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MULTI-VALUED, SINGULAR STOCHASTIC EVOLUTION INCLUSIONS

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Abstract. We provide an abstract variational existence and uniqueness result for multi-valued, monotone, non-coercive stochastic evolution inclusions in Hilbert spaces with general additive and Wiener multiplicative noise. As examples we discuss certain singular diffusion equations such as the stochastic 1-Laplacian evolution (total variation flow) in all space dimensions and the stochastic singular fast diffusion equation. In case of additive Wiener noise we prove the existence of a unique weak-$*$ mean ergodic invariant measure.

1. Introduction

We consider the following evolution inclusion in a separable Hilbert space H

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni dg_t, \quad t > 0, \\ X_0 &= x. \end{aligned} \tag{1.1}$$

Here A is a possibly multi-valued, singular, maximal monotone operator and g is a càdlàg path in H. The meaning of the expression dg_t will be specified below.

In particular, we are interested in stochastic evolution inclusions of the type

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni dN_t, \quad t > 0, \\ X_0 &= x, \end{aligned} \tag{1.2}$$

where $\{N_t\}_{t\geq 0}$ is a càdlàg, adapted H-valued stochastic process on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and in inclusions of the form

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni B_t(X_t)\,dW_t, \quad t > 0, \\ X_0 &= x, \end{aligned} \tag{1.3}$$

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where for some separable Hilbert space U, $B : [0,T]\times\Omega\times S \to L_2(U;H)$ takes values in the space of Hilbert-Schmidt operators from U to H and $\{W_t\}$ is a cylindrical Wiener process.

We prove the unique existence of solutions to (1.1)–(1.3) and the unique existence of a weak-$*$ mean ergodic invariant measure for (1.3) with B_t being constant.

The standard variational approach to (S)PDE of type (1.3) requires the drift operator A to be single-valued an to extend to a hemicontinuous, coercive operator $A : V \to V^*$ for some Gelfand triple $V \subseteq H \subseteq V^*$ (cf. [Par75,KR79,PR07,RRW07]). The reflexivity of V and V^* is crucial for the construction of solutions. Therefore, the standard approach cannot be applied to many highly singular (S)PDE such as the total variation flow, the two phase Stefan problem, plasma diffusion and the curve shortening flow. In all of these examples the the space V degenerates in the sense that V or V^* fail to be reflexive. While recently increasing interest has been paid to this kind of singular, possibly multi-valued SPDE (cf. e.g. [Răş81, Răş82, Răş96, BR97, BDP05, BDPR09a, Cio11, Ste11]), the unique existence of solutions could only be shown for additive noise and under strong dimensional restrictions. The principal idea of most of these works is the concept of (stochastic) evolution variational inequalities (EVI), thus weakening the notion of solutions to (1.3). The approach via EVI has multiple drawbacks. First, it relies on the transformation of (1.3) to a random PDE and hence is restricted to simple structures of noise, such as additive or linear multiplicative noise. Second, due to the weaker notion of solutions it is hard to prove uniqueness. In fact, so far uniqueness of EVI could only be proven in case of sufficiently regular additive noise. Third, the construction of solutions to EVI still requires a coercivity condition of the type

$$ {}_{V^*}\langle A(u), u\rangle_V \geq c\,\|u\|_V^\alpha . \tag{1.4} $$

for some $\alpha \geq 1$, $c > 0$, which leads to restrictions on the dimension or the coercivity exponent α.

In order to remedy these obstacles, we introduce another Hilbert space S, embedded compactly and densely into H, such that

$$ S \subseteq V \subseteq H \equiv H^* \subseteq V^* \subseteq S^*. $$

Subsequently, we will drop the intermediate space V and formulate the conditions of our hypotheses solely with respect to S. We assume that the drift A is maximal monotone and of at most linear growth in S^*. We are able to replace the strong coercivity assumption (1.4) by weak dissipativity in S formulated in an approximative sense (cf. (A4) below). A similar condition has been used in [RW08, Liu10]. The main idea in the construction of solutions is to use a vanishing viscosity approximation and to apply standard results from the theory of multi-valued time-dependent evolution inclusions (cf. [HP97, HP00]) for the approximating viscous equations.

For additive noise, we choose a pathwise approach to construct the solutions, which encompasses general noise, as Lévy noise and noise driven by fractional Brownian motion with arbitrary Hurst parameter, since no Itō formula is needed. Another advantage of a pathwise construction is that the existence of a stochastic flow and a random dynamical system (RDS) are immediate consequences. Thus, assuming that the noise $\{N_t\}$ in (1.2) has càdlàg paths in S, satisfies some spatial regularity and has strictly stationary increments we prove that there is an RDS associated to (1.2). In case of additive Wiener noise this yields the Markov property for the associated semigroups $\{P_t\}$. Using stochastic calculus we prove that in case of multiplicative Wiener noise we can relax the spatial regularity assumptions on the noise while preserving the regularity of the solutions. For additive Wiener noise, we prove the existence and uniqueness of a weak-$*$ mean ergodic measure using recent methods by Komorowski, Peszat and Szarek [KPS10]. See also [ESvR10].

As examples we include certain singular diffusion equations such as the stochastic p-Laplacian evolution, $p \in [1,2)$ (including the total variation flow, i.e. $p = 1$, where the equation becomes a multi-valued inclusion) in all space dimensions

$$
\begin{aligned}
&dX_t = \operatorname{div}\left[|\nabla X_t|^{p-2}\nabla X_t\right] dt + \begin{cases} dN_t, \\ B_t(X_t)\,dW_t, \end{cases} \quad t>0, \\
&X_0 = x,
\end{aligned} \tag{1.5}
$$

generalizing the results of Barbu, Da Prato and Röckner [BDPR09a], known to hold only in space dimensions $d = 1,2$ and for Wiener additive noise. In addition, we solve the open problem posed in [BDPR09a] of uniqueness of the invariant measure for (1.5). See [Liu09] for other results on equation (1.5). The p-Laplace equation appears in geometry, quasi-regular mappings, fluid dynamics and plasma physics, see [DB93, Día85]. In [Lad67], Ladyženskaja suggests the p-Laplace evolution as a model of motion for non-Newtonian fluids. A typical 2-dimensional application can be found in image restoration (see [AVCM03, AK06]).

We also consider the stochastic singular fast diffusion equation for $r \in [0,1)$ (again including the critical, multi-valued case $r = 0$):

$$
\begin{aligned}
&dX_t = \Delta\left[|X_t|^{r-1}X_t\right] dt + \begin{cases} dN_t, \\ B_t(X_t)\,dW_t, \end{cases} \quad t>0, \\
&X_0 = x.
\end{aligned} \tag{1.6}
$$

This generalizes results given in [BDP10] where only additive case has been considered. Moreover, we solve the open problem posed in [BDP10] of uniqueness of the invariant measure for (1.6). All equations are treated with general additive càdlàg-noise and multiplicative Wiener noise in all space dimensions. The fast diffusion equation is, for example, used as a model in plasma physics (usually with more general nonlinearities) and self-organized criticality [BDPR09b]. See [BH80, Ros95, Váz06] and the references therein for further physical applications. Under dimensional restrictions, Liu and the second named author proved ergodicity and polynomial decay for equations (1.5), (1.6), excluding the limit cases $p = 1$, $r = 0$, see [LT11]. See also [Liu11, ESvRS10].

A further application of our general existence and uniqueness results, is the (1+1)-dim. stochastic curve shortening flow:

$$
\begin{aligned}
&dX_t(\xi) = \frac{\partial_\xi^2 X_t(\xi)}{1+(\partial_\xi X_t(\xi))^2}\,dt + \begin{cases} dN_t, \\ B_t(X_t)\,dW_t, \end{cases} \quad \xi \in [0,1], \quad t>0, \\
&X_0(\xi) = x(\xi),
\end{aligned} \tag{1.7}
$$

which has been studied by Es-Sarhir and von Renesse [ESvR10] (see also [ESvRS10]). We are able to generalize their result about unique existence to the case of general additive noise.

We are also able to treat the so called stochastic diffusion equation of plasma for any space dimension:

$$
\begin{aligned}
&dX_t = \Delta\left[\ln\left(|X_t|+1\right)\operatorname{sgn}(X_t)\right] dt + \begin{cases} dN_t, \\ B_t(X_t)\,dW_t, \end{cases} \quad t>0, \\
&X_0 = x,
\end{aligned} \tag{1.8}
$$

previously studied in an EVI setting in (1+1)-dimensions by Ciotir [Cio11]. See the references therein for the physical meaning of (1.8).

More generally, we are studying existence and uniqueness, as well as ergodicity (for additive Wiener noise), for generalized Φ-Laplacian equations with Neumann boundary conditions in a Riemannian manifold M of the type:

$$(1.9)\qquad \begin{aligned} &dX_t = \operatorname{div}[e^V\varphi(|\nabla X_t|)\operatorname{sgn}(\nabla X_t)]\,dt + \begin{cases} dN_t, \\ B_t(X_t)\,dW_t, \end{cases} \quad t>0,\\ &X_0 = x, \end{aligned}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a monotone function ($\varphi(0) = 0$) with sublinear growth and $V \in C^2(M)$ is a scalar potential. Dirichlet boundary conditions on a bounded domain $\Lambda \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, are also treated.

For general symmetric negative-definite Dirichlet operators L on abstract separable measure spaces $(E, \mathscr{B}, \mu)$ that satisfy a spectral gap condition, we consider the so called generalized fast-diffusion equation

$$(1.10)\qquad \begin{aligned} &dX_t \in L\left[\varphi(X_t)\right]\,dt + \begin{cases} dN_t, \\ B_t(X_t)\,dW_t, \end{cases} \quad t>0,\\ &X_0 = x, \end{aligned}$$

where $\varphi : \mathbb{R} \to 2^{\mathbb{R}}$ is a monotone graph ($0 \in \varphi(0)$) with sublinear growth.

Future applications to RDS and random attractors similar to [GLR11, BGLR11, Ges11] as well as continuity results in the parameter p in (1.5) (r in (1.6) resp.), similar to [CT11], are in preparation.

2. Deterministic case

Let H be a separable Hilbert space with dual H^*. Suppose that there is another Hilbert space S embedded densely and compactly into H. We thus have a Gelfand triple

$$S \subseteq H \subseteq S^*$$

and it holds that

$${}_{S^*}\langle v, u\rangle_S = (v,u)_H, \quad \text{whenever } u \in S,\ v \in H.$$

Let $i_S : S \to S^*$ denote the Riesz map of S. We note that the scalar product $(\cdot,\cdot)_S$ defines a bilinear, S-bounded, S-coercive form on H. By the Lax-Milgram Theorem there is a linear, positive definite, self-adjoint operator $T : D(T) \subseteq H \to H$ with $D(T^{1/2}) = S$ and $(T^{1/2}u, T^{1/2}v)_H = (u,v)_S$. We define $J_n = (1 + \frac{T}{n})^{-1}$, $n \in \mathbb{N}$, to be the resolvent associated to T and $T_n = TJ_n = n(1 - J_n)$ to be the Yosida approximation of T. Then

$$(x,y)_n := (x, T_n y)_H, \quad x, y \in H,$$

form a sequence of new inner products on H, the induced norms $\|\cdot\|_n$ are all equivalent to $\|\cdot\|_H$ and

$$\forall x \in S \ : \quad \|x\|_n \uparrow \|x\|_S \quad \text{as } n \to \infty.$$

Let $H_n := (H, (\cdot,\cdot)_n)$. We get a sequence of new Gelfand triples

$$S \subseteq H_n \subseteq S^*.$$

Moreover,

$$T_n\restriction_S : S \to S$$

is continuous, the Riesz-map $i_S : S \to S^*$ is given by the extension of T to $T : S \to S^*$ and thus

$$2\,{}_{S^*}\langle i_S(x), T_n(x)\rangle_S = 2(x, T_n x)_S = (T^{1/2}x, T_n T^{1/2}x)_H \geq 0, \quad \forall x \in S.$$

Let $g : [0,T] \to S$ be a càdlàg function (for simplicity $g(0) = 0$). We consider the integral evolution inclusion

$$(2.1)\qquad X_t \in x - \int_0^t A(s, X_s)\,ds + g_t,$$

where $A : [0,T] \times S \to 2^{S^*}$ is a multi-valued operator. We will construct solutions to this equation by first using the transformation $Y_t = X_t - g_t$ which leads to the evolution inclusion

$$\begin{cases} dY_t + A(t, Y_t + g_t)\,dt \ni 0, & t > 0, \\ Y_0 = x. \end{cases} \tag{2.2}$$

We will prove unique existence for (2.2) which leads to unique existence for (2.1) by transforming back, i.e. by defining $X_t := Y_t + g_t$.

Let $W^{1,2}(0,T)$ be the Bochner-Sobolev space associated to the embedding $S \subseteq H$ (cf. e.g. [Sho97, §III.1]) and let $D([0,T];H)$ be the space of all càdlàg functions in H endowed topology of *uniform convergence*. A solution of (2.2) is a function $Y \in W^{1,2}(0,T)$ such that

$$\frac{d}{dt} Y_t = -\zeta_t, \qquad \text{for a.e. } t \in [0,T],$$

$Y_0 = x$ and $\zeta \in L^2([0,T];S^*)$ such that $\zeta_t \in A(t, Y_t + g_t)$ for a.e. $t \in [0,T]$.

Definition 2.1. *A* solution *of* (2.1) *is a càdlàg function $X \in D([0,T];H)$ such that*

$$X_t = x - \int_0^t \eta_s\, ds + g_t,$$

for all $t \in [0,T]$ as an equation in S^*, where $\eta \in L^2([0,T];S^*)$ such that $\eta_t \in A(t, X_t)$ for a.e. $t \in [0,T]$.

Definition 2.2. *Let $(E, \mathscr{B}, \mu)$ be a σ-finite complete measure space and Y be a Polish space. A map $F : E \to 2^Y$ with non-empty, closed values is called* measurable *if $\{F(\cdot) \cap O \neq \varnothing\} \in \mathscr{B}$ for each open set $O \subseteq Y$.*

Hypothesis 1. *Suppose that $A : [0,T] \times S \to 2^{S^*}$ satisfies the following conditions: There is a constant $C > 0$ such that*

(A1) *For all $t \in [0,T]$, the map $x \mapsto A(t,x)$ is maximal monotone with nonempty values.*

(A2) *Linear growth: For all $x \in S$, for Lebesgue a.a. $t \in [0,T]$, for all $y \in A(t,x)$:*
$$\|y\|_{S^*} \le f(t) + C\,\|x\|_S\,,$$
with some $f \in L^2([0,T])$.

(A3) *Weak dissipativity in S: For all $x \in S$, for Lebesgue a.a. $t \in [0,T]$, for all $y \in A(t,x)$, and for all $n \in \mathbb{N}$:*
$$2\,{}_{S^*}\langle y, T_n(x)\rangle_S \ge -\gamma(t) - C\,\|x\|_S^2\,,$$
with some $\gamma \in L^1([0,T])$.

(A4) *Measurability: The map $t \mapsto A(t,x)$ is measurable w.r.t. the Lebesgue σ-algebra for all $x \in S$.*

Note that (A4) can be dropped if A is independent of time.

Hypothesis 2. $g \in L^2([0,T]; D(T^{3/2}))$.

Here, the operator $(T^{3/2}, D(T^{3/2}))$ is defined in terms of the spectral theorem and the Hilbert space $D(T^{3/2})$ is equipped with the graph norm.

