, which are the only ones contributing to the resonant term. One finds

$$\begin{aligned}
\tilde{g}_0^{(2)}(\dots, 0, z_{-1}, 0, z_1, 0, \dots) &= -2a_3 z_1 z_{-1} \,, \\
\tilde{g}_2^{(2)}(\dots, 0, z_{-1}, 0, z_1, 0, \dots) &= -a_3 z_{-1}^2 \,, \\
\tilde{g}_{-2}^{(2)}(\dots, 0, z_{-1}, 0, z_1, 0, \dots) &= -a_3 z_1^2 \,,
\end{aligned} \tag{4.59}$$

and substituting in (4.42) yields the result. □

This result has important consequences for the behaviour of the potential near bifurcation points. In the case of Neumann b.c., $\lambda_0 = -1$ and $\lambda_2 = (4\pi^2/L^2) - 1$. Thus the coefficient C_4 of the term $z_1^2 z_{-1}^2$ is given by

$$C_4(L) = \frac{3}{2}a_4 + \frac{8\pi^2 - 3L^2}{4\pi^2 - L^2}a_3^2 = \frac{1}{4L}\Bigl[U^{(4)}(0) + \frac{8\pi^2 - 3L^2}{4\pi^2 - L^2}U'''(0)^2\Bigr] \,. \tag{4.60}$$

In particular, at the bifurcation point we have

$$C_4(\pi) = \frac{3}{2}a_4 + \frac{5}{3}a_3^2 = \frac{1}{4L}\Bigl[U^{(4)}(0) + \frac{5}{3}U'''(0)^2\Bigr] \,. \tag{4.61}$$

The expression (4.33) for the normal form shows that if $C_4(\pi) > 0$, the system undergoes a subcritical pitchfork bifurcation at $L = \pi$. This means that the origin is an isolated stationary point if $L < \pi$, while for $L > \pi$ two new stationary points appear at a distance of order $\sqrt{L - \pi}$ from the origin. They correspond to the functions we denoted $u_{1,\pm}^*$. As a consequence, the period $T(E)$ defined in (2.16) must grow for small positive E, to be compatible with the existence of nonconstant stationary solutions for $L > \pi$. An analysis of the Hessian matrices of $\widehat{V}$ at $u_{1,\pm}^*$ shows that they have one negative eigenvalue for L slightly larger than π. This must remain true for all $L > \pi$ because we know that the stationary solutions $u_{1,\pm}^*$ remain isolated when L grows.

In the case of periodic b.c., $\lambda_0 = -1$ and $\lambda_2 = (16\pi^2/L^2) - 1$. Thus the coefficient C_4 of the term $z_1^2 z_{-1}^2$ is given by

$$C_4(L) = \frac{3}{2}a_4 + \frac{32\pi^2 - 3L^2}{16\pi^2 - L^2}a_3^2 = \frac{1}{4L}\Big[U^{(4)}(0) + \frac{32\pi^2 - 3L^2}{16\pi^2 - L^2}U'''(0)^2\Big] \,. \tag{4.62}$$

The value $C_4(2\pi)$ at the bifurcation point is equal to the value (4.61) of $C_4(\pi)$ for Neumann b.c. Thus the condition on the bifurcation being supercritical is exactly the same as before. The difference is that instead of being equal, z_1 and z_{-1} are only complex conjugate, and thus the centre manifold at the bifurcation point is two-dimensional. The invariance of the potential under translations $u(x) \mapsto u(x+\varphi)$ for any $\varphi \in \mathbb{R}$ implies that $\widehat{V}(z)$ is invariant under $z_k \mapsto \mathrm{e}^{\mathrm{i}\, k\varphi 2\pi/L}\, z_k$. This and the expression (4.33) for the normal form show that for $L > 2\pi$, there is a closed curve of stationary solutions at distance of order $\sqrt{L - 2\pi}$ from the origin. It corresponds to the family of solutions we denoted $u^*_{1,\varphi}$. An analysis of the Hessian of $\widehat{V}$ at any $u^*_{1,\varphi}$ shows that it has one negative and one vanishing eigenvalue (due to translation symmetry).

Finally note that a similar normal-form analysis can be made for the other bifurcations, at subsequent multiples of π or 2π. We do not detail this analysis, since only saddles with one negative eigenvalue are important for metastable transition times.

4.4 The truncated potential

Let $\widehat{V}^{(d)}$ be the restriction of the potential $\widehat{V}$ to the subspace of Fourier modes z_k such that $|k| \leqslant d$. For given d, let us write $z = (v, w)$, where v is the vector of Fourier components with $|k| \leqslant d$ and w contains the vector of remaining components. Then

$$\widehat{V}^{(d)}(v) = \widehat{V}(v, 0) \,. \tag{4.63}$$

Proposition 4.9. *There exists $d_0 < \infty$ such that for $d \geqslant d_0$, the potentials $\widehat{V}^{(d)}$ and $\widehat{V}$ have the same number of nondegenerate critical points, and with the same number of negative eigenvalues.*

Proof: A critical point (v^*, w^*) of $\widehat{V}$ has to satisfy the conditions

$$\partial_v \widehat{V}(v^*, w^*) = 0 \,, \qquad \partial_w \widehat{V}(v^*, w^*) = 0 \,, \tag{4.64}$$

while a critical points v_* of $\widehat{V}^{(d)}$ has to satisfy

$$\partial_v \widehat{V}(v_*, 0) = 0 \,. \tag{4.65}$$

Lemma 4.6 implies that all critical points of $\widehat{V}$ have an H^1-norm bounded by some constant M. Let us prove that

$$\|\partial_w \widehat{V}(v, 0)\|^2_{\ell^2} = \mathcal{O}(d^{-1}) \tag{4.66}$$

for $\|v\|_{H^1} \leqslant M$. Indeed it follows from (4.19) that

$$\frac{\partial \widehat{V}}{\partial w_k}(v, 0) = \int_0^L U'(u(x))\, \mathrm{e}^{\mathrm{i}\, bk\pi x/L}\, \mathrm{d}x \,, \tag{4.67}$$

where $u(x) = \sum_{|\ell| \leqslant d} v_\ell\, \mathrm{e}^{\mathrm{i}\, b\ell\pi x/L}$. Since $\|v\|_{H^1} \leqslant M$ and U is at least continuously differentiable, the Fourier components of $U'(u(x))$ decay like k^{-1} at least, which implies (4.66).

Let (v^*, w^*) be a critical point of $\widehat{V}$ and consider the function

$$F(\xi, w) = \partial_v \widehat{V}(v^* + \xi, w) \; . \tag{4.68}$$

Then $F(0, w^*) = 0$ and $\partial_\xi F(0, w^*) = \partial_{vv} \widehat{V}(v^*, w^*)$. Thus if (v^*, w^*) is nondegenerate, the implicit function theorem implies that in a neighbourhood of $w = w^*$, there exists a continuously differentiable function h with $h(w^*) = 0$ and such that all solutions of $F(\xi, w) = 0$ in a neighbourhood of $(0, w^*)$ are given by $\xi = h(w)$. In particular, choosing d large enough, we can assume that h is defined for $w = 0$, and we get

$$0 = F(h(0), 0) = \partial_v \widehat{V}(v^* + h(0), 0) \; . \tag{4.69}$$

This shows that $v_* = v^* + h(0)$ is a stationary point of $\widehat{V}^{(d)}$, which is unique in the neighbourhood of (v^*, w^*).

Conversely, let v_* be a stationary point of $\widehat{V}^{(d)}$. The same implicit-function-theorem argument shows that if v_* is nondegenerate, then there exists a continuously differentiable function h, with $h(0) = 0$, such that all solutions of $\partial_v \widehat{V}(v, w) = 0$ near $(v_*, 0)$ satisfy $v = v_* + h(w)$. Now let us consider the function

$$g(w) = \partial_w \widehat{V}(v_* + h(w), w) \; . \tag{4.70}$$

Then $g(0) = \partial_w \widehat{V}(v_*, 0)$ has an ℓ^2-norm of order $d^{-1/2}$ by (4.66). Furthermore,

$$\partial_w g(w) = \partial_{ww} \widehat{V}(v_* + h(w), w) + \partial_{wv} \widehat{V}(v_* + h(w), w) \partial_w h(w) \; . \tag{4.71}$$

The first matrix on the right-hand side has eigenvalues of order d^2, while the second one is small as a consequence of (4.66). Thus $\partial_w g$ is invertible near $w = 0$ for sufficiently large d, and the local inversion theorem shows that $g(w)$ has an isolated zero at a point w^* near $w = 0$. This yields the existence of a unique stationary point $(v^* = h(w^*), w^*)$ of $\widehat{V}$ in the vicinity of $(v_*, 0)$. □

5 A priori estimates

This section has two major aims:

- Show that the first-hitting time of a given set B admits a second moment, bounded uniformly in the dimension d;
- Derive a priori bounds on the equilibrium potential $h_{A,B}(x) = \mathbb{P}^x\{\tau_A < \tau_B\}$.

We start in Section 5.1 by recalling some general bounds involving sup and Hölder norms of solutions of the SPDE (2.1). In order to estimate moments of first-hitting times, the space being unbounded, we repeatedly need the Markov property to restart the process when it hits certain sets. This is most efficiently done using Laplace transforms, and we prove some useful inequalities in Section 5.2. Section 5.3 recalls some large-deviation results. Sections 5.4 and 5.5 contain the main estimates on moments, respectively, for the infinite-dimensional system and for its Galerkin approximation. Finally, Section 5.6 contains the estimates of the equilibrium potential.

5.1 A priori bounds on solutions of the SPDE

The solution of the heat equation $\partial_t u = \Delta u$ with initial condition $u_0 \in L^2(\mathbb{T}^1)$ can be written

$$u_t = \mathrm{e}^{\Delta t} \, u_0 \; , \tag{5.1}$$

where $\mathrm{e}^{\Delta t}$ stands for convolution with the heat kernel

$$G_t(x,y) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-k^2 t}\, e_k(x) e_k(y) 1_{\{t>0\}} \,. \tag{5.2}$$

Here the e_k are the eigenfunctions of the Laplacian, defined in (4.2).

Lemma 5.1 (Smoothing effect of the heat semigroup)**.** *For any $s \geqslant 0$, there is a finite constant $C(s)$ such that for all $u_0 \in L^2(\mathbb{T}^1)$*

$$\|\mathrm{e}^{\Delta t}\, u_0\|_{H^s} \leqslant (1 + C(s) t^{-s/2}) \|u_0\|_{L^2} \qquad \forall t > 0 \,. \tag{5.3}$$

Proof: We have

$$\mathrm{e}^{\Delta t}\, u_0 = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-k^2 t}\, y_k(0) e_k \,. \tag{5.4}$$

The result follows by computing the H^s-norm, and using the fact that $(2xt)^s\, \mathrm{e}^{-2xt}$ is bounded by a constant, depending only on s. □

Note that by Lemma 4.1, this implies

$$\|\mathrm{e}^{\Delta t}\, u_0\|_{\mathcal{C}^\alpha} \leqslant C(1 + t^{-s/2}) \|u_0\|_{L^2} \leqslant C(1 + t^{-s/2}) \|u_0\|_{L^\infty} \quad \forall t > 0 \,,\ \forall s > \alpha + \frac{1}{2} \,, \tag{5.5}$$

where the constant C depends only on α and s.

Consider now the stochastic convolution

$$W_\Delta(t) = \int_0^t \mathrm{e}^{\Delta(t-s)}\, \mathrm{d}W(s) \,, \tag{5.6}$$

where $W(t)$ is a cylindrical Wiener process on $L^2(\mathbb{T}^1)$. It is known that

$$W_\Delta(t) \in H^s(\mathbb{T}^1) \qquad \text{and} \qquad W_\Delta(t) \in \mathcal{C}^\alpha(\mathbb{T}^1) \tag{5.7}$$

almost surely, for all $t > 0$ and all $s < 1/2$ and $\alpha < 1/2$ (see e.g. [Hai09, p. 50]).

We will need to control the sup and Hölder norms of the rescaled process $\sqrt{2\varepsilon} W_\Delta(t)$.

Proposition 5.2 (Large-deviation estimate for W_Δ)**.** *For any $\alpha \in [0, 1/2)$ and $T > 0$, there exists a constant $\kappa > 0$ such that for all $H, \eta > 0$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*

$$\mathbb{P}\left\{ \sup_{0\leqslant t\leqslant T} \left\| \sqrt{2\varepsilon}\, W_\Delta(t) \right\|_{\mathcal{C}^\alpha} > H \right\} \leqslant \mathrm{e}^{-(\kappa H^2 - \eta)/2\varepsilon} \,. \tag{5.8}$$

Proof: Let $\mathcal{H}$ denote the Cameron–Martin space of the cylindrical Wiener process, defined by

$$\mathcal{H} = \left\{ \varphi \colon \varphi_t(x) = \int_0^t \int_0^x \dot\varphi(u,z)\, \mathrm{d}u\, \mathrm{d}z \,,\ \dot\varphi \in L^2([0,T] \times \mathbb{T}^1) \right\} . \tag{5.9}$$

Schilder's theorem for Gaussian fields shows that the family $\{\sqrt{2\varepsilon}\, W\}_{\varepsilon>0}$ satisfies a large-deviation principle with good rate function

$$I_0(\varphi) = \begin{cases} \frac{1}{2} \|\dot\varphi\|^2_{L^2([0,T]\times\mathbb{T}^1)} & \text{if } \varphi \in \mathcal{H} \,, \\ +\infty & \text{otherwise} \,. \end{cases} \tag{5.10}$$

Define a map $Z : \mathcal{H} \to L^2([0,T] \times \mathbb{T}^1)$ by

$$\varphi \mapsto Z[\varphi] \;, \quad Z[\varphi]_t = \int_0^t \mathrm{e}^{\Delta(t-s)}\, \dot\varphi_s \, \mathrm{d}s \;. \tag{5.11}$$

From the large-deviation principle for parabolic SPDEs established in [FJL82, CM97], it follows in particular, that the family $\{\sqrt{2\varepsilon}W_\Delta\}_{\varepsilon>0}$ satisfies a large-deviation principle with good rate function

$$I(\psi) = \begin{cases} \inf\Big\{ I_0(\varphi) \colon Z[\varphi] = \psi \Big\} & \text{if } \psi \in \mathrm{Im}(Z) \;, \\ +\infty & \text{otherwise} \;. \end{cases} \tag{5.12}$$

Now observe that if $\psi = Z[\varphi]$, for any $T_1 \in [0,T]$ and any $s \in [0,1)$ one has by Lemma 5.1

$$\begin{aligned} \|\psi_{T_1}\|_{H^s(\mathbb{T}^1)} &\leqslant \int_0^{T_1} \|\mathrm{e}^{\Delta(T_1-t)}\, \dot\varphi_t\|_{H^s(\mathbb{T}^1)} \, \mathrm{d}t \\ &\leqslant \int_0^{T_1} \biggl(1 + \frac{C(s)}{(T_1-t)^{s/2}}\biggr) \|\dot\varphi_t\|_{L^2(\mathbb{T}^1)} \, \mathrm{d}t \\ &\leqslant \biggl(\int_0^{T_1} \biggl(1 + \frac{C(s)}{(T_1-t)^{s/2}}\biggr)^2 \mathrm{d}t \biggr)^{1/2} \bigl(2I_0(\varphi)\bigr)^{1/2} \;. \end{aligned} \tag{5.13}$$

Since $s < 1$, the integral is finite (and increasing in T_1). Together with Lemma 4.1, this proves that

$$I(\psi) \geqslant \frac{1}{C_1(T_1,\alpha)} \|\psi_{T_1}\|^2_{\mathcal{C}^\alpha(\mathbb{T}^1)} \tag{5.14}$$

for all $T_1 \in [0,T]$ and $0 \leqslant \alpha < 1/2$, where C_1 is increasing in T_1. By a standard application of the large-deviation principle (see e.g. [FJL82, CM97])

$$\limsup_{\varepsilon\to 0} 2\varepsilon \log \mathbb{P}\biggl\{ \sup_{0\leqslant t\leqslant T} \bigl\|\sqrt{2\varepsilon}W_\Delta(t)\bigr\|_{\mathcal{C}^\alpha} > H \biggr\} \leqslant -\inf\bigl\{ I(\psi) \colon \exists T_1 \in [0,T] : \|\psi_{T_1}\|_{\mathcal{C}^\alpha} > H \bigr\} \;. \tag{5.15}$$

The bound (5.14) implies that the right-hand side is bounded above by $-H^2/C_1(T,\alpha)$, which concludes the proof. □

We now turn to properties of mild solutions of the full nonlinear SPDE, given by

$$u_t = \mathrm{e}^{\Delta t}\, u_0 + \sqrt{2\varepsilon}\, W_\Delta(t) + \int_0^t \mathrm{e}^{\Delta(t-s)}\, U'(u_s) \, \mathrm{d}s \;. \tag{5.16}$$

Results in [Cer96, Cer99] provide estimates on the sup norm of u_t:

Proposition 5.3 (Uniform bounds on the sup norm)**.** *For any $u_0 \in \mathcal{C}^0(\mathbb{T}^1)$ and any $T > 0$, there exists a unique mild solution on $[0,T]$ such that $\mathbb{E}\{\sup_{t\in[0,T]} \|u_t\|^2_{L^\infty}\} < \infty$. Furthermore, there is a constant c depending only on U' such that the following bounds hold:*

1. *There exists $\gamma > 0$ such that for any u_0 and any $t \geqslant 0$,*

$$\|u_t\|_{L^\infty} \leqslant \mathrm{e}^{\gamma t}\|u_0\|_{L^\infty} + \sqrt{2\varepsilon} \sup_{0\leqslant s\leqslant t} \|W_\Delta(s)\|_{L^\infty} + c\,\mathrm{e}^{\gamma t} \int_0^t \Bigl(1 + (2\varepsilon)^{3/2} \|W_\Delta(s)\|^3_{L^\infty}\Bigr) \, \mathrm{d}s \;, \tag{5.17}$$

2. For any $t > 0$,

$$\sup_{u_0 \in \mathcal{C}^0(\mathbb{T}^1)} \|u_t\|_{L^\infty} \leqslant c\biggl(1 + \sqrt{2\varepsilon} \sup_{0 \leqslant s \leqslant t} \|W_\Delta(s)\|_{L^\infty}\biggr) \frac{1}{\sqrt{t}} + \sqrt{2\varepsilon}\, \|W_\Delta(t)\|_{L^\infty} \,. \tag{5.18}$$

PROOF: Existence and uniqueness of the mild solution are proved in [DPZ92, Theorem 7.13]. The estimate (5.17) is Proposition 3.2 of [Cer99], with $m = 1$, while the uniform estimate (5.18) is Proposition 3.4 of [Cer99], c.f. also [Cer96, Lemma 3.4]. □

Observe that in the case $\varepsilon = 0$, we can find a constant M *uniform* in t such that $\|u_t\|_{L^\infty} \leqslant M(1 + \|u_0\|_{L^\infty})$ for all $t \geqslant 0$. Hence Proposition 5.2 shows that for all $H_1 > 0$,

$$\mathbb{P}^{u_0}\biggl\{ \sup_{0 \leqslant t \leqslant T} \|u_t\|_{L^\infty} \geqslant M(1 + \|u_0\|_{L^\infty}) + H_1 \biggr\} \leqslant \mathrm{e}^{-\kappa(T) f(H_1)/2\varepsilon} \,, \tag{5.19}$$

where

$$f(H_1) = \min\{H_1^2, H_1^{2/3}\} \,, \tag{5.20}$$

for some $\kappa(T) > 0$ and ε small enough.

