HARNACK INEQUALITIES FOR SUBORDINATE BROWNIAN MOTIONS

PANKI KIM AND ANTE MIMICA

ABSTRACT. In this paper, we consider transient subordinate Brownian motion X in $\mathbb{R}^d$, $d \geq 1$, where the Laplace exponent ϕ of the corresponding subordinator satisfies some mild conditions. The scale invariant Harnack inequality is proved for X. We first give new forms of asymptotical properties of the Lévy and potential density of the subordinator near zero. Using these results we find asymptotics of the Lévy density and potential density of X near the origin, which is essential to our approach. The examples which are covered by our results include geometric stable processes and relativistic geometric stable processes, i.e. the cases when the subordinator has the Laplace exponent

$$\phi(\lambda) = \log(1 + \lambda^{\alpha/2}) \quad (0 < \alpha \leq 2, d > \alpha)$$

and

$$\phi(\lambda) = \log(1 + (\lambda + m^{\alpha/2})^{2/\alpha} - m) \quad (0 < \alpha < 2,\, m > 0, d > 2)\,.$$

INTRODUCTION

Consider a Brownian motion $B = (B_t, \mathbb{P}_x)$ in $\mathbb{R}^d$, $d \geq 1$, and an independent subordinator $S = (S_t\colon t \geq 0)$. It is known that the stochastic process $X = (X_t, \mathbb{P}_x)$ defined by $X_t = B(S_t)$ is a Lévy process. The process X is called the subordinate Brownian motion.

A non-negative function $h\colon \mathbb{R}^d \to [0, \infty)$ is said to be harmonic with respect to X in an open set $D \subset \mathbb{R}^d$ if for all open sets $B \subset \mathbb{R}^d$ whose closure is compact and contained in D the following mean value property holds

$$h(x) = \mathbb{E}_x[h(X_{\tau_B})] \quad \text{for all} \quad x \in B\,,$$

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where $\tau_B = \inf\{t > 0\colon X_t \notin B\}$ denotes the first exit time from the set B .

The Harnack inequality holds for the process X if there exists a constant $c > 0$ such that for any $r \in (0,1)$ and any non-negative function h on $\mathbb{R}^d$ which is harmonic in ball $B_r = \{z \in \mathbb{R}^d\colon |z| < r\}$ the following inequality is true

$$h(x) \leq c\, h(y) \quad \text{for all} \quad x, y \in B_{\frac{r}{2}} \,. \tag{0.1}$$

Space homogeneity of Lévy processes implies that the same inequality is true on any ball $B(x_0, r) = \{z \in \mathbb{R}^d\colon |z - x_0| < r\}$. This type of Harnack inequality is sometimes called scale invariant (or geometric) Harnack inequality, since the constant c in (0.1) stays the same for any $r \in (0,1)$.

A very successful technique for proving Harnack inequality for stable-like Markov jump processes was developed in [BL02]. The proof relied on an estimate of Krylov and Safonov type. More precisely, this estimate says that there exists a constant $c > 0$ such that

$$\mathbb{P}_x(T_A < \tau_{B(0,r)}) \geq c\, \frac{|A|}{|B(0,r)|}$$

for any $r \in (0,1)$, $x \in B(0, \frac{r}{2})$ and $A \subset \mathbb{R}^d$ closed, where $T_A = \tau_{A^c}$ denotes the first hitting time of the set A and $|A|$ denotes its Lebesgue measure.

Although this technique is quite general and can be applied to a much larger class of Markov jump processes, there are situations when it is not applicable even to a rotationally invariant Lévy process.

For example, in [ŠSV06] the non-scale invariant Harnack inequality was proved for geometric stable and iterated geometric stable processes. It was not known whether scale invariant version of this inequality held. Recently this turned to be the case in dimension $d = 1$ (see [GR11]). In [GR11] the authors used theory of fluctuation of one-dimensional Lévy processes and it was not clear how to generalize this technique to higher dimensions. Nevertheles, this result suggests that the scale invariant version of Harnack inequality may hold in higher dimensions.

Our aim is to prove the scale invariant Harnack inequality for a class of Lévy processes which includes geometric stable and iterated geometric stable processes in higher dimensions.

The Krylov-Safonov type estimate was indispensable in the proof of the Harnack inequality in [ŠSV06]. Contrary to the case of stable-like processes, this estimate is not uniform in $r \in (0,1)$. For example, for a geometric stable process it is possible to find a sequence of radii (r_n) and closed sets $A_n \subset B(0, r_n)$ such that $r_n \to 0$, $\frac{|A_n|}{|B(0,r_n)|} \geq \frac{1}{4}$ and

$$\mathbb{P}_0(T_{A_n} < \tau_{B(0,r_n)}) \to 0, \quad \text{as} \quad n \to \infty \,.$$

This shows that investigation of Harnack inequality becomes interesting even in the case of a Lévy process. We have not encountered a technique so far that would cover cases of a more general jump process in this direction.

The main ingredient in our proof of Harnack inequality is a good estimate of the Green function $G_{B(0,r)}(x,y)$ of the ball $B(0,r)$ when y is near its boundary. To be more precise, we will prove that there are a function $\xi\colon (0,1) \to (0,\infty)$ and constants $c_1, c_2 > 0$ and $0 < \kappa_1 < \kappa_2 < 1$ such that for every $r \in (0,1)$,

$$c_1 \xi(r) r^{-d}\, \mathbb{E}_y \tau_{B(0,r)} \le G_{B(0,r)}(x,y) \le c_2 \xi(r) r^{-d}\, \mathbb{E}_y \tau_{B(0,r)}, \tag{0.2}$$

for $x \in B(0,\kappa_1 r)$ and $y \in B(0,r) \setminus B(0,\kappa_2 r)$ (see Corollary 4.9).

Depending on the considered process, the function $r \to \xi(r)$ can have two different types of behavior. For example, in the case of stable processes $\xi(r) \asymp 1$ as $r \to 0+$, while in the case of geometric stable processes $\xi(r) \to 0$ as $r \to 0+$.

Let us be more precise now. In this paper we consider subordinate Brownian motions X in $\mathbb{R}^d$ $(d \ge 1)$, for which the Laplace exponent ϕ of the corresponding subordinator S satisfies (see Sections 1 and 2 for details concerning these conditions):

(A-1) ϕ is a complete Bernstein function;
(A-2) the Lévy measure of S is infinite;
(A-3) there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta \in (0,1]$ such that $d + 2\delta - 2 > 0$ and

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \le \sigma\, x^{-\delta} \quad \text{for all} \quad x \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0$$

and in the case $d \le 2$ we assume further that there are $\sigma' > 0$ and $\delta' \in \big(1 - \frac{d}{2}, (1+\frac{d}{2}) \wedge (2\delta + \frac{d-2}{2})\big)$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \ge \sigma'\, x^{-\delta'} \quad \text{for all} \quad x \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0\,;$$

(A-4) there exists $R > 0$ so that

$$\int_0^R \frac{\lambda^{\frac{d}{2}-1}}{\phi(\lambda)}\, d\lambda < \infty\,.$$

Our main result is the following scale invariant Harnack inequality.

Theorem 0.1 (Harnack inequality)**.** *Suppose X is a subordinate Brownian motion satisfying* **(A-1)**–**(A-4)**. *There exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$h(x_1) \le c\, h(x_2) \quad \textit{for all} \quad x_1, x_2 \in B(x_0, \tfrac{r}{2})$$

and for every non-negative function $h\colon \mathbb{R}^d \to [0,\infty)$ which is harmonic in $B(x_0,r)$.

This theorem is a new result for the following interesting examples (ϕ or ϕ_n denote the Laplace exponents of the corresponding subordinators):

Example 1 (Geometric stable processes)

$$\phi(\lambda) = \log(1+\lambda^{\beta/2}), \quad (0<\beta\le 2,\, d>\beta).$$

Note that the case $0<\beta\le 2$ and $d=1$ is considered in [GR11].

Example 2 (Iterated geometric stable processes)

$$\begin{aligned} \phi_1(\lambda) &= \log(1+\lambda^{\beta/2}) \quad (0<\beta\le 2) \\ \phi_{n+1} &= \phi_1\circ\phi_n \quad n\in\mathbb{N}, \end{aligned}$$

with an additional condition $d>2^{1-n}\beta^n$.

Example 3 (Relativistic geometric stable processes)

$$\phi(\lambda) = \log\Big(1+\big(\lambda+m^{\beta/2}\big)^{2/\beta}-m\Big) \quad (m>0,\, 0<\beta<2,\, d>2).$$

The condition **(A-3)** is implied by the following stronger condition

$$\forall\, x>0 \qquad \lim_{\lambda\to\infty}\frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\frac{\alpha}{2}-1} \qquad (0\le\alpha<2)\,. \tag{0.3}$$

In other words, (0.3) says that ϕ' varies regularly at infinity with index $\frac{\alpha}{2}-1$. The examples above satisfy this condition with $\alpha=0$.

