

A COMBINATORIAL PROOF OF THE DEGREE THEOREM IN AUTER SPACE

ROBERT MCEWEN AND MATTHEW C. B. ZAREMSKY

ABSTRACT. We use discrete Morse theory to give a new proof of the Degree Theorem in Auter space A_n . There is a filtration of A_n into subspaces $A_{n,k}$ using the *degree* of a graph, and the Degree Theorem is essentially the statement that each $A_{n,k}$ is $(k-1)$ -connected. This result is useful, for example to calculate stability bounds for the homology of $\text{Aut}(F_n)$. The standard proof of the Degree Theorem is global in nature. Here we give a proof that $A_{n,k}$ is $(k-1)$ -connected that only uses local considerations, and lends itself more readily to generalization.

1. INTRODUCTION

In this note we provide an alternate proof of the Degree Theorem in Auter space, with the advantage that the proof only relies on local data, and also lends itself more readily to certain generalizations. Auter space A_n was introduced in [HV98] as the space of basepointed marked metric graphs, along with a related combinatorial model called the spine L_n of Auter space. In the same paper, a measurement called the degree of a graph was used to filtrate A_n into highly connected sublevel sets $A_{n,k}$, which were then used to produce stability bounds for the rational and integral homology of $\text{Aut}(F_n)$. The proof of the Degree Theorem in [HV98], which shows that $A_{n,k}$ is $(k-1)$ -connected, is done by globally deforming disks in A_n via an iterated process. Here we construct a height function on L_n that reduces the problem to local questions, and then using discrete Morse theory we prove the connectivity of $L_{n,k}$, and hence $A_{n,k}$, relatively easily. This viewpoint is also advantageous as it lends itself more readily to generalization, for example a similar method has been used in [Zar] to get stability results for the groups ΣAut_n^m of *partially symmetric* automorphisms.

In Section 2 we describe the spine of Auter space L_n , and define the notion of the *degree* d_0 of a graph. We use the degree to filtrate L_n into sublevel sets $L_{n,k}$, as in [HV98]. We then define a height function h on L_n refining d_0 , and consider the descending links of vertices in L_n with respect to h . The descending link of a vertex decomposes as a join of two complexes, called the *down-link* and *up-link*. In Section 3 we analyze the connectivity of the down-link, and in Section 4 we do the same for the up-link. The upshot of this is that the descending links are all highly connected; see Corollary 4.6. From this we quickly obtain that $L_{n,k}$, and hence $A_{n,k}$ is $(k-1)$ -connected; see Theorem 4.7.

Acknowledgments. The authors would like to thank Kai-Uwe Bux, who helped with a preliminary version of this paper, and Allen Hatcher for his comments and suggestions.

Date: May 1, 2012.

2010 Mathematics Subject Classification. Primary 20F65; Secondary 57M07, 20F28.

Key words and phrases. Auter space, Degree Theorem, automorphisms of free groups.

The second author was supported by CRC 701 of the DFG.

2. OUTER SPACE, DEGREE, AND A HEIGHT FUNCTION

We begin by describing the *spine of Outer space* L_n introduced in [HV98]. Let R_n be the rose with n edges, i.e., the graph with a single vertex p_0 and n edges. Here by a *graph* we always mean a connected one-dimensional CW-complex, with the usual notions of vertices and edges. If Γ is a rank n graph with basepoint vertex p , a homotopy equivalence $\rho : R_n \rightarrow \Gamma$ taking p_0 to p is called a *marking* on Γ . Two markings are equivalent if there is a basepoint-preserving homotopy between them. We only consider graphs such that p is at least bivalent and all other vertices are at least trivalent. The spine L_n of Outer space is then the complex of marked basepointed rank n graphs (Γ, p, ρ) , up to equivalence of markings.