Combining (5.16) and Proposition 5.3, we obtain the following estimate on the Hölder norm of u_T at a given time $T > 0$.

Proposition 5.4 (Bound on the Hölder norm)**.** *For any $T > 0$ and $0 < \alpha < 1/2$, there exist constants $\kappa_1(T, \alpha), \kappa_2(T, \alpha) > 0, c(\alpha) > 0$ such that*

$$\mathbb{P}^{u_0}\bigl\{\|u_T\|_{\mathcal{C}^\alpha} > H\bigr\} \leqslant \exp\biggl\{ -\frac{\kappa_1}{2\varepsilon} \min\Bigl\{ H^2, f\Bigl((\kappa_2 H - 1)^{1/3} - M(1 + \|u_0\|_{L^\infty}) \Bigr) \Bigr\} \biggr\} \tag{5.21}$$

for all $u_0 \in L^\infty$ and $H > c(\alpha)(1 + T^{-s/2})\|u_0\|_{L^\infty}$ such that $(\kappa_2 H - 1)^{1/3} - M(1 + \|u_0\|_{L^\infty}) > 0$, and all $\varepsilon < \varepsilon_0(\alpha, T, H)$.

PROOF: Denote by $u_t^{(0)}$, $u_t^{(1)}$ and $u_t^{(2)}$ the three summands on the right-hand side of (5.16). Then the probability (5.21) can be bounded by $\sum_{i=0}^2 P_i$, where $P_i = \mathbb{P}^{u_0}\{\|u_T^{(i)}\|_{\mathcal{C}^\alpha} > H/3\}$.

Pick $s > \alpha + 1/2$. Then Lemma 4.1 and Lemma 5.1 show that there exists $C_1(\alpha, s)$ such that $P_0 = 0$, provided we choose $H/3 > C_1(1 + T^{-s/2})\|u_0\|_{L^\infty}$. Furthermore, Proposition 5.2 provides a bound on P_1 of order $\mathrm{e}^{-\kappa_1 H^2/2\varepsilon}$.

As for P_2, it can be bounded as follows. Since $|U'(u)| \leqslant M_0(1 + |u|^3)$ for some constant M_0, we have by (5.5)

$$\|u_T^{(2)}\|_{\mathcal{C}^\alpha} \leqslant \int_0^T \|\mathrm{e}^{\Delta(T-t)}\, U'(u_t)\|_{\mathcal{C}^\alpha}\, \mathrm{d}t \leqslant \int_0^T C(\alpha, s)(1 + (T-t)^{-s/2})\, \mathrm{d}t\, M_0\Bigl(1 + \sup_{t \in [0,T]} \|u_t\|_{L^\infty}^3\Bigr) \,. \tag{5.22}$$

The integral is bounded provided $s < 2$. The result then follows by using (5.19). □

5.2 Laplace transforms

Let $(E, \|\cdot\|)$ be a Banach space, and let $(x_t)_{t \geqslant 0}$ be an E-valued Markov process with continuous sample paths. All subsets of E considered below are assumed to be measurable with respect to the Borel σ-algebra on E.

Recall that the Laplace transform of an almost surely finite positive random variable τ is given by

$$\mathbb{E}\bigl\{\mathrm{e}^{\lambda\tau}\bigr\} = 1 + \int_0^\infty \lambda\, \mathrm{e}^{\lambda t}\, \mathbb{P}\bigl\{\tau > t\bigr\}\, \mathrm{d}t \tag{5.23}$$

for any $\lambda \in \mathbb{C}$. There exists a $c \in [0,\infty]$ such that the Laplace transform is analytic in λ for $\operatorname{Re}\lambda < c$.

To control first-hitting times of bounded sets $B \subset E$, we will introduce an auxiliary set C with bounded complement, $B \cap C = \emptyset$, such that the process is unlikely to hit C before B. On the rare occasions the process does hit C before B, we will use the strong Markov property to restart the process on the boundary ∂C. The following proposition recalls how the restart procedure is encoded in Laplace transforms.

Proposition 5.5 (Effect of restart on Laplace transform)**.** *Let $B, C \subset E$ be disjoint sets, and let $x \notin B \cup C$. Then*

$$\begin{aligned}\mathbb{E}^x\big\{\mathrm{e}^{\lambda\tau_B}\big\} &= \mathbb{E}^x\big\{\mathrm{e}^{\lambda\tau_{B\cup C}}\big\} + \mathbb{E}^x\big\{\mathrm{e}^{\lambda\tau_{B\cup C}}\,1_{\{\tau_C<\tau_B\}}\big[\mathbb{E}^{x_{\tau_C}}\big\{\mathrm{e}^{\lambda\tau_B}\big\}-1\big]\big\} &(5.24)\\ &= \mathbb{E}^x\big\{\mathrm{e}^{\lambda\tau_{B\cup C}}\,1_{\{\tau_B<\tau_C\}}\big\} + \mathbb{E}^x\big\{\mathrm{e}^{\lambda\tau_{B\cup C}}\,1_{\{\tau_C<\tau_B\}}\mathbb{E}^{x_{\tau_C}}\big\{\mathrm{e}^{\lambda\tau_B}\big\}\big\}\,. &(5.25)\end{aligned}$$

In the same way, or by differentiating (5.24) with respect to λ and evaluating in $\lambda = 0$, the moments of first-hitting times can be expressed. Assuming their existence, for the first two moments we find

$$\begin{aligned}\mathbb{E}^x\big\{\tau_B\big\} &= \mathbb{E}^x\big\{\tau_{B\cup C}\big\} + \mathbb{E}^x\big\{1_{\{\tau_C<\tau_B\}}\mathbb{E}^{x_{\tau_C}}\,\{\tau_B\}\big\}\,, &(5.26)\\ \mathbb{E}^x\big\{\tau_B^2\big\} &= \mathbb{E}^x\big\{\tau_{B\cup C}^2\big\} + 2\mathbb{E}^x\big\{\tau_{B\cup C}1_{\{\tau_C<\tau_B\}}\mathbb{E}^{x_{\tau_C}}\,\{\tau_B\}\big\} + \mathbb{E}^x\big\{1_{\{\tau_C<\tau_B\}}\mathbb{E}^{x_{\tau_C}}\,\{\tau_B^2\}\big\} &(5.27)\end{aligned}$$

for any choice of disjoint sets B and C, and any $x \notin (B \cup C)$.

Below we will use the notations

$$\mathbb{P}^A\{X \in \cdot\} = \sup_{y\in A}\mathbb{P}^y\{X \in \cdot\} \qquad \text{and} \qquad \mathbb{E}^A\{X\} = \sup_{y\in A}\mathbb{E}^y\{X\}\,. \tag{5.28}$$

It follows that for any three pairwise disjoint sets A, B and C,

$$\begin{aligned}\mathbb{E}^A\big\{\tau_B\big\} &\leqslant \mathbb{E}^A\big\{\tau_{B\cup C}\big\} + \mathbb{E}^A\big\{1_{\{\tau_C<\tau_B\}}\mathbb{E}^{x_{\tau_C}}\,\{\tau_B\}\big\}\\ &\leqslant \mathbb{E}^A\big\{\tau_{B\cup C}\big\} + \mathbb{P}^A\big\{\tau_C<\tau_B\big\}\mathbb{E}^{\partial C}\big\{\tau_B\big\}\,, &(5.29)\end{aligned}$$

and a similar relation holds for the second moment.

Lemma 5.6 (Moment estimate based on the Markov property)**.** *Let $B \subset E$ be such that*

$$\mathbb{P}^{B^c}\big\{\tau_B > T\big\} < 1 \tag{5.30}$$

for some $T > 0$. Then for any $n \in \mathbb{N}$,

$$\mathbb{E}^{B^c}\big\{\tau_B^n\big\} \leqslant \frac{n!T^n}{\big(1-\mathbb{P}^{B^c}\big\{\tau_B>T\big\}\big)^n}\,. \tag{5.31}$$

Proof: The Markov property implies that for any $m \in \mathbb{N}$ and any $x \in B^c$,

$$\mathbb{P}^x\big\{\tau_B > (m+1)T\big\} = \mathbb{E}^x\big\{1_{\{\tau_B>mT\}}\mathbb{P}^{y_{mT}}\{\tau_B>T\}\big\} \leqslant \mathbb{P}^{B^c}\{\tau_B>T\}\mathbb{P}^x\{\tau_B>mT\}\,, \tag{5.32}$$

so that $\mathbb{P}^x\{\tau_B > mT\} \leqslant (\mathbb{P}^{B^c}\{\tau_B > T\})^m$. Integration by parts shows that

$$\mathbb{E}^x\big\{\tau_B^n\big\} = n\int_0^\infty t^{n-1}\mathbb{P}^x\big\{\tau_B>t\big\}\,\mathrm{d}t \leqslant nT^n\sum_{m=0}^\infty (m+1)^{n-1}\mathbb{P}^x\big\{\tau_B>mT\big\}\,. \tag{5.33}$$

The result is thus a consequence of the inequality

$$\sum_{m=0}^{\infty}(m+1)^{n-1}p^m \leqslant \frac{(n-1)!}{(1-p)^n} \qquad \forall p\in[0,1)\;, \tag{5.34}$$

which follows from properties of the polylogarithm function and Eulerian numbers. □

Remark 5.7. Equation (5.31) implies that if (5.30) holds, the Laplace transform of τ_B exists for

$$\lambda < \frac{1}{T}\big(1-\mathbb{P}^{B^c}\big\{\tau_B > T\big\}\big) \tag{5.35}$$

and satisfies

$$\mathbb{E}^{B^c}\big\{\mathrm{e}^{\lambda\tau_B}\big\} \leqslant \frac{1}{1-\lambda T/(1-\mathbb{P}^{B^c}\{\tau_B>T\})}\;. \tag{5.36}$$

A sharper bound on the Laplace transform can be obtained by a direct integration by parts, but this does not automatically lead to better bounds on the moments.

Next, we will iterate the estimate (5.29) in order to get a better bound on the moments of first hitting times.

Corollary 5.8 (Three-set argument)**.** *Let $A,B,C\subset E$ be such that A,B and C are pairwise disjoint, and assume $\mathbb{P}^A\big\{\tau_C<\tau_B\big\}<1$, $\mathbb{E}^A\big\{\tau_B^k\big\}<\infty$ and $\mathbb{E}^{\partial C}\big\{\tau_B^k\big\}<\infty$ for $k=1,2$. Then*

$$\mathbb{E}^A\big\{\tau_B\big\} \leqslant \frac{\mathbb{E}^A\big\{\tau_{B\cup C}\big\}+\mathbb{P}^A\big\{\tau_C<\tau_B\big\}\mathbb{E}^{\partial C}\big\{\tau_{A\cup B}\big\}}{1-\mathbb{P}^A\big\{\tau_C<\tau_B\big\}}\;, \tag{5.37}$$

$$\mathbb{E}^A\big\{\tau_B^2\big\} \leqslant 4\frac{\mathbb{E}^A\big\{\tau_{B\cup C}^2\big\}+\mathbb{E}^{\partial C}\big\{\tau_{A\cup B}^2\big\}}{\big(1-\mathbb{P}^A\big\{\tau_C<\tau_B\big\}\big)^2}\;. \tag{5.38}$$

Proof: We introduce the shorthands

$$X_k=\mathbb{E}^A\big\{\tau_{B\cup C}^k\big\}\;,\qquad Y_k=\mathbb{E}^{\partial C}\big\{\tau_{A\cup B}^k\big\}\;,\qquad p=\mathbb{P}^A\big\{\tau_C<\tau_B\big\}\;, \tag{5.39}$$

for $k=1,2$. Note that $X_k,Y_k<\infty$ and $p<1$ according to our assumptions. Applying (5.29), once to the triple (A,B,C) and once to the triple $(\partial C,B,A)$, yields

$$\begin{aligned}\mathbb{E}^A\big\{\tau_B\big\} &\leqslant X_1+p\mathbb{E}^{\partial C}\big\{\tau_B\big\}\;,\\ \mathbb{E}^{\partial C}\big\{\tau_B\big\} &\leqslant Y_1+\mathbb{E}^{\partial A}\big\{\tau_B\big\}=Y_1+\mathbb{E}^A\big\{\tau_B\big\}\;,\end{aligned} \tag{5.40}$$

where we have bounded $\mathbb{P}^{\partial C}\{\tau_A<\tau_B\}$ by 1. In addition, we used that hitting B requires first exiting from A which is necessarily realized by passing through ∂A. This implies

$$\begin{aligned}\mathbb{E}^A\big\{\tau_B\big\} &\leqslant \frac{X_1+pY_1}{1-p}\;,\\ \mathbb{E}^{\partial C}\big\{\tau_B\big\} &\leqslant \frac{X_1+Y_1}{1-p}\;,\end{aligned} \tag{5.41}$$

which proves (5.37). Starting from (5.27), we find

$$\begin{aligned}\mathbb{E}^A\big\{\tau_B^2\big\} &\leqslant X_2+2X_1\mathbb{E}^{\partial C}\big\{\tau_B\big\}+p\mathbb{E}^{\partial C}\big\{\tau_B^2\big\}\;,\\ \mathbb{E}^{\partial C}\big\{\tau_B^2\big\} &\leqslant Y_2+2Y_1\mathbb{E}^A\big\{\tau_B\big\}+\mathbb{E}^A\big\{\tau_B^2\big\}\;.\end{aligned} \tag{5.42}$$

Together with (5.41), this gives

$$(1-p)^2\mathbb{E}^A\big\{\tau_B^2\big\} \leqslant (1-p)(X_2+pY_2)+2(X_1^2+(1+p)X_1Y_1+p^2Y_1^2)\;, \tag{5.43}$$

and the result follows after some algebra, using Jensen's inequality. Note that we have overestimated some terms in order to to obtain a more compact expression. □

5.3 Large deviations

As shown in [FJL82, Fre88], the family $\{u_t\}_{\varepsilon>0}$ of mild solutions of the SPDE with initial condition $u_0 \in E = \mathcal{C}^0(\mathbb{T}^1)$ satisfies a large-deviation principle in E, equipped with the sup norm, with rate function

$$I_{[0,T]}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \int_0^L \Big[\dot\varphi_t(x) - \varphi_t''(x) + U'(\varphi_t(x))\Big]^2 \,\mathrm{d}x\,\mathrm{d}t & \text{if the integral is finite ,} \\ +\infty & \text{otherwise .} \end{cases} \tag{5.44}$$

For $u_0 \in E$ and $A \subset E$, let

$$H(u_0, A) = \frac{1}{2} \inf_{T>0} \Bigg(\inf_{\varphi:\varphi(0)=u_0, \exists t \leqslant T \text{ s.t. } \varphi(t)\in A} I_{[0,T]}(\varphi) \Bigg) , \tag{5.45}$$

where the second infimum runs over all continuous paths $\varphi : [0, T] \to E$ connecting u_0 in a time $t \leqslant T$ to a point in A (if u_0 is a local minimum, then $u \mapsto 2H(u_0, u)$ is called *quasipotential*).

We define the *relative communication height* between u_0 and A by

$$\overline{V}(u_0, A) = \inf_{\psi:\psi(0)=u_0, \psi(1)\in A} \Bigg(\sup_{t\in[0,1]} V[\psi_t] - V[u_0] \Bigg) , \tag{5.46}$$

where the infimum now runs over all continuous paths $\psi : [0, 1] \to E$ connecting u_0 to an endpoint in A (the parameter t need not be associated to time in this definition). Note that $\overline{V}(u_0, A) = 0$ if and only if one can find a path from u_0 to A along which the potential is nonincreasing. This holds in particular when u_0 lies in the basin of attraction of A. If $\overline{V}(u_0, A) > 0$, then one has to cross a potential barrier in order to reach A from u_0.

The following classical result shows that $H(u_0, A)$ can be estimated below in terms of the relative communication height.

Lemma 5.9. *For any $u_0 \in E$ and any $A \subset E$, we have*

$$H(u_0, A) \geqslant \overline{V}(u_0, A) . \tag{5.47}$$

Proof: Let φ be a path connecting u_0 to A in time T. By definition of the communication height, the potential on any such path has to reach the value $\overline{V}(u_0, A) + V(u_0)$ at least once. Denoting by T_1 the first time this happens, we have

$$\begin{aligned} I_{[0,T]}(\varphi) &\geqslant \frac{1}{2}\int_0^{T_1} \int_0^L \Big[\dot\varphi_t(x) + \varphi_t''(x) - U'(\varphi_t(x))\Big]^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + 2\int_0^{T_1} \int_0^L \Big[-\varphi_t''(x) + U'(\varphi_t(x))\Big]\dot\varphi_t(x) \,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant 2\int_0^{T_1} \int_0^L \Big[\varphi_t'(x)\dot\varphi_t'(x) + U'(\varphi_t(x))\dot\varphi_t(x)\Big] \,\mathrm{d}x\,\mathrm{d}t \\ &= 2\Big[V(\varphi_{T_1}) - V(\varphi_0)\Big] = 2\overline{V}(u_0, A) . \end{aligned} \tag{5.48}$$

The right-hand side being independent of T, the result follows. □

5.4 Bounds on moments of τ_B in infinite dimension

We will now apply the results of the previous sections to the mild solution u_t of the SPDE, with $E = \mathcal{C}^0(\mathbb{T}^1)$ equipped with the sup norm. We fix $\alpha \in (0, 1/2)$ and introduce two families of sets

$$A_1(R) = \bigl\{u \in \mathcal{C}^0(\mathbb{T}^1)\colon \|u\|_{L^\infty} \leqslant R\bigr\}\,,$$
$$A_2(R) = \bigl\{u \in \mathcal{C}^0(\mathbb{T}^1)\colon \|u\|_{\mathcal{C}^\alpha} \leqslant R\bigr\}\,. \tag{5.49}$$

Note that $A_2 \subset A_1$, and that A_2 is a compact subset of E, while A_1 is not compact as a subset of E.