Theorem 0.1 covers also processes for which the Harnack inequality was known before (see [RSV06, Mim10, KSVa]):

Example 4 Assume that ϕ satisfies **(A-1)**, **(A-2)**, **(A-4)** and

$$\phi(\lambda)\asymp\lambda^{\alpha/2}\ell(\lambda),\ \lambda\to\infty \quad (0<\alpha<2\wedge d)$$

where ℓ varies slowly at infinity, i.e.

$$\forall\, x>0 \qquad x>0\lim_{\lambda\to\infty}\frac{\ell(\lambda x)}{\ell(\lambda)} = 1\,.$$

For example, $\ell(\lambda)=\log(1+\lambda)$ or $\ell(\lambda)=\log(1+\log(1+\lambda))$.

The function ξ that appeared in (0.2) is of the form $\xi(r) = \frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})}$. Simple calculation shows that in **Example 1**

$$\xi(r)\asymp\tfrac{1}{\log(r^{-1})} \ \text{ as } \ r\to 0+\,,$$

while in **Example 4**

$$\xi(r)\asymp 1 \ \text{ as } \ r\to 0+\ .$$

Harnack inequalities for symmetric stable Lévy processes were obtained in [BSS02, BS05]. A new technique on Harnack inequalities for stable like jump processes was developed in [BL02] and generalized in [SV04]. Similar technique was used for various jump processes in [CK03, CK08, BK05]. In [KS07] the Harnack inequality was proved for truncated stable processes and it was generalized in [Mim10]. Harnack inequality for some classes subordinate Brownian motions was also considered in [KSVa].

The paper is organized as follows. In Section 1 we give basic notions which we use in sections that follow. Asymptotical properties of the Lévy and the potential densities of subordinators are obtained in Section 2. Technical lemmas concerning asymptotic inversion of the Laplace transform used in this section are deferred to Appendix A. These results in Appendix A can be also of independent interest, since they represent an alternative to the Tauberian theorems, which were mainly used in previous works.

Using results of the Section 2 we obtain the behavior of the Lévy measure and the Green function (potential) of the process X in Section 3. In Section 4 we obtain pointwise estimates of the Green functions of small balls needed to prove the main result, which is proved in Section 5.

Notation. Throughout the paper we use the notation $f(r) \asymp g(r),\ r \to a$ to denote that $f(r)/g(r)$ stays between two positive constants as $r \to a$. Simply, $f \asymp g$ means that the quotient $f(r)/g(r)$ stays bounded between two positive numbers on their common domain of definition. We say that $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if $s \le t$ implies $f(s) \le f(t)$ and analogously for a decreasing function. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. We will use ":=" to denote a definition, which is read as "is defined to be". For any $a, b \in \mathbb{R}$, we use the notations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The values of the constants $c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from location to location. The labeling of the constants $c_1, c_2, \ldots$ starts anew in the proof of each result.

1. Preliminaries

A stochastic process $X = (X_t, \mathbb{P}_x)$ in $\mathbb{R}^d$ is said to be a pure jump Lévy process if it has stationary and independent increments, its trajectories are right-continuous with left limits and the characteristic exponent Φ in

$$\mathbb{E}_x \left[\exp\left\{i\langle \xi, X_t - X_0\rangle\right\}\right] = \exp\left\{-t\Phi(\xi)\right\},\ \xi \in \mathbb{R}^d$$

is of the form

$$\Phi(\xi) = \int_{\mathbb{R}^d} \left(1 - \exp\{i\langle \xi, x\rangle\} + i\langle \xi, x\rangle \, 1_{\{|x|<1\}}\right) \Pi(dx). \tag{1.1}$$

The measure Π in (1.1) is called the Lévy measure of X and it satisfies

$$\Pi(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (1 \wedge |x|^2)\Pi(dx) < \infty \,.$$

Let $S = (S_t \colon t \geq 0)$ be a subordinator, i.e. a Lévy process taking values in $[0, \infty)$ and starting at 0. It is more convenient to consider the Laplace transform in this case

$$\mathbb{E} \exp\{-\lambda S_t\} = \exp\{-t\phi(\lambda)\}. \tag{1.2}$$

The function ϕ in (1.2) is called the Laplace exponent of S and it is of the form

$$\phi(\lambda) = \gamma t + \int_{(0,\infty)} (1 - e^{-\lambda t}) \, \mu(dt) \,, \tag{1.3}$$

where $\gamma \geq 0$ and the Lévy measure μ of S is now a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\mu(dt) < \infty$ (see p. 72 in [Ber96]).

The function ϕ is an example of a Bernstein function, i.e. $\phi \in C^\infty(0, \infty)$ and $(-1)^n \phi^{(n)} \leq 0$ for all $n \in \mathbb{N}$ (see p. 15 in [SSV10]). Here $\phi^{(n)}$ denotes the n-th derivative of ϕ. Conversely, every Bernstein function ϕ satisfying $\phi(0+) = 0$ has a representation (1.3) and there exists a subordinator with the Laplace exponent ϕ .

The potential measure of the subordinator S is defined by

$$U(A) = \int_0^\infty \mathbb{P}(S_t \in A) \, dt. \tag{1.4}$$

The Laplace transform of U is then

$$\mathcal{L}U(\lambda) = \mathbb{E} \int_{(0,\infty)} e^{-\lambda S_t} \, dt = \frac{1}{\phi(\lambda)}, \qquad \lambda > 0 \,. \tag{1.5}$$

A Bernstein function ϕ is said to be a complete Bernstein function if the Lévy measure μ has a completely monotone denisty, i.e. $\mu(dt) = \mu(t) \, dt$ with $\mu \in C^\infty(0, \infty)$ satisfying $(-1)^n \mu^{(n)} \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$. The corresponding subordinator is called a complete subordinator.

In this case we can control large jumps of Lévy density μ in the follwing way. There exists a constant $c > 0$ such that

$$\mu(t) \leq c \, \mu(t + 1) \quad \text{for all} \quad t \geq 1 \tag{1.6}$$

(see Lemma 2.1 in [KSVb]). If, in addition, $\mu(0,\infty)=\infty$, the potential measure U has a decreasing density, i.e. there exists a decreasing function $u\colon (0,\infty)\to(0,\infty)$ such that $U(dt)=u(t)\,dt$ (see Corollary 10.7 in [SSV10]).

Let $B=(B_t,\mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$ and let $S=(S_t\colon t\geq 0)$ be an independent subordinator. We define a new process $X=(X_t,\mathbb{P}_x)$ by $X_t=B(S_t)$ and call it subordinate Brownian motion. This process is a Lévy process with the characteristic exponent $\Phi(\xi)=\phi(|\xi|^2)$. Moreover, Φ has representation (1.1), with the Lévy measure of the form $\Pi(dx)=j(|x|)\,dx$ and

$$j(r)=\int_{(0,\infty)}(4\pi t)^{-d/2}\exp\left(-\tfrac{r^2}{4t}\right)\mu(dt),\ r>0 \tag{1.7}$$

(see Theorem 30.1 in [Sat99]).

The process X has a transition density $p(t,x,y)$ given by

$$p(t,x,y)=\int_0^\infty(4\pi t)^{-d/2}\exp\left(-\tfrac{|x-y|^2}{4t}\right)\mathbb{P}(S_t\in ds)\,. \tag{1.8}$$

Transience of the process X plays important role in our approach. The process X is said to be transient if $\mathbb{P}_0(\lim_{t\to\infty}|X_t|=\infty)=1$.

Since the characteristic exponent of X is symmetric we have the following Chung-Fuchs type criterion for transience

$$\begin{aligned} X \quad \text{is transient} \quad &\Longleftrightarrow \quad \int_{B(0,R)}\frac{d\xi}{\phi(|\xi|^2)}<\infty \ \text{ for some } \ R>0\\ &\Longleftrightarrow \quad \int_0^R\frac{\lambda^{\frac d2-1}}{\phi(\lambda)}\,d\lambda<\infty \ \text{ for some } \ R>0 \qquad (1.9)\\ &\Longleftrightarrow \ \mathbb{E}_0\left[\int_0^\infty 1_{\{|X_t|<R\}}\,dt\right]<\infty \quad \text{for every } \ R>0 \qquad (1.10)\end{aligned}$$

(see Corollary 37.6 and Theorem 35.4 in [Sat99]).

In this case we can define the Green function (potential) by

$$G(x,y)=\int_0^\infty p(t,x,y)\,dt\,.$$

Then (1.4) and (1.8) give us a useful formula $G(x,y)=G(y-x)=g(|y-x|)$, where

$$g(r)=\int_{(0,\infty)}(4\pi t)^{-d/2}\exp\left(-\tfrac{r^2}{4t}\right)U(dt),\ \ r>0\,. \tag{1.11}$$

Note that g and j are decreasing.

Let $D \subset \mathbb{R}^d$ be a bounded open subset. We define killed process X^D by $X^D_t = X_t$ if $t < \tau_D$ and $X^D_t = \Delta$ otherwise, where Δ is some point adjoined to D (usually called cemetery).

The transition density and the Green function of X^D are given by

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}_x\left[p(t-\tau_D, X(\tau_D), y); \tau_D < t\right]$$

and $G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt$. In the transient case we have the following formula

$$G_D(x,y) = G(x,y) - \mathbb{E}_x[G(X(\tau_D),y)]\,. \tag{1.12}$$

Also, $G_D(x,y)$ is symmetric and, for fixed $y \in D$, $G_D(\cdot,y)$ is harmonic in $D \setminus \{y\}$. Furthermore, $G_D\colon (D\times D)\setminus\{(x,x)\colon x \in D\} \to [0,\infty)$ and $x \mapsto \mathbb{E}_x\tau_D$ are continuous functions.