In order to construct solutions to (2.2) we will consider a viscosity approximation. Let $\varepsilon > 0$ and consider the perturbed problem

$$\begin{cases} \dfrac{d}{dt} Y_t^\varepsilon + \varepsilon i_S(Y_t^\varepsilon) \in -A(t, Y_t^\varepsilon + g_t), & t > 0, \\ Y_0^\varepsilon = x, \end{cases} \tag{2.3}$$

The unique existence of variational solutions to these approximating problems is proved in Proposition 6.1 below. Application of the back transformation for the approximating equation yields

$$X_t^\varepsilon \in x - \int_0^t A(s, X_s^\varepsilon) - \varepsilon i_S(X_s^\varepsilon - g_s)\, ds + g_t, \quad t > 0. \tag{2.4}$$

Letting $\varepsilon \to 0$ we will prove the existence of solutions to (2.2). Transforming back we obtain

Theorem 2.3. *Assume Hypotheses 1 and 2 and let $x \in S$. Then* (2.1) *has a unique solution in the sense of Definition 2.1 satisfying $X \in L^\infty([0,T]; S)$ with*

$$\sup_{t\in[0,T]} \|X_t^\varepsilon - X_t\|_H^2 + \int_0^T \|X_t^\varepsilon - X_t\|_S^2\, dt \to 0$$

and X is right continuous in S.

Proof. See section 6.1.1. □

By monotonicity of the drift we can extend the unique existence of solutions to all initial conditions $x \in H$ at the dispense of allowing for limiting solutions in the sense

Definition 2.4 (Limit solution)**.** *We say that a function $X \in D([0,T]; H)$ is a* limit solution *to* (2.1) *with starting point $x \in H$, if $X_0 = x$ and for each approximation $x^\delta \in S$ with $x^\delta \to x$ in H the associated solution X^δ converges to X in $D([0,T]; H)$.*

We obtain

Theorem 2.5 (Extension to all initial conditions $x \in H$)**.** *Suppose that Hypotheses 1 and 2 hold and let $x \in H$. Then there is a unique limit solution X.*

3. Stochastic evolution inclusions with additive noise

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space (not necessarily complete, nor right-continuous), $N : [0,T] \times \Omega \to S$ be an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted stochastic process with càdlàg paths in S and $N_0 = 0$. We define $L^0(\Omega, \mathscr{F}_0; H)$ to be the space of all $\mathscr{F}_0$-measurable, H-valued random variables and let $x \in L^0(\Omega, \mathscr{F}_0; H)$. For each $\omega \in \Omega$ we consider the following integral equation in S^*

$$X_t(\omega) \in x(\omega) - \int_0^t A(s, X_s(\omega))\, ds + N_t(\omega). \tag{3.1}$$

Definition 3.1. *An $\{\mathscr{F}_t\}_{t\in[0,T]}$-adapted stochastic process $X : [0,T] \times \Omega \to H$ is a* pathwise (limit) solution *to* (3.1) *with starting point $x \in L^0(\Omega, \mathscr{F}_0; H)$ if for all $\omega \in \Omega$: $X(\omega)$ is a (limit) solution for* (3.1) *with $g_\cdot = N_\cdot(\omega)$.*

Setting $g_t := N_t(\omega)$ for fix $\omega \in \Omega$, Theorem 2.3 and Theorem 2.5 yield the existence of a pathwise (limit) solution X as long as A satisfies Hypothesis 1 and $N_\cdot(\omega)$ satisfies Hypothesis 2 for each $\omega \in \Omega$. The $\{\mathscr{F}_t\}_{t\in[0,T]}$-adaptedness of X is proved in Section 6.2 below. We obtain

Theorem 3.2. *Assume that A satisfies Hypothesis 1 and N satisfies Hypothesis 2 pathwisely. For $x \in L^0(\Omega, \mathscr{F}_0; S)$ there is a unique pathwise solution to* (3.1) *in the sense of Definition 3.1 satisfying $X(\omega) \in L^\infty([0,T]; S)$ and*

$$\sup_{t\in[0,T]} \|X_t^\varepsilon(\omega) - X_t(\omega)\|_H^2 + \int_0^T \|X_t^\varepsilon(\omega) - X_t(\omega)\|_S^2\, dt \to 0, \quad \forall \omega \in \Omega.$$

Moreover, $X(\omega)$ is right continuous in S. For $x \in L^0(\Omega, \mathscr{F}_0; H)$ there is a unique pathwise limit solution to (3.1).

If the noise is two-sided and strictly stationary then the solutions generate a random dynamical system (RDS). Let $((\Omega, \mathscr{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system, i.e. $(t, \omega) \mapsto \theta_t(\omega)$ is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F})$ measurable, $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t \circ \theta_s$ and θ_t is $\mathbb{P}$-preserving, for all $s, t \in \mathbb{R}$. We assume that $N : \mathbb{R} \times \Omega \to S$ satisfies

Hypothesis 3.

(N) *For all $t \geq s$ and $\omega \in \Omega$*

$$N_t(\omega) - N_s(\omega) = N_{t-s}(\theta_s \omega).$$

By [GLR11, Lemma 3.1] for each S valued process $\widetilde{N}_t$ with $\widetilde{N}_0 = 0$ a.s., stationary increments and a.s. càdlàg paths there exists a metric dynamical system $((\Omega, \mathscr{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ and a version N_t of $\widetilde{N}_t$ on $((\Omega, \mathscr{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ such that N_t satisfies (N). In particular, applications include all Lévy processes and fractional Brownian motion with arbitrary Hurst parameter.

Since we constructed the solution pathwisely we obtain

Corollary 3.3. *Assume that A satisfies Hypothesis 1, is independent of time t and N satisfies Hypothesis 2 and 3 for all $\omega \in \Omega$. Then*

$$\varphi(t, \omega)x := X_t^x(\omega), \quad x \in H,\ t \in \mathbb{R}_+,\ \omega \in \Omega$$

defines a continuous RDS associated to (1.2)*, where $X_t^x(\omega)$ is the pathwise limit solution starting at x obtained in Theorem 3.2.*

4. Stochastic evolution inclusions with multiplicative noise

If the random perturbation is given by a stochastic integral with respect to a Wiener process we can use Itō's formula to derive solutions to the corresponding stochastic partial differential equation even if the diffusion coefficients only take values in $L_2(U, H)$ (i.e. Hypothesis 2 is not satisfied). In case of $L_2(U, S)$ valued noise the S regularity of the solution is preserved. This allows much rougher noise, since Hypothesis 2 is not required anymore.

Using a fixed point argument we can then extend the existence of solutions to the case of multiplicative noise. For noise taking values in $L_2(U, S)$ and initial data $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; S)$ we will obtain variational solutions, while for less regular initial data and noise these will be extended to limit solutions.

In order to be able to use stochastic calculus we require a normal filtered probability space $(\Omega, \widehat{\mathscr{F}}, \{\widehat{\mathscr{F}_t}\}_{t\geq 0}, \mathbb{P})$ with a cylindrical Wiener process $\{W_t\}_{t\geq 0}$ in U, where U is some separable Hilbert space. We further require the diffusion coefficients $B : [0, T] \times \Omega \times S \to L_2(U, H)$ to be progressively measurable (i.e. for every $t \in [0, T]$ the map $B : [0, t] \times \Omega \times S \to L_2(U; H)$ is $\mathscr{B}([0, t]) \otimes \widehat{\mathscr{F}_t} \otimes \mathscr{B}(S)$-measurable) and the following random version of Hypothesis 2:

Hypothesis 4. (B1) *There is an $h \in L^1([0, T] \times \Omega)$ such that*

$$\|B_t(x)\|^2_{L_2(U,H)} \leq C\|x\|^2_S + h_t, \quad \textit{(Growth)},$$

for all $t \in [0, T]$, $x \in S$ and $\omega \in \Omega$.

(B2) *There is a $C > 0$ such that*

$$\|B_t(x) - B_t(y)\|^2_{L_2(U,H)} \leq C\|x - y\|^2_H, \quad \textit{(Lipschitz continuity)},$$

for all $t \in [0, T]$, $x, y \in S$ and $\omega \in \Omega$.

For the existence of variational solutions we further require

Hypothesis 5. (B3) *There is an $h \in L^1([0, T] \times \Omega)$ such that*

$$\|B_t(x)\|^2_{L_2(U,S)} \leq C\|x\|^2_S + h_t, \quad \textit{(S-Growth)},$$

for all $t \in [0, T]$, $x \in S$ and $\omega \in \Omega$.

Definition 4.1. *We say that a continuous $\{\widehat{\mathscr{F}_t}\}_{t\geq 0}$-adapted stochastic process $X : [0,T]\times\Omega \to H$ is a solution to*

$$\begin{aligned} dX_t + A(t, X_t)\,dt &\ni B_t(X_t)\,dW_t \\ X(0) &= x \end{aligned} \tag{4.1}$$

if $X \in L^2(\Omega; C([0,T]; H)) \cap L^2([0,T]\times\Omega; S)$ and X solves the following integral equation in S^*

$$X_t = X_0 - \int_0^t \eta_s\,ds + \int_0^t B_s(X_s)\,dW_s,$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, where $\eta \in A(\cdot, X)$, $dt \otimes \mathbb{P}$-a.e.

Theorem 4.2 (Multiplicative noise)**.** *Let $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; S)$. Assume that A satisfies Hypothesis 1 and B satisfies Hypothesis 4 and 5. Then there exists a unique solution X to* (4.1) *in the sense of Definition 4.1 satisfying*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_S^2 < \infty$$

and X is $\mathbb{P}$-a.s. right-continuous in S.

Proof. See section 6.3.2. □

Using monotonicity of the drift and Lipschitz continuity of the noise, we can extend the existence result to every initial condition $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; H)$ and driving noise taking values in $L_2(U, H)$ in a limiting sense.

Definition 4.3. *An $\{\widehat{\mathscr{F}_t}\}_{t\in[0,T]}$-adapted stochastic process $X \in L^2(\Omega; C([0,T]; H))$ is a limit solution to* (4.1) *with starting point $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; H)$ if for all approximations $x^\delta \in L^2(\Omega, \widehat{\mathscr{F}_0}; S)$ with $x^\delta \to x$ in $L^2(\Omega, \widehat{\mathscr{F}_0}; H)$ and B^δ satisfying Hypothesis 4 and 5 with $B^\delta(u) \to B(u)$ in $L^2([0,T]\times\Omega; L_2(U,H))$ for all $u \in S$ we have*

$$X^\delta \to X, \quad \textit{in } L^2(\Omega; C([0,T]; H)).$$

Theorem 4.4 (Multiplicative noise for all initial conditions)**.** *Let $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; H)$. Assume that A satisfies Hypothesis 1 and B satisfies Hypothesis 4. Then there exists a unique limit solution X to* (4.1) *in the sense of Definition 4.3.*

Proof. See section 6.3.3. □

5. Ergodicity

In the following assume A, B to be independent of $(t,\omega) \in [0,T]\times\Omega$ and to satisfy Hypotheses 1 and 5 with f, γ and h being constant. We further restrict to the case of additive Wiener noise. Since B satisfies Hypothesis 5 the process BW_t is a trace class Wiener process in S in the following denoted by W_t^B. We consider the canonical realization of its two-sided extension: $\Omega := C(\mathbb{R}; S)$, $\mathscr{F}_t := \sigma\{\pi_s |\ s \in (-\infty, t]\}$, $\mathscr{F} := \sigma\left\{\bigcup_{t\in\mathbb{R}} \mathscr{F}_t\right\}$ and let $\mathbb{P}$ be the law of W^B on Ω. Define the Wiener shift to be $\theta_t(\omega) := \omega(t+\cdot) - \omega(t)$. Then $(\Omega, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, \{\theta_t\}_{t\in\mathbb{R}}, \mathbb{P})$ is a metric dynamical system, the evaluation process π_t is a trace-class Wiener process, which by abuse of notation is again denoted by W_t^B and W_t^B satisfies Hypothesis 3. Let $\{\widehat{\mathscr{F}_t}\}$ be the right-continuous completion of $\{\mathscr{F}_t\}$. We consider evolution inclusions of the form

$$\begin{aligned} dX_t + A(X_t)\,dt &\ni dW_t^B, \quad t > 0 \\ X_0 &= x. \end{aligned} \tag{5.1}$$

We denote by $\mathscr{B}(H)$ the set of all Borel measurable subsets of H, by $\mathscr{B}_b(H)$ (resp. $C_b(H)$) the Banach space of all bounded, measurable (resp. continuous) functions on H equipped with the supremum norm and by $\mathrm{Lip}_b(H)$ the space of all bounded

Lipschitz continuous functions on H. By $\mathscr{M}_1$ we denote the set of all Borel probability measures on H. For a semigroup $\{P_t\}$ on $\mathscr{B}_b(H)$ we define the dual semigroup $\{P_t^*\}$ on $\mathscr{M}_1$ by $P_t^*\mu(B) := \int_H P_t \mathbb{1}_B d\mu$, for $B \in \mathscr{B}(H)$. A measure $\mu \in \mathscr{M}_1$ is said to be invariant for the semigroup P_t if $P_t^*\mu = \mu$, for all $t \geq 0$. For $T > 0$ and $\mu \in \mathscr{M}_1$ we define

$$Q^T\mu := \frac{1}{T}\int_0^T P_r^*\mu\, dr$$

and we write $Q^T(x,\cdot)$ for $\mu = \delta_x$. We recall:

Definition 5.1. *A semigroup $\{P_t\}$ is called weak-$*$ mean ergodic if there exists a measure $\mu_* \in \mathscr{M}_1$ such that*

$$w\text{-}\lim_{T\to\infty} Q^T\nu = \mu_*,$$

for all $\nu \in \mathscr{M}_1$.

Let $X(t,s;\omega)x$ be the solution to (5.1) starting at time $s \in \mathbb{R}$ in $x \in H$, with respect to the Wiener process $\widetilde{W}_t^B = W_{t+s}^B - W_s^B$ (cf. [PR07, p. 105]). We define

$$P_x := (X(\cdot,0,\cdot)x)_*\mathbb{P},$$

to be the law on $C([0,\infty),H)$ of the solution $X(\cdot,0,\cdot)x$ viewed as a random variable taking values in $C([0,\infty),H)$. By Itō's formula

$$\mathbb{E}\sup_{t\in[0,T]} \|X(t,0,\cdot)x - X(t,0,\cdot)y\|_H^2 \leq \|x-y\|_H^2. \tag{5.2}$$

Proposition 5.2. *The family $\{P_x\}_{x\in H}$ defines a time-homogeneous Markov process on $C([0,\infty),H)$ with respect to the filtration $\{\widehat{\mathscr{F}}_t\}_{t\in\mathbb{R}_+}$, i.e.*

$$\mathbb{E}_x[F(\pi_{t+s})|\widehat{\mathscr{F}}_s] = \mathbb{E}_{\pi_s}[F(\pi_t)], \quad \forall F \in \mathscr{B}_b(H),\ P_x\text{-a.s.},$$

where $\mathbb{E}_x$, $\mathbb{E}_x[\cdot|\widehat{\mathscr{F}}_s]$ denote (conditional) expectation with respect to P_x.

