

SOME EXAMPLES OF FUCHSIAN GROUPS SITTING IN $SL_2(\mathbb{Q})$

E. B. VINBERG

ABSTRACT. Answering a question of A. Rapinchuk, we construct infinitely many non-commensurable (up to conjugacy) cocompact Fuchsian groups sitting in $SL_2(\mathbb{Q})$, and also infinitely many non-commensurable non-cocompact Fuchsian groups of finite covolume sitting in $SL_2(\mathbb{Q})$.

1. COCOMPACT GROUPS

The construction of the first series of examples is based on quadrilaterals with all angles $\pi/3$ on the Lobachevski plane L^2 .

We will use the vector model of L^2 as a connected component of the hyperboloid $(x, x) = -1$ in the 3-dimensional Minkowski space $\mathbb{R}^{2,1}$. The group of motions in this model is an index 2 subgroup $O'_{2,1}$ of the group $O_{2,1}$ of (pseudo)orthogonal transformations of $\mathbb{R}^{2,1}$. The group of proper motions is $SO'_{2,1} = O'_{2,1} \cap SO_{2,1}$.

The doubled Gram matrix of the unit normals e_1, e_2, e_3, e_4 to the sides of a quadrilateral P with all angles $\pi/3$ has the form

$$2G = \begin{pmatrix} 2 & -1 & -a & -1 \\ -1 & 2 & -1 & -b \\ -a & -1 & 2 & -1 \\ -1 & -b & -1 & 2 \end{pmatrix},$$

where $a, b > 2$ are the doubled hyperbolic cosines of the distances between opposite sides of P . The condition $\det G = 0$ results in the following relation between a and b :

$$(1) \quad (a - 2)(b - 2) = 4.$$

Let Γ be the group generated by the reflections in the sides of P . It is a discrete group of motions with the fundamental domain P . Let Γ_+ be the even subgroup of Γ (constituted by the proper motions). It is an index 2 subgroup generated by the rotations by $2\pi/3$ around the vertices of P (whose product is equal to the identity transformation).

Assume now that $a, b \in \mathbb{Q}$. Then the \mathbb{Q} -linear span of e_1, e_2, e_3, e_4 is a Γ -invariant \mathbb{Q} -form of the space $\mathbb{R}^{2,1}$. Denote it by $\mathbb{Q}^{2,1}$, and the group of its orthogonal transformations by $O_{2,1}(\mathbb{Q})$. Set also $SO_{2,1}(\mathbb{Q}) = O_{2,1}(\mathbb{Q}) \cap SO_{2,1}$. Then

$$\Gamma \subset O_{2,1}(\mathbb{Q}), \quad \Gamma_+ \subset SO_{2,1}(\mathbb{Q}).$$

The vectors e_1, e_2, e_3 constitute a basis of $\mathbb{Q}^{2,1}$. Computing the corner minors of the Gram matrix of this basis, one can obtain that the scalar square in $\mathbb{Q}^{2,1}$ in a suitable basis is proportional to the quadratic form

$$(2) \quad f(x) = 3x_1^2 + x_2^2 - \frac{a+2}{a-1}x_3^2.$$

We are going to choose a so that the quadratic space $\mathbb{Q}^{2,1}$ is isotropic. This will take place, for example, if $\frac{a+2}{a-1}$ is a square of some rational number t , which means that

$$(3) \quad a = \frac{t^2 + 2}{t^2 - 1},$$

where $t \in \mathbb{Q}$, $1 < t < 2$. By (1) we then obtain

$$(4) \quad b = \frac{2(t^2 + 2)}{4 - t^2}.$$

The groups Γ and Γ_+ constructed as above, with a and b defined by formulas (3) and (4), will be denoted by $\Gamma(t)$ and $\Gamma_+(t)$, resp. Replacing t with $\frac{2}{t}$ corresponds to interchanging a and b , so one may assume that

$$1 < t < \sqrt{2}.$$

There is a covering homomorphism

$$\pi : \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SO}_{2,1}(\mathbb{C})$$

such that $\pi(\mathrm{SL}_2(\mathbb{R})) = \mathrm{SO}'_{2,1}$. If the quadratic space $\mathbb{Q}^{2,1}$ is isotropic, then π can be defined over \mathbb{Q} so that

$$\pi(\mathrm{SL}_2(\mathbb{Q})) \subset \mathrm{SO}_{2,1}(\mathbb{Q}).$$

The group $\pi(\mathrm{SL}_2(\mathbb{Q}))$ consists of the orthogonal transformations of spinor norm 1. By the construction, the group $\Gamma(t)$ is generated by reflections corresponding to unit vectors in $\mathbb{Q}^{2,1}$. The group $\Gamma_+(t)$ is generated by products of pairs of such reflections, which have the spinor norm 1. This implies that

$$\Gamma_+(t) \subset \pi(\mathrm{SL}_2(\mathbb{Q})).$$

Set

$$\Delta(t) = \pi^{-1}(\Gamma_+(t)) \subset \mathrm{SL}_2(\mathbb{Q}).$$

We are now to determine if the subgroup $\Delta(t_1)$ is commensurable to some conjugate of $\Delta(t_2)$ in $\mathrm{GL}_2(\mathbb{Q})$ (or, equivalently, in $\mathrm{SL}_2(\mathbb{C})$) for different t_1, t_2 . To this end, we will use the notion of the least ring of definition introduced in [1].