To be more precise, L_n is a simplicial complex with a vertex for every equivalence class of triples (Γ, p, ρ) . An r -simplex is given by a chain of *forest collapses* $\Gamma_r \xrightarrow{d_r} \Gamma_{r-1} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_1} \Gamma_0$ and markings $\rho_i : R_n \rightarrow \Gamma_i$ with the following diagram commuting up to homotopy.

$$\begin{array}{ccccccc}
 \Gamma_r & \xrightarrow{d_r} & \Gamma_{r-1} & \xrightarrow{d_{r-1}} & \cdots & \xrightarrow{d_2} & \Gamma_1 & \xrightarrow{d_1} & \Gamma_0 \\
 & & \swarrow \rho_r & & \swarrow \rho_{r-1} & & \swarrow \rho_1 & & \swarrow \rho_0 \\
 & & & & R_n & & & &
 \end{array}$$

Here a *forest collapse* or *blow-down* $d : \Gamma \rightarrow \Gamma'$ is a (basepoint-preserving) homotopy equivalence of graphs that amounts to collapsing each component of a forest F in Γ to a point. We will write the resulting graph as Γ/F . The reverse of a blow-down is, naturally, called a *blow-up*.

Let Γ be a graph with rank n , basepoint p and vertex set $V(\Gamma)$. The *degree* of Γ can be defined as $d_0(\Gamma) := \sum_{p \neq v \in V(\Gamma)} (\text{val}(v) - 2)$ or equivalently as $d_0(\Gamma) = 2n - \text{val}(p)$; see

[HV98, Section 3]. Here $\text{val}(v)$ is the valency of v , that is the number of half-edges incident to v . For $k \geq 0$, let $L_{n,k}$ be the subcomplex of L_n spanned by vertices represented by triples (Γ, p, ρ) with $d_0(\Gamma) \leq k$. The Degree Theorem in [HV98] is essentially the statement that $L_{n,k}$ is $(k-1)$ -connected; see [HV98, Corollary 3.2 and Section 5].

We now define some other measurements on Γ . For $v \in V(\Gamma)$ let $d(p, v)$ denote the minimum length of an edge path in Γ from v to p , and call $d(p, v)$ the *level* of v . Here we are treating each edge in the graph as having length 1. Define $\Lambda_i(\Gamma) := \{v \in V(\Gamma) \mid d(p, v) = i\}$, $n_i(\Gamma) := -|\Lambda_i(\Gamma)|$ and

$$d_i(\Gamma) := \sum_{v \in V(\Gamma) \setminus \Lambda_i(\Gamma)} (\text{val}(v) - 2)$$

for $i \geq 0$. Note that $\Lambda_0(\Gamma) = \{p\}$, $n_0(\Gamma) = -1$, and $d_0(\Gamma)$ agrees with the definition of degree, so this is not an abuse of notation. Finally, define

$$h(\Gamma) := (d_0(\Gamma), n_1(\Gamma), d_1(\Gamma), n_2(\Gamma), d_2(\Gamma), \dots)$$

to be the *height* of the graph Γ , considered with the lexicographic ordering. This height function is a refinement of the degree function. Extend the definition of h to the vertices of L_n via $h(\Gamma, p, \rho) = h(\Gamma)$. This is well defined up to equivalence since h does not depend on ρ . For brevity, in the future we will often just refer to vertices in L_n as being graphs, rather than equivalence classes of triples (Γ, p, ρ) .

For a vertex Γ in L_n , the *descending star* $\text{st}\downarrow(\Gamma)$ with respect to h is the set of simplices in the star of Γ whose vertices other than Γ all have strictly lower height than Γ . The *descending link* $\text{lk}\downarrow(\Gamma)$ is the set of faces of simplices in $\text{st}\downarrow(\Gamma)$ that do not themselves contain Γ . There are two types of vertex in $\text{lk}\downarrow(\Gamma)$: those obtained

from Γ by a descending blow-up, and those obtained by a descending blow-down. Here we say that a blow-up or blow-down is *descending* if the resulting graph has a lower height than the starting graph. Call the full subcomplex of $\text{lk}\downarrow(\Gamma)$ spanned by vertices of the first type the *up-link*, and the subcomplex spanned by vertices of the second type the *down-link*. Any vertex in the up-link is related to every vertex in the down-link by a blow-down, so $\text{lk}\downarrow(\Gamma)$ is the simplicial join of the up- and down-links. Note that any blow-down necessarily increases some n_i , and so adjacent vertices in L_n have different heights. Hence, the descending stars of two vertices with the same height must be disjoint.

