GLUING DERIVED EQUIVALENCES TOGETHER

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ABSTRACT. The Grothendieck construction of a diagram X of categories can be seen as a process to construct a single category $\mathrm{Gr}(X)$ by gluing categories in the diagram together. Here we formulate diagrams of categories as oplax functors from a small category I to the 2-category $\Bbbk$-**Cat** of small $\Bbbk$-categories for a fixed commutative ring $\Bbbk$. In our previous paper we defined derived equivalences of those oplax functors. Roughly speaking two oplax functors $X, X' \colon I \to \Bbbk$-**Cat** are derived equivalent if there exist a family of triangle equivalences $F(i)$ from the derived categories $\mathcal{D}(\mathrm{Mod}\, X(i))$ to $\mathcal{D}(\mathrm{Mod}\, X'(i))$ indexed by the objects i in I such that for each morphism $a \colon i \to j$ in I $F(j)\mathcal{D}(\mathrm{Mod}\, X)(a)$ is naturally isomorphic to $\mathcal{D}(\mathrm{Mod}\, X')(a)F(i)$, where $\mathcal{D}(\mathrm{Mod}\, X)(a) \colon \mathcal{D}(\mathrm{Mod}\, X(i)) \to \mathcal{D}(\mathrm{Mod}\, X'(i))$ is given by the left derived tensor product with a $X(i)$-$X'(i)$-bimodule $\overline{X(a)}$ naturally defined by the functor $X(a) \colon X(i) \to X(j)$. In this paper we glue the derived equivalences $F(i)$ together to obtain a derived equivalence between Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$, which shows that if oplax functors are derived equivalent, then so are their Grothendieck constructions. This generalizes and well formulates the fact that if two $\Bbbk$-categories with a G-action for a group G are "G-equivariantly" derived equivalent, then their orbit categories are derived equivalent. As an easy application we see by a unified proof that if two $\Bbbk$-algebras A and A' are derived equivalent, then so are the path categories AQ and $A'Q$ for any quiver Q; so are the incidence algebras AS and $A'S$ for any poset S; and so are the semigroup algebras AG and $A'G$ for any semigroup G. Also we will give examples of gluing of many smaller derived equivalences together to have a larger derived equivalence.

1. INTRODUCTION

This is a continuation of the paper [6]. We fix a small category I and a commutative ring $\Bbbk$ and denote by $\Bbbk$-**Cat** (resp. $\Bbbk$-**Ab**, $\Bbbk$-**Tri**) the 2-category of small $\Bbbk$-categories (resp. abelian $\Bbbk$-categories, triangulated $\Bbbk$-categories). For a $\Bbbk$-category $\mathcal{C}$ a (right) $\mathcal{C}$-*module* is a contravariant functor from $\mathcal{C}$ to the category $\mathrm{Mod}\,\Bbbk$ of $\Bbbk$-modules, and we denote by $\mathrm{Mod}\,\mathcal{C}$ (resp. $\mathrm{Prj}\,\mathcal{C}$, $\mathrm{prj}\,\mathcal{C}$) the category of $\mathcal{C}$-modules (resp. projective $\mathcal{C}$-modules, finitely generated projective $\mathcal{C}$-modules).

A $\Bbbk$-category $\mathcal{C}$ with an action of a group G have been well investigated in connection with a so-called covering technique in representation theory of algebras (see e.g., [8]). The orbit category $\mathcal{C}/G$ and the canonical functor $\mathcal{C} \to \mathcal{C}/G$ are naturally constructed from these data, and one studied relationships between $\mathrm{Mod}\,\mathcal{C}$ and $\mathrm{Mod}\,\mathcal{C}/G$. We brought this point of view to the derived equivalence classification problem of algebras in [1], and a main tool obtained there was fully used in the derived equivalence classifications in [2, 3]. The main tool was extended in [4] in the following form:

Theorem 1.1. *Let G be a group acting on categories $\mathcal{C}$ and $\mathcal{C}'$. Assume the condition*

> (∗) *There exists a G-stable tilting subcategory E of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathcal{C})$ such that there is a G-equivariant equivalence $\mathcal{C}' \to E$.*

Then the orbit categories $\mathcal{C}/G$ and $\mathcal{C}'/G$ are derived equivalent.

(In the above, E is said to be *G-stable* if the set of objects in E is stable under the G-action on $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathcal{C})$ induced from that on $\mathcal{C}$.) Observe that if we regard G as a category with a single object $*$, then a G-action on a category $\mathcal{C}$ is nothing but a functor $X : G \to \Bbbk\text{-}\mathbf{Cat}$ with $X(*) = \mathcal{C}$; and the orbit category $\mathcal{C}/G$ coincides with (the $\Bbbk$-linear version of) the Grothendieck construction $\mathrm{Gr}(X)$ of X defined in [10].

The purpose of this paper is to generalize this theorem to an arbitrary category I and to any *oplax functors* $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ (roughly speaking an oplax functor X is a family $(X(i))_{i \in I_0}$ of $\Bbbk$-categories indexed by the objects i of I with an action of I, the precise definition is given in Definition 2.1). Recall that if $\mathcal{C}$ is a category with an action of a group G, then the module category $\mathrm{Mod}\,\mathcal{C}$ (resp. the derived category $\mathcal{D}(\mathrm{Mod}\,\mathcal{C})$) has the induced G-action; thus both of them are again categories with G-actions. Hence for an oplax functor X the "module category" $\mathrm{Mod}\,X$ (resp. the "derived category" $\mathcal{D}(\mathrm{Mod}\,X)$) should again be a family of categories with an I-action, i.e., an oplax functor from I to $\Bbbk\text{-}\mathbf{Ab}$ (resp. to $\Bbbk\text{-}\mathbf{Tri}$). In addition, we need a notion of equivalences between oplax functors for two purposes:

> (a) to generalize the statement (∗); and
> (b) to define a derived equivalence of oplax functors X, X' by an existence of an equivalence between oplax functors $\mathcal{D}(\mathrm{Mod}\,X)$ and $\mathcal{D}(\mathrm{Mod}\,X')$.

To define equivalences of objects we need notions of 1-morphisms and 2-morphisms, thus we need a 2-categorical structure on the collection of oplax functors, i.e., it is needed to define a 2-category $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$ of all oplax functors from I to a 2-category $\mathbf{C}$, which can be used for both (a) and (b) above. Having these things in mind we see that to generalize the theorem above we have to solve the following problems:

> (1) Define the "module category" of an oplax functor again as an oplax functor.
> (2) Define the "derived category" of an oplax functor as an oplax functor.
> (3) Give a natural definition of an equivalence between oplax functors using 2-morphisms of the 2-category of oplax functors.
> (4) Give a condition on a 1-morphism between oplax functors to be an equivalence.
> (5) Give a natural definition of a derived equivalence between oplax functors by the equivalence (defined in (3)) of their "derived categories" defined in (2).
> (6) Characterize the existence of derived equivalences of oplax functors by tilting subcategories, which turns out to be a generalization of Rickard's Morita theorem for oplax functors.
> (7) Induce a derived equivalence of Grothendieck constructions of oplax functors from the existence of tilting subcategories, which will be a generalization of the theorem above.

In our previous paper [6] we have solved the problems (1) – (6) and made clear the meaning of the condition (∗) in the setting of oplax functors. In this paper we solve the problem (7), and in addition we give a unified way to solve (1) and (2) using the

following general statement on compositions with pseudofunctors (cf. Gordon–Power–Street [9, Subsection 5.6]):

Theorem (Theorem 6.5)**.** *Let* $\mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be 2-categories and* $V\colon \mathbf{C} \to \mathbf{D}$ *a pseudofunctor. Then the obvious correspondence* (*see subsection 9.1 for details*)

$$\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)\colon \overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{D})$$

turns out to be a pseudofunctor.

The solutions of (1) and (2) use the correspondence on objects given by the pseudofunctor $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)$. The correspondence on 1-morphisms is needed also to solve (7). The following is our main result (see Definition 7.4 for definitions):

Theorem (Theorem 8.1)**.** *Let* $X, X' \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. *Assume that there exists a tilting oplax functor* $\mathcal{T}$ *for* X *such that* $\mathcal{T}$ *and* X' *are equivalent in* $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. *Then* $\mathrm{Gr}(X)$ *and* $\mathrm{Gr}(X')$ *are derived equivalent.*

Note that there is an easier way (Lemma 7.1, a solution of (4)) to verify that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ in the above. This theorem can be used to glue many derived equivalences together as shown in Example 8.5.

The paper is organized as follows. In section 2 we recall the definition of the 2-category $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$ of oplax functors from a category I to a 2-category $\mathbf{C}$. In section 3 we first define a diagonal 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ in an obvious way, and introduce a notion of I-coverings $(F, \psi)\colon X \to \Delta(\mathcal{C})$ for an oplax functor $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ and $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$ (the subscript 0 stands for objects) as a generalization of G-coverings for a group G. In section 4 we define a $\Bbbk$-linear version of Grothendieck construction as a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$ and introduce the canonical morphism $(P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$. In section 5 we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor with counit given by the family of canonical morphisms, in particular, this shows that the canonical morphism $(P, \phi)\colon X \to \Delta(\mathrm{Gr}(X))$ is an I-covering and any other I-covering $X \to \Delta(\mathcal{C})$ is given as the composite of this followed by $\Delta(H)$ for an equivalence $H\colon \mathrm{Gr}(X) \to \mathcal{C}$. This will be used in the proof of the main result. In section 6 we redefine the module oplax functor $\mathrm{Mod}\, X\colon I \to \Bbbk\text{-}\mathbf{Ab}$ and its derived oplax functor $\mathcal{D}(\mathrm{Mod}\, X)\colon I \to \Bbbk\text{-}\mathbf{Tri}$ for an oplax functor $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ by using Theorem 6.5. In addition, we also define $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ for $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ and show that this construction preserves I-precoverings, which is also used in the proof of the main result. It is obvious that the definitions given here coincide with those given in our previous paper [6]. In section 7 we recall the definition of derived equivalences of oplax functors in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and the theorem characterizing the derived equivalence by tilting oplax functors (Theorem 7.5). In section 8 we give a proof of Theorem 8.1, and give some applications including an example of gluing of pieces of derived equivalences together to have larger one. In the last section we give a proof of Theorem 6.5.

2. Preliminaries

Throughout this paper I stands for a category. In this section we recall the definition of the 2-category of oplax functors from I to a 2-category from [6] (see also Tamaki [13]).