By the result of Ikeda and Watanabe (see Theorem 1 in [IW62])

$$\mathbb{P}_x(X_{\tau_D} \in F) = \int_F \int_D G_D(x,y) j(|z-y|)\,dy\,dz \tag{1.13}$$

for any $F \subset \overline{D}^c$. If we define the Poisson kernel of the set D by

$$K_D(x,z) = \int_D G_D(x,y) j(|z-y|)\,dy, \tag{1.14}$$

then $\mathbb{P}_x(X_{\tau_D} \in F) = \int_F K_D(x,z)\,dz$ for any $F \subset \overline{D}^c$. In other words, the Poisson kernel is the density of the exit distribution.

Since a subordinate Brownian motion is a rotationally invariant Lévy process, it follows

$$\mathbb{P}_x(X_{\tau_{B(x_0,r)}} \in \partial B(x_0,r)) = 0$$

(see [Szt00]) and thus if $h\colon \mathbb{R}^d \to [0,\infty)$ is a measurable function, then we have

$$\mathbb{E}_x[h(X_{\tau_{B(z_0,s)}})] = \int_{\overline{B(z_0,s)}^c} K_{B(z_0,s)}(x,z) h(z)\,dz \tag{1.15}$$

for any ball $B(z_0,s)$.

2. Subordinators

Let $S = (S_t\colon t \geq 0)$ be a subordinator with the Laplace exponent ϕ satisfying the following conditions:

(A-1) ϕ is a complete Bernstein function;
(A-2) the Lévy density μ of S_t is infinite, i.e. $\mu(0,\infty) = \infty$;

(A-3) there exist constants $\sigma > 0$, $\lambda_0 > 0$ and $\delta > 0$ such that $d + 2\delta - 2 > 0$ and

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \le \sigma\, x^{-\delta} \quad \text{for all} \quad x \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0 \tag{2.1}$$

and in the case $d \le 2$ we assume further that there are $\sigma' > 0$ and

$$\delta' \in \big(1 - \tfrac{d}{2}, (1 + \tfrac{d}{2}) \wedge (2\delta + \tfrac{d-2}{2})\big) \tag{2.2}$$

such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \ge \sigma'\, x^{-\delta'} \quad \text{for all} \quad x \ge 1 \quad \text{and} \quad \lambda \ge \lambda_0\,; \tag{2.3}$$

(A-4) there exists $R > 0$ so that

$$\int_0^R \frac{\lambda^{\frac{d}{2}-1}}{\phi(\lambda)}\, d\lambda < \infty\,.$$

Remark 2.1. (i) **(A-1)** and **(A-2)** imply that the potential measure of U of S has a decreasing density, i.e. there is a decreasing function $u\colon (0,\infty) \to (0,\infty)$ so that $U(dt) = u(t)\, dt$.

(ii) Since ϕ is a complete Bernstein function,

$$\phi(\lambda) = \gamma\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \mu(t)\, dt\,.$$

Note that **(A-3)** implies $\gamma = 0$, by letting $x \to +\infty$.

(iii) Condition **(A-4)** is equivalent to the transience of the corresponding subordinate Brownian motion.

(iv) If $d \ge 2$, then $d + 2\delta - 2 > 0$ in **(A-3)** is always true.

First we prove a simple result that holds for any Bernstein function, which will be used in Section 4.

Lemma 2.2. *Let ϕ be a Bernstein function.*

(i) *For every $x \ge 1$,*

$$\phi(\lambda x) \le x\phi(\lambda) \quad \textit{for all} \quad \lambda > 0\,.$$

(ii) (2.1) *implies that for every $\varepsilon > 0$ there is a constant $c = c(\varepsilon) > 0$ so that*

$$\frac{\phi(\lambda x)}{\phi(\lambda)} \le c\, x^{1-\delta+\varepsilon} \quad \textit{for all} \quad \lambda \ge \lambda_0 \quad \textit{and} \quad x \ge 1\,.$$

Proof. (i) Since ϕ' is decreasing and $x \ge 1$,

$$\phi(\lambda x) = \int_0^{\lambda x} \phi'(s)\, ds \le \int_0^{\lambda x} \phi'(\tfrac{s}{x})\, ds = x\phi(\lambda)\,.$$

(ii) Without loss of generality we may assume that $\sigma \geq 2$ in **(A-3)**. Using **(A-3)**, for any $k \geq 2$ the following recursive inequality holds

$$\begin{aligned}\phi(\lambda\sigma^{\frac{k}{\varepsilon}}) - \phi(\lambda\sigma^{\frac{k-1}{\varepsilon}}) &= \int_{\lambda\sigma^{\frac{k-1}{\varepsilon}}}^{\lambda\sigma^{\frac{k}{\varepsilon}}} \phi'(s)\,ds \leq \sigma^{1-\frac{\delta}{\varepsilon}} \int_{\lambda\sigma^{\frac{k-1}{\varepsilon}}}^{\lambda\sigma^{\frac{k}{\varepsilon}}} \phi'(s\sigma^{-\frac{1}{\varepsilon}})\,ds \\ &= \sigma^{1+\frac{1-\delta}{\varepsilon}} \left(\phi(\lambda\sigma^{\frac{k-1}{\varepsilon}}) - \phi(\lambda\sigma^{\frac{k-2}{\varepsilon}})\right).\end{aligned}$$

Iteration yields

$$\phi(\lambda\sigma^{\frac{k}{\varepsilon}}) - \phi(\lambda\sigma^{\frac{k-1}{\varepsilon}}) \leq \sigma^{(k-1)(1+\frac{1-\delta}{\varepsilon})} \left(\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) - \phi(\lambda)\right) \tag{2.4}$$

for every $k \geq 2$.

Let $n \in \mathbb{N}$ be chosen so that $\sigma^{\frac{n-1}{\varepsilon}} \leq x < \sigma^{\frac{n}{\varepsilon}}$.

If $n = 1$, then by (i),

$$\phi(\lambda\sigma^{-\frac{1}{\varepsilon}}) \leq \sigma^{\frac{1}{\varepsilon}}\phi(\lambda) \leq \sigma^{\frac{1}{\varepsilon}+\frac{2\delta}{\varepsilon}} x^{-\delta}\phi(\lambda)$$

which, by monotonicity of ϕ, implies that $\frac{\phi(\lambda x)}{\phi(\lambda)} \leq \sigma^{\frac{1+2\delta}{\varepsilon}} x^{-\delta}$.

Let us consider now the case $n \geq 2$. Using (2.4) and (i) we deduce

$$\begin{aligned}\phi(\lambda\sigma^{\frac{n}{\varepsilon}}) - \phi(\lambda) &= \left(\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) - \phi(\lambda)\right) \sum_{k=2}^{n} \sigma^{(k-1)(1+\frac{1-\delta}{\varepsilon})} \\ &\leq \left(\phi(\lambda\sigma^{\frac{1}{\varepsilon}}) - \phi(\lambda)\right) \frac{\sigma^{n(1+\frac{1-\delta}{\varepsilon})}}{\sigma^{1+\frac{1-\delta}{\varepsilon}} - 1} \\ &\leq \sigma^{\frac{1}{\varepsilon}}\phi(\lambda)\sigma^{n(1+\frac{1-\delta}{\varepsilon})}.\end{aligned}$$

Therefore

$$\phi(\lambda x) \leq \phi(\lambda\sigma^{\frac{n}{\varepsilon}}) \leq 2\sigma^{1+\frac{2-\delta}{\varepsilon}}\phi(\lambda)\left(\sigma^{\frac{n-1}{\varepsilon}}\right)^{\varepsilon+1-\delta} \leq 2\sigma^{1+\frac{2-\delta}{\varepsilon}}\phi(\lambda)x^{\varepsilon+1-\delta}.$$

□

Proposition 2.3. *If the Lévy measure μ of S has a deceasing density $t \to \mu(t)$ and* **(A-3)** *holds, then*

$$\mu(t) \asymp t^{-2}\phi'(t^{-1}),\ t \to 0+ .$$

Proof. Note that

$$\phi(\lambda + \varepsilon) - \phi(\varepsilon) = \int_0^\infty \left(e^{-\lambda t} - e^{-\lambda(t+\varepsilon)}\right)\mu(t)\,dt$$

for any $\lambda > 0$ and $\varepsilon > 0$ and thus the condition (A.1) in Appendix A holds with $f = \phi$ and $\nu = \mu$. Since ϕ is a Bernstein function, it follows that $\phi' \geq 0$ and ϕ' is decreasing. Now we can apply Lemma A.1 and Lemma A.2. □

Proposition 2.4. *If the potential measure U of S has a decreasing density u and* **(A-3)** *holds, then*

$$u(t) \asymp t^{-2}\frac{\phi'(t^{-1})}{\phi(t^{-1})^2},\ t \to 0+\ .$$

Proof. By (1.5)

$$\int_0^\infty e^{-\lambda t} u(t)\, dt = \psi(\lambda),$$

with $\psi(\lambda) = \frac{1}{\phi(\lambda)}$. Note that, for $\lambda \geq \lambda_0$ and $x \geq 1$, **(A-3)** implies

$$\left|\frac{\psi'(\lambda x)}{\psi'(\lambda)}\right| = \left(\frac{\phi(\lambda)}{\phi(\lambda x)}\right)^2 \frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq c x^{-\delta},$$

since ϕ is increasing.

