Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise.

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Abstract

Unique existence of solutions to porous media equations driven by continuous linear multiplicative space-time rough signals is proven for initial data in $L^1(\mathcal{O})$ on bounded domains $\mathcal{O}$. The generation of a continuous, order-preserving random dynamical system (RDS) on $L^1(\mathcal{O})$ and the existence of a random attractor for stochastic porous media equations perturbed by linear multiplicative noise in *space and time* is obtained. The random attractor is shown to be compact and attracting in $L^\infty(\mathcal{O})$ norm. Uniform L^∞ bounds and uniform space-time continuity of the solutions is shown. General noise including fractional Brownian motion for all Hurst parameters is treated. A pathwise Wong-Zakai result for driving noise given by a continuous semimartingale is obtained. For fast diffusion equations driven by continuous linear multiplicative space-time rough signals existence of solutions is proven for initial data in $L^{m+1}(\mathcal{O})$.

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0 Introduction

The qualitative study of stochastic dynamics induced by stochastic partial differential equations (SPDE) especially in the case of non-Markovian noise is based on the theory of RDS (cf. e.g. [1]). Since the foundational work [32, 11, 10] the long-time behaviour of several quasilinear SPDE has been investigated by means of the existence of random attractors. However, all these results are restricted to simple models of the noise (e.g. additive or real multiplicative), not including the important case of linear multiplicative space-time noise. This is mainly due to the difficulty to even define an associated RDS for more general SPDE. The generation of an RDS is usually shown by use of a transformation of the SPDE into a random PDE. Depending on the structure of the noise monotonicity and coercivity properties of the drift are preserved under this transformation. For example, this is the case for additive, real multiplicative and first order linear multiplicative noise [15] with suitable assumptions on the diffusion coefficients. For linear multiplicative noise, however, this is not the case, thus making the analysis of the random PDE much harder. The generation of an RDS and the existence of random attractors for stochastic porous media equations (SPME) with additive noise has been obtained in [8, 18]. A first approach to tackle the generation of an RDS for SPME with linear multiplicative space-time noise, i.e. for equations of the form

$$dX_t = \Delta\big(|X_t|^m sgn(X_t)\big)dt + \sum_{k=1}^{N} \mu_k e_k X_t d\beta_t^{(k)}, \quad 1 < m < \infty \tag{0.0}$$

has been given in [7] by proving the unique existence of pathwise solutions to a corresponding random PDE for essentially bounded initial conditions $x \in L^\infty(\mathcal{O})$. The existence and uniqueness up to indistinguishability of probabilistically strong solutions to (0.0) even including $0 < m < 1$ and all initial conditions $x \in (H_0^1(\mathcal{O}))^*$ has been obtained in [30]. However, this does not yield the existence of an RDS. The pathwise solutions to the transformed equation constructed in [7] yield a stochastic flow $\varphi(t,\omega)x$ satisfying the perfect cocycle property on $L^\infty(\mathcal{O})$. However, neither continuity of $x \mapsto \varphi(t,\omega)x$ nor continuity of $t \mapsto \varphi(t,\omega)x$ has been obtained. These properties of RDS are crucial to obtain the existence of random attractors. Due to the strong norm on the state space $L^\infty(\mathcal{O})$ especially the continuity in the initial condition is not clear. In this paper we prove the generation of an RDS corresponding to SPME driven by multiplicative space-time rough signals for all initial conditions $X_0 \in L^1(\mathcal{O})$, i.e. to equations of the form

$$\begin{aligned} dX_t &= \Delta\left(|X_t|^m sgn(X_t)\right)dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \\ X(0) &= X_0, \text{ on } \mathcal{O} \end{aligned} \tag{0.1}$$

with Dirichlet boundary conditions, rough driving signals $z^{(k)} \in C([0,T];\mathbb{R})$ and with $f_k \in C^\infty(\bar{\mathcal{O}})$. We assume the number of signals N to be finite and high regularity for f_k for simplicity only. In fact, most of the proofs only require $\sum_{k=1}^{\infty} f_k(\xi) z_t^k \in C([0,T]; C^2(\bar{\mathcal{O}}))$. The stochastic Stratonovich integral occurring in (0.1) is informal, the rigorous justification of this notation is part of our results. The resulting stochastic flow is proven to be an RDS φ on $L^1(\mathcal{O})$ which is continuous in the initial condition and in time. Generalizing the notion of quasi-continuity of RDS we show that φ is quasi-weakly-continuous on $L^p(\mathcal{O})$ for all $p \in [1,\infty)$ and quasi-weakly*-continuous on $L^\infty(\mathcal{O})$. Moreover, we prove the existence of an absorbing random set $F \subseteq X$ which even is bounded in $L^\infty(\mathcal{O})$, as well as asymptotic compactness of φ on each $L^p(\mathcal{O})$, $p \in [1,\infty]$ (requiring a uniform convexity condition for $\mathcal{O}$ if $p = \infty$). Generalizing an existence result for random attractors of quasi-continuous RDS we deduce the existence of a random attractor A for φ (as an RDS on $L^1(\mathcal{O})$), which is compact and attracting in each $L^p(\mathcal{O})$ with $p \in [1,\infty]$.

We obtain new spatial and temporal regularity properties for solutions to (0.1) analogous to those proved for deterministic porous media type equations by De Giorgi, Nash, Moser type iteration techniques in [14]. More precisely, we prove that the solution X is locally equicontinuous on $\mathcal{O}_T$ (i.e. continuous on each compact set $K \subseteq (0,T] \times \mathcal{O}$ with a modulus of continuity independent of the initial condition). Under appropriate assumptions on the boundary $\partial\mathcal{O}$ (on the initial data X_0 resp.), also equicontinuity up to the boundary (continuity up to initial time $t = 0$ resp.) is obtained. Applied to driving signals given by independent Brownian motions this implies a new regularity result for the variational stochastic solution X corresponding to (0.1), namely $\mathbb{P}$-a.s. local equicontinuity on $\mathcal{O}_T$. This complements the regularity results given in [17], where it is shown that $\Phi(X) \in L^2([0,T] \times \Omega; H_0^1(\mathcal{O}))$ and $X \in L^\infty([0,T]; L^{m+1}(\Omega \times \mathcal{O}))$ if the initial condition is regular enough.

Recently, increasing attention has been paid to PDE driven by rough signals (RPDE). Starting from the theory of rough paths [27, 28, 16] several distinct approaches to RPDE have been suggested (cf. [25, 26, 19, 20, 9] and references therein). We construct solutions to porous media equations (PME) driven by rough paths, assuming only continuity of the driving signal which again acts linear multiplicatively in space and time. As usual in the theory of PDE driven by rough paths the construction proceeds by a Wong-Zakai approximation of the driving noise, proving the existence of a limit solution independent of the chosen approximating sequence. If the driving signal is given by a continuous semimartingale we prove that this limit solution solves the corresponding SPDE and thus we prove a pathwise Wong-Zakai result for SPME driven by linear multiplicative space-time semimartingale noise.

The long-time behaviour of SPDE can be analyzed in terms of the associated Markovian semigroup and its ergodicity or in terms of the associated RDS and its random attractor. As soon as the driving noise lacks the Markov property the SPDE does not induce a Markovian semigroup anymore. In contrast, analyzing the associated RDS merely requires the noise to have stationary increments and some path regularity (cf. e.g. [18]). In particular, RDS can be used to study long-time behaviour of SPDE driven by fractional Brownian Motion (fBm). The characteristic long-range dependence of fBm makes an investigation of the induced stochastic dynamics especially intriguing. In this paper we only assume that the noise has stationary increments and continuous paths, thus including fBm for all Hurst parameters.

Our methods to prove the existence of solutions to (0.1) for initial conditions $X_0 \in L^{m+1}(\mathcal{O})$ also apply in the case of fast diffusions (i.e. for $0 < m < 1$) driven by continuous signals. In particular, this generalizes results given in [7] since no restrictions on the dimension d nor on the exponent $0 < m < 1$ are assumed. In order not to overload the presentation, the case of fast diffusion equations is treated as a remark (Remark 1.5) only.

SPME and stochastic fast diffusion equations (SFDE) have been intensively investigated in recent years (cf. e.g. [12, 21, 13, 30, 31, 2, 3, 4] and references therein). The long-time behaviour of SPME with Brownian additive noise in terms of the existence of random attractors has first been treated in [8] which then has been partially extended to more generally distributed additive noise in [18]. The SFDE ($0 < m < 1$) with linear multiplicative space-time noise has been first solved in [30]. Subsequently, extinction in finite time with positive probability has been shown in [5] and in a more singular case which is used as model to study self-organized criticality in [6].

Survey of the construction of the RDS and of the proofs of its properties:
Let $1 < m < \infty$ and $\Phi \in C(\mathbb{R})$ be given by

$$\Phi(r) := |r|^m sgn(r).$$

First part: In the first part we construct “pathwise“ solutions to the rough partial differential equation (0.1). Step by step we will allow rougher signals $z^{(k)}$ and initial conditions X_0 at the expense of weaker notions of solutions. The construction of solutions to (0.1) for signals of bounded variation proceeds by first transforming the equation into a PDE and then constructing solutions to this transformed equation. Let

$$\mu_t(\xi) := -\sum_{k=1}^{N} f_k(\xi) z_t^{(k)}. \tag{0.2}$$

Defining $Y = e^{\mu} X$ we obtain the transformed equation (which was first studied in [6, 7])

$$\begin{aligned} \partial_t Y_t &= e^{\mu_t} \Delta \Phi(e^{-\mu_t} Y_t), \text{ on } \mathcal{O}_T \\ Y(0) &= Y_0, \text{ on } \mathcal{O}, \end{aligned} \tag{0.3}$$

with Dirichlet boundary conditions. This transformation is rigorous for driving signals of bounded variation (Theorem 1.2) as well as for signals given by paths of a continuous semi-martingale (Theorem 1.18). Next, we prove uniqueness of essentially bounded solutions to (0.3) (Theorem 1.3).
For continuous driving signals we construct weak solutions to (0.3) as limits of solutions to a non-degenerate, smooth approximation, i.e. we approximate Φ by $\Phi^{(\delta)}$ with $0 < C(\delta) \leq \dot{\Phi}^{(\delta)}$ and the signal z by $z^{(\delta)} \in C^{\infty}([0,T]; \mathbb{R}^N)$. Solutions to these non-degenerate approximations are obtained via classical existence results for quasilinear equations. Passing to the limit

$\delta \to 0$ in order to obtain weak solutions to (0.3) requires a-priori $L^\infty(\mathcal{O})$ bounds for the approximating solutions $Y^{(\delta)}$. Such bounds are obtained by construction of bounded supersolutions to (0.3). Thereby, unique existence of *weak solutions* to (0.3) satisfying analogous $L^\infty(\mathcal{O})$ bounds is obtained for essentially bounded initial conditions (Theorem 1.4). In case of signals of bounded variation this yields weak solutions to (0.1) by transformation.
Next, we approximate general continuous driving signals z by continuous signals of bounded variation $z^{(\varepsilon)}$ and prove that the corresponding weak solutions $X^{(\varepsilon)}$ converge to a limit X independent of the chosen approximating sequence $z^{(\varepsilon)}$. We call the limit X a *rough weak solution* to (0.1) and observe $X = e^{-\mu}Y$, where Y denotes the weak solution to (0.3) for the continuous driving signal z (Theorem 1.7).
In order to construct solutions for general initial data $X_0 \in L^1(\mathcal{O})$ we prove Lipschitz continuity of X in the initial condition with respect to the $L^1(\mathcal{O})$ norm. For $X_0 \in L^1(\mathcal{O})$ solutions are then obtained as *limit solutions* by approximation of X_0 by essentially bounded initial conditions (Theorem 1.9). Using an $L^1 - L^\infty$ regularizing property of the flow, these limit solutions are characterized as unique *generalized weak solutions* to (0.3) (Theorem 1.17). This regularization property also builds the foundation of the proof of bounded absorption. The key idea is to combine an interval splitting technique as in [7, proof of Lemma 3.3] with the known deterministic case, where it is known that there is a function of the form

$$U(t) := At^{-\frac{1}{m-1}}(R^2 - |\xi|^2)^{\frac{1}{m}}$$

that is a uniform (i.e. independent of the initial condition) supersolution to the PME. Combining these ideas we construct a new uniform supersolution for (0.3). The resulting construction is quite different from the one given in [7].
Based on continuity results presented in [14] we then prove that the limit solutions are uniformly continuous on each compact set $K \subseteq (0,T] \times \mathcal{O}$ (Theorem 1.12). This finishes the treatment of the pathwise case.

Second part: In the second part we consider SPME driven by signals given by stochastic processes:

$$\begin{aligned} dX_t &= \Delta\Phi(X_t)dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \\ X(0) &= X_0, \text{ on } \mathcal{O}. \end{aligned} \tag{0.4}$$

with Dirichlet boundary conditions, where z is an $\mathbb{R}^N$ valued stochastic process with stationary increments and continuous paths. Defining

$$\varphi(t,\omega)x = X(t,0;\omega)x$$

yields an RDS on $L^1(\mathcal{O})$ satisfying a comparison principle (Theorem 1.31), where $X(t,0;\omega)x$ is the solution obtained in the first part driven by the signal $z = z(\omega)$. The uniform $L^\infty(\mathcal{O})$ bound and the regularity results obtained for the rough PDE (0.1) continue to hold for φ, which induces asymptotic compactness of φ in each $L^p(\mathcal{O})$ and thereby the existence of a random attractor (Theorem 1.32).

In Section 1 we introduce the detailed setup and present the main results. Proofs of the pathwise results are given in Section 2 while the ones for the stochastic case and the RDS φ are given in Section 3.

As usual in probability theory we denote the time-dependency of functions by a subscript X_t rather than by $X(t)$ in order to keep the equations at a bearable length. We would like to apologize to the readers with a more analytical background for this maybe unfamiliar notation.

1 Setup and Main Results

Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a smooth, bounded domain, $T > 0$ and $\mathcal{O}_T := [0,T] \times \mathcal{O}$. By $\mathcal{P}\mathcal{O}_T$ we denote the parabolic boundary $[0,T] \times \partial\mathcal{O} \cup \{0\} \times \mathcal{O}$. Let $C(\mathcal{O})$ be the set of continuous functions on $\mathcal{O}$, $C^{m,n}(\bar{\mathcal{O}}_T) \subseteq C(\bar{\mathcal{O}}_T)$ be the set of all continuous functions on $\mathcal{O}_T$ having m-th continuous derivatives in time and n-th continuous derivative in space. By $C^{1-var}([0,T];H)$ we denote the set of all continuous functions of bounded variation and by $C^w([0,T];H)$ the weakly continuous functions taking values in H. As usual, $W^{m,p}(\mathcal{O})$ denotes the Sobolev space of order m in $L^p(\mathcal{O})$, $W_0^{m,p}(\mathcal{O})$ the subspace of functions vanishing on $\partial\mathcal{O}$. We denote by $H_0^1(\mathcal{O})$ the first order Sobolev space with zero boundary in $L^2(\mathcal{O})$ and its dual by H. For a subset K of a Banach space X we define $\|K\|_X := \sup_{k\in K} \|k\|_X$.

1.1 Porous Medium Equation driven by rough signals

Let us first define what we mean by a solution to (0.1) and (0.3). Setting $B(x)(z) := \sum_{k=1}^N f_k x z^{(k)}$ for $x \in L^1(\mathcal{O})$ and $z \in \mathbb{R}^N$ we can rewrite

$$B(X_t) \circ dz_t = \sum_{k=1}^N f_k X_t \circ dz_t^{(k)}.$$

As outlined in the introduction, we will introduce several notions of solutions to (0.1) and (0.3), corresponding to the intermediate steps in the construction of the solution for initial values in $L^1(\mathcal{O})$ and continuous driving signals. The final result will be the unique existence of a function $X \in L^1(\mathcal{O}_T)$ such that the transformation $Y = e^{\mu} X$ is a generalized weak solution of (0.3) (cf. Definition 1.15 below) as well as its continuity properties (cf. Theorem 1.12 below). Defining X to be a solution to (0.1) is further justified by the construction since X is obtained as the unique limit of solutions to approximating equations, independent of the chosen approximating sequence. In order to underline this fact, to explain the structure of the construction and to point out the higher regularity of solutions for more regular initial data and driving signals, we explicitly formulate the intermediate existence and uniqueness results. We will use the usual notation for (very) weak solutions as in [14].

Definition 1.1 (weak & very weak solutions).

i. *Let $Y_0 \in L^1(\mathcal{O})$. A function $Y \in L^1(\mathcal{O}_T)$ with $\Phi(e^{-\mu}Y) \in L^1(\mathcal{O}_T)$ is called a very weak solution to* (0.3) *if*

$$-\int_{\mathcal{O}_T} Y_r \partial_r \eta \, d\xi dr - \int_{\mathcal{O}} Y_0 \eta_0 \, d\xi = \int_{\mathcal{O}_T} \Phi(e^{-\mu_r} Y_r) \Delta(e^{\mu_r} \eta_r) \, d\xi dr, \tag{1.5}$$

for all $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $\{T\} \times \mathcal{O}$ and on $[0,T] \times \partial\mathcal{O}$. If in addition $\Phi(e^{-\mu}Y) \in L^1([0,T]; W_0^{1,1}(\mathcal{O}))$ then Y is said to be a weak solution to (0.3).

ii. *Let $z \in C^{1-var}([0,T];\mathbb{R}^N)$ and $X_0 \in L^1(\mathcal{O})$. A function $X \in L^1(\mathcal{O}_T)$ such that $t \mapsto \left(\int_{\mathcal{O}} B(X_t)\eta_t \, d\xi\right)$ is continuous and $\Phi(X) \in L^1(\mathcal{O}_T)$ is called a very weak solution to* (0.1) *if*

$$-\int_{\mathcal{O}_T} X_r \partial_r \eta \, d\xi dr - \int_{\mathcal{O}} X_0 \eta_0 \, d\xi = \int_{\mathcal{O}_T} \Phi(X_r) \Delta \eta_r \, d\xi dr + \int_0^T \left(\int_{\mathcal{O}} B(X_r) \eta_r \, d\xi \right) dz_r,$$

for all $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $\{T\} \times \mathcal{O}$ and on $[0,T] \times \partial\mathcal{O}$. If in addition $\Phi(X) \in L^1([0,T]; W_0^{1,1}(\mathcal{O}))$ then X is said to be a weak solution to (0.1).

A function $Y \in L^1(\mathcal{O}_T) \cap C([0,T];H)$ with $\Phi(e^{-\mu}Y) \in L^1([0,T]; H_0^1(\mathcal{O}))$ is a weak solution to (0.3) iff

$$\frac{dY_t}{dt} = e^{\mu_t} \Delta \Phi(e^{-\mu_t} Y_t),$$

for a.e. $t \in [0,T]$ as an equation in H. Similarly, $X \in L^1(\mathcal{O}_T) \cap C([0,T];H)$ with $\Phi(X) \in L^1([0,T]; H_0^1(\mathcal{O}))$ is a weak solution to (0.1) iff

$$X_t = X_0 + \int_0^t \Delta \Phi(X_r) dr + \int_0^t B(X_r) dz_r,$$

for all $t \in [0,T]$ as an equation in H. If we replace H by some weaker space $H^{-k} \supseteq L^1(\mathcal{O})$ then similar equivalences hold for very weak solutions in $C^w([0,T]; L^1(\mathcal{O}))$.

For very weak solutions we will prove that the equations (0.1) and (0.3) are indeed in one to one correspondence under the transformation $Y = e^{\mu}X$.

Theorem 1.2. *Let $X_0 \in L^1(\mathcal{O})$, $z \in C^{1-var}([0,T];\mathbb{R}^N)$ and $X \in L^1(\mathcal{O}_T)$ such that $t \mapsto \left(\int_{\mathcal{O}} B(X_t)\eta_t \, d\xi\right)$ is continuous for all $\eta \in C^{0,2}(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $\{T\} \times \mathcal{O}$ and on $[0,T] \times \partial\mathcal{O}$. Then X is a very weak solution to* (0.1) *iff $Y := e^{\mu}X$ is a very weak solution to* (0.3).