Definition 2.1. Let $\mathbf{C}$ be a 2-category. An *oplax functor* from I to $\mathbf{C}$ is a triple (X, η, θ) of data:

- a quiver morphism $X \colon I \to \mathbf{C}$, where I and $\mathbf{C}$ are regarded as quivers by forgetting additional data such as 2-morphisms or compositions;
- a family $\eta := (\eta_i)_{i \in I_0}$ of 2-morphisms $\eta_i \colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a} \colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b, a) \in \mathrm{com}(I) := \{(b, a) \in I_1 \times I_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(a) For each $a \colon i \to j$ in I the following are commutative:

$$X(a\mathbb{1}_i) \overset{\theta_{a,\mathbb{1}_i}}{\Longrightarrow} X(a)X(\mathbb{1}_i) \qquad \text{and} \qquad X(\mathbb{1}_j a) \overset{\theta_{\mathbb{1}_j,a}}{\Longrightarrow} X(\mathbb{1}_j)X(a)$$

$$\Big\Downarrow X(a)\eta_i \qquad\qquad \Big\Downarrow \eta_j X(a) \quad ; \text{ and}$$

$$X(a)\mathbb{1}_{X(i)} \qquad\qquad \mathbb{1}_{X(j)}X(a)$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in I the following is commutative:

$$
\begin{array}{ccc}
X(cba) & \overset{\theta_{c,ba}}{\Longrightarrow} & X(c)X(ba) \\
\theta_{cb,a} \Big\Downarrow & & \Big\Downarrow X(c)\theta_{b,a} \\
X(cb)X(a) & \underset{\theta_{c,b}X(a)}{\Longrightarrow} & X(c)X(b)X(a).
\end{array}
$$

Definition 2.2. Let $\mathbf{C}$ be a 2-category and $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be oplax functors from I to $\mathbf{C}$. A 1-*morphism* (called a *left transformation*) from X to X' is a pair (F, ψ) of data

- a family $F := (F(i))_{i \in I_0}$ of 1-morphisms $F(i) \colon X(i) \to X'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- a family $\psi := (\psi(a))_{a \in I_1}$ of 2-morphisms $\psi(a) \colon X'(a)F(i) \Rightarrow F(j)X(a)$ in $\mathbf{C}$ indexed by $a \colon i \to j$ in I_1:

$$
\begin{array}{ccc}
X(i) & \xrightarrow{\ F(i)\ } & X'(i) \\
{\scriptstyle X(a)}\Big\downarrow & {\scriptstyle \psi(a)}\ \Big/\!\!\Big/ & \Big\downarrow{\scriptstyle X'(a)} \\
X(j) & \xrightarrow[\ F(j)\ ]{} & X'(j)
\end{array}
$$

satisfying the axioms

(a) For each $i \in I_0$ the following is commutative:

$$
\begin{array}{ccc}
X'(\mathbb{1}_i)F(i) & \xRightarrow{\ \psi(\mathbb{1}_i)\ } & F(i)X(\mathbb{1}_i) \\
{\scriptstyle \eta'_i F(i)}\Big\| & & \Big\|{\scriptstyle F(i)\eta_i} \\
\mathbb{1}_{X'(i)}F(i) & =\!=\!=\!= & F(i)\mathbb{1}_{X(i)}
\end{array}
\quad ;\ \text{and}
$$

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in I the following is commutative:

$$
\begin{array}{ccccc}
X'(ba)F(i) & \xRightarrow{\ \theta'_{b,a}F(i)\ } & X'(b)X'(a)F(i) & \xRightarrow{\ X'(b)\psi(a)\ } & X'(b)F(j)X(a) \\
{\scriptstyle \psi(ba)}\Big\| & & & & \Big\|{\scriptstyle \psi(b)X(a)} \\
F(k)X(ba) & \multicolumn{3}{c}{\xRightarrow[\ \ \ \ \ \ \ \ F(k)\,\theta_{b,a}\ \ \ \ \ \ \ \ ]{}} & F(k)X(b)X(a).
\end{array}
$$

Definition 2.3. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ be oplax functors from I to $\mathbf{C}$, and (F, ψ), (F', ψ') 1-morphisms from X to X'. A 2-morphism from (F, ψ) to (F', ψ') is a family $\zeta = (\zeta(i))_{i \in I_0}$ of 2-morphisms $\zeta(i) \colon F(i) \Rightarrow F'(i)$ in $\mathbf{C}$ indexed by $i \in I_0$ such that the following is commutative for each $a \colon i \to j$ in I:

$$
\begin{array}{ccc}
X'(a)F(i) & \xRightarrow{\ X'(a)\zeta(i)\ } & X'(a)F'(i) \\
{\scriptstyle \psi(a)}\Big\| & & \Big\|{\scriptstyle \psi'(a)} \\
F(j)X(a) & \xRightarrow{\ \zeta(j)X(a)\ } & F'(j)X(a).
\end{array}
$$

Definition 2.4. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta)$, $X' = (X', \eta', \theta')$ and $X'' = (X'', \eta'', \theta'')$ be oplax functors from I to $\mathbf{C}$, and let $(F, \psi) \colon X \to X'$, $(F', \psi') \colon X' \to X''$ be 1-morphisms. Then the composite $(F', \psi')(F, \psi)$ of (F, ψ) and (F', ψ') is a 1-morphism from X to X'' defined by

$$
(F', \psi')(F, \psi) := (F'F, \psi' \circ \psi),
$$

where $F'F := ((F'(i)F(i))_{i \in I_0}$ and for each $a\colon i \to j$ in I, $(\psi' \circ \psi)(a) := F'(j)\psi(a) \circ \psi'(a)F(i)$ is the pasting of the diagram

$$
\begin{array}{ccccc}
X(i) & \xrightarrow{F(i)} & X'(i) & \xrightarrow{F'(i)} & X''(i) \\
{\scriptstyle X(a)}\downarrow & {\scriptstyle \psi(a)} & {\scriptstyle X'(a)}\downarrow & {\scriptstyle \psi'(a)} & \downarrow{\scriptstyle X''(a)} \\
X(j) & \xrightarrow[F(j)]{} & X'(j) & \xrightarrow[F'(j)]{} & X''(j).
\end{array}
$$

The following is straight forward to verify.

Proposition 2.5. *Let* $\mathbf{C}$ *be a* 2*-category. Then oplax functors* $I \to \mathbf{C}$, 1*-morphisms between them, and* 2*-morphisms between* 1*-morphisms* (*defined above*) *define a* 2*-category, which we denote by* $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$.

3. I-COVERINGS

In this section we introduce the notion of I-coverings that is a generalization of that of G-coverings for a group G introduced in [4]. This will be used in the proof of our main theorem.

Definition 3.1. We define a 2-functor $\Delta\colon \Bbbk\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ as follows, which is called the *diagonal* 2-functor:

- Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$. Then $\Delta(\mathcal{C})$ is defined to be a functor sending each morphism $a\colon i \to j$ in I to $\mathbb{1}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$.
- Let $E\colon \mathcal{C} \to \mathcal{C}'$ be a 1-morphism in $\Bbbk\text{-}\mathbf{Cat}$. Then $\Delta(E)\colon \Delta(\mathcal{C}) \to \Delta(\mathcal{C}')$ is a 1-morphism (F, ψ) in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ defined by $F(i) := E$ and $\psi(a) := \mathbb{1}_E$ for each $i \in I_0$ and each $a \in I_1$:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{E} & \mathcal{C}' \\
{\scriptstyle 1_{\mathcal{C}}}\downarrow & {\scriptstyle 1_E} & \downarrow{\scriptstyle 1_{\mathcal{C}'}} \\
\mathcal{C} & \xrightarrow[E]{} & \mathcal{C}'.
\end{array}
$$

- Let $E, E'\colon \mathcal{C} \to \mathcal{C}'$ be 1-morphisms in $\Bbbk\text{-}\mathbf{Cat}$, and $\alpha\colon E \to E'$ a 2-morphism in $\Bbbk\text{-}\mathbf{Cat}$. Then $\Delta(\alpha)\colon \Delta(E) \to \Delta(E')$ is a 2-morphism in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ defined by $\Delta(\alpha) := (\alpha)_{i \in I_0}$.

Remark 3.2. Let $\mathbf{C}$ be a 2-category, $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$, and $C \in \mathbf{C}_0$. Further let

- F be a family of 1-morphisms $F(i)\colon X(i) \to C$ in $\mathbf{C}$ indexed by $i \in I_0$; and
- ψ be a family 2-morphisms $\psi(a)\colon F(i) \Rightarrow F(j)X(a)$ indexed by $a\colon i \to j$ in I:

$$
\begin{array}{ccc}
X(i) & \xrightarrow{F(i)} & C \\
{\scriptstyle X(a)}\downarrow & & \| \\
X(j) & \xrightarrow[F(j)]{} & C
\end{array}
$$

Then (F, ψ) is in $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})(X, \Delta(C))$ if and only if the following hold.

(a) For each $i \in I_0$ the following is commutative:

$$
\begin{array}{ccc}
F(i) & \overset{\psi(\mathbb{1}_i)}{\Longrightarrow} & F(i)X(\mathbb{1}_i) \\
 & \diagdown & \Big\Vert F(i)\eta_i \quad ; \text{ and} \\
 & & F(i)\mathbb{1}_{X(i)}
\end{array}
$$

(b) For each $i \overset{a}{\to} j \overset{b}{\to} k$ in I the following is commutative:

$$
\begin{array}{ccc}
F(i) & \overset{\psi(a)}{\Longrightarrow} & F(j)X(a) \\
{\scriptstyle\psi(ba)}\Big\Vert & & \Big\Vert {\scriptstyle\psi(b)X(a)} \\
F(k)X(ba) & \underset{F(k)\,\theta_{b,a}}{\Longrightarrow} & F(k)X(b)X(a).
\end{array}
$$

Definition 3.3. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ and $(F, \psi)\colon X \to \Delta(\mathcal{C})$ be in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then

(1) (F, ψ) is called an *I-precovering* if the homomorphism

$$
(F, \psi)^{(1)}_{x,y}\colon \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y) \to \mathcal{C}(F(i)x, F(j)y)
$$

of $\Bbbk$-modules defined by $(f_a\colon X(a)x \to y)_{a \in I(i,j)} \mapsto \sum_{a \in I(i,j)} F(j)(f_a) \circ \psi(a)(x)$ is an isomorphism.

(2) (F, ψ) is called an *I-covering* if it is an *I*-precovering and is *dense*, i.e., for each $c \in \mathcal{C}_0$ there exists an $i \in I_0$ and $x \in X(i)_0$ such that $F(i)(x)$ is isomorphic to c in $\mathcal{C}$.

4. GROTHENDIECK CONSTRUCTIONS

In this section we define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$ whose correspondence on objects is a $\Bbbk$-linear version of (the opposite version of) the original Grothendieck construction (cf. [13]).

Definition 4.1. We define a 2-functor $\mathrm{Gr}\colon \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$, which is called the *Grothendieck construction*.

On objects. Let $X = (X, \eta, \theta) \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$. Then $\mathrm{Gr}(X) \in \Bbbk\text{-}\mathbf{Cat}_0$ is defined as follows.

- $\mathrm{Gr}(X)_0 := \bigcup_{i \in I_0}\{i\} \times X(i)_0 = \{{}_ix := (i, x) \mid i \in I_0, x \in X(i)_0\}$.
- For each ${}_ix, {}_jy \in \mathrm{Gr}(X)_0$, we set

$$
\mathrm{Gr}(X)({}_ix, {}_jy) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y).
$$

- For each $_ix, _jy, _kz \in \mathrm{Gr}(X)_0$ and each $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)(_ix, _jy)$, $g = (g_b)_{b \in I(j,k)} \in \mathrm{Gr}(X)(_jy, _kz)$, we set

$$
g \circ f := \left(\sum_{\substack{a \in I(i,j) \\ b \in I(j,k) \\ c = ba}} g_b \circ X(b) f_a \circ \theta_{b,a} x \right)_{c \in I(i,k)},
$$

where each summand is the composite of

$$
X(ba)x \xrightarrow{\theta_{b,a}x} X(b)X(a)x \xrightarrow{X(b)f_a} X(b)y \xrightarrow{g_b} z.
$$

On 1-morphisms. Let $X = (X, \eta, \theta), X' = (X', \eta', \theta')$ be objects of $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and $(F, \psi) \colon X \to X'$ a 1-morphism in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then a 1-morphism

$$
\mathrm{Gr}(F, \psi) \colon \mathrm{Gr}(X) \to \mathrm{Gr}(X')
$$

in $\Bbbk\text{-}\mathbf{Cat}$ is defined as follows.