We see that (A.1) in Appendix A is satisfied with $f = \frac{1}{\phi}$ and $\nu = u$. Since ϕ is a Bernstein function, $\phi' \geq 0$ and ϕ' is a decreasing function. Thus $|f'| = \frac{\phi'}{\phi^2}$ is also a decreasing function. The result follows now from Lemma A.1 and Lemma A.2. □

3. Lévy density and potential

In Section 2 we have established asymptotic behavior of the Lévy and potential density of S near zero. In this section we are going to use these results to give new forms of asymptotic behavior of the Lévy density and potential of the process X near the origin.

Lemma 3.1. *Suppose that ϕ is a special Bernstein function, i.e., $\lambda \to \lambda/\phi(\lambda)$ is also a Bernstein function. Then the functions $\eta_1, \eta_2 \colon (0,\infty) \to (0,\infty)$ given by*

$$\eta_1(\lambda) = \lambda^2\phi'(\lambda) \quad \text{and} \quad \eta_2(\lambda) = \lambda^2\frac{\phi'(\lambda)}{\phi^2(\lambda)}$$

are increasing.

Proof. It is enough to prove that η_2 is increasing, because $\eta_1 = \eta_2 \cdot \phi^2$ is then a product of two increasing functions.

Since ϕ is a special Bernstein function,

$$\frac{\lambda}{\phi(\lambda)} = \theta + \int_{(0,\infty)} (1 - e^{-\lambda t})\nu(dt),$$

for some $\theta \geq 0$ and a Lévy measure ν (see pp. 92-93 in [SSV10]). Then

$$\lambda^2 \frac{\phi'(\lambda)}{\phi(\lambda)^2} = \lambda \left(-\frac{\lambda}{\phi(\lambda)} \right)' + \frac{\lambda}{\phi(\lambda)}$$
$$= \theta + \int_{(0,\infty)} \left(1 - (1+\lambda t)e^{-\lambda t}\right) \nu(dt)\,.$$

Now the claim follows, since $\lambda \mapsto 1 - (1+\lambda t)e^{-\lambda t}$ is increasing for any $t > 0$. □

Proposition 3.2. *If the Lévy measure μ of S has a deceasing density $t \to \mu(t)$ and and* **(A-3)** *holds, then*

$$j(r) \asymp r^{-d-2}\phi'(r^{-2}),\ \ r \to 0+\ .$$

Proof. We use formula (1.7), i.e.

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right)\mu(t)\,dt\,.$$

Proposition 2.3 implies that $\mu(t) \asymp t^{-2}\phi'(t^{-1}),\ t \to 0+$.

We are going to use Proposition A.3 in Appendix A with $A = 2$, $\eta = \mu$ and $\psi = \phi'$. In order to do this, we need to check conditions (a), (b) and (c)-(ii). The condition (a) follows from the fact that ϕ is a Bernstein function and Lemma 3.1, while (b) follows from

$$\int_1^\infty t^{-d/2}\mu(t)\,dt \leq \int_1^\infty \mu(t)\,dt = \mu(1,\infty) < \infty,$$

since μ is a Lévy measure. Finally the condition (c)-(ii) follows from (2.2)–(2.3). □

Lemma 3.3. *If the potential measure U of S has a decreasing density u and* **(A-4)** *holds, then*

$$\int_1^\infty t^{-d/2}u(t)\,dt < \infty\,.$$

Proof. It follows from (1.5) and (1.9) that for any $t \geq 1$ and $R > 0$

$$\infty > \int_0^R \frac{\lambda^{\frac{d}{2}-1}}{\phi(\lambda)}\,d\lambda = \int_0^R \int_0^\infty \lambda^{\frac{d}{2}-1}e^{-\lambda t}u(t)\,dt\,d\lambda$$
$$= \int_0^\infty \int_0^{tR} s^{\frac{d}{2}-1}t^{-\frac{d}{2}}e^{-s}u(t)\,ds\,dt \geq \int_1^\infty \int_0^{tR} s^{\frac{d}{2}-1}t^{-\frac{d}{2}}e^{-s}u(t)\,ds\,dt$$
$$\geq \left(\int_0^R s^{\frac{d}{2}-1}e^{-s}\,ds\right)\cdot\left(\int_1^\infty t^{-d/2}u(t)\,dt\right)\,.$$

□

Proposition 3.4. *Assume that the potential measure of S has a decreasing density and that* **(A-3)** *and* **(A-4)** *hold. Then then*

$$g(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2},\ r \to 0+\ .$$

Proof. By (1.11) we have

$$g(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) u(t)\, dt\,.$$

Proposition 2.4 implies that $u(t) \asymp t^{-2}\frac{\phi'(t^{-1})}{\phi(t^{-1})^2},\ t \to 0+$.

We are going to use Proposition A.3 in Appendix with $A = 2$, $\eta = u$ and $\psi = \frac{\phi'}{\phi^2}$.

In order to use it, we need to check conditions (a), (b) and (c)-(ii). The condition (a) follows from the fact that ϕ' and $\frac{1}{\phi^2}$ are decreasing, since ϕ is a Bernstein function. The condition (b) follows from Lemma 3.3.

Now we check the condition (c)-(ii) when $d \le 2$; Note that by (2.2), $1 - \frac{d}{2} < \delta' < 2\delta - 1 + \frac{d}{2}$ (and $\delta \le 1 < 1 + \frac{\delta'}{2}$). Thus $0 < \delta' + 2 - 2\delta < 1 + \frac{d}{2}$. Choose $\varepsilon > 0$ small so that $0 < \delta' + 2 - 2\delta + 2\varepsilon < 1 + \frac{d}{2}$, then applying (2.3) and Lemma 2.2 (ii), we get

$$\frac{\psi(\lambda x)}{\psi(\lambda)} = \frac{\phi'(\lambda x)}{\phi'(\lambda)}\frac{\phi(\lambda)^2}{\phi(\lambda x)^2} \ge c_1 x^{-\delta'} c_2 x^{-2+2\delta-2\varepsilon} = c_1 c_2 x^{-(\delta'+2-2\delta+2\varepsilon)},$$

Thus (A.7) holds. □

4. Green function estimates

The purpose of this section is to establish pointwise Green function estimates More precisely, we are interested in estimate of $G_{B(x_0,r)}(x,y)$ for $x \in B(x_0, b_1 r)$ and $y \in A(x_0, b_2 r, r) := \{y \in \mathbb{R}^d : b_2 r \le |y - x_0| < r\}$, for some $b_1, b_2 \in (0,1)$. As a starting point we need an estimate of $G_{B(x_0,r)}(x,y)$ away from the boundary.

For the remainder of this paper we always assume that $S = (S_t : t \ge 0)$ is a subordinator with the Laplace exponent ϕ satisfying **(A-1)**–**(A-4)** and assume that $X = (X_t, \mathbb{P}_x)$ is the subordinate process defined by $X_t = B(S_t)$ where $B = (B_t, \mathbb{P}_x)$ is a Brownian motion in $\mathbb{R}^d$ independent of S.

Since ϕ is a complete Bernstein function, using Lemma 4.2 in [RSV06], our (1.6) and Proposition 3.2 we see that there is a constant $c > 0$ such that

$$j(r+1) \le j(r) \le c j(r+1) \quad \text{for all} \quad r \ge 1\,. \tag{4.1}$$

Recall that X is transient so that its potential G is finite.

Lemma 4.1. *There exists $a \in (0, \frac{1}{3})$ and $c_1 > 0$ such that for any $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$G_{B(x_0,r)}(x,y) \geq c_1 \frac{|x-y|^{-d-2}\phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2} \quad \text{for all} \quad x,y \in B(x_0, ar)\,. \tag{4.2}$$

In particular, there is a constant $c_2 \in (0,1)$ so that

$$G_{B(x_0,r)}(x,y) \geq c_2 g(|x-y|) \quad \text{for all} \quad x,y \in B(x_0, ar)\,.$$

Proof. Let $x, y \in B(x_0, ar)$ with $a \in (0,1)$ chosen in the course of the proof. We use (1.12), i.e.

$$G_{B(x_0,r)}(x,y) = g(|x-y|) - \mathbb{E}_x[g(|X(\tau_{B(0,r)}) - y|)].$$

Since $|X(\tau_{B(x_0,r)}) - y| \geq (1-a)r$ and $|x-y| \leq 2ar$, we get

$$|X(\tau_{B(x_0,r)}) - y| \geq \tfrac{1-a}{2a}|x-y|.$$

This together with the fact that g is decreasing yields

$$G_{B(x_0,r)}(x,y) \geq g(|x-y|) - g(\tfrac{1-a}{2a}|x-y|)\,. \tag{4.3}$$

By Proposition 3.4 there exist constants $0 < c_1 < c_2$ such that

$$c_1 s^{-d+2}\psi(s^{-2}) \leq g(s) \leq c_2 s^{-d+2}\psi(s^{-2}),\ \ s \in (0,1)\,, \tag{4.4}$$

with

$$\psi(\lambda) = \lambda^2 \tfrac{\phi'(\lambda)}{\phi(\lambda)^2},\ \ \lambda > 0\,.$$

Considering only $a < \frac{1}{3}$ it follows that $\frac{2a}{1-a} < 1$. Combining (4.3), (4.4) we arrive at

$$\begin{aligned} &G_{B(x_0,r)}(x,y) \\ &\geq c_1|x-y|^{-d+2}\psi(|x-y|^{-2}) \left[1 - c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d-2} \frac{\psi\left(\left(\frac{2a}{1-a}\right)^2 |x-y|^{-2}\right)}{\psi(|x-y|^{-2})}\right]. \end{aligned} \tag{4.5}$$

When $d \geq 3$, choose $a < \frac{1}{3}$ small enough so that $c_2 c_1^{-1}\left(\frac{2a}{1-a}\right)^{d-2} \leq \frac{1}{2}$.

