

# THE MULTINORM PRINCIPLE FOR LINEARLY DISJOINT GALOIS EXTENSIONS

TIMOTHY P. POLLIO AND ANDREI S. RAPINCHUK

ABSTRACT. Let  $L_1$  and  $L_2$  be finite separable extensions of a global field  $K$ , and let  $E_i$  be the Galois closure of  $L_i$  over  $K$  for  $i = 1, 2$ . We establish a local-global principle for the product of norms from  $L_1$  and  $L_2$  (so-called *multinorm principle*) provided that the extensions  $E_1$  and  $E_2$  are linearly disjoint over  $K$ .

## 1. INTRODUCTION

Let  $K$  be a global field. Given a finite extension  $L/K$ , we let  $J_K$  and  $J_L$  denote the groups of ideles of  $K$  and  $L$  respectively, and let  $N_{L/K} : J_L \rightarrow J_K$  denote the natural extension of the norm map associated with  $L/K$  (cf. [2, p. 73-75]). Then the extension  $L/K$  is said to satisfy the *Hasse norm principle* if

$$K^\times \cap N_{L/K}(J_L) = N_{L/K}(L^\times).$$

The classical result of Hasse states that this is always the case if  $L/K$  is a cyclic Galois extension. For general extensions (even Galois extensions), the Hasse principle does not necessarily hold, and its investigation has received a lot of attention. The obstruction to the Hasse principle is given by the quotient

$$\text{III}(L/K) = \frac{K^\times \cap N_{L/K}(J_L)}{N_{L/K}(L^\times)}$$

which is a finite group called the *Tate-Shafarevich group* of the extension  $L/K$ . (We note that it coincides with the Tate-Shafarevich group of the corresponding norm torus  $R_{L/K}^{(1)}(\text{GL}_1)$ , cf. [17], §11).

In [2, p. 198], Tate gave the following cohomological computation of  $\text{III}(L/K)$  for a Galois extension  $L/K$ : *Let  $G = \text{Gal}(L/K)$ , and for a valuation  $v$  of  $K$ , let  $G^v$  be the decomposition group of (a fixed extension of)  $v$ . Then  $\text{III}(L/K)$  is the dual of (hence is isomorphic to) the kernel of the map  $H^3(G, \mathbb{Z}) \rightarrow \prod_v H^3(G^v, \mathbb{Z})$  induced by restriction.* Various aspects of the Hasse principle were investigated in [6], [11], and [12], and a computation of  $\text{III}(L/K)$  for an arbitrary finite extension  $L/K$  in terms of so-called representation groups of the relevant Galois groups was given by Drakokhrust [4].

In [8], Hürlimann considered the tori of norm type associated with a pair of finite extensions  $L_1, L_2$  of a global field  $K$ . The triviality of the Tate-Shafarevich group for this torus is equivalent to the fact that

$$(M) \quad K^\times \cap N_{L_1/K}(J_{L_1})N_{L_2/K}(J_{L_2}) = N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times).$$

Following [13], we say that the pair  $L_1, L_2$  satisfies the *multinorm principle* if (M) holds. It was shown in [8] that this is indeed the case if  $L_1$  is a cyclic Galois extension of  $K$  and  $L_2$  is an arbitrary Galois extension (a similar result was independently obtained by Colliot-Thélène and Sansuc [3]). A more general sufficient condition for the multinorm principle was given in [13], Proposition 6.11. This result was used to give a simplified proof of the Hasse principle for Galois cohomology of simply connected outer forms of type  $A_n$  over number fields (cf. [13], Ch. VI) and in the analysis of the Margulis-Platonov conjecture for anisotropic inner forms of type  $A_n$  (*loc. cit.*, §9.2); it was also employed in [15] in the computation of the metaplectic kernel. More recently, another sufficient condition for the multinorm principle was given in [16] (cf. Proposition 4.2) in order to study the

local-global principle for embedding fields with an involutive automorphism into simple algebras with involution; some further applications of this result can be found in [5].

It should be emphasized that in *all* of these results it was assumed that one of the extensions satisfies the Hasse principle. In this light, the main result of this note looks quite surprising: we show that no assumption of this nature is actually needed.

**Theorem.** *Let  $L_1$  and  $L_2$  be two finite separable extensions of a global field  $K$ , and let  $E_i$  be the Galois closure of  $L_i$  over  $K$  for  $i = 1, 2$ . If  $E_1 \cap E_2 = K$  (i.e.,  $E_1$  and  $E_2$  are linearly disjoint over  $K$ ) then the pair  $L_1, L_2$  satisfies the multinorm principle.*

We notice that the conclusion of the theorem can be false for non-linearly disjoint extensions. For example, if  $L_1 = L_2 =: L$ , then the multinorm principle is equivalent to the norm principle for  $L/K$ , hence may fail. See §4 for more sophisticated examples and a discussion of a more general conjecture.

The proof of the theorem is based on the following sufficient condition for the multinorm principle.

**Proposition 1.** *Let  $L_1$  and  $L_2$  be two finite separable extensions of  $K$  such that their Galois closures  $E_1$  and  $E_2$  satisfy  $E_1 \cap E_2 = K$ . Set  $L = L_1 L_2$ . If the map*

$$\phi : \text{III}(L/K) \rightarrow \text{III}(L_1/K) \times \text{III}(L_2/K)$$

*induced by the diagonal embedding  $K^\times \hookrightarrow K^\times \times K^\times$  is surjective, then the pair  $L_1, L_2$  satisfies the multinorm principle.*

In §2, we prove the proposition and also reduce the proof of the theorem to the case where both  $L_1$  and  $L_2$  are Galois extensions of  $K$ . Then, to complete the proof of the theorem, we verify that the map  $\phi$  is in fact surjective for any two linearly disjoint Galois extensions - cf. Proposition 3 in §3. Finally, §4 contains some additional results and examples related to the multinorm principle.

## 2. PROOF OF PROPOSITION 1

The following statement will enable us to prove Proposition 1, but is also of independent interest.

**Proposition 2.** *Let  $L_1$  and  $L_2$  be finite extensions of  $K$  such that their Galois closures  $E_1$  and  $E_2$  satisfy  $E_1 \cap E_2 = K$ . Let  $L = L_1 L_2$ , and let*

$$S = K^\times \cap N_{L/K}(J_L) \quad \text{and} \quad T = N_{L_1/K}(L_1^\times) N_{L_2/K}(L_2^\times).$$

*Then the following conditions are equivalent:*

- (1) *The pair  $L_1, L_2$  satisfies the multinorm principle;*
- (2)  *$K^\times \cap N_{L_i/K}(J_{L_i}) \subset T$  for  $i = 1$  and  $2$ ;*
- (3)  *$K^\times \cap N_{L_i/K}(J_{L_i}) \subset T$  for at least one index  $i \in \{1, 2\}$ ;*
- (4)  *$S \subset T$ .*

*Proof.* The implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) are obvious, while the nontrivial implication (4)  $\Rightarrow$  (1) is a consequence of the following statement which is extracted from the the proof of Proposition 6.11 in [13].

**Lemma 3.** *Let  $L_1$  and  $L_2$  be as in Proposition 2. Then in the above notations we have*

$$K^\times \cap N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2}) = ST.$$

*Proof.* For completeness, we (succinctly) reproduce the argument given in [13]. Let  $M_i$  be the maximal abelian extension of  $K$  contained in  $L_i$  for  $i = 1, 2$ , and  $M$  be the maximal abelian extension of  $K$  contained in  $L$ . Then by Galois theory the fact that  $E_1 \cap E_2 = K$  implies that

- $M = M_1 M_2$  and  $\text{Gal}(M/K)$  is naturally isomorphic to  $\text{Gal}(M/M_1) \times \text{Gal}(M/M_2)$ ;
- the maximal abelian extension of  $L_i$  contained in  $L$  is  $L_i M_{3-i}$  for  $i = 1, 2$ .