As an immediate consequence we obtain that X is a weak solution to (0.1) iff $Y := e^{\mu}X$ is a weak solution to (0.3). We will prove the following uniqueness of very weak solutions:

Theorem 1.3. *Essentially bounded very weak solutions to* (0.1) *and* (0.3) *are unique.*

Note that in the case of very weak solutions to (0.1) we implicitly assume $z \in C^{1-var}([0,T];\mathbb{R}^N)$. As outlined in the introduction by a non-degenerate approximation of (0.3) we obtain:

Theorem 1.4. *i. Let $Y_0 \in L^\infty(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. Then there exists a unique weak solution $Y \in C([0,T];H) \cap L^\infty(\mathcal{O}_T)$ to* (0.3) *satisfying $\Phi(e^{-\mu}Y) \in L^2([0,T];H_0^1(\mathcal{O}))$. There is a function $U : [0,T] \times \mathcal{O} \to \bar{\mathbb{R}}$ (taking the value ∞ at $t = 0$) that is piecewise smooth on $(0,T]$ such that for all $Y_0 \in L^\infty(\mathcal{O})$*

$$Y_t \le U_t, \ \ a.e. \ in \ \mathcal{O}, \ \forall t \in [0,T].$$

(U is more explicitly defined in the proof below).

ii. Let $z \in C^{1-var}([0,T];\mathbb{R}^N)$ and $X_0 \in L^\infty(\mathcal{O})$. Then there exists a unique weak solution $X \in C([0,T];H) \cap L^\infty(\mathcal{O}_T)$ to (0.1) *satisfying $\Phi(X) \in L^2([0,T];H_0^1(\mathcal{O}))$ and $X_t \le U_t$ a.e. in $\mathcal{O}$, $\forall t \in [0,T]$ with a function U as in (i).*

The existence of such an upper bound U_t that is independent of the initial condition is due to the nonlinearity of the porous medium operator and is well known in the deterministic case (cf. [34] and references therein) with U_t being of the form $U_t = At^{-\frac{1}{m-1}}(R^2 - |\xi|^2)^{\frac{1}{m}}$.

Remark 1.5. *For the case of fast diffusion equations, i.e. for $0 < m < 1$ we obtain:*

i. For $Y_0 \in L^{m+1}(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$ there exists a weak solution $Y \in C([0,T];H)$ to (0.3) *satisfying $\Phi(e^{-\mu}Y) \in L^2([0,T];H_0^1(\mathcal{O}))$. If $Y_0 \in L^\infty(\mathcal{O})$ then*

$$Y_t \le K_t, \ \ a.e. \ in \ \mathcal{O}, \ \forall t \in [0,T],$$

with $K = K(\|Y_0\|_{L^\infty(\mathcal{O})}) : [0,T] \times \mathcal{O} \to \mathbb{R}_+$ being a piecewise smooth function on $[0,T]$. The map $t \mapsto Y_t$ is weakly continuous in each $L^p(\mathcal{O})$, $p \in [1,\infty)$.

ii. Let $z \in C^{1-var}([0,T];\mathbb{R}^N)$ and $X_0 \in L^{m+1}(\mathcal{O})$. Then there exists a weak solution $X \in C([0,T];H)$ to (0.1) *satisfying $\Phi(X) \in L^2([0,T];H_0^1(\mathcal{O}))$. If $X_0 \in L^\infty(\mathcal{O})$ then $X_t \le K_t$ a.e. in $\mathcal{O}$, $\forall t \in [0,T]$ with a function K as in (i). The map $t \mapsto X_t$ is weakly continuous in each $L^p(\mathcal{O})$, $p \in [1,\infty)$.*

No uniqueness is obtained for the fast diffusion case.

So far we can solve (0.3) for driving signals being merely continuous while for (0.1) we require continuous signals of bounded variation. Since we aim to include rough paths (as they occur for example as sample paths of fractional Brownian motion) we need to allow rougher signals

$z \in C([0,T];\mathbb{R}^N)$ for (0.1) as well. Similar as in the theory of rough paths such solutions will be constructed as limits of solutions to smoothed signals $z^{(\varepsilon)} \in C^{1-var}([0,T];\mathbb{R}^N)$ with $z^\varepsilon \to z$ in $C([0,T];\mathbb{R}^N)$. We prove that the solutions $X^{(\varepsilon)}$ to (0.1) driven by these smoothed signals converge to $X := e^{-\mu}Y$, i.e. to a limit not depending on the chosen approximating sequence. In other words, X is the limit obtained by any Wong-Zakai approximation of (0.1).

Definition 1.6. *Let $z \in C([0,T];\mathbb{R}^N)$. We call $X \in C([0,T];H)$ a rough weak solution to* (0.1) *if $X(0) = X_0$ and for all approximations $z^{(\varepsilon)} \in C^{1-var}([0,T];\mathbb{R}^N)$ of the driving signal z with $z^{(\varepsilon)} \to z$ in $C([0,T];\mathbb{R}^N)$ and corresponding weak solutions $X^{(\varepsilon)}$ to* (0.1) *driven by $z^{(\varepsilon)}$ we have*

$$X_t^{(\varepsilon)} \to X_t, \text{ in } H, \ \forall t \in [0,T]. \tag{1.6}$$

Theorem 1.7. *Let $X_0 \in L^\infty(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. Then there exists a unique rough weak solution X to* (0.1) *with $\Phi(X) \in L^2([0,T];H_0^1(\mathcal{O}))$ given by $X = e^{-\mu}Y$, where Y is the corresponding weak solution to* (0.3)*. X satisfies $X_t \leq U_t$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$, with U as in Theorem 1.4.*

Since the weak solutions to (0.1) obtained in Theorem 1.4 are also given by $X = e^{-\mu}Y$, the notions of rough weak solutions and weak solutions to (0.1) coincide for continuous driving signals of bounded variation and essentially bounded initial conditions.

Definition 1.8. *Let $X_0 \in L^1(\mathcal{O})$ and $z \in C([0,T];\mathbb{R}^N)$. A function $X \in C^w([0,T];L^1(\mathcal{O}))$ is said to be a limit solution to* (0.1) *if $X(0) = X_0$ and for all approximations $X_0^{(\delta)} \in L^\infty(\mathcal{O})$ with $X_0^{(\delta)} \to X_0$ in $L^1(\mathcal{O})$ and corresponding rough weak solutions $X^{(\delta)}$ to* (0.1) *we have $X_t^{(\delta)} \to X_t$ in $L^1(\mathcal{O})$ uniformly in time.*

These limit solutions play an important role for allowing initial conditions in $L^1(\mathcal{O})$. In Lemma 2.6 below we will establish uniform $L^1(\mathcal{O})$ continuity in the initial condition for rough weak solutions. This will allow to construct limit solutions for initial values in $L^1(\mathcal{O})$ by approximation in the initial condition.

Theorem 1.9. *Let $z \in C([0,T];\mathbb{R}^N)$. For each $X_0 \in L^1(\mathcal{O})$ there is a unique limit solution X satisfying $\Phi(X) \in L^1(\mathcal{O}_T)$. For $X_0^{(i)} \in L^1(\mathcal{O})$, $i = 1,2$ the corresponding limit solutions satisfy*

$$\sup_{t\in[0,T]} \|(X_t^{(1)} - X_t^{(2)})^+\|_{L^1(\mathcal{O})} + \|(\Phi(X^{(1)}) - \Phi(X^{(2)}))^+\|_{L^1(\mathcal{O}_T)} \leq C\|(X_0^{(1)} - X_0^{(2)})^+\|_{L^1(\mathcal{O})},$$

and

$$\sup_{t\in[0,T]} \|X_t^{(1)} - X_t^{(2)}\|_{L^1(\mathcal{O})} + \|\Phi(X^{(1)}) - \Phi(X^{(2)})\|_{L^1(\mathcal{O}_T)} \leq C\|X_0^{(1)} - X_0^{(2)}\|_{L^1(\mathcal{O})}.$$

We further have $X_t \leq U_t$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$, where U is as in Theorem 1.4.

As a special case we obtain the following comparison principle

Corollary 1.10. *Let $X_0^{(1)}, X_0^{(2)} \in L^1(\mathcal{O})$ with $X_0^{(1)} \leq X_0^{(2)}$ a.e. in $\mathcal{O}$. Then*

$$X_t^{(1)} \leq X_t^{(2)},$$

for all $t \in [0,T]$, a.e. in $\mathcal{O}$. In particular, if $0 \leq X_0$ then $0 \leq X$.

Let X be a limit solution. By Theorem 1.9 there are rough weak solutions with $X^{(\delta)} \to X$ in $L^\infty([0,T]; L^1(\mathcal{O}))$ and $\Phi(X^{(\delta)}) \to \Phi(X)$ in $L^1(\mathcal{O}_T)$. Hence, there are weak solutions $Y^{(\delta)} = e^\mu X^{(\delta)}$ converging in $L^\infty([0,T]; L^1(\mathcal{O}))$ to $Y := e^\mu X$ and $\Phi(e^{-\mu} Y^{(\delta)}) \to \Phi(e^{-\mu} Y)$ in $L^1(\mathcal{O}_T)$. Passing to the limit $\delta \to 0$ in (1.5) yields

Remark 1.11. *Let $X_0 \in L^1(\mathcal{O})$ and X be the corresponding limit solution. Then $Y := e^\mu X$ is a very weak solution of* (0.3).

The limit solution X turns out to be in fact more regular. The proof proceeds by choosing the approximations used in the construction of weak solutions in a way that allows to apply the regularity results presented in [14]. We say that a quantity depends only on the data if it is a function of d, m, T.

Theorem 1.12. *Let $z \in C([0,T]; \mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$ and X be the corresponding limit solution. Then*

i. X is uniformly continuous on every compact set $K \subseteq (0,T] \times \mathcal{O}$, with modulus of continuity depending only on the data and dist$(K, \partial\mathcal{O}_T)$.

ii. If $X_0 \in L^\infty(\mathcal{O})$ is continuous on a compact set $K \subseteq \mathcal{O}$, then X is uniformly continuous on $[0,T] \times K'$ for every compact set $K' \subseteq \mathring{K}$, with modulus of continuity depending only on the data, $\|X_0\|_{L^\infty(\mathcal{O})}$, dist$(K, \partial\mathcal{O})$, dist$(K', \partial K)$ and the modulus of continuity of X_0 over K.

iii. Assume:

($\mathcal{O}1$) There exist $\theta^* > 0, R_0 > 0$ such that $\forall x_0 \in \partial\mathcal{O}$ and every $R \leq R_0$

$$|\mathcal{O} \cap B_R(x_0)| < (1 - \theta^*)|B_R(x_0)|.$$

Then for every $\tau > 0$, X is uniformly continuous on $[\tau, T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, θ^* and τ.

By dominated convergence we obtain:

Corollary 1.13. *Let $z \in C([0,T]; \mathbb{R}^N)$.*

i. If $X_0 \in L^1(\mathcal{O})$ then $X \in C([0,T]; L^1(\mathcal{O})) \cap C((0,T]; L^p(\mathcal{O}))$ for every $p \in [1, \infty)$.

ii. *If* $X_0 \in L^\infty(\mathcal{O})$ *then* $X \in C([0,T]; L^p(\mathcal{O}))$ *for every* $p \in [1, \infty)$.

The continuity obtained in Theorem 1.12 together with the L^∞-bounds from Theorem 1.4 imply that the convergence of the various approximating solutions used to construct limit solutions driven by rough signals in fact holds locally uniformly. For example we obtain

Corollary 1.14. *Let* $z \in C([0,T]; \mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$. *Then the convergence in* (1.6) *holds uniformly on compact sets* $K \subseteq (0,T] \times \mathcal{O}$.

In Remark 1.11 we have shown that the limit solutions X are solutions to (0.1) in the sense that their transformations $Y := e^\mu X$ are very weak solutions to (0.3). However, since uniqueness of very weak solutions has only been obtained in the essentially bounded case, this does not yield a characterization of limit solutions. To overcome this problem we recall that the limit solutions constructed in Theorem 1.9 enjoy an $L^1 - L^\infty$ regularizing property. This regularization can be used in order to characterize the transformation $Y := e^\mu X$ of limit solutions X as generalized weak solutions.

Definition 1.15. *Let* $z \in C([0,T]; \mathbb{R}^N)$. *A map* $Y \in C([0,T]; L^1(\mathcal{O}))$ *is said to be a generalized weak solution to* (0.3) *if* Y *is an essentially bounded weak solution to* (0.3) *on each interval* $[\tau, T]$ *with* $\tau > 0$. *I.e.* $Y \in L^\infty([\tau,T] \times \mathcal{O})$, $\Phi(e^{-\mu} Y) \in L^1([\tau,T]; W_0^{1,1}(\mathcal{O}))$ *and*

$$-\int_{[\tau,T]\times\mathcal{O}} Y_r \, \partial_r \eta \, d\xi dr - \int_{\mathcal{O}} Y_\tau \eta_\tau \, d\xi = -\int_{[\tau,T]\times\mathcal{O}} \nabla\Phi(e^{-\mu_r} Y_r) \cdot \nabla(e^{\mu_r} \eta_r) \, d\xi dr,$$

for all $\eta \in C^1([\tau,T] \times \bar{\mathcal{O}})$ *with* $\eta = 0$ *on* $[\tau,T] \times \partial\mathcal{O}$ *and on* $\{T\} \times \mathcal{O}$.
$X \in C([0,T]; L^1(\mathcal{O}))$ *is said to be a generalized weak solution to* (0.1) *if* $Y = e^\mu X$ *is a generalized weak solution to* (0.3).

Using the continuity $X \in C([0,T]; L^1(\mathcal{O}))$ of generalized weak solutions and Lipschitz continuity of weak solutions in the initial condition (Theorem 1.9) we obtain

Proposition 1.16 (Uniqueness of generalized weak solutions)**.** *Let* $X^{(i)}$ *be generalized weak solutions with initial conditions* $X_0^{(i)}$, $i = 1, 2$. *Then*

$$\sup_{t\in[0,T]} \|X_t^{(1)} - X_t^{(2)}\|_{L^1(\mathcal{O})} \leq C \|X_0^{(1)} - X_0^{(2)}\|_{L^1(\mathcal{O})}.$$

In Theorem 1.9 we have obtained that every limit solution X is essentially bounded on $[\tau, T] \times \mathcal{O}$ for all $\tau > 0$. By uniqueness of limit solutions this implies that X is a rough weak solution on $[\tau, T]$. Thus $Y = e^\mu X$ is a generalized weak solution.

Theorem 1.17. *Let* $X_0 \in L^1(\mathcal{O})$ *and let* X *be the corresponding limit solution to* (0.1). *Then* X *is the unique generalized weak solution to* (0.3).

1.2 Stochastic Porous Medium Equation and RDS

So far we did not require the driving signal to be given by a stochastic process. We aim to study the qualitative behaviour, in particular the long-time behaviour of solutions to PME driven by rough noise. If the rough signal is given by a stationary random process this additional structure can be used to significantly simplify this task. This approach is nicely captured by the theory of RDS.

For signals given by the paths of a continuous semimartingale stochastic calculus may be used to give a meaning to the integral over the rough signal occurring in (0.1). This allows to further justify the notion of a rough weak solution which was based on a Wong-Zakai approximation of the noise (Definition 1.6).

Theorem 1.18. *Let $z : [0,T] \times \Omega \to \mathbb{R}^N$ be a continuous semimartingale on a normal filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $X_0 \in L^0(\Omega, \mathcal{F}_0; L^1(\mathcal{O}))$ and $X(\omega)$ be the corresponding (pathwise) limit solution to* (0.1)*. Then*

$$\int_{\mathcal{O}} X_t \varphi \, d\xi = \int_{\mathcal{O}} X_s \varphi \, d\xi + \int_s^t \int_{\mathcal{O}} \Phi(X_r) \Delta\varphi \, d\xi dr + \int_s^t \left(\int_{\mathcal{O}} B(X_r) \varphi \, d\xi \right) \circ dz_r, \tag{1.7}$$

for all $\varphi \in C_0^2(\bar{\mathcal{O}})$ and all $0 \le s \le t \le T$, $\mathbb{P}$-almost surely.

The process $X : \Omega \to C([0,T]; L^1(\mathcal{O}))$ is adapted to the filtration generated by z by continuity in the driving process (Theorem 1.7). Hence, the stochastic integral in (1.7) is well defined for every $\varphi \in L^\infty(\mathcal{O})$.

Remark 1.19. *By Theorem 1.7 we know that for any approximation $z^{(\varepsilon)} \in C^{1-var}([0,T]; \mathbb{R}^N)$ with $z^{(\varepsilon)} \to z$ in $C([0,T]; \mathbb{R}^N)$ (pathwise) we have $X_t^{(\varepsilon)}(\omega) \to X_t(\omega)$ in H for all $t \in [0,T]$ and all $\omega \in \Omega$. Since X is a solution to* (1.7)*, this yields a pathwise Wong-Zakai result.*

1.2.1 Quasi-continuity of random dynamical systems

We will now first recall basic notions from the theory of RDS and then develop an existence result for random attractors based on weakened continuity assumptions for RDS and asymptotic compactness. This generalized result is needed since the RDS corresponding to (0.1), while being continuous on $L^1(\mathcal{O})$, is only continuous in some weaker sense on $L^p(\mathcal{O})$, $p \in (1, \infty]$. For more details on the theory of RDS and random attractors we refer to [32, 11, 10, 1].

Definition 1.20. *Let (X, d) be a complete and separable metric space. A random dynamical system (RDS) over $(\theta_t)_{t \in \mathbb{R}}$ is a measurable map $\varphi : \mathbb{R}_+ \times X \times \Omega \to X$, such that $\varphi(0, \omega) = id$ and*

$$\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \quad \textit{(cocycle property)}$$

for all $t, s \in \mathbb{R}_+$ and $\omega \in \Omega$. φ is said to be a continuous RDS if $x \mapsto \varphi(t, \omega)x$ is continuous for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

We now recall the stochastic generalization of notions of absorption, attraction and Ω-limit sets.

Definition 1.21. *i. A set-valued map $D : \Omega \to 2^X$ is called measurable if $\omega \mapsto D(\omega)$ takes values in the closed subsets of X and for all $x \in X$ the map $\omega \mapsto d(x, D(\omega))$ is measurable, where for nonempty sets $A, B \in 2^X$ we set*

$$d(A, B) := \sup_{x \in A} \inf_{y \in B} d(x, y)$$

and $d(x, B) = d(\{x\}, B)$. A measurable set-valued map is also called a (closed) random set.

ii. A set universe $\mathcal{D}$ is a collection of families of subsets $(D(\omega))_{\omega \in \Omega}$ of X such that if $D \in \mathcal{D}$ and $\hat{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ then $\hat{D} \in \mathcal{D}$. A universe of random sets is a set universe consisting of random closed sets.

iii. Let A, B be random sets. A is said to absorb B if there exists an absorption time $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

A is said to attract B if

$$d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t \to \infty]{} 0, \ \forall \omega \in \Omega.$$

iv. Let $\mathcal{D}$ be a universe of random sets and $D \in \mathcal{D}$. Then D is said to be a $\mathcal{D}$-absorbing set for φ if D absorbs every set $D \in \mathcal{D}$. $\mathcal{D}$-attracting sets are defined analogously.

We require absorption and attraction to hold for all $\omega \in \Omega$ in order to state our results in their full strength. This is stronger than usual in the theory of RDS where an exceptional $\mathbb{P}$-zero set is allowed.

Definition 1.22. *Let $\mathcal{D}$ be a universe of random sets. Then φ is said to be $\mathcal{D}$-asymptotically compact in X if the sequence $\varphi(t_n, \theta_{-t_n}\omega)x_n$ has a convergent subsequence in X, for all $\omega \in \Omega$, $t_n \to \infty$, $x_n \in D(\theta_{-t_n}\omega)$ and $D \in \mathcal{D}$.*

Definition 1.23. *Let $\mathcal{D}$ be a universe of random sets. A $\mathcal{D}$-random attractor for an RDS φ is a compact random set $A \in \mathcal{D}$ satisfying:*

(i) A is invariant, i.e. $\varphi(t, \omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.

(ii) A is $\mathcal{D}$-attracting.

Since we require $A \in \mathcal{D}$ the random attractor for an RDS is uniquely determined.

In [23] the assumption of continuity of RDS has been weakened while preserving sufficient criteria for the existence of random attractors. This allowed to study RDS on subspaces of their "original" state spaces. We prove generalizations of these results and identify some underlying structures, which will allow to prove the existence of random attractors for φ seen as an RDS on $L^p(\mathcal{O})$ for all $p \in [1, \infty)$. If condition $(\mathcal{O}1)$ is satisfied we will also obtain the existence of random attractors with respect to the L^∞ norm.