Let $B \subset \mathcal{C}^0(\mathbb{T}^1)$ be a non-empty, bounded open set in the $\|\cdot\|_{L^\infty}$-topology. We let

$$H_0 = H_0(B) = H(u_-^*, B) \vee H(u_+^*, B) \tag{5.50}$$

be the cost, in terms of the rate function, to reach the set B from either one of the local minima. Our aim is to estimate the first two moments of τ_B, using the three-set argument Corollary 5.8 for $A = A_2(R_2) \setminus B$ and $C = A_1(R_0)^c$, with appropriately chosen R_0 and R_2. We thus proceed to estimating the quantities appearing in the right-hand side of (5.37) and (5.38).

Proposition 5.10 (Bounds on moments of τ_{A_2})**.** *For any sufficiently large R_2, there exist $T_0 < \infty$, $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$,*

$$\mathbb{E}^{A_2(R_2)^c}\bigl\{\tau_{A_2(R_2)}^n\bigr\} \leqslant n! T_0^n \tag{5.51}$$

holds for all $n \geqslant 1$.

Proof: Choose a fixed $T_1 > 0$. Then the Markov property applied at time $T_1/2$ shows that for any $R_1 > 0$,

$$\mathbb{P}^{A_2(R_2)^c}\bigl\{\tau_{A_2(R_2)} > T_1\bigr\} \leqslant \mathbb{P}^{A_2(R_2)^c}\bigl\{\|u_{T_1/2}\|_{L^\infty} > R_1\bigr\} + \mathbb{P}^{A_1(R_1)}\bigl\{\|u_{T_1/2}\|_{\mathcal{C}^\alpha} > R_2\bigr\}\,. \tag{5.52}$$

Proposition 5.2 and (5.18) show that for $R_1 = c\sqrt{2/T_1} + \eta$ with $\eta > 0$, the first term on the right-hand side is smaller than $1/4$ for $\varepsilon \leqslant \varepsilon_0(T_1)$, uniformly in the initial condition. By Proposition 5.4, we can find R_2 such that the second term is also smaller than $1/4$. This shows

$$\mathbb{P}^{A_2(R_2)^c}\bigl\{\tau_{A_2(R_2)} > T_1\bigr\} \leqslant \frac{1}{2}\,. \tag{5.53}$$

By Lemma 5.6, this yields (5.51) with $T_0 = 2T_1$. □

Proposition 5.11. *For any $R_2, \eta > 0$, there exists a constant $T(\eta) \in (0, \infty)$ such that*

$$\mathbb{P}^{A_2(R_2)}\bigl\{\tau_B > T\bigr\} \leqslant 1 - \frac{1}{2}\,\mathrm{e}^{-(H_0+\eta)/\varepsilon} \tag{5.54}$$

for sufficiently small ε.

Proof: We start by fixing an initial condition $u_0 \in A_2(R_2)$. By the large-deviation principle, we have

$$\liminf_{\varepsilon\to 0} 2\varepsilon \log \mathbb{P}^{u_0}\bigl\{\tau_B \leqslant T\bigr\} \geqslant -\inf\bigl\{I(\varphi)\colon \varphi_0 = u_0,\ \varphi_T \in B\bigr\}\,. \tag{5.55}$$

Following a classical procedure similar to the one in the proof of [FJL82, Theorem 9.1], we construct a path $\varphi^*_{u_0}$ connecting u_0 to a point $u^* \in B$ such that $I(\varphi^*_{u_0}) \leqslant 2H_0+\eta$. This can be done by following the deterministic flow from u_0 to the neighbourhood of a stationary solution of the deterministic PDE at zero cost, then connecting to that stationary solution at finite cost. Any two stationary solutions and u^* can also be connected at finite cost, and $\varphi^*_{u_0}$ is obtained by concatenation. It follows that there exists $\varepsilon_0(\eta, u_0) > 0$ such that

$$\mathbb{P}^{u_0}\big\{\tau_B \leqslant T\big\} \geqslant \mathrm{e}^{-(I(\varphi^*_{u_0})+\eta)/2\varepsilon} \qquad \forall \varepsilon < \varepsilon_0(\eta, u_0)\;. \tag{5.56}$$

The set $A_2(R_2)$ being compact, we can find, for any $\delta > 0$, a finite cover of $A_2(R_2)$ with $N(\delta)$ balls of the form $D_n = \{u \in \mathcal{C}^0(\mathbb{T}^1)\colon \|u - u_n\|_{L^\infty} < \delta\}$. Hence

$$\max_{1\leqslant n\leqslant N(\delta)} \mathbb{P}^{u_n}\big\{\tau_B \leqslant T\big\} \geqslant \mathrm{e}^{-(I^*(\delta)+\eta)/2\varepsilon} \qquad \forall \varepsilon < \varepsilon_1(\eta, \delta)\;, \tag{5.57}$$

where $I^*(\delta) = \max_n I(\varphi^*_{u_n}) < 2H_0 + \eta$ and $\varepsilon_1(\eta, \delta) = \min_n \varepsilon_0(\eta, u_n) > 0$.

Consider now two solutions $u^{(1)}_t, u^{(2)}_t$ of the SPDE with initial conditions $u^{(1)}_0, u^{(2)}_0 \in D_n$. By a Gronwall-type argument similar to the one given in [FJL82, Theorem 5.10] and the bound (5.21), for any $\kappa_1 > 0$ there exist $K(\kappa_1) > 0$ such that

$$\mathbb{P}\biggl\{\sup_{0\leqslant t\leqslant T} \|u^{(1)}_t - u^{(2)}_t\|_{L^\infty} > \mathrm{e}^{KT}\|u^{(1)}_0 - u^{(2)}_0\|_{L^\infty}\biggr\} \leqslant \mathrm{e}^{-\kappa_1/2\varepsilon}\;. \tag{5.58}$$

The set B being open, it contains a ball $\{u \in \mathcal{C}^0(\mathbb{T}^1)\colon \|u - u^*\|_{L^\infty} < \rho\}$ with $\rho > 0$. Thus choosing $\delta = \mathrm{e}^{-KT}\rho$, combining (5.57) and (5.58), we get

$$\mathbb{P}^{A_2(R_2)}\big\{\tau_B > T\big\} \leqslant 1 - \mathrm{e}^{-(H_0+\eta)/\varepsilon} + \mathrm{e}^{-\kappa_1/2\varepsilon} \tag{5.59}$$

for all sufficiently small ε. Now choosing, e.g., $\kappa_1 = I(\varphi^*_{u_1})$ and $\delta = \mathrm{e}^{-K(\kappa_1)T}\rho$ guarantees that the term $\mathrm{e}^{-\kappa_1/2\varepsilon}$ is negligible. □

Proposition 5.12. *For every $\eta > 0$ and sufficiently large R_0, R_2 satisfying $R_0 > R_2$, there exists $T_0(\eta) \in (0, \infty)$ such that*

$$\mathbb{E}^{A_1(R_0)\cap B^c}\big\{\tau^n_{B\cup A_1(R_0)^c}\big\} \leqslant n! T_0^n\, \mathrm{e}^{n(H_0+\eta)/\varepsilon} \tag{5.60}$$

for all $n \geqslant 1$.

Proof: For any $T_1 > 1$ and $R_1 > 0$, we have

$$\begin{aligned}\mathbb{P}^{A_1(R_0)\cap B^c}\big\{\tau_{B\cup A_1(R_0)^c} > T_1\big\} &\leqslant \mathbb{P}^{A_1(R_0)\cap B^c}\big\{\|u_{T_1/3}\|_{L^\infty} > R_1\big\} \\ &\quad + \mathbb{P}^{A_1(R_1)}\big\{\|u_{T_1/3}\|_{\mathcal{C}^\alpha} > R_2\big\} \\ &\quad + \mathbb{P}^{A_2(R_2)}\big\{\tau_B > T_1/3\big\}\;.\end{aligned} \tag{5.61}$$

The third term on the right-hand side is bounded by $1 - \frac{1}{2}\,\mathrm{e}^{-(H_0+\eta)/\varepsilon}$ by Proposition 5.11. Proposition 5.4 shows that

$$\mathbb{P}^{A_1(R_1)}\big\{\|u_{T_1/3}\|_{\mathcal{C}^\alpha} > R_2\big\} \leqslant \exp\biggl\{-\frac{\kappa_1}{2\varepsilon}\min\Bigl\{R_2^2, f\Bigl((\kappa_2 R_2 - 1)^{1/3} - M(1+R_1)\Bigr)\Bigr\}\biggr\}\;, \tag{5.62}$$

while (5.18) shows that there exists $\kappa_3(T_1) > 0$ such that

$$\mathbb{P}^{A_1(R_0)\cap B^c}\big\{\|u_{T_1/3}\|_{L^\infty} > R_1\big\} \leqslant \exp\biggl\{-\frac{\kappa_3}{2\varepsilon}\biggl(R_1 - \frac{c}{\sqrt{T_1/3}}\biggr)^2\biggr\}\;. \tag{5.63}$$

We thus choose first R_1 large enough so that the exponent in (5.63) is strictly smaller than $-(H_0+\eta)/\varepsilon$, and then R_2 suffiently large for the exponent in (5.62) to be smaller than $-(H_0+\eta)/\varepsilon$ as well. This shows that the probability in (5.61) is smaller than $1-\frac{1}{4}\,\mathrm{e}^{-(H_0+\eta)/\varepsilon}$ for sufficiently small ε, and the result follows from Lemma 5.6 with $T_0 = 4T_1$. □

Proposition 5.13. *For any $R_2 > 0$, there exists a constant $R_0 > R_2$ such that*

$$\mathbb{P}^{A_2(R_2)}\big\{\tau_{A_1(R_0)^c} \leqslant \tau_B\big\} \leqslant \frac{1}{2} \tag{5.64}$$

for sufficiently small ε.

Proof: For any $T > 0$ and $n \in \mathbb{N}$, we have

$$\mathbb{P}^{A_2(R_2)}\big\{\tau_{A_1(R_0)^c} \leqslant \tau_B\big\} \leqslant \mathbb{P}^{A_2(R_2)}\big\{\tau_{A_1(R_0)^c} \leqslant nT\big\} + \mathbb{P}^{A_2(R_2)}\big\{\tau_B > nT\big\}\;. \tag{5.65}$$

We introduce the quantities

$$\begin{aligned} p_n &= \mathbb{P}^{A_2(R_2)}\big\{\tau_B > nT\big\}\;,\\ q_n &= \mathbb{P}^{A_2(R_2)}\big\{u_{nT} \notin A_2(R_2)\big\}\;,\\ r_n &= \mathbb{P}^{A_2(R_2)}\big\{\tau_{A_1(R_0)^c} \leqslant nT\big\}\;. \end{aligned} \tag{5.66}$$

Using the Markov property, they can all be expressed in terms of p_1, q_1 and r_1. Namely,

$$\begin{aligned} q_{n+1} &\leqslant \mathbb{P}^{A_2(R_2)}\big\{u_{nT} \notin A_2(R_2)\big\} + \mathbb{E}^{A_2(R_2)}\big\{1_{\{u_{nT}\in A_2(R_2)\}}\mathbb{P}^{u_{nT}}\{u_T \notin A_2(R_2)\}\big\}\\ &\leqslant q_n + q_1(1-q_n)\;, \end{aligned} \tag{5.67}$$

and one easily shows by induction that

$$q_n \leqslant 1-(1-q_1)^n \leqslant nq_1\;. \tag{5.68}$$

Splitting again according to whether u_{nT} belongs to $A_2(R_2)$ or not, we get

$$r_{n+1} \leqslant q_n + r_1(1-q_n) \leqslant r_1 + nq_1\;. \tag{5.69}$$

In a similar way, we have

$$\begin{aligned} p_{n+1} &\leqslant p_1\mathbb{P}^{A_2(R_2)}\big\{u_{nT} \in A_2(R_2), \tau_B > nT\big\} + \mathbb{P}^{A_2(R_2)}\big\{u_{nT} \notin A_2(R_2), \tau_B > nT\big\}\\ &= p_1p_n + (1-p_1)\mathbb{P}^{A_2(R_2)}\big\{u_{nT} \notin A_2(R_2), \tau_B > nT\big\}\\ &\leqslant p_1p_n + (1-p_1)q_n\;. \end{aligned} \tag{5.70}$$

It follows by induction that

$$p_n \leqslant p_1^n + n^2q_1(1-p_1)\;. \tag{5.71}$$

Putting together the different estimates, we obtain

$$\mathbb{P}^{A_2(R_2)}\big\{\tau_{A_1(R_0)^c} \leqslant \tau_B\big\} \leqslant p_n + r_n \leqslant p_1^n + r_1 + nq_1\big[1+n(1-p_1)\big]\;. \tag{5.72}$$

It remains to estimate p_1, q_1 and r_1. Proposition 5.11 shows that

$$p_1 \leqslant 1 - \frac{1}{2}\,\mathrm{e}^{-H_1/\varepsilon}\;, \tag{5.73}$$

where $H_1 = H_0 + \eta$. We can estimate q_1 by

$$q_1 \leqslant \mathbb{P}^{A_2(R_2)}\big\{\|u_{T/2}\|_{L^\infty} > R_1\big\} + \mathbb{P}^{A_1(R_1)}\big\{\|u_{T/2}\|_{\mathcal{C}^\alpha} > R_2\big\}\,, \tag{5.74}$$

and both terms can be bounded as in the proof of Proposition 5.12. An appropriate choice of R_1, R_2 ensures that $q_1 \leqslant \mathrm{e}^{-H_1/\varepsilon}$. Finally, by (5.19) we also have

$$r_1 = \mathbb{P}^{A_2(R_2)}\big\{\tau_{A_1(R_0)^c} \leqslant T\big\} \leqslant \exp\left\{-\frac{\kappa(T)}{2\varepsilon} f(R_0 - M(1+R_2))\right\} \leqslant \mathrm{e}^{-H_1/\varepsilon} \tag{5.75}$$

for sufficiently large R_0. The choice

$$n = \left\lceil (4\log 2)\,\mathrm{e}^{H_1/\varepsilon} \right\rceil \tag{5.76}$$

yields $\log(p_1^n) \leqslant -\frac{1}{2} n\,\mathrm{e}^{-H_1/\varepsilon} \leqslant -2\log(2)$ so that $p_1^n \leqslant 1/4$, while the other terms in (5.72) are exponentially small. This concludes the proof. □

Combining Propositions 5.10, 5.12 and 5.13, we finally get the main result of this section.

Corollary 5.14 (Main estimate on the moments of τ_B)**.** *Let $B \subset \mathcal{C}^0(\mathbb{T}^1)$ be a non-empty, bounded open set in the $\|\cdot\|_{L^\infty}$-topology. Then for all $R_0, \eta > 0$, there exist constants $\varepsilon_0 > 0$ and $T_0 < \infty$ such that*

$$\mathbb{E}^{A_1(R_0)}\big\{\tau_B\big\} \leqslant T_0\,\mathrm{e}^{(H_0(B)+\eta)/\varepsilon} \qquad \text{and} \qquad \mathbb{E}^{A_1(R_0)}\big\{\tau_B^2\big\} \leqslant T_0^2\,\mathrm{e}^{2(H_0(B)+\eta)/\varepsilon} \tag{5.77}$$

for all $\varepsilon < \varepsilon_0$.

Proof: Making R_0 larger if necessary, we choose R_2 and $R_0 > R_2$ large enough for the three previous results to hold, and such that $B \subsetneq A_1(R_0)$. We apply the three-set argument Corollary 5.8 with $A = A_2(R_2) \setminus B$ and $C = A_1(R_0)^c$. Noting that $A \subset A_2(R_2)$ and $\tau_{A\cup B} \leqslant \tau_{A_2(R_2)}$, one can easily bound $\mathbb{E}^C\left\{\tau^k_{A\cup B}\right\}$ with Proposition 5.10, $\mathbb{E}^A\left\{\tau^k_{B\cup C}\right\}$ with Proposition 5.12, and $\mathbb{P}^A\{\tau_C < \tau_B\}$ with Proposition 5.13. This shows the result for initial conditions in $A_2(R_2) \setminus B$. Now we can easily extend these bounds to all initial conditions in $A_1(R_0)$ by using Proposition 5.10 and restarting the process when it first hits $A_2(R_2) \setminus B$. □

Note that in the proof of Proposition 3.2, we apply this result when B is a neighbourhood of u_+^*. In that case, $H_0(B)$ is equal to the potential difference between the transition state and the local minimum u_-^*.

5.5 Uniform bounds on moments of τ_B in finite dimension

In this section, we derive bounds on the moments of first-hitting times, similar to those in Corollary 5.14, for the finite-dimensional process, uniformly in the dimension.

For $E = \mathcal{C}^0(\mathbb{T}^1)$ and $d \in \mathbb{N}$, we denote by E_d the finite-dimensional space

$$E_d = \left\{ u \in \mathcal{C}^0(\mathbb{T}^1)\colon u(x) = \sum_{k\colon |k|\leqslant d} y_k e_k(x)\,,\ y_k \in \mathbb{R} \right\}, \tag{5.78}$$

and by $\{u_t^{(d)}\}_{t\geqslant 0}$ the solution of the projected equation, cf. (3.17). Given a set $A \subset E$, we write $A_d = A \cap E_d$, and denote by $\tau_{A_d}^{(d)}$ the first time $u_t^{(d)}$ hits A_d, while τ_A denotes as before the first time the infinite-dimensional process u_t hits A.