The crucial observation is that the map

$$\varphi: J_{L_1}/L_1^\times N_{L/L_1}(J_L) \times J_{L_2}/L_2^\times N_{L/L_2}(J_L) \longrightarrow J_K/K^\times N_{L/K}(J_L),$$

induced by the product of the norm maps  $N_{L_1/K}$  and  $N_{L_2/K}$ , is an isomorphism, which is proved by showing that  $\varphi$  is surjective and that its domain and target have the same order. To this end, we consider the following commutative diagram

$$(1) \quad \begin{array}{ccc} J_{M_1}/M_1^\times N_{M/M_1}(J_M) \times J_{M_2}/M_2^\times N_{M/M_2}(J_M) & \xrightarrow{\psi} & J_K/K^\times N_{M/K}(J_M) \\ \theta_1 \times \theta_2 \downarrow & & \downarrow \theta \\ \text{Gal}(M/M_1) \times \text{Gal}(M/M_2) & \xrightarrow{\iota} & \text{Gal}(M/K) \end{array},$$

where  $\psi$  is constructed analogously to  $\varphi$ ,

$$\theta_i: J_{M_i}/M_i^\times N_{M/M_i}(J_M) \rightarrow \text{Gal}(M/M_i) \quad \text{and} \quad \theta: J_K/K^\times N_{M/K}(J_M) \rightarrow \text{Gal}(M/K)$$

are the isomorphisms given by the corresponding Artin maps (cf. [2, Ch. VII]), and  $\iota$  is induced by the canonical embeddings  $\text{Gal}(M/M_i) \rightarrow \text{Gal}(M/K)$ ; the commutativity of (1) follows from Proposition 4.3 in [2]. In our situation,  $\iota$  is an isomorphism, so  $\psi$  is also an isomorphism, implying that

$$(2) \quad J_K = K^\times N_{M_1/K}(J_{M_1}) N_{M_2/K}(J_{M_2}).$$

We now recall the fact that for any finite separable extension  $P/F$  of global fields we have

$$F^\times N_{P/F}(J_P) = F^\times N_{R/F}(J_R),$$

where  $R$  is the maximal abelian extension of  $F$  contained in  $P$  (cf. [2, Exercise 8]). Thus,

$$K^\times N_{L_i/K}(J_{L_i}) = K^\times N_{M_i/K}(J_{M_i}) \quad \text{for } i = 1, 2,$$

which in conjunction with (2) yields that

$$J_K = K^\times N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2}),$$

proving that  $\varphi$  is surjective. On the other hand, since  $L_1 M_2$  is the maximal abelian extension of  $L_1$  contained in  $L$ , using the fundamental isomorphism of global class field theory we obtain

$$\begin{aligned} |J_{L_1}/L_1^\times N_{L/L_1}(J_L)| &= |J_{L_1}/L_1^\times N_{L_1 M_2/L_1}(J_{L_1 M_2})| = [L_1 M_2 : L_1] = \\ &= [M_2 : K] = [M : M_1] = |J_{M_1}/M_1^\times N_{M/M_1}(J_M)|, \end{aligned}$$

and similarly

$$|J_{L_2}/L_2^\times N_{L/L_2}(J_L)| = |J_{M_2}/M_2^\times N_{M/M_2}(J_M)| \quad \text{and} \quad |J_K/K^\times N_{L/K}(J_L)| = |J_K/K^\times N_{M/K}(J_M)|.$$

Since  $\psi$  is an isomorphism, these equations imply that the domain and the target of  $\varphi$  have the same order, proving that  $\varphi$  is in fact an isomorphism.

Now, take any  $a \in K^\times \cap N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2})$ , and write it in the form

$$a = N_{L_1/K}(x_1) N_{L_2/K}(x_2) \quad \text{with } x_i \in J_{L_i}.$$

Then  $(x_1 L_1^\times N_{L/L_1}(J_L), x_2 L_2^\times N_{L/L_2}(J_L)) \in \text{Ker } \varphi$ . Using the injectivity of  $\varphi$  established above, we see that we can write

$$x_i = y_i N_{L/L_i}(z_i) \quad \text{with } y_i \in L_i^\times, z_i \in J_L \quad \text{for } i = 1, 2.$$

Then

$$a = (N_{L_1/K}(y_1) N_{L_2/K}(y_2)) N_{L/K}(z_1 z_2) \in TS.$$

This proves the inclusion

$$K^\times \cap N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2}) \subset ST,$$

while the reverse inclusion is obvious.  $\square$

**Remark.** If one of the  $L_i$ 's satisfies the usual Hasse norm principle then condition (3) of Proposition 2 obviously holds for this  $i$ . This yields the multinorm principle in this situation, which is precisely the assertion of Proposition 6.11 in [13]. Thus, the latter is a particular case of our Proposition 2.

Before proceeding with the proof of Proposition 1, we will now use Lemma 3 to give

*Reduction of the theorem to the Galois case.* Let  $L_1, L_2$  be as in the theorem, and let us assume that we already know that their Galois closures  $E_1, E_2$  satisfy the multinorm principle. We will now show that the pair  $L_1, L_2$  satisfies the multinorm principle as well. Generalizing the notions introduced in the proof of Proposition 2, for a pair of finite extensions  $P_1$  and  $P_2$  of  $K$ , we set

$$S_{P_1, P_2} = K^\times \cap N_{P_1 P_2 / K}(J_{P_1 P_2}) \quad \text{and} \quad T_{P_1, P_2} = N_{P_1 / K}(P_1^\times) N_{P_2 / K}(P_2^\times).$$

We also set

$$R_{P_1, P_2} = K^\times \cap N_{P_1 / K}(J_{P_1}) N_{P_2 / K}(J_{P_2}).$$

We note that for any other finite extensions  $P'_1$  and  $P'_2$  of  $K$  we have the inclusions

$$(3) \quad S_{P_1, P_2} \subset R_{P'_1, P_2} \quad \text{and} \quad S_{P_1, P_2} \subset R_{P_1, P'_2}.$$

Now, applying Lemma 3 twice in conjunction with (3), we obtain

$$(4) \quad R_{L_1, L_2} = T_{L_1, L_2} S_{L_1, L_2} \subset T_{L_1, L_2} R_{E_1, L_2} = T_{L_1, L_2} T_{E_1, L_2} S_{E_1, L_2} \subset T_{L_1, L_2} T_{E_1, L_2} R_{E_1, E_2}.$$

Since by our assumption the multinorm principle holds for the pair  $E_1, E_2$ , we have  $R_{E_1, E_2} = T_{E_1, E_2}$ , so (4) becomes

$$R_{L_1, L_2} \subset T_{L_1, L_2} T_{E_1, L_2} T_{E_1, E_2} = T_{L_1, L_2},$$

which means that the multinorm principle holds for the pair  $L_1, L_2$ .  $\square$

To complete the proof of Proposition 1, we need the following elementary group-theoretic lemma.