Definition 1.24. *An RDS φ on a Banach space X endowed with some topology τ is said to be quasi-τ-continuous if $\varphi(t_n, \omega)x_n \to^\tau \varphi(t, \omega)x$, whenever $(t_n, x_n) \in \mathbb{R}_+ \times X$ is a sequence such that $\varphi(t_n, \omega)x_n$ is bounded and $(t_n, x_n) \to (t, x)$ for $n \to \infty$. Here "$\to^\tau$" denotes convergence with respect to τ-topology.*

In [23] a general result proving quasi-continuity for restrictions of continuous RDS to subspaces of the state space has been proven. More precisely:

Proposition 1.25 (Proposition 3.3 [23])**.** *Let Y, X be Banach spaces such that $i : Y \hookrightarrow X$ and $i^* : X^* \hookrightarrow Y^*$ are dense and continuous. If φ is an RDS on X, Y (resp.) and φ is (norm-weak) continuous on X, then φ is quasi-weakly-continuous on Y, i.e. quasi-τ-continuous for τ being the weak topology on Y.*

If Y is a reflexive space then continuity and density of $i : Y \hookrightarrow X$ implies the same for $i^* : X^* \hookrightarrow Y^*$. For non-reflexive spaces the situation may be more involved and in general one may only conclude the existence of the continuous map $i^* : X^* \hookrightarrow Y^*$. However, even in the non-reflexive case ${}_{Y^*}\langle , \rangle_Y : i^*(X^*) \times Y \to \mathbb{R}$ defines a duality mapping, i.e.

i. ${}_{Y^*}\langle i^*(x^*), y\rangle_Y = 0$ for all $y \in Y$ implies $i^*(x^*) = 0$,

ii. ${}_{Y^*}\langle i^*(x^*), y\rangle_Y = 0$ for all $x^* \in X^*$ implies $y = 0$.

Since $i^*(X^*) \subseteq Y^*$ is a linear subspace and ${}_{Y^*}\langle , \rangle_Y : i^*(X^*) \times Y \to \mathbb{R}$ is a duality mapping, the corresponding weak topology $\sigma(Y, \overline{i^*(X^*)})$ on Y is Hausdorff, where $\overline{i^*(X^*)}$ denotes the closure of $i^*(X^*)$ with respect to $\| \cdot \|_{Y^*}$. Norm-weak continuity of φ in X just means continuity of $(t, x) \mapsto {}_{X^*}\langle x^*, \varphi(t, \omega)x\rangle_X$ for all $x^* \in X^*$, $\omega \in \Omega$. Hence, norm-weak continuity of φ in X implies norm-$\sigma(Y, i^*(X^*))$ continuity on Y. On bounded sets $B \subseteq Y$ we have $\sigma(Y, i^*(X^*)) \cap B = \sigma(Y, \overline{i^*(X^*)}) \cap B$. This is the precise idea of quasi-continuity. We obtain

Proposition 1.26. *Let X, Y be Banach spaces such that $i : Y \hookrightarrow X$ is dense and continuous. If φ is an RDS on X, Y and φ is (norm-weak) continuous on X, then φ is quasi-$\sigma(Y, \overline{i^*(X^*)})$-continuous on Y.*

In the following let $\mathcal{D}$ be a universe of random sets and κ be the Kuratowski measure of non-compactness. We will prove that in the proof of existence of random attractors the assumption of omega-limit-compactness can be replaced by asymptotic compactness. This indeed weakens the assumptions since every $\mathcal{D}$-omega-limit compact RDS φ, i.e. satisfying

$$\lim_{T\to\infty} \kappa\left(\bigcup_{t\geq T} \varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega)\right) = 0,$$

for all $\omega \in \Omega$ and $D \in \mathcal{D}$ is $\mathcal{D}$-asymptotically compact.

For a topology τ on a Banach space X and a random set B we define the Ω-limit set

$$\Omega^\tau(B,\omega) = \{y \in X | \ \exists t_n \to \infty, \ x_n \in B(\theta_{-t_n}\omega), \ \varphi(t_n,\theta_{-t_n}\omega)x_n \to^\tau y\}.$$

Ω-limit sets with respect to the norm topology are simply denoted by $\Omega(B,\omega)$. One of the ideas in [23] in order to allow quasi-weak-continuity of φ is to consider Ω-limit sets with respect to the weak topology replacing the usual norm topology. For asymptotically compact RDS these notions actually coincide:

Lemma 1.27. *Let φ be a $\mathcal{D}$-asymptotically compact RDS on the Banach space X endowed with a Hausdorff topology τ that is weaker than the norm topology. Then*

$$\Omega(B,\omega) = \Omega^\tau(B,\omega), \quad \forall B \in \mathcal{D}.$$

In the proof of existence of random attractors we can replace $\mathcal{D}$-omega-limit-compactness by $\mathcal{D}$-asymptotic compactness due to the following observation

Lemma 1.28. *Let φ be a $\mathcal{D}$-asymptotically compact, quasi-τ-continuous RDS on the Banach space X endowed with a Hausdorff topology τ that is weaker than the norm topology. Further assume that there is a bounded $\mathcal{D}$-absorbing set F. Then $\Omega(B,\omega)$ is a nonempty, compact, invariant set for each $B \in \mathcal{D}$, $B \neq \emptyset$, $\omega \in \Omega$.*

If we work with the weaker notion of absorption occuring only $\mathbb{P}$-a.s. then invariance in Lemma 1.28 is satisfied only crudely. That is $\varphi(t,\omega)\Omega(B,\omega) = \Omega(B,\theta_t\omega)$ on a $\mathbb{P}$-zero that may depend on t. In the proof of the existence of random attractors this obstacle can be resolved by a "perfection" result proving that there is an indistinguishable, perfectly invariant modification of $\Omega(B,\omega)$.

With these preparations it is easy to see that the proof of [23, Theorem 4.1] can be modified so that only quasi-τ-continuity and asymptotic compactness with respect to the universe of all bounded deterministic sets has to be assumed.
In our case the universe of absorbed sets will be much larger than just deterministic bounded sets. This allows to drop the assumption of ergodicity of the underlying metric dynamical system. In conclusion we obtain the following:

Theorem 1.29. *Let φ be a quasi-τ-continuous RDS on a Banach space X, where τ is a Hausdorff topology that is weaker than the norm topology. Then φ has a $\mathcal{D}$-random attractor iff*

i. *φ has a bounded $\mathcal{D}$-absorbing random set $F \in \mathcal{D}$.*

ii. *φ is $\mathcal{D}$-asymptotically compact in X.*

1.3 RDS and random attractors for (0.1)

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued adapted stochastic process and $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system, i.e. $(t, \omega) \mapsto \theta_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}/\mathcal{F}$ measurable, $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t \circ \theta_s$ and θ_t is $\mathbb{P}$-preserving, for all $s, t \in \mathbb{R}$. We assume

$(S1)$ (Strictly stationary increments) For all $t, s \in \mathbb{R}$, $\omega \in \Omega$:

$$z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s\omega).$$

We assume $z_0 = 0$ for notational convenience only.

$(S2)$ (Regularity) z_t has continuous paths.

Adaptedness and $(S2)$ imply joint measurability of z, i.e. $z : \mathbb{R}\times\Omega \to \mathbb{R}^N$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{F}/\mathcal{B}(\mathbb{R}^N)$ measurable. Note:

$$\mu_t(\omega) - \mu_s(\omega) = \sum_{k=1}^{N} f_k(z_t^k(\omega) - z_s^k(\omega)) = \mu_{t-s}(\theta_s\omega) \tag{1.8}$$

and recall that f_k are functions depending on the space variable.

By [18, Lemma 3.1] for each $\mathbb{R}^N$ valued process $\tilde{z}_t$ with $\tilde{z}_0 = 0$ a.s., stationary increments and a.s. continuous paths there exists a metric dynamical system $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ and a version z_t of $\tilde{z}_t$ on $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ such that z_t satisfies $(S1)$, $(S2)$. In particular, applications include fractional Brownian motion with arbitrary Hurst parameter.

Using the pathwise results obtained in Section 1.1 we define the RDS φ on $X := L^1(\mathcal{O})$ associated to (0.4). For $t \geq s$, $\omega \in \Omega$ and $x \in L^1(\mathcal{O})$ let $X(t, s; \omega)x$ denote the unique limit solution to (0.1) on $[s, \infty)$ with $X_s = x$ and driving signal $z = z(\omega)$.

Definition 1.30. *For $t \geq s$, $\omega \in \Omega$ and $x \in L^1(\mathcal{O})$ define*

$$\varphi(t - s, \theta_s\omega)x := X(t, s; \omega)x.$$

Theorem 1.31. *The map φ from Definition 1.30 is a continuous RDS on $X = L^1(\mathcal{O})$ and thus a quasi-weakly-continuous RDS on each $L^p(\mathcal{O})$, $p \in [1,\infty)$. In addition, φ is a quasi-weakly*-continuous RDS on $L^\infty(\mathcal{O})$. φ satisfies comparison, i.e. for $x_1, x_2 \in X$ with $x_1 \le x_2$ a.e. in $\mathcal{O}$*

$$\varphi(t,\omega)x_1 \le \varphi(t,\omega)x_2, \quad \text{a.e. in } \mathcal{O}.$$

Moreover, φ satisfies $\varphi(t,\omega)0 = 0$ and

i. $x \mapsto \varphi(t,\omega)x$ is Lipschitz continuous on X, locally uniformly in t.

ii. $t \mapsto \varphi(t,\omega)x$ is continuous in X.

iii. $\varphi(t,\omega)x \le U_t(\omega)$ a.e. in $\mathcal{O}$ for all $t \ge 0$, $\omega \in \Omega$, with U as in Theorem 1.4.

iv. φ satisfies the same regularity properties as for the pathwise solutions obtained in Theorem 1.12.

Let $\mathcal{D}$ be the universe of all random closed sets in X. Using the uniform L^∞ bound obtained in Theorem 1.31 we obtain the existence of a $\mathcal{D}$-absorbing set F which is bounded even in $L^\infty(\mathcal{O})$. In fact, the absorption time $t_D(\omega)$ can be chosen independent of ω and D (cf. Proposition 3.1 below).
If the domain $\mathcal{O}$ satisfies condition ($\mathcal{O}$1), by combining the uniform $L^\infty(\mathcal{O})$ estimate and Theorem 1.31 (iii) we will conclude that the set $\varphi(\delta,\omega)F(\omega)$ with $\delta > 0$ is compact in $C^0(\bar{\mathcal{O}})$ and $\mathcal{D}$-absorbing in $\mathcal{D}$. By Theorem 1.29 this implies the existence of a $\mathcal{D}$-random attractor. If the domain $\mathcal{O}$ does not necessarily satisfy condition ($\mathcal{O}$1) we only get inner continuity, i.e. equicontinuity of $\varphi(\delta,\omega)K(\omega)$ on each compact set $K \subseteq \mathcal{O}$. In this case we cannot conclude the existence of a compact $\mathcal{D}$-absorbing set, but we can still prove $\mathcal{D}$-asymptotic compactness for φ. By Theorem 1.29 we arrive at the following:

Theorem 1.32. *Let $\mathcal{D}$ be the universe of all random closed sets. The RDS φ has a $\mathcal{D}$-random attractor A (as an RDS on $L^1(\mathcal{O})$). A is compact in each $L^p(\mathcal{O})$ and attracts all sets in $\mathcal{D}$ in L^p-norm, $p \in [1,\infty)$.*

Moreover, $A(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $A(\omega)$ are equicontinuous on every compact set $K \subseteq \mathcal{O}$.

If ($\mathcal{O}$1) is satisfied then $A(\omega)$ is a compact set in $C^0(\bar{\mathcal{O}})$ and attracts all sets in $\mathcal{D}$ in L^∞-norm.

2 Porous Medium Equation driven by rough signals

2.1 Transformation for signals of bounded variations

In this section we prove Theorem 1.2. Let $z \in C^{1-var}([0,T];\mathbb{R}^N)$, $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $\{T\} \times \mathcal{O}$ and on $[0,T] \times \partial\mathcal{O}$ and let X be a very weak solution to (0.1). We prove that $Y := e^{\mu} X$ is a very weak solution to (0.3). Let $z^{\varepsilon} \in C^1([0,T];\mathbb{R}^N)$ such that $z^{(\varepsilon)} \to z$ in $C([0,T];\mathbb{R}^N)$ with uniformly bounded variation, i.e. $\sup_{\varepsilon>0} \|z^{(\varepsilon)}\|_{C^{1-var}} < \infty$. Define μ^{ε} as in (0.2). Then

$$-\int_{\mathcal{O}_T} Y_r \partial_r \eta_r \; d\xi dr = -\int_{\mathcal{O}_T} X_r e^{\mu_r} \partial_r \eta_r \; dr d\xi = -\lim_{\varepsilon \to 0} \int_{\mathcal{O}_T} X_r e^{\mu_r^{(\varepsilon)}} \partial_r \eta_r \; d\xi dr$$

and

$$\begin{aligned}
-\int_{\mathcal{O}_T} X_r e^{\mu_r^{(\varepsilon)}} \partial_r \eta_r \; d\xi dr &= -\int_{\mathcal{O}_T} X_r \partial_r \left(e^{\mu_r^{(\varepsilon)}} \eta_r \right) \; d\xi dr + \int_{\mathcal{O}_T} X_r \eta_r \partial_r e^{\mu_r^{(\varepsilon)}} \; d\xi dr \\
&= \int_{\mathcal{O}} X_0 e^{\mu_0^{(\varepsilon)}} \eta_0 \; d\xi + \int_{\mathcal{O}_T} \Phi(X_r) \Delta \left(e^{\mu_r^{(\varepsilon)}} \eta_r \right) \; d\xi dr + \int_0^T \left(\int_{\mathcal{O}} B(X_r) \left(e^{\mu_r^{(\varepsilon)}} \eta_r \right) \; d\xi \right) dz_r \\
&\quad - \int_{\mathcal{O}_T} X_r \eta_r e^{\mu_r^{(\varepsilon)}} \partial_r \mu^{(\varepsilon)} \; d\xi dr \\
&= \int_{\mathcal{O}} X_0 e^{\mu_0^{(\varepsilon)}} \eta_0 \; d\xi + \int_{\mathcal{O}_T} \Phi(X_r) \Delta \left(e^{\mu_r^{(\varepsilon)}} \eta_r \right) \; d\xi dr + \int_0^T \left(\int_{\mathcal{O}} B(X_r) \left(e^{\mu_r^{(\varepsilon)}} \eta_r \right) \; d\xi \right) dz_r \\
&\quad - \int_0^T \left(\int_{\mathcal{O}} B(X_r) \left(\eta_r e^{\mu_r^{(\varepsilon)}} \right) \; d\xi \right) dz_r^{(\varepsilon)}.
\end{aligned}$$

By continuity of the Riemann-Stieltjes integral with respect to the convergence $z^{(\varepsilon)} \to z$ specified above and uniform convergence of the integrands (cf. [16, Proposition 2.7]) we can take the limit $\varepsilon \to 0$ to obtain the assertion. The other implication follows by similar arguments.

2.2 Uniqueness of essentially bounded very weak solutions

We prove Theorem 1.3. Let $Y^{(1)}$, $Y^{(2)}$ be two essentially bounded very weak solutions to (0.3) with the same initial condition $Y_0 \in L^1(\mathcal{O})$ and let $Y = Y^{(1)} - Y^{(2)}$. Then

$$\begin{aligned}
\int_{\mathcal{O}_T} Y_r \partial_r \eta \; d\xi dr &= -\int_{\mathcal{O}_T} \left(\Phi(e^{-\mu_r} Y_r^{(1)}) - \Phi(e^{-\mu_r} Y_r^{(2)}) \right) \Delta(e^{\mu_r} \eta_r) \; d\xi dr \\
&= -\int_{\mathcal{O}_T} a_r Y_r \Delta(e^{\mu_r} \eta_r) \; d\xi dr,
\end{aligned}$$

for all $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $\{T\} \times \mathcal{O}$ and on $[0,T] \times \partial\mathcal{O}$, where

$$a_t := \begin{cases} \frac{\Phi(e^{-\mu_t} Y_t^{(1)}) - \Phi(e^{-\mu_t} Y_t^{(2)})}{Y_t^{(1)} - Y_t^{(2)}} & \text{, for } Y_t^{(1)} \neq Y_t^{(2)} \\ 0 & \text{, otherwise.} \end{cases}$$

Let $z^\varepsilon \in C^\infty([0,T];\mathbb{R}^N)$ with $z^{(\varepsilon)} \to z$ in $C([0,T];\mathbb{R}^N)$ such that for $\mu^{(\varepsilon)}$ as in (0.2) we have $\sup_{t\in[0,T]} \|e^{\mu_t^{(\varepsilon)}} - e^{\mu_t}\|_{C^2(\mathcal{O})} \leq \varepsilon^2$. By equicontinuity of $z^{(\varepsilon)}$ we can choose a partition $0 = \tau_0 < ... < \tau_N = T$ such that

$$\begin{aligned} \delta :=& \|e^{\mu}\big(2|\nabla(\mu^{(\varepsilon)} - \mu_{\tau_i})|^4 + 2|\Delta(\mu^{(\varepsilon)} - \mu_{\tau_i})|^2 + |\nabla(\mu^{(\varepsilon)} - \mu_{\tau_i})|^2\big)\|_{L^\infty([\tau_i,\tau_{i+1}]\times\mathcal{O})} \\ <& \frac{1}{16C}, \end{aligned} \tag{2.9}$$

for all $i = 0, ..., N-1$, $\varepsilon > 0$, where C is a constant that will be specified below. Let $\gamma = \max_i\{|\tau_{i+1} - \tau_i|\}$. We prove $Y = 0$ a.e. via induction over $i = 0, ..., N-1$. Thus, assume $Y = 0$ on $[0, \tau_i] \times \mathcal{O}$ almost everywhere. We can modify τ_i so that (2.9) is preserved and $Y(\tau_i) = 0$ a.e. in $\mathcal{O}$. Define $\mathcal{O}_i := [\tau_i, \tau_{i+1}] \times \mathcal{O}$. Then

$$\int_{\mathcal{O}_i} Y_r\big(\partial_r \eta_r + a_r \Delta(e^{\mu_r}\eta_r)\big)\, d\xi dr = 0,$$

for all $\eta \in C^{1,2}([\tau_i, \tau_{i+1}] \times \bar{\mathcal{O}})$ with $\eta = 0$ on $\{\tau_{i+1}\} \times \mathcal{O}$ and on $[\tau_i, \tau_{i+1}] \times \partial\mathcal{O}$.

For $Y_t^{(1)} \neq Y_t^{(2)}$ we have $a_t = e^{-\mu_t}\dot{\Phi}(\zeta_t)$ with $\zeta_t \in [e^{-\mu_t}Y_t^{(1)}, e^{-\mu_t}Y_t^{(2)}]$ and thus $\|a\|_{L^\infty(\mathcal{O}_T)} < \infty$ by essential boundedness of $Y^{(i)}$. We consider a non-degenerate, smooth approximation of a. Set $\hat{a}_\varepsilon := a \vee \varepsilon$ and let $a_{\varepsilon,\delta}$ be a smooth approximation of $\hat{a}_\varepsilon$ such that $a_{\varepsilon,\delta} \geq \varepsilon$ and $\int_{\mathcal{O}_T} |Y|^2(\hat{a}_\varepsilon - a_{\varepsilon,\delta})^2\, dxdr \leq \delta$. Then choose $a_\varepsilon = a_{\varepsilon,\varepsilon^2}$.