If blowing down the forest F is a descending blow-down, we will call the forest itself *descending*, and similarly a forest can be ascending. It will be a good idea to describe precisely which forests in a graph are ascending and descending. For a forest F in Γ define $D(F) := \min\{i \mid F \text{ has a vertex in } \Lambda_i\}$ to be the *level* of F .

Lemma 2.1. If F connects vertices in $\Lambda_{D(F)}$ then F is ascending. Otherwise F is descending.

Proof. Let $i := D(F)$. Observe that blowing down F does not change any n_j for $j < i$, and it is also easy to see that blowing down F does not change d_j for any $j < i$. If F connects vertices in Λ_i , then blowing down F increases n_i , so F is ascending. If F does not connect any vertices in Λ_i , then blowing down F will not change n_i , but since each non-basepoint vertex of Γ is at least trivalent, d_i will be smaller in Γ/F than in Γ , and so F is descending. \square

As a corollary to the proof we obtain

Corollary 2.2. A blow-up at a vertex $v \in \Lambda_i$ is descending if and only if it decreases n_i , that is increases $|\Lambda_i|$. \square

An example of a descending blow-up is given in Figure 1. Here d_0 stays constant 4, and n_1 decreases from -1 to -2 .

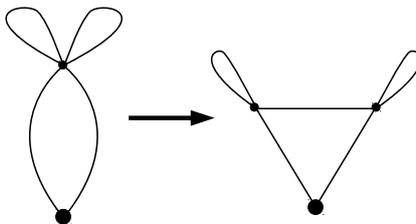


FIGURE 1. A descending blow-up.

We close this section with some definitions regarding edges in graphs. Let ϵ be an edge in Γ , with vertices v_1 and v_2 . We call ϵ *horizontal* if $d(p, v_1) = d(p, v_2)$, and *vertical* if $d(p, v_1) \neq d(p, v_2)$. Let ϵ be a vertical edge with vertices v_1 and v_2 such that $d(p, v_1) > d(p, v_2)$. We call v_1 the *top* of ϵ and v_2 the *bottom*. A half-edge can also have a top or a bottom (or neither, e.g., if it comes from a horizontal edge). We say that ϵ is *decisive* if it is the unique vertical edge having v_1 as its top, that is if any minimal length edge path from v_1 to p must begin with ϵ .

3. CONNECTIVITY OF THE DOWN-LINK

In this section we analyze the down-link of Γ . In order for a certain induction to run, it will become necessary to consider (connected) graphs with vertices of valency 1 and 2. It turns out that h does not “work correctly” on such graphs, for instance

Lemma 2.1 no longer holds. Therefore in this section we will use Lemma 2.1 as a guide for which forests we want to consider.

Definition 3.1. Let Γ be a connected graph with basepoint p , and with no restriction on the valency of vertices. Let F be a subforest of Γ , with level $D(F)$. We will call F *bad* if it connects vertices in $\Lambda_{D(F)}$, and *good* if it does not.

Of course, thanks to Lemma 2.1, if all non-basepoint vertices in Γ are at least trivalent and p is at least bivalent, then a forest in Γ is good if and only if it is descending. Let $P(\Gamma)$ be the poset of good forests in Γ , ordered by inclusion, so for a vertex Γ in L_n , the geometric realization $|P(\Gamma)|$ of $P(\Gamma)$ is the down-link of Γ . Let V be the number of vertices in Γ and E the number of edges. In what follows we will suppress the bars indicating geometric realization, so posets themselves will be said to have a homotopy type. Recall that an empty wedge of spheres is a single point.

Proposition 3.2. $P(\Gamma)$ is homotopy equivalent to a (possibly empty) wedge of spheres of dimension $V - 2$.