**Lemma 4.** *Let  $\mathcal{A}$  be an abelian group with subgroups  $\mathcal{B}$  and  $\mathcal{C}$ . Then the sequence*

$$\mathcal{A} \xrightarrow{f} \frac{\mathcal{A}}{\mathcal{B}} \times \frac{\mathcal{A}}{\mathcal{C}} \xrightarrow{g} \frac{\mathcal{A}}{\mathcal{BC}} \longrightarrow 1,$$

where  $f$  and  $g$  are defined by

$$f(x) = (x\mathcal{B}, x\mathcal{C}) \quad \text{and} \quad g(x\mathcal{B}, y\mathcal{C}) = xy^{-1}\mathcal{BC},$$

is exact.

*Proof of Proposition 1.* Applying Lemma 4 to the group  $\mathcal{A} = K^\times \cap N_{L/K}(J_L)$  and its subgroups

$$\mathcal{B} = N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L) \quad \text{and} \quad \mathcal{C} = N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L),$$

we obtain the following exact sequence

$$(5) \quad K^\times \cap N_{L/K}(J_L) \xrightarrow{f} \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L)} \times \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L)} \xrightarrow{g} \\ \longrightarrow \frac{K^\times \cap N_{L/K}(J_L)}{(N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L))(N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L))} \longrightarrow 1.$$

By our assumption, the composite homomorphism

$$\text{III}(L/K) = \frac{K^\times \cap N_{L/K}(J_L)}{N_{L/K}(L^\times)} \xrightarrow{\bar{f}} \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L)} \times \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L)} \xrightarrow{h} \\ \longrightarrow \frac{K^\times \cap N_{L_1/K}(J_{L_1})}{N_{L_1/K}(L_1^\times)} \times \frac{K^\times \cap N_{L_2/K}(J_{L_2})}{N_{L_2/K}(L_2^\times)} = \text{III}(L_1/K) \times \text{III}(L_2/K),$$

where  $\bar{f}$  is induced by  $f$  and  $h$  by the inclusions  $K^\times \cap N_{L/K}(J_L) \subset K^\times \cap N_{L_i/K}(J_{L_i})$  for  $i = 1, 2$ , is surjective. Since  $h$  is obviously injective, we conclude that  $\bar{f}$ , hence  $f$ , is surjective. So, the exact sequence (5) yields that its third term is trivial, i.e.

$$\begin{aligned} S &= K^\times \cap N_{L/K}(J_L) = (N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L))(N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L)) \subset \\ &\subset N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times) = T. \end{aligned}$$

This verifies condition (4) of Proposition 2, thereby yielding the validity of the multinorm principle for the pair  $L_1, L_2$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

As we have seen in § 2, it is enough to prove the main theorem assuming that both  $L_1$  and  $L_2$  are Galois extensions of  $K$ . In this case, the claim is a consequence of Proposition 1 combined with the following statement.

**Proposition 5.** *Let  $L_1$  and  $L_2$  be Galois extensions of  $K$  with  $L_1 \cap L_2 = K$ , and let  $L = L_1L_2$ . Then the map*

$$\phi: \text{III}(L/K) \rightarrow \text{III}(L_1/K) \times \text{III}(L_2/K)$$

*induced by the diagonal embedding  $K^\times \hookrightarrow K^\times \times K^\times$  is surjective.*

Our proof relies on properties of the deflation and residuation maps for the Tate cohomology groups, introduced in [18] and [7], and their interaction with the fundamental isomorphisms of class field theory. Since these maps are rarely used, we briefly recall in the appendix their construction, which is needed to prove the key Lemma 8.

Given a finite group  $G$  and a  $G$ -module  $A$ , we let  $\hat{H}^i(G, A)$  denote the  $i$ th Tate cohomology group (cf., for example, [2, Ch. IV, § 6]). For a normal subgroup  $H$  of  $G$  and any  $i \geq 0$ , one can define the *deflation map*

$$\text{Def}_{G/H}^G: \hat{H}^{-i}(G, A) \rightarrow \hat{H}^{-i}(G/H, A^H).$$

The deflation map is natural; in particular, it has the following properties.

**Lemma 6.** *For any  $G$ -module homomorphism  $f: A \rightarrow B$  and any  $i \geq 0$ , the diagram*

$$\begin{array}{ccc} \hat{H}^{-i}(G, A) & \longrightarrow & \hat{H}^{-i}(G, B) \\ \downarrow \text{Def}_{G/H}^G & & \downarrow \text{Def}_{G/H}^G \\ \hat{H}^{-i}(G/H, A^H) & \longrightarrow & \hat{H}^{-i}(G/H, B^H) \end{array}$$

*in which the horizontal maps are induced by  $f$ , is commutative.*

*Proof.* This is Proposition 8 in [18].  $\square$

**Lemma 7.** *Let*

$$(6) \quad 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

*be an exact sequence of  $G$ -modules, and assume that the induced sequence of  $G/H$ -modules*

$$(7) \quad 0 \rightarrow A^H \rightarrow B^H \rightarrow C^H \rightarrow 0$$

*is also exact. Then for any  $i \geq 1$  the diagram*

$$\begin{array}{ccc} \hat{H}^{-i}(G, C) & \longrightarrow & \hat{H}^{-i+1}(G, A) \\ \downarrow \text{Def}_{G/H}^G & & \downarrow \text{Def}_{G/H}^G \\ \hat{H}^{-i}(G/H, C^H) & \longrightarrow & \hat{H}^{-i+1}(G/H, A^H) \end{array}$$

*in which the horizontal maps are the coboundary maps arising from the exact sequences (6) and (7), is commutative.*

*Proof.* This is Proposition 4 in [18].  $\square$

Our proof also makes use of the *residuation map*  $\text{Rsd}_H^G$  – see the appendix. The key property that we need is that in the case of interest to us, the residuation map is the dual of the usual inflation map. More precisely, we have the following.

**Lemma 8.** *Let  $G = H \times K$  and identify  $G/K$  with  $H$ . Then for  $i \geq 2$  the residuation and inflation maps in the following diagram*

$$\begin{array}{ccccc} \hat{H}^{-i}(G, \mathbb{Z}) \times \hat{H}^i(G, \mathbb{Z}) & \xrightarrow{\cup} & \hat{H}^0(G, \mathbb{Z}) & & \\ \downarrow \text{Rsd}_H^G & & \uparrow \text{Inf}_H^G & & \uparrow \text{Cor}_H^G \\ \hat{H}^{-i}(H, \mathbb{Z}) \times \hat{H}^i(H, \mathbb{Z}) & \xrightarrow{\cup} & \hat{H}^0(H, \mathbb{Z}) & & \end{array}$$

are adjoint with respect to the pairings given by the  $\cup$ -products. That is,

$$f \cup \text{Inf}_H^G(\psi) = \text{Cor}_H^G(\text{Rsd}_H^G(f) \cup \psi)$$

for every  $f \in \hat{H}^{-i}(G, \mathbb{Z})$  and  $\psi \in \hat{H}^i(H, \mathbb{Z})$ .

*Proof.* This uses an explicit construction of the residuation map and will be given in the appendix.  $\square$

Another critical ingredient of the proof of Proposition 5 is the following result of K. Horie and M. Horie [7] that shows how the deflation and residuation maps interact with the isomorphisms from class field theory. For a global field  $K$ , we let  $C_K = J_K/K^\times$  denote the idele class group. Furthermore, given a Galois extension  $F/K$  of global fields, for any  $\text{Gal}(F/K)$ -module  $A$  we write  $\hat{H}(F/K, A)$  instead of  $\hat{H}(\text{Gal}(F/K), A)$ , and then for any  $i \in \mathbb{Z}$  there is a canonical isomorphism  $\Phi_F: \hat{H}^{i-2}(F/K, \mathbb{Z}) \rightarrow \hat{H}^i(F/K, C_F)$  called the *Tate isomorphism* (cf. [2], Ch. VII).