Let $\eta = e^{-\mu_{\tau_i}}\varphi$ with φ being the classical solution to

$$\begin{aligned} \partial_t\varphi + a_\varepsilon e^{\mu_{\tau_i}}\Delta(e^{\mu^{(\varepsilon)} - \mu_{\tau_i}}\varphi) - \theta = 0, &\text{ on } \mathcal{O}_T \\ \varphi = 0, &\text{ on } [\tau_i, \tau_{i+1}] \times \partial\mathcal{O} \\ \varphi(\tau_{i+1}) = 0, &\text{ on } \mathcal{O}, \end{aligned} \tag{2.10}$$

where θ is an arbitrary smooth testfunction. Time inversion transforms (2.10) into a uniformly parabolic linear equation with smooth coefficients. Thus, unique existence of classical solutions follows from [22, Theorem 6.2, p. 457]. Then

$$
\begin{aligned}
0 &= \int_{\mathcal{O}_i} Y_r\big(\partial_r\eta + a_r\Delta(e^{\mu_r}\eta_r)\big)\ d\xi dr \\
&= \int_{\mathcal{O}_i} Y_r\big(\partial_r\eta + a_{\varepsilon,r}\Delta(e^{\mu_r^{(\varepsilon)}}\eta)\big)\ d\xi dr + \int_{\mathcal{O}_i} Y_r(a_r - a_{\varepsilon,r})\Delta(e^{\mu_r^{(\varepsilon)}}\eta_r)\ d\xi dr \\
&\quad + \int_{\mathcal{O}_i} Y_r a_r \Delta((e^{\mu_r} - e^{\mu_r^{(\varepsilon)}})\eta_r)\ d\xi dr \qquad (2.11)\\
&= \int_{\mathcal{O}_i} e^{-\mu_{\tau_i}} Y_r\theta_r\ d\xi dr + \int_{\mathcal{O}_i} Y_r(a_r - a_{\varepsilon,r})\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)\ d\xi dr \\
&\quad + \int_{\mathcal{O}_i} Y_r a_r \Delta((e^{\mu_r} - e^{\mu_r^{(\varepsilon)}})e^{-\mu_{\tau_i}}\varphi_r)\ d\xi dr.
\end{aligned}
$$

We need to prove that the last two terms vanish for $\varepsilon \to 0$. For this we first derive a bound for $\int_{\mathcal{O}_i} a_{\varepsilon,r}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2\ d\xi dr$. Let $\zeta \in C^\infty(\mathbb{R})$ with $\zeta(\tau_i) = 0$, $\zeta \le 1$ on $[0,T]$ and $\dot\zeta \ge c > 0$, for some $c \le \frac{1}{4\gamma}$. Multiplying (2.10) by $\zeta\Delta\varphi$ and integrating yields

$$
\int_{\mathcal{O}_i} (\partial_r\varphi_r)\,\zeta_r\Delta\varphi_r\ d\xi dr = \int_{\mathcal{O}_i} \Big(-a_{\varepsilon,r}e^{\mu_{\tau_i}}\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi)\zeta_r\Delta\varphi_r + \theta_r\zeta_r\Delta\varphi_r\Big)\ d\xi dr.
$$

Note that

$$
\begin{aligned}
\Delta\varphi &= \Delta\left(e^{-(\mu^{(\varepsilon)}-\mu_{\tau_i})}e^{\mu^{(\varepsilon)}-\mu_{\tau_i}}\varphi\right) \\
&= \varphi(-|\nabla(\mu^{(\varepsilon)}-\mu_{\tau_i})|^2 - \Delta(\mu^{(\varepsilon)}-\mu_{\tau_i})) - 2\nabla(\mu^{(\varepsilon)}-\mu_{\tau_i})\nabla\varphi + e^{-(\mu^{(\varepsilon)}-\mu_{\tau_i})}\Delta e^{\mu^{(\varepsilon)}-\mu_{\tau_i}}\varphi.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}_i} |\nabla\varphi_r|^2\dot\zeta_r\ d\xi dr + \int_{\mathcal{O}_i} a_{\varepsilon,r}e^{2\mu_{\tau_i}-\mu_r^{(\varepsilon)}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2\zeta_r\ d\xi dr \\
&= \int_{\mathcal{O}_i} a_{\varepsilon,r}\zeta_r e^{\mu_{\tau_i}}|\varphi_r\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|\big(|\nabla(\mu_r^{(\varepsilon)}-\mu_{\tau_i})|^2 + |\Delta(\mu_r^{(\varepsilon)}-\mu_{\tau_i})|\big)\ d\xi dr \\
&\quad + \int_{\mathcal{O}_i} 2\left(a_{\varepsilon,r}\zeta_r e^{\mu_{\tau_i}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)||\nabla(\mu_r^{(\varepsilon)}-\mu_{\tau_i})||\nabla\varphi_r| + \theta_r\zeta_r\Delta\varphi_r\right)\ d\xi dr.
\end{aligned}
$$

The first term on the right hand side is bounded by

$$
\begin{aligned}
&\int_{\mathcal{O}_i} a_{\varepsilon,r}\zeta_r e^{\mu_{\tau_i}}|\varphi_r\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|\big(|\nabla(\mu_r^{(\varepsilon)}-\mu_{\tau_i})|^2 + |\Delta(\mu_r^{(\varepsilon)}-\mu_{\tau_i})|\big)\ d\xi dr \\
&\le \int_{\mathcal{O}_T} \frac{1}{4}a_{\varepsilon,r}\zeta_r e^{2\mu_{\tau_i}-\mu_r^{(\varepsilon)}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2 d\xi dr + C\delta\int_{\mathcal{O}_T} \dot\zeta_r|\nabla\varphi_r|^2\ d\xi dr
\end{aligned}
$$

and the second by

$$
\begin{aligned}
&\int_{\mathcal{O}_i} (2a_{\varepsilon,r}\zeta_r e^{\mu_{\tau_i}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)||\nabla(\mu_r^{(\varepsilon)}-\mu_{\tau_i})||\nabla\varphi_r| + \theta_r\zeta_r\Delta\varphi_r)\ d\xi dr \\
&\leq \int_{\mathcal{O}_i} \frac{1}{4}a_{\varepsilon,r}\zeta_r e^{2\mu_{\tau_i}-\mu_r^{(\varepsilon)}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2\ d\xi dr + (C\delta+\frac{1}{8})\int_{\mathcal{O}_i}\dot{\zeta}_r|\nabla\varphi_r|^2\ d\xi dr + C\int_{\mathcal{O}_i}|\nabla\theta_r|^2\ d\xi dr.
\end{aligned}
$$

Using this we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}_i}|\nabla\varphi_r|^2\dot{\zeta}_r\ d\xi dr + \int_{\mathcal{O}_i} a_{\varepsilon,r}e^{2\mu_{\tau_i}-\mu_r^{(\varepsilon)}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2\zeta_r\ d\xi dr \\
&\leq \int_{\mathcal{O}_i} \frac{1}{2}a_{\varepsilon,r}\zeta_r e^{2\mu_{\tau_i}-\mu_r^{(\varepsilon)}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2\ d\xi dr + (2C\delta+\frac{1}{8})\int_{\mathcal{O}_i}\dot{\zeta}_r|\nabla\varphi_r|^2\ d\xi dr + C\int_{\mathcal{O}_i}|\nabla\theta_r|^2\ d\xi dr,
\end{aligned}
$$

where $C = C(\|Y^{(i)}\|_{L^\infty([0,T]\times\mathcal{O})}, T, \|e^\mu\|_{L^\infty(\mathcal{O}_T)})$ is a generic constant. Since C is independent of the choice of ζ, using Fatou's Lemma and (2.9) we obtain

$$
\int_{\mathcal{O}_i} a_{\varepsilon,r}e^{2\mu_{\tau_i}-\mu_r^{(\varepsilon)}}|\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2 d\xi dr \leq C\int_{\mathcal{O}_i}|\nabla\theta_r|^2\ d\xi dr. \tag{2.12}
$$

By the choice a_ε we have

$$
\begin{aligned}
\int_{\mathcal{O}_i}|Y_r|^2\frac{(a_r-a_{\varepsilon,r})^2}{a_{\varepsilon,r}}\ d\xi dr &\leq \frac{1}{\varepsilon}\left(\int_{\mathcal{O}_i}2|Y_r|^2(a_r-\hat{a}_{\varepsilon,r})^2\ d\xi dr + \int_{\mathcal{O}_i}2|Y_r|^2(\hat{a}_{\varepsilon,r}-a_{\varepsilon,r})^2\ d\xi dr\right) \\
&\leq 4\varepsilon\int_{\mathcal{O}_i}|Y_r|^2\ d\xi dr.
\end{aligned}
$$

For the second term in (2.11) we obtain

$$
\begin{aligned}
&\int_{\mathcal{O}_i} Y_r(a_r-a_{\varepsilon,r})\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)\ d\xi dr \\
&\quad\leq \left(\int_{\mathcal{O}_i}|a_{\varepsilon,r}||\Delta(e^{\mu_r^{(\varepsilon)}-\mu_{\tau_i}}\varphi_r)|^2\ d\xi dr\right)^{\frac{1}{2}}\left(\int_{\mathcal{O}_i}|Y_r|^2\frac{(a_r-a_{\varepsilon,r})^2}{a_{\varepsilon,r}}\ d\xi dr\right)^{\frac{1}{2}} \\
&\quad\leq C\sqrt{\varepsilon}\|\nabla\theta\|_{L^2(\mathcal{O}_i)}\|Y\|_{L^2(\mathcal{O}_i)} \to 0,
\end{aligned}
$$

for $\varepsilon \to 0$. For the third term in (2.11) we use (2.12) and $a_\varepsilon \geq \varepsilon$ to get

$$
\|\varphi\|_{H^2(\mathcal{O})} \leq \frac{C\|\nabla\theta\|_{L^2(\mathcal{O}_i)}}{\varepsilon}.
$$

Hence

$$
\begin{aligned}
\int_{\mathcal{O}_i} Y_r a_r\Delta((e^{\mu_r}-e^{\mu_r^{(\varepsilon)}})e^{-\mu_{\tau_i}}\varphi_r)\ d\xi dr &\leq C\|\nabla\theta\|_{L^2(\mathcal{O}_i)}\frac{\|e^{\mu_r}-e^{\mu_r^{(\varepsilon)}}\|_{C^2(\mathcal{O})}}{\varepsilon} \\
&\leq \varepsilon C\|\nabla\theta\|_{L^2(\mathcal{O}_i)} \to 0,
\end{aligned}
$$

for $\varepsilon \to 0$. Taking $\varepsilon \to 0$ in in (2.11) yields

$$0 = \int_{\mathcal{O}_i} e^{-\mu_{\tau_i}} Y_r \theta_r \; d\xi dr,$$

for any smooth testfunction θ. Thus $Y = 0$ in $\mathcal{O}_i = [\tau_i, \tau_{i+1}] \times \mathcal{O}$ almost everywhere. Induction now finishes the proof.

Remark 2.1. *The method to prove uniqueness used above fails for fast diffusion equations, since the difference quotient*

$$a_t := \begin{cases} \frac{\Phi(e^{-\mu_t} Y_t^{(1)}) - \Phi(e^{-\mu_t} Y_t^{(2)})}{Y_t^{(1)} - Y_t^{(2)}} & , \text{ for } Y_t^{(1)} \neq Y_t^{(2)} \\ 0 & , \text{ otherwise.} \end{cases}$$

it not known to remain bounded.

2.3 Weak solutions and uniform bounds

We will now prove Theorem 1.4. In order to construct weak solutions to (0.3) several steps are needed. First we will consider approximating equations, where the degenerate nonlinearity Φ is replaced by non-degenerate functions $\Phi^{(\delta)}$ and the driving signals z are approximated by smooth signals $z^{(\delta)}$ (Section 2.3.1). Existence of classical solutions to these equations follows from well-known existence results (cf. e.g. [22]). Then we will prove uniform L^∞ bounds for these approximating solutions (Section 2.3.2) which will be used in Section 2.3.3 to finally construct weak solutions to (0.3) by monotonicity methods.

2.3.1 Non-degenerate, smooth approximation and classical solutions

For $\delta > 0$ we choose an approximating function $\Phi^{(\delta)} \in C^\infty(\mathbb{R})$ such that

i. $\Phi^{(\delta)}(0) = 0$ and $\Phi^{(\delta)}$ is anti-symmetric in 0,

ii. $\Phi^{(\delta)}(r) = \Phi(r)$, for all $\delta \le |r| \le \frac{1}{\delta}$,

iii. For all $r \in \mathbb{R}$:

$$\begin{aligned} 0 < C_1(\delta) \le \dot{\Phi}^{(\delta)}(r) &\le C_2(\delta) < \infty, \\ \ddot{\Phi}^{(\delta)}(r) &\le C_2(\delta) < \infty. \end{aligned}$$

In particular $\Phi^{(\delta)}(r) = \int_0^r \dot{\Phi}^{(\delta)}(s)ds \le C_2(\delta)r$. We further choose smooth approximations $z^{(\delta)} \in C^\infty([0,T];\mathbb{R}^N)$ of the driving signal z. Using the homogeneity of Φ we can rewrite (0.3) as

$$\partial_t Y_t = e^{\mu_t} \Delta \left(\Phi(e^{-\mu_t}) \Phi(Y_t) \right), \text{ on } \mathcal{O}_T. \tag{2.13}$$

One advantage of rewriting (0.3) in this form prior to approximating Φ by $\Phi^{(\delta)}$ is that the substitution $Z^{(\delta)} := \Phi^{(\delta)}(Y^{(\delta)})$ can still be used in the approximating equation so that the continuity results obtained in [14] can be applied. We construct a solution to (2.13) by considering approximating equations

$$\begin{aligned} \partial_t Y_t^{(\delta)} &= e^{\mu_t}\Delta\left(\Phi(e^{-\mu_t})\Phi^{(\delta)}(Y_t^{(\delta)})\right), \text{ on } \mathcal{O}_T \\ Y^{(\delta)}(0) &= Y_0, \text{ on } \mathcal{O}, \end{aligned} \tag{2.14}$$

with Dirichlet boundary conditions and smooth signals $z \in C^\infty([0,T];\mathbb{R}^N)$. (2.14) is a quasilinear, uniformly parabolic equation with smooth coefficients. From standard results the unique existence of a classical solution follows (cf. e.g. [22, Theorem 6.2, p. 457]).

2.3.2 Uniform $L^\infty(\mathcal{O}_T)$ bound for classical solutions to (2.14)

Lemma 2.2. *Let $Y_0 \in L^\infty(\mathcal{O})$, $\{z^{(\varepsilon)} \in C^\infty([0,T];\mathbb{R}^N) | \, \varepsilon > 0\}$ be a compact set in $C([0,T];\mathbb{R}^N)$ and $Y^{(\delta,\varepsilon)}$ be a classical solution to (2.14) driven by $z^{(\varepsilon)}$. There are constants $\sigma_0 = \sigma_0(\|Y_0\|_{L^\infty(\mathcal{O})}) > 0$, $M > 0$ depending only on $\|Y_0\|_{L^\infty(\mathcal{O})}$, the uniform bound and uniform modulus of continuity of $\{z^{(\varepsilon)}\}$, piecewise smooth functions $K^{(\sigma_0,\varepsilon)}$ and a $\delta_0 = \delta_0(\sup_{\varepsilon>0} \|z^{(\varepsilon)}\|_{L^\infty(\mathcal{O}_T)}, \|Y_0\|_{L^\infty(\mathcal{O})}) > 0$ such that*

$$Y^{(\delta,\varepsilon)} \le K^{(\sigma_0,\varepsilon)} \le M, \text{ on } [0,T]\times\mathcal{O},$$

for all $\delta \le \delta_0$.

Proof. We will construct a piecewise smooth (thus bounded) supersolution to

$$\partial_t Y_t^{(\delta,\varepsilon)} = e^{\mu_t^{(\varepsilon)}}\Delta\left(\Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\right), \text{ on } \mathcal{O}_T, \tag{2.15}$$

with initial condition Y_0 and Dirichlet boundary conditions. Let $R > 0$ such that $\bar{\mathcal{O}} \subseteq B_R(0)$. Since $\{z^{(\varepsilon)}\}$ is a set of equicontinuous functions, there exists a $\gamma > 0$ and a partition $0 = \tau_0 < \tau_1 < ... < \tau_L = T$ with $1 > \tau_i - \tau_{i-1} > \gamma$ (hence $L \le \frac{T}{\gamma}$) such that

$$\begin{aligned} \frac{1}{2} \le &\left(\inf_{\substack{\xi\in\bar{\mathcal{O}} \\ t\in[\tau_i,\tau_{i+1})}} e^{\mu_t^{(\varepsilon)}-\mu_{\tau_i}^{(\varepsilon)}}\Phi(e^{\mu_{\tau_i}^{(\varepsilon)}-\mu_t^{(\varepsilon)}})\right)\left(1-\frac{mR}{d}\sup_{t\in[\tau_i,\tau_{i+1})}\Big(2\|\nabla(\mu_{\tau_i}^{(\varepsilon)}-\mu_t^{(\varepsilon)})\|_{L^\infty(\mathcal{O})}\right. \\ &\left.+\frac{Rm}{2}\|\nabla(\mu_{\tau_i}^{(\varepsilon)}-\mu_t^{(\varepsilon)})\|^2_{L^\infty(\mathcal{O})}+\frac{R}{2}\|\Delta(\mu_{\tau_i}^{(\varepsilon)}-\mu_t^{(\varepsilon)})\|_{L^\infty(\mathcal{O})}\Big)\right), \end{aligned} \tag{2.16}$$

and

$$\frac{1}{2} \le \inf_{\xi\in\bar{\mathcal{O}},\ t\in[\tau_i,\tau_{i+1})} e^{(m-1)(\mu_t^{(\varepsilon)}-\mu_{\tau_i}^{(\varepsilon)})}, \tag{2.17}$$

for all $i = 0, ..., L-1$, $\varepsilon > 0$. Let $A^{\frac{m-1}{m}} := \frac{R^{\frac{2}{m}}}{(m-1)d}$, $C_4 := \inf_{\xi\in\mathcal{O}}(R^2 - |\xi|^2)$ and consider the inverse $\beta := \Phi^{-1}$. For $\sigma > 0$ we define

$$K_0^{(\sigma,\varepsilon)}(t,\xi) := \beta\left(A(t+\sigma)^{-\frac{m}{m-1}}(R^2 - |\xi|^2)\Phi(e^{\mu^{(\varepsilon)}_{\tau_i}})\right)$$

and choose $\sigma_0 = \sigma_0(\|Y_0\|_{L^\infty(\mathcal{O})})$ so that $\|Y_0\|_{L^\infty(\mathcal{O})} \le K_0^{(\sigma_0,\varepsilon)}(0)$. Then inductively define $\sigma_{i+1} = \frac{1}{2}(\sigma_i + \gamma)$ for $i = 0, \dots, L-1$ (we can thus regard σ_i as a function of σ_0) and let

$$\begin{aligned} K_i^{(\sigma_0,\varepsilon)}(t,\xi) &:= \beta\left(A(t-\tau_i+\sigma_i)^{-\frac{m}{m-1}}(R^2 - |\xi|^2)\Phi(e^{\mu^{(\varepsilon)}_{\tau_i}})\right) \\ &= A^{\frac{1}{m}}(t-\tau_i+\sigma_i)^{-\frac{1}{m-1}}(R^2-|\xi|^2)^{\frac{1}{m}}e^{\mu^{(\varepsilon)}_{\tau_i}}, \quad t\in[\tau_i,\tau_{i+1}],\ \xi\in\mathcal{O}. \end{aligned} \tag{2.18}$$

By the choice of σ_i, $i = 1, \dots, L-1$ we have $K_i^{(\sigma_0,\varepsilon)}(\tau_{i+1}) \le K_{i+1}^{(\sigma_0,\varepsilon)}(\tau_{i+1})$. We note

$$\begin{aligned} &A^{\frac{1}{m}}\left(1+\max_{i=0,\dots,L-1}\sigma_i\right)^{-\frac{1}{m-1}} C_4^{\frac{1}{m}} e^{-\sup_{\varepsilon>0}\|\mu^{(\varepsilon)}\|_{L^\infty(\mathcal{O}_T)}} \\ &\le K_i^{(\sigma_0,\varepsilon)}(t) \le A^{\frac{1}{m}}\left(\min_{i=0,\dots,L-1}\sigma_i\right)^{-\frac{1}{m-1}} R^{\frac{2}{m}} e^{\sup_{\varepsilon>0}\|\mu^{(\varepsilon)}\|_{L^\infty(\mathcal{O}_T)}}, \end{aligned} \tag{2.19}$$

for all $t \in [\tau_i, \tau_{i+1}]$. Hence, we can choose $\delta_0 > 0$ (depending only on σ_0, $\sup_{\varepsilon>0}\|z^{(\varepsilon)}\|_{L^\infty(\mathcal{O}_T)}$) such that

$$K_i^{(\sigma_0,\varepsilon)}(t) \in [\delta, \frac{1}{\delta}], \tag{2.20}$$

for all $t \in [\tau_i, \tau_{i+1}]$ and $\delta \le \delta_0$. Then $\Phi^{(\delta)}(K_i(t)) = \Phi(K_i(t))$ and we compute (for simplicity we drop the ε dependencies and the σ_0 dependency of K_i)

$$\begin{aligned} &\Delta\left(\Phi(e^{-\mu_t})\Phi^{(\delta)}(K_i(t))\right) = \Delta\left(A(t-\tau_i+\sigma_i)^{-\frac{m}{m-1}}(R^2-|\xi|^2)\Phi(e^{\mu_{\tau_i}-\mu_t})\right) \\ &= A(t-\tau_i+\sigma_i)^{-\frac{m}{m-1}}\Phi(e^{\mu_{\tau_i}-\mu_t})\Big(-2d - 4m\xi\cdot\nabla(\mu_{\tau_i}-\mu_t) \\ &\qquad + (R^2-|\xi|^2)(m^2|\nabla(\mu_{\tau_i}-\mu_t)|^2 + m\Delta(\mu_{\tau_i}-\mu_t))\Big) \end{aligned}$$

and

$$\partial_t K_i(t) = -\frac{A^{\frac{1}{m}}}{m-1}(t-\tau_i+\sigma_i)^{-\frac{m}{m-1}}(R^2-|\xi|^2)^{\frac{1}{m}}e^{\mu_{\tau_i}}.$$