Proof. See section 6.4. □

We define

$$P_tF(x) := \mathbb{E}\left[F(X(t,0,\cdot)x)\right], \quad F \in C_b(H),\ x \in H$$

to be the Feller semigroup associated to the stochastic flow $X(t,s,\omega)x$. By Proposition 5.2 the semigroup property is satisfied. The so-called e-property

$$|P_tF(x) - P_tF(y)| \leq \mathrm{Lip}(F)\,\|x-y\|_H\,, \quad x,y \in H, \tag{5.3}$$

for all $F \in \mathrm{Lip}_b(H)$ follows from (5.2). Consider the assumption

Hypothesis 6. (A5) *There are constants $C, c > 0$ such that*

$$2\,{}_{S^*}\langle y,x\rangle_S \geq c\|y\|_{S^*} - C,$$

for all $[x,y] \in A$.

Let $u(\cdot,0)x \in C([0,T];H)$ denote the unique solution to

$$\begin{aligned} du_t + A(u_t)dt &\ni 0,\\ u_0 &= x \in H. \end{aligned} \tag{5.4}$$

The unique existence of such a solution u follows from Theorem 2.3 and Theorem 2.5 with $g \equiv 0$.

Definition 5.3. *We say that* finite time extinction *holds for equation* (5.4)*, if for all bounded sets $B \subseteq H$ there exists an extinction time $T_B \geq 0$ for the solution $\{u(\cdot,0)x\}$ to* (5.4) *such that $u(t,0)x \equiv 0$ for all $t \geq T_B$ and for all $x \in B$.*

Hypothesis 7. *Suppose that there exists a measurable* Lyapunov function $\Theta : H \to [0, +\infty]$ *satisfying*

$$_{S^*}\langle y, x\rangle_S \geq \Theta(x), \quad \forall x \in S, \ \forall y \in A(x).$$

and consider the following conditions

(L1) *Finite time extinction holds for* (5.4) *and* Θ *has bounded sublevel sets.*

(L2) *The solutions* $\{u(\cdot, 0)x\}$ *to* (5.4) *satisfy* $u(t, 0)x \to 0$ *for* $x \in H$ *as* $t \to \infty$ *and* Θ *has compact sublevel sets.*

A simple sufficient condition for Hypothesis 7, (L1) is given by

Remark 5.4. *Suppose that there is a constant* $c > 0$ *such that*

$$2\,{}_{S^*}\langle y, x\rangle_S \geq c\|x\|_H^\alpha, \tag{5.5}$$

for some $1 \leq \alpha < 2$ *and all* $[x, y] \in A$*. Then Hypothesis 7,* (L1) *holds.*

Proof. See section 6.4. □

Then, using Theorem [KPS10, Theorem 2], we prove

Theorem 5.5. *Assume Hypotheses 1, 5, 6 and* (L1) *or* (L2) *in Hypothesis 7. Then, there exists a unique invariant measure* μ_* *for* $\{P_t\}$ *and*

(1) The semigroup $\{P_t\}$ *is weak-$*$ mean ergodic.*

(2) For any $\psi \in \mathrm{Lip}_b(H)$ *and* $\mu \in \mathscr{M}_1$ *the weak law of large number holds, i.e.*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \psi(\pi_s)\, ds = \int_H \psi\, d\mu_*,$$

in $\mathbb{P}_\mu$*-probability, where* $\mathbb{P}_\mu$ *is the law of the Markov process* π *started with initial distribution* μ*.*

Proof. See section 6.4. □

6. Proofs

6.1. **Deterministic case.**

6.1.1. *Deterministic case with initial data* $x \in S$ *(Theorem 2.3).*

We first prove that the approximating equation (2.3) has a unique solution:

Proposition 6.1. *Assume Hypothesis 1:* (A1), (A2), (A4), *Hypothesis 2 and let* $x \in H$*. Then equation* (2.3) *has a unique solution* $Y^\varepsilon \in C([0, T]; H)$ *in the sense that*

$$Y_t^\varepsilon = x + \int_0^t \zeta_s^\varepsilon\, ds + g_t,$$

for all $t \in [0, T]$ *as an equation in* S^**, where* $\zeta^\varepsilon \in L^2([0, T]; S^*)$ *such that* $\zeta_t^\varepsilon \in -\varepsilon i_S(Y_t^\varepsilon) - A(t, Y_t^\varepsilon + g_t)$ *for a.e.* $t \in [0, T]$*.*

Proof. We aim to apply Theorem A.3 with

$$F(t, x) = -A(t, x + g_t) - \frac{\varepsilon}{2} i_S(x),$$

$$J(t, x) = \frac{\varepsilon}{2} i_S(x),$$

and $u_0 = x$. Obviously (J1)–(J4) are satisfied. We check the conditions on F:

(F1): A consequence of maximal monotonicity (A1), see [Bar93, Ch. 2, Proposition 1.1].

(F2): Let $x \in S$. By [HP97, Ch. 3, Theorem 1.28] and maximal monotonicity of $x \mapsto A(t, x)$ for all $t \in [0, T]$, $x \mapsto A(t, x)$ is strongly-to-weakly uppersemicontinuous. (A4) and [Zyg92, Theorem 1, Theorem 2] imply that $(t, x) \mapsto$

$A(t,x)$ is product measurable and hence superpositionally measurable. $F+(\varepsilon/2)i_S$ is the composition of the càdlàg function $t \mapsto (t, x+g_t)$ and A, thus $t \mapsto F(t,x)$ is measurable.

(F3): Since $A(t,\cdot)$ is maximal monotone so is $F(t,\cdot)$ for each $t \in [0,T]$, which implies sequential closedness in $S \times S^*_w$, see [Bar93, Ch. 2, Proposition 1.1].

(F4): Obvious by (A2) and Hypothesis 2.

(F5): We first note that (A1) combined with (A2) implies a weak coercivity property for A:

$$\begin{aligned} {}_{S^*}\langle y, x\rangle_S &= {}_{S^*}\langle y-z, x-0\rangle_S + {}_{S^*}\langle z, x\rangle_S \\ &\geq -\|z\|_{S^*}\|x\|_S \geq -f(t)\|x\|_S, \end{aligned} \tag{6.1}$$

for all $t \in [0,T]$ and $[x,y] \in A(t)$ and $z \in A(t,0)$. For $t \in [0,T]$, $[x,y] \in F(t)$, $z \in A(t, x+g_t)$:

$$\begin{aligned} {}_{S^*}\langle y, x\rangle_S &= -{}_{S^*}\langle z, x\rangle_S - \frac{\varepsilon}{2}\|x\|_S^2 = -{}_{S^*}\langle z, x+g_t\rangle_S - {}_{S^*}\langle z, g_t\rangle_S - \frac{\varepsilon}{2}\|x\|_S^2 \\ &\leq f(t)\|x+g_t\|_S + \|z\|_{S^*}\|g_t\|_S - \frac{\varepsilon}{2}\|x\|_S^2 \\ &\leq C(\varepsilon)(f(t)^2 + f(t)\|g_t\|_S + \|g_t\|_S^2) \end{aligned}$$

and the last term is in $L^1([0,T])_+$ by (A2) and Hypothesis 2.

Application of Theorem A.3 thus yields the existence of a solution and monotonicity of F, J implies uniqueness. $\square$

Lemma 6.2. *Under Hypothesis 1 and 2 for starting points $x \in S$, we have that*

$$\|Y_t^\varepsilon\|_S^2 \leq C\,\|x\|_S^2 + C\int_0^t e^{-Cs}\left(\gamma(s) + f_s^2 + \|T^{3/2}g_s\|_H^2\right) ds < \infty.$$

Proof. By Proposition 6.1 we know that $Y^\varepsilon \in L^2([0,T];S)$. Apply the chain-rule [Sho97, §III.4] to $\|\cdot\|_n^2$ to obtain that

$$\begin{aligned} \|Y_t^\varepsilon\|_n^2 e^{-Kt} &\leq \|x\|_n^2 - 2\int_0^t e^{-Ks}\,{}_{S^*}\langle \eta_s^\varepsilon + \varepsilon i_S(Y_s^\varepsilon), T_n(Y_s^\varepsilon)\rangle_S\,ds - K\int_0^t e^{-Ks}\,\|Y_s^\varepsilon\|_n^2\,ds \\ &= \|x\|_n^2 - 2\int_0^t e^{-Ks}\,{}_{S^*}\langle \eta_s^\varepsilon, T_n(Y_s^\varepsilon + g_s)\rangle_S\,ds + 2\int_0^t e^{-Ks}\,{}_{S^*}\langle \eta_s^\varepsilon, T_n(g_s)\rangle_S\,ds \\ &\quad - 2\varepsilon\int_0^t e^{-Ks}(Y_s^\varepsilon, T_n(Y_s^\varepsilon))_S\,ds - K\int_0^t e^{-Ks}\,\|Y_s^\varepsilon\|_n^2\,ds \\ &\leq \|x\|_n^2 + \int_0^t e^{-Ks}\left(\gamma(s) + C\|Y_s^\varepsilon + g_s\|_S^2\right) ds + \int_0^t e^{-Ks}\|\eta_s^\varepsilon\|_{S^*}^2\,ds \\ &\quad + \int_0^t e^{-Ks}\|T_n g_s\|_S^2\,ds - K\int_0^t e^{-Ks}\,\|Y_s^\varepsilon\|_n^2\,ds \\ &\leq \|x\|_n^2 + C\int_0^t e^{-Ks}\|Y_s^\varepsilon\|_S^2\,ds - K\int_0^t e^{-Ks}\,\|Y_s^\varepsilon\|_n^2\,ds \\ &\quad + C\int_0^t e^{-Ks}\left(\gamma(s) + f_s^2 + \|Tg_s\|_S^2\right) ds, \end{aligned}$$

where $\eta_s^\varepsilon \in A(Y_s^\varepsilon + g_s)$ ds-a.e. Taking $K = C$ and $n \to \infty$ yields

$$\|Y_t^\varepsilon\|_S^2 e^{-Ct} \leq \|x\|_S^2 + C\int_0^t e^{-Cs}\left(\gamma(s) + f_s^2 + \|T^{3/2}g_s\|_H^2\right) ds. \tag{6.2}$$

$\square$

We are now ready to prove unique existence of solutions for initial data $x \in S$, i.e.

Proof of Theorem 2.3. For $\varepsilon > 0$ let Y^ε denote the solution to (2.3) with corresponding selection $\eta^\varepsilon \in A(\cdot, Y^\varepsilon + g)$. By Lemma 6.2 $\|Y^\varepsilon\|_{L^\infty([0,T];S)} \le C$. Hence, there is a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ such that

$$Y^{\varepsilon_n} \rightharpoonup^* Y$$

weakly-$*$ in $L^\infty([0,T];S)$. By the chain-rule (cf. [Sho97, §III.4])

$$\begin{aligned}(6.3)\quad \left\|Y_t^\varepsilon - Y_t^\delta\right\|_H^2 \le -2\int_0^t {}_{S^*}\left\langle \eta_s^\varepsilon - \eta_s^\delta, Y_s^\varepsilon - Y_s^\delta\right\rangle_S \, ds \\ - 2\int_0^t {}_{S^*}\left\langle \varepsilon i_S(Y_s^\varepsilon) - \delta i_S(Y_s^\delta), Y_s^\varepsilon - Y_s^\delta\right\rangle_S \, ds, \quad t \in [0,T],\end{aligned}$$

for all $0 < \delta < \varepsilon$. Hence,

$$\int_0^T {}_{S^*}\left\langle \varepsilon i_S(Y_s^\varepsilon) - \delta i_S(Y_s^\delta), Y_s^\varepsilon - Y_s^\delta\right\rangle_S \, ds \le 0.$$

By [BCP70, Lemma 1.4], $Y^{\varepsilon_n} \rightharpoonup Y$ weakly in $L^2([0,T];S)$ and

$$(6.4)\qquad \int_0^T \|Y_t^{\varepsilon_n}\|_S^2 \, dt \to \int_0^T \|Y_t\|_S^2 \, dt.$$

Hence $Y^{\varepsilon_n} \to Y$ strongly in $L^2([0,T];S)$. Then (6.3) implies $Y^{\varepsilon_n} \to Y$ in $C([0,T];H)$. By (A2) and Lemma 6.2 we can choose a further subsequence (again denoted by ε_n) such that

$$\eta^{\varepsilon_n} \rightharpoonup \eta, \quad \text{in } L^2([0,T];S^*).$$

By Proposition B.2 the operator $A : L^2([0,T];S) \to 2^{L^2([0,T];S^*)}$ is strongly-weakly closed and thus $[Y_t + g_t, \eta_t] \in A(t)$ for a.e. $t \in [0,T]$. By monotonicity the solution is unique and thus the whole sequence (net) $\{Y^\varepsilon\}$ converges strongly to Y in $L^2([0,T];S)$. Since Y is a unique solution to (2.2) $X_t := Y_t + g_t$ is a unique solution to (2.1).

We now prove $X \in L^\infty([0,T];S)$ and X is right-continuous in S. By the same calculation as for (6.2) we obtain

$$(6.5)\qquad \|Y_t\|_S^2 e^{-Ct} \le \|Y_s\|_S^2 e^{-Cs} + C\int_s^t e^{-Cs}\left(\gamma(s) + f_s^2 + \|T^{3/2} g_s\|_H^2\right) ds.$$

In particular

$$\sup_{t \in [0,T]} \|Y_t\|_S^2 < \infty,$$

which together with continuity in H implies weak continuity of Y. Let $t_n \in [0,T]$ with $t_n \downarrow t$. By (6.5)

$$\|Y_{t_n}\|_S^2 e^{-Ct_n} \le \|Y_t\|_S^2 e^{-Ct} + C\int_t^{t_n} e^{-Cs}\left(\gamma(s) + f_s^2 + \|T^{3/2} g_s\|_H^2\right) ds.$$

Hence, $\overline{\lim}_{n\to\infty} \|Y_{t_n}\|_S^2 \le \|Y_t\|_S^2$. By weak continuity we conclude $Y_{t_n} \to Y_t$ in S.

□

6.1.2. *Deterministic case with initial data $x \in H$ (Theorem 2.5).*

We proof Theorem 2.5. Let x^δ, $\bar{x}^\delta \in S$ be two approximations of x in H and Y^δ, $\bar{Y}^\delta$ be the variational solutions constructed in Theorem 2.3 corresponding to $x^\delta, \bar{x}^\delta \in S$. By the chain-rule

$$\begin{aligned}\|Y_t^\delta - \bar{Y}_t^\varepsilon\|_H^2 &= \|x^\delta - \bar{x}^\varepsilon\|_H^2 - 2\int_0^t {}_{S^*}\langle \eta_r^\delta - \bar{\eta}_r^\varepsilon, Y_r^\delta - \bar{Y}_r^\varepsilon\rangle_S \\ &\le \|x^\delta - \bar{x}^\varepsilon\|_H^2,\end{aligned}$$

where $\eta_r^\delta \in A(Y_r^\delta + g_r)$, $\bar{\eta}_r^\varepsilon \in A(\bar{Y}_r^\varepsilon + g_r)$. Thus Y^δ is a Cauchy sequence in $C([0,T];H)$ and every sequence of approximative solutions $\bar{Y}^\delta$ converges to the same limit Y in $C([0,T];H)$.

6.2. **Stochastic evolution inclusions with additive noise.**

Let $x \in L^0(\Omega, \mathscr{F}_0; S)$ and X be the corresponding pathwise solution to (3.1). It remains to prove $\{\mathscr{F}_t\}_{t\in[0,T]}$-adaptedness of X.