Proof. Our proof is similar to the proof of Proposition 2.2 in [Vog90]. We induct on the number of edges E . We can assume that Γ has no loops, since they do not affect V or $P(\Gamma)$. We remark that already after this reduction the vertices may have arbitrary valency, so it is important that we are considering “good” forests instead of “descending” forests. The base case is $E = 0$, for which $V = 1$ and $P(\Gamma) = \emptyset = S^{V-2}$ as desired.

Now suppose $E > 0$. Choose an edge ϵ with endpoints v_1, v_2 maximizing the quantity $d(p, v_1) + d(p, v_2)$. In other words, ϵ is as far as possible from the basepoint. Let $P_1(\Gamma) \subseteq P(\Gamma)$ be the poset of all good forests in Γ except the forest just consisting of the edge ϵ . Also let $P_0(\Gamma) \subseteq P_1(\Gamma)$ be the poset of good forests that do not contain ϵ . For any $F \in P_1(\Gamma)$, $F \setminus \{\epsilon\}$ is again a good forest by definition, so the poset map $g : P_1(\Gamma) \rightarrow P_1(\Gamma)$ given by $F \mapsto F \setminus \{\epsilon\}$ is well defined. Clearly g is the identity on its image $P_0(\Gamma)$, and $g(F) \leq F$ for all $F \in P_1(\Gamma)$, so g induces a homotopy equivalence between $P_1(\Gamma)$ and $P_0(\Gamma)$ by [Qui78, Section 1.3].

If ϵ is a separating edge then $P(\Gamma)$ is contractible with cone point ϵ . Assume ϵ is not separating, so $\Gamma \setminus \epsilon$ is a connected graph. Consider the map $\iota : P(\Gamma \setminus \epsilon) \rightarrow P_0(\Gamma)$ induced by $\Gamma \setminus \epsilon \hookrightarrow \Gamma$. Since $D(\epsilon)$ is maximized, ϵ cannot be decisive, so adding ϵ to the graph does not change the levels Λ_i . In particular adding ϵ cannot affect whether a forest F in $\Gamma \setminus \epsilon$ is good or bad, so g is an isomorphism. Since $\Gamma \setminus \epsilon$ has $E - 1$ edges and V vertices, by induction we get that $P(\Gamma \setminus \epsilon)$ is homotopy equivalent to $\bigvee S^{V-2}$. Hence the same is true of $P_1(\Gamma)$.

If ϵ is horizontal then it is bad, so $P_1(\Gamma) = P(\Gamma)$ and we are done. Assume instead that ϵ is vertical, hence good, which means $P(\Gamma) = P_1(\Gamma) \cup \text{st}(\epsilon)$ with $P_1(\Gamma) \cap \text{st}(\epsilon) = \text{lk}(\epsilon)$, where link and star are taken in $P(\Gamma)$. It now suffices to show that $\text{lk}(\epsilon) \simeq \bigvee S^{V-3}$. We first claim that for a forest F in Γ , $F \in P(\Gamma)$ if and only if $F/\epsilon \in P(\Gamma/\epsilon)$, where F/ϵ is the image of F in Γ/ϵ . If $D(F) < D(\epsilon)$ then this is trivial, so assume $D(F) \geq D(\epsilon)$. By our choice of ϵ , $D(F) = D(\epsilon)$, and it is then immediate that F is good if and only if F/ϵ is. Now consider the map $c : \text{lk}(\epsilon) \rightarrow P(\Gamma/\epsilon)$ sending F to F/ϵ . This is well-defined by the previous claim, and is clearly injective. We claim that c is also surjective, and hence an isomorphism. Let $\Phi \in P(\Gamma/\epsilon)$. There are precisely two forests in Γ that map to Φ under blowing down ϵ , one that contains ϵ and one that does not. Let Φ' be the one that does. If Φ was good then so is Φ' , again by the previous claim, so $\Phi' \in \text{lk}(\epsilon)$. Hence c is an isomorphism, and since Γ/ϵ has $E - 1$ edges and $V - 1$ vertices we have by induction that $\text{lk}(\epsilon) \simeq \bigvee S^{V-3}$ as desired. \square

It will also be convenient to establish one specific case when $P(\Gamma)$ is contractible.