In order to show that $K_i(t)$ is a supersolution to (2.15) on $[\tau_i, \tau_{i+1}]$ we thus have to show

$$\begin{aligned} 0 &\le \partial_t K_i(t) - e^{\mu_t}\Delta\left(\Phi(e^{-\mu_t})\Phi^{(\delta)}(K_i(t))\right) \\ &= -\frac{A^{\frac{1}{m}}}{m-1}(t-\tau_i+\sigma_i)^{-\frac{m}{m-1}}(R^2-|\xi|^2)^{\frac{1}{m}}e^{\mu_{\tau_i}} - A(t-\tau_i+\sigma_i)^{-\frac{m}{m-1}}e^{\mu_t}\Phi(e^{\mu_{\tau_i}-\mu_t}) \\ &\quad \Big(-2d - 4m\xi\cdot\nabla(\mu_{\tau_i}-\mu_t) + (R^2-|\xi|^2)\left(m^2|\nabla(\mu_{\tau_i}-\mu_t)|^2 + m\Delta(\mu_{\tau_i}-\mu_t)\right)\Big), \end{aligned}$$

for all $t \in [\tau_i, \tau_{i+1}]$. Equivalently

$$\frac{(R^2-|\xi|^2)^{\frac{1}{m}}}{m-1} \le A^{\frac{m-1}{m}} e^{\mu_t-\mu_{\tau_i}} \Phi(e^{\mu_{\tau_i}-\mu_t}) \Big(2d + 4m\xi \cdot \nabla(\mu_{\tau_i}-\mu_t) \\ - (R^2-|\xi|^2) \left(m^2 |\nabla(\mu_{\tau_i}-\mu_t)|^2 + m\Delta(\mu_{\tau_i}-\mu_t) \right) \Big).$$

It is thus sufficient to show

$$\frac{R^{\frac{2}{m}}}{m-1} \le A^{\frac{m-1}{m}} \left(\inf_{\substack{\xi\in\bar{\mathcal{O}} \\ t\in[\tau_i,\tau_{i+1}]}} e^{\mu_t-\mu_{\tau_i}} \Phi(e^{\mu_{\tau_i}-\mu_t}) \right) \Big(2d - 4mR \|\nabla(\mu_{\tau_i}-\mu_t)\|_{L^\infty(\mathcal{O})} \\ - R^2 \left(m^2 \|\nabla(\mu_{\tau_i}-\mu_t)\|^2_{L^\infty(\mathcal{O})} + m\|\Delta(\mu_{\tau_i}-\mu_t)\|_{L^\infty(\mathcal{O})} \right) \Big),$$

for all $t \in [\tau_i, \tau_{i+1}]$, which is satisfied by the choice of A and τ_i in (2.16). In conclusion, $K_i^{(\sigma_0,\varepsilon)}(t)$ is a supersolution to (2.15) on $[\tau_i, \tau_{i+1}]$ for each $\delta \le \delta_0$. We define

$$K^{(\sigma_0,\varepsilon)}(t) := \sum_{i=0}^{L-1} \mathbb{1}_{[\tau_i,\tau_{i+1})}(t) K_i^{(\sigma_0,\varepsilon)}(t). \tag{2.21}$$

Since the comparison principle [24, Theorem 9.7] applies on each interval $[\tau_i, \tau_{i+1}]$, by induction we have

$$Y^{(\delta,\varepsilon)}(t,\xi) \le K^{(\sigma_0,\varepsilon)}(t,\xi), \quad \forall t \in [0,T],\ \xi \in \mathcal{O},\ \delta \le \delta_0.$$

The upper bound in (2.19) yields a uniform bound M for $K^{(\sigma_0,\varepsilon)}$. M depends on σ_0, $\sup_\varepsilon \|z^{(\varepsilon)}\|_{L^\infty(\mathcal{O})}$ and via the bound of the partition size γ and the definition of σ_i, on the uniform modulus of continuity of $\{z^{(\varepsilon)}\}$. □

2.3.3 Existence of weak solutions

We will now take the limit $\delta \to 0$ in (2.14) in order to obtain weak solutions to (0.3) in the sense of Definition 1.1.

Lemma 2.3. *Let $Y_0 \in L^\infty(\mathcal{O})$, $\{z^{(\varepsilon)} \in C^\infty([0,T];\mathbb{R}^N) |\ \varepsilon > 0\} \subseteq C([0,T];\mathbb{R}^N)$ be compact and $Y^{(\delta,\varepsilon)}$ be a classical solution to (2.14) driven by $z^{(\varepsilon)}$. Then*

$$\sup_{t\in[0,T]} \left(\|Y_t^{(\delta,\varepsilon)}\|_{m+1}^{m+1} + \|Y_t^{(\delta,\varepsilon)}\|_H^2 \right) + C_1 \left\|\nabla \left(\Phi(e^{-\mu^{(\varepsilon)}}) \Phi^\delta(Y^{(\delta,\varepsilon)}) \right) \right\|_{L^2(\mathcal{O}_T)} \le C_2, \tag{2.22}$$

for all $\varepsilon > 0$, $\delta \le \delta_0$ (with δ_0 from Lemma 2.2) and for some constants $0 < C_1, C_2$ independent of δ and ε. C_2 may depend on $\|Y_0\|_{L^\infty(\mathcal{O})}$, the uniform bound and the uniform modulus of continuity of $\{z^{(\varepsilon)}\}$.

Proof. Let $\Psi^{(\delta)} \in C^1(\mathbb{R})$ so that $\dot{\Psi}^{(\delta)} = \Phi^{(\delta)}$. We compute

$$
\begin{aligned}
\partial_t \int_{\mathcal{O}} \Psi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\, d\xi &= \int_{\mathcal{O}} \frac{\Phi(e^{-\mu_t^{(\varepsilon)}})}{\Phi(e^{-\mu_t^{(\varepsilon)}})} \Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)}) \partial_t Y^{(\delta,\varepsilon)}\, d\xi \\
&= -\int_{\mathcal{O}} \frac{e^{\mu_t^{(\varepsilon)}}}{\Phi(e^{-\mu_t^{(\varepsilon)}})} \nabla \left(\Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\right) \nabla \left(\Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\right) d\xi \\
&\quad - \int_{\mathcal{O}} \Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)}) \nabla \left(\frac{e^{\mu_t^{(\varepsilon)}}}{\Phi(e^{-\mu_t^{(\varepsilon)}})}\right) \nabla \left(\Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\right)\, d\xi \qquad (2.23) \\
&\leq \sup_{(t,\xi)\in\bar{\mathcal{O}}_T} \left(\varepsilon_1 \left|\nabla \frac{e^{\mu_t^{(\varepsilon)}}}{\Phi(e^{-\mu_t^{(\varepsilon)}})}\right|^2 - \frac{e^{\mu_t^{(\varepsilon)}}}{\Phi(e^{-\mu_t^{(\varepsilon)}})}\right) \int_{\mathcal{O}} |\nabla \Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})|^2\, d\xi \\
&\quad + C_{\varepsilon_1} \int_{\mathcal{O}} \left(\Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\right)^2 d\xi,
\end{aligned}
$$

for all $\varepsilon_1 > 0$ and some $C_{\varepsilon_1} > 0$. Choosing ε_1 small enough and using the uniform L^∞ bound derived in Lemma 2.2 we conclude

$$
\sup_{t\in[0,T]} \int_{\mathcal{O}} \Psi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\, d\xi + C_1 \int_{\mathcal{O}_T} |\nabla \Phi(e^{-\mu_r^{(\varepsilon)}})\Phi^{(\delta)}(Y_r^{(\delta,\varepsilon)})|^2\, d\xi dr \leq \int_{\mathcal{O}} \Psi^{(\delta)}(Y_0)\, d\xi + C_2,
$$

for all $\delta \leq \delta_0$ and for some constants $C_1, C_2 > 0$ independent of δ and ε, where C_2 may depend on $\|Y_0\|_{L^\infty(\mathcal{O})}$, the uniform bound and the uniform modulus of continuity of $\{z^{(\varepsilon)}\}$.

It remains to prove the bound of $\|Y^{(\delta)}\|_H^2$. By the chain rule we have

$$
\frac{d}{dt}\|Y_t^{(\delta,\varepsilon)}\|_H^2 = 2\int_{\mathcal{O}} (-\Delta)^{-1} \left(Y_t^{(\delta,\varepsilon)}\right) e^{\mu_t^{(\varepsilon)}} \Delta \left(\Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)})\right) d\xi.
$$

Since for f, g, h sufficiently smooth and $h_{|\partial\mathcal{O}} = 0$ we have

$$
\int_{\mathcal{O}} fg\Delta h\, d\xi = \int_{\mathcal{O}} (f\Delta(gh) + 2h\nabla f \cdot \nabla g + fh\Delta(g))\, d\xi.
$$

We obtain

$$
\begin{aligned}
\frac{d}{dt}\|Y_t^{(\delta,\varepsilon)}\|_H^2 &= -2\int_{\mathcal{O}} Y_t^{(\delta,\varepsilon)} e^{\mu_t^{(\varepsilon)}} \Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)}) d\xi \\
&\quad + 2\int_{\mathcal{O}} \Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)}) \left(2\nabla \left((-\Delta)^{-1}(Y_t^{(\delta,\varepsilon)})\right) \cdot \nabla \left(e^{\mu_t^{(\varepsilon)}}\right)\right) d\xi \qquad (2.24) \\
&\quad + 2\int_{\mathcal{O}} \Phi(e^{-\mu_t^{(\varepsilon)}})\Phi^{(\delta)}(Y_t^{(\delta,\varepsilon)}) \left((-\Delta)^{-1}(Y_t^{(\delta,\varepsilon)}) \Delta \left(e^{\mu_t^{(\varepsilon)}}\right)\right) d\xi \\
&\leq C(1 + \|Y_t^{(\delta,\varepsilon)}\|_H^2), \quad \forall \delta \leq \delta_0
\end{aligned}
$$

where $0 < C$ is a constant independent of δ, ε, possibly depending on $\|Y_0\|_{L^\infty(\mathcal{O})}$, the uniform bound and the uniform modulus of continuity of $\{z^{(\varepsilon)}\}$. Gronwall's inequality then yields the bound. $\square$

Proof of Theorem 1.4. We approximate the initial condition Y_0 by smooth functions $Y_0^{(\delta)} \in C^2(\bar{\mathcal{O}})$ such that $Y_0^{(\delta)} \to Y_0$ almost everywhere and $\|Y_0^{(\delta)}\|_{L^\infty(\mathcal{O})} \le \|Y_0\|_{L^\infty(\mathcal{O})}$. The continuous driving signal z is approximated by smooth signals $z^{(\delta)} \in C^\infty([0,T];\mathbb{R}^N)$ such that $z^{(\delta)} \to z$ in $C([0,T];\mathbb{R}^N)$. In particular $\{z^{(\delta)} |\ \delta > 0\}$ is a compact set in $C([0,T];\mathbb{R}^N)$. Let $Y^{(\delta)}$ be classical solutions to (2.14) with initial condition $Y_0^{(\delta)}$ and driving signal $z^{(\delta)}$. In the following let $\delta \le \delta_0$ with δ_0 as in Lemma 2.3.

By Lemma 2.3 we know that $Y^{(\delta)}$ is uniformly bounded in $L^\infty([0,T];L^{m+1}(\mathcal{O}))$ and in $L^\infty([0,T];H)$. By Sobolev embedding, for $k \ge \frac{n}{2}(\frac{1-m}{1+m}) \vee 1$ we have $H_0^k(\mathcal{O}) \hookrightarrow L^{\frac{m+1}{m}}(\mathcal{O})$. Consequently, $L^{m+1}(\mathcal{O}) \hookrightarrow H^{-k} := (H_0^k(\mathcal{O}))^*$ and $H \hookrightarrow H^{-k}$. Hence, weak* limits obtained in $L^\infty([0,T];L^{m+1}(\mathcal{O}))$ and $L^\infty([0,T];H)$ coincide.
We further know that $\Phi(e^{-\mu_t^{(\delta)}})\Phi^{(\delta)}(Y^{(\delta)})$ is uniformly bounded in $L^2([0,T];H_0^1(\mathcal{O}))$ and boundedness of $Y^{(\delta)}$ in $L^\infty([0,T];L^{m+1}(\mathcal{O}))$ implies boundedness of $\Phi(e^{-\mu_t^{(\delta)}})\Phi^{(\delta)}(Y^{(\delta)})$ in $L^\infty([0,T];L^{\frac{m+1}{m}}(\mathcal{O}))$.

Hence, we can choose a subsequence (again denoted by δ) such that

$$\begin{aligned} Y^{(\delta)} &\rightharpoonup^* Y, \text{ in } L^\infty([0,T];L^{m+1}(\mathcal{O})) \text{ and in } L^\infty([0,T];H), \\ Z^{(\delta)} := \Phi(e^{-\mu_t^{(\delta)}})\Phi^{(\delta)}(Y^{(\delta)}) &\rightharpoonup Z, \text{ in } L^2([0,T];H_0^1(\mathcal{O})), \\ Z^{(\delta)} &\rightharpoonup^* Z, \text{ in } L^\infty([0,T];L^{\frac{m+1}{m}}(\mathcal{O})). \end{aligned}$$

Since

$$-\int_{\mathcal{O}_T} Y_r^{(\delta)}\partial_r\eta_r \, d\xi dr - \int_{\mathcal{O}} Y_0^{(\delta)}\eta_0 \, d\xi = -\int_{\mathcal{O}_T} \nabla\left(\Phi(e^{-\mu_r^{(\delta)}})\Phi^{(\delta)}(Y_r^{(\delta)})\right)\nabla\left(e^{\mu_r^{(\delta)}}\eta_r\right) \, d\xi dr,$$

we obtain

$$-\int_{\mathcal{O}_T} Y_r\partial_r\eta \, d\xi dr - \int_{\mathcal{O}} Y_0\eta_0 \, d\xi = -\int_{\mathcal{O}_T} \nabla Z_r \nabla\left(e^{\mu_r}\eta_r\right) \, d\xi dr,$$

for all $\eta \in C^1(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $[0,T]\times\partial\mathcal{O}$ and on $\{T\}\times\mathcal{O}$.

First we will prove that $Y_t^{(\delta)} \rightharpoonup Y_t$ in H, for all $t \in [0,T]$. We consider the set $\mathcal{K} = \{(Y^{(\delta)},h)_H |\ h \in H,\ \|h\|_H \le 1,\ \delta > 0\} \subseteq C([0,T])$. By Lemma 2.3, $\mathcal{K}$ is bounded in $C([0,T])$. Moreover,

$$(Y_{t+s}^{(\delta)} - Y_t^{(\delta)}, h)_H = \int_t^{t+s}\left(\frac{dY^{(\delta)}}{dr}, h\right)_H dr \le \|h\|_H s^{\frac{1}{2}}\left\|\frac{dY^{(\delta)}}{dr}\right\|_{L^2([0,T];H)} \le C\|h\|_H s^{\frac{1}{2}}.$$

Hence, $\mathcal{K}$ is a set of equibounded, equicontinuous functions and thus is relatively compact in $C([0,T])$. For every $h \in H$, $\|h\|_H \le 1$ there is a subsequence (again denoted by δ) such that $(Y^{(\delta)},h)_H \to g$ in $C([0,T])$. Since also $Y^{(\delta)} \rightharpoonup Y$ in $L^2([0,T];H)$ (thus $(Y^{(\delta)},h)_H \rightharpoonup (Y,h)_H$ in $L^2([0,T])$) we have $g = (Y,h)$ which implies $Y_t^{(\delta)} \rightharpoonup Y_t$ in H for all $t \in [0,T]$.

We need to prove $Z = \Phi(e^{-\mu_t} Y)$ almost everywhere. This will be done by considering the equation on $H = (H_0^1(\mathcal{O}))^*$. Since $Y^{(\delta)}$ solves (2.14) we conclude

$$\frac{dY^{(\delta)}}{dt} \rightharpoonup \frac{dY}{dt}, \text{ in } L^2([0,T];H),$$

and

$$\begin{aligned} \frac{dY}{dt} &= e^{\mu_t}\Delta Z, \text{ for a.e. } t \in [0,T] \\ Y(0) &= Y_0. \end{aligned}$$

In particular, since also $Y \in L^\infty([0,T];H)$ we have $Y \in C([0,T];H)$. By the chain rule we obtain

$$\begin{aligned} \|Y_t\|_H^2 = \|Y_0\|_H^2 &- 2\int_0^t \int_{\mathcal{O}} e^{\mu_r} Z_r Y_r d\xi\, dr \\ &+ 2\int_0^t \int_{\mathcal{O}} Z_r \Big(2\nabla(e^{\mu_r})\nabla((-\Delta)^{-1}(Y_r)) + \Delta(e^{\mu_r})(-\Delta)^{-1}(Y_r)\Big) d\xi\, dr. \end{aligned} \tag{2.25}$$

Applying the chain rule to (2.14) yields

$$\begin{aligned} \|Y_t^{(\delta)}\|_H^2 = \|Y_0^{(\delta)}\|_H^2 &- 2\int_0^t \int_{\mathcal{O}} e^{\mu_r^{(\delta)}} Z_r^{(\delta)} Y_r^{(\delta)} d\xi\, dr \\ + 2\int_0^t \int_{\mathcal{O}} Z_r^{(\delta)} &\Big(2\nabla(e^{\mu_r^{(\delta)}})\nabla((-\Delta)^{-1}(Y_r^{(\delta)})) + \Delta(e^{\mu_r^{(\delta)}})(-\Delta)^{-1}(Y_r^{(\delta)})\Big) d\xi\, dr. \end{aligned} \tag{2.26}$$

Since $(-\Delta)^{-1}(Y^{(\delta)}) \in L^2([0,T];H_0^1(\mathcal{O}))$ and $\frac{d(-\Delta)^{-1}(Y^{(\delta)})}{dt} \in L^2([0,T];H_0^1(\mathcal{O})) \subseteq L^2([0,T];L^2(\mathcal{O}))$ are uniformly bounded and $H_0^1(\mathcal{O}) \hookrightarrow\hookrightarrow L^2(\mathcal{O}) \hookrightarrow L^2(\mathcal{O})$, by the Aubin-Lions compactness Theorem we have (for a subsequence again denoted by δ)

$$(-\Delta)^{-1}(Y^{(\delta)}) \to (-\Delta)^{-1}(Y), \text{ strongly in } L^2([0,T];L^2(\mathcal{O})).$$

Note that also $Z^{(\delta)} \rightharpoonup Z$ in $L^2([0,T];H_0^1(\mathcal{O}))$. Taking the limit $\delta \to 0$ in (2.26) yields

$$\begin{aligned} \|Y_t\|_H^2 \le \|Y_0\|_H^2 &- \limsup_{\delta\to 0} 2\int_0^t \int_{\mathcal{O}} e^{\mu_r^{(\delta)}} Z_r^{(\delta)} Y_r^{(\delta)}\, d\xi dr \\ &+ 2\int_0^t \int_{\mathcal{O}} Z_r \Big(2\nabla(e^{\mu_r})\nabla((-\Delta)^{-1}(Y_r)) + \Delta(e^{\mu_r})(-\Delta)^{-1}(Y_r)\Big)\, d\xi dr. \end{aligned}$$

Substracting (2.25) we arrive at

$$\limsup_{\delta\to 0} \int_{\mathcal{O}_T} e^{\mu_r^{(\delta)}} Z_r^{(\delta)} Y_r^{(\delta)} d\xi\, dr \le \int_{\mathcal{O}_T} e^{\mu_r} Z_r Y_r\, d\xi dr. \tag{2.27}$$

By monotonicity of $\Phi^{(\delta)}$ we have

$$\int_{\mathcal{O}_T} e^{\mu_r^{(\delta)}}\Phi(e^{-\mu_r^{(\delta)}})(\Phi^{(\delta)}(Y_r^{(\delta)})-\Phi^{(\delta)}(z_r))(Y_r^{(\delta)}-z_r)\ d\xi dr \geq 0,$$

for all $z \in C^1(\bar{\mathcal{O}}_T)$. Using (2.27) we can take $\delta \to 0$ to obtain

$$\int_{\mathcal{O}_T} e^{\mu_r}(Z_r - \Phi(e^{-\mu_r})\Phi(z_r))(Y_r - z_r)\ d\xi dr \geq 0,$$

for all $z \in C^1(\bar{\mathcal{O}}_T)$, hence by approximation for all $z \in L^{m+1}(\mathcal{O}_T)$. Taking $z = Y - \varepsilon h$ with $h \in C^0(\bar{\mathcal{O}}_T)$, dividing by ε and letting $\varepsilon \to 0$ yields

$$\int_{\mathcal{O}_T} e^{\mu_r}(Z_r - \Phi(e^{-\mu_r})\Phi(Y_r))h\ d\xi dr \geq 0,$$

for all $h \in C^0(\mathcal{O}_T)$. This implies $Z = \Phi(e^{-\mu})\Phi(Y)$ almost everywhere.