**Lemma 9.** ([7], Theorem 1) *Let  $E \subset F$  be Galois extensions of a global field  $K$ . Then for any  $i \geq 0$ , the following diagram*

$$(8) \quad \begin{array}{ccc} \hat{H}^{-i-2}(F/K, \mathbb{Z}) & \xrightarrow{\Phi_F} & \hat{H}^{-i}(F/K, C_F) \\ \downarrow \text{Rsd}_{\text{Gal}(E/K)}^{\text{Gal}(F/K)} & & \downarrow \text{Def}_{\text{Gal}(E/K)}^{\text{Gal}(F/K)} \\ \hat{H}^{-i-2}(E/K, \mathbb{Z}) & \xrightarrow{\Phi_E} & \hat{H}^{-i}(E/K, C_E) \end{array}$$

commutes.

(We will only use this lemma for  $i = 1$ .)

*Proof of Proposition 5.* For a finite Galois extension  $F/K$ , we let

$$\kappa_F: \hat{H}^0(F/K, F^\times) \rightarrow \hat{H}^0(F/K, J_F)$$

denote the map induced by the inclusion  $F^\times \rightarrow J_F$ . Then clearly  $\text{III}(F/K) = \text{Ker } \kappa_F$ . Now, let  $G_j = \text{Gal}(L_j/K)$  for  $j = 1, 2$ . Since  $L_1$  and  $L_2$  are assumed to be linearly disjoint, for  $L = L_1L_2$  and  $G = \text{Gal}(L/K)$  there is a natural isomorphism

$$G = G_1 \times G_2,$$

which in particular allows us to identify  $G/G_{3-j}$  with  $G_j$  for  $j = 1, 2$ . Considering the inclusion  $L^\times \rightarrow J_L$  as part of the exact sequence of  $G$ -modules  $1 \rightarrow L^\times \rightarrow J_L \rightarrow C_L \rightarrow 1$  and applying Lemmas 6 and 7 to  $H = G_{3-j}$  with  $i = 1$  we obtain (observing that the corresponding sequence (7)

is  $1 \rightarrow L_j^\times \rightarrow J_{L_j} \rightarrow C_{L_j} \rightarrow 1$ , cf. [2, Ch. VII, Prop. 8.1]) the following commutative diagram with exact rows:

$$(9) \quad \begin{array}{ccccc} \hat{H}^{-1}(G, C_L) & \longrightarrow & \hat{H}^0(G, L^\times) & \xrightarrow{\kappa_L} & \hat{H}^0(G, J_L) \\ \downarrow \text{Def}_{G_j}^G & & \downarrow \text{Def}_{G_j}^G & & \downarrow \text{Def}_{G_j}^G \\ \hat{H}^{-1}(G_j, C_{L_j}) & \longrightarrow & \hat{H}^0(G_j, L_j^\times) & \xrightarrow{\kappa_{L_j}} & \hat{H}^0(G_j, J_{L_j}) \end{array}$$

for each  $j = 1, 2$ . Since the deflation map in dimension 0 is induced by the identity map (cf. the appendix), we see that the map  $\phi$  in Proposition 5 is the map  $\text{Ker}(\kappa_L) \rightarrow \text{Ker}(\kappa_{L_1}) \times \text{Ker}(\kappa_{L_2})$  induced by  $\text{Def}_{G_1}^G \times \text{Def}_{G_2}^G$ . So, it follows from (9) that  $\phi$  is surjective if

$$(10) \quad \text{Def}_{G_1}^G \times \text{Def}_{G_2}^G : \hat{H}^{-1}(G, C_L) \rightarrow \hat{H}^{-1}(G_1, C_{L_1}) \times \hat{H}^{-1}(G_2, C_{L_2})$$

is such. Now, using Lemma 9 with  $i = 1$ , we obtain the following commutative diagram

$$\begin{array}{ccc} \hat{H}^{-3}(G, \mathbb{Z}) & \xrightarrow{\Phi_L} & \hat{H}^{-1}(G, C_L) \\ \downarrow \text{Rsd}_{G_j}^G & & \downarrow \text{Def}_{G_j}^G \\ \hat{H}^{-3}(G_j, \mathbb{Z}) & \xrightarrow{\Phi_{L_j}} & \hat{H}^{-1}(G_j, C_{L_j}) \end{array}$$

for each  $j = 1, 2$ . So, the surjectivity of (10) is equivalent to that of

$$(11) \quad \text{Rsd}_{G_1}^G \times \text{Rsd}_{G_2}^G : \hat{H}^{-3}(G, \mathbb{Z}) \rightarrow \hat{H}^{-3}(G_1, \mathbb{Z}) \times \hat{H}^{-3}(G_2, \mathbb{Z}).$$

For this, we will use the duality between the residuation and inflation maps provided by Lemma 8. More precisely, it is well-known (cf., for example, [1, Theorem 6.6, p. 250]) that for any finite group  $H$  and any  $i \in \mathbb{Z}$ , the  $\cup$ -product defines a perfect pairing

$$\alpha_H : \hat{H}^{-i}(H, \mathbb{Z}) \times \hat{H}^i(H, \mathbb{Z}) \rightarrow \hat{H}^0(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}.$$

On the other hand, in our situation,  $\text{Cor}_{G_j}^G$  identifies  $H^0(G_j, \mathbb{Z}) = \mathbb{Z}/|G_j|\mathbb{Z}$  with

$$|G_{3-j}|\mathbb{Z}/|G|\mathbb{Z} \subset \mathbb{Z}/|G|\mathbb{Z} = \hat{H}^0(G, \mathbb{Z}).$$

It follows that  $\alpha = \text{Cor}_{G_1}^G \circ \alpha_{G_1} + \text{Cor}_{G_2}^G \circ \alpha_{G_2}$  defines a perfect pairing

$$(\hat{H}^{-i}(G_1, \mathbb{Z}) \times \hat{H}^{-i}(G_2, \mathbb{Z})) \times (\hat{H}^i(G_1, \mathbb{Z}) \times \hat{H}^i(G_2, \mathbb{Z})) \rightarrow H^0(G, \mathbb{Z}).$$

Furthermore, by Lemma 8, we have the following commutative diagram

$$\begin{array}{ccc} \hat{H}^{-3}(G, \mathbb{Z}) & \times & H^3(G, \mathbb{Z}) \\ \text{Rsd}_{G_1}^G \times \text{Rsd}_{G_2}^G \downarrow & & \uparrow \text{Inf}_{G_1}^G + \text{Inf}_{G_2}^G \\ (\hat{H}^{-3}(G_1, \mathbb{Z}) \times \hat{H}^{-3}(G_2, \mathbb{Z})) & \times & (\hat{H}^3(G_1, \mathbb{Z}) \times \hat{H}^3(G_2, \mathbb{Z})) \end{array} \begin{array}{c} \searrow \cup \\ \nearrow \alpha \\ \rightarrow \hat{H}^0(G, \mathbb{Z}) \end{array}$$

Thus, the surjectivity of (15) is equivalent to the injectivity of  $\text{Inf}_{G_1}^G + \text{Inf}_{G_2}^G$ , and the proof of the proposition is completed by the following statement.