- For each $_ix \in \mathrm{Gr}(X)_0$, $\mathrm{Gr}(F, \psi)(_ix) := _i(F(i)x)$.
- For each $_ix, _jy \in \mathrm{Gr}(X)_0$ and each $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)(_ix, _jy)$, we set $\mathrm{Gr}(F, \psi)(f) := (F(j)f_a \circ \psi(a)x)_{a \in I(i,j)}$, where each entry is the composite of

$$
X'(a)F(i)x \xrightarrow{\psi(a)x} F(j)X(a)x \xrightarrow{F(j)f_a} F(j)y.
$$

On 2-morphisms. Let $X = (X, \eta, \theta), X' = (X', \eta', \theta')$ be objects of $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ and $(F, \psi), (F', \psi') \colon X \to X'$ 1-morphisms in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$, and let $\zeta \colon (F, \psi) \Rightarrow (F', \psi')$ be a 2-morphism in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then a 2-morphism

$$
\mathrm{Gr}(\zeta) \colon \mathrm{Gr}(F, \psi) \Rightarrow \mathrm{Gr}(F', \psi')
$$

in $\Bbbk\text{-}\mathbf{Cat}$ is defined by

$$
\mathrm{Gr}(\zeta)_ix := (\delta_{a,\mathbf{1}_i}\zeta(i)x)_{a \in I(i,i)} \colon _i(F(i)x) \to _i(F'(i)x)
$$

in $\mathrm{Gr}(X')$ for each $_ix \in \mathrm{Gr}(X)_0$.

Definition 4.2. Let $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. We define a left transformation $(P_X, \phi_X) := (P, \phi) \colon X \to \Delta(\mathrm{Gr}(X))$ (called the *canonical morphism*) as follows.

- For each $i \in I_0$, the functor $P(i) \colon X(i) \to \mathrm{Gr}(X)$ is defined by

$$
\begin{cases}
P(i)x := _ix \\
P(i)f := (\delta_{a,\mathbf{1}_i}f \circ (\eta_ix))_{a \in I(i,i)} \colon _ix \to _iy \text{ in } \mathrm{Gr}(X)
\end{cases}
$$

for each $f \colon x \to y$ in $X(i)$.
- For each $a \colon i \to j$ in I, the natural transformation $\phi(a) \colon P(i) \Rightarrow P(j)X(a)$

$$
\begin{array}{ccc}
X(i) & \xrightarrow{P(i)} & \mathrm{Gr}(X) \\
{\scriptstyle X(a)}\downarrow & {\scriptstyle \phi(a)}\nearrow\nearrow & \| \\
X(j) & \xrightarrow[P(j)]{} & \mathrm{Gr}(X)
\end{array}
$$

is defined by $\phi(a)x := (\delta_{b,a}\mathbb{1}_{X(a)x})_{b\in I(i,j)}$ for all $x \in X(i)_0$.

Lemma 4.3. *The (P,ϕ) defined above is a 1-morphism in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.*

Proof. Straightforward. □

Proposition 4.4. *Let $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then the canonical morphism $(P,\phi)\colon X \to \Delta(\mathrm{Gr}(X))$ is an I-covering. More precisely, the morphism*

$$(P,\phi)^{(1)}_{x,y}\colon \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \to \mathrm{Gr}(X)(P(i)x, P(j)y)$$

is the identity for each $i, j \in I_0$ and for each $x \in X(i)$, $y \in X(j)$.

Proof. By the definitions of $\mathrm{Gr}(X)_0$ and of P it is obvious that (P,ϕ) is dense. Let $i, j \in I_0$ and $x \in X(i)$, $y \in X(j)$. We only have to show that

$$(P,\phi)^{(1)}_{x,y}\colon \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y) \to \mathrm{Gr}(X)(P(i)x, P(j)y)$$

is the identity. Let $f = (f_a)_{a\in I(i,j)} \in \bigoplus_{a\in I(i,j)} X(j)(X(a)x, y)$. Then

$$\begin{aligned}
(P,\phi)^{(1)}_{x,y}(f) &= \sum_{a\in I(i,j)} P(j)(f_a) \circ \phi(a)x \\
&= \sum_{a\in I(i,j)} (\delta_{b,\mathbf{1}_j} f_a \circ (\eta_j x))_{b\in I(j,j)} \circ (\delta_{c,a}\mathbb{1}_{X(a)x})_{c\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\sum_{\substack{b\in I(j,j) \\ c\in I(i,j) \\ d=bc}} \delta_{b,\mathbf{1}_j} f_a \circ (\eta_j x) \circ \delta_{c,a}\mathbb{1}_{X(b)X(a)x} \circ \theta_{b,c}x \right)_{d\in I(i,j)} \\
&= \sum_{a\in I(i,j)} \left(\delta_{d,a} f_a \circ (\eta_j x) \circ \mathbb{1}_{X(\mathbf{1}_j)X(a)x} \circ \theta_{\mathbf{1}_j,a}x \right)_{d\in I(i,j)} \\
&= (f_a \circ (\eta_j x) \circ \theta_{\mathbf{1}_j,a}x)_{a\in I(i,j)} = (f_a)_{a\in I(i,j)} \\
&= f,
\end{aligned}$$

as required. □

Lemma 4.5. *Let $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})_0$ and $H\colon \mathrm{Gr}(X) \to \mathcal{C}$ be in $\Bbbk\text{-}\mathbf{Cat}$ and consider the composite 1-morphism $(F,\psi)\colon X \xrightarrow{(P,\phi)} \Delta(\mathrm{Gr}(X)) \xrightarrow{\Delta(H)} \Delta(\mathcal{C})$. Then (F,ψ) is an I-covering if and only if H is an equivalence.*

Proof. Obviously (F,ψ) is dense if and only if so is H. Further for each $i, j \in I_0$, $x \in X(i)$ and $y \in X(j)$, $(F,\psi)^{(1)}_{x,y}$ is an isomorphism if and only if so is $H_{ix,jy}$ because

we have a commutative diagram

$$\bigoplus_{a \in I(i,j)} X(j)(X(a)x,y) \xrightarrow{(F,\psi)^{(1)}_{x,y}} \mathcal{C}(F(i)x,F(j)y).$$

$$(P,\phi)^{(1)}_{x,y} \left\| \qquad H_{ix,jy} \right.$$

$$\mathrm{Gr}(X)(_ix,_jy)$$

$$\square$$

5. Adjoints

In this section we will show that the Grothendieck construction is a strict left adjoint to the diagonal 2-functor, and that I-coverings are essentially given by the counit of the adjoint.

Definition 5.1. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$. We define a functor $Q_{\mathcal{C}} \colon \mathrm{Gr}(\Delta(\mathcal{C})) \to \mathcal{C}$ by

- $Q_{\mathcal{C}}(_ix) := x$ for all $_ix \in \mathrm{Gr}(\Delta(\mathcal{C}))_0$; and
- $Q_{\mathcal{C}}((f_a)_{a \in I(i,j)}) := \sum_{a \in I(i,j)} f_a$ for all $(f_a)_{a \in I(i,j)} \in \mathrm{Gr}(\Delta(\mathcal{C}))(_ix,_jy)$ and for all $_ix,_jy \in \mathrm{Gr}(\Delta(\mathcal{C}))_0$.

It is easy to verify that $Q_{\mathcal{C}}$ is a $\Bbbk$-functor.

Theorem 5.2. *The 2-functor* $\mathrm{Gr} \colon \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat}) \to \Bbbk\text{-}\mathbf{Cat}$ *is a strict left 2-adjoint to the 2-functor* $\Delta \colon \Bbbk\text{-}\mathbf{Cat} \to \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})$. *The counit is given by the family of canonical morphisms* $(P_X,\phi_X) \colon X \to \Delta(\mathrm{Gr}(X))$ *indexed by* $X \in \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})$, *and the unit is given by the family of* $Q_{\mathcal{C}} \colon \mathrm{Gr}(\Delta(\mathcal{C})) \to \mathcal{C}$ *defined as above indexed by* $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$.

In particular, (P_X,ϕ_X) *has a strict universality in the comma category* $(X \downarrow \Delta)$, *i.e., for each* $(F,\psi) \colon X \to \Delta(\mathcal{C})$ *in* $\overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})$ *with* $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$, *there exists a unique* $H \colon \mathrm{Gr}(X) \to \mathcal{C}$ *in* $\Bbbk\text{-}\mathbf{Cat}$ *such that the following is a commutative diagram in* $\overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})$:

$$
\begin{array}{ccc}
X & \xrightarrow{(F,\psi)} & \Delta(\mathcal{C}). \\
{\scriptstyle (P_X,\phi_X)}\Big\downarrow & {\scriptstyle \Delta(H)} \nearrow & \\
\Delta(\mathrm{Gr}(X)) & &
\end{array}
$$

Proof. For simplicity set $\varepsilon := ((P_X,\phi_X))_{X \in \overleftarrow{\mathrm{Oplax}}(I,\Bbbk\text{-}\mathbf{Cat})_0}$ and $\eta := (Q_{\mathcal{C}})_{\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0}$.

Claim 1. $\Delta\eta \cdot \varepsilon\Delta = \mathbb{1}_{\Delta}$.

Indeed, Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$. It is enough to show that $\Delta(Q_{\mathcal{C}}) \cdot (P_{\Delta(\mathcal{C})},\phi_{\Delta(\mathcal{C})}) = \mathbb{1}_{\Delta(\mathcal{C})}$. Now

$$\text{LHS} = \left((Q_{\mathcal{C}}P_{\Delta(\mathcal{C})}(i))_{i \in I_0}, (Q_{\mathcal{C}}\phi_{\Delta(\mathcal{C})}(a))_{a \in I_1}\right), \text{ and}$$

$$\text{RHS} = ((\mathbb{1}_{\mathcal{C}})_{i \in I_0}, (\mathbb{1}_{\mathbb{1}_{\mathcal{C}}})_{a \in I_1}).$$

First entry: Let $i \in I$. Then $Q_{\mathcal{C}}P_{\Delta(\mathcal{C})}(i) = \mathbb{1}_{\mathcal{C}}$ because for each $x,y \in \mathcal{C}_0$ and each $f \in \mathcal{C}(x,y)$ we have $(Q_{\mathcal{C}}P_{\Delta(\mathcal{C})}(i))(x) = Q_{\mathcal{C}}(_ix) = x$; and $(Q_{\mathcal{C}}P_{\Delta(\mathcal{C})}(i))(f) = (\delta_{a,\mathbf{1}_i}f \cdot ((\eta_{\Delta(\mathcal{C})})_i x))_{a \in I_1} = \sum_{a \in I(i,i)} \delta_{a,\mathbf{1}_i}f = f$.

Second entry: Let $a\colon i \to j$ in I. Then $Q_\mathcal{C}\phi_{\Delta(\mathcal{C})}(a) = \mathbb{1}_{\mathbf{1}_\mathcal{C}}$ because for each $x \in \mathcal{C}_0$ we have $Q_\mathcal{C}\left(\phi_{\Delta(\mathcal{C})}(a)x\right) = Q_\mathcal{C}\left((\delta_{b,a}\mathbb{1}_{\Delta(\mathcal{C})(a)x})_{b\in I(i,j)}\right) = \sum_{b\in I(i,j)}\delta_{b,a}\mathbb{1}_x = \mathbb{1}_x = \mathbb{1}_{\mathbf{1}_\mathcal{C}x}$. This shows that LHS = RHS.