Proposition 5.15 (Main estimate on moments of $\tau_{B_d}^{(d)}$)**.** *Let $B \subset E$ be an open ball of radius r in the $\|\cdot\|_{L^\infty}$-norm. We assume that the centre of B is some $w \in E_d$. As in Corollary 5.14, we define $A = A_2(R_2) \setminus B$. Then, there exist constants $\varepsilon_0 > 0$ and $H_1, T_1 < \infty$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there is a $d_0(\varepsilon) \in \mathbb{N}$ such that*

$$\mathbb{E}^{A_d}\{\tau_{B_d}^{(d)}\} \leqslant T_1\, \mathrm{e}^{H_1/\varepsilon} \qquad \text{and} \qquad \mathbb{E}^{A_d}\{(\tau_{B_d}^{(d)})^2\} \leqslant T_1^2\, \mathrm{e}^{2H_1/\varepsilon} \tag{5.79}$$

for all $d \geqslant d_0(\varepsilon)$.

Proof: We fix constants $\delta, T > 0$, and let Ω_d be the event

$$\Omega_d = \left\{ \sup_{0 \leqslant t \leqslant T} \|u_t^{(d)} - u_t\|_{L^\infty} \leqslant \delta \right\}. \tag{5.80}$$

Theorem 3.1 shows that for given $\gamma < 1/2$, there exists an almost surely finite random variable Z such that

$$\mathbb{P}(\Omega_d^c) \leqslant \mathbb{P}\{Z > \delta d^\gamma\}. \tag{5.81}$$

Given $D \subset \mathcal{C}^0(\mathbb{T}^1)$, we define the sets

$$\begin{aligned} D_{d,+} &= \{u \in E_d \colon \exists v \in D \text{ s.t. } \|v - u\|_{L^\infty} \leqslant \delta\}, \\ D_{d,-} &= \{u \in E_d \colon \{v \in E \colon \|v - u\|_{L^\infty} \leqslant \delta\} \subset D\}, \end{aligned} \tag{5.82}$$

which satisfy $D_{d,-} \subset D_d \subset D_{d,+}$. Then for any initial condition $u_0 \in E_d$, we have the two inequalities

$$\begin{aligned} \mathbb{P}^{u_0}\{\tau_{D_{d,+}}^{(d)} > T\} &\leqslant \mathbb{P}^{u_0}\{\tau_D > T\} + \mathbb{P}(\Omega_d^c), \\ \mathbb{P}^{u_0}\{\tau_{D_{d,-}}^{(d)} \leqslant T\} &\leqslant \mathbb{P}^{u_0}\{\tau_D \leqslant T\} + \mathbb{P}(\Omega_d^c). \end{aligned} \tag{5.83}$$

Let R_0 be as in the proof of Corollary 5.14, and define the sets

$$\begin{aligned} C &= \{u \in E \colon \|u\|_{L^\infty} \geqslant R_0\}, \\ C' &= \{u \in E \colon \|u\|_{L^\infty} \geqslant R_0 + 2\delta\}, \\ B' &= \{u \in E \colon \{v \in E \colon \|v - u\|_{L^\infty} \leqslant \delta\} \subset B\}. \end{aligned} \tag{5.84}$$

We assume δ to be small enough for B' to be non-empty. Note that C and C' are the complements of open balls in the $\|\cdot\|_{L^\infty}$-norm while B' is the open ball of radius $r - \delta$ around the center w of B. $\rho > 0$.

Applying the three-set argument (5.37) to the triple $(A_{d,+} \setminus B_d, B_d, C_{d,-})$, where the sets are disjoint for sufficiently large R_0, we get

$$\mathbb{E}^{A_d}\{\tau_{B_d}^{(d)}\} \leqslant \frac{\mathbb{E}^{A_{d,+}}\{\tau_{B_d \cup C_{d,-}}^{(d)}\} + \mathbb{P}^{A_{d,+}}\{\tau_{C_{d,-}}^{(d)} < \tau_{B_d}^{(d)}\}\mathbb{E}^{C_{d,-}}\{\tau_{A_{d,+} \cup B_d}^{(d)}\}}{1 - \mathbb{P}^{A_{d,+}}\{\tau_{C_{d,-}}^{(d)} < \tau_{B_d}^{(d)}\}}. \tag{5.85}$$

Using the facts that $A_d \subset A_{d,+}$, $(B')_{d,+} = B_d$ and $(C')_{d,+} \subset C_{d,-}$, we now reduce the estimation of each of the terms on the right-hand side of (5.85) to probabilities that can be controlled, via (5.83), in terms of the infinite-dimensional process.

Since $C_{d,-} \subset E_d \setminus (A_2(R_2))_{d,+}$ for sufficiently large R_0 and $A_{d,+} \cup B_d \supset (A_2(R_2))_{d,+}$, we have by Lemma 5.6

$$\mathbb{E}^{C_{d,-}}\bigl\{\tau^{(d)}_{A_{d,+}\cup B_d}\bigr\} \leqslant \mathbb{E}^{E_d\setminus(A_2(R_2))_{d,+}}\bigl\{\tau^{(d)}_{(A_2(R_2))_{d,+}}\bigr\} \leqslant \frac{T}{1-\mathbb{P}^{E_d\setminus(A_2(R_2))_{d,+}}\bigl\{\tau^{(d)}_{(A_2(R_2))_{d,+}} > T\bigr\}} \,. \tag{5.86}$$

By (5.83) we have for any $u_0 \in E_d \setminus (A_2(R_2))_{d,+}$

$$\mathbb{P}^{u_0}\bigl\{\tau^{(d)}_{(A_2(R_2))_{d,+}} > T\bigr\} \leqslant \mathbb{P}^{u_0}\bigl\{\tau_{A_2(R_2)} > T\bigr\} + \mathbb{P}(\Omega_d^c) \,. \tag{5.87}$$

As we have seen in (5.53), the first term on the right-hand side can be bounded by $1/2$. As for the second term, (5.81) shows that it is smaller than $1/4$ for $d \geqslant d_0(\varepsilon)$ large enough. Hence the right-hand side of (5.86) can be bounded by $4T/3$.

The term $\mathbb{E}^{A_{d,+}}\bigl\{\tau^{(d)}_{B_d\cup C_{d,-}}\bigr\} \leqslant \mathbb{E}^{A_{d,+}}\bigl\{\tau^{(d)}_{(B')_{d,+}\cup(C')_{d,+}}\bigr\}$ can be estimated in a similar way, by comparing with $\mathbb{P}^{A_{d,+}}\{\tau_{B'\cup C'} > T\}$ and proceeding as in the proof of Proposition 5.12, cf. (5.61).

Finally, we have the bounds

$$\begin{aligned}\mathbb{P}^{A_{d,+}}\bigl\{\tau^{(d)}_{C_{d,-}} < \tau^{(d)}_{B_d}\bigr\} &\leqslant \mathbb{P}^{A_{d,+}}\bigl\{\tau^{(d)}_{C_{d,-}} \leqslant T\bigr\} + \mathbb{P}^{A_{d,+}}\bigl\{\tau^{(d)}_{(B')_{d,+}} > T\bigr\} \\ &\leqslant \mathbb{P}^{A_{d,+}}\bigl\{\tau_C \leqslant T\bigr\} + \mathbb{P}^{A_{d,+}}\bigl\{\tau_{B'} > T\bigr\} + 2\mathbb{P}(\Omega_d^c) \,. \end{aligned} \tag{5.88}$$

Proposition 5.11 and (5.75) show that the sum of the first two terms on the right-hand side can be bounded by $1 - \frac{1}{2}\,\mathrm{e}^{-H_1/\varepsilon}$. The third term can be bounded by $\frac{1}{4}\,\mathrm{e}^{-H_1/\varepsilon}$, provided d is larger than some (possibly large) $d_0(\varepsilon)$.

This completes the bound on the first moment, and the second moment can be estimated in the same way. □

5.6 Bounds on the equilibrium potential in finite dimension

The aim of this subsection is to obtain bounds on the equilibrium potential

$$h^{(d)}_{A_d,B_d}(u_0) = \mathbb{P}^{u_0}\bigl\{\tau^{(d)}_{A_d} < \tau^{(d)}_{B_d}\bigr\} \,, \tag{5.89}$$

when A and B are small open balls, in the L^∞-norm, around the local minima $u^*_\pm$ of the potential V, and as before $A_d = A \cap E_d$ and $B_d = B \cap E_d$. We denote the centre of A by u_1^* and the centre of B by u_2^*, where either $u_1^* = u_-^*$ and $u_2^* = u_+^*$ or vice versa.

We now derive a bound on $h^{(d)}_{A_d,B_d}(u_0)$, which is useful when u_0 lies in the basin of attraction of B_d.

Proposition 5.16. *Let u_1^* and u_2^* be two different local minima of V and consider the balls $A = \{\|u - u_1^*\|_{L^\infty} < r\}$ and $B = \{\|u - u_2^*\|_{L^\infty} < r\}$, where r is small enough to guarantee $\bar{A} \cap \bar{B} = \emptyset$. Then for any $\eta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\eta) > 0$, $d_0 = d_0(\eta,\varepsilon) < \infty$ and $H_0 = H_0(\eta) > 0$ such that*

$$h^{(d)}_{A_d,B_d}(u_0) \leqslant 4\,\mathrm{e}^{-H_0/\varepsilon} \tag{5.90}$$

holds for all $\varepsilon < \varepsilon_0$, all $d \geqslant d_0$ and all u_0 satisfying $H(u_0,A) \geqslant \eta$ and $H(u_0,B) = 0$. The result holds uniformly for u_0 from a $\|\cdot\|_{L^\infty}$-bounded subset of E_d.

PROOF: Fix a u_0 such that $H(u_0, A) \geqslant \eta$ and $H(u_0, B) = 0$. For any constant $T > 0$, we can write

$$h^{(d)}_{A_d,B_d}(u_0) \leqslant \mathbb{P}^{u_0}\{\tau^{(d)}_{A_d} \leqslant T\} + \mathbb{P}^{u_0}\{\tau^{(d)}_{B_d} > T\} \,. \tag{5.91}$$

For $0 < \delta < r$, we define $\Omega_d = \Omega_d(\delta)$ as in (5.80). Then we have, in a way similar to (5.83),

$$\begin{aligned} \mathbb{P}^{u_0}\{\tau^{(d)}_{A_d} \leqslant T\} &\leqslant \mathbb{P}^{u_0}\{\tau_{A_+} \leqslant T\} + \mathbb{P}(\Omega_d^c) \,, \\ \mathbb{P}^{u_0}\{\tau^{(d)}_{B_d} > T\} &\leqslant \mathbb{P}^{u_0}\{\tau_{B_-} > T\} + \mathbb{P}(\Omega_d^c) \,, \end{aligned} \tag{5.92}$$

where $A_+ = \{\|u - u_1^*\|_{L^\infty} < r + \delta\}$ and $B_- = \{\|u - u_2^*\|_{L^\infty} < r - \delta\}$. We choose δ small enough that $H(u_0, A_+) \geqslant \eta/2$. The large-deviation principle shows that

$$\limsup_{\varepsilon\to 0} 2\varepsilon \log \mathbb{P}^{u_0}\{\tau_{A_+} \leqslant T\} = -2H(u_0, A_+) \leqslant -\eta \,. \tag{5.93}$$

Thus there exists $\varepsilon_0(\eta) > 0$, independent of T, such that

$$\mathbb{P}^{u_0}\{\tau_{A_+} \leqslant T\} \leqslant \mathrm{e}^{-\eta/4\varepsilon} \tag{5.94}$$

holds for all $\varepsilon < \varepsilon_0$. Choosing $d \geqslant d_0(\eta, \varepsilon)$ where d_0 is large enough that $\mathbb{P}(\Omega^c_{d_0}) \leqslant \mathrm{e}^{-\eta/4\varepsilon}$, we have

$$\mathbb{P}^{u_0}\{\tau^{(d)}_{A_d} \leqslant T\} \leqslant 2\,\mathrm{e}^{-\eta/4\varepsilon} \tag{5.95}$$

for $\varepsilon < \varepsilon_0$ and $d \geqslant d_0$.

To estimate the second term in (5.92), we assume $\delta < r/2$ and let $B'_- = \{\|u - u_2^*\|_{L^\infty} < r - 2\delta\}$. Assume T is large enough that the deterministic solution starting in u_0 reaches B'_- in time T. Then the large-deviation principle in [Fre88] shows that the stochastic sample path starting in u_0 is unlikely to leave a tube of size δ in the L^∞-norm around the deterministic solution before time T, which implies that there exists $\kappa > 0$ such that

$$\mathbb{P}^{u_0}\{\tau_{B_-} > T\} \leqslant \mathrm{e}^{-\kappa\delta^2/\varepsilon} \tag{5.96}$$

for ε small enough. This implies the result, with $H_0 = \kappa\delta^2 \wedge \eta/4$.

As for the uniformity in u_0, note that after a first finite time T_1 we may assume that the process has reached a compact subset, cf. the proof of Proposition 5.12. Restarting from this compact subset a standard compactness argument yields the uniformity of ε_0, δ_0 and H_0 in u_0. □

Next we derive a more precise bound, which is useful in situations where we know $V[u_0]$ explicitly.

Proposition 5.17. *Let u_1^* and u_2^* be two different local minima of V and consider the balls $A = \{\|u - u_1^*\|_{L^\infty} < r\}$ and $B = \{\|u - u_2^*\|_{L^\infty} < r\}$. Assume that r is small enough that $\overline{A} \cap \overline{B} = \emptyset$. Then for any $\eta, M > 0$, there exist $\varepsilon_0 = \varepsilon_0(\eta, M) > 0$ and $d_0 = d_0(\eta, M, \varepsilon) < \infty$ such that*

$$h^{(d)}_{A_d,B_d}(u_0) \leqslant 3\Big(\mathrm{e}^{-[\overline{V}(u_0,A)-\eta]/\varepsilon} + \mathrm{e}^{-1/\eta\varepsilon}\Big) \tag{5.97}$$

holds for all $\varepsilon < \varepsilon_0$, all $d \geqslant d_0$ and all $u_0 \in E_d$ such that $V[u_0] \leqslant M$.

PROOF: Fix a u_0 with $V[u_0] \leqslant M$. We decompose the equilibrium potential in the same way as in (5.91) and (5.92). It follows from (5.93) and Lemma 5.9 that there exists $\varepsilon_0(\eta) > 0$, independent of T, such that

$$\mathbb{P}^{u_0}\big\{\tau_{A_+} \leqslant T\big\} \leqslant \mathrm{e}^{-(\overline{V}(u_0,A_+)-\eta/2)/\varepsilon} \tag{5.98}$$

holds for all $\varepsilon < \varepsilon_0$. We choose δ in the definition of A_+ small enough that $\overline{V}(u_0, A_+) \geqslant \overline{V}(u_0, A) - \eta/2$, and finally $d \geqslant d_0(\eta, \varepsilon)$ where d_0 is large enough that $\mathbb{P}(\Omega_{d_0}^c) \leqslant \mathrm{e}^{-1/\eta\varepsilon}$. This shows that

$$\mathbb{P}^{u_0}\big\{\tau_{A_d}^{(d)} \leqslant T\big\} \leqslant \mathrm{e}^{-(\overline{V}(u_0,A)-\eta)/\varepsilon} + \mathrm{e}^{-1/\eta\varepsilon} \tag{5.99}$$

for $\varepsilon < \varepsilon_0$ and $d \geqslant d_0$.

In order to estimate $\mathbb{P}^{u_0}\{\tau_{B_-} > T\}$, we let $D(\kappa)$ be the set of $u \in E$ such that $\overline{V}(u, A_+) > 0$ and $\|\nabla V[u]\|_{L^2} > \kappa$. Then we can decompose

$$\mathbb{P}^{u_0}\big\{\tau_{B_-} > T\big\} \leqslant \mathbb{P}^{u_0}\big\{\tau_{D(\kappa)^c} > T\big\} + \mathbb{P}^{u_0}\big\{\tau_{F(\kappa)} \leqslant T\big\} \,. \tag{5.100}$$

where $F(\kappa) = D(\kappa)^c \cap B_-^c$. Now the same argument as above shows that

$$\mathbb{P}^{u_0}\big\{\tau_{F(\kappa)} \leqslant T\big\} \leqslant \mathrm{e}^{-(\overline{V}(u_0,F(\kappa))-\eta/2)/\varepsilon} \,. \tag{5.101}$$

Note that $\lim_{\delta\to 0} \overline{V}(u_0, F(\delta)) = \overline{V}(u_0, F(0))$. Let us show that $\overline{V}(u_0, F(0)) = \overline{V}(u_0, A_+)$ provided δ is small enough. First observe that $F(0)^c \subset D(0) \cup B_-$, and thus the fact that A and B have disjoint closure and the definition of $D(0)$ imply that $A_+ \subset F(0)$ for sufficiently small δ. As a consequence, $\overline{V}(u_0, F(0)) \leqslant \overline{V}(u_0, A_+)$. Assume by contradiction that $\overline{V}(u_0, F(0)) < \overline{V}(u_0, A_+)$. Then there must exist a path φ, connecting u_0 to a point $u \in F(0)$, on which the potential remains strictly smaller than $\overline{V}(u_0, A_+) + V[u_0]$. By definition of $D(0)$, we must have $\overline{V}(u, A_+) = 0$ since by assumption u_2^* is the only stationary point in $\{u\colon \widehat{V}(u, A_+) > 0\}$. But this implies that we can connect u_0 to A_+, via u, by a path on which the potential remains strictly smaller than $\overline{V}(u_0, A_+) + V[u_0]$, contradicting the definition of $\overline{V}(u_0, A)$. We have thus proved that $\overline{V}(u_0, F(0)) = \overline{V}(u_0, A_+)$, and it follows that there exists a $\delta_0(\eta)$ such that for $\delta < \delta_0(\eta)$

$$\mathbb{P}^{u_0}\big\{\tau_{F(\kappa)} \leqslant T\big\} \leqslant \mathrm{e}^{-(\overline{V}(u_0,A_+)-\eta)/\varepsilon} \,. \tag{5.102}$$

for all $\kappa < \kappa_0$.