**Lemma 10.** *For any finite group  $G$  of the form  $G = G_1 \times G_2$  and any  $i \geq 1$ , the map*

$$\text{Inf}_{G_1}^G + \text{Inf}_{G_2}^G : \hat{H}^i(G_1, \mathbb{Z}) \times \hat{H}^i(G_2, \mathbb{Z}) \rightarrow \hat{H}^i(G, \mathbb{Z})$$

*is injective.*

*Proof.* For a subgroup  $H \subset G$ , we let  $\text{Res}_H^G : \hat{H}^i(G, \mathbb{Z}) \rightarrow \hat{H}^i(H, \mathbb{Z})$  denote the corresponding restriction map. Identifying  $G/G_{3-j}$  with  $G_j$  as above, it is easy to see that the composition

$$\text{Res}_{G_j}^G \circ \text{Inf}_{G_j}^G : \hat{H}^i(G_j, \mathbb{Z}) \rightarrow \hat{H}^i(G_j, \mathbb{Z})$$

is the identity map, while the composition  $\text{Res}_{G_{3-j}}^G \circ \text{Inf}_{G_j}^G$  is zero, and our assertion follows.  $\square$

**Remark.** We note that the deflation map in the context of Tate-Shafarevich groups and its connection with the inflation map was used in [10, p. 97] for a different purpose.

#### 4. EXAMPLES AND EXTENSIONS

In this section we give examples where the multinorm principle fails and prove some results that compliment and extend the main theorem.

**Example 1.** For non-Galois extensions, the condition  $L_1 \cap L_2 = K$  may not imply the multinorm principle for the pair  $L_1, L_2$ . Indeed, let  $F/K$  be a Galois extension with Galois group  $G = \text{Gal}(F/K)$  isomorphic to  $A_6$  as in Lemma 2 of [11], and let  $H$  be a subgroup of  $G$  of index 10 (see *loc. cit.* or [13], p. 311). Since  $A_6$  is simple, we can choose  $\sigma \in G$  such that  $\sigma H \sigma^{-1} \neq H$ . Set

$$L_1 = F^H \quad \text{and} \quad L_2 = F^{\sigma H \sigma^{-1}} = \sigma(L_1).$$

Clearly,  $A_6$  does not have any subgroups of index 2 or 5, so  $\langle H, \sigma H \sigma^{-1} \rangle = G$  and therefore

$$(12) \quad L_1 \cap L_2 = K.$$

On the other hand, since  $L_1$  and  $L_2$  are Galois-conjugate over  $K$ , we have

$$N_{L_1/K}(L_1^\times) = N_{L_2/K}(L_2^\times) \quad \text{and} \quad N_{L_1/K}(J_{L_1}) = N_{L_2/K}(J_{L_2}).$$

This means that the multinorm principle for the pair  $L_1, L_2$  is equivalent to the Hasse norm principle for  $L_1/K$ . However, according to Theorem 1 of [11], the latter actually fails for  $L_1/K$ . Thus, the pair  $L_1, L_2$  does not satisfy the Hasse norm principle despite (12).  $\square$

We note that the extensions  $L_1$  and  $L_2$  in Example 1 are not linearly disjoint. However, even for linearly disjoint extensions  $L_1, L_2$  their Galois closures  $E_1$  and  $E_2$  need not satisfy  $E_1 \cap E_2 = K$  (e.g. for the linearly disjoint extensions  $L_1 = \mathbb{Q}(\sqrt[3]{5})$  and  $L_2 = \mathbb{Q}(\sqrt[3]{7})$  of  $\mathbb{Q}$ , we have  $E_1 \cap E_2 = \mathbb{Q}(\zeta_3)$  where  $\zeta_3$  is a primitive 3rd root of unity), which is required to apply our Main Theorem. So, the question of whether any pair  $L_1, L_2$  of linearly disjoint extensions of  $K$  satisfies the multinorm principle remains open.

On the other hand, it would be interesting to analyze the multinorm principle for at least pairs of Galois extensions  $L_1, L_2$  such that  $L_1 \cap L_2 \neq K$ . This case is not well-understood as of now, but the following proposition clarifies the nature of additional conditions one needs to impose to avoid obvious counter-examples.

**Proposition 11.** *Let  $L_1$  and  $L_2$  be finite Galois extensions of  $K$  satisfying  $L_1 \cap L_2 = K$ , and let  $L_3$  be any finite extension of  $L_1$ . If  $L_1/K$  fails to satisfy the norm principle, then the pair  $L_1 L_2, L_3$  fails to satisfy the multinorm principle.*

*Proof.* It follows from Proposition 5 that the natural homomorphism

$$\text{III}(L_1 L_2 / K) \rightarrow \text{III}(L_1 / K)$$

is surjective. Since  $\text{III}(L_1 / K)$  is non-trivial, this means that there exists  $x \in K^\times \cap N_{L_1 L_2 / K}(J_{L_1 L_2})$  that is not in  $N_{L_1 / K}(L_1^\times)$ . Then  $x$  lies in  $K^\times \cap N_{L_1 L_2 / K}(J_{L_1 L_2}) N_{L_3 / K}(J_{L_3})$ , but cannot be contained in  $N_{L_1 L_2 / K}((L_1 L_2)^\times) N_{L_3 / K}(L_3^\times) \subseteq N_{L_1 / K}(L_1^\times)$ .  $\square$

Based on the (negative) result of the proposition, we would like to propose the following.

**Conjecture.** *Let  $L_1$  and  $L_2$  be finite Galois extensions of  $K$ . If every extension  $P$  of  $K$  contained in  $L_1 \cap L_2$  satisfies the norm principle then the pair  $L_1, L_2$  satisfies the multinorm principle. (It may be enough to require that only the intersection  $L_1 \cap L_2$  satisfies the norm principle.)*

We note that, if proved, this conjecture would imply that a pair  $L_1, L_2$  of finite Galois extensions of  $K$  satisfies the multinorm whenever the intersection  $L_1 \cap L_2$  is a cyclic extension of  $K$ .

Next, we would like to point out that in some simple cases the Main Theorem can be proved without any use of group cohomology. The first such instance is when both extensions are biquadratic.

**Proposition 12.** *Let  $L_1$  and  $L_2$  be biquadratic extensions of  $K$  satisfying  $L_1 \cap L_2 = K$ . Then the pair  $L_1, L_2$  satisfies the multinorm principle.*

*Proof.* Write  $L_1 = K(\sqrt{a}, \sqrt{b})$  and  $L_2 = K(\sqrt{c}, \sqrt{d})$ . If at least one of the extensions satisfies the norm principle then the result follows from Proposition 2 (see the remark after the proposition). So, we only need to consider the case where both extensions fail to satisfy the norm principle. Using Tate's computation of the Tate-Shafarevich group for a Galois extension mentioned in the introduction, one readily sees that all local degrees of  $L_i$  over  $K$  are either 1 or 2, and then  $\text{III}(L_i/K)$  is of order 2 for both  $i = 1, 2$ . We let  $S$  and  $T$  denote the sets of places of  $K$  that split in  $K(\sqrt{a})$  and  $K(\sqrt{c})$  respectively. Following [2, Exercise 5], consider the following homomorphisms of  $K^\times$  to  $\{\pm 1\}$ :

$$\varphi_1(x) = \prod_{v \in S} (x, b)_v \quad \text{and} \quad \varphi_2(x) = \prod_{v \in T} (x, d)_v,$$

where  $(x, y)_v$  denotes the Hilbert symbol at  $v$ . Clearly  $\ker \varphi_i$  is an index two subgroup in  $K^\times$  that according to *loc. cit.* admits the following description

$$(13) \quad \ker \varphi_i = \{x \in K^\times \mid x^2 \in N_{L_i/K}(L_i^\times)\}$$

for  $i = 1, 2$ . Since  $b$  and  $d$  define different cosets modulo  $K^{\times 2}$ , it follows from properties of the Hilbert symbol (cf. [2, Exercise 2.6]) that the homomorphisms  $\varphi_1$  and  $\varphi_2$  are distinct, hence  $(\ker \varphi_1)(\ker \varphi_2) = K^\times$ . Using (13), we obtain the inclusion

$$(14) \quad K^{\times 2} \subset N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times).$$