Claim 2. $\eta\,\mathrm{Gr} \cdot \mathrm{Gr}\,\varepsilon = \mathbb{1}_{\mathrm{Gr}}$.

Indeed, let $X \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. It is enough to show that $Q_{\mathrm{Gr}(X)} \cdot \mathrm{Gr}(P_X, \phi_X) = \mathbb{1}_{\mathrm{Gr}(X)}$.

On objects: Let ${}_ix \in \mathrm{Gr}(X)_0$. Then $Q_{\mathrm{Gr}(X)}\left(\mathrm{Gr}(P_X, \phi_X)(x)\right) = Q_{\mathrm{Gr}(X)}({}_i(P_X(i)x)) = {}_ix$.

On morphisms: Let $f = (f_a)_{a\in I(i,j)}\colon {}_ix \to {}_jy$ be in $\mathrm{Gr}(X)$. Then $Q_{\mathrm{Gr}(X)}\,\mathrm{Gr}(P_X, \phi_X)(f) = Q_{\mathrm{Gr}(X)}((P_X(j)(f_a) \circ \phi_X(a)x)_{a\in I(i,j)}) = \sum_{a\in I(i,j)} P_X(j)(f_a) \circ \phi_X(a)x = (P_X, \phi_X)_{x,y}^{(1)}(f) = f$. Thus the claim holds.

The two claims above prove the assertion. $\qquad\square$

Corollary 5.3. *Let* $(F, \psi)\colon X \to \Delta(\mathcal{C})$ *be in* $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. *Then the following are equivalent.*

(1) (F, ψ) *is an* I-*covering;*

(2) *There exists an equivalence* $H\colon \mathrm{Gr}(X) \to \mathcal{C}$ *such that the diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\;(F,\psi)\;} & \Delta(\mathcal{C}) \\
{\scriptstyle (P_X,\phi_X)}\Big\downarrow & \nearrow{\scriptstyle \Delta(H)} & \\
\Delta(\mathrm{Gr}(X)) & &
\end{array}
$$

is strictly commutative.

Proof. This immediately follows by Theorem 5.2 and Lemma 4.5. $\qquad\square$

6. The Module oplax functor

Let $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ be an oplax functor. In this section we simplify the definition of the "module category" $\mathrm{Mod}\,X$ of X as an oplax functor $I \to \Bbbk\text{-}\mathbf{Cat}$ given in our previous paper [6]. Recall that the *module category* $\mathrm{Mod}\,\mathcal{C}$ of a category $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ is defined to be the functor category $\Bbbk\text{-}\mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}\,\Bbbk)$, where $\mathrm{Mod}\,\Bbbk$ denotes the category of $\Bbbk$-modules. Since $\Bbbk\text{-}\mathbf{Cat}$ is a 2-category, this is extended to a representable 2-functor

$$\mathrm{Mod}' := \Bbbk\text{-}\mathbf{Cat}((\text{-})^{\mathrm{op}}, \mathrm{Mod}\,\Bbbk)\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}.$$

As is easily seen the composite $\mathrm{Mod}' \circ X$ turns out to be an oplax functor $I \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$, i.e., a contravariant lax functor $I \to \Bbbk\text{-}\mathbf{Ab}$. When X is a group action, namely when I is a group G and $X\colon G \to \Bbbk\text{-}\mathbf{Cat}$ is a functor, the usual module category $\mathrm{Mod}\,X$ with a G-action of X was defined to be the composite functor $\mathrm{Mod}\,X := \mathrm{Mod}' \circ X \circ i$, where $i\colon G \to G$ is the group anti-isomorphism defined by $x \mapsto x^{-1}$ for all $x \in G$. In this way we can change $\mathrm{Mod}' \circ X$ to a covariant one. But in general we cannot assume the existence of such an isomorphism i. Instead in this paper we will use a covariant "pseudofunctor" $\mathrm{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}$ defined below and will define $\mathrm{Mod}\,X$ as the composite $\mathrm{Mod} \circ X$, which can be seen as a "lax" extended version of the module

category construction of a category with a G-action stated above. We start with a notion of oplax functors from a 2-category to a 2-category.

Definition 6.1. Let $\mathbf{B}$ and $\mathbf{C}$ be 2-categories.

(1) An *oplax functor* from $\mathbf{B}$ to $\mathbf{C}$ is a triple (X, η, θ) of data:

- a triple $X = (X_0, X_1, X_2)$ of maps $X_i \colon \mathbf{B}_i \to \mathbf{C}_i$ ($\mathbf{B}_i$ denotes the collection of i-morphisms of $\mathbf{B}$ for each $i = 0, 1, 2$) preserving domains and codomains of all 1-morphisms and 2-morphisms (i.e. $X_1(\mathbf{B}_1(i, j)) \subseteq \mathbf{C}_1(X_0 i, X_0 j)$ for all $i, j \in \mathbf{B}_0$ and $X_2(\mathbf{B}_2(a, b)) \subseteq \mathbf{C}_2(X_1 a, X_1 b)$ for all $a, b \in \mathbf{B}_1$ (we omit the subscripts of X below));
- a family $\eta := (\eta_i)_{i \in \mathbf{B}_0}$ of 2-morphisms $\eta_i \colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{C}$ indexed by $i \in \mathbf{B}_0$; and
- a family $\theta := (\theta_{b,a})_{(b,a)}$ of 2-morphisms $\theta_{b,a} \colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{C}$ indexed by $(b, a) \in \mathrm{com}(\mathbf{B}) := \{(b, a) \in \mathbf{B}_1 \times \mathbf{B}_1 \mid ba \text{ is defined}\}$

satisfying the axioms:

(i) $(X_1, X_2) \colon \mathbf{B}_{1,2} \to \mathbf{C}_{1,2}$ is a functor ($\mathbf{B}_{1,2}$ denotes the category defined by $\mathbf{B}_1$ and $\mathbf{B}_2$);

(ii) For each $a \colon i \to j$ in $\mathbf{B}_1$ the following are commutative:

$$
\begin{array}{ccc}
X(a\mathbb{1}_i) \xRightarrow{\theta_{a,1_i}} X(a)X(\mathbb{1}_i) & & X(\mathbb{1}_j a) \xRightarrow{\theta_{1_j,a}} X(\mathbb{1}_j)X(a) \\
\Big\Downarrow X(a)\eta_i \quad \text{and} & & \Big\Downarrow \eta_j X(a) \quad ; \\
X(a)\mathbb{1}_{X(i)} & & \mathbb{1}_{X(j)}X(a)
\end{array}
$$

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{B}_1$ the following is commutative:

$$
\begin{array}{ccc}
X(cba) & \xRightarrow{\theta_{c,ba}} & X(c)X(ba) \\
\theta_{cb,a} \Big\Downarrow & & \Big\Downarrow X(c)\theta_{b,a} \quad ; \text{ and} \\
X(cb)X(a) & \xRightarrow[\theta_{c,b}X(a)]{} & X(c)X(b)X(a)
\end{array}
$$

(iv) For each $a, a' \colon i \to j$ and $b, b' \colon j \to k$ in $\mathbf{B}_1$ and each $\alpha \colon a \to a'$, $\beta \colon b \to b'$ in $\mathbf{B}_2$ the following is commutative:

$$
\begin{array}{ccc}
X(ba) & \xRightarrow{\theta_{b,a}} & X(b)X(a) \\
X(\beta)*X(\alpha) \Big\Downarrow & & \Big\Downarrow X(\beta*\alpha) \\
X(b'a') & \xRightarrow{\theta_{b',a'}} & X(b')X(a').
\end{array}
$$

(2) A *lax functor* from $\mathbf{B}$ to $\mathbf{C}$ is an oplax functor from $\mathbf{B}$ to $\mathbf{C}^{\mathrm{co}}$. ($\mathbf{C}^{\mathrm{co}}$ denotes the 2-category obtained from $\mathbf{C}$ by reversing the 2-morphisms.)

(3) A *pseudofunctor* from $\mathbf{B}$ to $\mathbf{C}$ is an oplax functor with all η_i and $\theta_{b,a}$ 2-isomorphisms.

(4) We define a 2-category $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C})$ consisting of all oplax functors $\mathbf{B} \to \mathbf{C}$ by the same way as in the definition of $\overleftarrow{\mathrm{Oplax}}(I, \mathbf{C})$.

Remark 6.2. (1) Note that a (strict) 2-functor from $\mathbf{B}$ to $\mathbf{C}$ is a pseudofunctor with all η_i and $\theta_{b,a}$ identities.

(2) By regarding the category I as a 2-category with all 2-morphisms identities, the definition above coincides with Definition 2.1.

Example 6.3. (1) Since $\Bbbk\text{-}\mathbf{Cat}$ is a 2-category, $\mathrm{Mod}' := \Bbbk\text{-}\mathbf{Cat}((\text{-})^{\mathrm{op}}, \mathrm{Mod}\,\Bbbk)\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}^{\mathrm{coop}}$ is a 2-functor, which we can regard as a contravariant lax functor

$$\mathrm{Mod}' := \Bbbk\text{-}\mathbf{Cat}((\text{-})^{\mathrm{op}}, \mathrm{Mod}\,\Bbbk)\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}.$$

(2) We define a pseudofunctor $\mathrm{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}$ as follows.

- For each $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$ we set $\mathrm{Mod}\,\mathcal{C} := \mathrm{Mod}'\,\mathcal{C}$.
- For each $F\colon \mathcal{C} \to \mathcal{C}'$ in $\Bbbk\text{-}\mathbf{Cat}_1$ we set $\mathrm{Mod}\,F := \text{-}\otimes_{\mathcal{C}}\overline{F}\colon \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}'$, where $\overline{F}$ is the $\mathcal{C}$-$\mathcal{C}'$-bimodule defined by $\overline{F}(x,y) := \mathcal{C}'(y, F(x))$ for all $x \in \mathcal{C}_0$, $y \in \mathcal{C}'_0$, which we sometimes write as $\overline{F} := \mathcal{C}'(?, F(\text{-}))$.
- For each $\alpha\colon F \Rightarrow G$ in $\Bbbk\text{-}\mathbf{Cat}_2$ (with $F, G\colon \mathcal{C} \to \mathcal{C}'$ in $\Bbbk\text{-}\mathbf{Cat}_1$) we define $\mathrm{Mod}\,\alpha\colon \mathrm{Mod}\,F \Rightarrow \mathrm{Mod}\,G$ by setting $(\mathrm{Mod}\,\alpha)x := \mathcal{C}'(?, \alpha x)\colon \mathcal{C}'(?, Fx) \Rightarrow \mathcal{C}'(?, Gx)$ for all $x \in \mathcal{C}_0$.
- For each $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ we define $\eta_{\mathcal{C}}\colon \mathrm{Mod}\,\mathbb{1}_{\mathcal{C}} \Rightarrow \mathbb{1}_{\mathrm{Mod}\,\mathcal{C}}$ by setting $\eta_{\mathcal{C}}M\colon M \otimes_{\mathcal{C}} \mathcal{C}(?, \text{-}) \to M$ to be the canonical isomorphisms for all $M \in \mathrm{Mod}\,\mathcal{C}$.
- For each pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$ in $\Bbbk\text{-}\mathbf{Cat}$ we define $\theta_{G,F}\colon \mathrm{Mod}\,GF \Rightarrow \mathrm{Mod}\,G \circ \mathrm{Mod}\,F$ as the inverse of the canonical isomorphism

$$\text{-}\otimes_{\mathcal{C}}\mathcal{C}'(?, F(\text{-})) \otimes_{\mathcal{C}'} \mathcal{C}''(?, G(\text{-})) \Rightarrow \text{-}\otimes_{\mathcal{C}}\mathcal{C}''(?, GF(\text{-})).$$

It is straightforward to check that this defines a pseudofunctor.