Under the assumptions Theorem 2.3 we have shown that there is a unique solution Y to (2.2). We now prove that the solution map

$$\begin{aligned}F : S \times \Big(L^2([0,T];S) \cap L^2_w([0,T];D(T^{3/2}))\Big) &\to C([0,T];H) \\ (x,g) &\mapsto Y,\end{aligned}$$

is continuous (here, the subscript "$_w$" denotes the weak Hilbert topology). Let $x^n \to x$ in S and $g^n \to g$ in $L^2([0,T];S)$ with $g^n \rightharpoonup g$ weakly in $L^2([0,T];D(T^{3/2}))$. The bound in Lemma 6.2 for Y^n, Y does not depend on n since g^n is uniformly bounded in $L^2([0,T];D(T^{3/2}))$. Hence $\eta_t^n \in A(t, Y_t^n + g_t^n)$, $\eta_t \in A(t, Y_t + g_t)$ are uniformly bounded in $L^2([0,T];S^*)$. By the chain rule ([Sho97, §III.4])

$$\begin{aligned}\|Y_t^n - Y_t\|_H^2 &= -2\int_0^t {}_{S^*}\langle \eta_s^n - \eta_s, Y_s^n - Y_s\rangle_S \, ds \\ &\le -2\int_0^t {}_{S^*}\langle \eta_s^n - \eta_s, g_s - g_s^n\rangle_S \, ds \\ &\le 2\|\eta_s^n - \eta_s\|_{L^2([0,T];S^*)}\|g - g^n\|_{L^2([0,T];S)} \to 0,\end{aligned}$$

which proves the claimed continuity.

By Kuratowski's Theorem the map $N\restriction_{[0,t]} : \Omega \to L^2([0,t];D(T^{3/2}))$ is $\mathscr{F}_t$-measurable for all $t \in [0,T]$. Hence, continuity of the solution map F implies $\{\mathscr{F}_t\}_{t\in[0,T]}$-adaptedness of Y and thus of X.

6.3. **Stochastic evolution inclusions with multiplicative noise.**

We will now prove Theorem 4.2 and Theorem 4.4. The proof consists of two main steps. First we consider the case of additive noise satisfying the weaker regularity properties in Hypothesis 4, 5 as compared to Hypothesis 2. In the second step we use the unique existence of solutions for additive noise in order to construct solutions to the case of multiplicative noise, using a fixed point argument.

6.3.1. *Stochastic evolution inclusions with additive Wiener noise.*

For the rest of this Section assume B to be independent of $x \in S$.

Proposition 6.3. *Let $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; S)$ and assume Hypotheses 1, 4 and 5. Then there is a solution X to* (4.1) *satisfying*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_S^2 < \infty$$

and X is $\mathbb{P}$-a.s. right continuous in S. If $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; H)$ and Hypotheses 1 and 4 are satisfied, then there is a unique limit solution to (4.1)*.*

Proof. We first make some general remarks concerning the approximation of elements $x \in S$. Since $T : D(T) \subseteq H \to H$ is an anticompact, self-adjoint operator, there is an orthonormal basis of eigenvectors $e_i \in S$ of H. Let $H_m = \operatorname{span}\{e_1, ..., e_m\}$ and $P_m : H \to H_m$ be the orthogonal projection onto H_m in H. Since $(\cdot,\cdot)_S = (T^{1/2}\cdot, T^{1/2}\cdot)_H$ the restriction of P_m to S is the orthogonal projection onto H_m in S. Moreover, P_m can be extended continuously to $P_m : S^* \to S$ with $\|P_m y\|_S \le C_m \|y\|_{S^*}$.

We consider the approximating equations

$$\begin{cases} dX_t^m + A(t, X_t^m)\, dt \ni B_t^m\, dW_t \\ \qquad\qquad\qquad\qquad X_0 = x, \end{cases} \tag{6.6}$$

where $B^m := P_m B$. Then

$$\|P_m B\|^2_{L^2([0,T]\times\Omega; L_2(U,S))} \le \|B\|^2_{L^2([0,T]\times\Omega; L_2(U,S))}$$

and $\|TP_m B\|^2_{L^2([0,T]\times\Omega; L_2(U,S))} \le C_m \|B\|^2_{L^2([0,T]\times\Omega; L_2(U,S))}$. Hence

$$N_t^m(\omega) := \int_0^t B_s^m\, dW_s(\omega)$$

satisfies Hypothesis 2 for a.a. $\omega \in \Omega$. By dominated convergence $B^m \to B$ in $L^2([0,T]\times\Omega; L_2(U,H))$ and by Theorem 2.3 there is a solution $X^m(\omega)$ to (6.6) for a.a. $\omega \in \Omega$.

Lemma 6.4. *There is $C > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^m\|_S^2\right] \le 4e^{CT}\left(\mathbb{E}\|x\|_S^2 + \mathbb{E}\int_0^T (h_s + \gamma_s)\, ds\right),$$

where γ is as in Hypothesis 1 and h as in Hypothesis 5.

Proof. We first note that the process $t \mapsto \int_0^t \eta_s^m\, ds$ (where $\eta^m \in A(\cdot, X^m)$ $dt \otimes \mathbb{P}$-a.s.) is progressively measurable and that under this weaker assumption Itō's formula (cf. [PR07, Theorem 4.2.5]) for the approximating norm $\|\cdot\|_n^2$ may still be applied. In fact, the same proof as for [PR07, Theorem 4.2.5] can be used. We obtain by (A3):

$$\begin{aligned}
\|X_t^m\|_n^2\, e^{-Ct} = {} & \|x\|_n^2 - \int_0^t 2e^{-Cs}\, {}_{S^*}\langle \eta_s^m, T_n(X_s^m)\rangle_S\, ds + \int_0^t 2e^{-Cs}(X_s^m, B_s^m\, dW_s)_n \\
& + \int_0^t e^{-Cs}\|B_s^m\|^2_{L_2(U,H_n)}\, ds - C\int_0^t e^{-Cs}\|X_s^m\|_n^2\, ds \\
\le {} & \|x\|_n^2 + \int_0^t e^{-Cs}\gamma_s\, ds + C\int_0^t e^{-Cs}\, \|X_s^m\|_S^2\, ds \\
& + \int_0^t 2e^{-Cs}(X_s^m, B_s^m\, dW_s)_n + \int_0^t e^{-Cs}\|B_s^m\|^2_{L_2(U,H_n)}\, ds \\
& - C\int_0^t e^{-Cs}\|X_s^m\|_n^2\, ds.
\end{aligned}$$

By Burkholder's inequality, choosing K large enough and taking $n \to \infty$, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^m\|_S^2\, e^{-Ct}\right] \le 2\left(\mathbb{E}\|x\|_S^2 + \mathbb{E}\int_0^T e^{-Cs}(2h_s+\gamma_s)\,ds\right).$$

□

We proceed with the proof of Proposition 6.3. By Itō's formula and Burkholder's inequality

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^n - X_t^m\|_H^2\right] \le C\mathbb{E}\int_0^T \|B_s^n - B_s^m\|_{L_2(U,H)}^2\,ds \to 0,$$

for $n,m\to\infty$. Hence, $X^n \to X$ in $L^2(\Omega; C([0,T];H))$. By lower semi-continuity of the norm $\|\cdot\|_{L^2(\Omega;C([0,T];S))}$ on $L^2(\Omega;C([0,T];H))$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_S^2\right] \le C\left(\mathbb{E}\|x\|_S^2 + \mathbb{E}\int_0^T (h_s+\gamma_s)\,ds\right). \tag{6.7}$$

By Lemma 6.4 and (A2), for some subsequence $\eta^m \rightharpoonup \bar\eta$ in $L^2([0,T]\times\Omega;S^*)$ and we obtain $\mathbb{P}$-a.s.

$$X_t = x - \int_0^t \bar\eta_s ds + \int_0^t B_s dW_s, \quad \forall t\in[0,T].$$

Applying Itō's formula to this equation and to (6.6), then subtracting the resulting equations yields

$$\begin{aligned}\mathbb{E}\left[\|X_t^m\|_H^2 - \|X_t\|_H^2\right] = &-\mathbb{E}\left[\int_0^t 2\,{}_{S^*}\langle \eta_r^m, X_r^m\rangle_S - 2\,{}_{S^*}\langle\bar\eta_r, X_r\rangle_S\,dr\right]\\ &+\mathbb{E}\int_0^t \|B_r^m - B_r\|_{L_2(U;H)}^2.\end{aligned}$$

Thus

$$\varlimsup_{m\to\infty} \mathbb{E}\int_0^t {}_{S^*}\langle\eta_r^m, X_r^m\rangle_S \le \mathbb{E}\int_0^t {}_{S^*}\langle\bar\eta_r, X_r\rangle_S\,dr.$$

By Proposition B.4, $A : L^2([0,T]\times\Omega;S) \to L^2([0,T]\times\Omega;S^*)$ is maximal monotone. Thus, Minty's trick implies $[X,\bar\eta]\in A$ $dt\otimes\mathbb{P}$-a.e.

Right continuity of X in S is shown as in the proof of Theorem 2.3, i.e. (6.7) yields weak continuity and repeating the calculations from Lemma 6.4 for all initial times $s\le t$ we can then deduce right-continuity.

For general initial conditions $x\in L^2(\Omega,\widehat{\mathscr{F}_0};H)$ and noise satisfying Hypothesis 4 only we consider approximations $x^m \in L^2(\Omega,\widehat{\mathscr{F}_0};S)$ and $B^m := P_m B$ with corresponding variational solutions X^m. Applying Itō's formula for the difference $\|X^m - X^n\|_H^2$ and using Burkholder's inequality yields

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^m - X_t^n\|_H^2\right] \le C\left(\mathbb{E}\|x^m - x^n\|_H^2 + \mathbb{E}\int_0^T \|B_s^n - B_s^m\|_{L_2(U,H)}^2\,ds\right),$$

which implies the existence of a limit X of X^m in $L^2(\Omega;C([0,T];H))$. For two approximating solutions X^m, $\bar X^m$ by the same argument we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t^m - \bar X_t^m\|_H^2\right] \le C\left(\mathbb{E}\|x^m - \bar x^m\|_H^2 + \mathbb{E}\int_0^T \|B_s^m - \bar B_s^m\|_{L_2(U,H)}^2\,ds\right),$$

This implies that the limit X does not depend on the approximating sequence. □

6.3.2. *Proof of Theorem 4.2.*

First let $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; S)$ and B satisfy Hypothesis 4 and Hypothesis 5. We construct a solution by freezing the noise. For $K > 0$ let

$$\mathscr{W}^K := \left\{ Z \in L^2(\Omega; C([0,T]; H)) | \ \|Z\|^2_{L^2(\Omega; C([0,T];S))} \leq K \right\}.$$

Since $\|\cdot\|^2_{L^2(\Omega;C([0,T];S))}$ is a lower semi-continuous function on $L^2(\Omega; C([0,T]; H))$ the subsets $\mathscr{W}^K$ are closed in $L^2(\Omega; C([0,T]; H))$. For $Z \in L^2(\Omega; C([0,T]; S))$ we have

$$\|B_t(Z_t)\|^2_{L_2(U,S)} \leq C\|Z_t\|^2_S + h_t \in L^1([0,T] \times \Omega).$$

By Proposition 6.3 there exists a unique corresponding solution $X = F(Z) \in L^2(\Omega; C([0,T]; S))$ driven by the diffusion coefficients $B(Z)$. We will prove that the solution map

$$F : L^2(\Omega; C([0,T]; H)) \to L^2(\Omega; C([0,T]; H))$$

is a contraction for $T > 0$ small enough. Let $Z^{(1)}, Z^{(2)} \in L^2(\Omega; C([0,T]; H))$. Then by Itō's formula und Burkholder's inequality

$$\begin{aligned}
\|F(Z^{(1)}) - F(Z^{(2)})\|^2_{L^2(\Omega;C([0,T];H))} &\leq \mathbb{E} \int_0^T \|B_s(Z_s^{(1)}) - B_s(Z_s^{(2)})\|^2_{L_2(U,H)}\, ds \\
&\leq C\mathbb{E} \int_0^T \|Z_s^{(1)} - Z_s^{(2)}\|^2_H \, ds \\
&\leq CT\|Z_s^{(1)} - Z_s^{(2)}\|^2_{L^2(\Omega;C([0,T];H))}.
\end{aligned}$$

Thus, for $T > 0$ small enough F is a contractive mapping. For $Z \in \mathscr{W}^K$, by (6.7)

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|^2_S\right] &\leq 4e^{CT} \left(\mathbb{E}\|x\|^2_S + \mathbb{E}\int_0^T C\|Z_s\|^2_S ds + \mathbb{E}\int_0^T (h_s + \gamma_s)\ ds \right) \\
&\leq 4e^{CT} \left(\mathbb{E}\|x\|^2_S + CT\|Z_s\|^2_{L^2(\Omega;C([0,T];S))} + \mathbb{E}\int_0^T (h_s + \gamma_s)\ ds \right) \\
&\leq 4e^{CT} \left(\mathbb{E}\|x\|^2_S + CTK + \mathbb{E}\int_0^T (h_s + \gamma_s)\ ds \right)
\end{aligned}$$

Choosing $K \geq 8e^{CT} \left(\mathbb{E}\|x\|^2_S + \mathbb{E}\int_0^T (h_s + \gamma_s)\ ds \right)$ thus yields

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|^2_S\right] \leq \frac{K}{2} + 4Ce^{CT}TK.$$

Hence, for $T \geq 0$ small enough, F leaves $\mathscr{W}^K$ invariant and is contractive. By Banach's fixed point theorem, there is a unique fixed point $X \in L^2(\Omega; C([0,T]; H))$, i.e. $F(X) = X$ or in other words

$$dX_t + A(X_t)\, dt \ni B(X_t)\, dW_t.$$

By Theorem 6.3 we have

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|^2_S < \infty$$

and X is $\mathbb{P}$-a.s. right-continuous in S.

6.3.3. *Proof of Theorem 4.4.*

Let $x \in L^2(\Omega, \widehat{\mathscr{F}_0}; H)$, B satisfying Hypothesis 4 only, $x^n \in L^2(\Omega, \widehat{\mathscr{F}_0}; S)$ with $x^n \to x$ in $L^2(\Omega, \widehat{\mathscr{F}_0}; H)$ and $B^m := P_m B$, thus satisfying Hypothesis 4 and Hypothesis 5. By Itō's formula and Burkholder's inequality

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \|X_t^n - X_t^m\|_H^2 \, e^{-Ct} \\
&\le C\left(\mathbb{E}\|x^n - x^m\|_H^2 + \mathbb{E}\int_0^T \|B_s^n(X_s^n) - B_s^m(X_s^m)\|_{L_2(U;H)}^2 ds\right) \\
&\le C\mathbb{E}\|x^n - x^m\|_H^2 + C\mathbb{E}\int_0^T \|B_s^n(X_s^n) - B_s(X_s^n)\|_{L_2(U;H)}^2 ds \\
&\quad + C\mathbb{E}\int_0^T \|B_s(X_s^m) - B_s^m(X_s^m)\|_{L_2(U;H)}^2 ds \to 0,
\end{aligned}$$

by dominated convergence. Hence, there is a limit $X^n \to X \in L^2(\Omega; C([0,T]; H))$. Similar arguments yield the independence of X from the approximating sequence.

6.4. **Markov processes and ergodicity.**
In the following assume that the assumptions considered in Section 5 are satisfied.