Lemma 3.3. If Γ has a decisive edge then $P(\Gamma)$ is contractible.

Proof. The proof is almost the same as the proof of the previous lemma. We again induct on E . If $E = 0$ then Γ does not have any edges, much less any decisive edges, and so the claim is vacuously true. Now assume $E > 0$ and suppose Γ has a decisive edge η . If η has maximum distance to the base point among edges in Γ then it is separating and $P(\Gamma)$ is contractible with the forest consisting just of η serving as a cone point. Otherwise, let $\epsilon \neq \eta$ be an edge in Γ that has maximum distance to the basepoint, and define $P_1(\Gamma)$ and $P_0(\Gamma)$ as in the previous proof.

As before we have $P_1(\Gamma) \simeq P_0(\Gamma) \cong P(\Gamma \setminus \epsilon)$, which is contractible by induction since $\Gamma \setminus \epsilon$ still contains the decisive edge η . If ϵ is horizontal, $P(\Gamma) = P_1(\Gamma)$ and we are done, so assume ϵ is vertical. As in the previous proof, it then suffices to show that $\text{lk}(\epsilon)$ has the appropriate homotopy type, i.e., is contractible. We still have the isomorphism $c : \text{lk}(\epsilon) \rightarrow P(\Gamma/\epsilon)$. Let η' be the image of η in Γ/ϵ . Since η is decisive, ϵ and η have different tops. Since ϵ is at maximal distance from p , it is clear that η' is a decisive edge in Γ/ϵ . Hence $\text{lk}(\epsilon)$ is contractible by induction. \square

4. CONNECTIVITY OF THE UP-LINK

We now inspect the up-link. Let $\text{BU}(v)$ be the poset of all blow-ups at the vertex v . We can describe $\text{BU}(v)$ using the combinatorial framework for graph blow-ups described in [CV86] and [Vog90], namely $\text{BU}(v)$ is the poset of *compatible partitions* of the set of incident half-edges, which we now describe.

Let $[n] := \{1, 2, \dots, n\}$, and consider partitions of $[n]$ into two blocks. Denote such a partition by $\alpha = \{a, \bar{a}\}$, where $1 \in a$. Define the *size* of α be $s(\alpha) := |\bar{a}|$. Recall that distinct partitions $\{a, \bar{a}\}$ and $\{b, \bar{b}\}$ are said to be *compatible* if either $a \subset b$ or $b \subset a$. For $n \geq 3$ let $\Sigma(n)$ denote the simplicial complex of partitions $\alpha = \{a, \bar{a}\}$ of $[n]$ into blocks a and \bar{a} such that a and \bar{a} each have at least two elements, so $2 \leq s(\alpha) \leq n - 2$. That is, the vertices of $\Sigma(n)$ are such partitions, and a j -simplex is given by a collection of $j + 1$ distinct, pairwise compatible partitions. Note that $\Sigma(3) = \emptyset$. Also define a similar complex $\Sigma'(n)$ for $n \geq 2$, identical to $\Sigma(n)$ except that we allow partitions $\alpha = \{a, \bar{a}\}$ with $|\bar{a}| = 1$. We do not allow $|a| = 1$ though, so for example $\Sigma'(2) = \emptyset$.

It can be seen that for $v \neq p$ the geometric realization of $\text{BU}(v)$ is isomorphic to the barycentric subdivision of $\Sigma(n)$, where $n = \text{val}(v)$. The idea is that a partition describes an *ideal edge*, i.e., an edge blow-up at a vertex with n incident half-edges, and the blocks a and \bar{a} indicate which half-edges attach to which endpoints of the new edge. See [CV86] and [Vog90] for more details.

Thanks to Corollary 2.2 we know precisely when a blow-up at v is descending, namely when more than one resulting vertex is at the same level as v . Hence a blow-up at v is descending if and only if it separates the set of half-edges at v with top v . We say that such a blow-up *separates at* v . Let $\text{SBU}(v)$ be the poset of blow-ups at v that separate at v . Note that blow-ups at the basepoint p are never separating, so $\text{SBU}(p) = \emptyset$.