When $d \leq 2$, using (2.2), first choose $\varepsilon > 0$ small enough so that $d - 2 - 2\delta' + 4\delta - 4\varepsilon > 0$ then choose $a < \frac{1}{3}$ small enough so that $c_2 c_1^{-1}\left(\frac{2a}{1-a}\right)^{d-2-2\delta'+4\delta-4\varepsilon} \leq \frac{1}{2}$.

Then using the fact that $\lambda \to \psi(\lambda)$ is increasing (Lemma 3.1) when $d \geq 3$, and using (2.1), (2.3) and Lemma 2.2 (ii) when $d \leq 2$, we get

$$\begin{aligned}
& c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d-2} \frac{\psi\left(\left(\frac{2a}{1-a}\right)^2 |x-y|^{-2}\right)}{\psi(|x-y|^{-2})} \\
\leq & \begin{cases} c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d-2} & \text{when } d \geq 3 \\ c_2 c_1^{-1} \left(\tfrac{2a}{1-a}\right)^{d+2} \frac{\phi'\left(\left(\frac{2a}{1-a}\right)^2 |x-y|^{-2}\right) \phi(|x-y|^{-2})^2}{\phi'(|x-y|^{-2}) \phi\left(\left(\frac{2a}{1-a}\right)^2 |x-y|^{-2}\right)^2} \leq \left(\tfrac{2a}{1-a}\right)^{(d+2)-2\delta'-4+4\delta-4\varepsilon} & \text{when } d \leq 2 \end{cases} \\
\leq & \frac{1}{2}. \qquad (4.6)
\end{aligned}$$

Therefore (4.5) and (4.6) yield

$$G_{B(x_0,r)}(x,y) \geq \tfrac{c_1}{2c_2} |x-y|^{-d+2} \psi(|x-y|^{-2}) \text{ for all } \ x,y \in B(x_0, ar)\,.$$

□

Proposition 4.2. *There exists a constant $c > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$\mathbb{E}_x \tau_{B(x_0,r)} \geq \frac{c}{\phi(r^{-2})} \quad \textit{for all} \ \ x \in B(x_0, \tfrac{ar}{2})\,,$$

where $a \in (0, \frac{1}{3})$ as in Lemma 4.1.

Proof. Take a as in Lemma 4.1 and set $b = \frac{a}{2}$. For any $x \in B(x_0, br)$ we have $B(x, br) \subset B(x_0, ar)$ and so it follows from Lemma 4.1 that

$$\begin{aligned}
\mathbb{E}_x \tau_{B(x_0,r)} &\geq \int_{B(x,br)} G_{B(x_0,r)}(x,y)\, dy \\
&\geq c_1 \int_{B(x,br)} \frac{|x-y|^{-d-2} \phi'(|x-y|^{-2})}{\phi(|x-y|^{-2})^2}\, dy = \frac{c_2}{\phi(b^{-2} r^{-2})} \geq \frac{b^2 c_2}{\phi(r^{-2})}\,.
\end{aligned}$$

The last inequality follows Lemma 2.2, since $b < 1$. □

Remark 4.3. Note that, by (1.12), for any $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$ we have

$$G_{B(x_0,r)}(x,y) \leq g(|x-y|) \ \text{ for all } \ x,y \in B(x_0, r)$$

and, consequently, $\mathbb{E}_x \tau_{B(x_0,r)} \leq \frac{c}{\phi(r^{-2})}$ for any $x \in B(x_0, r)$.

Our approach in obtaining pointwise estimates of Green function of balls uses maximum principle for certain operators (in a similar way as in [BS05]).

More precisely, for $r > 0$ we define a Dynkin-like operator $\mathcal{U}_r$ by

$$(\mathcal{U}_r f)(x) = \frac{\mathbb{E}_x[f(X(\tau_{B(x,r)}))] - f(x)}{\mathbb{E}_x \tau_{B(x,r)}}$$

for measurable functions $f\colon \mathbb{R}^d \to \mathbb{R}$ whenever it is well-defined.

Example 4.4. Let $x \in \mathbb{R}^d$ and $r > 0$. Define

$$\eta(z) := \mathbb{E}_z \tau_{B(x,r)}, \;\; z \in \mathbb{R}^d .$$

By the strong Markov property, for any $y \in B(x,r)$ and $s < r - |y-x|$

$$\eta(y) = \mathbb{E}_y[\tau_{B(y,s)} + \tau_{B(x,r)} \circ \theta_{\tau_{B(y,s)}}] = \mathbb{E}_y \tau_{B(y,s)} + \mathbb{E}_y \eta(X(\tau_{B(y,s)})) .$$

Therefore

$$(\mathcal{U}_s \eta)(y) = -1 \quad \text{ for any } y \in B(x,r) \text{ and } s < r - |y-x|. \tag{4.7}$$

Remark 4.5. Let $h\colon \mathbb{R}^d \to [0,\infty)$ be a non-negative function that is harmonic in bounded open set $D \subset \mathbb{R}^d$. Then for $x \in D$ and $s < \operatorname{dist}(x, \partial D)$ we have $h(x) = \mathbb{E}_x[h(X(\tau_{B(x,s)}))]$. Thus

$$(\mathcal{U}_s h)(x) = 0 \;\; \text{for all} \;\; x \in D.$$

Proposition 4.6 (Maximum principle)**.** *Assume that there exist $x_0 \in \mathbb{R}^d$ and $r > 0$ such that $(\mathcal{U}_r f)(x_0) < 0$. Then*

$$f(x_0) > \inf_{x \in \mathbb{R}^d} f(x) . \tag{4.8}$$

Proof. If (4.8) is not true, then $f(x_0) \le f(x)$ for all $x \in \mathbb{R}^d$. This implies $(\mathcal{U}_r f)(x_0) \ge 0$, which is in contradiction with the assumption. □

Proposition 4.7. *There exists a constant $c > 0$ such that for all $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$*

$$G_{B(x_0,r)}(x,y) \le c\, \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)}$$

for all $x \in B(x_0, \frac{br}{2})$ and $y \in A(x_0, br, r)$, where $b = \frac{a}{2}$ with $a \in (0, \frac{1}{3})$ from Lemma 4.1.

Proof. Take $a \in (0, \frac{1}{3})$ as in Proposition 4.2 (which is the same one as in Lemma 4.1), set $b = \frac{a}{2}$ and let $x \in B(x_0, \frac{br}{2})$ and $y \in A(x_0, br, r)$. Define functions

$$\eta(z) := \mathbb{E}_z \tau_{B(x_0,r)} \;\; \text{and} \;\; h(z) := G_{B(x_0,r)}(x,z)$$

and choose $s < (r - |y|) \wedge \frac{br}{4}$. Note that h is harmonic in $B(x_0,r) \setminus \{x\}$.

Since $h(z) \le g(\frac{br}{8})$ for $z \in B(x, \frac{br}{8})^c$ and $y \in A(x_0, br) \subset B(x, br/8)^c$, (1.13), (1.15) and Remark 4.5 yield

$$\begin{aligned}
\mathcal{U}_s(h \wedge g(\tfrac{br}{8}))(y) &= \mathcal{U}_s(h \wedge g(\tfrac{br}{8}) - h)(y) \\
&= \frac{1}{\mathbb{E}_y \tau_{B(y,s)}} \int_{\overline{B(y,s)}^c} \int_{B(y,s)} G_{B(y,s)}(y,v) j(|z-v|)(h(z) \wedge g(\tfrac{br}{8}) - h(z))\, dv\, dz \\
&= \frac{1}{\mathbb{E}_y \tau_{B(y,s)}} \int_{B(x,\frac{br}{8})} \int_{B(y,s)} G_{B(y,s)}(y,v) j(|z-v|)(h(z) \wedge g(\tfrac{br}{8}) - h(z))\, dv\, dz \\
&\ge -\frac{1}{\mathbb{E}_y \tau_{B(y,s)}} \int_{B(x,\frac{br}{8})} \int_{B(y,s)} G_{B(y,s)}(y,v) j(|z-v|) h(z)\, dv\, dz.
\end{aligned}$$

Note that $|z-v| \ge |x-y| - |x-z| - |y-v| \ge \frac{br}{8}$ for $v \in B(x, \frac{br}{8})$ and $z \in B(y,s)$ implies $-j(|z-v|) \ge -j(\frac{br}{8})$. Thus

$$\begin{aligned}
&\mathcal{U}_s(h \wedge g(\tfrac{br}{8}) - h)(y) \\
&\ge -\frac{j(\frac{br}{8})}{\mathbb{E}_y \tau_{B(y,s)}} \left(\int_{B(x,\frac{br}{8})} G_{B(x_0,r)}(x,z)\, dz \right) \cdot \left(\int_{B(y,s)} G_{B(y,s)}(y,v)\, dv \right) \\
&\ge -\frac{j(\frac{br}{8})}{\mathbb{E}_y \tau_{B(y,s)}} \left(\int_{B(x_0,r)} G_{B(x_0,r)}(x,z)\, dz \right) \mathbb{E}_y \tau_{B(y,s)} \\
&= -j(\tfrac{br}{8}) \eta(x) \ge -c_1 \left(\tfrac{b}{8}\right)^{-d-2} \frac{r^{-d-2} \phi'(r^{-2})}{\phi(r^{-2})},
\end{aligned} \tag{4.9}$$

where in the last inequality we have used Proposition 3.2, Remark 4.3 and the fact that ϕ' is decreasing.