It remains to estimate the first term on the right-hand side of (5.100). Let φ be a continuous path starting in u_0 and remaining in $\mathcal{D}(\kappa)$ up to time T. Then its rate function satisfies

$$\begin{aligned} I(\varphi) &\geqslant \int_0^T \int_0^L \Big[-\varphi_t''(x) + U'(\varphi_t(x))\Big]\dot{\varphi}_t(x)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_0^L \Big[-\varphi_t''(x) + U'(\varphi_t(x))\Big]^2 \mathrm{d}x\,\mathrm{d}t \\ &\geqslant V[\varphi_T] - V[u_0] + \frac{1}{2}\kappa^2 T \,, \end{aligned} \tag{5.103}$$

where we have used the fact that the second integral is proportional to $\|\nabla V\|_{L^2}^2$, cf. (4.25). The large-deviation principle implies that

$$\limsup_{\varepsilon\to 0} 2\varepsilon \log \mathbb{P}^{u_0}\big\{\tau_{D(\kappa)^c} > T\big\} \leqslant -\left[\frac{1}{2}\kappa^2 T + \inf_{D(\kappa)} V - V[u_0]\right] . \tag{5.104}$$

Since $V[u_0] \leqslant M$, we can find for any $\kappa > 0$, a $T = T(\kappa, M)$ such that the right-hand side is smaller than $-\overline{V}(u_0, A)$.

Finally note that $\{u_0\colon V[u_0] \leqslant M\}$ is contained in a closed ball in the $\mathcal{C}^{1/2}$-norm, so that a standard compactness argument allows to choose ε_0 and d_0 uniformly in u_0 from this set. This concludes the proof. □

6 Estimating capacities

6.1 Neumann b.c.

We consider the potential energy

$$V[u] = \int_0^L \left[\frac{1}{2}u'(x)^2 + U(u(x))\right] \mathrm{d}x \tag{6.1}$$

for functions $u(x)$ containing at most $2d+1$ nonvanishing Fourier modes and satisfying Neumann boundary conditions, that is

$$u(x) = \sum_{k=-d}^{d} z_k \frac{\mathrm{e}^{ik\pi x/L}}{\sqrt{L}} = y_0 \frac{1}{\sqrt{L}} + \sum_{k=1}^{d} y_k \sqrt{\frac{2}{L}} \cos(k\pi x/L) \;, \tag{6.2}$$

where $y_0 = z_0$ and $y_k = \sqrt{2}z_k = \sqrt{2}z_{-k}$ for $k \geqslant 1$. The expression $\widehat{V}$ of the potential in Fourier variables follows from (4.31) and (4.32), with the sums restricted to $-d \leqslant k \leqslant d$.

Note that

$$u(L-x) = y_0 \frac{1}{\sqrt{L}} + \sum_{k=1}^{d} (-1)^k y_k \sqrt{\frac{2}{L}} \cos(k\pi x/L) \;, \tag{6.3}$$

so that the fact that $V[u] = V[u(L-\cdot)]$ implies the symmetry

$$\widehat{V}(y_0, y_1, \dots, y_d) = \widehat{V}(y_0, -y_1, \dots, (-1)^d y_d) \;. \tag{6.4}$$

Our aim is to estimate the capacity $\mathrm{cap}_A(B)$, where A is a ball of radius r in the L^∞-norm around the stationary point u_-^*, and B is a ball of radius r around u_+^*. Note that u_-^* has y-coordinates $(u_-\sqrt{L}, 0, \dots, 0)$ and u_+^* has y-coordinates $(u_+\sqrt{L}, 0, \dots, 0)$. We will rely on the variational representation of capacities

$$\mathrm{cap}_A(B) = \inf_{h \in \mathcal{H}_{A,B}} \Phi_{(A\cup B)^c}(h) \;, \tag{6.5}$$

in terms of the Dirichlet form

$$\Phi_D(h) = \varepsilon \int_D \mathrm{e}^{-\widehat{V}(y)/\varepsilon} \|\nabla h(y)\|_{\ell^2}^2 \, \mathrm{d}y \;, \tag{6.6}$$

where in (6.5), $\mathcal{H}_{A,B}$ denotes the set of functions h satisfying the boundary conditions $h = 1$ on ∂A and $h = 0$ on ∂B for which $\Phi_{(A\cup B)^c}(h)$ is defined and finite.

6.1.1 $L < \pi$

We consider first the case where $L \leqslant \pi - c$ for some constant $c > 0$. We know that in this case, $\widehat{V}$ has only three stationary points, all lying on the y_0-axis. One of them is the origin O, where the Hessian of $\widehat{V}$ has eigenvalues

$$\lambda_k = -1 + \left(\frac{k\pi}{L}\right)^2 , \qquad k = 0, \dots, d \;. \tag{6.7}$$

Thus O is a saddle with one-dimensional unstable manifold, which in this case is contained in the y_0-axis. Let $\mathcal{W}^{\mathrm{s}}(O)$ denote the d-dimensional stable manifold of the origin.

Lemma 6.1 (Growth of the potential on the stable manifold)**.** *There exists a constant $m_0 > 0$ such that for all $y \in \mathcal{W}^{\mathrm{s}}(O)$,*

$$\widehat{V}(y) \geqslant m_0 \|y\|_{H^1}^2 \,. \tag{6.8}$$

PROOF: Let $y_\perp = (y_1, \dots, y_d)$. The centre-stable manifold theorem for differential equations in Banach spaces [Gal93, Theorem 1.1] shows that $\mathcal{W}^{\mathrm{s}}(O)$ can be locally described by a graph of the form $y_0 = g(y_\perp)$. More precisely, the nonlinear part of $\nabla \widehat{V}(y)$ being of order $\|y\|_{H^s}^2$ for $s > 1/4$, there exist constants $\rho, M >$ such that

$$\bigl|g(y_\perp)\bigr| \leqslant M \|y_\perp\|_{H^s}^2 \qquad \forall y_\perp \colon \|y_\perp\|_{H^s} \leqslant \rho \,. \tag{6.9}$$

Since

$$\widehat{V}(y) = \frac{1}{2} \sum_{k=0}^{d} \lambda_k y_k^2 + \mathcal{O}\bigl(\|y\|_{H^s}^3\bigr) \tag{6.10}$$

holds for all $s > 1/4$, we have in particular

$$\widehat{V}(g(y_\perp), y_\perp) = -\frac{1}{2} g(y_\perp)^2 + \frac{1}{2} \sum_{k=1}^{d} \lambda_k y_k^2 + \mathcal{O}\bigl(\|y\|_{H^1}^3\bigr) \tag{6.11}$$

whenever $\|y_\perp\|_{H^1} \leqslant \rho$. Thus using (6.9) to bound $g(y_\perp)^2$, we obtain the existence of constants $m_1, \rho_1 > 0$ such that

$$\widehat{V}(y) \geqslant m_1 \|y\|_{H^1}^2 \qquad \forall y \in \mathcal{W}^{\mathrm{s}}(O) \colon \|y\|_{H^1} \leqslant \rho_1 \,. \tag{6.12}$$

We have used the fact that $\|y\|_{H^1}^2 = |y_0|^2 + \|y_\perp\|_{H^1}^2$ and estimated $|y_0|^2$ on the stable manifold by applying (6.9) once more. A similar computation shows that

$$-\nabla \widehat{V}(y) \cdot \nabla(\|y\|_{H^1}^2) < 0 \qquad \forall y \in \mathcal{W}^{\mathrm{s}}(O) \colon \|y\|_{H^1} \leqslant \rho_1 \,, \tag{6.13}$$

that is, the vector field $-\nabla \widehat{V}(y)$ points inward the ball of radius ρ_1 on the stable manifold. By definition of the stable manifold, $\widehat{V}$ has to decrease on $\mathcal{W}^{\mathrm{s}}(O)$ along orbits of the gradient flow $\dot{y} = -\nabla \widehat{V}(y)$. We thus conclude from (6.12) and (6.13) that

$$\widehat{V}(y) \geqslant m_1 \rho_1^2 \qquad \forall y \in \mathcal{W}^{\mathrm{s}}(O) \colon \|y\|_{H^1} \geqslant \rho_1 \,. \tag{6.14}$$

Next, recall that by Lemma 4.5, there exist constants $\alpha, \beta > 0$ such that

$$\widehat{V}(y) \geqslant -\alpha + \beta \|y\|_{H^1}^2 \tag{6.15}$$

for all $y \in \mathbb{R}^{d+1}$. Define $\gamma > 0$ by $-\alpha + \beta\gamma^2 = 1$. Then for all $y \in \mathcal{W}^{\mathrm{s}}(O)$ such that $\rho_1 \leqslant \|y\|_{H^1} \leqslant \gamma$, we have

$$\widehat{V}(y) \geqslant m_1 \rho_1^2 = m_1 \frac{\rho_1^2}{\gamma^2} \gamma^2 \geqslant m_1 \frac{\rho_1^2}{\gamma^2} \|y\|_{H^1}^2 \,. \tag{6.16}$$

Together with (6.12) and (6.15) for $\|y\|_{H^1} > \gamma$, this proves (6.8), with the choice $m_0 = \min\{m_1, (m_1\rho_1^2/\gamma^2), (1/\gamma^2)\}$. □

Proposition 6.2 (Upper bound on the capacity)**.** *There exist constants $r_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$\mathrm{cap}_A(B) \leqslant \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \Bigg(\prod_{k=1}^{d} \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \Bigg) \big[1 + c_+ \varepsilon^{1/2} |\log \varepsilon|^{3/2} \big] \tag{6.17}$$

holds for all $r < r_0$, all $\varepsilon < \varepsilon_0$ and all $d \geqslant 1$, where the constant c_+ is independent of ε and d.

Proof: Choosing the radius r of the balls A and B small enough, we can ensure that A and B lie at a L^∞-distance of order 1 from the stable manifold $\mathcal{W}^{\mathrm{s}}(O)$.

By the variational principle (6.5), it is sufficient to construct a particular function $h_+ \in \mathcal{H}_{A,B}$ for which the claimed upper bound holds. We define h_+ separately in different sets D, S defined below, and the remaining part of $\mathbb{R}^{d+1}$. Let

$$\delta_k = \sqrt{\frac{c_k \varepsilon |\log \varepsilon|}{|\lambda_k|}} \,, \qquad k = 0, \dots d \,, \tag{6.18}$$

with $c_k = c_0(1 + \log(1 + k))$. We will choose c_0 sufficiently large below. We set

$$D = \prod_{k=0}^{d} [-\delta_k, \delta_k] \,. \tag{6.19}$$

Note that for any $s < 1/2$ one has

$$\|y\|_{H^s}^2 = \sum_{k=0}^{d} \frac{c_k (1 + k^2)^s}{|\lambda_k|} \varepsilon |\log \varepsilon| = \mathcal{O}(\varepsilon |\log \varepsilon|) \tag{6.20}$$

for any $y \in D$, uniformly in d. By (6.10) we thus have

$$\widehat{V}(y) = \frac{1}{2} \sum_{k=0}^{d} \lambda_k y_k^2 + \mathcal{O}\big(\varepsilon^{3/2} |\log \varepsilon|^{3/2}\big) \tag{6.21}$$

for all $y \in D$, where again the remainder is uniformly bounded in the dimension d. On D, we define h_+ by

$$h_+(y) = f(y_0) \coloneqq \int_{y_0}^{\delta_0} \frac{\mathrm{e}^{\widehat{V}(t,0,\dots,0)/\varepsilon}}{\displaystyle\int_{-\delta_0}^{\delta_0} \mathrm{e}^{\widehat{V}(s,0,\dots,0)/\varepsilon} \, \mathrm{d}s} \, \mathrm{d}t \,. \tag{6.22}$$

The contribution of h_+ on D to the Dirichlet form is given by

$$\begin{aligned} \Phi_D(h_+) &= \varepsilon \int_D f'(y_0)^2 \, \mathrm{e}^{-\widehat{V}(y)/\varepsilon} \, \mathrm{d}y \\ &= \varepsilon \int_D \frac{\mathrm{e}^{-\widehat{V}(y)/\varepsilon + 2\widehat{V}(y_0,0,\dots,0)/\varepsilon}}{\displaystyle\Bigg(\int_{-\delta_0}^{\delta_0} \mathrm{e}^{\widehat{V}(t,0,\dots,0)/\varepsilon} \, \mathrm{d}t \Bigg)^2} \, \mathrm{d}y \,. \end{aligned} \tag{6.23}$$

Using the expression (6.21) of the potential, one readily gets

$$\Phi_D(h_+) \leqslant \varepsilon \Bigg(\int_{-\delta_0}^{\delta_0} \mathrm{e}^{-y_0^2/2\varepsilon} \, \mathrm{d}y_0 \Bigg)^{-1} \prod_{k=1}^{d} \int_{-\delta_k}^{\delta_k} \mathrm{e}^{-\lambda_k y_k^2/2\varepsilon} \, \mathrm{d}y_k \big[1 + \mathcal{O}\big(\varepsilon^{1/2} |\log \varepsilon|^{3/2}\big) \big] \,, \tag{6.24}$$

which implies that $\Phi_D(h_+)$ its bounded above by the right-hand side of (6.17), provided c_0 is chosen large enough.

We now continue h_+ outside the set D. Let S be a layer of thickness of order $\sqrt{\varepsilon|\log\varepsilon|}$ in $\|\cdot\|_{\ell_2}$-norm around the stable manifold $\mathcal{W}^{\mathrm{s}}(O)$. We set $h_+ = 1$ in the connected component of $\mathbb{R}^{d+1} \setminus S$ containing A, $h_+ = 0$ in the connected component of $\mathbb{R}^{d+1} \setminus S$ containing B, and interpolate h_+ in an arbitrary way inside S, requiring only $\|\nabla h_+(y)\|_{\ell^2}^2 \leqslant M/(\varepsilon|\log\varepsilon|)$ for some constant M. Then the contribution $\Phi_{\mathbb{R}^{d+1}\setminus S}(h_+)$ to the capacity is zero, and it remains to estimate $\Phi_{S\setminus D}(h_+)$. By Lemma 6.1, we have

$$\begin{aligned}
\Phi_{S\setminus D}(h_+) &\leqslant \frac{M\varepsilon}{\varepsilon|\log\varepsilon|} \int_{S\setminus D} \mathrm{e}^{-m_0\sum_{k=0}^d (1+k^2)y_k^2/\varepsilon}\,\mathrm{d}y \\
&\leqslant \frac{M\varepsilon}{\varepsilon|\log\varepsilon|} \sum_{k=0}^d \prod_{j\neq k} \int_{-\infty}^{\infty} \mathrm{e}^{-m_0(1+j^2)y_j^2/\varepsilon}\,\mathrm{d}y_j \cdot 2\int_{\delta_k}^{\infty} \mathrm{e}^{-m_0(1+k^2)y_k^2/\varepsilon}\,\mathrm{d}y_k \\
&\leqslant \frac{M\varepsilon}{\varepsilon|\log\varepsilon|} \prod_{j=0}^d \sqrt{\frac{\pi\varepsilon}{m_0(1+j^2)}} \sum_{k=0}^d \varepsilon^{\kappa c_k}\,, \qquad (6.25)
\end{aligned}$$

where $\kappa > 0$ depends only on m_0 and L. Recalling the choice $c_k = c_0(1+\log(1+k))$, we find

$$\begin{aligned}
\sum_{k=0}^d \varepsilon^{\kappa c_k} &\leqslant \varepsilon^{\kappa c_0} + \int_0^d \varepsilon^{\kappa c_0(1+\log(1+x))}\,\mathrm{d}x \\
&= \varepsilon^{\kappa c_0} + \varepsilon^{\kappa c_0} \int_1^{d+1} x^{-\kappa c_0|\log\varepsilon|}\,\mathrm{d}x \\
&\leqslant \varepsilon^{\kappa c_0}\left[1 + \frac{1}{\kappa c_0|\log\varepsilon| - 1}\right], \qquad (6.26)
\end{aligned}$$

uniformly in d, provided $\kappa c_0|\log\varepsilon| > 1$. Thus we can ensure that $\Phi_{S\setminus D}(h_+)$ is negligible by making c_0 large enough. □

Proposition 6.3 (Lower bound on the capacity)**.** *There exist $r_0 > 0$, $\varepsilon_0 > 0$ and $d_0(\varepsilon) < \infty$ such that*

$$\mathrm{cap}_A(B) \geqslant \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \left(\prod_{k=1}^d \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\right)\left[1 - c_-\varepsilon^{1/2}|\log\varepsilon|^{3/2}\right] \qquad (6.27)$$

holds for all $r < r_0$, all $\varepsilon < \varepsilon_0$ and all $d \geqslant d_0(\varepsilon)$, where the constant c_- is independent of ε and d.