Now, let  $x_i \in K^\times$  be such that  $\varphi_i(x_i) = -1$ . Then  $x_i^2 \notin N_{L_i/K}(L_i^\times)$ . On the other hand, since all the local degrees of  $L_i$  over  $K$  are either 1 or 2, we see that  $x_i^2 \in K^\times \cap N_{L_i/K}(J_{L_i})$ . This means that the coset  $x_i^2 N_{L_i/K}(L_i^\times)$  is a generator of  $\text{III}(L_i/K) \simeq \mathbb{Z}/2\mathbb{Z}$ , hence

$$K^\times \cap N_{L_i/K}(J_{L_i}) = \{1, x_i^2\}N_{L_i/K}(L_i^\times).$$

Now, taking into account (14), we see that

$$K^\times \cap N_{L_i/K}(L_i^\times) \subset N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times),$$

verifying thereby condition (2) of Proposition 2 and completing the proof of the multinorm principle for the pair  $L_1, L_2$ .  $\square$

Another instance is when both extensions are of a prime degree  $p$ . We recall that any extension  $L/K$  of degree  $p$  satisfies the norm principle (cf. [13, Proposition 6.10]). The following proposition provides an analog of this fact for the multinorm principle.

**Proposition 13.** *Let  $L_1$  and  $L_2$  be two separable extensions of  $K$  of a prime degree  $p$ . Then the pair  $L_1, L_2$  satisfies the multinorm principle.*

(Note that in this proposition we don't need to assume that our extensions or their Galois closures are linearly disjoint.)

**Lemma 14.** *Let  $L_1$  and  $L_2$  be finite extensions of  $K$ . For any finite extension  $P$  of  $K$  of degree relatively prime to both  $[L_1 : K]$  and  $[L_2 : K]$ , the validity of the multinorm principle for the pair  $L_1P, L_2P$  of extensions of  $P$  implies its validity for the pair  $L_1, L_2$ .*

*Proof.* For  $i = 1, 2$ , since  $[L_i : K]$  is coprime to  $[P : K]$ , the extensions  $L_i$  and  $P$  are linearly disjoint over  $K$ , which implies that the norm map  $N_{L_i/K}$  coincides (on  $J_{L_i}$  and  $L_i^\times$ ) with the restriction of the norm map  $N_{L_iP/P}$ . Now, suppose that the multinorm principle holds for the pair  $L_1P, L_2P$  over  $P$ , and let

$$x \in K^\times \cap N_{L_1/K}(J_{L_1})N_{L_2/K}(J_{L_2}).$$

Then it follows from the above remark that  $x \in P^\times \cap N_{L_1P/P}(J_{L_1P})N_{L_2P/P}(J_{L_2P})$ , and hence

$$x = N_{L_1P/P}(y_1)N_{L_2P/P}(y_2) \text{ for some } y_i \in (L_iP)^\times, i = 1, 2.$$

Applying  $N_{P/K}$ , we obtain

$$x^{[P:K]} = N_{L_1/K}(N_{L_1P/L_1}(y_1))N_{L_2/K}(N_{L_2P/L_2}(y_2)) \in N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times).$$

Since  $x^{[L_1:K]} \in N_{L_1/K}(L_1^\times)$  and the degrees  $[L_1 : K]$  and  $[P : K]$  are relatively prime, we conclude that

$$x \in N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times),$$

proving the multinorm principle for  $L_1, L_2$ .  $\square$

*Proof of Proposition 13.* We first reduce the proof to the case where both  $L_1$  and  $L_2$  are Galois extensions of  $K$ . Let  $E_1$  be the Galois closure of  $L_1$  and let  $G = \text{Gal}(E_1/K)$ . Then  $G$  is isomorphic to a subgroup of the symmetric group  $S_p$ , so its Sylow  $p$ -subgroup  $G_p$  is a cyclic group of order  $p$ . Set  $P = E_1^{G_p}$ ; then  $E_1 = L_1P$ . Since the degree  $[P : K]$  is coprime to  $p$ , according to Lemma 14, it suffices to prove the multinorm principle for the pair  $L_1P, L_2P$  of extensions of  $P$ . This enables us to assume without any loss of generality that one of the extensions is Galois. Repeating the argument for the other extension, we can assume that both extensions are Galois.

Now, let us consider the case where  $L_1$  and  $L_2$  are cyclic Galois extensions of  $K$  of degree  $p$ . By the Hasse theorem,  $L_i/K$  satisfies the norm principle for  $i = 1, 2$ . So, if  $L_1 \cap L_2 = K$  then the multinorm principle for  $L_1, L_2$  follows from Proposition 2 as condition (2) therein obviously holds. In the remaining case  $L_1 = L_2$ , the multinorm principle reduces to the norm principle for  $L_i$ , and therefore holds as well.  $\square$

**Remark.** If  $L_1$  and  $L_2$  are two separable extensions of  $K$  of a prime degree  $p$ , and  $E_1$  and  $E_2$  are their Galois closures, then one of the following occurs: either the degree of  $E := E_1 \cap E_2$  is prime to  $p$ , or  $E_1 = E_2$ . To see this, one first proves the following elementary lemma from group theory: *Let  $G$  be a transitive subgroup of  $S_p$ . If  $N \neq \{1\}$  is a normal subgroup of  $G$  then the order  $|N|$  is divisible by  $p$ .* Then, if  $E_1 \neq E_2$ , for at least one  $i \in \{1, 2\}$ , the group  $\text{Gal}(E_i/E)$  is a nontrivial normal subgroup of  $\text{Gal}(E_i/K) \subset S_p$ , hence has order divisible by  $p$ . Since the order of  $S_p$  is not divisible by  $p^2$ , we obtain that  $[E : K]$  is prime to  $p$ , as claimed.

Now, if  $[E : K]$  is prime to  $p$  then by Lemma 14 it is enough to prove the multinorm principle for the pair of extensions  $L'_1 := L_1E, L'_2 := L_2E$  of  $E$ . But the Galois closures of  $L'_1$  and  $L'_2$  coincide with  $E_1$  and  $E_2$  respectively, hence are linearly disjoint over  $E$ . So, the multinorm principle for  $L'_1, L'_2$  immediately follows from Proposition 2 as  $L'_1/E$  and  $L'_2/E$  satisfy the norm principle.

An obvious way to construct distinct degree  $p > 2$  extensions  $L_1$  and  $L_2$  of  $K$  such that  $E_1 = E_2$  is to pick an arbitrary non-Galois degree  $p$  extension  $L_1$  and take for  $L_2$  its suitable Galois conjugate. We note, however, that the group-theoretic constructions in [9] allow one to produce *non-conjugate* extensions with this property. In any case, letting  $P$  denote the fixed field of a Sylow  $p$ -subgroup of  $\text{Gal}(E/K)$ , we will have  $L_1P = L_2P = E$ . Then arguing as in Lemma 14 one shows that

$$N_{L_1/K}(L_1^\times) = N_{L_2/K}(L_2^\times) \text{ and } N_{L_1/K}(J_{L_1}) = N_{L_2/K}(J_{L_2})$$

(even when  $L_1$  and  $L_2$  are not Galois conjugate!). Thus, in this case the multinorm principle for  $L_1, L_2$  reduces to the norm principle for  $L_i/K$ . This provides a somewhat more detailed perspective on the result of Proposition 13.