(3) Denote by $\Bbbk\text{-}\mathbf{ModCat}$ the 2-subcategory of $\Bbbk\text{-}\mathbf{Ab}$ consisting of the following:

- objects: $\mathrm{Mod}\,\mathcal{C}$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$,
- 1-morphisms: functors between objects having exact right adjoints, and
- 2-morphisms: all natural transformations between 1-morphisms.

Then note that the pseudofunctor $\mathrm{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab}$ defined above can be seen as a pseudofunctor $\Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{ModCat}$. For each $\mathrm{Mod}\,\mathcal{C}$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$ we denote by $\mathcal{K}_p(\mathrm{Mod}\,\mathcal{C})$ the full subcategory of the homotopy category $\mathcal{K}(\mathrm{Mod}\,\mathcal{C})$ of $\mathrm{Mod}\,\mathcal{C}$ consisting of *homotopically projective* objects M, i.e., objects M such that $\mathcal{K}(\mathrm{Mod}\,\mathcal{C})(M, A) = 0$ for all acyclic objects A. Recall that there is a natural embedding $\mathbf{j}_{\mathcal{C}}\colon \mathcal{K}_p(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ having a left adjoint $\mathbf{p}_{\mathcal{C}}$ such that there exists a quasi-isomorphism $\eta_{\mathcal{C}}M\colon \mathbf{j}_{\mathcal{C}}\mathbf{p}_{\mathcal{C}}M \to M$ for each $M \in \mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ and that $\mathbf{p}_{\mathcal{C}}\mathbf{j}_{\mathcal{C}} = \mathbb{1}_{\mathcal{K}_p(\mathrm{Mod}\,\mathcal{C})}$. Then we can define a pseudofunctor $\mathcal{D}\colon \Bbbk\text{-}\mathbf{ModCat} \to \Bbbk\text{-}\mathbf{Tri}$ as follows.

- For each $\mathrm{Mod}\,\mathcal{C}$ in $\Bbbk\text{-}\mathbf{ModCat}_0$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$ we set $\mathcal{D}(\mathrm{Mod}\,\mathcal{C})$ to be the derived category of $\mathrm{Mod}\,\mathcal{C}$.
- For each $F\colon \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}'$ in $\Bbbk\text{-}\mathbf{ModCat}_1$, F naturally induces a functor $\mathcal{K}F\colon \mathcal{K}(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{K}(\mathrm{Mod}\,\mathcal{C}')$, which restricts to a functor $\mathcal{K}_pF\colon \mathcal{K}_p(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{K}_p(\mathrm{Mod}\,\mathcal{C}')$ because F has an exact right adjoint. Then we set $\mathcal{D}F$ to be the left derived functor $\mathbf{L}F\colon \mathcal{D}(\mathrm{Mod}\,\mathcal{C}) \to \mathcal{D}(\mathrm{Mod}\,\mathcal{C}')$ of F, which is defined as the composite $\mathbf{L}F := \mathbf{j}_{\mathcal{C}'}(\mathcal{K}_pF)\mathbf{p}_{\mathcal{C}}$.

- For each $\alpha\colon F \Rightarrow F'$ in $\Bbbk\text{-}\mathbf{ModCat}_2$ with $F, F'\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}'$ in $\Bbbk\text{-}\mathbf{ModCat}_1$, α naturally induces a natural transformation $\mathcal{K}_p\alpha\colon \mathcal{K}_pF \Rightarrow \mathcal{K}_pF'$. Then we define $\mathcal{D}\alpha := \mathbf{j}_{\mathcal{C}'}(\mathcal{K}_p\alpha)\mathbf{p}_{\mathcal{C}}$.
- We define $\eta_{\operatorname{Mod}\mathcal{C}}\colon \mathcal{D}(\mathbb{1}_{\operatorname{Mod}\mathcal{C}})(= \mathbf{j}_{\mathcal{C}}\mathbf{p}_{\mathcal{C}}) \Rightarrow \mathbb{1}_{\mathcal{D}(\operatorname{Mod}\mathcal{C})}$ by $\eta_{\operatorname{Mod}\mathcal{C}} := (\eta_{\mathcal{C}}M)_{M \in \mathcal{D}(\operatorname{Mod}\mathcal{C})}$.
- Note that for each $\operatorname{Mod}\mathcal{C} \xrightarrow{F} \operatorname{Mod}\mathcal{C}' \xrightarrow{F'} \operatorname{Mod}\mathcal{C}''$ in $\Bbbk\text{-}\mathbf{ModCat}_1$ we have $\mathbf{L}(F' \circ F) = \mathbf{L}F' \circ \mathbf{L}F$ because $\mathbf{p}_{\mathcal{C}'}\mathbf{j}_{\mathcal{C}'} = \mathbb{1}_{\mathcal{K}_p(\operatorname{Mod}\mathcal{C})}$. We define $\theta_{F',F}\colon \mathbf{L}(F' \circ F) \Rightarrow \mathbf{L}F' \circ \mathbf{L}F$ as the identity $\mathbb{1}_{\mathbf{L}(F'\circ F)}$.

It is straightforward to check that this defines a pseudofunctor.

Example 6.4. (1) We define a pseudofunctor $\operatorname{prj}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{add}$ as the subpseudofunctor of $\operatorname{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Ab} \hookrightarrow \Bbbk\text{-}\mathbf{add}$ by setting $\operatorname{prj}\mathcal{C}$ to be the full subcategory of $\operatorname{Mod}\mathcal{C}$ consisting of finitely generated projective $\mathcal{C}$-modules for each $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$, where $\Bbbk\text{-}\mathbf{add}$ is the full 2-subcategory of $\Bbbk\text{-}\mathbf{Cat}$ consisting of additive $\Bbbk$-categories. Then for each $F\colon \mathcal{C} \to \mathcal{C}'$ in $\Bbbk\text{-}\mathbf{Cat}_1$ and each $x \in \mathcal{C}_0$ we have

$$(\operatorname{prj}F)(\mathcal{C}(\text{-},x)) = \mathcal{C}(\text{-},x) \otimes_{\mathcal{C}} \overline{F} \cong \mathcal{C}'(\text{-},F(x)). \tag{6.1}$$

Note that we can define two 2-functors $\oplus\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{add}$ and $\operatorname{sic}\colon \Bbbk\text{-}\mathbf{add} \to \Bbbk\text{-}\mathbf{add}$ by forming formal additive hulls (see e.g., [2, Subsection 4.1]) and by taking split idempotent completions (see e.g., [4, Definition 3.1]), respectively. Then the Yoneda embeddings $Y_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{prj}\mathcal{C}$, $x \mapsto \mathcal{C}(\text{-},x)$ ($\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$) induce a natural 2-isomorphism $Y\colon \operatorname{sic} \circ \oplus \Rightarrow \operatorname{prj}$:

(2) A 2-functor $\mathcal{K}^{\mathrm{b}}\colon \Bbbk\text{-}\mathbf{add} \to \Bbbk\text{-}\mathbf{Tri}$ is canonically defined by setting $\mathcal{K}^{\mathrm{b}}(\mathcal{C})$ to be the homotopy category of bounded complexes in $\mathcal{C}$ for each $\mathcal{C} \in \Bbbk\text{-}\mathbf{add}$. Then the composite pseudofunctor $\mathcal{K}^{\mathrm{b}} \circ \operatorname{prj}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Tri}$ turns out to be a subpseudofunctor of $\mathcal{D} \circ \operatorname{Mod}\colon \Bbbk\text{-}\mathbf{Cat} \to \Bbbk\text{-}\mathbf{Tri}$.

The following is a useful tool to define new oplax functors from an old one by composing with pseudofunctors. The proof will be given in the last section.

Theorem 6.5. *Let* $\mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *be 2-categories and* $V\colon \mathbf{C} \to \mathbf{D}$ *a pseudofunctor. Then the obvious correspondence* (*see subsection 9.1 for details*)

$$\overleftarrow{\operatorname{Oplax}}(\mathbf{B}, V)\colon \overleftarrow{\operatorname{Oplax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\operatorname{Oplax}}(\mathbf{B}, \mathbf{D})$$

turns out to be a pseudofunctor.

Definition 6.6. Let $X = (X, \eta, \theta) \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.

(1) We define the *module oplax functor* $\operatorname{Mod}X = (\operatorname{Mod}X, \operatorname{Mod}\eta, \operatorname{Mod}\theta)\colon I^{\mathrm{op}} \to \Bbbk\text{-}\mathbf{ModCat}$ of X as the composite $\operatorname{Mod}X := \operatorname{Mod}\circ X = \overleftarrow{\operatorname{Oplax}}(I, \operatorname{Mod})(X)\colon I \xrightarrow{X} \Bbbk\text{-}\mathbf{Cat} \xrightarrow{\operatorname{Mod}} \Bbbk\text{-}\mathbf{ModCat}$. By applying Theorem 6.5 to $\mathbf{B} := I$, $\mathbf{C} := \Bbbk\text{-}\mathbf{Cat}$, $\mathbf{D} := \Bbbk\text{-}\mathbf{ModCat}$ and $V := \operatorname{Mod}$ (Example 6.3(2)) we see that $\operatorname{Mod}X \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{ModCat})$. Then we have

- for each $i \in I_0$, $(\operatorname{Mod} X)(i) = \operatorname{Mod}(X(i))$; and
- for each $a\colon i \to j$ in I the functor $(\operatorname{Mod} X)(a)\colon (\operatorname{Mod} X)(i) \to (\operatorname{Mod} X)(j)$ is given by $(\operatorname{Mod} X)(a) = \text{-} \otimes_{X(i)} \overline{X(a)}$, where $\overline{X(a)}$ is an $X(i)$-$X(j)$-bimodule defined by
$$\overline{X(a)}(x, y) := X(j)(y, X(a)(x))$$
for all $x \in X(i)_0$ and $y \in X(j)_0$.

(2) By Theorem 6.5 and Example 6.3 we can define an oplax functor $\mathcal{D}(\operatorname{Mod} X) \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$ as the composite $\mathcal{D}(\operatorname{Mod} X) := \mathcal{D} \circ \operatorname{Mod} X$, which we call the *derived module oplax functor* of X. Then for each $a\colon i \to j$ in I, $\mathcal{D}(\operatorname{Mod} X)(i) \xrightarrow{\mathcal{D}(\operatorname{Mod} X)(a)} \mathcal{D}(\operatorname{Mod} X)(j)$ is equal to

$$\mathcal{D}(\operatorname{Mod} X(i)) \xrightarrow{\text{-}\overset{\mathbf{L}}{\otimes}_{X(i)} \overline{X(a)}} \mathcal{D}(\operatorname{Mod} X(j)).$$

(3) By Theorem 6.5 and Example 6.4 we can define a pseudofunctor

$$\overleftarrow{\operatorname{Oplax}}(I, \mathcal{K}^{\mathrm{b}} \circ \operatorname{prj})\colon \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat}) \to \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$$

sending each $X \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ to $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$. By the remark in Example 6.4(2) $\mathcal{K}^{\mathrm{b}}(\operatorname{prj} X)$ is an oplax subfunctor of $\mathcal{D}(\operatorname{Mod} X)$.