Proof of Proposition 5.2: As in [PR07, Proposition 4.3.5] we note that by (5.2) it is enough to show

$$\mathbb{E}_x[G(\pi_{t_1}, ..., \pi_{t_n})F(\pi_{t+s})] = \int_\Omega G(\pi_{t_1}(\omega), ..., \pi_{t_n}(\omega))\mathbb{E}_{\pi_s(\omega)}[F(\pi_t)] \, dP_x(\omega),$$

for all $0 \le t_1 \le ... \le t_n \le s$, $G : H^n \to \mathbb{R}$ continuous, bounded and $F \in C_b(H)$. Equivalently

$$\begin{aligned}
&\mathbb{E}[G(X(t_1, 0; \cdot)x, ..., X(t_n, 0; \cdot)x)F(X(t+s, 0; \cdot)x)] \\
&= \int_\Omega G(X(t_1, 0; \omega)x, ..., X(t_n, 0; \omega)x)\mathbb{E}[F(X(t, 0; \cdot)X(s, 0; \omega)x)] \, d\mathbb{P}(\omega).
\end{aligned} \tag{6.8}$$

Let us first consider the case of regular initial conditions $x \in S$ and additive, pathwise $D(T^{3/2})$-regular noise (Theorem 3.2). By Corollary 3.3, $X(t, s; \omega)x$ is a stochastic flow, i.e.

$$X(t, s; \omega)x = X(t, r; \omega)X(r, s; \omega)x, \quad \forall s \le r \le t$$

and a cocycle

$$X(t, s; \omega)x = X(t - s, 0; \theta_s\omega), \quad \forall s \le t.$$

Thus:

$$X(t, 0, \cdot)X(s, 0, \omega)x = X(t + s, s, \theta_{-s}\cdot)X(s, 0, \omega)x.$$

Since $X(t + s, s, \cdot)$ is independent of $\widehat{\mathscr{F}_s}$ we conclude

$$\begin{aligned}
\mathbb{E}[F(X(t, 0, \cdot)X(s, 0, \omega)x)] &= \mathbb{E}[F(X(t + s, s, \cdot)X(s, 0, \omega)x)] \\
&= \mathbb{E}[F(X(t + s, s, \cdot)X(s, 0, \cdot)x)|\widehat{\mathscr{F}_s}](\omega) \\
&= \mathbb{E}[F(X(t + s, 0, \cdot)x)|\widehat{\mathscr{F}_s}](\omega)
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_\Omega G(X(t_1,0,\omega)x,...,X(t_n,0,\omega)x)\mathbb{E}[F(X(t,0,\cdot)X(s,0,\omega)x)]\ d\mathbb{P}(\omega)\\
&=\int_\Omega G(X(t_1,0,\omega)x,...,X(t_n,0,\omega)x)\mathbb{E}[F(X(t+s,0,\cdot)x)|\widehat{\mathscr{F}_s}](\omega)\ d\mathbb{P}(\omega)\\
&=\mathbb{E}[G(X(t_1,0,\cdot)x,...,X(t_n,0,\cdot)x)F(X(t+s,0,\cdot)x)].
\end{aligned}$$

For initial conditions $x\in H$ and noise B satisfying Hypothesis 4 only, solutions were constructed as limits of pathwise solutions $X^m(\cdot,0,\cdot)x\to X(\cdot,0,\cdot)x$ in $L^2(\Omega;C([0,T];H))$. Using uniform Lipschitz continuity of the pathwise solutions in the initial condition we realize that (6.8) is preserved in the limit. $\square$

Recall that $u(\cdot,0)x\in C([0,T];H)$ denotes the unique solution to (5.4).

Proof of Remark 5.4: The Lyapunov function $\Theta(x):=c\|x\|_H^\alpha$ is measurable and has bounded sublevel sets. It remains to prove finite time extinction.

First let $x\in S\cap B$. By the chain rule of calculus

$$\frac{d}{dt}\|u_t\|_H^2=2\,{}_{S^*}\langle-\eta_t,u_t\rangle_S\le -c\|u_t\|_H^\alpha=-c\left(\|u_t\|_H^2\right)^{\frac{\alpha}{2}},\qquad \text{for a.e. } t\in[0,\infty),$$

where α is as in (5.5) and $\eta_t\in A(u_t)$ for a.e. $t\in[0,T]$. Hence, informally $f(t):=\|u_t\|_H^2$ is a subsolution to the ordinary differential equation

$$f'(t)=-cf(t)^{\frac{\alpha}{2}},\quad \text{for a.e. } t\in[0,T].$$

Hence

$$\|u_t\|_H^2\le\left(\left(\|x\|_H^{2-\alpha}-ct\frac{2-\alpha}{2}\right)\vee 0\right)^{\frac{2}{2-\alpha}}\le\left(\left(\|B\|_H^{2-\alpha}-ct\frac{2-\alpha}{2}\right)\vee 0\right)^{\frac{2}{2-\alpha}}.$$

By continuity in the initial condition the same inequality holds for all $x\in B$. We conclude $u_t\equiv 0$ for all $t\ge T_B:=\|B\|_H^{2-\alpha}\frac{2}{c(2-\alpha)}$. $\square$

In order to prove Theorem 5.5, we need some preparation. We start by proving stochastic stability for equation (5.1).

Lemma 6.5. *Suppose that Hypotheses 1, 5, 6 hold. For each $T>0$, $\varepsilon>0$ and $B\subseteq H$ bounded we have*

$$\mathbb{P}[\|X(T,0;\cdot)x-u(T,0)x\|_H^2\le\varepsilon]>0,$$

uniformly for all $x\in B$.

Proof. Since W_t^B is a trace class Wiener process on S, for each $\delta>0$, $T>0$ we can find a subset $\Omega^\delta\subseteq\Omega$ of positive mass such that $\sup\{\|W_t^B(\omega)\|_S|\ t\in[0,T]\}<\frac{\delta}{2}$ for all $\omega\in\Omega^\delta$. Let $x\in S$. For $\omega\in\Omega^\delta$ we have by (A5)

$$\begin{aligned}
\|Y(t,0;\omega)x\|_H^2&=\|x\|_H^2-2\int_0^t {}_{S^*}\langle\eta_r,X(r,0;\omega)x-W_r^B(\omega)\rangle_S\,dr\\
&\le\|x\|_H^2-c\int_0^t\|\eta_r\|_{S^*}dr-2\int_0^t {}_{S^*}\langle\eta_r,-W_r^B(\omega)\rangle_S\,dr+Ct\\
&\le\|x\|_H^2-\left(c-2\sup_{r\in[0,T]}\|W_r^B(\omega)\|_S\right)\int_0^t\|\eta_r\|_{S^*}dr+Ct\\
&\le\|x\|_H^2-(c-\delta)\int_0^t\|\eta_r\|_{S^*}dr+Ct,
\end{aligned}$$

where $\eta_r \in A(X(r,0;\omega)x)$ for a.e. $r \in [0,T]$ and $X = Y + W^B$, as before. The same argument can be applied for $u(t,0)x$ to obtain

$$\int_0^T \|\widetilde{\eta}_r\|_{S^*}\,dr + \int_0^T \|\eta_r\|_{S^*}\,dr \le C(1+\|x\|_H^2),$$

where $\widetilde{\eta}_r \in A(u(t,0)x)$ and $\eta_r \in A(X(r,0;\omega)x)$ for a.e. $r \in [0,T]$ and all δ small enough. Using the monotonicity of A we estimate the difference to the deterministic solution by

$$\begin{aligned}\|Y(t,0;\omega)x - u(t,0)x\|_H^2 &= \int_0^t {}_{S^*}\langle \eta_r - \widetilde{\eta}_r, Y(r,0;\omega)x - u(r,0)x\rangle_S\,dr\\ &\le \|\eta - \widetilde{\eta}\|_{L^1([0,T];S^*)}\|W^B(\omega)\|_{L^\infty([0,T];S)}\\ &\le C(1+\|x\|_H^2)\delta,\end{aligned}$$

for $\omega \in \Omega^\delta$. By continuity in x this inequality remains true for all $x \in H$. Hence

$$\begin{aligned}\|X(t,0;\omega)x - u(t,0)x\|_H^2 &= \|Y(t,0;\omega)x + W_t^B(\omega) - u(t,0)x\|_H^2\\ &\le C(1+\|x\|_H^2)\delta, \quad \forall \omega \in \Omega^\delta.\end{aligned}$$

Choosing $\delta > 0$ small enough thus yields the claim. □

Next, we prove the asymptotic concentration of the average mass on bounded (compact resp.) sets.

Lemma 6.6. *Suppose that Hypotheses 1, 5, 7,* (L1) *(or* (L2) *resp.) hold. For each $\varepsilon > 0$ and each bounded set $A \subseteq H$ there exists a bounded (compact resp.) measurable set $B = B(\varepsilon, A) \subseteq H$ such that*

$$\inf_{x\in A} \varliminf_{T\to\infty} Q^T(x,B) > 1-\varepsilon$$

Proof. Let $\varepsilon > 0$. For $R > 0$ set $K := K_R := \{\Theta \le R\}$, which is a bounded (compact resp.) measurable set by (L1) ((L2) resp.). By Itō's formula,

$$\frac{1}{T}\mathbb{E}\int_0^T \Theta(X(s,0;\cdot)x)\,ds \le C(\|x\|_H^2+1), \quad \text{for } T \ge 1.$$

Let $A \subseteq H$ be bounded and $x \in A$. Then

$$\begin{aligned}Q^T(x,K) &= \frac{1}{T}\int_0^T P_s(x,K)\,ds\\ &\ge \frac{1}{T}\int_0^T \left(1 - \frac{\mathbb{E}\left[\Theta(X(s,0;\cdot)x)\right]}{R}\right)ds \ge 1 - \frac{C}{R}(\|A\|_H^2+1).\end{aligned}$$

Choosing $R(\varepsilon,A) > \varepsilon^{-1}C(\|A\|_H^2+1)$ yields the claim with $B := K_{R(\varepsilon,A)}$. □

We are now ready to prove a locally uniform lower bound for the average mass at 0.

Lemma 6.7. *Suppose that Hypotheses 1, 5, 6, 7,* (L1) *hold. For each $\delta > 0$ and each bounded set $A \subseteq H$*

$$\inf_{x\in A} \varliminf_{T\to\infty} Q^T(x,B_\delta(0)) > 0.$$

Proof. Let $A \subseteq H$ be bounded, $x \in A$ and $B = B(\frac{1}{2},A)$ be as in Lemma 6.6. By (L1), there exists an extinction time T_0 corresponding to B such that $u_{T_0}^z = 0$ for all $z \in B$. Using Lemma 6.5 yields

$$P_{T_0}(z,B_\delta(0)) = \mathbb{P}(\|X_{T_0}^z\|_H \le \delta) = \mathbb{P}(\|X_{T_0}^z - u(T_0,0)z\|_H \le \delta) \ge \gamma_1 > 0, \tag{6.9}$$

where $\gamma_1 = \gamma_1(T_0, \delta)$ is independent of $z \in B$. Thus

$$\begin{aligned}
\varliminf_{T\to\infty} Q^T(x, B_\delta(0)) &= \varliminf_{T\to\infty} \frac{1}{T}\int_0^T P_s(x, B_\delta(0))\, ds \\
&= \varliminf_{T\to\infty} \frac{1}{T}\int_0^T P_{s+T_0}(x, B_\delta(0)) ds \\
&= \varliminf_{T\to\infty} \frac{1}{T}\int_0^T \int_H P_s(x, dz) P_{T_0}(z, B_\delta(0))\, ds \\
&\geq \varliminf_{T\to\infty} \frac{1}{T}\int_0^T \int_B P_s(x, dz) P_{T_0}(z, B_\delta(0))\, ds \\
&\geq \varliminf_{T\to\infty} \gamma_1 \frac{1}{T}\int_0^T \int_B P_s(x, dz)\, ds \\
&\geq \gamma_1 \varliminf_{T\to\infty} Q^T\big(x, B\big) \geq \frac{\gamma_1}{2} > 0,
\end{aligned}$$

where $\gamma_1 = \gamma_1(T_0, \delta) = \gamma_1(\|A\|_H, \delta)$. □

For the next proposition, see [ESvR10, Proposition 4.9]. We shall adapt their proof to our setting and include it for convenience.

Proposition 6.8 (Es-Sarhir–von Renesse)**.** *Suppose that Hypotheses 1, 5, 6, 7,* (L2) *hold. Then for every $\delta > 0$ and every $x \in H$:*

$$\varliminf_{T\to\infty} Q^T(x, B_\delta(0)) > 0.$$

Proof. Let $\delta > 0$ and $x \in H$. Repeating the proof of Lemma 6.6, we may pick some $R > 0$ such that for the compact set $K := K_R := \{\Theta \leq R\}$ it holds that

$$Q^T(x, K) > \frac{1}{2}, \quad \forall T > 1.$$

Claim 1: There exists a $\gamma_1 > 0$ and a finite sequence $T_1, T_2, \ldots, T_k$, $T_i > 0$ for $i \in \{1, \ldots, k\}$ such that

$$\frac{1}{k}\sum_{i=1}^k P_{T_i}(y, B_\delta(0)) > \gamma_1, \quad \forall y \in K.$$

By (L2), for each $z \in H$ there exists a $T_z < \infty$ such that $u(t, 0)z \in B_{\delta/4}(0)$ for all $t \geq T_z$. Hence, using Lemma 6.5 we obtain

$$P_{T_z}(z, B_{\delta/2}(0)) \geq \mathbb{P}\left\{\|X(T_z, 0; \cdot)z - u(T_z, 0)z\|_H \leq \frac{\delta}{4}\right\} =: \gamma_z > 0.$$

Let φ be a bounded Lipschitz function on H such that $\mathbb{1}_{B_{\delta/2}(0)} \leq \varphi \leq \mathbb{1}_{B_\delta(0)}$ and $\mathrm{Lip}(\varphi) \leq 2/\delta$. Defining $r_z := \frac{\delta\gamma_z}{4}$, the e-property (5.3) implies

$$\begin{aligned}
P_{T_z}(y, B_\delta(0)) &\geq P_{T_z}\varphi(y) \geq P_{T_z}\varphi(z) - \frac{2}{\delta}\|z - y\|_H \\
&\geq P_{T_z}(z, B_{\delta/2}(0)) - \frac{2}{\delta}\|z - y\|_H \geq \gamma_z - \frac{2}{\delta}\|z - y\|_H \geq \frac{\gamma_z}{2}, \quad \forall y \in B_{r_z}(z).
\end{aligned}$$

Since K is compact, we may select a finite sequence (z_i, r_{z_i}), $i \in \{1, \ldots, k\}$, such that $K \subseteq \bigcup_{i=1}^k B_{r_{z_i}}(z_i)$. Setting $T_i := T_{z_i}$ the claim follows with $\gamma_1 := \frac{1}{2}\min_{1\leq i\leq k} \gamma_{x_i}$.

Using Fatou's lemma we obtain

$$\begin{aligned}
\varliminf_{T\to\infty} Q^T(x, B_\delta(0)) &= \varliminf_{T\to\infty} \frac{1}{T}\int_0^T P_s(x, B_\delta(0))\, ds \\
&= \varliminf_{T\to\infty} \frac{1}{k}\sum_{i=1}^k \frac{1}{T}\int_0^T P_{s+T_i}(x, B_\delta(0)) ds \\
&\ge \varliminf_{T\to\infty} \frac{1}{k}\sum_{i=1}^k \frac{1}{T}\int_0^T \int_K P_s(x, dy) P_{T_i}(y, B_\delta(0))\, ds \\
&\ge \gamma_1 \varliminf_{T\to\infty} \frac{1}{T}\int_0^T P_s(x, K)\, ds \\
&= \gamma_1 \varliminf_{T\to\infty} Q^T(x, K) \\
&\ge \frac{\gamma_1}{2} > 0.
\end{aligned}$$

□

We see that Theorem [KPS10, Theorem 2] can be applied to yield the claim of Theorem 5.5.