Similarly, by Proposition 3.4 and Proposition 4.2 we see that there is a constant $c_2 > 0$ such that

$$g(\tfrac{br}{8}) \le c_2 \left(\tfrac{b}{8}\right)^{-d-2} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \eta(z) \qquad \text{for all} \ \ z \in B(x_0, br)\,.$$

Setting $c_3 := (c_1 \vee c_2) \left(\frac{b}{8}\right)^{-d-2} + 1$ we obtain

$$c_3 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \eta(z) - h(z) \wedge g(\tfrac{br}{8}) \ge c_3 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \eta(z) - g(\tfrac{br}{8}) \ge 0$$

for all $z \in B(x_0, br)$. Therefore, the function

$$u(\cdot) := c_3 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \eta(\cdot) - h(\cdot) \wedge g(\tfrac{br}{8})$$

is nonnegative for $z \in B(x_0, br)$, vanishes on $B(x_0,r)^c$ and, by (4.7) and (4.9),

$$\mathcal{U}_s u(y) \le -c_3 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} + c_1 \left(\tfrac{b}{8}\right)^{-d-2} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} < 0 \quad \text{for} \quad y \in A(x_0, br, r)\,.$$

If $\inf_{y\in\mathbb{R}^d} u(y) < 0$, then by continuity of u on $B(x_0,r)$ there would exist $y_0 \in A(x_0, br, r)$ such that $u(y_0) = \inf_{y\in\mathbb{R}^d} u(y)$. But then $\mathcal{U}_s u(y_0) \ge 0$, by the maximum principle (see Proposition 4.6), which is not true. Therefore $\inf_{y\in\mathbb{R}^d} u(y) \ge 0$.

Finally, since $h \le g(\frac{br}{8})$ on $A(x_0, br, r)$ it follows that

$$G_{B(x_0,r)}(x,y) \le c_4 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\eta(y) \quad \text{for all} \quad y \in A(x_0, br, r)\,.$$

□

Proposition 4.8. *There exist constants $c > 0$ and $b \in (0,1)$ such that for any $r \in (0,1)$ and $x_0 \in \mathbb{R}^d$*

$$G_{B(x_0,r)}(x,y) \ge c\, \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)}$$

for all $x \in B(x_0, br)$ and $y \in B(x_0, r)$.

Proof. Choose $a \in (0, \frac{1}{3})$ as in Lemma 4.1. Then

$$G_{B(x_0,r)}(x,v) \ge c_1 \frac{|x-v|^{-d-2}\phi'(|x-v|^{-2})}{\phi(|x-v|^{-2})^2} \quad \text{for} \quad x, v \in B(x_0, ar)\,. \tag{4.10}$$

By Proposition 4.7 we know that there exists a constant $c_2 > 0$ so that

$$G_{B(x_0,r)}(x,v) \le c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \quad \text{for} \quad x \in B(x_0, \tfrac{ar}{4}),\ v \in A(x_0, \tfrac{ar}{2}, r)\,. \tag{4.11}$$

Also, by Remark 4.3 there is a constant $c_3 > 0$ such that

$$\mathbb{E}_v \tau_{B(x_0,r)} \le \frac{c_3}{\phi(r^{-2})} \quad \text{for} \quad v \in B(x_0, r)\,. \tag{4.12}$$

We take

$$b \le \min\left\{ \tfrac{1}{4}\left(\tfrac{c_1}{2c_2c_3}\right)^{1/d}, \tfrac{a}{8} \right\}$$

and fix it. Note that $c_2c_3 \le \frac{c_1}{2}(4b)^{-d}$, i.e. $b \le \frac{1}{4}\left(\frac{c_1}{2c_2c_3}\right)^{1/d}$. Thus by Lemma 3.1

$$c_2c_3 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})^2} \le \tfrac{c_1}{2} \frac{(4br)^{-d-2}\phi'((4b)^{-2}r^{-2})}{\phi((4b)^{-2}r^{-2})^2}\,.$$

Now, by (4.10) and (4.12), for all $x \in B(x_0, br)$ and $v \in B(x, br)$

$$c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \le \tfrac{1}{2} G_{B(x_0,r)}(x, v)\,. \tag{4.13}$$

For the rest of the proof, we fix $x \in B(x_0, br)$ and define a function

$$h(v) = G_{B(x_0,r)}(x, v) \wedge \left(c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \right) .$$

Let $y \in A(x_0, \frac{ar}{2}, r)$ and take $s < (r - |y|) \wedge \frac{br}{8}$. Note that, by (4.13),

$$h(v) \le \tfrac{1}{2} G_{B(x_0,r)}(x, v) \quad \text{for} \quad v \in B(x, br).$$

Therefore, (1.13) and Remark 4.5 yield

$$\begin{aligned}
(\mathcal{U}_s h)(y) &= \mathcal{U}_s \left(h - G_{B(x_0,r)}(x, \cdot) \right)(y) \\
&= \frac{1}{\mathbb{E}_y \tau_{B(y,s)}} \int_{\overline{B(y,s)}^c} \int_{B(y,s)} G_{B(y,s)}(y, v) j(|z - v|) \left(h(z) - G_{B(x_0,r)}(x, z) \right) dv\, dz \\
&\le -\frac{1}{2\mathbb{E}_y \tau_{B(y,s)}} \int_{B(x,br)} \int_{B(y,s)} G_{B(y,s)}(y, v) j(|z - v|) G_{B(x_0,r)}(x, z)\, dv\, dz\,.
\end{aligned}$$

Note that in the second equality we have used that $h(y) = G_{B(x_0,r)}(x, y)$, which follows from (4.11).

Since $|z - v| \le 2r$ we obtain

$$(\mathcal{U}_s h)(y) \le -\frac{j(2r)}{2\mathbb{E}_y \tau_{B(y,s)}} \left(\int_{B(x,br)} G_{B(x_0,r)}(x, z)\, dz \right) \mathbb{E}_y \tau_{B(y,s)}$$

By Proposition 3.2, (4.10) and the fact that $\lambda \mapsto \frac{\lambda}{\phi(\lambda)}$ is increasing (by Lemma 2.2 or using the fact that ϕ is a complete Bernstein function) and ϕ' decreasing we finally arrive at

$$(\mathcal{U}_s h)(y) \le -c_4 \frac{r^{-d-2}\phi'(r^{-2})}{\phi\left((br)^{-2}\right)} \le -c_5 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\,.$$

Define $u = h - \kappa\eta$, where $\eta(v) = \mathbb{E}_v \tau_{B(x_0,r)}$ and

$$\kappa = \min\left\{ \frac{c_5}{2}, \frac{c_1}{2c_3}, \frac{c_2}{2} \right\} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\,.$$

For $y \in A(x_0, \frac{ar}{2}, r)$ we have by (4.7),

$$(\mathcal{U}_s u)(y) \le -c_5 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} + \kappa \le -\tfrac{c_5}{2} \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} < 0\,.$$

On the other hand, by (4.10)–(4.12), for all $v \in B(x_0, \frac{ar}{2})$,

$$u(v) \geq \left(\tfrac{c_1}{c_3} \wedge c_2\right) \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} - \kappa \, \mathbb{E}_v \tau_{B(x_0,r)}$$
$$\geq \left(\tfrac{c_1}{2c_3} \wedge \tfrac{c_2}{2}\right) \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \mathbb{E}_v \tau_{B(x_0,r)} \geq 0 \, .$$

Similarly as in Proposition 4.7, by the maximum principle it follows that

$$u(y) \geq 0 \quad \text{for all} \quad y \in B(x_0, r) \, .$$

□

Combining Propositions 4.7 and 4.8 we obtain an important estimate for the Green function.