In the context of the present paper, let $\Delta \subset \mathrm{SL}_2(\mathbb{Q})$ be a subgroup, which is Zariski dense in $\mathrm{SL}_2(\mathbb{C})$. A subring $A \subset \mathbb{Q}$ (with unit) is called a ring of definition of Δ , if Δ is conjugate in $\mathrm{GL}_2(\mathbb{Q})$ to a subgroup of $\mathrm{SL}_2(A)$. It follows from general results of [1] that, for any subgroup Δ as above, there exists a least ring of definition, and that it is the same for commensurable subgroups. It is also clear that the least field of definition is the same for conjugate subgroups.

Moreover, if $\pi(\Delta) \subset \mathrm{SO}_{2,1}$ is the even subgroup of the group generated by reflections

$$R_i : x \rightarrow x - 2(e_i, x)e_i \quad (i = 1, \dots, m),$$

where $(e_1, e_1) = \dots = (e_m, e_m) = 1$, then the least ring of definition of Δ is generated by the cyclic products of the doubled Gram matrix of the vectors e_1, \dots, e_m : see [1], Theorem 3. (The cyclic products of a matrix $C = (c_{ij})$ are the products $c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_{k-1} i_k} c_{i_k i_1}$.)

Note that any subring of \mathbb{Q} has the form

$$A = \mathbb{Z}[p_1^{-1}, p_2^{-1}, \dots],$$

where $\{p_1, p_2, \dots\}$ is the (finite or infinite) set of primes invertible in A .

It follows from the above that the least ring of definition of the subgroup $\Delta(t) \subset \mathrm{SL}_2(\mathbb{Q})$ is

$$A(t) = \mathbb{Z}[a, b],$$

where a and b are defined by formulas (3) and (4).

Let $t = \frac{u}{v}$, where u and v are coprime positive integers. Then

$$a = \frac{u^2 + 2v^2}{u^2 - v^2}, \quad b = \frac{2(u^2 + 2v^2)}{4v^2 - u^2}.$$

Analyzing possible common divisors of the enumerators and denominators of these fractions, one comes to the conclusion that the invertible primes in $A(t)$ are the prime divisors of the integers $u + v$, $u - v$, $2v + u$, $2v - u$, with the exceptions of 2, if $4|u$, and 3, if $u \equiv \pm 4v \pmod{9}$. It follows that there are infinitely many different rings $A(t)$ and, thereby, infinitely many mutually non-commensurable (up to conjugacy) subgroups $\Delta(t)$.

There remain, however, interesting open questions, namely, whether all the constructed groups $\Delta(t)$ with $t \in \mathbb{Q}$, $1 < t < \sqrt{2}$, are mutually non-commensurable and whether all the corresponding rings $A(t)$ are different. It is also not clear, if the least rings of definition of non-commensurable groups are different.

Choosing a so that the quadratic space $\mathbb{Q}^{2,1}$ is non-isotropic, one can in a similar way construct infinitely many non-commensurable Fuchsian groups sitting in the group of quaternions of norm 1 of some quaternion division algebra over \mathbb{Q} . It looks quite plausible that such families of Fuchsian groups exist in all quaternion division algebras over \mathbb{Q} .

2. NON-COCOMPACT GROUPS

Our second series of examples is based on quadrilaterals with all angles 0 (i.e., with all vertices at infinity) on L^2 .

The doubled Gram matrix of the unit normals e_1, e_2, e_3, e_4 to the sides of such quadrilateral P has the form

$$2G = \begin{pmatrix} 2 & -2 & -a & -2 \\ -2 & 2 & -2 & -b \\ -a & -2 & 2 & -2 \\ -2 & -b & -2 & 2 \end{pmatrix}.$$

The condition $\det G = 0$ results in the following relation between a and b :

$$(5) \quad (a - 2)(b - 2) = 16.$$

Let Γ be the group generated by the reflections in the sides of P , and let Γ_+ be its even subgroup.

Assume that $a, b \in \mathbb{Q}$. Then the \mathbb{Q} -linear span of e_1, e_2, e_3, e_4 is a Γ -invariant \mathbb{Q} -form of the space $\mathbb{R}^{2,1}$. Denote it by $\mathbb{Q}^{2,1}$, and the group of its orthogonal transformations by $\mathrm{O}_{2,1}(\mathbb{Q})$. Set also $\mathrm{SO}_{2,1}(\mathbb{Q}) = \mathrm{O}_{2,1}(\mathbb{Q}) \cap \mathrm{SO}_{2,1}$. Then

$$\Gamma \subset \mathrm{O}_{2,1}(\mathbb{Q}), \quad \Gamma_+ \subset \mathrm{SO}_{2,1}(\mathbb{Q}).$$

The quadratic space $\mathbb{Q}^{2,1}$ is isotropic for any a . Let

$$\pi : \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{SO}_{2,1}(\mathbb{C})$$

be a covering homomorphism defined over \mathbb{Q} . As in the previous section, we obtain

$$\Gamma_+ \subset \pi(\mathrm{SL}_2(\mathbb{Q})).$$

Set

$$\Delta = \pi^{-1}(\Gamma_+) \subset \mathrm{SL}_2(\mathbb{Q}).$$

The least ring of definition of Δ is

$$A = \mathbb{Z}[a, b].$$

For different rational numbers $a > 2$, one obtains infinitely many different rings A and, thereby, infinitely many non-commensurable subgroups $\Delta \subset \mathrm{SL}_2(\mathbb{Q})$.

REFERENCES

- [1] E. B. Vinberg. Rings of definition of dense subgroups of semisimple linear groups. *Math. USSR, Izv.* **5** (1972), 45–55.

PROF. E. B. VINBERG, DEPARTMENT OF MECHANICS AND MATHEMATICS, MOSCOW STATE UNIVERSITY, MOSCOW 119992, GSP-2, RUSSIA