7. Examples

In the following we present several examples of singular SPDE which can be treated by the general results discussed above. Let us first consider the case in which the drift is given as the subgradient of a convex function. Let $S \subseteq H \subseteq S^*$ be a Gelfand triple of Hilbert spaces and T, T_n, J_n as in Section 2.

Proposition 7.1. *Let $\varphi : S \to \mathbb{R}$ be a lower semi-continuous, convex function such that $\inf_{u\in S}\varphi(u) > -\infty$. Then $A := \partial\varphi : S \to 2^{S^*}$ satisfies* (A1)*,* (A4)*,* (A5)*.*

If

$$\varphi(u) \le C\left(1 + \|u\|_S^2\right), \quad \forall u \in S, \tag{7.1}$$

for some $C > 0$, then A satisfies (A2)*.*

If in addition, φ is non-expansive with respect to J_n, i.e. $\varphi(J_n u) \le \varphi(u)$ for all $u \in S$, $n \in \mathbb{N}$ then A satisfies (A3)*.*

Proof. By replacing φ with $\varphi - \inf_{u\in S}\varphi(u)$, we may assume $\varphi \ge 0$. By [Phe89, Proposition 3.3], φ is continuous on S. (A4) is satisfied since φ is independent of time t.

(A1): By [Bar76, Theorem 2.1, Proposition 2.7] A is maximal monotone with nonempty values.

(A5): As noted above, φ is continuous. Hence, there is $\delta > 0$ such that $\varphi(w) \le \varphi(0) + 1$ for all $\|w\|_S \le \delta$. For $v \in \partial\varphi(u)$ we observe

$$\begin{aligned}
\|v\|_{S^*} &= \frac{1}{\delta}\sup_{w\in S,\ \|w\|_S=\delta} {}_{S^*}\langle v, w\rangle_S \\
&= \frac{1}{\delta}\sup_{w\in S,\ \|w\|_S=\delta} {}_{S^*}\langle v, w-u\rangle_S + \frac{1}{\delta}\, {}_{S^*}\langle v, u\rangle_S \\
&\le \frac{1}{\delta}\sup_{w\in S,\ \|w\|_S=\delta} \varphi(w) - \varphi(u) + \frac{1}{\delta}\, {}_{S^*}\langle v, u\rangle_S \\
&\le \frac{1}{\delta}(\varphi(0)+1) + \frac{1}{\delta}\, {}_{S^*}\langle v, u\rangle_S\,.
\end{aligned}$$

(A2): Let $u \in S$, $v \in \partial\varphi(u)$ and φ^* be the Legendre-Fenchel transform of φ. By (7.1) it follows that

$$\varphi^*(v) \geq \frac{\|v\|_{S^*}^2}{4C} - C. \quad \forall v \in S^*,$$

compare e.g. with [RW98, eq. 11(3)]. Hence by Young's equality,

$$\begin{aligned}\|v\|_{S^*}^2 &\leq 4C\left(\varphi^*(v) + C\right) = 4C\left({}_{S^*}\langle v, u\rangle_S - \varphi(u) + C\right)\\ &\leq 4C\left(\|v\|_{S^*}\|u\|_S + C\right),\end{aligned}$$

which implies (A2).

(A3): For $u \in S$, $v \in \partial\varphi(u)$

$${}_{S^*}\langle v, T_n u\rangle_S = -n\, {}_{S^*}\langle v, J_n u - u\rangle_S \geq n\left(\varphi(u) - \varphi(J_n u)\right) \geq 0,$$

for all $u \in S$ and $n \in \mathbb{N}$. □

For any convex function $\varphi : S \to \mathbb{R}$ we define its extension onto H by $\varphi(u) := +\infty$, for all $u \in H \setminus S$.

Proposition 7.2. *Let $\varphi : S \to \mathbb{R}$ be an even, convex function such that $\varphi(u) = \inf_{v\in S}\varphi(v)$ iff $u = 0$ and such that its extension onto H has compact sublevel sets. Then $A := \partial\varphi$ satisfies condition* (L2) *in Hypothesis 7.*

Proof. By the subgradient property we have

$${}_{S^*}\langle v, u\rangle_S \geq \varphi(u) - \varphi(0), \quad \forall v \in \partial\varphi(u).$$

Hence, we may choose $\Theta := \varphi - \varphi(0)$ in Hypothesis 7, (L2). Closedness of the sublevel sets on H implies lower semi-continuity of Θ on H. Then [Bru75, Theorem 5] implies (L2). □

In the following let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a (not necessarily complete nor right-continuous) filtered probability space and $N : [0,T] \times \Omega \to S$ be an $\{\mathscr{F}_t\}$-adapted process satisfying $N_\cdot(\omega) \in L^2([0,T]; D(T^{\frac{3}{2}}))$ for all $\omega \in \Omega$ and strict stationarity, i.e. (N). Furthermore, let $(\Omega, \widehat{\mathscr{F}}, \{\widehat{\mathscr{F}_t}\}_{t\geq 0}, \mathbb{P})$ be a normal filtered probability space and $B : [0,T] \times \Omega \times S \to L_2(U, H)$ be an $\widehat{\mathscr{F}_t}$-progressively measurable map. The choice of the underlying Gelfand triple $S \subseteq H \subseteq S^*$ will be specified in each example.

7.1. Stochastic generalized fast-diffusion equation.

We adopt a framework inspired by Röckner and Wang [RW08]. Let $(E, \mathscr{B}, \mu)$ be a finite measure space and $(\mathscr{E}, D(\mathscr{E}))$ be a symmetric Dirichlet form on $L^2(\mu)$ with associated Dirichlet operator $(L, D(L))$ (cf. [FOT94]). Assume that L is strictly coercive and anticompact. Then $D(\mathscr{E})$ is a Hilbert space with norm $\|\cdot\|_0 := \mathscr{E}^{1/2}(\cdot)$ and $D(\mathscr{E}) \subseteq L^2(\mu)$ is dense and compact.

We define the generalized fast diffusion operator in the Gelfand triple

$$S := L^2(\mu) \subseteq H := D(\mathscr{E})^* \subseteq S^*.$$

Let $\Psi : \mathbb{R} \to \mathbb{R}_+$ be an even, convex, continuous function with $\Psi(0) = 0$, subdifferential $\Phi = \partial\Psi : \mathbb{R} \to 2^{\mathbb{R}}$ and

$$\Psi(r) \leq C\left(|r|^2 + \mathbb{1}_{\mu(E)<\infty}\right), \quad \forall r \in \mathbb{R}$$

for some constant $C > 0$. Explicit examples are given by:

(1) Fast diffusion equation:

$$\Phi_p(r) := \partial\left(s \mapsto \tfrac{1}{p}|s|^p\right)(r) = |r|^{p-1}\operatorname{sgn}(r), \quad p \in [1,2].$$

Note that we include the limit case $p = 1$ for which

$$\Phi_1(r) = \begin{cases} \operatorname{sgn}(r), & r \neq 0 \\ [-1,1], & r = 0. \end{cases}$$

(2) Plasma diffusion:

$$\Phi_{\ln}(r) := \partial\Big(s \mapsto (|s|+1)\ln(|s|+1) - |s|\Big)(r) = \ln(|r|+1)\operatorname{sgn}(r).$$

We define

$$\varphi(u) := \int_E \Psi(u(\xi))\,d\mu(\xi), \quad u \in S$$

and $A := \partial\varphi : S \to 2^{S^*}$. As in [Bar76, Proposition 2.9] we obtain

$$A(u) = \left\{v \in S^* = L^2(\mu) \;\middle|\; v(\xi) \in \Phi(u(\xi)),\ \text{a.e. } \xi \in E\right\}. \tag{7.2}$$

Example 7.3. *Consider the stochastic generalized fast-diffusion equation*

$$\begin{aligned} dX_t &\in L\Phi(X_t)\,dt + \begin{cases} dN_t \\ B_t(X_t)\,dW_t \end{cases}, \quad t \in (0,T], \\ X_0 &= x. \end{aligned} \tag{7.3}$$

Then Theorem 3.2, Corollary 3.3, Theorem 4.2 and Theorem 4.4 apply, proving the unique existence of a (limit) solution to (7.3)*.*

Proof. By continuity, convexity and the growth condition for Ψ, $\varphi : S \to \mathbb{R}_+$ is a convex continuous function. By Proposition 7.1 it only remains to prove that φ is non-expansive with respect to J_n.

Recall that $-L$ equals the Riesz map of S as we dualize over $H = D(\mathscr{E})^*$. Since we have assumed L to be a symmetric Dirichlet operator, it holds that

$$\|J_n u\|_{L^\infty(\mu)} \le \|u\|_{L^\infty(\mu)}, \quad \forall u \in L^\infty(\mu),\ \ n \in \mathbb{N}$$

and

$$J_n u \le J_n v, \quad \forall u,v \in L^2(\mu),\ \ 0 \le u \le v\ \ \mu\text{-a.e.},\ \ n \in \mathbb{N}.$$

By an interpolation theorem due to Maligranda (cf. [Mal89, Theorem 3]), we obtain

$$\varphi(J_n u) \le \varphi(u), \quad \forall u \in \widetilde{L}^\Psi(\mu) \supseteq L^2(\mu) = S.$$

See Appendix C for the definition of $\widetilde{L}^\Psi(\mu)$. □

In order to prove ergodicity of the corresponding Markovian semigroup we require one of the following stronger coercivity conditions

(C1) For each $v \in L^2(\mu)$ with $v(\xi) \in \Phi(u(\xi))$ for a.e. $\xi \in E$ we have

$$\int_E vu\,d\mu \ge c\|u\|_H^p,$$

for some $p \in [1,2)$ and some constant $c > 0$.

For example, this is satisfied in the case of stochastic fast diffusion equations on bounded domains, i.e. for $E = \Lambda \subseteq \mathbb{R}^d$ being a bounded, smooth domain, $\mu = dx$ being the Lebesgue measure, $L = \Delta$ being the Dirichlet Laplacian and $\Psi(r) = \frac{1}{p}|r|^p$ with $p \in \left[1 \vee \frac{2d}{d+2}, 2\right)$ (cf. also [LT11, Liu11]).

(C2) The embedding $L^1(\mu) \subseteq H = D(\mathscr{E})^*$ is compact, $\Psi(r) = 0$ iff $r = 0$ and

$$\lim_{r \to +\infty} \frac{\Psi(r)}{r} = +\infty.$$

In particular, this is satisfied by the plasma diffusion in one space dimension.

Example 7.4. *Consider the generalized stochastic fast-diffusion equation,*

$$
\begin{aligned}
&dX_t \in L\Phi(X_t)\,dt + B\,dW_t, \quad t \in (0,T], \\
&X_0 = x
\end{aligned}
\tag{7.4}
$$

and assume that (C1) *or* (C2) *is satisfied. Then the associated Markovian semigroup* P_t *is weak-$*$ mean ergodic.*

Proof. First assume (C1). We prove that (L1) is satisfied by Remark 5.4. By (7.2) and (C1) we obtain

$$2\,{}_{S^*}\langle y,x\rangle_S = \int_E y(\xi)x(\xi)\ d\mu(\xi) \geq c\|x\|_H^p.$$

Hence, Theorem 5.5 applies.

Let us now assume (C2). By Fatou's lemma φ is lower semi-continuous on $L^1(\mu)$. Compactness of the embedding $L^1(\mu) \subseteq H$ implies that φ has relatively compact sublevel sets in H. It remains to show closedness of the sublevel sets $\{\varphi \leq \alpha\}$ in H. Let $x_n \in \{\varphi \leq \alpha\}$, $n \in \mathbb{N}$, such that $\|x_n - x\|_H \to 0$ for some $x \in H$. By de la Vallée-Poussin's theorem, $\{x_n\}$ is uniformly integrable in $L^1(\mu)$. By the Dunford–Pettis theorem, there is $y \in L^1(\mu)$ and a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'} \rightharpoonup y$ weakly in $L^1(\mu)$. Since $\{\varphi \leq \alpha\}$ is convex and closed in $L^1(\mu)$, it is also weakly closed in $L^1(\mu)$ by Mazur's lemma. Hence $y \in \{\varphi \leq \alpha\}$. By weak continuity of the embedding $L^1(\mu) \subseteq H$, we conclude $y = x$.

The claim follows now from Proposition 7.2. □

As pointed out above, explicit examples include the fast diffusion equation for $p \in \left[1 \vee \frac{2d}{d+2}, 2\right)$ and the generalized stochastic plasma-diffusion equation for $d = 1$. Note that in the limit case of the fast diffusion equation ($\Psi_1(r) = |r|$), we can still allow $d = 1, 2$. For $d = 2$ the embedding $L^1(\mu) \subseteq H$ still is well-defined but not compact. Therefore, only (L1) but not (L2) applies and thus the proof of ergodicity crucially relies on finite time extinction for the deterministic equation.

7.2. **The stochastic singular Φ-Laplacian equation.**

7.2.1. *Dirichlet boundary conditions in a bounded domain.*

Let $d \in \mathbb{N}$, $\Lambda \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Lambda$ and $-\Delta$ be the Dirichlet Laplacian on Λ. We define $|\cdot|$ and $\langle\cdot,\cdot\rangle$ to be the Euclidean norm and the inner-product on $\mathbb{R}^d$ respectively. Let

$$S := H_0^1(\Lambda) \subseteq H := L^2(\Lambda) \subseteq S^*,$$

where we endow S with the equivalent norm $\|u\|_S := \left(\int_\Lambda |\nabla u|^2\,d\xi\right)^{1/2}$.

Let $\Psi(x) = \widetilde{\Psi}(|x|) : \mathbb{R}^d \to \mathbb{R}_+$ be a radially symmetric function with $\widetilde{\Psi} : \mathbb{R} \to [0, +\infty)$ being an Orlicz function (cf. Definition C.1). We further assume

$$\Psi(x) \leq C(|x|^2 + 1), \quad \forall x \in \mathbb{R}^d$$

for some constant $C > 0$. Let $\Phi := \partial\Psi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$. Explicit examples are given by:

(1) Singular p-Laplacian: $\Phi_p(x) := \partial\left(\frac{1}{p}|\cdot|^p\right)(x) = |x|^{p-1}\operatorname{Sgn}(x)$, $p \in [1,2]$, where

$$\operatorname{Sgn}(x) := \begin{cases} \dfrac{x}{|x|}, & \text{if } x \in \mathbb{R}^d \setminus \{0\}, \\ \left\{y \in \mathbb{R}^d \mid |y| \leq 1\right\}, & \text{if } x = 0. \end{cases}$$

Note that we include the limit case $p = 1$.