It remains to prove that the uniform L^∞ bound obtained in Lemma 2.2 remains valid for weak solutions. We first note that by uniform continuity of $\{z^{(\delta)}|\delta > 0\}$ the partition τ_i in (2.16) can be chosen independent of δ. Thus $K_i^{(\sigma_0,\delta)}$ defined in (2.18) only depends on δ via the factor $e^{\mu_{\tau_i}^{(\delta)}}$ and converges uniformly to a piecewise smooth function $K_i^{(\sigma_0)}$ given by (2.18) with $\mu^{(\varepsilon)} = \mu$. We define $K^{(\sigma_0)}$ as in (2.21). By Lemma 2.2 we know that $Y_t^{(\delta)} \leq K^{(\delta,\sigma_0)}(t)$ for all $t \in [0,T]$ and all $\delta \leq \delta_0$. Since the cone of nonnegative distributions in H is convex, closed and $Y_t^{(\delta)} \rightharpoonup Y_t$ in H we conclude $Y_t \leq K^{(\sigma_0)}(t)$ a.e. in $\mathcal{O}$ for all $t \in [0,T]$. Note that $K^{(\sigma_0)}$ is increasing as σ_0 decreases. Defining $U := K^{(0)} : [0,T] \to \bar{\mathbb{R}}$ as in (2.18) with $\sigma_0 = 0$ (with the convention $\frac{1}{0} = \infty$) yields a piecewise smooth function on $(0,T]$ (taking the value ∞ at $t = 0$) with $Y_t \leq K^{(\sigma_0)}(t) \leq U_t$ a.e. in $\mathcal{O}$ and for all $t \in [0,T]$.

For later use we prove weak continuity of $t \mapsto Y_t$ in $L^p(\mathcal{O})$. Let $p \in (2,\infty)$ and $t_n \to t \in [0,T]$. Then Y_t is uniformly bounded in $L^p(\mathcal{O})$ and thus there is a weakly convergence subsequence $Y_{t_{n_k}}$. Since $Y \in C([0,T];H)$, the weak limit is Y_t and by arbitrarity of the sequence t_n we obtain $Y_{t_n} \rightharpoonup Y_t$ in $L^p(\mathcal{O})$.

Assume that $z \in C^{1-var}([0,T];\mathbb{R}^N)$, by Theorem 1.2 $X = e^{-\mu}Y$ is a weak solution to (0.1) and the bounds follow from the corresponding ones for Y. $\square$

Proof of Remark 1.5. The proof of existence of weak solutions to (0.1) and (0.3) proceeds with only minor modifications. The statements of Lemma 2.2 remain true, however, with a modified upper bound $K^{(\sigma_0,\varepsilon)}$.

Proof of Lemma 2.2 for fast diffusion equations: Again we construct a supersolution to (2.15) which is piecewise smooth (thus bounded) in $\bar{\mathcal{O}}_T$. Let R, β, C_4 and τ_i, $i = 0, ..., L-1$ as before and $A^{\frac{m-1}{m}} = \frac{R^{\frac{2}{m}}}{(1-m)d}$. We inductively define

$$K_i^{(\sigma_0,\varepsilon)}(t,\xi) = A^{\frac{1}{m}}(\sigma_i - t)^{\frac{1}{1-m}}(R^2 - |\xi|^2)^{\frac{1}{m}} e^{\mu^{(\varepsilon)}_{\tau_i}}, \quad t \in [\tau_i, \tau_{i+1}],\ \xi \in \mathcal{O},$$

where $\sigma_i > \tau_{i+1},\ i = 1, ..., L-1$ are chosen (large enough) such that $K_0^{(\sigma_0,\varepsilon)}(0) \geq Y_0$ and $K_i(\tau_{i+1}) \leq K_{i+1}(\tau_{i+1})$, which is satisfied if $\sigma_{i+1} \geq 2\sigma_i + \tau_{i+1}$. The remaining calculations and arguments are similar to those of the degenerate case. Note, however, the changing signs due to the changing sign of $1 - m$. $\square$

Next we prove a-priori estimates for the approximating classical solutions analogous to those given in Lemma 2.3. Here we can allow $Y_0 \in L^{m+1}(\mathcal{O})$ since in (2.23) and (2.24) the term $\int_{\mathcal{O}} \left(\Phi(e^{-\mu_t})\Phi^{(\delta)}(Y^{(\delta)})\right)^2 \, d\xi$ can be bounded by $C \int_{\mathcal{O}} \Psi^{(\delta)}(Y^{(\delta)}) \, d\xi$. Thus, the L^∞ bound is not needed to prove (2.22). The same proof as for Theorem 1.4 can then be used to construct weak solutions for all initial conditions $Y_0 \in L^{m+1}(\mathcal{O})$ (but without $L^\infty(\mathcal{O})$ bound). This finishes the proof of existence of weak solutions for the case of fast diffusions. If $Y_0 \in L^\infty(\mathcal{O})$ then Lemma 2.2 yields L^∞ boundedness of Y.
In order to obtain a uniform upper bound independent of the initial condition as in the degenerate case ($m > 1$) we would have to let $\sigma_0 \to \infty$ in $K^{(\sigma_0)}$ implying $U \equiv \infty$. Moreover, we do not have a uniqueness result for essentially bounded weak solutions in the case of fast diffusion equations. Therefore, it is not known whether each such weak solution is a limit of solutions to the non-degenerate approximating equations which will be needed for the proof of uniform continuity in the initial condition with respect to the L^1 norm. $\square$

2.4 Rough Weak Solutions

We prove Theorem 1.7. Let $Y_0 \in L^\infty(\mathcal{O})$ and $z^{(\varepsilon)} \in C^{1-var}([0,T]; \mathbb{R}^N)$ such that $z^{(\varepsilon)} \to z$ in $C([0,T]; \mathbb{R}^N)$. In particular $\{z^{(\varepsilon)} |\ \varepsilon > 0\}$ is compact in $C([0,T]; \mathbb{R}^N)$. We require uniform bounds for the corresponding weak solutions $Y^{(\varepsilon)}$ to (0.3) driven by $z^{(\varepsilon)}$.

Lemma 2.4. *Let $\{z^{(\varepsilon)} |\ \varepsilon > 0\} \subseteq C([0,T]; \mathbb{R}^N)$ compact and $Y^{(\varepsilon)}$ the weak solutions to* (0.3) *driven by $z^{(\varepsilon)}$. Then there exists a constant $M > 0$ (independent of ε) such that*

$$\sup_{t\in[0,T]} \|Y_t^{(\varepsilon)}\|_{L^\infty(\mathcal{O})} + \|\Phi(e^{-\mu^{(\varepsilon)}} Y^{(\varepsilon)})\|^2_{L^2([0,T]; H_0^1(\mathcal{O}))} \leq M.$$

Proof. For $\varepsilon > 0$ let $\{z^{(\tau,\varepsilon)} \in C^\infty([0,T]; \mathbb{R}^N) |\ \tau > 0\}$ be the sequence of smooth functions obtained by convolution of $z^{(\varepsilon)}$ with a standard Dirac sequence. Since $\{z^{(\varepsilon)} |\ \varepsilon > 0\}$ is a set of equicontinuous functions there is a uniform modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$. Uniform boundedness and the modulus of continuity are preserved under convolution with a Dirac sequence. Thus, the set $\{z^{(\tau,\varepsilon)} |\ \varepsilon > 0,\ \tau > 0\}$ is compact in $C([0,T]; \mathbb{R}^N)$.

Let now $Y_0^{(\delta)}$ be a smooth approximation of Y_0 as in the proof of Theorem 1.4 and let $Y^{(\delta,\varepsilon)}$ be the corresponding smooth solution to (2.14) driven by $z^{(\delta,\varepsilon)}$. By Lemma 2.2 and Lemma 2.3 there is a uniform constant $M > 0$ (depending only on $\|Y_0\|_{L^\infty(\mathcal{O})}$) bounding $\|Y^{(\delta,\varepsilon)}\|_{L^\infty(\mathcal{O})}$ and $\|\Phi(e^{-\mu^{(\delta,\varepsilon)}})\Phi^{(\delta)}(Y^{(\delta,\varepsilon)})\|^2_{L^2([0,T];H_0^1(\mathcal{O}))}$. By weak lower semicontinuity of the L^∞ norm on L^{m+1}, the convergence $Y^{(\delta,\varepsilon)} \rightharpoonup^* Y$ in $L^\infty([0,T];L^{m+1}(\mathcal{O}))$ and the convergence $\Phi(e^{-\mu^{(\delta,\varepsilon)}})\Phi^{(\delta)}(Y^{(\delta,\varepsilon)}) \rightharpoonup \Phi(e^{-\mu^{(\varepsilon)}}Y^{(\varepsilon)})$ in $L^2([0,T];H_0^1(\mathcal{O}))$ obtained in the proof of Theorem 1.4 these bounds continue to hold for $Y^{(\varepsilon)}$. $\square$

By Theorem 1.4 there is a weak solutions Y to (0.3) driven by z. Let $X := e^{-\mu}Y$ and $X^{(\varepsilon)} := e^{-\mu^{(\varepsilon)}}Y^{(\varepsilon)}$. Then $X^{(\varepsilon)}$ solves (0.1) and we need to prove $X_t^{(\varepsilon)} \to X_t$ in H for all $t \in [0,T]$. For this it is enough to prove $Y_t^{(\varepsilon)} \to Y_t$ in H for all $t \in [0,T]$.

Lemma 2.4 implies that $Y^{(\varepsilon)}$ is uniformly bounded in $L^\infty(\mathcal{O}_T)$, hence also in $L^\infty([0,T];H)$. Moreover, $Z^{(\varepsilon)} = \Phi(e^{-\mu^{(\varepsilon)}}Y^{(\varepsilon)})$ is uniformly bounded in $L^\infty(\mathcal{O}_T)$ and in $L^2([0,T];H_0^1(\mathcal{O}))$. By the same argument as in Theorem 1.4 we obtain the weak convergence $Y_t^{(\varepsilon)} \rightharpoonup Y_t$ in H for all $t \in [0,T]$ and $Z^{(\varepsilon)} \rightharpoonup Z = \Phi(e^{-\mu}Y)$ in $L^2([0,T];H_0^1(\mathcal{O}))$. Hence $X_t^{(\varepsilon)} \rightharpoonup X_t := e^{-\mu_t}Y_t$ in H for all $t \in [0,T]$. Since Y is the unique weak solution to (0.3) the uniform bounds for X follow from Theorem 1.4.

It remains to prove that the convergence $X_t^{(\varepsilon)} \rightharpoonup X_t$ is strong in H. As in (2.25) and (2.26) we have

$$
\begin{aligned}
\|Y_t\|_H^2 &= \|Y_s\|_H^2 - 2\int_s^t \int_{\mathcal{O}} e^{\mu_r} Z_r Y_r d\xi\, dr \\
&\quad + 2\int_s^t \int_{\mathcal{O}} Z_r\Big(2\nabla(e^{\mu_r})\nabla((-\Delta)^{-1}(Y_r)) + \Delta(e^{\mu_r})(-\Delta)^{-1}(Y_r)\Big)d\xi\, dr.
\end{aligned} \tag{2.28}
$$

and

$$
\begin{aligned}
\|Y_t^{(\varepsilon)}\|_H^2 &= \|Y_s^{(\varepsilon)}\|_H^2 - 2\int_s^t \int_{\mathcal{O}} e^{\mu_r^{(\varepsilon)}} Z_r^{(\varepsilon)} Y_r^{(\varepsilon)} d\xi\, dr \\
&\quad + 2\int_s^t \int_{\mathcal{O}} Z_r^{(\varepsilon)}\Big(2\nabla(e^{\mu_r^{(\varepsilon)}})\nabla((-\Delta)^{-1}(Y_r^{(\varepsilon)})) + \Delta(e^{\mu_r^{(\varepsilon)}})(-\Delta)^{-1}(Y_r^{(\varepsilon)})\Big)d\xi\, dr.
\end{aligned} \tag{2.29}
$$

Since $Y^{(\varepsilon)} \in L^2([0,T];L^2(\mathcal{O}))$ and $\frac{dY^{(\varepsilon)}}{dt} \in L^2([0,T];H)$ bounded and $L^2(\mathcal{O}) \hookrightarrow\hookrightarrow H$, by the Aubin-Lions compactness theorem we have

$$Y^{(\varepsilon)} \to Y, \text{ strongly in } L^2([0,T];H).$$

Integrating (2.28) and (2.29) over $s \in [0,t]$ and subtracting yields

$$t \limsup_{\varepsilon\to 0} \Big(\|Y_t^{(\varepsilon)}\|_H^2 - \|Y_t\|_H^2\Big) \le 0,$$

which implies strong convergence $Y_t^{(\varepsilon)} \to Y_t$ in H.

2.5 Limit solutions and dynamics on $L^1(\mathcal{O})$

2.5.1 L^1-continuity and a comparison principle

We will now prove uniform L^1 continuity in the initial condition for weak solutions to (0.3). Using this uniform continuity we can then construct limit solutions to (0.3).

Lemma 2.5. *Let $Y \in L^\infty([0,T]; L^1(\mathcal{O}))$ such that $t \mapsto Y_t(\xi)$ is continuously differentiable on $[0,T]$ for almost all $\xi \in \mathcal{O}$ and $\partial_t Y \in L^1(\mathcal{O}_T)$. Then*

$$\int_{\mathcal{O}} Y_t^+ \, d\xi - \int_{\mathcal{O}} Y_s^+ \, d\xi = \int_s^t \int_{\mathcal{O}} \partial_r Y_r \; sgn^+(Y_r) \, d\xi dr,$$

where $(\cdot)^+ = max(\cdot, 0)$ and $sgn^+(\cdot) = max(sgn(\cdot), 0)$.

Proof. Let $t \in [0,T]$. Since $Y \in L^\infty([0,T]; L^1(\mathcal{O}))$ there is a sequence $t_n \to t$ and a constant $M > 0$ such that $\|Y_{t_n}\|_{L^1(\mathcal{O})} \leq M$. By continuity of $t \mapsto Y_t(\xi)$ for almost all $\xi \in \mathcal{O}$ we have $Y_{t_n}(\xi) \to Y_t(\xi)$ almost everywhere. Fatou's Lemma yields $\|Y_t\|_{L^1(\mathcal{O})} \leq \liminf_{n\to\infty} \|Y_{t_n}\|_{L^1(\mathcal{O})} \leq M$. Thus, $Y_t \in L^1(\mathcal{O})$ for all $t \in [0,T]$. Let $\sigma^{(\tau)} \in C^\infty(\mathbb{R})$ be such that

$$\sigma^{(\tau)}(r) := \begin{cases} 0 & , \text{for } r \leq 0 \\ r & , \text{for } r \geq \tau, \end{cases}$$

with $0 \leq \dot{\sigma}^{(\tau)} \leq 1$ and $0 \leq \ddot{\sigma}^{(\tau)} \leq \frac{C}{\tau}$. For $0 \leq s < t \leq T$ we obtain

$$\int_{\mathcal{O}} \sigma^{(\tau)}(Y_t) \, d\xi - \int_{\mathcal{O}} \sigma^{(\tau)}(Y_s) \, d\xi = \int_s^t \int_{\mathcal{O}} \partial_r \sigma^{(\tau)}(Y_r) \, d\xi dr = \int_s^t \int_{\mathcal{O}} \dot{\sigma}^{(\tau)}(Y_r) \partial_r Y_r \, d\xi dr.$$

By dominated convergence this yields the assertion. ∎

Lemma 2.6. *Let $Y_0^i \in L^\infty(\mathcal{O})$, $i = 1, 2$ and $Y^{(i)}$ be the corresponding essentially bounded weak solution to (0.3). Then there exists a constant $C > 0$ such that*

$$\sup_{t\in[0,T]} \|(Y_t^{(1)} - Y_t^{(2)})^+\|_{L^1(\mathcal{O})} + \|(\Phi(e^{-\mu}Y^{(1)}) - \Phi(e^{-\mu}Y^{(2)}))^+\|_{L^1(\mathcal{O}_T)} \leq C\|(Y_0^{(1)} - Y_0^{(2)})^+\|_{L^1(\mathcal{O})},$$

and

$$\sup_{t\in[0,T]} \|Y_t^{(1)} - Y_t^{(2)}\|_{L^1(\mathcal{O})} + \|\Phi(e^{-\mu}Y^{(1)}) - \Phi(e^{-\mu}Y^{(2)})\|_{L^1(\mathcal{O}_T)} \leq C\|Y_0^{(1)} - Y_0^{(2)}\|_{L^1(\mathcal{O})}.$$

Proof. Let $\sigma^{(\tau)}$ be as in the proof of Lemma 2.5 and let $\varphi \in C^2(\bar{\mathcal{O}})$ be the unique classical solution to

$$\begin{aligned} \Delta\varphi &= -1, \text{ in } \mathcal{O} \\ \varphi &= 1, \text{ on } \partial\mathcal{O}. \end{aligned}$$

By the maximum principle we have $\varphi \geq 1$. By Theorem 1.3 the weak solutions $Y^{(i)}$ coincide with the weak solutions constructed in the proof of Theorem 1.4 by approximation with classical solutions $Y^{(i,\delta)}$ to (2.14). Let $z^{(\delta)} \in C^\infty([0,T];\mathbb{R}^N)$ be the corresponding smooth approximation of the driving signal z. By equicontinuity of $z^{(\delta)}$ we can find a partition $0 = \tau_0 < \tau_1 < ... < \tau_N = T$ of $[0,T]$ such that

$$\left(\inf_{\substack{\xi\in\bar{\mathcal{O}},\\ t\in[\tau_i,\tau_{i+1}]}} e^{\mu_t^{(\delta)}(\xi)-\mu_{\tau_i}^{(\delta)}(\xi)}\right)\Big(-1+2\|\varphi\|_{C^1(\mathcal{O})}\big(\|\nabla(\mu_t^{(\delta)}-\mu_{\tau_i}^{(\delta)})\|_{C^0(\mathcal{O})}+\|\nabla(\mu_t^{(\delta)}-\mu_{\tau_i}^{(\delta)})\|^2_{C^0(\mathcal{O})}$$
$$+\|\Delta(\mu_t^{(\delta)}-\mu_{\tau_i}^{(\delta)})\|_{C^0(\mathcal{O})}\big)\Big) \leq -\frac{1}{2},$$

for all $t\in[\tau_i,\tau_{i+1}]$, all $i = 0,...,N-1$ and all $\delta > 0$. Let now $\delta > 0$ be arbitrary, fixed. For simplicity we drop the δ dependency of the signal in the following calculation. Define

$$\eta_t(\xi) := \varphi(\xi)\sum_{i=0}^{N-1}\mathbb{1}_{[\tau_i,\tau_{i+1})}(t)e^{-\mu_{\tau_i}(\xi)}.$$

For $\tau_i \leq s < t < \tau_{i+1}$ by Lemma 2.5 we have

$$\int_{\mathcal{O}}(Y_t^{(1,\delta)}-Y_t^{(2,\delta)})^+\eta_t\,d\xi-\int_{\mathcal{O}}(Y_s^{(1,\delta)}-Y_s^{(2,\delta)})^+\eta_s\,d\xi$$
$$=\int_s^t\int_{\mathcal{O}}\partial_r(Y^{(1,\delta)}-Y^{(2,\delta)})sgn^+\left(Y_r^{(1,\delta)}-Y_r^{(2,\delta)}\right)\eta_r\,d\xi dr.$$