Corollary 4.9. *There exist constants $c_1, c_2 > 0$ and $b_1, b_2 \in (0, \frac{1}{2})$, $2b_1 < b_2$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \, \mathbb{E}_y \tau_{B(x_0,r)} \leq G_{B(x_0,r)}(x,y) \leq c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \, \mathbb{E}_y \tau_{B(x_0,r)}$$

for all $x \in B(x_0, b_1 r)$ and $y \in A(x_0, b_2 r, r)$. □

5. Poisson kernel and Harnack inequality

The first goal of this section is to estimate Poisson kernel of a ball given by

$$K_{B(x_0,r)}(x,z) = \int_{B(x_0,r)} G_{B(x_0,r)}(x,y) j(|z-y|) \, dy \, , \tag{5.1}$$

where $x \in B(x_0, br)$ and $z \notin B(x_0, r)$.

Proposition 5.1. *There exist constants $c > 0$ and $b \in (0,1)$ such that for all $x_0 \in \mathbb{R}^d$ and $r \in (0,1)$*

$$K_{B(x_0,r)}(x_1, z) \leq c K_{B(x_0,r)}(x_2, z)$$

for all $x_1, x_2 \in B(x_0, br)$ and $z \in B(x_0, r)^c$.

Proof. Take $b_1, b_2 \in (0, \frac{1}{2})$ as in Corollary 4.9, and let $x_0 \in \mathbb{R}^d$, $x_1, x_2 \in B(x_0, b_1 r)$ and $z \in B(x_0, r)^c$.

We split the integral in (5.1) in two parts

$$
\begin{aligned}
K_{B(x_0,r)}(x_1,z) = &\int_{B(x_0,b_2 r)} G_{B(x_0,r)}(x_1,y) j(|z-y|)\, dy \\
&+ \int_{A(x_0,b_2 r,r)} G_{B(x_0,r)}(x_1,y) j(|z-y|)\, dy \ =: \ I_1 + I_2 \,.
\end{aligned}
$$

To estimate I_2 we use Corollary 4.9 to get that for $y \in A(x_0, b_2 r, r)$

$$
c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)} \le G_{B(x_0,r)}(x_i,y) \le c_2 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})}\, \mathbb{E}_y \tau_{B(x_0,r)}, \quad i=1,2.
$$

Therefore

$$
\begin{aligned}
I_2 &= \int_{A(x_0,b_2 r,r)} G_{B(x_0,r)}(x_1,y) j(|z-y|)\, dy \\
&\le c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \int_{A(x_0,b_2 r,r)} \mathbb{E}_y\, \tau_{B(x_0,r)} j(|z-y|)\, dy \\
&\le \frac{c_1}{c_2} \int_{A(x_0,b_2 r,r)} G_{B(x_0,r)}(x_2,y) j(|z-y|)\, dy \ \le \ \frac{c_1}{c_2} K_{B(x_0,r)}(x_2,z)\,.
\end{aligned}
$$

To handle I_1 we consider two cases. If $z \in A(x_0, r, 2)$, then

$$
(1-b_2)|z-x_0| \le |z-y| \le (1+b_2)|z-x_0| \ \text{ for all } \ y \in B(x_0, b_2 r)\,.
$$

Since $1 - b_2 \ge \frac{1}{2}$ and $1 + b_2 \le 2$ we obtain

$$
j\left(2|z-x_0|\right) \le j(|z-y|) \le j\left(\tfrac{1}{2}|z-x_0|\right)\,. \tag{5.2}
$$

Using Proposition 3.2 and the fact that ϕ' is decreasing we see that

$$
j\left(\tfrac{1}{2}|z-x_0|\right) \le c_3 j\left(2|z-x_0|\right)\,. \tag{5.3}
$$

Lemma 4.1, Remark 4.3, (5.2) and (5.3) yield

$$
\begin{aligned}
I_1 &\le j\left(\tfrac{1}{2}|z-x_0|\right) \int_{B(x_0,b_2 r)} G_{B(x_0,r)}(x_1,y)\, dy \\
&\le c_3 j\left(2|z-x_0|\right) \frac{c_4}{\phi((b_2 r)^{-2})} \\
&\le c_5 \int_{B(x_0,b_2 r)} G_{B(x_0,r)}(x_2,y) j(|z-y|)\, dy \ \le \ c_5 K_{B(x_0,r)}(x_2,z)\,.
\end{aligned}
$$

When $z \in B(x_0,2)^c$ we use $|z-x_0| - br \le |z-y| \le |z-x_0| + br$ for all $y \in B(x_0, br)$. Since $b_2 \in (0, \frac{1}{2})$ and $r \in (0,1)$ we have

$$
j(|z-x_0| + \tfrac{1}{2}) \le j(|z-y|) \le j(|z-x_0| - \tfrac{1}{2})\,. \tag{5.4}
$$

By (4.1) we have

$$j(|z-x_0|-\tfrac{1}{2}) \le c_6 j(|z-x_0|+1) \le j(|z-x_0|+\tfrac{1}{2})\,. \tag{5.5}$$

Similar to the previous case, by (4.1), Lemma 4.1, (5.4) and (5.5) we have $I_2 \le c_7 K_{B(x_0,r)}(x_2,z)$. Therefore, $I_2 \le (c_5 \vee c_7) K_{B(x_0,r)}(x_2,z)$ which implies that

$$K_{B(x_0,r)}(x_1,z) \le \max\{c_5, c_7, \tfrac{c_1}{c_2}\} K_{B(x_0,r)}(x_2,z)\,.$$

□

Now we are ready to prove our main result.

Proof of Theorem 0.1. Take $b>0$ as in Proposition 5.1 and set $a=\frac{b}{4}$. Using representation

$$h(x) = \mathbb{E}_x[h(X_{\tau_{B(x_0,2ar)}})] = \int_{\overline{B(x_0,2ar)}^c} K_{B(x_0,2ar)}(x_1,z)h(z)\,dz$$

and Proposition 5.1 we have

$$\begin{aligned} h(x_1) &= \int_{\overline{B(x_0,2ar)}^c} K_{B(x_0,2ar)}(x_1,z)h(z)\,dz \\ &\le c \int_{\overline{B(x_0,2ar)}^c} K_{B(x_0,2ar)}(x_2,z)h(z)\,dz = c\,h(x_2). \end{aligned}$$

Then, using standard Harnack chain argument, we prove the theorem. □

Appendix A. Asymptotical properties

In this section we always assume that $f\colon (0,\infty)\to(0,\infty)$ is a differentiable function satisfying

$$|f(\lambda+\varepsilon)-f(\lambda)| = \int_0^\infty \left(e^{-\lambda t} - e^{-(\lambda+\varepsilon)t}\right)\,\nu(t)\,dt\,, \tag{A.1}$$

for all $\lambda>0$, $\varepsilon\in(0,1)$ and a decreasing function $\nu\colon (0,\infty)\to(0,\infty)$.

Lemma A.1. *For all $t>0$,*

$$\nu(t) \le (1-2e^{-1})^{-1}\, t^{-2} |f'(t^{-1})|\,.$$

Proof. Let $\varepsilon\in(0,1)$. Then

$$\begin{aligned} |f(\lambda+\varepsilon)-f(\lambda)| &= \int_0^\infty \left(e^{-\lambda t} - e^{-\lambda t-\varepsilon t}\right)\nu(t)\,dt \\ &= \lambda^{-1}\int_0^\infty e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz\,. \end{aligned}$$

Since ν is decreasing, for any $r > 0$ we conclude

$$\begin{aligned}|f(\lambda+\varepsilon)-f(\lambda)| &\geq \lambda^{-1}\int_0^r e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz \\ &\geq \lambda^{-1}\nu(\lambda^{-1}r)\int_0^r e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\,dz.\end{aligned}$$

Therefore

$$\left|\frac{f(\lambda+\varepsilon)-f(\lambda)}{\varepsilon}\right| \geq \lambda^{-2}\nu(\lambda^{-1}r)\int_0^r z\,e^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}\,dz. \tag{A.2}$$

By the Fatou's lemma and (A.2) we obtain

$$\begin{aligned}|f'(\lambda)| = \lim_{\varepsilon\to 0+}\left|\frac{f(\lambda+\varepsilon)-f(\lambda)}{\varepsilon}\right| &\geq \lambda^{-2}\nu(\lambda^{-1}r)\int_0^r ze^{-z}\,dz \\ &= \lambda^{-2}\nu(\lambda^{-1}r)\left(1-e^{-r}(r+1)\right).\end{aligned}$$

In particular, for $r = 1$ we deduce

$$\nu(t) \leq \left(1-2e^{-1}\right)^{-1} t^{-2}|f'(t^{-1})|,\ t>0\,.$$

□

Lemma A.2. *Assume that $|f'|$ is decreasing and there exist $c_1 > 0$, $\lambda_0 > 0$ and $\delta > 0$ such that*

$$\left|\frac{f'(\lambda x)}{f'(\lambda)}\right| \leq c_1 x^{-\delta} \quad \text{for all} \quad \lambda\geq\lambda_0 \quad \text{and} \quad x \geq 1\,. \tag{A.3}$$

Then there is a constant $c_2 = c_2(c_1,\lambda_0,\delta) > 0$ such that

$$\nu(t) \geq c_2 t^{-2}|f'(t^{-1})| \quad \text{for any} \quad t \leq 1/\lambda_0\,.$$