Proof: We write as before $y = (y_0, y_\perp)$, where $y_\perp = (y_1, \ldots, y_d)$. Let

$$\widehat{D}_\perp = \prod_{k=1}^d [-\hat{\delta}_k, \hat{\delta}_k] \qquad \text{with } \hat{\delta}_k = \sqrt{\frac{\hat{c}_k\varepsilon|\log\varepsilon|}{\lambda_k}}\,, \qquad (6.28)$$

where the constants $\hat{c}_k$ are of the form $\hat{c}_k = \hat{c}_0(1+\log(1+k))$. Note that as in the previous proof, this implies $\|y_\perp\|_{H^s} = \mathcal{O}(\sqrt{\varepsilon|\log\varepsilon|}\,)$ for $y_\perp \in \widehat{D}_\perp$ and all $s < 1$. Given $\rho > 0$, we set

$$\widehat{D} = [-\rho, \rho] \times \widehat{D}_\perp\,. \qquad (6.29)$$

Let $h^* = h_{A,B}$ denote the equilibrium potential defined by $\mathrm{cap}_A(B) = \Phi_{(A\cup B)^c}(h_{A,B})$, cf. (6.5). Then the capacity can be bounded below as follows:

$$\begin{aligned}
\mathrm{cap}_A(B) &= \Phi_{(A\cup B)^c}(h^*) \\
&\geqslant \Phi_{\widehat{D}}(h^*) \\
&= \varepsilon \int_{\widehat{D}_\perp} \int_{-\rho}^{\rho} \mathrm{e}^{-\widehat{V}(y_0,y_\perp)/\varepsilon} \|\nabla h^*(y_0,y_\perp)\|_{\ell^2}^2 \,\mathrm{d}y_0 \,\mathrm{d}y_\perp \\
&\geqslant \varepsilon \int_{\widehat{D}_\perp} \int_{-\rho}^{\rho} \mathrm{e}^{-\widehat{V}(y_0,y_\perp)/\varepsilon} \left| \frac{\partial h^*}{\partial y_0}(y_0,y_\perp) \right|^2 \mathrm{d}y_0 \,\mathrm{d}y_\perp \\
&\geqslant \varepsilon \int_{\widehat{D}_\perp} \left[\inf_{f: f(-\rho)=h^*(-\rho,y_\perp), f(\rho)=h^*(\rho,y_\perp)} \int_{-\rho}^{\rho} \mathrm{e}^{-\widehat{V}(y_0,y_\perp)/\varepsilon} f'(y_0)^2 \,\mathrm{d}y_0 \right] \mathrm{d}y_\perp \,. \qquad (6.30)
\end{aligned}$$

Solving a one-dimensional Euler–Lagrange problem, we obtain that the infimum is realised by the function f such that

$$f'(y_0) = \frac{\bigl[h^*(\rho,y_\perp) - h^*(-\rho,y_\perp)\bigr] \,\mathrm{e}^{\widehat{V}(y_0,y_\perp)/\varepsilon}}{\displaystyle\int_{a(y_\perp)}^{b(y_\perp)} \mathrm{e}^{\widehat{V}(t,y_\perp)/\varepsilon} \,\mathrm{d}t} \,. \qquad (6.31)$$

Substituting in (6.30) and carrying out the integral over y_0, we obtain

$$\mathrm{cap}_A(B) \geqslant \varepsilon \int_{\widehat{D}_\perp} \frac{\bigl[h^*(\rho,y_\perp) - h^*(-\rho,y_\perp)\bigr]^2}{\displaystyle\int_{a(y_\perp)}^{b(y_\perp)} \mathrm{e}^{\widehat{V}(y_0,y_\perp)/\varepsilon} \,\mathrm{d}y_0} \,\mathrm{d}y_\perp \,. \qquad (6.32)$$

By (6.9) (which also applies to the infinite-dimensional system) and the fact that $\|y_\perp\|_{H^s} = \mathcal{O}(\sqrt{\varepsilon|\log\varepsilon|})$, any point $(\rho, y_\perp)$ lies on the same side of the stable manifold $\mathcal{W}^{\mathrm{s}}(O)$ as u_+^*. This implies that $H((\rho,y_\perp),B) = 0$ while $H((\rho,y_\perp),A) \geqslant \eta$, where η is uniform in $y_\perp \in \widehat{D}_\perp$. We can thus apply Proposition 5.16 to obtain the existence of $H_0 > 0$ such that

$$h^*(\rho,y_\perp) = \mathbb{P}^{(\rho,y_\perp)}\bigl\{\tau_A < \tau_B\bigr\} \leqslant 4\,\mathrm{e}^{-H_0/\varepsilon} \,, \qquad (6.33)$$

provided ε is small enough and d is larger than some $d_0(\varepsilon)$. For similar reasons, we also have

$$h^*(-\rho,y_\perp) = 1 - \mathbb{P}^{(-\rho,y_\perp)}\bigl\{\tau_B < \tau_A\bigr\} \geqslant 1 - 4\,\mathrm{e}^{-H_0/\varepsilon} \,. \qquad (6.34)$$

Substituting in (6.32), we obtain

$$\mathrm{cap}_A(B) \geqslant \varepsilon \int_{\widehat{D}_\perp} \frac{\bigl[1 - 8\,\mathrm{e}^{-H_0/\varepsilon}\bigr]^2}{\displaystyle\int_{a(y_\perp)}^{b(y_\perp)} \mathrm{e}^{\widehat{V}(y_0,y_\perp)/\varepsilon} \,\mathrm{d}y_0} \,\mathrm{d}y_\perp \,. \qquad (6.35)$$

Consider now, for fixed $y_\perp \in \widehat{D}_\perp$, the function $y_0 \mapsto g(y_0) = \widehat{V}(y_0,y_\perp)$. It satisfies, for all $1/4 < s < 1/2$,

$$\begin{aligned}
g(y_0) &= -\frac{1}{2}y_0^2 + \frac{1}{2}\sum_{k=1}^{d} \lambda_k y_k^2 + \mathcal{O}\bigl(\|y\|_{H^s}^3\bigr) \,, \\
g'(y_0) &= -y_0 + \mathcal{O}\bigl(\|y\|_{H^s}^2\bigr) = -y_0 + \mathcal{O}\bigl(y_0^2\bigr) + \mathcal{O}\bigl(\varepsilon|\log\varepsilon|\bigr) \,, \\
g''(y_0) &= -1 + \mathcal{O}\bigl(\|y\|_{H^s}\bigr) = -1 + \mathcal{O}\bigl(y_0\bigr) + \mathcal{O}\bigl(\varepsilon^{1/2}|\log\varepsilon|^{1/2}\bigr) \,. \qquad (6.36)
\end{aligned}$$

The assumption on U being a double-well potential, the definitions of A, B and the implicit-function theorem imply that g admits a unique maximum at $y_0^* = \mathcal{O}\big(\varepsilon|\log\varepsilon|\big)$, and we have

$$g(y_0^*) = \frac{1}{2}\sum_{k=1}^{d}\lambda_k y_k^2 + \mathcal{O}\big(|y_0^*|^3\big) + \mathcal{O}\big(\|y_\perp\|_{H^s}^3\big) = \frac{1}{2}\sum_{k=1}^{d}\lambda_k y_k^2 + \mathcal{O}\big(\varepsilon^{3/2}|\log\varepsilon|^{3/2}\big)\ ,$$
$$g''(y_0^*) = -1 + \mathcal{O}\big(\varepsilon^{1/2}|\log\varepsilon|^{1/2}\big)\ . \tag{6.37}$$

Thus by applying standard Laplace asymptotics, we obtain

$$\int_{a(y_\perp)}^{b(y_\perp)} \mathrm{e}^{\widehat{V}(y_0,y_\perp)/\varepsilon}\,\mathrm{d}y_0 = \sqrt{2\pi\varepsilon}\exp\biggl\{\frac{1}{2\varepsilon}\sum_{k=1}^{d}\lambda_k y_k^2\biggr\}\big[1+\mathcal{O}\big(\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big)\big]\ . \tag{6.38}$$

Substituting in (6.35) yields

$$\begin{aligned}
\mathrm{cap}_A(B) &\geqslant \frac{\varepsilon}{\sqrt{2\pi\varepsilon}}\prod_{k=1}^{d}\int_{-\hat\delta_k}^{\hat\delta_k}\mathrm{e}^{-\lambda_k y_k^2/2\varepsilon}\,\mathrm{d}y_k\big[1-\mathcal{O}\big(\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big)\big] \\
&= \frac{\varepsilon}{\sqrt{2\pi\varepsilon}}\prod_{k=1}^{d}\biggl(\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\big[1-\mathcal{O}(\varepsilon^{\hat c_k/2})\big]\biggr)\big[1-\mathcal{O}\big(\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big)\big] \\
&= \frac{\varepsilon}{\sqrt{2\pi\varepsilon}}\biggl(\prod_{k=1}^{d}\sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\biggr)\biggl[1-\mathcal{O}\biggl(\sum_{k=1}^{d}\varepsilon^{\hat c_k/2}\biggr)\biggr]\big[1-\mathcal{O}\big(\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big)\big]\ ,
\end{aligned} \tag{6.39}$$

and the result follows from the same estimate as in (6.26), taking $\hat c_0 \geqslant 1$. □

6.1.2 L near π

We now turn to the case $|L-\pi| \leqslant c$, with c small. Then the eigenvalue λ_1 associated with the first Fourier mode satisfies $|\lambda_1| \leqslant \eta$, where we can assume η to be small by making c small.

Recall from Proposition 4.8 that if the local potential U is of class $\mathcal{C}^5$, there exists a change of variables $y = z + g(z)$, with $\|g(z)\|_{H^t} = \mathcal{O}(\|z\|_{H^s}^2)$ for all $5/12 < s < 1/2$ and $t < 2s - 1/2$, such that

$$\widehat{V}(z+g(z)) = \frac{1}{2}\sum_{k=0}^{d}\lambda_k z_k^2 + \frac{1}{2}C_4 z_1^4 + \mathcal{O}(\|z\|_{H^s}^5) \tag{6.40}$$

with $C_4 > 0$. Note that the factor $1/2$ in front of C_4 results from the change from complex to real Fourier series. In order to localise this change of variables, it will be convenient to introduce a $\mathcal{C}^\infty$ cut-off function $\theta : \mathbb{R}^{d+1} \to [0,1]$ satisfying

$$\theta(z) = \begin{cases} 1 & \text{for } \|z\|_{H^s} \leqslant 1\ , \\ 0 & \text{for } \|z\|_{H^s} \geqslant 2\ . \end{cases} \tag{6.41}$$

Given $\rho > 0$, we consider the potential

$$\widetilde{V}_\rho(z) = \widehat{V}\biggl(z + \theta\Big(\frac{z}{\rho}\Big)g(z)\biggr)\ , \tag{6.42}$$

which is equal to $\widehat{V}(z)$ for $\|z\|_{H^s} \geqslant 2\rho$, and to the normal form (6.40) for $\|z\|_{H^s} \leqslant \rho$. It what follows, we will always assume that $\rho > |\lambda_1|$.

The expression (6.40) of the normal form shows that for sufficiently small ρ,

- if $\lambda_1 \geqslant 0$, the origin O is the only stationary point of $\widetilde{V}_\rho$ in the ball $\|z\|_{H^1} < \rho$, and $\lambda_0 = -1$ is the only negative eigenvalue of the Hessian of $\widetilde{V}_\rho$ at O;
- if $\lambda_1 < 0$, the origin O is a stationary point with two negative eigenvalues, and there are two additional stationary points $P_\pm$ with coordinates

$$z_1^\pm = \pm\sqrt{2|\lambda_1|/C_4} + \mathcal{O}(\lambda_1)\;, \quad z_k^\pm = \mathcal{O}(\lambda_1^2/\lambda_k) \quad \text{for } k = 0, 2\ldots, d\;. \tag{6.43}$$

The symmetry (6.4) implies that $z_k^+ = (-1)^k z_k^-$ and $\widehat{V}(P_+) = \widehat{V}(P_-)$. The eigenvalues of the Hessian of the potential at $P_\pm$ are the same, owing to the symmetry, and of the form

$$\mu_1 = -2\lambda_1 + \mathcal{O}(|\lambda_1|^{3/2})\;, \quad \mu_k = \lambda_k + \mathcal{O}(|\lambda_1|^{3/2}) \quad \text{for } k = 0, 2\ldots, d\;, \tag{6.44}$$

which shows that P_+ and P_- are saddles with a one-dimensional unstable manifold, and a d-dimensional stable manifold. The unstable manifolds necessarily converge to the two local minima of the potential.

The basins of attraction of the two minima of the potential are separated by a d-dimensional manifold that we will denote $\mathcal{W}^{\mathrm{s}}$. For $\lambda_1 > 0$, $\mathcal{W}^{\mathrm{s}} = \mathcal{W}^{\mathrm{s}}(O)$ is the stable manifold of the origin. For $\lambda_1 < 0$, we have $\mathcal{W}^{\mathrm{s}} = \mathcal{W}^{\mathrm{s}}(O) \cup \mathcal{W}^{\mathrm{s}}(P_-) \cup \mathcal{W}^{\mathrm{s}}(P_+)$. See for instance [Jol89] for a picture of the situation.

Lemma 6.4 (Growth of the potential along $\mathcal{W}^{\mathrm{s}}$)**.** *Let $z_\perp = (z_2, \ldots, z_d)$. There exist constants $\rho > 0$, $m_0 > 0$ and $\eta > 0$ such that for $|\lambda_1| < \eta$ and all $z \in \mathcal{W}^{\mathrm{s}}$, one has*

$$\widetilde{V}_\rho(z) \geqslant m_0 \biggl[z_0^2 + z_1^2 \wedge \Bigl(\frac{1}{2}\lambda_1 z_1^2 + \frac{1}{2} C_4 z_1^4 \Bigr) + \|z_\perp\|_{H^1}^2 \biggr]\;. \tag{6.45}$$

Proof: The manifold $\mathcal{W}^{\mathrm{s}}$ can be locally described by a graph $z_0 = \psi(z_1, z_\perp)$, where

$$|\psi(z_1, z_\perp)| \leqslant M(z_1^4 + \|z_\perp\|_{H^s}^4) \qquad \text{whenever } z_1^2 + \|z_\perp\|_{H^s}^2 \leqslant \rho_0^2 \tag{6.46}$$

for any $s > 1/4$ and some $M > 0$ and $\rho_0 > 0$. This implies

$$\widetilde{V}_\rho(\psi(z_1, z_\perp), z_1, z_\perp) = \frac{1}{2}\lambda_1 z_1^2 + \frac{1}{2} C_4 z_1^4 + \frac{1}{2}\sum_{k=2}^d \lambda_k z_k^2 + \mathcal{O}(|z_1^5| + \|z_\perp\|_{H^s}^5) \tag{6.47}$$

for $z_1^2 + \|z_\perp\|_{H^s}^2 \leqslant (\rho_0 \wedge \rho)^2 = \rho_1^2$, and proves (6.45) for $\|z\|_{H^s} \leqslant \rho_1$. In particular, for $z_1^2 + \|z_\perp\|_{H^1}^2 = \rho_1^2$, we obtain the existence of a constant $m_2 > 0$ such that $\widetilde{V}_\rho(z) \geqslant m_2 \rho_1^4$, provided $|\lambda_1|$ is small enough. The remainder of the proof is similar to the proof of Lemma 6.1. □

Proposition 6.5 (Upper bound on the capacity)**.** *There exist constants $\varepsilon_0, \eta, c_+ > 0$ such that for $\varepsilon < \varepsilon_0$ and $d \geqslant 1$,*

1. *If $0 \leqslant \lambda_1 \leqslant \eta$, then*

$$\mathrm{cap}_A(B) \leqslant \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \int_{-\infty}^{\infty} \mathrm{e}^{-u_1(y_1)/\varepsilon}\, \mathrm{d}y_1 \biggl(\prod_{k=2}^d \sqrt{\frac{2\pi\varepsilon}{\lambda_k}} \biggr) \bigl[1 + c_+ R(\varepsilon, \lambda_1) \bigr]\;, \tag{6.48}$$

where

$$u_1(y_1) = \frac{1}{2}\lambda_1 y_1^2 + \frac{1}{2} C_4 y_1^4 \tag{6.49}$$

and

$$R(\varepsilon, \lambda) = \biggl[\frac{\varepsilon |\log \varepsilon|^3}{\lambda \vee \sqrt{\varepsilon |\log \varepsilon|}} \biggr]^{1/2}\;. \tag{6.50}$$

2. If $-\eta < \lambda_1 < 0$, then

$$\mathrm{cap}_A(B) \leqslant 2\varepsilon\sqrt{\frac{|\mu_0|}{2\pi\varepsilon}} \int_0^\infty \mathrm{e}^{-u_2(y_1)/\varepsilon}\,\mathrm{d}y_1 \left(\prod_{k=2}^d \sqrt{\frac{2\pi\varepsilon}{\mu_k}}\right) \mathrm{e}^{-\widehat{V}(P_\pm)/\varepsilon}\big[1 + c_+ R(\varepsilon,\mu_1)\big]\;, \tag{6.51}$$

where

$$u_2(y_1) = \frac{1}{2}C_4\left(y_1^2 - \frac{\mu_1}{4C_4}\right)^2\;. \tag{6.52}$$

Proof: The proof is similar to those of [BG10, Proposition 5.1 and Theorem 4.1], the main difference lying in the dimension-dependence of the domains of integration. The capacity can be bounded above by $\Phi_{A\cup B}(h_+)$ for any $h_+ \in \mathcal{H}_{A,B}$. The change of variables $y = z + g(z)$ and (4.36) lead to

$$\Phi_{A\cup B}(h_+) = \varepsilon \int_{(A\cup B)^c} \mathrm{e}^{-\widetilde{V}_\rho(z)/\varepsilon} \|\nabla h_+(z)\|_{\ell^2}^2 \Big[1 + 1_{\{\|z\|_{H^s}\leqslant\rho\}}\mathcal{O}\big(\|z\|_{H^s}\big)\Big]\,\mathrm{d}z\;. \tag{6.53}$$

Consider first the case $\lambda_1 \geqslant 0$. Let D be a box defined by (6.19), where we take δ_k as in (6.18) for $k \neq 1$, while δ_1 is the positive solution of $u_1(\delta_1) = c_1\varepsilon|\log\varepsilon|$, which satisfies

$$\delta_1^2 = \mathcal{O}\left(\frac{\varepsilon|\log\varepsilon|}{\lambda_1 \vee \sqrt{\varepsilon|\log\varepsilon|}}\right)\;. \tag{6.54}$$

Note that if $z \in D$, then $\|z\|_{H^s} = \mathcal{O}(\delta_1)$ for all $s < 1$. This ensures that the potential $\widetilde{V}_\rho$ is given by the normal form (6.40). The rest of the proof then proceeds exactly as in Proposition 6.2. We have slightly overestimated the logarithmic part of the error terms to get more compact expressions.

For $-c\sqrt{\varepsilon|\log\varepsilon|} \leqslant \lambda_1 < 0$, the proof is the same, with δ_1 of order $(\varepsilon|\log\varepsilon|)^{1/4}$. Note that in this case, the potential at the saddles $P_\pm$ has order $\varepsilon|\log\varepsilon|$, so that $\mathrm{e}^{-\widetilde{V}_\rho(P_\pm)/\varepsilon}$ is still close to 1 for small c.

Finally, for $-\eta \leqslant \lambda_1 < -c\sqrt{\varepsilon|\log\varepsilon|}$, we evaluate separately the capacities on each half-space $\{z_1 < 0\}$ and $\{z_1 > 0\}$. Each Dirichlet form is dominated by the integral over a box around P_+, respectively P_-, where the extension of the box in the z_1-direction is of order $\sqrt{\varepsilon|\log\varepsilon|/\mu_1}$. The main point is to notice that

$$u_1(z_1) = \frac{1}{2}C_4\left(y_1^2 - \frac{\mu_1}{48C_4}\right)^2 + \widetilde{V}_\rho(P_\pm) + \mathcal{O}(\mu_1^{3/2}y_1^2) \tag{6.55}$$

(see [BG10, Proposition 5.4]). □

Remark 6.6. As shown in [BG10, Section 5.4], the integrals of $\mathrm{e}^{-u_1(y_1)/\varepsilon}$ and $\mathrm{e}^{-u_2^\pm(y_1)/\varepsilon}$ can be expressed in terms of Bessel functions, yielding the functions $\Psi_\pm$ given in (2.28) and (2.29).