We will say that a partition $\alpha = \{a, \bar{a}\}$ of $[n]$ *splits* a subset $S \subseteq [n]$ if $S \not\subseteq a$ and $a \not\subseteq S$. Define the *splitting level* $\ell(\alpha)$ to be the minimum element of \bar{a} , i.e., the smallest ℓ such that α splits $[\ell]$. Note that $2 \leq \ell(\alpha) \leq n - 1$ for $\alpha \in \Sigma(n)$ and $2 \leq \ell(\alpha) \leq n$ for $\alpha \in \Sigma'(n)$. Let $\Sigma(n, r)$ be the sublevel set of $\Sigma(n)$ spanned by partitions α with $\ell(\alpha) \leq r$, and similarly define $\Sigma'(n, r)$.

Lemma 4.1. Let $v \neq p$ be a vertex in Γ with n incident half-edges. Let r be the number of half-edges with top v . Then $|\text{SBU}(v)| \simeq \Sigma(n, r)$.

Proof. It is clear that the geometric realization $|\text{SBU}(v)|$ contains the barycentric subdivision of $\Sigma(n, r)$ as a subcomplex. Also, any simplex in $|\text{SBU}(v)|$ has at

least one vertex in $\Sigma(n, r)$. Hence there is a map $|\text{SBU}(v)| \rightarrow |\text{SBU}(v)|$ sending each simplex to its face spanned by vertices in $\Sigma(n, r)$. This clearly induces a deformation retraction from $|\text{SBU}(v)|$ to $\Sigma(n, r)$. \square

We now want to calculate the homotopy type of $\Sigma(n, r)$, and perhaps unsurprisingly we will use Morse theory. Consider the height function $z(\alpha) := (\ell(\alpha), s(\alpha))$ on $\Sigma(n)$, with the lexicographic ordering. Since compatible partitions have different sizes, they also have different z -values. Note that $\Sigma(n, r)$ is a sublevel set with respect to z , namely $\Sigma(n, r) = \Sigma(n)^{z \leq (r, n-2)}$. We can also think of z as a height function on $\Sigma'(n)$, and before handling $\Sigma(n, r)$ it will be convenient to first calculate the homotopy type of $\Sigma'(n, r)$.

Lemma 4.2. For any $n \geq 2$ and $2 \leq r \leq n$, $\Sigma'(n, r) \simeq \bigvee S^{n-3}$.

Proof. We induct on n . Since $\Sigma'(2) = \emptyset$, we already know that $\Sigma'(2, r) = \emptyset = S^{2-3}$ for any r , which handles the base case. Now let $n > 2$ and consider the complex $\Sigma'(n, 2)$. This is spanned by partitions $\{a, \bar{a}\}$ in which the set $\{1, 2\}$ is split, and so any such a will be $a = \{1\} \cup T$ for T a non-empty subset of $\{3, 4, \dots, n\}$. Thus $\Sigma'(n, 2)$ is isomorphic to the barycentric subdivision of an $(n-3)$ -simplex, and so is contractible.

We now analyze the descending links of partitions with respect to z , and build up from $\Sigma'(n, 2)$ to $\Sigma'(n, r)$ for $r > 2$ by attaching new partitions along their descending links. Let $\alpha = \{a, \bar{a}\}$ be a partition in $\Sigma'(n, r) \setminus \Sigma'(n, 2)$ and set $\ell := \ell(\alpha) > 2$ and $s := s(\alpha)$. A partition $\beta = \{b, \bar{b}\}$ compatible with α is in the z -descending link $\text{lk}_{\downarrow z}(\alpha)$ of α precisely when either $\ell(\beta) < \ell$, or $\ell(\beta) = \ell$ and $a \subsetneq b$. Note that in the first case $b \subseteq a$, so any partition of the first type is compatible with every partition of the second type. Hence the z -descending link of α is a join, of an *in-link* and an *out-link*. The in-link is the full subcomplex of $\text{lk}_{\downarrow z}(\alpha)$ spanned by partitions of the first type, and the out-link is spanned by partitions of the second type. See Figure 2 for an example.