Finally, we observe that the multinorm can be considered not only for pairs but for any finite families of finite extensions of  $K$ . More precisely, we say that a family  $L_1, \dots, L_m$  ( $m \geq 2$ ) satisfies the multinorm principle if

$$K^\times \cap N_{L_1/K}(J_{L_1}) \cdots N_{L_m/K}(J_{L_m}) = N_{L_1/K}(L_1^\times) \cdots N_{L_m/K}(L_m^\times).$$

**Example 2.** *The multinorm principle may fail for a triple  $L_1, L_2, L_3$  of finite Galois extensions of  $K$  even when the fields  $L_i$  and  $L_j$  are pairwise linearly disjoint over  $K$ . Indeed, set  $K = \mathbb{Q}$  and*

$$L_1 = \mathbb{Q}(\sqrt{13}), \quad L_2 = \mathbb{Q}(\sqrt{17}), \quad \text{and } L_3 = \mathbb{Q}(\sqrt{13 \cdot 17}).$$

Then

$$K^\times \cap N_{L_1/K}(J_{L_1})N_{L_2/K}(J_{L_2})N_{L_3/K}(J_{L_3}) = K^\times,$$

but  $N_{L_1/K}(L_1^\times)N_{L_2/K}(L_2^\times)N_{L_3/K}(L_3^\times)$  is a subgroup of  $K^\times$  of index 2 (cf. [2, Exercise 5] and [16, Lemma 4.8]), hence the multinorm principle fails (see also [8, §2]).  $\square$

Generalizing the Main Theorem of this note, one can show that given finite Galois extensions  $L_1, \dots, L_m$  of  $K$  such that

$$\text{Gal}(L_1 \cdots L_m/K) \simeq \text{Gal}(L_1/K) \times \cdots \times \text{Gal}(L_m/K)$$

(in other words, the whole family  $L_1, \dots, L_m$  is linearly disjoint over  $K$ ) then the multinorm principle still holds for  $L_1, \dots, L_m$ . This, however, requires some new considerations which will be described in [14].

#### APPENDIX. DEFLATION AND RESIDUATION MAPS AND THEIR PROPERTIES.

In this appendix, we briefly sketch the construction of the deflation and residuation maps and prove Lemma 8 (note that our account, unlike that in [18] and [7], is based on homogeneous cochains).

Given a finite group  $G$ , we let  $X = \{X_i\}_{i \in \mathbb{Z}}$  denote the standard complex used to define the Tate cohomology groups (cf. [2, ch. IV, §6]). More precisely, for  $i \geq 0$ ,  $X_i = \mathbb{Z}[G^{i+1}]$  with the  $G$ -action  $s(g_0, \dots, g_i) = (sg_0, \dots, sg_i)$ , and the differential  $d: X_{i+1} \rightarrow X_i$  given by

$$d(g_0, \dots, g_{i+1}) = \sum_{j=0}^{i+1} (-1)^j (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_{i+1}).$$

Furthermore, for  $i \geq 1$ , we set  $X_{-i} = \text{Hom}_{\mathbb{Z}}(X_{i-1}, \mathbb{Z})$ , which is a free  $\mathbb{Z}$ -module with a basis  $(s_1^*, \dots, s_i^*)$ , where all  $s_j \in G$ , defined by

$$(s_1^*, \dots, s_i^*)(g_0, \dots, g_{i-1}) = \begin{cases} 1 & \text{if } s_j = g_{j-1} \text{ for all } j, \\ 0 & \text{otherwise,} \end{cases}$$

and the  $G$ -action  $g(s_1^*, \dots, s_i^*) = ((gs_1)^*, \dots, (gs_i)^*)$ . The differential  $d: X_{-i} \rightarrow X_{-i-1}$  is given by

$$d(s_1^*, \dots, s_i^*) = \sum_{j=1}^{i+1} \sum_{g \in G} (-1)^j (s_1^*, \dots, s_{j-1}^*, g^*, s_j^*, \dots, s_i^*).$$

Finally, the “special” differential  $d: X_0 \rightarrow X_{-1}$  is defined by

$$d(g_0) = \sum_{s \in G} s^*.$$

Then for any  $G$ -module  $A$  and all  $i \in \mathbb{Z}$  we have

$$\hat{H}^i(G, A) = H^i(\text{Hom}_G(X, A)).$$

*Deflation map.* Given any normal subgroup  $H$  of  $G$ , we let  $Y$  denote the standard complex for  $G/H$ . Then for any  $G$ -module  $A$  and each  $i \geq 1$  there is a map  $\delta_{-i}: \text{Hom}_G(X_{-i}, A) \rightarrow \text{Hom}_G(Y_{-i}, A)$  given by

$$(\delta_{-i}f)(\alpha_1^*, \dots, \alpha_i^*) = \sum_{g_i H = \alpha_i} f(g_1^*, \dots, g_i^*)$$

for  $f \in \text{Hom}_G(X_{-i}, A)$  and  $\alpha_1, \dots, \alpha_i \in G/H$ . One can check that the image of  $\delta_{-i}$  lies in  $\text{Hom}_{G/H}(Y_{-i}, A^H)$ , hence  $\delta_{-i}$  induces a map

$$\text{Def}_{G/H}^G: \hat{H}^{-i}(G, A) \rightarrow \hat{H}^{-i}(G/H, A^H)$$

called the *deflation map*. For  $i = 0$  one gives an ad hoc definition of the deflation map. Namely, for any group  $G$  and any  $G$ -module  $A$  we have  $\hat{H}^0(G, A) \simeq A^G/N_G(A)$ , where  $N_G$  is the norm map,  $N_G(a) = \sum_{g \in G} ga$ . Then

$$\text{Def}_{G/H}^G: \hat{H}^0(G, A) \rightarrow \hat{H}^0(G/H, A^H)$$

is induced by the identification  $A^G \rightarrow (A^H)^{G/H}$  and the inclusion  $N_G(A) \hookrightarrow N_{G/H}(A^H)$ . (In terms of homogeneous cochains, every element of  $\hat{H}^0(G, A)$  is represented by a function  $f \in \text{Hom}_G(\mathbb{Z}[G], A)$  with values in  $A^G$ . Then  $\text{Def}_{G/H}^G$  is induced by the map  $\delta: \text{Hom}_G(\mathbb{Z}[G], A^G) \rightarrow \text{Hom}_{G/H}(\mathbb{Z}[G/H], A^H)$  given by  $\delta(f)(g_0H) = f(g_0)$ .)