Let $Y^{(\delta)} := Y^{(1,\delta)}-Y^{(2,\delta)}$ and $w^{(\delta)} = \Phi(e^{-\mu_r})(\Phi^{(\delta)}(Y^{(1,\delta)})-\Phi^{(\delta)}(Y^{(2,\delta)}))$. We observe:

$$\int_s^t\int_{\mathcal{O}}\partial_r Y^{(\delta)}sgn^+\left(Y_r^{(\delta)}\right)\varphi e^{-\mu_{\tau_i}}\,d\xi dr = \int_s^t\int_{\mathcal{O}}(\Delta w_r^{(\delta)})sgn^+(w_r^{(\delta)})e^{\mu_r-\mu_{\tau_i}}\varphi\,d\xi\,dr$$
$$= \lim_{\tau\to0}\Big(-\int_s^t\int_{\mathcal{O}}\nabla w_r^{(\delta)}\nabla(\dot\sigma^{(\tau)}(w_r^{(\delta)}))e^{\mu_r-\mu_{\tau_i}}\varphi\,d\xi dr \tag{2.30}$$
$$-\int_s^t\int_{\mathcal{O}}\nabla w_r^{(\delta)}\nabla(e^{\mu_r-\mu_{\tau_i}}\varphi)\dot\sigma^{(\tau)}(w_r^{(\delta)})\,d\xi dr\Big).$$

Since $\nabla\dot\sigma^{(\tau)}(w_r^{(\delta)}) = \ddot\sigma^{(\tau)}(w_r^{(\delta)})\nabla w_r^{(\delta)}$, the first term has negative sign. Partial integration of the second term gives

$$-\int_{\mathcal{O}}\nabla w^{(\delta)}\nabla(e^{\mu_r-\mu_{\tau_i}}\varphi)\dot\sigma^{(\tau)}(w^{(\delta)})\,d\xi = \int_{\mathcal{O}}w^{(\delta)}\Delta(e^{\mu_r-\mu_{\tau_i}}\varphi)\dot\sigma^{(\tau)}(w^{(\delta)})\,d\xi$$
$$+\int_{\mathcal{O}}w^{(\delta)}\nabla(e^{\mu_r-\mu_{\tau_i}}\varphi)\nabla\dot\sigma^{(\tau)}(w^{(\delta)})\,d\xi.$$

For the second term on the right hand side we note

$$\int_{\mathcal{O}}w^{(\delta)}\nabla(e^{\mu_r-\mu_{\tau_i}}\varphi)\nabla\dot\sigma^{(\tau)}(w^{(\delta)})\,d\xi = \int_{\mathcal{O}\cap\{0<w^{(\delta)}<\tau\}}w^{(\delta)}\ddot\sigma^{(\tau)}(w^{(\delta)})\nabla(e^{\mu_r-\mu_{\tau_i}}\varphi)\cdot\nabla w^{(\delta)}\,d\xi \to 0,$$

for $\tau \to 0$, by $\ddot{\sigma}^{(\tau)} \le \frac{C}{\tau}$ and dominated convergence. Using dominated convergence we can take the limit $\tau \to 0$ in (2.30) to get

$$\int_s^t \int_{\mathcal{O}} \partial_r Y^{(\delta)} sgn^+(Y_r^{(\delta)}) \varphi e^{-\mu_{\tau_i}} \, d\xi dr \le \int_s^t \int_{\mathcal{O}} w_r^{(\delta)} \Delta(e^{\mu_r - \mu_{\tau_i}} \varphi) sgn^+(w_r^{(\delta)}) \, d\xi dr.$$

We note

$$\Delta \left(e^{\mu_r - \mu_{\tau_i}} \varphi \right) = e^{\mu_r - \mu_{\tau_i}} \left(\Delta \varphi + 2 \nabla \varphi \cdot \nabla (\mu_r - \mu_{\tau_i}) + \varphi (|\nabla (\mu_r - \mu_{\tau_i})|^2 + \Delta(\mu_r - \mu_{\tau_i})) \right) \le -\frac{1}{2},$$

by the choice of φ and τ_i. Thus,

$$\int_s^t \int_{\mathcal{O}} \partial_r Y^{(\delta)} sgn^+(Y_r^{(\delta)}) \varphi e^{-\mu_{\tau_i}} \, d\xi dr + \frac{1}{2} \int_s^t \int_{\mathcal{O}} (w_r^{(\delta)})^+ \, d\xi dr \le 0.$$

In conclusion:

$$\begin{aligned}
\int_{\mathcal{O}} (Y_t^{(1,\delta)} - Y_t^{(2,\delta)})^+ \eta_t \, d\xi - \int_{\mathcal{O}} (Y_s^{(1,\delta)} - Y_s^{(2,\delta)})^+ \eta_s \, d\xi + \frac{1}{2} \int_s^t \int_{\mathcal{O}} (w_r^{(\delta)})^+ \, d\xi dr \\
= \int_s^t \int_{\mathcal{O}} \partial_r Y^{(\delta)} sgn^+(Y_r^{(\delta)}) \eta_r \, d\xi dr + \frac{1}{2} \int_s^t \int_{\mathcal{O}} (w_r^{(\delta)})^+ \, d\xi dr \le 0,
\end{aligned}$$

for all $\tau_i \le s < t < \tau_{i+1}$ and hence for all $0 \le s < t \le T$. We have

$$\begin{aligned}
&\|(Y_t^{(1,\delta)} - Y_t^{(2,\delta)})^+\|_{L^1(\mathcal{O})} + \|\Phi(e^{-\mu^{(\delta)}})(\Phi^{(\delta)}(Y^{(1,\delta)}) - \Phi^{(\delta)}(Y^{(2,\delta)}))^+\|_{L^1(\mathcal{O}_T)} \\
&\le C \|(Y_0^{(1,\delta)} - Y_0^{(2,\delta)})^+\|_{L^1(\mathcal{O})},
\end{aligned}$$

for all $t \in [0, T]$, where the constant C does not depend on δ (using uniform boundedness of $z^{(\delta)}$). By the proof of Theorem 1.4 we know that $Y_t^{(i,\delta)} \rightharpoonup Y_t^{(i)}$ in $L^1(\mathcal{O})$ and $\Phi(e^{-\mu^{(\delta)}})\Phi^{(\delta)}(Y^{(i,\delta)}) \rightharpoonup \Phi(e^{-\mu} Y^{(i)})$ in $L^2([0,T]; H_0^1(\mathcal{O}))$. By weak lower semicontinuity of $\|(\cdot)^+\|_{L^1(\mathcal{O})}$ and $\|(\cdot)^+\|_{L^1(\mathcal{O}_T)}$, taking the limit $\delta \to 0$ we obtain

$$\|(Y_t^{(1)} - Y_t^{(2)})^+\|_{L^1(\mathcal{O})} + \|(\Phi(e^{-\mu} Y^{(1)}) - \Phi(e^{-\mu} Y^{(2)}))^+\|_{L^1(\mathcal{O}_T)} \le C \|(Y_0^{(1)} - Y_0^{(2)})^+\|_{L^1(\mathcal{O})}.$$

Since $Z^{(i)} := -Y^{(i)}$ again is an essentially bounded weak solution of (0.3), the same assertion follows for $\|(Y_t^{(1)} - Y_t^{(2)})^-\|_{L^1(\mathcal{O})}$. Adding both inequalities yields

$$\|Y_t^{(1)} - Y_t^{(2)}\|_{L^1(\mathcal{O})} + \|\Phi(e^{-\mu} Y^{(1)}) - \Phi(e^{-\mu} Y^{(2)})\|_{L^1(\mathcal{O}_T)} \le C \|Y_0^{(1)} - Y_0^{(2)}\|_{L^1(\mathcal{O})}.$$

□

Remark 2.7. *Following the same argument, but with $\Delta \varphi = -1$ with Dirichlet boundary conditions, the same result can be established in the weighted L^1-space L^1_φ. This then allows to construct limit solutions even for initial conditions in L^1_φ.*

Using this uniform L^1 continuity in the initial condition we can now construct limit solutions for all initial conditions in L^1.

Proof of Theorem 1.9. Let $Y_0 := e^{\mu_0} X_0 \in L^1(\mathcal{O})$ and $Y_0^{(\delta)} \to Y_0$ in $L^1(\mathcal{O})$ with $Y_0^{(\delta)} \in L^\infty(\mathcal{O})$. Let $Y^{(\delta)}$ be the essentially bounded weak solution corresponding to $Y_0^{(\delta)}$. By Lemma 2.6 we have

$$\sup_{t\in[0,T]} \|Y_t^{(\delta_1)} - Y_t^{(\delta_2)}\|_{L^1(\mathcal{O})} + \|\Phi(e^{-\mu}Y^{(\delta_1)}) - \Phi(e^{-\mu}Y^{(\delta_2)})\|_{L^1(\mathcal{O}_T)} \le C\|Y_0^{(\delta_1)} - Y_0^{(\delta_2)}\|_{L^1(\mathcal{O})},$$

for all $\delta_1, \delta_2 > 0$. Hence, $Y_t^{(\delta)}$ is a Cauchy sequence in $L^1(\mathcal{O})$ and thus uniformly convergent to some limit $Y_t \in L^1(\mathcal{O})$. Since $\Phi(e^{-\mu}Y^{(\delta_1)})$ is a Cauchy sequence in $L^1(\mathcal{O}_T)$ and Φ is continuous we obtain $\Phi(e^{-\mu}Y^{(\delta_1)}) \to \Phi(e^{-\mu}Y)$ in $L^1(\mathcal{O}_T)$.

By Theorem 1.7, $X^{(\delta)} = e^{-\mu}Y^{(\delta)}$ are rough weak solutions and we conclude $X_t^{(\delta)} \to X_t := e^{-\mu_t}Y_t$ uniformly in $L^1(\mathcal{O})$ and $\Phi(X^{(\delta)}) \to \Phi(X)$ in $L^1(\mathcal{O}_T)$. In the proof of Theorem 1.4 we have proven weak continuity of $t \mapsto Y_t^{(\delta)}$ in $L^p(\mathcal{O})$. Hence $t \mapsto X_t^{(\delta)}$ is weakly continuous in $L^1(\mathcal{O})$ and thus is $t \mapsto X_t$. The bound $X_t \le U_t$ follows immediately. $\square$

2.6 (Equi-)continuity of solutions

Proof of Theorem 1.12. We only prove (i). The proofs of (ii), (iii) are analogous. Let $X_0 \in L^1(\mathcal{O})$ and X be the corresponding limit solution. Since $K \subseteq (0,T] \times \mathcal{O}$ is compact, there is a $\tau > 0$ such that $K \subseteq [\tau, T] \times \mathcal{O}$. By Theorem 1.9 we know that $Y = e^{\mu}X \in L^\infty([\tau,T]\times\mathcal{O})$ and by Remark 1.11 Y is a very weak solution of (0.3). By Theorem 1.3 and Theorem 1.4 this implies that Y is an essentially bounded weak solution to (0.3) on $[\tau,T]\times\mathcal{O}$ with initial condition Y_τ. Due to the uniform L^∞ bound U established in Theorem 1.9, $\|Y_\tau\|_{L^\infty(\mathcal{O})}$ is bounded independent of the initial condition Y_0. It is thus sufficient to prove the claimed regularity for weak solutions Y of (0.3) with a modulus of continuity depending only on the data and $\|Y_0\|_{L^\infty(\mathcal{O})}$.

Let $Y^{(\delta)}$ the sequence of approximating solutions with initial condition $Y_0^{(\delta)}$ and driving signal $z^{(\delta)}$ used in Theorem 1.4. By Theorem 1.4 and Lemma 2.2, Y and $Y^{(\delta)}$ are uniformly bounded, i.e.

$$\|Y\|_{L^\infty(\mathcal{O}_T)}, \|Y^{(\delta)}\|_{L^\infty(\mathcal{O}_T)} \le M, \text{ for all } \delta \le \delta_0,$$

for some constant $M > 0$ depending on $\|Y_0\|_{L^\infty(\mathcal{O})}$. We aim to apply the continuity results for porous media type PDE given in [14] to the approximating equation (2.14). In [14] equations of the form

$$\frac{d}{dt}\beta(v) = \operatorname{div} a(t,\xi,v,\nabla v) + b(t,\xi,v,\nabla v), \text{ on } \mathcal{O}_T \tag{2.31}$$

with Dirichlet boundary conditions and initial value v_0 are considered. We first rewrite the approximating equations in the form of (2.31). The approximating equation (2.14) (driven by $z^{(\delta)}$) is equivalent to

$$\partial_t Y_t^{(\delta)} = \mathrm{div}\ a^{(\delta)}(t,\xi,\Phi^{(\delta)}(Y_t^{(\delta)}),\nabla\Phi^{(\delta)}(Y_t^{(\delta)})) + b^{(\delta)}(t,\xi,\Phi^{(\delta)}(Y_t^{(\delta)}),\nabla\Phi^{(\delta)}(Y_t^{(\delta)}))$$

with

$$\begin{aligned} a^{(\delta)}(t,\xi,z,p) &= e^{(1-m)\mu_t^{(\delta)}(\xi)}p, \\ b^{(\delta)}(t,\xi,z,p) &= e^{\mu_t^{(\delta)}(\xi)}\Delta(\Phi(e^{-\mu_t^{(\delta)}(\xi)}))z - (m+1)e^{(1-m)\mu_t^{(\delta)}(\xi)}\nabla\mu_t^{(\delta)}(\xi)\cdot p. \end{aligned}$$

Let $\beta^{(\delta)} := \left(\Phi^{\delta}\right)^{-1}$. For the approximating solutions $Y^{(\delta)}$ we define $Z^{(\delta)} := \Phi^{\delta}(Y^{(\delta)})$. Then $Z^{(\delta)}$ satisfies

$$\partial_t\beta^{(\delta)}(Z_t^{(\delta)}) = \mathrm{div}\ a^{(\delta)}(t,\xi,Z_t^{(\delta)},\nabla Z_t^{(\delta)}) + b^{(\delta)}(t,\xi,Z_t^{(\delta)},\nabla Z_t^{(\delta)}). \tag{2.32}$$

The continuity of solutions to equations of such type has been shown in [14] under the assumption of an a-priori $L^\infty([0,T]\times\mathcal{O})$-bound and a growth bound for b (among other assumptions). The growth bound on b used in [14] is not satisfied by (2.32). However, using the a-priori L^∞ bound on $Y^{(\delta)}$ we can cut-off b in the z variable without changing the solution property of $Y^{(\delta)}$, thus guaranteeing that the growth condition is satisfied. We modify $\Phi^{(\delta)}$ on $\mathbb{R}\setminus[-M,M]$ to obtain $\dot\Phi^{(\delta,M)} \le C_2$ uniformly in δ (while preserving the properties (i)-(iii) in (2.3.1)) and we modify b by

$$b^{(M,\delta)}(t,\xi,z,p) = e^{\mu_t^{(\delta)}(\xi)}\Delta(\Phi(e^{-\mu_t^{(\delta)}(\xi)}))z\mathbb{1}_{|z|\le M} - (m+1)e^{(1-m)\mu_t^{(\delta)}(\xi)}\nabla\mu_t^{(\delta)}(\xi)\cdot p.$$

Let $\beta^{(\delta,M)} := \left(\Phi^{(\delta,M)}\right)^{-1}$. Using the L^∞ bound we realize that $Z^{(\delta)}$ is a solution of

$$\begin{aligned} \partial_t\beta^{(\delta,M)}(Z_t^{(\delta)}) &= \mathrm{div}\ a^{(\delta)}(t,\xi,Z_t^{(\delta)},\nabla Z_t^{(\delta)}) + b^{(M,\delta)}(t,\xi,Z_t^{(\delta)},\nabla Z_t^{(\delta)}) \\ Z^{(\delta)}(0) &= Z_0^{(\delta)} := \Phi^{(\delta)}(Y_0^{(\delta)}),\ \text{on } \mathcal{O}, \end{aligned} \tag{2.33}$$

for M large enough. By [14], we obtain that $Z^{(\delta)}$ and thus $Y^{(\delta)}$ are equicontinuous on K with modulus of continuity depending only on the data, $\|Y_0\|_{L^\infty(\mathcal{O})}$ and $\mathrm{dist}(K,\partial\mathcal{O}_T)$. Hence, the set $\{Y^{(\delta)}|\ \delta>0\}$ is a compact subset of $C(K)$ and we can choose a uniformly convergent subsequence. By the proof of existence of weak solutions we know that $Y^{(\delta)} \rightharpoonup Y$ in $L^{m+1}(\mathcal{O}_T)$. Consequently $Y^{(\delta)} \to Y$ uniformly on K. This implies $Y \in C(K)$ with the same modulus of continuity. $\square$

Proof of Corollary 1.13. Let $X_0 \in L^1(\mathcal{O})$ and X be the corresponding limit solution. By Theorem 1.12, $t \mapsto X_t(\xi)$ is continuous on $(0,T]$ for each $\xi \in \mathcal{O}$. By Theorem 1.9 X is uniformly bounded on $[\tau,T]\times\mathcal{O}$ for all $\tau>0$. Dominated convergence implies $X \in$

$C((0,T];L^p(\mathcal{O}))$. We can approximate X_0 by $X_0^{(\delta)} \in C(\bar{\mathcal{O}})$ such that $X_0^{(\delta)} \to X_0$ in $L^1(\mathcal{O})$. Let $X^{(\delta)}$ be the weak solution corresponding to $X_0^{(\delta)}$. By Theorem 1.12 (ii), $t \mapsto X_t^{(\delta)}(\xi)$ is continuous on $[0,T]$ for each $\xi \in \mathcal{O}$ and by Theorem 1.4 $X^{(\delta)}$ is uniformly bounded in $[0,T] \times \mathcal{O}$. Dominated convergence implies $X^{(\delta)} \in C([0,T];L^1(\mathcal{O}))$. By Theorem 1.9 we have $\sup_{t\in[0,T]} \|X_t^{(\delta)} - X_t\|_{L^1(\mathcal{O})} \to 0$, hence also $X \in C([0,T];L^1(\mathcal{O}))$. If $X_0 \in L^\infty(\mathcal{O})$, then by uniqueness of essentially bounded weak solutions and Theorem 1.4 X is uniformly bounded in $[0,T] \times \mathcal{O}$. Since also $X \in C([0,T];L^1(\mathcal{O}))$ this implies $X \in C([0,T];L^p(\mathcal{O}))$ by dominated convergence. $\square$

3 Generation of an RDS and random attractors

3.1 Transformation in the semimartingale case

Proof of Theorem 1.18. Let z be a continuous semimartingale in $\mathbb{R}^N$, X be the limit solution to (0.1) and $Y := e^\mu X$. By Remark 1.11, Y is a very weak solution to (0.3). We will now prove that X satisfies (1.7).