Proof. Let $\varepsilon\in(0,1)$. For $r\in(0,1]$ we have

$$\begin{aligned}|f(\lambda+\varepsilon)-f(\lambda)| &= \lambda^{-1}\int_0^\infty e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz \\ &= I_1(\varepsilon)+I_2(\varepsilon)\,,\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}I_1(\varepsilon) &= \lambda^{-1}\int_0^r e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz \\ I_2(\varepsilon) &= \lambda^{-1}\int_r^\infty e^{-z}\left(1-e^{-\varepsilon\lambda^{-1}z}\right)\nu(\lambda^{-1}z)\,dz\,.\end{aligned}$$

Since ν is decreasing,

$$\frac{I_2(\varepsilon)}{\varepsilon} \le \lambda^{-2}\nu(\lambda^{-1}r)\int_r^\infty ze^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}\,dz\,,$$

and so by the dominated convergence theorem we deduce

$$\limsup_{\varepsilon\to 0+}\frac{I_2(\varepsilon)}{\varepsilon} \le \lambda^{-2}\nu(\lambda^{-1}r)\int_r^\infty ze^{-z}\,dz = (r+1)e^{-r}\lambda^{-2}\nu(\lambda^{-1}r)\,. \tag{A.5}$$

On the other hand, by Lemma A.1 and (A.3) we have

$$\begin{aligned}\frac{I_1(\varepsilon)}{\varepsilon} &\le \frac{\lambda^{-2}}{1-2e^{-1}}\int_0^r ze^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}\,\frac{|f'(\lambda z^{-1})|}{\lambda^{-2}z^2}\,dz\\ &\le \frac{c_1}{1-2e^{-1}}\,|f'(\lambda)|\int_0^r e^{-z}\,\frac{1-e^{-\varepsilon\lambda^{-1}z}}{\varepsilon\lambda^{-1}z}z^{\delta-1}\,dz\,.\end{aligned}$$

The dominated convergence implies

$$\limsup_{\varepsilon\to 0+}\frac{I_1(\varepsilon)}{\varepsilon} \le \frac{c_1}{1-2e^{-1}}\,|f'(\lambda)|\int_0^r e^{-z}z^{\delta-1}\,dz \quad \text{for any} \quad \lambda\ge\lambda_0. \tag{A.6}$$

Combining (A.4), (A.5) and (A.6) we deduce

$$|f'(\lambda)| \le \frac{c_1}{1-2e^{-1}}\,|f'(\lambda)|\int_0^r e^{-z}z^{\delta-1}\,dz + (r+1)e^{-r}\lambda^{-2}\nu(\lambda^{-1}r)$$

for all $\lambda\ge\lambda_0$.

Choosing $r_0\in(0,1]$ so that

$$\frac{c_1}{1-2e^{-1}}\int_0^{r_0} e^{-z}z^{\delta-1}\,dz \le \frac{1}{2}\,.$$

we have

$$\nu(\lambda^{-1}r_0) \ge \frac{e^{r_0}}{2(r_0+1)}\lambda^2|f'(\lambda)| \quad \text{for all} \quad \lambda\ge\lambda_0.$$

Since $|f'|$ is decreasing, we see that

$$\begin{aligned}\nu(t) &\ge \frac{e^{r_0}}{2(r_0+1)}\frac{|f'(r_0/t)|}{(t/r_0)^2}\\ &\ge \frac{r_0^2e^{r_0}}{2(r_0+1)}t^{-2}|f'(t^{-1})| \quad \text{for all} \quad t\le r_0\lambda_0^{-1}.\end{aligned}$$

On the other hand, for $r_0\lambda_0^{-1}\le t\le\lambda_0^{-1}$ we have

$$\nu(t) \ge \nu(\lambda_0^{-1}) \ge t^{-2}|f'(t^{-1})|\frac{(r_0/\lambda_0)^2}{|f'(\lambda_0)|},$$

since ν and $|f'|$ are decreasing.

Setting

$$c_2 = \frac{r_0^2 e^{r_0}}{2(r_0+1)} \wedge \frac{\nu(\lambda_0^{-1})\lambda_0^{-2} r_0^2}{|f'(\lambda_0)|}$$

we get

$$\nu(t) \geq c_2 t^{-2} |f'(t^{-1})| \quad \text{for all} \quad t \leq \lambda_0^{-1}\,.$$

□

Proposition A.3. *Let $A > 0$ and $\eta\colon (0,\infty) \to (0,\infty)$ be a decreasing function satisfyng the following conditions:*

(a) *there exists a decreasing function $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lambda \mapsto \lambda^2\psi(\lambda)$ is increasing and satisfies*

$$\eta(t) \asymp t^{-A}\psi(t^{-1}),\ \ t \to 0+\,;$$

(b) $\int_1^\infty t^{-d/2}\eta(t)\,dt < \infty$

(c) *either (i) $A > 3 - \frac{d}{2}$ or (ii) $A > 3 - \frac{d}{2}$ when $d \geq 3$ and in the case $d \leq 2$ there exist $\delta > 0$ and $c > 0$ such that $A - \delta > 1 - \frac{d}{2}$ and*

$$\frac{\psi(\lambda x)}{\psi(\lambda)} \geq c x^{-\delta} \quad \text{for all} \quad x \geq 1 \quad \text{and} \quad \lambda \geq 1\,. \tag{A.7}$$

If

$$I(r) = \int_0^\infty (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right)\eta(t)\,dt$$

exists for $r \in (0,1)$ small enough, then

$$I(r) \asymp r^{-d-2A+2}\psi(r^{-2}),\ \ r \to 0+\ .$$

Proof. Write for $r > 0$

$$\begin{aligned} I(r) &= \int_0^{r^2} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right)\eta(t)\,dt + \int_{r^2}^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right)\eta(t)\,dt \\ &= I_1(r) + I_2(r)\,. \end{aligned} \tag{A.8}$$

By condition (a),

$$\begin{aligned} I_1(r) &\leq c_1 \int_0^{r^2} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) t^{-A}\psi(t^{-1})\,dt \\ &\leq c_2\psi(r^{-2}) \int_0^{r^2} t^{-\frac{d}{2}-A} \exp\left(-\tfrac{r^2}{4t}\right)dt \\ &= c_3 r^{-d-2A+2}\psi(r^{-2}) \int_{\frac14}^\infty t^{A-2+\frac{d}{2}} e^{-t}\,dt\ . \end{aligned} \tag{A.9}$$

Similarly,

$$\begin{aligned} I_2(r) &\le c_1 \int_{r^2}^1 (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) t^{-A}\psi(t^{-1})\, dt + \int_1^\infty (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right)\eta(t)\, dt \\ &\le c_1 \int_{r^2}^1 (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\, dt + \int_1^\infty (4\pi t)^{-d/2}\eta(t)\, dt\,. \end{aligned}$$

The following inequality holds

$$\int_{r^2}^1 (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\, dt \le c_4 r^{-d-2A+2}\psi(r^{-2})\,, \tag{A.10}$$

since

(1) if condition (c)-(i) holds, then by conditions (a) and (c)-(i)

$$\begin{aligned} \int_{r^2}^1 (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\, dt &\le r^{-4}\psi(r^{-2}) \int_{r^2}^1 (4\pi t)^{-d/2} t^{2-A}\, dt \\ &\le c_5 r^{-d-2A+2}\psi(r^{-2})\,; \end{aligned}$$

(2) if condition (c)-(ii) holds and $d \le 2$, (A.7) implies

$$\begin{aligned} \int_{r^2}^1 (4\pi t)^{-d/2} t^{-A}\psi(t^{-1})\, dt &\le \psi(r^{-2}) r^{-2\delta} \int_{r^2}^1 (4\pi t)^{-\frac{d}{2}} t^{-A+\delta}\, dt \\ &\le c_6 r^{-d-2A+2}\psi(r^{-2})\,. \end{aligned}$$

In particular, (A.10) implies

$$r^{-d-2A+2}\psi(r^{-2}) \ge c_7 > 0 \quad \text{for all} \quad r \in (0,1)$$

and thus

$$I_2(r) \le c_6 r^{-d-2A+2}\psi(r^{-2}) + c_8 \le c_9 r^{-d-2A+2}\psi(r^{-2})\,. \tag{A.11}$$

Combining (A.8), (A.9) and (A.11) we get the upper bound

$$I(r) \le c_7 r^{-d+2-2A}\psi(r^{-2}) \quad \text{for all} \quad r \in (0,1)\,.$$

To get the lower bound we estimate $I(r)$ from below by $I_1(r)$ and use (a) to get

$$\begin{aligned} j(r) \ge I_1(r) &\ge c_8 \int_0^{r^2} (4\pi t)^{-d/2} \exp\left(-\tfrac{r^2}{4t}\right) t^{-A}\psi(t^{-1})\, dt \\ &\ge c_8 r^{-4}\psi(r^{-2}) \int_0^{r^2} (4\pi t)^{-d/2} t^{2-A} \exp\left(-\tfrac{r^2}{4t}\right) dt \\ &= c_9 r^{-d-2A+2}\psi(r^{-2}) \int_{\frac14}^\infty s^{-\frac{d}{2}+A-4} e^{-s}\, ds \\ &= c_{10} r^{-d-2A+2}\psi(r^{-2}) \quad \text{for all} \quad r \in (0,1). \end{aligned}$$

□

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Department of Mathematical Sciences, Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 151-747, Republic of Korea

E-mail address: pkim@snu.ac.kr

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: amimica@math.uni-bielefeld.de