*Residuation map.* Let  $G, H, X, Y$ , and  $A$  be as above. We let  $I_H$  denote the augmentation ideal of  $\mathbb{Z}[H]$ , and set  $A_H = A/I_H A$ . For each  $i \geq 1$  there is a map  $\delta'_{-i}: \text{Hom}_G(X_{-i}, A) \rightarrow \text{Hom}_{G/H}(Y_{-i}, A_H)$  given by

$$(\delta'_i f)(\alpha^*, \alpha_2^*, \dots, \alpha_i^*) = \sum_{g_i H = \alpha_i} f(g^*, g_2^*, \dots, g_i^*) + I_H,$$

where  $g$  is an arbitrary (single) element such that  $gH = \alpha$ ; since  $f$  is a  $G$ -map, this definition does not depend on the choice of  $g$ . Then for  $i \geq 2$ ,  $\delta'_{-i}$  induces a map on cohomology

$$\text{Rsd}_{G/H}^G: \hat{H}^{-i}(G, A) \rightarrow \hat{H}^{-i}(G/H, A_H),$$

called the *residuation map*. We note that in the special case where  $A$  is a trivial  $G$ -module, we have  $A = A^H = A_H$ , and

$$(15) \quad |H| \cdot \text{Rsd}_{G/H}^G = \text{Def}_{G/H}^G.$$

We will make use of this fact below for  $A = \mathbb{Z}$ .

*Proof of Lemma 8.* Fix  $i \geq 2$ , and to simplify notation we will write  $\text{Inf}$ ,  $\text{Def}$ , ... instead of  $\text{Inf}_H^G$ ,  $\text{Def}_H^G$ , etc. Let  $\bar{f} \in \hat{H}^{-i}(G, \mathbb{Z})$  and  $\bar{\psi} \in \hat{H}^i(H, \mathbb{Z})$  be represented by the homogeneous cocycles  $f \in \text{Hom}_G(\mathbb{Z}[(G^*)^i], \mathbb{Z})$ , where  $(G^*)^i = \{(s_1^*, \dots, s_i^*) \mid s_j \in G\}$ , and  $\psi \in \text{Hom}_H(\mathbb{Z}[H^{i+1}], \mathbb{Z})$ . Furthermore,  $\text{Def}(\bar{f})$  and  $\text{Rsd}(\bar{f})$  are represented respectively by  $\tilde{f}_1$  and  $\tilde{f}_2 \in \text{Hom}_H(\mathbb{Z}[(H^*)^i], \mathbb{Z})$  defined by

$$\tilde{f}_1(h_1^*, \dots, h_i^*) = \sum_{k_j \in K} f((h_1 k_1)^*, \dots, (h_i k_i)^*) \text{ and } \tilde{f}_2(h_1^*, h_2^*, \dots, h_i^*) = \sum_{k_j \in K} f(h_1^*, (h_2 k_2)^*, \dots, (h_i k_i)^*),$$

and  $\text{Inf}(\bar{\psi})$  is represented by  $\tilde{\psi} \in \text{Hom}_G(\mathbb{Z}[G^{i+1}], \mathbb{Z})$  given by

$$\tilde{\psi}(h_0 k_0, \dots, h_i k_i) = \psi(h_0, \dots, h_i).$$

Next, as shown in [2, p. 105-108], the cup-product  $\bar{a} \cup \bar{b}$  of classes  $\bar{a} \in \hat{H}^{-i}(G, \mathbb{Z})$  and  $\bar{b} \in \hat{H}^i(G, \mathbb{Z})$  that are represented by the cocycles  $a$  and  $b$ , is represented by the function

$$g_0 \mapsto \sum_{s_1, \dots, s_i \in G} a(s_1^*, \dots, s_i^*) b(s_i, \dots, s_1, g_0),$$

and the cup-product of classes in  $\hat{H}^{-i}(H, \mathbb{Z})$  and  $\hat{H}^i(H, \mathbb{Z})$  is described similarly. Finally, the corestriction map from  $H^0(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}$  to  $H^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$  is given by multiplication by  $[G : H] = |K|$ .

Putting this information together, we obtain that  $\text{Cor}(\text{Rsd}(\bar{f}) \cup \bar{\psi})$  is represented by the function

$$h_0 k_0 \mapsto |K| \sum_{h_1, \dots, h_i \in H} \tilde{f}_2(h_1^*, \dots, h_i^*) \psi(h_i, \dots, h_1, h_0),$$

and therefore in view of (15) by the function

$$\begin{aligned} h_0 k_0 &\mapsto \sum_{h_1, \dots, h_i \in H} \tilde{f}_1(h_1^*, \dots, h_i^*) \psi(h_i, \dots, h_1, h_0) = \sum_{h_j \in H} \sum_{k_j \in K} f((h_1 k_1)^*, \dots, (h_i k_i)^*) \psi(h_i, \dots, h_1, h_0) \\ &= \sum_{h_j \in H, k_j \in K} f((h_1 k_1)^*, \dots, (h_i k_i)^*) \tilde{\psi}(h_1 k_1, \dots, h_i k_i, h_0 k_0) = \sum_{s_j \in G} f(s_1^*, \dots, s_i^*) \tilde{\psi}(s_i, \dots, s_1, h_0 k_0). \end{aligned}$$

But the function

$$h_0 k_0 \mapsto \sum_{s_j \in G} f(s_1^*, \dots, s_i^*) \tilde{\psi}(s_i, \dots, s_1, h_0 k_0)$$

also represents  $\bar{f} \cup \text{Inf}(\bar{\psi})$ , yielding our claim.  $\square$

**Acknowledgements.** The second-named author was partially supported by NSF grant DMS-0965758, BSF grant 2010149 and the Humboldt Foundation. During the preparation of the final version of this paper, he was visiting the Mathematics Department of the University of Michigan as a Gehring Professor; the hospitality and generous support of this institution are thankfully acknowledged.

#### REFERENCES

1. H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.
2. J.W.S. Cassels, A. Fröhlich (Eds.), *Algebraic Number Theory*, Thompson Book Company Inc., Washington D.C., 1967.
3. J.-L. Colliot-Thélène, J.-J. Sansuc, Private Communication.
4. Yu. A. Drakokhrust, *On the complete obstruction to the Hasse principle*, Amer. Math. Soc. Transl.(2) **143** (1989), 29-34.
5. S. Garibaldi, P. Gille, *Algebraic groups with few subgroups*, J. London Math. Soc. **80(2)**(2009), 405-430.
6. S. Gurak, *On the Hasse norm principle*, J. reine und angew. Math **299/300**(1978), 16-27.
7. K. Horie, M. Horie, *Deflation and residuation for class formation*, Journal of Algebra **245**(2001), 607-619.
8. W. Hürlimann, *On algebraic tori of norm type*, Comment. math. Helv. **59**(1984), 539-549.
9. N. Ito, *On permutation groups of prime degree  $p$  which contain (at least) two classes of conjugate subgroups of index  $p$* , Rendiconti del Seminario Matematico della Università di Padova **38**(1967), 287-292.
10. H. Opolka, *Zur Auflösung zahlentheoretischer Knoten*, Math. Z. **173**(1980), 95-103.
11. V.P. Platonov, A. Drakokhrust, *On The Hasse principle for algebraic number fields*, Soviet Math. Dokl. **31**(1985), No.2, 349-353.
12. V.P. Platonov, A. Drakokhrust, *The Hasse principle for primary extensions of algebraic number fields*, Soviet Math. Dokl. **32**(1985), No.3, 789-792.
13. V.P. Platonov, A.S. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, 1994.
14. T. Pollio, *On the multinorm principle for families of finite extensions* (in preparation).
15. G. Prasad, A.S. Rapinchuk, *Computation of the metaplectic kernel*, Publ. Math. IHES **84**(1996), 91-187.
16. G. Prasad, A.S. Rapinchuk, *Local-global principles for embedding of fields with involution into simple algebras with involution*, Comment. math. Helv. **85**(2010), 583-645.
17. V.E. Voskresenskii, *Algebraic Groups and Their Birational Invariants*, AMS, 1998.
18. E. Weiss, *A deflation map*, J. Math. Mech. **8**(1959), 309-329.