Proposition 6.7 (Lower bound on the capacity)**.** *There exist constants $\varepsilon_0, \eta, c_- > 0$ and $d_0(\varepsilon) < \infty$ such that for $\varepsilon < \varepsilon_0$ and $d \geqslant d_0(\varepsilon)$,*

1. If $0 \leqslant \lambda_1 \leqslant \eta$, then

$$\mathrm{cap}_A(B) \geqslant \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \int_{-\infty}^\infty \mathrm{e}^{-u_1(y_1)/\varepsilon}\,\mathrm{d}y_1 \left(\prod_{k=2}^d \sqrt{\frac{2\pi\varepsilon}{\lambda_k}}\right)\big[1 - c_- R(\varepsilon,\lambda_1)\big]\;, \tag{6.56}$$

where u_1 and R are defined in (6.49) *and* (6.50)*.*

2. If $-\eta < \lambda_1 < 0$, then

$$\mathrm{cap}_A(B) \geqslant 2\varepsilon\sqrt{\frac{|\mu_0|}{2\pi\varepsilon}} \int_0^\infty \mathrm{e}^{-u_2(y_1)/\varepsilon}\,\mathrm{d}y_1 \left(\prod_{k=2}^d \sqrt{\frac{2\pi\varepsilon}{\mu_k}}\right) \mathrm{e}^{-\widehat{V}(P_\pm)/\varepsilon}\big[1 - c_- R(\varepsilon,\mu_1)\big] \;, \tag{6.57}$$

where u_2 is the function defined in (6.52)*.*

PROOF: For $-c\sqrt{\varepsilon|\log\varepsilon|} \leqslant \lambda_1 \leqslant \eta$, the proof is exactly the same as the proof of Proposition 6.3, except that $\hat{\delta}_1$ is defined in a similar way as δ_1 in (6.54), and thus the error terms are larger. For $-\eta \leqslant \lambda_1 < c\sqrt{\varepsilon|\log\varepsilon|}$, the definition of the set $\widehat{D}$ has to be slightly modified. Since the same modification is needed for all $L - \pi$ of order 1, we postpone that part of the proof to the next subsection. □

6.1.3 $L > \pi$

We finally consider the case $L \geqslant \pi + c$. Recall from Section 2.2 the following properties of the deterministic system:

1. The infinite-dimensional system has exactly two saddles of index 1, given by functions $u_\pm^*(x)$ of class $\mathcal{C}^2$ (at least). The fact that $u_\pm^*(x) \in \mathcal{C}^2$ implies that their Fourier components decrease like k^{-2}.
2. The Hessian of V corresponds to the second Fréchet derivative of V at $u_\pm^*$, given by the map

$$\begin{aligned}(v_1, v_2) &\mapsto \int_0^L \Big[-v_1''(x) + U''(u_\pm^*(x))v_1(x)\Big]v_2(x)\,\mathrm{d}x \\ &= \int_0^L \Big[v_1'(x)v_2'(x) + U''(u_\pm^*(x))v_1(x)v_2(x)\Big]\,\mathrm{d}x \;. \end{aligned} \tag{6.58}$$

The eigenvalues μ_k of the Hessian are solutions of the Sturm–Liouville problem $v''(x) = -U''(u_\pm^*(x))v(x)$. They satisfy $\mu_0 < 0 < \mu_1 < \dots$ and

$$-\gamma_1 + \gamma_2 k^2 \leqslant \mu_k \leqslant \gamma_3 k^2 \qquad \forall k \;, \tag{6.59}$$

for some constants $\gamma_1, \gamma_2, \gamma_3 > 0$ (this follows from expressions for the asymptotics of the eigenvalues of Sturm–Liouville equations, see for instance [VS00]). In Fourier variables, we have

$$\nabla^2\widehat{V}(u_\pm^*) = \Lambda + Q(u_\pm^*) \;, \tag{6.60}$$

where Λ is a diagonal matrix with entries λ_k, and the matrix Q represents the second summand in the integral (6.58). Thus if v has Fourier coefficients z, we have

$$\langle z, Q(u_\pm^*)z\rangle = \int_0^L U''(u_\pm^*(x))v(x)^2\,\mathrm{d}x \;. \tag{6.61}$$

If M is a constant such that $|U''(u_\pm^*(x))| \leqslant M$, for all x, we get

$$\big|\langle z, Q(u_\pm^*)z\rangle\big| \leqslant M\|v\|_{L^2}^2 = M\|z\|_{\ell^2}^2 \;. \tag{6.62}$$

3. As shown in Section 4.4, similar statements hold true for the finite-dimensional potential for sufficiently large d. As above we denote the two saddles by $P_\pm$, and the eigenvalues of the Hessian by $\mu_k = \mu_k(d)$. Let $S_\pm$ be the orthogonal change-of-basis matrices such that

$$S_\pm\nabla^2\widehat{V}(u_\pm^*)S_\pm^{\mathrm{T}} = \mathrm{diag}(\mu_0, \dots, \mu_d) \;. \tag{6.63}$$

Lemma 6.8 (Equivalence of norms)**.** *There exists a constant $\beta_0 > 0$, independent of d, such that*

$$\beta_0 \|y\|_{H^1}^2 \leqslant \|S_\pm y\|_{H^1}^2 \leqslant \frac{1}{\beta_0} \|y\|_{H^1}^2 \,. \tag{6.64}$$

Proof: On one hand, the $S_\pm$ being orthogonal, y and $z = S_\pm y$ have the same ℓ^2-norm and we have the obvious bound

$$\|y\|_{H^1}^2 \geqslant \|y\|_{\ell^2}^2 = \|z\|_{\ell^2}^2 \,. \tag{6.65}$$

On the other hand, using $1 + k^2 = 1 + (1 + \lambda_k)L^2/\pi^2$ and again equality of the ℓ^2-norms, we get

$$\|y\|_{H^1}^2 = \left(1 + \frac{L^2}{\pi^2}\right) \|z\|_{\ell^2}^2 + \frac{L^2}{\pi^2} \sum_{k=0}^{d} \lambda_k y_k^2 \,. \tag{6.66}$$

Now by (6.60) and (6.62), we have

$$\sum_{k=0}^{d} \mu_k z_k^2 = \langle y, \nabla^2 \widehat{V}(P_+) y \rangle = \sum_{k=0}^{d} \lambda_k y_k^2 + \langle y, Qy \rangle \leqslant \sum_{k=0}^{d} \big(\lambda_k + M\big) y_k^2 \,. \tag{6.67}$$

It follows that

$$\|y\|_{H^1}^2 \geqslant \sum_{k=0}^{d} \left(1 + \frac{L^2}{\pi^2}\big[1 + \mu_k - M\big]\right) z_k^2 =: \sum_{k=0}^{d} c_k z_k^2 \,. \tag{6.68}$$

The lower bound (6.59) implies that

$$c_k \geqslant 1 + \frac{L^2}{\pi^2}\big[1 - M - \gamma_1\big] + \gamma_2 \frac{L^2}{\pi^2} k^2 \,. \tag{6.69}$$

Let k_0 be the smallest integer such that $c_{k_0} \geqslant 1$. We may assume $d > k_0$, since otherwise there is nothing to prove. It is easy to check that

$$c_k \geqslant \begin{cases} -\beta_1 & \text{for } 0 \leqslant k \leqslant k_0 \,, \\ 1 + \beta_2 k^2 & \text{for } k_0 + 1 \leqslant k \leqslant d \,, \end{cases} \tag{6.70}$$

where $\beta_1 = (M + \gamma_1 - 1)(L^2/\pi^2) - 1$ and $\beta_2 = \gamma_2 L^2/((k_0^2 + 1)\pi^2)$. Thus setting

$$a = \sum_{k=0}^{k_0} z_k^2 \,, \qquad b_1 = \sum_{k=k_0+1}^{d} z_k^2 \,, \qquad b_2 = \sum_{k=k_0+1}^{d} (1 + \beta_2 k^2) z_k^2 \,, \tag{6.71}$$

we can write the bounds (6.65) and (6.68) in the form

$$\|y\|_{H^1}^2 \geqslant a + b_1 \qquad \text{and} \qquad \|y\|_{H^1}^2 \geqslant -\beta_1 a + b_2 \,. \tag{6.72}$$

By distinguishing the cases $(1 + \beta_1)a \leqslant -b_1 + b_2$ and $(1 + \beta_1)a > -b_1 + b_2$, one can deduce from these two inequalities that

$$\|y\|_{H^1}^2 \geqslant \frac{a + b_2}{2 + \beta_1} \,, \tag{6.73}$$

which implies $\|y\|_{H^1}^2 \geqslant \beta_0 \|z\|_{H^1}^2$ for some $\beta_0 > 0$. The inequality $\|z\|_{H^1}^2 \geqslant \beta_0 \|y\|_{H^1}^2$ can be proved in a similar way, using the upper bound on the μ_k. □

We denote again by $\mathcal{W}^{\mathrm{s}}$ the basin boundary, which is formed by the closure of the stable manifolds of P_+ and P_-.

Lemma 6.9 (Growth of the potential along $\mathcal{W}^{\mathrm{s}}$)**.** *There exists a constant $m_0 > 0$ such that for all $y \in \mathcal{W}^{\mathrm{s}}$,*

$$\widehat{V}(y) - \widehat{V}(P_+) = \widehat{V}(y) - \widehat{V}(P_-) \geqslant m_0\big(\|y - P_+\|_{H^1}^2 \wedge \|y - P_-\|_{H^1}^2\big) \,. \tag{6.74}$$

Proof: We have

$$\widehat{V}(P_+ + S_+^{\mathrm{T}} z) = \widehat{V}(P_+) + \frac{1}{2}\sum_{k=0}^{d} \mu_k z_k^2 + \mathcal{O}(\|z\|_{H^s}^3) \tag{6.75}$$

for any $s > 1/4$. Since the stable manifold can be described locally by an equation of the form $z_0 = g(z_\perp)$, we obtain, as in the proof of Lemma 6.1, the existence of constants $m_1, \rho_1 > 0$ such that

$$\widehat{V}(P_+ + S_+^{\mathrm{T}} z) \geqslant \widehat{V}(P_+) + m_1 \|z\|_{H^1}^2 \qquad \forall z \colon P_+ + S^{\mathrm{T}} z \in \mathcal{W}^{\mathrm{s}} \,,\, \|z\|_{H^1} \leqslant \rho_1 \,. \tag{6.76}$$

By Lemma 6.8, this implies

$$\widehat{V}(y) \geqslant \widehat{V}(P_+) + \beta_0 m_1 \|y - P_+\|_{H^1}^2 \qquad \forall y \in \mathcal{W}^{\mathrm{s}} \colon \|y - P_+\|_{H^1} \leqslant \sqrt{\beta_0}\, \rho_1 \,. \tag{6.77}$$

A similar bound holds in the neighbourhood of P_-.

Now choose a $\gamma > 0$ such that $-\alpha + \beta\gamma^2/4 \geqslant 1$, where α and β are the constants appearing in (6.15), and such that $\gamma \geqslant 3\|P_+\|_{H^1}$. We want to consider the case of $y \in \mathcal{W}^{\mathrm{s}}$ satisfying $\sqrt{\beta_0}\rho_1 \leqslant \|y - P_+\|_{H^1} \wedge \|y - P_-\|_{H^1} \leqslant \gamma$. Without loss of generality we may assume $\|y - P_+\|_{H^1} \geqslant \|y - P_-\|_{H^1}$. As in the proof of Lemma 6.1, we use the fact that the vector field $-\nabla\widehat{V}(y)$ is pointing inward. Thus,

$$\widehat{V}(y) - \widehat{V}(P_+) \geqslant m_1\beta_0(\sqrt{\beta_0}\rho_1)^2 \geqslant m_1\beta_0^2 \frac{\rho_1^2}{\gamma^2}\big(\|y - P_+\|_{H^1}^2 \wedge \|y - P_-\|_{H^1}^2\big) \,. \tag{6.78}$$

Together with (6.15) for $\|y - P_+\|_{H^1} \wedge \|y - P_-\|_{H^1} > \gamma$, this proves (6.74) for all $y \in \mathcal{W}^{\mathrm{s}}$. □

Proposition 6.10 (Upper bound on the capacity)**.** *There exist $r_0, \varepsilon_0 > 0$ and $d_0 < \infty$ such that for $r < r_0$, $\varepsilon < \varepsilon_0$ and $d \geqslant d_0$,*

$$\mathrm{cap}_A(B) \leqslant 2\varepsilon\sqrt{\frac{|\mu_0|}{2\pi\varepsilon}}\left(\prod_{k=1}^{d}\sqrt{\frac{2\pi\varepsilon}{\mu_k}}\right) \mathrm{e}^{-\widehat{V}(P_\pm)/\varepsilon}\big[1 + c_+\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big] \,, \tag{6.79}$$

where the constant c_+ is independent of ε and d.

Proof: The proof is similar to the proof of Proposition 6.2. We first compute the Dirichlet form over a box D_+, defined in rotated coordinates $z = S_+(y - P_+)$ by $|z_k| \leqslant \sqrt{c_k\varepsilon|\log\varepsilon|/|\mu_k|}$. Constructing h_+ as a function of z_0 as before yields a contribution equal to half the expression in (6.79). The other half comes from a similar contribution from a box D_- centred in P_-. The remaining part of the Dirichlet form can be shown to be negligible with the help of Lemmas 6.8 and 6.9. □

Proposition 6.11 (Lower bound on the capacity)**.** *There exist $r_0, \varepsilon_0 > 0$ and $d_0(\varepsilon) < \infty$ such that for $r < r_0$, $\varepsilon < \varepsilon_0$ and $d \geqslant d_0(\varepsilon)$,*

$$\mathrm{cap}_A(B) \geqslant 2\varepsilon \sqrt{\frac{|\mu_0|}{2\pi\varepsilon}} \left(\prod_{k=1}^{d} \sqrt{\frac{2\pi\varepsilon}{\mu_k}} \right) \mathrm{e}^{-\widehat{V}(P_\pm)/\varepsilon} \big[1 - c_- \varepsilon^{1/2} |\log \varepsilon|^{3/2} \big] \,, \tag{6.80}$$

where the constant c_- is independent of ε and d.

Proof: We perform the change of variables $y = P_+ + S_+^{\mathrm{T}} z$ in the Dirichlet form, which is an isometry, and thus of unit Jacobian. Let A', B' denote the images of A and B under the inverse isometry.

Let $\widetilde{V}(z) = \widehat{V}(S_+(y - P_+))$ be the expression of the potential in the new variables, given by (6.75). We define $\widehat{D}_\perp$ as in (6.28) and set

$$\widehat{D}_+ = \big\{ z = (z_0, z_\perp) \colon z_\perp \in \widehat{D}_\perp, -\rho < z_0 < \rho \big\} \,. \tag{6.81}$$

Then by the same computation as in (6.30)–(6.35), we have

$$\Phi_{\widehat{D}_+}(h^*) \geqslant \varepsilon \int_{\widehat{D}_\perp} \frac{\big[1 - \mathcal{O}(\mathrm{e}^{-H_0/\varepsilon})\big]^2}{\displaystyle\int_{-\rho}^{\rho} \mathrm{e}^{\widetilde{V}(z_0, z_\perp)/\varepsilon} \, \mathrm{d}z_0} \, \mathrm{d}z_\perp \,. \tag{6.82}$$

The function $z_0 \mapsto \widetilde{V}(z_0, z_\perp)$ admits its maximum in a point $z_0^* = \mathcal{O}(\varepsilon |\log \varepsilon|)$. We can thus apply the Laplace method to obtain

$$\int_{-\rho}^{\rho} \mathrm{e}^{\widetilde{V}(z_0, \varphi(z_0) + z_\perp)/\varepsilon} \, \mathrm{d}z_0 = \sqrt{\frac{2\pi\varepsilon}{|\mu_0|}} \, \mathrm{e}^{\widehat{V}(P_+)/\varepsilon} \exp\left\{ \frac{1}{2\varepsilon} \sum_{k=1}^{d} \mu_k z_k^2 \right\} \big[1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{3/2}) \big] \,. \tag{6.83}$$

Substituting this into (6.82), the Dirichlet form $\Phi_{\widehat{D}_+}(h^*)$ can be estimated as in (6.39).

Now a similar estimate holds for the Dirichlet form $\Phi_{\widehat{D}_-}(h^*)$ on a set $\widehat{D}_-$ constructed around P_-. The two sets may overlap, but the contribution of the overlap to the capacity is negligible. □

6.2 Periodic b.c.

We turn now to the study of capacities for periodic b.c. Since most arguments are the same as for Neumann b.c., we only give the main results and briefly comment on a few differences.

The potential energy (6.1) is invariant under translations $u \mapsto u(\cdot + \varphi)$. As a consequence, when expressed in Fourier variables it satisfies the symmetry

$$\widehat{V}\big(\{z_k\}_{-d \leqslant k \leqslant d}\big) = \widehat{V}\big(\{\mathrm{e}^{2\,\mathrm{i}\,\pi k \varphi / L} \, z_k\}_{-d \leqslant k \leqslant d}\big) \,. \tag{6.84}$$

The eigenvalues of the Hessian of $\widehat{V}$ at the origin are of the form

$$\lambda_k = -1 + \left(\frac{2k\pi}{L} \right)^2 \,, \qquad k = -d, \dots, d \,, \tag{6.85}$$

and are thus doubly degenerate for $k \neq 0$.

The case $L \leqslant 2\pi - c$ is treated in exactly the same way as the case $L \leqslant \pi - c$ for Neumann b.c., with the result

$$\mathrm{cap}_A(B) = \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \Bigg(\prod_{k=1}^{d} \frac{2\pi\varepsilon}{\lambda_k} \Bigg) \big[1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{3/2}) \big] \,, \tag{6.86}$$

where the error term is uniform in d.

For $2\pi - c < L \leqslant 2\pi$, the capacity can again be estimated by using the normal form. The only difference is that the centre manifold is now two-dimensional, which leads to the expression

$$\mathrm{cap}_A(B) = \frac{\varepsilon}{\sqrt{2\pi\varepsilon}} \int_0^{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-u(r_1,\varphi_1)/\varepsilon} \, r_1 \, \mathrm{d}r_1 \, \mathrm{d}\varphi_1 \Bigg(\prod_{k=2}^{d} \frac{2\pi\varepsilon}{\lambda_k} \Bigg) \big[1 + \mathcal{O}(R(\varepsilon, \lambda_1)) \big] \,, \tag{6.87}$$

where

$$u(r_1, \varphi_1) = \frac{1}{2} \lambda_1 r_1^2 + C_4 r_1^4 \tag{6.88}$$

results from the terms in $z_{\pm 1}$ written in polar coordinates, and $R(\varepsilon, \lambda_1)$ is the same as in (6.50). The integral can be expressed in terms of the distribution function of a Gaussian random variable, cf. [BG10, Section 5.4].