Proposition 6.7. *The pseudofunctor $\mathcal{K}^{\mathrm{b}} \circ \operatorname{prj}$ preserves I-precoverings, that is, if $(F, \psi)\colon X \to \Delta(\mathcal{C})$ is an I-precovering in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ with $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}$, then so is $\mathcal{K}^{\mathrm{b}}(\operatorname{prj}(F, \psi))\colon \mathcal{K}^{\mathrm{b}}(\operatorname{prj} X) \to \Delta(\mathcal{K}^{\mathrm{b}}(\operatorname{prj} \mathcal{C}))$ in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$.*

Proof. It is straightforward to verify that the 2-functors $\oplus$, sic and $\mathcal{K}^{\mathrm{b}}$ defined in Example 6.4 preserve I-precoverings. Then the assertion follows from the natural 2-isomorphism $Y\colon \operatorname{sic} \circ \oplus \Rightarrow \operatorname{prj}$. $\qquad\square$

7. Derived equivalences of oplax functors

In this section we recall necessary terminologies and the main theorem in our previous paper [6]. First we cite the following. See [6] for the proof.

Lemma 7.1. *Let $\mathbf{C}$ be a 2-category and $(F, \psi)\colon X \to X'$ a 1-morphism in the 2-category $\overleftarrow{\operatorname{Oplax}}(I, \mathbf{C})$. Then (F, ψ) is an equivalence in $\overleftarrow{\operatorname{Oplax}}(I, \mathbf{C})$ if and only if*
   (1) *For each $i \in I_0$, $F(i)$ is an equivalence in $\mathbf{C}$; and*
   (2) *For each $a \in I_1$, $\psi(a)$ is a 2-isomorphism in $\mathbf{C}$ (namely, (F, ψ) is I-equivariant).*

Definition 7.2. Let $X, X' \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then X and X' are said to be *derived equivalent* if $\mathcal{D}(\operatorname{Mod} X)$ and $\mathcal{D}(\operatorname{Mod} X')$ are equivalent in the 2-category $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$.

By Lemma 7.1 we obtain the following.

Proposition 7.3. *Let $X, X' \in \overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Then X and X' are derived equivalent if and only if there exists a 1-morphism $(F, \psi)\colon \mathcal{D}(\operatorname{Mod} X) \to \mathcal{D}(\operatorname{Mod} X')$ in $\overleftarrow{\operatorname{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$ such that*
   (1) *For each $i \in I_0$, $F(i)$ is a triangle equivalence; and*

(2) *For each $a \in I_1$, $\psi(a)$ is a natural isomorphism (i.e., (F, ψ) is I-equivariant).*

A $\Bbbk$-category $\mathcal{A}$ is called $\Bbbk$-*projective* if $\mathcal{A}(x, y)$ are projective $\Bbbk$-modules for all $x, y \in \mathcal{A}_0$.

Definition 7.4. Let $X \colon I \to \Bbbk\text{-}\mathbf{Cat}$ be an oplax functor.
  (1) X is called $\Bbbk$-*projective* if $X(i)$ are $\Bbbk$-projective for all $i \in I_0$.
  (2) An oplax subfunctor $\mathcal{T}$ of of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ is called *tilting* if for each $i \in I_0$, $\mathcal{T}(i)$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$, namely,
    - $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))(U, V[n]) = 0$ for all $U, V \in \mathcal{T}(i)_0$ and $0 \neq n \in \mathbb{Z}$; and
    - the smallest thick subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$ containing $\mathcal{T}(i)$ is equal to $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$.
  (3) A tilting oplax subfunctor $\mathcal{T}$ of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ with an I-equivariant inclusion $(\sigma, \rho) \colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ is called a *tilting oplax functor* for X.

The following was our main theorem in [6] that gives a generalization of the Morita type theorem characterizing derived equivalences of categories by Rickard [12] and Keller [11] in our setting.

Theorem 7.5. *Let $X, X' \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Consider the following conditions.*
  (1) *X and X' are derived equivalent.*
  (2) *$\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ and $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X')$ are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Tri})$.*
  (3) *There exists a tilting oplax functor $\mathcal{T}$ for X such that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$.*

Then
  (a) *(1) implies (2).*
  (b) *(2) implies (3).*
  (c) *If X' is $\Bbbk$-projective, then (3) implies (1).*

8. Derived equivalences of Grothendieck constructions

The following is our main result in this paper.

Theorem 8.1. *Let $X, X' \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Assume that there exists a tilting oplax functor $\mathcal{T}$ for X such that $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$ (the condition (3) in Theorem 7.5). Then $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent.*

Proof. Let $\mathcal{T}$ be a tilting oplax subfunctor of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$ with an I-equivariant inclusion $(\sigma, \rho) \colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$. Put $(P, \phi) := (P_X, \phi_X)$ for short. Let $\mathcal{T}'$ be the full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, \mathrm{Gr}(X))$ consisting of the objects $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(U)$ with $i \in I_0$ and $U \in \mathcal{T}(i)_0$. Then $\mathcal{T}'$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, \mathrm{Gr}(X))$. Indeed, let $L, M \in \mathcal{T}'_0$ and $0 \neq p \in \mathbb{Z}$. Then $L = \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(i))(U)$ and $M = \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, P(j))(V)$ for some $i, j \in I_0$ and some $U \in \mathcal{T}(i)_0$, $V \in \mathcal{T}'(j)_0$. Since

$$\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P, \phi)) \colon \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X) \to \Delta(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, \mathrm{Gr}(X)))$$

is an I-precovering by Proposition 6.7, we have

$$\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))(L, M[p]) \cong \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P,\phi))(U), \mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P,\phi))(V[p]))$$

$$\cong \bigoplus_{a\in I(i,j)} \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X(j))(\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X)(a)(U), V[p])$$

$$\overset{(a)}{\cong} \bigoplus_{a\in I(i,j)} \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X(j))(\mathcal{T}(a)U, V[p]) \overset{(b)}{=} 0,$$

where the isomorphism (a) follows using the natural isomorphism $\rho(a)$:

$$
\begin{array}{ccc}
U \in \mathcal{T}(i) \lhook\joinrel\longrightarrow & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X(i)) \\
{\scriptstyle \mathcal{T}(a)}\big\downarrow \quad {\scriptstyle\rho(a)} \nearrow \; {\scriptstyle \cong} & \big\downarrow {\scriptstyle \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X)(a)} \\
\mathcal{T}(i) \lhook\joinrel\longrightarrow & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,X(j))
\end{array}
$$

and the equality (b) follows by assumption from the fact that $\mathcal{T}(a)U, V \in \mathcal{T}(j)$. Now for a triangulated category $\mathcal{U}$ and a class of objects $\mathcal{V}$ in $\mathcal{U}$ we denote by thick $\mathcal{V}$ the smallest thick subcategory of $\mathcal{U}$ containing $\mathcal{V}$. Then for each $i \in I_0$ and $x \in X(i)$ we have $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,P(i))(X(i)(\text{-}, x)) \cong (\mathrm{prj}\,P(i))(X(i)(\text{-}, x)) \cong \mathrm{Gr}(X)(\text{-}, P(i)(x)) = \mathrm{Gr}(X)(\text{-}, {}_i x)$ by the formula (6.1), and hence

$$\mathrm{Gr}(X)(\text{-}, {}_i x) \cong \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,P(i))(X(i)(\text{-}, x))$$

$$\in \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,P(i))(\mathrm{thick}\,\mathcal{T}(i))$$

$$\subseteq \mathrm{thick}\{\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,P(i))(U) \mid U \in \mathcal{T}(i)\}$$

$$\subseteq \mathrm{thick}\,\mathcal{T}'.$$

Therefore thick $\mathcal{T}' = \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))$, and hence $\mathcal{T}'$ is a tilting subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\,\mathrm{Gr}(X))$, as desired. Hence $\mathrm{Gr}(X)$ and $\mathcal{T}'$ are derived equivalent. Let (F, ψ) be the restriction of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}(P, \phi))$ to $\mathcal{T}$. Then $(F, \phi)\colon \mathcal{T} \to \mathcal{T}'$ is a dense functor and an I-precovering, thus it is an I-covering, which shows that $\mathcal{T}' \simeq \mathrm{Gr}(\mathcal{T})$ by Corollary 5.3. Since $\mathcal{T}$ and X' are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$, we have $\mathrm{Gr}(\mathcal{T}) \simeq \mathrm{Gr}(X')$. As a consequence, $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are derived equivalent. $\square$

Corollary 8.2. *Let $X, X' \in \overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. If X and X' are derived equivalent, then so are $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.* $\square$

The following is easy to verify.

Lemma 8.3. *Let C, C' be in $\Bbbk\text{-}\mathbf{Cat}$. If C and C' are derived equivalent, then so are $\Delta(C)$ and $\Delta(C')$.* $\square$

The lemma above and Corollary 8.2 give us a unified proof of the following fact.

Theorem 8.4. *Assume that $\Bbbk$-algebras A and A' are derived equivalent. Then the following pairs are derived equivalent as well:*

(1) *path-algebras AQ and $A'Q$ for any quiver Q;*
(2) *incidence algebras AS and $A'S$ for any poset S; and*

(3) *semigroup algebras AG and $A'G$ for any semigroup G.*

$\square$

Example 8.5. Let n be a natural number ≥ 3, and I the free category defined by the quiver $Q\colon 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} n$. Define functors $X, X'\colon I \to \Bbbk\text{-}\mathbf{Cat}$ as follows.

For each $i \in I_0 = \{2, \ldots, n\}$ let $X(i)$ be the $\Bbbk$-category defined by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \cdots \underset{\beta_{i-1}}{\overset{\alpha_{i-1}}{\rightleftarrows}} i$$

with relations $\alpha_{j+1}\alpha_j = 0$, $\beta_j\beta_{j+1} = 0$, $\alpha_j\beta_j = \beta_{j+1}\alpha_{j+1}$ for all $j = 1, \ldots, i-1$ and $\alpha_1\beta_1\alpha_1 = 0$, $\beta_{i-1}\alpha_{i-1}\beta_{i-1} = 0$. For each $a_i\colon i \to i+1$ in I_1 let $X(a_i)\colon X(i) \to X(i+1)$ be the inclusion functor. This defines a functor $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$.

For each $i \in I_0 = \{2, \ldots, n\}$ let $X'(i)$ be the $\Bbbk$-category defined by the quiver

$$1 \xleftarrow{\gamma_1} 2 \xrightarrow{\gamma_2} 3 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_{i-1}} i$$
$$\underset{\gamma_i}{\curvearrowleft}$$

with relations $\gamma_{j+i} \cdots \gamma_{j+1}\gamma_j = 0$ for all $j \in \mathbb{Z}/i\mathbb{Z}$. For each $a_i\colon i \to i+1$ in I_1 let $X(a_i)\colon X(i) \to X(i+1)$ be the functor defined by the correspondence $1 \mapsto 1$, $j \mapsto j+1$ and $\alpha_1 \mapsto \alpha_2\alpha_1$, $\alpha_j \mapsto \alpha_{j+1}$ for all $j = 2, \ldots i$. This defines a functor $X'\colon I \to \Bbbk\text{-}\mathbf{Cat}$.