(2) Curve shortening flow:

$$\Phi_{\arctan}(r) := \partial\left(s \mapsto s\arctan(s) - \frac{1}{2}\ln(s^2+1)\right)(r) = \arctan(r), \quad r \in \mathbb{R}.$$

We define

$$\varphi(u) := \int_\Lambda \Psi(\nabla u(\xi))\ d\xi, \quad u \in S$$

and $A := \partial\varphi : S \to 2^{S^*}$. Since $\nabla : S = H_0^1(\Lambda) \to L^2(\Lambda;\mathbb{R}^d)$ is a linear, continuous operator with adjoint $\nabla^* = (-\Delta)^{-1} \circ \operatorname{div}$, the chain-rule for subgradients (cf. [Sho97, Proposition 7.8]) implies $\partial\varphi = \nabla^* \circ \partial\left(\int_\Lambda \Psi(\cdot)\ d\xi\right) \circ \nabla$, i.e. $\nabla^* y \in \partial\varphi(x) \subseteq S^*$ iff $y \in L^2(\Lambda;\mathbb{R}^d)$ and $y(\xi) \in \Phi(\nabla x(\xi))$ for almost every $\xi \in \Lambda$. In this case

$$ {}_{S^*}\langle \nabla^* y, z\rangle_S = \int_\Lambda \langle y(\xi), \nabla z\rangle\ d\xi. \tag{7.5}$$

Hence A is an S-realization of the singular Φ-Laplace operator

$$\text{“}A(u) := -\operatorname{div}\Phi(\nabla u)\text{”}.$$

Example 7.5. *Consider the stochastic, singular Φ-Laplace equation*

$$\begin{aligned} dX_t &\in \operatorname{div}\Phi(\nabla X_t)\,dt + \begin{cases} dN_t \\ B_t(X_t)dW_t \end{cases}, \quad t \in (0,T], \\ X_0 &= x, \end{aligned} \tag{7.6}$$

with Dirichlet boundary conditions. Then Theorem 3.2, Corollary 3.3, Theorem 4.2 and Theorem 4.4 apply, proving the unique existence of a (limit) solution to (7.6).

Proof. By continuity, convexity and the growth bound of Ψ, $\varphi : S \to \mathbb{R}_+$ is a convex, continuous function. By Proposition 7.1 it only remains to prove that φ is non-expansive with respect to J_n. This will be done in Lemma 7.6 below. □

Lemma 7.6. *For all $u \in S = H_0^1(\Lambda)$ and all $n \in \mathbb{N}$*

$$\varphi(J_n u) \le \varphi(u).$$

Proof. Let $n \in \mathbb{N}$. We first note that

$$\left(1 - \frac{\Delta}{n}\right)\operatorname{div}\eta = \operatorname{div}\left(\left(1-\frac{\Delta}{n}\right)\eta\right), \quad \forall \eta \in C_0^\infty(\Lambda;\mathbb{R}^d).$$

Hence, $J_n(\operatorname{div}\eta) = \operatorname{div} J_n\eta$. Since $-\Delta$ is a symmetric Dirichlet operator, by Proposition C.3, we conclude as in Section 7.1

$$\int_\Lambda \Psi^*(J_n v)\ d\xi \le \int_\Lambda \Psi^*(v)\ d\xi,$$

for all $v \in \widetilde{L}^{\Psi^*}$. Using Proposition C.2, we compute

$$\begin{aligned} \varphi(J_n u) &= \int_\Lambda \Psi(\nabla J_n u)\,d\xi \\ &= \sup\left\{ \left|\int_\Lambda \langle \nabla J_n u, v\rangle\ d\xi\right| - \int_\Lambda \Psi^*(v)\,d\xi \ \middle|\ v \in C_0^\infty(\Lambda;\mathbb{R}^d)\right\} \\ &= \sup\left\{ \left|\int_\Lambda u\ \operatorname{div} J_n v\,d\xi\right| - \int_\Lambda \Psi^*(v)\,d\xi \ \middle|\ v \in C_0^\infty(\Lambda;\mathbb{R}^d)\right\} \\ &\le \sup\left\{ \left|\int_\Lambda \langle \nabla u, J_n v\rangle\ d\xi\right| - \int_\Lambda \Psi^*(J_n v)\,d\xi \ \middle|\ J_n v \in C_0^\infty(\Lambda;\mathbb{R}^d)\right\} = \varphi(u). \end{aligned}$$

□

In order to prove ergodicity of the associated Markovian semigroup for dimensions ($d \geq 2$) we need to assume

(C3) For some $p \in [1,2)$ and for all $\nabla^* v \in \partial\varphi(u) \subseteq S^*$

$$\int_\Lambda \langle v(\xi), \nabla u(\xi)\rangle \; d\xi \geq \|u\|_H^p.$$

For example, this is the case for the stochastic singular p-Laplace equation, i.e. for $\Phi(x) := |x|^{p-1}\operatorname{Sgn}(x)$ with $p \in \left[1 \vee \frac{2d}{2+d}, 2\right)$ (cf. also [LT11]). For $p = 1$, the result was conjectured in [CT11].

Example 7.7. *Consider the stochastic, singular Φ-Laplace equation*

$$\begin{aligned} dX_t &\in \operatorname{div}\Phi(\nabla X_t)\,dt + B\,dW_t, \quad t \in (0,T], \\ X_0 &= x, \end{aligned} \tag{7.7}$$

with Dirichlet boundary conditions and assume (C3) *or $d = 1$. Then the associated Markovian semigroup P_t is weak-$*$ mean ergodic.*

Proof. First assume (C3). We prove that (L1) is satisfied by Remark 5.4. Using (7.2.1) and (7.5) we observe

$$2\,{}_{S^*}\langle \nabla^* y, x\rangle_S = \int_\Lambda \langle y(\xi), \nabla x(\xi)\rangle \; d\xi \geq \|x\|_H^p,$$

for all $y \in L^2(\Lambda;\mathbb{R}^d)$ with $y(\xi) \in \Phi(\nabla x(\xi))$ for a.e. $\xi \in \Lambda$. Thus, we may apply Theorem 5.5.

If $d = 1$, then Hypothesis 7, (L2) is satisfied: To see this, note that the embedding $W_0^{1,1}(\Lambda) \subseteq L^2(\Lambda)$ is compact and hence the sub-levels of φ are relatively compact in $L^2(\Lambda)$. Also φ is lower semi-continuous on $W_0^{1,1}(\Lambda)$ by Fatou's lemma. But φ is also lower semi-continuous on $L^2(\Lambda)$ and thus possesses compact sub-level sets: Indeed, let $u_n, u \in L^2(\Lambda)$, $n \in \mathbb{N}$, such that $\|u_n - u\|_{L^2(\Lambda)} \to 0$ and $\varphi(u_n) \leq \alpha$. Using Proposition C.2, we compute

$$\begin{aligned} \varphi(u) &= \sup\left\{ \left|\int_\Lambda u \; \operatorname{div} v \; d\xi\right| - \int_\Lambda \Psi^*(v) \; d\xi \;\middle|\; v \in C_0^\infty(\Lambda;\mathbb{R}^d)\right\} \\ &= \sup\left\{ \lim_{n\to\infty}\left|\int_\Lambda u_n \; \operatorname{div} v \; d\xi\right| - \int_\Lambda \Psi^*(v) \; d\xi \;\middle|\; v \in C_0^\infty(\Lambda;\mathbb{R}^d)\right\} \\ &\leq \varliminf_{n\to\infty} \sup\left\{ \left|\int_\Lambda u_n \; \operatorname{div} v \; d\xi\right| - \int_\Lambda \Psi^*(v) \; d\xi \;\middle|\; v \in C_0^\infty(\Lambda;\mathbb{R}^d)\right\} \\ &= \varliminf_{n\to\infty} \int_\Lambda \Psi(\nabla u_n) \; d\xi \leq \alpha. \end{aligned}$$

Hence we can apply Proposition 7.2. □

In particular, this applies to the stochastic curve shortening flow in one space dimension.

7.2.2. *Neumann boundary conditions in a Riemannian manifold.*

Let M be a connected, complete Riemannian manifold of dimension $d \in \mathbb{N}$ with either convex or empty boundary ∂M and volume element dx. For $x \in M$ we define $\langle \cdot,\cdot\rangle_x$ and $|\cdot|_x$ to be the Riemannian inner-product and norm on T_xM. Furthermore, let $V \in C^2(M)$, $Z := \nabla V$. Assume that $\int_M e^{V(x)}dx < +\infty$, set $L := \Delta + Z$ and

$$\mu(dx) := \frac{e^{V(x)}dx}{\int_M e^{V(x)}dx},$$

which is a symmetrizing probability measure for $-L$. We consider the Gelfand triple

$$S := H^1(M,\mu) \subseteq H := L^2(M,\mu) \subseteq S^*,$$

where $H^1(M,\mu) := \{u \in L^2(M,\mu) \mid \int_M |\nabla u|^2\,d\mu < +\infty\}$. Let $\{P_t\}$ be the Feller semigroup associated to $-L$, assume that $S \subseteq H$ is compact (for criteria cf. e.g. [Wan05, Theorem 1.4.15, p. 62]) and assume that there exists a real constant $K_Z \leq 0$ such that

$$\mathrm{Ric}(X,X) - \langle \nabla_X Z, X\rangle \geq -K_Z|X|^2, \quad \forall X \in TM.$$

Then the (reflecting) L-diffusion process is nonexplosive. Let $\Psi(x,\xi) = \widetilde{\Psi}(|\xi|_x) : M \times TM \to \mathbb{R}_+$ with $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}_+$ being an Orlicz function. We further assume

$$\Psi(x,\xi) \leq C(|\xi|_x^2 + 1),$$

for some constant $C > 0$. Let $\Phi := \partial_\xi \Psi$. An explicit example is given by the singular p-Laplacian nonlinearity

$$\Psi_p(x,\xi) := \tfrac{1}{p}|\xi|_x^p, \quad \forall x \in M,\ \xi \in T_xM,$$

for some $p \in [1,2]$. The nonlinear diffusion operator "$A(u) := -\operatorname{div}\left(e^V \Phi(\nabla u)\right)$" is defined rigorously as the subdifferential of the convex potential

$$\varphi(u) := \int_M \Psi(x, \nabla u(x))\,d\mu(x), \quad u \in S.$$

As in Section 7.2.1, $A := \partial\varphi$ admits the variational characterization: $\nabla^* y \in A(u)$ iff $|y| \in H$ and $y(x) \in \Phi(x, \nabla u(x))$ for almost every $x \in M$. In this case

$$_{S^*}\langle \nabla^* y, v\rangle_S = \int_M \langle y(x), \nabla v(x)\rangle_x \; d\mu(x), \quad u, v \in S.$$

Example 7.8. *Consider the stochastic Φ-Laplace equation with Neumann boundary conditions*

$$\begin{aligned} dX_t &\in \operatorname{div}(e^V \Phi(\nabla X_t))\,dt + \begin{cases} dN_t \\ B_t(X_t)dW_t \end{cases}, \quad t \in (0,T], \\ X_0 &= x. \end{aligned} \tag{7.8}$$

Then Theorem 3.2, Corollary 3.3, Theorem 4.2 and Theorem 4.4 apply, proving the unique existence of a (limit) solution to (7.8).

Proof. The proof proceeds as before. It only remains to prove that φ is non-expansive with respect to J_n. By [Wan05, Proposition 2.5.1, p. 108]

$$|\nabla P_t u| \leq e^{K_Z t} P_t|\nabla u| \leq P_t|\nabla u|, \quad \forall u \in C_b^1(M),\ t \geq 0.$$

Hence

$$|\nabla J_n u| = \left|\nabla \int_0^\infty e^{-t} P_{t/n}(u)\,dt\right| \leq \int_0^\infty e^{-t} P_{t/n}|\nabla u|\,dt = J_n|\nabla u|, \quad \forall u \in C_b^1(M).$$

Since $\{P_t\}$ is Markovian symmetric on $L^2(M,\mu)$, by [Mal89, Theorem 3]

$$\int_M \widetilde{\Psi}(J_n u(x))d\mu(x) \leq \int_M \widetilde{\Psi}(u(x))d\mu(x), \ \ \mu\text{-a.e.} \quad \forall u \in \widetilde{L}^\Psi(M,\mu).$$

Combining these two inequalities we obtain

$$\begin{aligned} \varphi(J_n u) &= \int_M \widetilde{\Psi}(|\nabla J_n u(x)|_x)\; d\mu(x) \\ &\leq \int_M \widetilde{\Psi}(J_n|\nabla u|(x))\; d\mu(x) \\ &\leq \int_M \widetilde{\Psi}(|\nabla u|(x))\; d\mu(x) = \varphi(u), \quad \forall u \in S, \end{aligned}$$

by density and dominated convergence. □

To prove ergodicity we again need to require a stronger coercivity condition, i.e. we assume

$$\int_M \langle \Phi(x, \nabla u(x)), \nabla u(x)\rangle_x \; d\mu(x) \geq c\|u\|_H^p, \tag{7.9}$$

for some $c > 0$ and some $p \in [1, 2)$. For example, this is satisfied by the standard p-Laplacian for $p \in \left[1 \vee \frac{2d}{2+d}, 2\right)$. By the same proof as for Example 7.7 we obtain

Example 7.9. *Consider the stochastic Φ-Laplace equation with Neumann boundary conditions*

$$\begin{aligned} dX_t &\in \operatorname{div}(e^V \Phi(\nabla X_t))\, dt + B\, dW_t, \quad t \in (0, T], \\ X_0 &= x, \end{aligned} \tag{7.10}$$

for Φ satisfying (7.9)*. Then the associated Markovian semigroup P_t is weak-$*$ mean ergodic.*

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Appendix A. A deterministic existence result

Consider the following multi-valued Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}u(t) + J(t, u(t)) \in F(t, u(t)) & \text{for a.e } t \in [0, T], \\ u(0) = u_0. \end{cases} \tag{A.1}$$

Here $J : [0, T] \times S \to S^*$ and $F : [0, T] \times S \to 2^{S^*}$.

Definition A.1. *A* solution *of* (A.1) *is a function $u \in W^{1,2}(0, T)$ such that*

$$\frac{d}{dt}u(t) + J(t, x(t)) = f(t)$$

a.e. on $[0, T]$, $u(0) = u_0$ and $f \in L^2([0, T]; S^*)$ such that $f(t) \in F(t, u(t))$ for a.e. $t \in [0, T]$.

We need the following definition:

Definition A.2. *An operator $A : S \to S^*$ is said to be of* type $(S)_+$*, if $u_n \rightharpoonup u$ weakly in S and*

$$\overline{\lim_n}\; {}_{S^*}\langle A(u_n) - A(u), u_n - u\rangle_S \leq 0$$

together imply that $u_n \to u$ strongly in S.

Certainly, a *strongly monotone* operator $A : S \to S^*$, i.e. for some $c > 0$:

$${}_{S^*}\langle A(u) - A(v), u - v\rangle_S \geq c\, \|u - v\|_S^2 \quad \forall u, v \in S,$$

is of type $(S)_+$.

Consider the following conditions on J and F:

(J1) $t \mapsto J(t, x)$ is measurable for all $x \in S$.
(J2) $x \mapsto J(t, x)$ is demicontinuous and of type $(S)_+$ for almost all $t \in [0, T]$.
(J3) $\|J(t, x)\|_{S^*} \leq j_1(t) + c_1 \|x\|_S$ a.e. on $[0, T]$ for all $x \in S$ with $j_1 \in L^2([0, T])$, $c_1 > 0$.

(J4) ${}_{S^*}\langle J(t,x),x\rangle_S \geq c_2\,\|x\|_S^2 - c_2'\,\|x\|_S - j_2(t)$ a.e. on $[0,T]$ for all $x\in S$ with $c_2,c_2'>0$ and $j_2\in L^1([0,T])$.

(F1) The values of F are non-empty, closed and convex.

(F2) $t\mapsto F(t,x)$ is measurable for all $x\in S$.

(F3) $\operatorname{Gr} F(t,\cdot)$ is sequentially closed in $S\times S_w^*$ for almost all $t\in[0,T]$.