For $2\pi \leqslant L \leqslant 2\pi + c$, the expression for the capacity is given by (6.87) with an extra term $\mathrm{e}^{V/\varepsilon}$, where $V \simeq -\lambda_1^2/(16 C_4 \varepsilon)$ is the value of the potential at the transition state.

Finally in the case $L \geqslant 2\pi + c$, we have to take into account the fact that instead of isolated transition states, there is a whole family of transition states $\{P(\varphi)\}_{0 \leqslant \varphi < L}$, satisfying by symmetry

$$P_k(\varphi) = \mathrm{e}^{2\,\mathrm{i}\,\pi k \varphi / L} \, P_k(0) \,. \tag{6.89}$$

The eigenvalues μ_k of the Hessian at any transition state satisfy

$$\mu_0 < \mu_{-1} = 0 < \mu_1 < \mu_2, \mu_{-2} < \dots \tag{6.90}$$

When evaluating the Dirichlet form, we construct an approximation of the equilibrium potential in a neighbourhood of the transition states in a way which is invariant under the symmetry. The result is

$$\mathrm{cap}_A(B) = \frac{\varepsilon}{\sqrt{2\pi\varepsilon |\mu_0|}} \ell_{\mathrm{saddle}} \Bigg(\frac{2\pi\varepsilon}{\mu_1} \prod_{|k| \geqslant 2}^{d} \frac{2\pi\varepsilon}{\mu_k} \Bigg)^{1/2} \mathrm{e}^{-\widehat{V}(P(0))/\varepsilon} \big[1 + \mathcal{O}(\varepsilon^{1/2} |\log \varepsilon|^{3/2}) \big] \,, \tag{6.91}$$

where ℓ_{saddle} is the "length of the saddle", due to the integration along the direction with vanishing eigenvalue μ_{-1}. It is given by

$$\ell_{\mathrm{saddle}} = \int_0^L \left\| \frac{\partial P}{\partial \varphi} \right\|_{\ell^2} \mathrm{d}\varphi \,, \tag{6.92}$$

where (6.89) shows that

$$\left\| \frac{\partial P}{\partial \varphi} \right\|_{\ell^2}^2 = \sum_{k=-d}^{d} \left(\frac{2\pi k}{L} \right)^2 |P_k(0)|^2 \,, \tag{6.93}$$

which converges as $d \to \infty$, by Parseval's identity, to $\|(u^*_{1,0})'\|^2_{L^2}$. Hence we have

$$\lim_{d\to\infty} \ell_{\text{saddle}}(d) = L\|(u^*_{1,0})'\|_{L^2} \; . \tag{6.94}$$

When L is close to 2π, the normal form shows that $|P_1(0)|^2 = |\lambda_1|/(2C_4) + \mathcal{O}(\lambda_1^2)$, while the other components of $P(0)$ are of order λ_1^2. Also the eigenvalue μ_1 satisfies $\mu_1 = -2\lambda_1 + \mathcal{O}(|\lambda_1|^{3/2})$. This shows that $\ell_{\text{saddle}} = 2\pi\sqrt{\mu_1/(8C_4)} + \mathcal{O}(\mu_1)$, and allows to check that the expressions (6.91) and (6.87) for the capacity are indeed compatible.

7 Uniform bounds on expected first-hitting times

7.1 Integrating the equilibrium potential against the invariant measure

We define as before the sets $A, B \subset E$ as the open balls

$$\begin{aligned} A &= \big\{u \in E\colon \|u - u^*_-\|_{L^\infty} < r\big\} \; , \\ B &= \big\{u \in E\colon \|u - u^*_+\|_{L^\infty} < \rho\big\} \; . \end{aligned} \tag{7.1}$$

The aim of this subsection is to obtain sharp upper and lower bounds on the integral

$$J_d(A,B) = \int_{E_d\setminus B_d} h^{(d)}_{A_d,B_d}(y)\, \mathrm{e}^{-\widehat{V}(y)/\varepsilon}\, \mathrm{d}y \; , \tag{7.2}$$

where $h^{(d)}_{A_d,B_d}$ is the equilibrium potential

$$h^{(d)}_{A_d,B_d}(y) = \mathbb{P}^y\big\{\tau^{(d)}_{A_d} < \tau^{(d)}_{B_d}\big\} \; . \tag{7.3}$$

Recall that the local minima $u^*_\pm$ of V are also local minima of the truncated potential, and that the eigenvalues of the Hessian of the potential at u^*_- are given by $\nu^-_k = (bk\pi/L)^2 + U''(u_-)$, where $b = 1$ and $k \in \mathbb{N}_0$ for Neumann b.c., and $b = 2$ and $k \in \mathbb{Z}$ for periodic b.c. Recall that $u_\pm$ denote the minima of the local potential U.

Proposition 7.1 (Upper bound on the integral)**.** *There exist constants $r_0 > 0$, $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, there exists a $d_0 = d_0(\varepsilon) < \infty$ such that*

$$J_d(A,B) \leqslant \prod_{|k|\leqslant d} \sqrt{\frac{2\pi\varepsilon}{\nu^-_k}}\, \mathrm{e}^{-V[u^*_-]/\varepsilon}\big[1 + c_+\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big] \tag{7.4}$$

for all $0 < r, \rho < r_0$, and all $d \geqslant d_0$, where the constant c_+ is independent of ε and d.

Proof: Let $\delta_k = \sqrt{c_k\varepsilon|\log\varepsilon|/\nu^-_k}$, where $c_k = c_0(1 + \log(1 + |k|))$. We introduce two sets

$$\begin{aligned} C_d &= [u_- - \delta_0, u_- + \delta_0] \times \prod_{0<|k|\leqslant d} [-\delta_k, \delta_k] \; , \\ D_d &= \big\{y \in E_d\colon \overline{V}(y, u^*_-) > 0\big\} \; , \end{aligned} \tag{7.5}$$

and split the domain of integration into C_d, $D_d \setminus B_d$, and the remaining part of $E_d \setminus B_d$. By Laplace asymptotics (cf. the quadratic approximation argument used in the proof of Proposition 6.2) one obtains that

$$\int_{C_d} h^{(d)}_{A_d,B_d}(y)\, \mathrm{e}^{-\widehat{V}(y)/\varepsilon}\, \mathrm{d}y \leqslant \int_{C_d} \mathrm{e}^{-\widehat{V}(y)/\varepsilon}\, \mathrm{d}y \tag{7.6}$$

satisfies the upper bound (7.4). To bound the integral over D_d, we use the bound on the equilibrium potential in Proposition 5.17 to get

$$\int_{D_d} h^{(d)}_{A_d,B_d}(y)\,\mathrm{e}^{-\widehat{V}(y)/\varepsilon}\,\mathrm{d}y \leqslant 3\int_{D_d}\left(\mathrm{e}^{-[\overline{V}(y,A)-\eta+\widehat{V}(y)]/\varepsilon}+\mathrm{e}^{-[1/\eta+\widehat{V}(y)]/\varepsilon}\right). \tag{7.7}$$

If u^*_{ts} denotes a transition state, we have $\overline{V}(y,A) = V[u^*_{\mathrm{ts}}] - \widehat{V}(y)$. Choosing η small enough that $1/\eta \geqslant V[u^*_{\mathrm{ts}}] - V[u^*_+]$, we thus obtain

$$\int_{D_d} h^{(d)}_{A_d,B_d}(y)\,\mathrm{e}^{-\widehat{V}(y)/\varepsilon}\,\mathrm{d}y \leqslant 6\,\mathrm{e}^{-(V[u^*_{\mathrm{ts}}]-\eta)/\varepsilon}\int_{D_d}\mathrm{d}y\,, \tag{7.8}$$

The lower bound (4.20) on the potential implies that D_d is contained in a set $\{\|y\|_{H^1} \leqslant M\}$ for some M. The scaling $y_k = \sqrt{M/(1+k^2)}z_k$ shows that

$$\int_{D_d}\mathrm{d}y \leqslant M^{d+1/2}\prod_{|k|\leqslant d}\frac{1}{\sqrt{1+k^2}}\int_{\mathbb{S}^{2d}}\mathrm{d}z\,. \tag{7.9}$$

The volume of the sphere $\mathbb{S}^{2d}$ is given by $2\pi^d/\Gamma(d)$, which by Stirling's formula is bounded by $(M_1/d)^d$ for some constant M_1. Thus choosing d_0 of order $1/\varepsilon$ or larger ensures that the integral (7.7) is negligible if we take η small enough.

Finally, we can bound the integral of $\mathrm{e}^{-\widehat{V}(y)/\varepsilon}$ over the remaining space in the same way as in the proof of Proposition 6.2, using again (4.20) to bound the potential below by a quadratic form. Choosing c_0 large enough ensures that this integral is negligible as well. □

Proposition 7.2 (Lower bound on the integral)**.** *There exist constants $r_0 > 0$, $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, there exists a $d_0 = d_0(\varepsilon) < \infty$ such that*

$$J_d(A,B) \geqslant \prod_{|k|\leqslant d}\sqrt{\frac{2\pi\varepsilon}{\nu^-_k}}\,\mathrm{e}^{-V[u^*_-]/\varepsilon}\bigl[1-c_-\varepsilon^{1/2}|\log\varepsilon|^{3/2}\bigr] \tag{7.10}$$

for all $0 < r, \rho < r_0$, and all $d \geqslant d_0$, where the constant c_- is independent of ε and d.

Proof: We define C_d as in the previous proof. The fact that $h^{(d)}_{A_d,B_d}(y) = 1 - h^{(d)}_{B_d,A_d}(y)$ shows that

$$\begin{aligned} J_d(A,B) &\geqslant \int_{C_d} h^{(d)}_{A_d,B_d}(y)\,\mathrm{e}^{-\widehat{V}(y)/\varepsilon}\,\mathrm{d}y \\ &= \int_{C_d}\mathrm{e}^{-\widehat{V}(y)/\varepsilon}\,\mathrm{d}y - \int_{C_d} h^{(d)}_{B_d,A_d}(y)\,\mathrm{e}^{-\widehat{V}(y)/\varepsilon}\,\mathrm{d}y\,. \end{aligned} \tag{7.11}$$

The first term on the right-hand side satisfies the claimed lower bound, by a computation similar to the one in the proof of Proposition 6.3, cf. (6.39). Proposition 5.16 shows that the second term on the right-hand side is smaller than the first one by an exponentially small term. □

7.2 Averaged bounds on expected first-hitting times

We define the sets A and B as in (7.1). According to (3.11),

$$\nu_{A,B}^{(d)}(dz) = \frac{-e_{A_d,B_d}(dz)\,\mathrm{e}^{-\widehat{V}(z)/\varepsilon}}{\mathrm{cap}_{A_d}(B_d)} \tag{7.12}$$

is a probability measure on ∂A_d.

The following result implies Proposition 3.3.

Proposition 7.3. *There exist $r_0, \varepsilon_0 > 0$ such that for $0 < r, \rho < r_0$ and $0 < \varepsilon < \varepsilon_0$, there exists a $d_0 = d_0(\varepsilon) < \infty$ such that for all $d \geqslant d_0$,*

$$C(d,\varepsilon)\,\mathrm{e}^{H(d)/\varepsilon}\big[1 - R_{d,B}^{-}(\varepsilon)\big] \leqslant \int_{\partial A_d} \mathbb{E}^z\big\{\tau_{B_d}^{(d)}\big\}\nu_{A,B}^{(d)}(dz) \leqslant C(d,\varepsilon)\,\mathrm{e}^{H(d)/\varepsilon}\big[1 + R_{d,B}^{+}(\varepsilon)\big]\,, \tag{7.13}$$

where the quantities $C(d,\varepsilon)$, $H(d)$ and $R_{d,B}^{\pm}(\varepsilon)$ are detailed below.

Proof: By (3.12), we have

$$\int_{\partial A_d} \mathbb{E}^z\big\{\tau_{B_d}^{(d)}\big\}\nu_{A,B}^{(d)}(dz) = \frac{J_d(A,B)}{\mathrm{cap}_{A_d}(B_d)}\,. \tag{7.14}$$

Hence the result follows immediately from Propositions 7.1, 7.2 and the bounds on capacities obtained in Section 6. □

We end by listing the expressions of the quantities appearing in (7.13). In the case of Neumann b.c., they are of the following form, depending on the value of L.

- For $L < \pi - c$, Propositions 6.2 and 6.3 yield a prefactor

$$C(d,\varepsilon) = 2\pi\bigg(\frac{1}{|\lambda_0|\nu_0^-}\prod_{k=1}^{d}\frac{\lambda_k}{\nu_k^-}\bigg)^{1/2} \tag{7.15}$$

  (recall that $\lambda_0 = -1$). As $d \to \infty$, the product converges to an infinite product which is finite, due to the fact that both λ_k and ν_k^- grow like $(k\pi/L)^2$. Since the transition state is u_0^*, the exponent is given by

$$H(d) = V[u_0^*] - V[u_-^*] = |U(u_-)| \tag{7.16}$$

  and is independent of d. The error terms satisfy

$$R_{d,B}^{\pm}(\varepsilon) = \mathcal{O}\big(\varepsilon^{1/2}|\log\varepsilon|^{3/2}\big) \tag{7.17}$$

  uniformly in d.
- For $L > \pi + c$, Propositions 6.10 and 6.11 yield a prefactor

$$C(d,\varepsilon) = \pi\bigg(\frac{1}{|\mu_0(d)|\nu_0^-}\prod_{k=1}^{d}\frac{\mu_k(d)}{\nu_k^-}\bigg)^{1/2}\,, \tag{7.18}$$

  where the eigenvalues $\mu_k(d)$ depend on d. They converge, as $d \to \infty$, to those of the Hessian at the transition state $u_{1,+}^*$, by the implicit-function theorem argument given in Proposition 4.9. The exponent is given by

$$H(d) = \widehat{V}(P_\pm(d)) - V[u_-^*]\,, \tag{7.19}$$

  and converges to $V[u_{1,+}^*] - V[u_-^*]$ as $d \to \infty$. The error terms satisfy (7.17) as well.

- For $L \leqslant \pi$, the value of the prefactor follows from Propositions 6.5 and 6.7. Using the computations of [BG10, Section 5.4] to determine the integral in (6.48) (note that our C_4 is equal to half the C_4 in that reference), we get

$$C(d,\varepsilon) = 2\pi \Bigg(\frac{1}{|\lambda_0|\nu_0^-} \frac{\lambda_1 + \sqrt{C_4\varepsilon}}{\nu_1^-} \prod_{k=2}^{d} \frac{\lambda_k}{\nu_k^-} \Bigg)^{1/2} \frac{1}{\Psi_+(\lambda_1/\sqrt{C_4\varepsilon}\,)} \,, \tag{7.20}$$

  where Ψ_+ is the function defined in (2.28). The exponent is still given by (7.16), while the error terms are of the form

$$R_{d,B}^{\pm}(\varepsilon) = \mathcal{O}\Bigg(\Bigg[\frac{\varepsilon |\log \varepsilon|^3}{\lambda_1 \vee \sqrt{\varepsilon |\log \varepsilon|}} \Bigg]^{1/2} \Bigg) \,. \tag{7.21}$$

- For $L \geqslant \pi$, again by Propositions 6.5 and 6.7 and [BG10, Section 5.4],

$$C(d,\varepsilon) = 2\pi \Bigg(\frac{1}{|\mu_0(d)|\nu_0^-} \frac{\mu_1(d) + \sqrt{C_4\varepsilon}}{\nu_1^-} \prod_{k=2}^{d} \frac{\mu_k(d)}{\nu_k^-} \Bigg)^{1/2} \frac{1}{\Psi_-(\mu_1(d)/\sqrt{C_4\varepsilon}\,)} \,, \tag{7.22}$$

  where Ψ_+ is the function defined in (2.29). The exponent is again given by (7.19), and the error terms satisfy (7.21).

The expressions are similar for periodic b.c.

A Monotonicity of the period

Consider the Hamiltonian system defined by the Hamiltonian (2.14). Let $T(E)$ be the period of its periodic solution with energy E, given by (2.16). The following lemma provides a sufficient condition for T being increasing in E.

Lemma A.1. *Assume that*

$$U'(u)^2 - 2U(u)U''(u) > 0 \qquad \text{for all } u \in (u_-, u_+) \setminus \{0\} \,. \tag{A.1}$$

Then $T(E)$ is strictly increasing on $[0, E_0)$.

Proof: We parametrize the upper half of the periodic orbit by

$$\begin{aligned} u' &= \sqrt{2E} \sin\varphi \,, \\ -U(u) &= E \cos^2\varphi \,, \end{aligned} \tag{A.2}$$

where $\varphi \in [0,\pi]$. The second relation can be inverted, writing $u = f_E(\varphi)$, where the function $f_E : [0,\pi] \to [u_2(E), u_3(E)]$ is increasing and maps $[0,\pi/2]$ on $[u_2, 0]$ and $[\pi/2,\pi]$ on $[0,u_3]$. Differentiating the relation $E\cos^2\varphi = -U(f_E(\varphi))$ shows that

$$\frac{\partial f_E}{\partial \varphi} = \frac{2E \sin\varphi\cos\varphi}{U'(f_E(\varphi))} \,, \qquad \frac{\partial f_E}{\partial E} = -\frac{\cos^2\varphi}{U'(f_E(\varphi))} \,. \tag{A.3}$$

The period is given by

$$\frac{T(E)}{2} = \int_{u_2(E)}^{u_3(E)} \frac{\mathrm{d}u}{u'} = \int_0^\pi \frac{\sqrt{2E}\cos\varphi}{U'(f_E(\varphi))} \,\mathrm{d}\varphi \,. \tag{A.4}$$

By (A.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{\sqrt{2E}\cos\varphi}{U'(f_E(\varphi))}\right) = \left[U'(f_E(\varphi))^2 - 2U(f_E(\varphi))U''(f_E(\varphi))\right]\frac{\cos\varphi}{\sqrt{2E}\,U'(f_E(\varphi))^3}\,. \qquad \text{(A.5)}$$

Since $\cos\varphi$ and $-U'(f_E(\varphi)$ have the same sign, the assumption (A.1) implies that the integral (A.4) is strictly increasing in E. □

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