As is explained in [1] we have a tilting spectroid $\mathcal{T}(i)$ for $X(i)$ that is a full subcategory of $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i))$ consisting of the following i objects

$$T(i)_1 := (\underline{P_1}),$$
$$T(i)_2 := (\underline{P_2} \xrightarrow{P(\alpha_2)} P_3 \xrightarrow{P(\alpha_3)} \cdots \xrightarrow{P(\alpha_{i-1})} P_i),$$
$$T(i)_3 := (\underline{P_2} \xrightarrow{P(\alpha_2)} P_3 \xrightarrow{P(\alpha_3)} \cdots \xrightarrow{P(\alpha_{i-2})} P_{i-1}),$$
$$\vdots$$
$$T(i)_i := (\underline{P_2}),$$

where $P_j := X(i)(\text{-},j) \in \mathrm{prj}\, X(i)$ for each $i \in X(i)_0$, $P(\alpha) := X(i)(\text{-},\alpha)$ for each $\alpha \in X(i)_1$ and the underline indicates the place of degree zero. Again by [1], $\mathcal{T}(i)$ is presented by the same quiver with relations as $X'(i)$ and we have an isomorphism $F(i)\colon X'(i) \to \mathcal{T}(i)$ sending j to $T(i)_j$ for all $j = 1, \ldots, i$ and γ_j to a morphism $\delta(i)_j\colon T(i)_j \to T(i)_{j+1}$ for all $j \in \mathbb{Z}/i\mathbb{Z}$, where $\delta(i)_1 := (\underline{P(\alpha_1)})$, $\delta(i)_j := (\mathbb{1}_{P_2}, \ldots, \mathbb{1}_{P_{i-j+1}}, 0)$ for all $j = 2, \ldots, i-1$ and $\delta(i)_i := (\underline{P(\beta_1)})$. Thus $\mathcal{T}(i)$ gives a derived equivalence between $X(i)$ and $X'(i)$.

For each $a_i\colon i \to i+1$ in I_1 define a functor $\mathcal{T}(a_i)\colon \mathcal{T}(i) \to \mathcal{T}(i+1)$ by the correspondence $T(i)_1 \mapsto T(i+1)_1$, $T(i)_j \mapsto T(i+1)_{j+1}$ and $\delta(i)_1 \mapsto \delta(i+1)_2\delta(i+1)_1$, $\delta(i)_j \mapsto \delta(i+1)_{j+1}$ for all $j = 2, \ldots, i$. This defines a functor $\mathcal{T}\colon I \to \Bbbk\text{-}\mathbf{Cat}$. Then we

have a strict commutative diagram

$$
\begin{array}{ccc}
X'(i) & \xrightarrow{\;F(i)\;} & \mathcal{T}(i) \\
{\scriptstyle X'(a_i)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{T}(a_i)} \\
X'(i+1) & \xrightarrow[\;F(i+1)\;]{} & \mathcal{T}(i+1)
\end{array}
$$

in $\Bbbk$-**Cat** for all $i \in I_0$, which shows that X' and $\mathcal{T}$ are equivalent in $\overleftarrow{\mathrm{Oplax}}(I, \Bbbk\text{-}\mathbf{Cat})$. Finally by definition of $\mathcal{T}(a_i)$'s it is easy to see that we have an I-equivariant inclusion $(\sigma, \rho) \colon \mathcal{T} \hookrightarrow \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X)$:

$$
\begin{array}{ccc}
\mathcal{T}(i) & \xhookrightarrow{\;\sigma(i)\;} & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i)) \\
{\scriptstyle \mathcal{T}(a_i)}\Big\downarrow & {\scriptstyle \rho(a_i)}\;\;{\scriptstyle \cong} & \Big\downarrow{\scriptstyle \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(a_i))} \\
\mathcal{T}(i+1) & \xhookrightarrow[\;\sigma(i+1)\;]{} & \mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, X(i+1)).
\end{array}
$$

Hence by Theorem 8.1 we can glue derived equivalences between $X(i)$'s and $X'(i)$'s together to have an derived equivalence between $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. For example when $n = 5$, these are presented by the following quivers

with suitable relations as calculated in [7]. Note that if we start with I presented by the same quiver Q as above with relations $a_{i+1}a_i = 0$ for all $i = 2, \ldots, n-2$, then both $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$ are presented by the same quivers with relations consisting of the same relations as before together with the additional relations that the vertical paths of length 2 are zero, respectively.

9. THE COMPOSITE OF OPLAX FUNCTORS AND PSEUDOFUNCTORS

In this section we prove Theorem 6.5. Throughout this section $\mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are 2-categories.

Notation 9.1. When we denote an oplax functor by a letter X the 1-st (resp. 2-nd and 3-rd) entry of X is denoted by $X_{012} := (X_0, X_1, X_2)$ (resp. η^X and θ^X), thus we set $X = (X_{012}, \eta^X, \theta^X)$, and sometimes we simply write X for X_d for all $d = 0, 1, 2$ if this seems to make no confusion.

9.1. Correspondences on cells.

Lemma 9.2. *Let* $X\colon \mathbf{B} \to \mathbf{C}$ *and* $V\colon \mathbf{C} \to \mathbf{D}$ *be oplax functors. We define the composite* $VX\colon \mathbf{B} \to \mathbf{D}$ *as follows.*

- $(VX)_d := V_d X_d\colon \mathbf{B}_d \xrightarrow{X_d} \mathbf{C}_d \xrightarrow{V_d} \mathbf{D}_d$ *for all* $d = 0, 1, 2$.

- $\eta_i^{VX} := \eta_{X(i)}^V \circ V\eta_i^X\colon \ VX(\mathbb{1}_i) \overset{V\eta_i^X}{\Longrightarrow} V(\mathbb{1}_{X(i)}) \overset{\eta_{X(i)}^V}{\Longrightarrow} \mathbb{1}_{(VX)(i)}$ *for all* $i \in \mathbf{B}_0$.

- $\theta_{b,a}^{VX} := \theta_{X(b),X(a)}^V \circ V\theta_{b,a}^X\colon \ VX(ba) \overset{V\theta_{b,a}^X}{\Longrightarrow} V(X(b) \circ X(a)) \overset{\theta_{X(b),X(a)}^V}{\Longrightarrow} VX(b) \circ VX(a)$ *for all* $(b,a) \in \mathrm{com}(\mathbf{B})$.

Namely, $VX := ((V_0 X_0, V_1 X_1, V_2, X_2), (\eta_{X(i)}^V \circ V\eta_i^X)_{i \in \mathbf{B}_0}, (\theta_{X(b),X(a)}^V \circ V\theta_{b,a}^X)_{(b,a) \in \mathrm{com}(\mathbf{B})})$.
Then the composite $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)(X) := VX\colon \mathbf{B} \to \mathbf{D}$ *is again an oplax functor.*

Proof. It is enough to verify the axioms (i) – (iv) in Definition 6.1.

(i) Since both $(X_1, X_2)\colon \mathbf{B}_{12} \to \mathbf{C}_{12}$ and $(V_1, V_2)\colon \mathbf{C}_{12} \to \mathbf{D}_{12}$ are functors, so is $((VX)_1, (VX)_2) = (V_1, V_2)(X_1, X_2)$.

(ii) For each $a\colon i \to j$ in $\mathbf{B}$ we have the following commutative diagram:

$$
\begin{array}{c}
VX(a)\mathbb{1}_{VX(i)} \xLeftarrow{\ VX(a)\eta_{X(i)}^V\ } VX(a)V(\mathbb{1}_{X(i)}) \xLeftarrow{\ VX(a)V(\eta_i^X)\ } VX(a)VX(\mathbb{1}_i) \\[2em]
\Big\Uparrow \theta_{X(a),\mathbf{1}_{X(i)}}^V \qquad\qquad \Big\Uparrow \theta_{X(a),X(\mathbb{1}_i)}^V \\[1.5em]
V(X(a)\mathbb{1}_{X(i)}) \xLeftarrow{\ V(X(a)\eta_i^X)\ } V(X(a)X(\mathbb{1}_i)) \\[2em]
\Big\Uparrow V(\theta_{a,\mathbf{1}_i}^X) \\[1.5em]
VX(a\mathbb{1}_i).
\end{array}
$$

The commutativity of the square follows from the axiom (iv) for θ^V. The remaining commutative diagram is obtained similarly. These two commutative diagrams verify the axiom (ii) of oplax functors.

(iii) For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{B}$ we have the following commutative diagram:

$$
\begin{array}{ccccc}
VX(cba) & \xRightarrow{\ V\theta_{c,ba}^X\ } & V(X(c)X(ba)) & \xRightarrow{\ \theta_{X(c),X(ba)}^V\ } & VX(c) \cdot VX(ba) \\[1em]
\Big\Downarrow V\theta_{cb,a}^X & & \Big\Downarrow V(\mathbf{1}_{X(c)}\theta_{b,a}^X) & & \Big\Downarrow VX(c)\cdot V\theta_{b,a}^X \\[1em]
V(X(cb)X(a)) & \xRightarrow[\ V(\theta_{c,b}^X \mathbf{1}_{X(a)})\ ]{} & V(X(c)X(b)X(a)) & \xRightarrow[\ \theta_{X(c),X(b)X(a)}^V\ ]{} & VX(c)V(X(b)X(a)) \\[1em]
\Big\Downarrow \theta_{X(cb),X(a)}^V & & \Big\Downarrow \theta_{X(c)X(b),X(a)}^V & & \Big\Downarrow VX(c)\theta_{X(b),X(a)}^V \\[1em]
VX(cb) \cdot VX(a) & \xRightarrow[\ V(\theta_{c,b}^X)VX(a)\ ]{} & V(X(c)X(b))VX(a) & \xRightarrow[\ \theta_{X(c),X(b)}^V VX(a)\ ]{} & VX(c) \cdot VX(b) \cdot VX(a),
\end{array}
$$

which verifies the axiom (iii) of oplax functors.

(iv) Let $a, a' \colon i \to j$; $b, b' \colon j \to k$; $\alpha \colon a \Rightarrow a'$ and $\beta \colon b \Rightarrow b'$ be in $\mathbf{B}$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
VX(ba) \xRightarrow{\;V(\theta^X_{b,a})\;} V(X(b) \cdot X(a)) \xRightarrow{\;\theta^V_{X(b),X(a)}\;} VX(b) \cdot VX(a) \\[2mm]
\Big\Vert{\scriptstyle VX(\beta\cdot\alpha)} \qquad\qquad \Big\Vert{\scriptstyle V(X\beta \cdot X\alpha)} \qquad\qquad \Big\Vert{\scriptstyle VX\beta \cdot VX\alpha} \\[2mm]
VX(b'a') \xRightarrow[V(\theta^X_{b',a'})]{} V(X(b') \cdot V(a')) \xRightarrow[\theta^V_{X(b'),X(a')}]{} VX(b') \cdot VX(a'),
\end{array}
$$

which verifies the axiom (iv) of oplax functors. $\qquad\qquad\qquad\qquad\square$

Lemma 9.3. *Let $X, X' \colon \mathbf{B} \to \mathbf{C}$ and $V \colon \mathbf{C} \to \mathbf{D}$ be oplax functors and $(F, \psi) \colon X \to X'$ a 1-morphism in $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C})$, and consider the diagram*

$$\tag{9.2}$$

Assume that $\theta^V_{d,c}$ are isomorphisms for all $(d, c) \in \mathrm{com}(\mathbf{C})$ (e.g., that V is a pseudofunctor). Then we can define a 1-morphism $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)(F, \psi) := V(F, \psi) \colon VX \to VX'$ in $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{D})$ by

$$
V(F, \psi) := \big((V(F(i)))_{i \in \mathbf{B}_0}, (\theta^V_{F(j),X(a)} \cdot V(\psi(a)) \cdot {\theta^V_{X'(a),F(i)}}^{-1})_{a \colon i \to j}\big).
$$

Proof. We set $X = (X, \eta, \theta)$ and $X' = (X', \eta', \theta')$ for short.