We consider the Sobolev spaces $H_0^{2k}(\mathcal{O})$ with the norm $\|\cdot\|_{H_0^{2k}(\mathcal{O})} := \|(-\Delta)^k \cdot\|_2$. By Sobolev embeddings there is a $k \in \mathbb{N}$ (w.l.o.g. k odd) such that $H_0^{2k}(\mathcal{O}) \hookrightarrow C^0(\mathcal{O})$ continuously. Hence $L^1(\mathcal{O}) \hookrightarrow (H_0^{2k}(\mathcal{O}))^* =: H^{-2k}$ and $Y \in C([0,T];L^1(\mathcal{O})) \subseteq C([0,T];H^{-2k})$. Let $\varphi \in H_0^{2(k+1)}(\mathcal{O})$ and $\tilde{e}_j$ be an orthonormal basis of H_0^{2k} given by $\tilde{e}_j = \frac{e_j}{\lambda_j^k} = (-\Delta)^{-k} e_j$, where e_j is an orthonormal basis of eigenvectors of $-\Delta$ on $L^2(\mathcal{O})$ with Dirichlet boundary conditions and λ_k are the corresponding eigenvalues. Further, let $P_M : H^{-2k} \to span\{e_1, ..., e_M\}$ be the orthogonal projection. Then $P_{M|L^2(\mathcal{O})}$, $P_{M|H_0^{2k}(\mathcal{O})}$ are the orthogonal projections onto $span\{e_1, ..., e_M\}$ in $L^2(\mathcal{O})$, $H_0^{2k}(\mathcal{O})$ respectively. We have

$$\int_{\mathcal{O}} X_t \varphi \, d\xi = {}_{H^{-2k}}\langle Y_t, e^{-\mu_t}\varphi\rangle_{H_0^{2k}} = \sum_{j=1}^\infty \left(\int_{\mathcal{O}} Y_t e_j \, d\xi \right) (e_j, e^{-\mu_t}\varphi)_2.$$

By the very weak solution property and continuity in $L^1(\mathcal{O})$

$$\int_{\mathcal{O}} Y_t e_j \, d\xi = \int_{\mathcal{O}} Y_s e_j \, d\xi + \int_s^t \int_{\mathcal{O}} \Phi(e^{-\mu_r} Y_r) \Delta(e^{\mu_r} e_j) \, d\xi dr, \quad \forall s \le t$$

hence $t \mapsto \int_{\mathcal{O}} Y_t e_j \, d\xi$ is an absolutely continuous map with derivative $\int_{\mathcal{O}} \Phi(e^{-\mu_t} Y_t) \Delta(e^{\mu_t} e_j) \, d\xi$ for a.e. $t \in [0,T]$. By Theorem 1.7 Y_t is adapted. As in [6, p. 22], by use of the stochastic Fubini Theorem (cf. e.g. [33]) we prove

$$(e_j, e^{-\mu_t}\varphi)_2 = (e_j, e^{-\mu_0}\varphi)_2 + \sum_{k=1}^N \int_0^t (e_j, f_k e^{-\mu_r}\varphi)_2 \circ dz_r^{(k)}.$$

In particular $(e_j, e^{-\mu_t}\varphi)_2$ is a real-valued continuous semimartingale. Hence, we can apply the Itô product rule (cf. [29, p. 83]) to get

$$
\begin{aligned}
\left(\int_{\mathcal{O}} Y_t e_j \, d\xi\right) (e_j, e^{-\mu_t}\varphi)_2 = {} & \left(\int_{\mathcal{O}} Y_s e_j \, d\xi\right) (e_j, e^{-\mu_s}\varphi)_2 \\
& + \int_s^t (e_j, e^{-\mu_r}\varphi)_2 \left(\int_{\mathcal{O}} \Phi(X_r)\Delta(e^{\mu_r}\tilde{e}_j) \, d\xi\right) dr \\
& + \sum_{k=1}^N \int_s^t \left(\int_{\mathcal{O}} Y_r e_j \, d\xi\right) (e_j, f_k e^{-\mu_r}\varphi)_2 \circ dz_r^{(k)}
\end{aligned}
\tag{3.34}
$$

for all $0 \le s \le t \le T$, $\mathbb{P}$-almost surely. Note

$$
\int_{\mathcal{O}} \Phi(X_r)\Delta(e^{\mu_r}\tilde{e}_j) \, d\xi = \int_{\mathcal{O}} \Phi(X_r)(\tilde{e}_j \Delta e^{\mu_r} + 2\nabla e^{\mu_r} \cdot \nabla \tilde{e}_j + e^{\mu_r}\Delta\tilde{e}_j) \, d\xi.
$$

We aim to sum over j in (3.34). For this we have to rewrite the second summand on the right hand side of the equation above. Due to the lack of regularity of $\Phi(X)$ this requires an additional approximation:

$$
\int_{\mathcal{O}} \Phi(X_r)\nabla e^{\mu_r} \cdot \nabla\tilde{e}_j \, d\xi = -\lim_{M\to\infty} \int_{\mathcal{O}} \big(\nabla P_M \Phi(X_r) \cdot \nabla e^{\mu_r} + P_M \Phi(X_r)\Delta(e^{\mu_r})\big)\tilde{e}_j \, d\xi.
$$

Hence

$$
\begin{aligned}
& \sum_{j=1}^K (\tilde{e}_j, e^{-\mu_r}\varphi)_{H_0^{2k}} \int_{\mathcal{O}} \Phi(X_r) 2\nabla e^{\mu_r} \cdot \nabla\tilde{e}_j \, d\xi \\
& = -2 \lim_{M\to\infty} \bigg(\int_{\mathcal{O}} \big(\nabla P_M \Phi(X_r) \cdot \nabla e^{\mu_r}\big) P_K(e^{-\mu_r}\varphi) \, d\xi + \int_{\mathcal{O}} \big(P_M \Phi(X_r)\Delta(e^{\mu_r})\big) P_K(e^{-\mu_r}\varphi) \, d\xi \bigg).
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
& \sum_{j=1}^\infty (\tilde{e}_j, e^{-\mu_r}\varphi)_{H_0^{2k}} \int_{\mathcal{O}} \Phi(X_r)\Delta(e^{\mu_r}\tilde{e}_j) \, d\xi \\
& = {}_{H^{-2k}}\langle \Phi(X_r)\Delta e^{\mu_r}, e^{-\mu_r}\varphi\rangle_{H_0^{2k}} + {}_{H^{-2k}}\langle \Phi(X_r) e^{\mu_r}, \Delta(e^{-\mu_r}\varphi)\rangle_{H_0^{2k}} \\
& \quad + 2\, {}_{H^{-2k}}\langle \Phi(X_r), \nabla e^{\mu_r} \cdot \nabla(e^{-\mu_r}\varphi)\rangle_{H_0^{2k}} \\
& = {}_{H^{-2k}}\langle \Phi(X_r), \Delta\varphi\rangle_{H_0^{2k}}.
\end{aligned}
$$

Summing up $j = 1, ..., \infty$ in (3.34) yields

$$
\int_{\mathcal{O}} X_t \varphi \, d\xi = \int_{\mathcal{O}} X_s \varphi \, d\xi + \int_s^t \int_{\mathcal{O}} \Phi(X_r)\Delta\varphi \, d\xi dr + \int_s^t \left(\int_{\mathcal{O}} B(X_r)\varphi \, d\xi\right) \circ dz_r,
$$

for all $0 \le s \le t \le T$ and all $\varphi \in H_0^{2(k+1)}(\mathcal{O})$ (thus by approximation for all $\varphi \in C_0^2(\bar{\mathcal{O}})$) $\mathbb{P}$-almost surely. $\square$

3.2 Quasi-continuous random dynamical systems

Proof of Lemma 1.27. Since τ is weaker than the norm topology $\Omega(B,\omega) \subseteq \Omega^\tau(B,\omega)$. Let now $y \in \Omega^\tau(B,\omega)$. Then there are $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ such that $\varphi(t_n,\theta_{-t_n}\omega)x_n \to^\tau y$. By $\mathcal{D}$ asymptotic compactness there is a convergent subsequence $\varphi(t_{n_k},\theta_{-t_{n_k}}\omega)x_{n_k}$. Since τ is weaker than norm topology and Hausdorff we conclude $\varphi(t_{n_k},\theta_{-t_{n_k}}\omega)x_{n_k} \to y \ni \Omega(B,\omega)$. $\square$

Proof of Lemma 1.28: Let $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$. Then there is a convergent subsequence $\varphi(t_{n_l},\theta_{t_{n_l}}\omega)x_{t_{n_l}} \to x \in \Omega(B,\omega)$. Hence, $\Omega(B,\omega)$ is nonempty.

Compactness: Let $x_n \in \Omega(B,\omega)$. For every $n \in \mathbb{N}$ there are sequences $t_{k(n)} \to \infty$ and $y_{k(n)} \in B(\theta_{-t_{k(n)}}\omega)$ such that $\varphi(t_{k(n)},\theta_{-t_{k(n)}}\omega)y_{k(n)} \to x_n$ for $k(n) \to \infty$. Therefore, we can find sequences $t_n \to \infty$, $y_n \in B(\theta_{-t_n}\omega)$ such that $\|\varphi(t_n,\theta_{-t_n}\omega)y_n - x_n\|_X < \frac{1}{n}$. By $\mathcal{D}$-asymptotic compactness there is a convergent subsequence $\varphi(t_{n_l},\theta_{-t_{n_l}}\omega)y_{n_l} \to x \ni \Omega(B,\omega)$. Hence, $x_{n_l} \to x \ni \Omega(B,\omega)$.

Invariance: First let $x \in \Omega(B,\omega)$. We need to prove $\varphi(t,\omega)x \in \Omega(B,\theta_t\omega)$. Since $x \in \Omega(B,\omega)$ there are sequences $t_n \to \infty$, $x_n \in B(\theta_{-t_n}\omega)$ such that $\varphi(t_n,\theta_{-t_n}\omega)x_n \to x$. By the cocycle property $\varphi(t+t_n,\theta_{-t_n}\omega)x_n = \varphi(t,\omega)\varphi(t_n,\theta_{-t_n}\omega)x_n$ and by bounded absorption $\varphi(t+t_n,\theta_{-t_n}\omega)x_n = \varphi(t+t_n,\theta_{-(t+t_n)}\theta_t\omega)x_n \in F(\theta_t\omega)$ for n large enough. By quasi-τ-continuity we conclude $\varphi(t+t_n,\theta_{-t_n}\omega)x_n \to^\tau \varphi(t,\omega)x$. Hence $\varphi(t,\omega)x \in \Omega^\tau(B,\theta_t\omega) = \Omega(B,\theta_t\omega)$.

Let now $z \in \Omega(B,\theta_t\omega)$, i.e.

$$\varphi(t_n,\theta_{-t_n}\theta_t\omega)x_n \to z \tag{3.35}$$

for some $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\theta_t\omega)$. By $\mathcal{D}$-asymptotic compactness of φ there is a subsequence $\varphi(t_{n_l}-t,\theta_{-(t_{n_l}-t)}\omega)x_{n_l} \to x \ni \Omega(B,\omega)$. By (3.35), quasi-$\tau$-continuity and the cocycle property we have $\varphi(t_{n_l},\theta_{-t_{n_l}}\theta_t\omega)x_{n_l} = \varphi(t,\omega)\varphi(t_{n_l}-t,\theta_{-(t_{n_l}-t)}\omega)x_{n_l} \to^\tau \varphi(t,\omega)x$. Since τ is weaker than norm topology and Hausdorff, we conclude $z = \varphi(t,\omega)x$ with $x \in \Omega(B,\omega)$. $\square$

Proof of Theorem 1.29: Necessity of the conditions follows from compactness of A and its attraction property. To prove sufficiency we first observe that by Lemma 1.28,

$$A(\omega) := \Omega(F,\omega)$$

is compact and invariant. Since $F \in \mathcal{D}$ and F is $\mathcal{D}$-absorbing we have $A(\omega) \subseteq F(\omega)$ for all $\omega \in \Omega$ and thus $A \in \mathcal{D}$. We only need to prove attraction. We first observe that

$$\Omega(D,\omega) \subseteq \Omega(F,\omega) = A(\omega), \quad \forall D \in \mathcal{D},\ \omega \in \Omega.$$

Indeed: By absorption we have $\Omega(B,\omega) \subseteq F(\omega)$. By Lemma 1.28 we know that $\Omega(B,\omega) = \varphi(t,\theta_{-t}\omega)\Omega(B,\theta_{-t}\omega) \subseteq \varphi(t,\theta_{-t}\omega)F(\theta_{-t}\omega)$. Hence

$$\Omega(B,\omega) \subseteq \bigcap_{t\geq 0} \varphi(t,\theta_{-t}\omega)F(\theta_{-t}\omega) \subseteq \Omega(F,\omega) = A(\omega).$$

Assume that A is not attracting. Then there is a set $B \in \mathcal{D}$, an $\omega \in \Omega$, sequences $t_n \to \infty$, $x_n \in B(\theta_{-t_n}\omega)$ and a $\delta > 0$ such that

$$d(\varphi(t_n,\theta_{-t_n}\omega)x_n, A(\omega)) \geq \delta$$

for all $n \in \mathbb{N}$. By asymptotic compactness there is a convergent subsequence $\varphi(t_{n_l},\theta_{-t_{n_l}}\omega)x_{n_l} \to x \in \Omega(B,\omega) \subseteq A(\omega)$, which implies a contradiction. □

3.3 RDS and random attractors for (0.1)

Proof of Theorem 1.31. By Theorem 1.9 the map $x \mapsto X(t,s;\omega)x$ is Lipschitz continuous in $X = L^1(\mathcal{O})$, locally uniformly in s,t. Uniqueness of essentially bounded very weak solutions implies the flow property

$$X(t,s;\omega)x = X(t,r;\omega)X(r,s;\omega)x, \quad \forall \omega \in \Omega,\ s \leq r \leq t$$

and cocycle property

$$X(t,s;\theta_r\omega)x = X(t+r,s+r;\omega)x, \quad \forall \omega \in \Omega,\ s \leq t,\ r \in \mathbb{R}$$

for all $x \in L^\infty(\mathcal{O})$. By Lipschitz continuity in the initial condition these properties remain true for all $x \in X = L^1(\mathcal{O})$.
Next, we prove measurability of the map $(t,s,\omega,x) \mapsto X(t,s;\omega)x$. First let $t \geq s$ and $x \in L^\infty(\mathcal{O})$. By Theorem 1.7 the map $\mu \to X_t(\mu)$ from $C(\mathbb{R};\mathbb{R}^N)$ to H is continuous. Since also $\omega \mapsto \mu(\omega)$ is a measurable map this implies measurability of $\omega \mapsto X(t,s;\omega)x$ in H. Hence

$$\omega \mapsto \int_{\mathcal{O}} (X(t,s;\omega)x)\, h\; d\xi$$

is measurable for all $h \in H_0^1(\mathcal{O})$. Since $X = L^1(\mathcal{O})$ is separable, by Pettis measurability theorem this implies measurability of $X(t,s;\cdot)x$. By approximation, this remains true for all $x \in X$. Since $X(\cdot,s;\omega)x \in C([0,T];X)$, for all $\omega \in \Omega$ we deduce joint measurability of $(t,\omega) \mapsto X(t,s;\omega)x$ in X. Using $X(t,s;\omega)x = X(t-s,0;\theta_s\omega)x$ and joint measurability of $(s,\omega) \mapsto (t-s,\theta_s\omega)$ this implies measurability of $(s,\omega) \mapsto X(t,s;\omega)x$. Hence, measurability of $(t,s,\omega,x) \mapsto X(t,s;\omega)x$ follows and φ defines a continuous RDS on $L^1(\mathcal{O})$.

By Theorem 1.9, $\varphi(t,\omega)x \in L^p(\mathcal{O})$ for all $t \in \mathbb{R}_+$ if $x \in L^p(\mathcal{O})$, $p \in [1,\infty]$. Since $L^p(\mathcal{O})$ is reflexive for $p \in (1,\infty)$ this implies quasi-weak-continuity of φ on $L^p(\mathcal{O})$ for all $p \in (1,\infty)$ by Proposition 1.26. For $p = \infty$ we note that $\sigma(L^\infty, \overline{i^*(L^\infty)})$ is the weak* topology. By Proposition 1.26 quasi-weak*-continuity of φ on $L^\infty(\mathcal{O})$ follows. □

3.4 Bounded Absorption, Asymptotic Compactness and Random Attractors for φ

In the following let $\mathcal{D}$ be the universe of all random closed sets.

Proposition 3.1 (Bounded absorption)**.** *There is an $L^\infty(\mathcal{O})$-bounded (i.e. $\|F(\omega)\|_{L^\infty(\mathcal{O})} < \infty$) $\mathcal{D}$-absorbing random set $F \in \mathcal{D}$. The absorption time for $D \in \mathcal{D}$, $\omega \in \Omega$ can be chosen independent of ω and D.*

Proof. Recall that by Theorem 1.31 we have $\varphi(t,\omega)x \le U_t(\omega)$ a.e. in $\mathcal{O}$ for all $t \ge 0$ and all $x \in X$. For $D \in \mathcal{D}$:

$$\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega) = \varphi(1,\theta_{-1}\omega)\varphi(t-1,\theta_{-t}\omega)D(\theta_{-t}\omega) \le U_1(\theta_{-1}\omega),$$

a.e. in $\mathcal{O}$ for all $t \ge 1$. Hence $F(\omega) = \{x \in L^\infty(\mathcal{O}) |\ \|x\|_{L^\infty(\mathcal{O})} \le \|U_1(\theta_{-1}\omega)\|_{L^\infty(\mathcal{O})}\}$ is a $\mathcal{D}$-absorbing set with absorption time $t \equiv 1$. ☐

Lemma 3.2 (Asymptotic compactness)**.** *i. The RDS φ is $\mathcal{D}$-asymptotically compact on each $L^p(\mathcal{O})$, $p \in [1,\infty)$.*

ii. If $(\mathcal{O}1)$ is satisfied, then there exists a compact $\mathcal{D}$-absorbing set K with $K(\omega) \subseteq C^0(\bar{\mathcal{O}})$ compact for each $\omega \in \Omega$. In particular, φ is $\mathcal{D}$-asymptotically compact on $L^\infty(\mathcal{O})$.

Proof. (i): Let $t_n \to \infty$, $D \in \mathcal{D}$ and $x_n \in D(\theta_{-t_n}\omega)$. In Proposition 3.1 we have proved the existence of a $\mathcal{D}$-absorbing random set F. Note

$$\begin{aligned}\varphi(t_n,\theta_{-t_n}\omega)x_n &= \varphi(1,\theta_{-1}\omega)\varphi(t_n-1,\theta_{-(t_n-1)}\theta_{-1}\omega)x_n \\ &\subseteq \varphi(1,\theta_{-1}\omega)F(\omega),\end{aligned}$$

for all $t_n \ge 2$. Since $F(\omega)$ is bounded in $L^\infty(\mathcal{O})$, by Theorem 1.31 $\varphi(1,\theta_{-1}\omega)F(\omega)$ is a set of uniformly continuous functions on each compact set $K \subseteq \mathcal{O}$ with modulus of continuity depending only on m, $\mathrm{dist}(K,\partial\mathcal{O})$ and $\|F(\omega)\|_{L^\infty(\mathcal{O})}$. Let $\{K_k | k \in \mathbb{N}\}$ be a sequence of compact sets in $\mathcal{O}$, such that $\mathcal{O} = \bigcup_{k\in\mathbb{N}} K_k$. For each $k \in \mathbb{N}$ we can choose a convergent subsequence of $\varphi(t_n,\theta_{-t_n}\omega)x_n \in C^0(K_k)$. Passing to a diagonal sequence, we can thus choose a subsequence (again denoted by n) such that $\varphi(t_n,\theta_{-t_n}\omega)x_n$ is convergent in each $C^0(K_k)$ and in particular pointwisely convergent in all of $\mathcal{O}$. By the uniform $L^\infty(\mathcal{O})$ bound on $\varphi(t_n;\theta_{-t_n}\omega)x_n$ this implies convergence of $\varphi(t_n;\theta_{-t_n}\omega)x_n$ in $L^p(\mathcal{O})$, for each $p \in [1,\infty)$.

(ii): By Theorem 1.31, (iii) the set $K(\omega) := \varphi(1,\theta_{-1}\omega)F(\omega)$ is uniformly bounded and equicontinuous in $C^0(\bar{\mathcal{O}})$. Since $F(\omega)$ is absorbing, so is $\varphi(1,\theta_{-1}\omega)F(\omega)$. ☐

Proof of Theorem 1.32. Let $\mathcal{D}^p$ be the universe of all random sets in $L^p(\mathcal{O})$, $p \in [1,\infty]$.

The (unique) existence of a $\mathcal{D}^p$-random attractor A^p in $L^p(\mathcal{O})$ follows from $\mathcal{D}^p$-absorption, $\mathcal{D}^p$-asymptotic compactness, quasi-weak-continuity of φ on $L^p(\mathcal{O})$ and Theorem 1.29 for each

$p \in [1, \infty)$. Since F in Proposition 3.1 is an L^∞ bounded set absorbing all sets in $\mathcal{D}^1$, all these attractors coincide.

By the invariance property of the random attractor and Proposition 3.1 we have $A(\omega) = \varphi(t, \theta_{-t}\omega)A(\theta_{-t}\omega) \subseteq F(\omega)$, for all $t \geq 1$ and thus L^∞ boundedness of A. Again by invariance of A, $A(\omega) = \varphi(1, \theta_{-1}\omega)A(\theta_{-1}\omega) \subseteq \varphi(1, \theta_{-1}\omega)F(\theta_{-1}\omega)$. Invoking Theorem 1.31 yields equicontinuity on each compact set $K \subseteq \mathcal{O}$.

If $(\mathcal{O}1)$ is satisfied, then we can argue as above for $p = \infty$. $\square$

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