(F4) $\|F(t,x)\|_{S^*}\leq f_1(t)+c_3\,\|x\|_S$ for a.e. $t\in[0,T]$ with $f_1\in L^2([0,T])$, $c_3>0$.

(F5) ${}_{S^*}\langle y,x\rangle_S \leq \gamma(t)$ for a.e. $t\in[0,T]$, all $x\in S$, $y\in F(t,x)$ and some $\gamma\in L^1([0,T])_+$.

Here "Gr" denotes the graph of a multi-valued map, $L^1([0,T])_+ := \{f\in L^1([0,T]) \mid f\geq 0 \text{ a.e. on } [0,T]\}$ and "S_w^*" denotes the weak Hilbert topology of S^*.

The following theorem can be found in [HP00, Theorem I.2.40] or [PPY00, Theorem 3]:

Theorem A.3 (Hu–Papageorgiou–Papalini–Yannakakis)**.** *Suppose that J and F satisfy* (J1)–(J4) *and* (F1)–(F5) *resp. Then the set of solutions to inclusion* (A.1) *for initial point $u_0\in H$ is nonempty, weakly compact in $W^{1,2}(0,T)$ and compact in $C([0,T];H)$.*

Appendix B. Maximal monotone operators depending on a parameter

Let S be a separable Hilbert space, $(E,\mathscr{B},\mu)$ be a σ-finite measure space and $A: E\times S\to 2^{S^*}$ satisfy:

(MM) For μ-a.a. $u\in E$ the map $x\mapsto A(u,x)$ is maximal monotone with nonempty values.

Definition B.1. *A map $A:S\to 2^{S^*}$ with non-empty values is called* strongly-to-weakly upper semi-continuous *if for each $x\in S$ and for each weakly open set V in S^* such that $A(x)\subseteq V$ and for all $\{x_n\}\subseteq S$ with $x_n\to x$ strongly, we have that $A(x_n)\subseteq V$ for large n.*

A map $A:S\to 2^{S^*}$ with non-empty values is called weakly upper hemicontinuous *if for each $x,y\in S$, $\lambda\in[0,1]$ and for each weakly open set V in S^* such that $A(x+\lambda y)\subseteq V$ and for all $\{\lambda_n\}\subseteq[0,1]$ with $\lambda_n\to\lambda$, we have that $A(x+\lambda_n y)\subseteq V$ for large n.*

A map $A:S\to 2^{S^*}$ with non-empty values is weakly upper hemicontinuous whenever it is strongly-to-weakly upper semi-continuous.

Let $\bar{A}$ be the mapping defined by:

$$\bar{A}: L^2(E,\mathscr{B},\mu;S)\to 2^{L^2(E,\mathscr{B},\mu;S^*)},$$
$$\bar{A}: x\mapsto \{y\in L^2(E,\mathscr{B},\mu;S^*) \mid y(u)\in A(u,x(u)),\ \mu\text{-a.e. } u\in E\}$$

Note that $\bar{A}(x)$ might be empty for some (or all) $x\in L^2(E,\mathscr{B},\mu;S)$.

Proposition B.2. *Suppose that* (MM) *holds. Then the graph of $\bar{A}$ is sequentially closed in $L^2(E,\mathscr{B},\mu;S)\times L_w^2(E,\mathscr{B},\mu;S^*)$.*

Proof. Our proof is inspired by [AD72, proof of Proposition (3.4)]. Let $x_n\to x$ in $L^2(E,\mathscr{B},\mu;S)$, such that $[x_n,y_n]\in\operatorname{Gr}\bar{A}$ exists for each $n\in\mathbb{N}$ and assume that $y_n\rightharpoonup\eta$ weakly in $L^2(E,\mathscr{B},\mu;S^*)$ for some $\eta\in L^2(E,\mathscr{B},\mu;S^*)$. Let us extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}(u)\to x(u)$ for μ-a.e. $u\in E$. By the Banach-Saks theorem,

$$w_n := \frac{1}{n}\sum_{i=1}^n y_i$$

converges to η strongly in $L^2(E,\mathscr{B},\mu;S^*)$ (and hence $\{w_{n_k}\}$, too). By extracting another subsequence, if necessary, there is a μ-nullset $N\in\mathscr{B}$ such that for all

$u \in E \setminus N$

$$x_{n_k}(u) \to x(u) \text{ strongly in } S, \quad w_{n_k}(u) \to \eta(u) \text{ strongly in } S^*$$

and $A(u,\cdot)$ is maximal monotone. By [HP97, Ch. 3, Theorem 1.28], $A(u,\cdot)$ is strongly-to-weakly upper semi-continuous. Let $u \in E \setminus N$ and V be any weakly open neighborhood of $A(u,x(u))$. Then $y_{n_k}(u) \in V$ for large k and hence $w_{n_k}(u) \in \operatorname{co}V$. As a result, $\eta(u) = \lim_k w_{n_k}(u) \in \overline{\operatorname{co}V}$. Since V was chosen arbitrarily and $A(u,x(u))$ is a weakly closed, convex set (see e.g. [HP97, Ch. 3, Proposition 1.14]), we conclude

$$\eta(u) \in \bigcap_{\substack{V \text{ weakly open} \\ V \supseteq A(u,x(u))}} \overline{\operatorname{co}V} = A(u,x(u)).$$

Since we can repeat the argument for all $u \in E \setminus N$, we get that $[x,\eta] \in \operatorname{Gr}\bar{A}$. □

Lemma B.3. *Suppose that $A : S \to 2^{S^*}$ is a map with non-empty values. Then the following statements are equivalent.*

(i) A is maximal monotone.
(ii) A is monotone, has weakly compact and convex values and is weakly upper hemicontinuous.

Proof. Follows from [CZ05, Propositions 2.6.4 and 2.6.5]. □

Proposition B.4. *Let $(E,\mathscr{B},\mu)$ be a complete σ-finite measure space. Suppose that $A : E \times S \to 2^{S^*}$ is a map with non-empty values such that*

(1) *$u \mapsto A(u,x)$ is $\mathscr{B}$-measurable for all $x \in S$.*
(2) *$x \mapsto A(u,x)$ is maximal monotone for all $u \in E$.*
(3) *$\sup_{y \in A(x,u)} \|y\|_{S^*} \le f(u) + C\|x\|_S$ for μ-a.a. $u \in E$ for all $x \in S$ and some $C > 0$, $f \in L^2(E,\mathscr{B},\mu)_+$.*

Then $\bar{A} : L^2(E,\mathscr{B},\mu;S) \to 2^{L^2(E,\mathscr{B},\mu;S^*)}$ is a maximal monotone map with non-empty values.

The above conditions mimic those in [Pap92, cond. H(A)], except for an additional coercivity condition that is assumed therein.

Proof of Proposition B.4. Compare with [Pap92, Proof of Lemma 3.3]. By condition (2) and [HP97, Ch. 3, Theorem 1.28] $x \mapsto A(u,x)$ is strongly-to-weakly upper semi-continuous for all $u \in E$. By condition (1) and [Zyg92, Theorem 1, Theorem 2] the map $u \mapsto A(u,x(u))$ is $\mathscr{B}$-measurable and non-empty valued for all $x \in L^2(E,\mathscr{B},\mu;S)$. Condition (3) implies that each measurable selection $y \in A(\cdot,x(\cdot))$ is contained in $L^2(E,\mathscr{B},\mu;S^*)$, thus $\bar{A}(x)$ is non-empty valued. Because of condition (2), $\bar{A}$ is monotone. We are left to prove that $\bar{A}$ is weakly upper hemicontinuous and has weakly compact, convex values in $L^2(E,\mathscr{B},\mu;S^*)$.

We note that by condition (2) and Lemma B.3, for all $u \in E$ every $x \in L^2(E,\mathscr{B},\mu;S)$, $A(u,x(u))$ is a non-empty, convex and weakly compact set in 2^{S^*}. It is easy to see that $A(\cdot,x(\cdot))$ is a convex subset of $L^2(E,\mathscr{B},\mu;S^*)$. As in the proof of Proposition B.2 we see that $A(\cdot,x(\cdot))$ is weakly closed in $L^2(E,\mathscr{B},\mu;S^*)$. Weak compactness follows from condition (3) and the Banach-Alaoglu theorem.

Now, let $x,v \in L^2(E,\mathscr{B},\mu;S)$, $\lambda \in [0,1]$, V be a weakly open set such that $\bar{A}(x+\lambda v) \subseteq V$ and $\{\lambda_n\} \subseteq [0,1]$ such that $\lambda_n \to \lambda$. Clearly, $x + \lambda_n v \to x + \lambda v$ strongly in $L^2(E,\mathscr{B},\mu;S)$. Suppose that "$\bar{A}(x+\lambda_n v) \subseteq V$ for large n" is not valid. Then there exists a subsequence $y_{n_k} \in \bar{A}(x+\lambda_{n_k} v)$ such that $y_{n_k} \notin V$. Passing to a further subsequence, by condition (3) and the Banach-Alaoglu theorem, we may assume that $y_{n_k} \rightharpoonup \eta$ weakly for some $\eta \in L^2(E,\mathscr{B},\mu;S^*)$. By Proposition B.2, $\eta \in \bar{A}(x+\lambda v) \subseteq V$ which gives a contradiction, since V was assumed to be weakly open. Hence $\bar{A}$ is weakly upper hemicontinuous and the proof is completed by Lemma B.3. $\square$

Appendix C. Vector-valued Orlicz Spaces

Definition C.1. *A function $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}_+$ is said to be an Orlicz function, if $\widetilde{\Psi}$ is even, convex, continuous, non-decreasing and $\widetilde{\Psi}(0) = 0$.*

For a lower semi-continuous, proper, convex function $\Psi : \mathbb{R}^d \to \overline{\mathbb{R}}_+$ we define the conjugate function by $\Psi^*(y) := \sup_{x\in\mathbb{R}^d}\{x \cdot y - \Psi(x)\}$. Then

$$x \cdot y \le \Psi(x) + \Psi^*(y), \quad \forall x, y \in \mathbb{R}^d$$

with equality iff $y \in \partial\Psi(x)$ iff $x \in \partial\Psi^*(y)$. Let $(E, \mathscr{B}, \mu)$ be a σ-finite measure space and $\Psi : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function. We define

$$\widetilde{L}^\Psi := \left\{u : E \to \mathbb{R}^d, \text{ measurable, } \varphi(u) := \int_E \Psi(u)\ d\mu < \infty\right\}.$$

Proposition C.2. *Let $\Psi : \mathbb{R}^d \to \mathbb{R}_+$ be an even, proper, lower semi-continuous, convex function with $\Psi(0) = 0$. Then*

$$\begin{aligned}\int_E \Psi(u)\ d\mu &= \sup\left\{ \left|\int_E u \cdot v\ d\mu\right| - \int \Psi^*(v)\ d\mu \ \middle|\ v \in \widetilde{L}^{\Psi^*}\right\} \\ &= \sup\left\{ \left|\int_E u \cdot v\ d\mu\right| - \int \Psi^*(v)\ d\mu \ \middle|\ v \text{ bounded, } \mu(\mathrm{supp}(v)) < \infty\right\}, \end{aligned} \tag{C.1}$$

for all $u \in \widetilde{L}^\Psi$.

Proof. By Young's inequality

$$\int_E u \cdot v\ d\mu \le \int_E \Psi(u)\ d\mu + \int_E \Psi^*(v)\ d\mu,$$

for all $u, v : E \to \mathbb{R}^d$ measurable with $u \cdot v \in L^1(E, \mu)$. Since Ψ is even we conclude "$\ge$" in (C.1).

Let now $u \in \widetilde{L}^\Psi$. Since E is σ-finite we can choose an exhaustion $E_m \subseteq E$ with $\mu(E_m) < \infty$. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable selection of $\partial\Psi$ and define

$$v_{m,n} := g(u)\mathbb{1}_{E_m \cap \{u \le n\}}.$$

Then $v_{m,n}$ is bounded and supported on $E_{m,n} := E_m \cap \{u \le n\}$, since subgradients of convex functions with full domain are locally bounded. We compute

$$\int_E u \cdot v_{m,n}\ d\mu = \int_{E_{m,n}} u \cdot g(u)\ d\mu = \int_{E_{m,n}} \Psi(u) + \Psi^*(g(u))\ d\mu.$$

Hence,

$$\int_{E_{m,n}} \Psi(u)\ d\mu \le \left|\int_E u \cdot v_{m,n}\ d\mu\right| - \int_E \Psi^*(v_{m,n})\ d\mu.$$

Consequently,

$$\int_E \Psi(u)\ d\mu \le \sup_{m,n\in\mathbb{N}} \left\{ \left|\int_E u \cdot v_{m,n}\ d\mu\right| - \int_E \Psi^*(v_{m,n})\ d\mu\right\}.$$

$\square$

By $\langle\cdot,\cdot\rangle_2, |\cdot|_2$ we denote the Euclidean scalar product and norm on $\mathbb{R}^d$. The following Proposition is a vector-valued generalization of [Mal89, Theorem 3].

Proposition C.3. *Let $X := L^1(\mu) + L^\infty(\mu)$ and $T : X \to X$ be a linear operator such that $T : L^1(\mu) \to L^1(\mu)$, $T : L^\infty(\mu) \to L^\infty(\mu)$ and*

$$\begin{aligned} Tx \le Ty, \quad \mu\text{-a.e.} \quad \forall\, 0 \le x \le y,\ x, y \in X, \\ \|Tx\|_1 \le M\|x\|_1, \quad \forall x \in L^1(\mu), \\ \|Tx\|_\infty \le M\|x\|_\infty, \quad \forall x \in L^\infty(\mu). \end{aligned}$$

Moreover, let $\Psi(\xi) = \widetilde{\Psi}(|\xi|_2)$, $\forall \xi \in \mathbb{R}^d$ for an Orlicz function $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}_+$. The operator T acts componentwisely on vector-valued functions $u \in L^1(\mu; \mathbb{R}^d) + L^\infty(\mu; \mathbb{R}^d)$, i.e. $(Tu)_i := Tu_i$ for $i = 1, ..., d$. Then

$$\int_E \Psi\left(\frac{Tu}{M}\right)\, d\mu \le \int_E \Psi(u)\, d\mu,$$

for all $u \in \widetilde{L}^\Psi(\mu)$.

Proof. Let $D \subseteq \mathbb{R}^d$ be a countable, dense subset of the unit ball. Then $|x|_2 = \sup_{y\in D}\langle x, y\rangle_2$. Hence,

$$\left|\frac{Tu}{M}\right|_2 = \sup_{y\in D}\left|\left\langle \frac{Tu}{M}, y\right\rangle_2\right| \le \sup_{y\in D}\frac{T|\langle u, y\rangle_2|}{M} \le \frac{T|u|_2}{M},$$

μ-almost everywhere, for all $u \in L^1(\mu; \mathbb{R}^d) + L^\infty(\mu; \mathbb{R}^d)$. Using [Mal89, Theorem 3] we conclude

$$\begin{aligned} \int_E \Psi\left(\frac{Tu}{M}\right)\, d\mu &= \int_E \widetilde{\Psi}\left(\left|\frac{Tu}{M}\right|_2\right)\, d\mu \le \int_E \widetilde{\Psi}\left(\frac{T|u|_2}{M}\right)\, d\mu \\ &\le \int_E \widetilde{\Psi}\left(|u|_2\right)\, d\mu = \int_E \Psi\left(u\right)\, d\mu. \end{aligned}$$

□

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