(a) For each $i \in \mathbf{B}_0$ we have the following commutative diagram:

which verifies the axiom (a) of 1-morphisms.

(b) For each $i \xrightarrow{a} j \xrightarrow{b} k$ in $\mathbf{B}$ we have the following commutative diagrams:

$$
\begin{array}{c}
VX'(ba) \cdot VF(i) \xRightarrow{V(\theta'_{b,a})VF(i)} V(X'(b)X'(a)) \cdot VF(i) \xRightarrow{\theta^V_{X'(b),X'(a)}VF(i)} VX'(b) \cdot VX'(a) \cdot VF(i)
\end{array}
$$

with vertical maps $\theta^V_{X'(ba),F(i)}$, $\theta^V_{X'(b)X'(a),F(i)}$, $VX'(b)\theta^{V}{}^{-1}_{X'(a),F(i)}$, and others including
$VX'(b)V(X'(a)F(i)) \xRightarrow{V(\mathbf{1}_{X'(b)})V(\psi(a))} VX'(b) \cdot V(F(j)X(a))$,
$\theta^V_{X(b),X'(a)F(i)}$, $\theta^V_{X'(b),F(j)X(a)}$, $VX'(b)\cdot\theta^V_{F(j),X(a)}$,
$VX'(b) \cdot VF(j) \cdot VX(a)$, $\theta^{V}{}^{-1}_{X'(b),F(j)}VX(a)$,
$V(X'(ba)F(i)) \xRightarrow{V(\theta'_{b,a}F(i))} V(X'(b)X'(a)F(i)) \xRightarrow{V(X'(b)\psi(a))} V(X'(b)F(j)X(a)) \xRightarrow{\theta^V_{X'(b)F'(j),X(a)}} V(X'(b)F(j))VX(a)$

and

$$
\begin{array}{ccccc}
V(X'(ba)F(i)) & \xRightarrow{V(\theta'_{b,a}F(i))} & V(X'(b)X'(a)F(i)) & \xRightarrow{V(X'(b)\psi(a))} & V(X'(b)F(j)X(a)) & \xRightarrow{\theta^V_{X'(b)F'(j),X(a)}} & V(X'(b)F(j))VX(a) \\
\Big\Downarrow V\psi(ba) & & & & \Big\Downarrow V(\psi(b)X(a)) & & \Big\Downarrow V(\psi(b))VX(a) \\
V(F(k)X(ba)) & \xRightarrow{V(F(k)\theta_{b,a})} & & & V(F(k)X(b)X(a)) & \xRightarrow{\theta^V_{X(b)F'(j),X(a)}} & V(F(k)X(b))VX(a) \\
\Big\Downarrow \theta^V_{F(k),X(ba)} & & & & \Big\Downarrow \theta^V_{F(k),X(b)X(a)} & & \Big\Downarrow \theta^V_{F(k),X(b)}VX(a) \\
VF(k) \cdot VX(ba) & \xRightarrow{VF(k)\cdot V\theta_{b,a}} & & & VF(k) \cdot V(X(b)X(a)) & \xRightarrow{VF(k)\cdot\theta^V_{X(b),X(a)}} & VF(k) \cdot VX(b) \cdot VX(a).
\end{array}
$$

Glue these two diagrams together along the common row to get a large diagram, which verifies the axiom (b) of 1-morphisms. $\qquad\square$

Lemma 9.4. *Let* $X, X'\colon \mathbf{B} \to \mathbf{C}$ *and* $V\colon \mathbf{C} \to \mathbf{D}$ *be oplax functors,* $(F, \psi), (F', \psi')\colon X \to X'$ *1-morphisms, and* $\alpha\colon (F,\psi) \Rightarrow (F', \psi')$ *a 2-morphism in* $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C})$. *Assume that all* $\theta^V_{d,c}$ *are isomorphisms (e.g., that* V *is a pseudofunctor). Then we can define a 2-morphism* $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)(\alpha) := V\alpha\colon V(F, \psi) \Rightarrow V(F', \psi')$ *in* $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{D})$ *by*

$$
V\alpha := (V\alpha_i)_{i \in \mathbf{B}_0}.
$$

Proof. Let $a\colon i \to j$ be in $\mathbf{B}$. It is enough to show the commutativity of the following diagram:

$$
\begin{array}{ccccccc}
VX'(a) \cdot VF(i) & \xLeftarrow{\theta^V_{X'(a),F(i)}} & V(X'(a)F(i)) & \xRightarrow{V(\psi(a))} & V(F(j)X(a)) & \xRightarrow{\theta^V_{F(j),X(a)}} & VF(j) \cdot VX(a) \\
\Big\Downarrow VX'(a)\cdot V\alpha_i & & \Big\Downarrow V(X'(a)\alpha_i) & & \Big\Downarrow V(\alpha_j X(a)) & & \Big\Downarrow V\alpha_i \cdot VX(a) \\
VX'(a) \cdot VF'(i) & \xLeftarrow{\theta^V_{X'(a),F'(i)}} & V(X'(a)F'(i)) & \xRightarrow{V(\psi'(a))} & V(F'(j)X(a)) & \xRightarrow{\theta^V_{F'(j),X(a)}} & VF'(j) \cdot VX(a).
\end{array}
$$

Since $\alpha = (\alpha_i \colon F(i) \Rightarrow F'(i))_{i \in \mathbf{B}_0}$ is a 2-morphism in $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C})$, we have the commutative diagram

$$
\begin{array}{ccc}
X'(a)F(i) & \xLongrightarrow{\psi(a)} & F(j)X(a) \\
{\scriptstyle X'(a)\alpha_i}\big\Vert\Big\downarrow & & \Big\Vert\Big\downarrow{\scriptstyle \alpha_j X(a)} \\
X'(a)F'(i) & \xLongrightarrow[\psi'(a)]{} & F'(j)X(a).
\end{array}
$$

This gives the commutativity of the central square of the diagram above by applying the functor (V_1, V_2) to it. The axiom (iv) of oplax functors for V shows the commutativity of the remaining squares. $\qquad\square$

9.2. **Proof of Theorem 6.5.** By the three lemmas above we can define a correspondence

$$
\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)_{012} \colon \overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{D})
$$

sending i-cells to i-cells for all $i = 0, 1, 2$ preserving domains and codomains. It remains to define families $H = (H_X)_{X \in \overleftarrow{\mathrm{Oplax}}(\mathbf{B},\mathbf{C})_0}$ and $\Theta = (\Theta_{F',F})_{(F',F) \in \mathrm{com}(\overleftarrow{\mathrm{Oplax}}(\mathbf{B},\mathbf{C}))}$ and to show that $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V) := (\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)_{012}, H, \Theta)$ becomes a pseudofunctor $\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C}) \to \overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{D})$.

For each $X \in \overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C})_0$ we define $H_X \colon V(\mathbb{1}_X) \Rightarrow \mathbb{1}_{VX}$ by setting

$$
H_X := (\eta^V_{X(i)} \colon V(\mathbb{1}_{X(i)}) \to \mathbb{1}_{VX(i)})_{i \in \mathbf{B}_0}.
$$

Then H_X turns out to be a 2-morphism because by definitions of θ^V and η^V we have a commutative diagram

$$
\begin{array}{ccccccc}
VX(a) \cdot V(\mathbb{1}_{X(i)}) & \xLongrightarrow{(\theta^V_{X(a),1_{X(i)}})^{-1}} & V(X(a) \cdot \mathbb{1}_{X(i)}) & =\!=\!= & V(\mathbb{1}_{X(j)}X(a)) & \xLongrightarrow{\theta^V_{1_{X(j)},X(a)}} & V(\mathbb{1}_{X(j)}) \cdot VX(a) \\
{\scriptstyle VX(a)\cdot\eta^V_{X(i)}}\big\Vert\big\downarrow & & & & & & \big\Vert\big\downarrow{\scriptstyle \eta^V_{X(j)}\cdot VX(a)} \\
VX(a) \cdot \mathbb{1}_{VX(i)} & =\!= & & & & & \mathbb{1}_{VX(j)}VX(a)
\end{array}
$$

for all $a \colon i \to j$ in $\mathbf{B}$. Note that H_X are isomorphisms because η^V_k are for all $k \in \mathbf{C}_0$.

For each $(F', F) \in \mathrm{com}(\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, \mathbf{C}))$, say $F \colon X \Rightarrow X'$ and $F' \colon X' \Rightarrow X''$, we define $\Theta_{F',F} \colon V(F'F) \Rightarrow VF' \circ VF$ by setting

$$
\Theta_{F',F} := (\theta^V_{F'(i),F(i)} \colon V(F'(i)F(i)) \to VF'(i) \cdot VF(i))_{i \in \mathbf{B}_0}.
$$

Then $\Theta_{F',F}$ turns out to be a 2-morphism. Indeed, it is enough to show the commutativity of the diagram

$$
\begin{array}{ccc}
VX''(a) \cdot V(F'(i)F(i)) & \xLongrightarrow{\Psi(a)} & V(F'(j)F(j)) \cdot VX(a) \\
{\scriptstyle VX''(a)\cdot\theta^V_{F'(i),F(i)}}\big\Vert\big\downarrow & & \big\Vert\big\downarrow{\scriptstyle \theta^V_{F'(j),F(j)}\cdot VX(a)} \\
VX''(a) \cdot VF'(i) \cdot VF(i) & \xLongrightarrow[\Psi'(a)]{} & VF'(j)VF(j)VX(a)
\end{array}
$$

for each $a\colon i \to j$ in $\mathbf{B}$, where we set $V(F'F) = ((V(F'(i)F(i))_{i\in\mathbf{B}_0}, (\Psi(a))_{a\in\mathbf{B}_1})$ and $VF' \cdot VF = ((VF(i) \cdot VF(i))_{i\in\mathbf{B}_0}, (\Psi'(a))_{a\in\mathbf{B}_1})$, namely

$$\Psi(a) := \theta^V_{F'(j)F(j),X(a)} \cdot V((F'(j) \cdot \psi(a)) \cdot V(\psi'(a) \cdot F(i)) \cdot (\theta'^V_{X''(a),F'(i)F(i)})^{-1}$$

$$\Psi'(a) := (VF'(j) \cdot (\theta^V_{F(j),X(a)} \cdot V\psi(a) \cdot (\theta'^V_{X'(a),F(i)})^{-1}) \circ (\theta^V_{F'(j),X'(a)} \cdot V\psi'(a) \cdot (\theta'^V_{X''(a),F'(i)})^{-1} \cdot VF(i))$$

for each $a\colon i \to j$ in $\mathbf{B}$. This follows from the coassociativity of V and the naturality of θ^V. Note that $\Theta_{F',F}$ are isomorphisms because $\theta^V_{b,a}$ are for all $a, b \in \mathbf{C}_0$.

Now the defining conditions of θ^V and η^V directly show that $(\overleftarrow{\mathrm{Oplax}}(\mathbf{B}, V)_{012}, H, \Theta)$ is an oplax functor, hence a pseudofunctor because all H_X and $\Theta_{F',F}$ are isomorphisms. $\square$

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