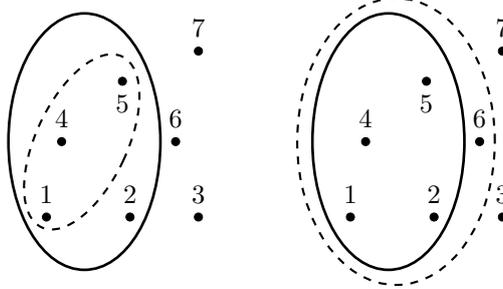


FIGURE 2. A partition in the in-link, and one in the out-link, of a partition with size $s = 3$ and splitting level $\ell = 3$.

First consider the out-link. Partitions $\beta = \{b, \bar{b}\}$ in the out-link are characterized by the property that $a \subsetneq b$ and $\ell \in \bar{b}$. Treating a as a single point, this amounts to saying that $a \subsetneq b$ and β splits $\{a, \ell\}$. Hence the out-link is isomorphic to $\Sigma'(s+1, 2)$. If $s = 1$ this is empty, and if $s > 1$ this is contractible as explained above. In particular if $s > 1$ then $\text{lk}_{\downarrow z}(\alpha)$ is already contractible. Now assume $s = 1$, so the out-link is empty and $\text{lk}_{\downarrow z}(\alpha)$ just equals the in-link. Then the in-link is isomorphic to the complex of partitions of $[n-1]$ that split $[\ell-1]$, and so is given by $\Sigma'(n-1, \ell-1)$. This is $(n-1-3)$ -spherical by induction, so we conclude that all descending links are either contractible or $(n-4)$ -spherical. Since $\Sigma'(n, 2)$ is $(n-3)$ -spherical this implies that $\Sigma'(n, r)$ is also $(n-3)$ -spherical. \square

Proposition 4.3. For any $n \geq 3$ and $2 \leq r \leq n - 1$, $\Sigma(n, r) \simeq \bigvee S^{n-4}$.

Proof. As in the previous proof we induct on n . When $n = 3$ we only consider $r = 2$, and $\Sigma(3, 2)$ is clearly empty. Now let $n > 3$ and consider $\Sigma(n, 2)$. As with $\Sigma'(n, 2)$, $\Sigma(n, 2)$ is spanned by partitions $\{a, \bar{a}\}$ in which the set $\{1, 2\}$ is split, and so any such a will be $a = \{1\} \cup T$, for T now a *proper* non-empty subset of $\{3, 4, \dots, n\}$. Thus $\Sigma(n, 2)$ is the surface of a barycentrically subdivided $(n - 3)$ -simplex, and so is homeomorphic to S^{n-4} . Now consider the descending link $\text{lk}_{\downarrow z}(\alpha)$ of $\alpha = \{a, \bar{a}\}$ with $\ell := \ell(\alpha) > 2$. The descending link decomposes as before as the join of an in-link and out-link. By the same argument as in the previous proof, the out-link is isomorphic to $\Sigma(s + 1, 2)$, which is homeomorphic to S^{s-3} . The in-link is isomorphic to the complex of partitions of $[n - s]$ that split $[\ell - 1]$ and have size at least 1. (Since \bar{a} has at least one element, we do have to consider partitions of $[n - s]$ that have size 1 as a partition of $[n - s]$.) So, the in-link is isomorphic to $\Sigma'(n - s, \ell - 1)$, and hence is homotopy equivalent to $\bigvee S^{n-s-3}$ by the previous lemma. Then $\text{lk}_{\downarrow z}(\alpha)$ is the join of the in- and out-links, and so is homotopy equivalent to $(\bigvee S^{n-s-3}) * S^{s-3} = \bigvee S^{n-5}$. Since $\Sigma(n, 2)$ is $(n - 4)$ -spherical and the descending links of partitions in $\Sigma(n, r) \setminus \Sigma(n, 2)$ are all $(n - 5)$ -spherical, we conclude that $\Sigma(n, r)$ is $(n - 4)$ -spherical. \square

We remark that since $\Sigma(n, n - 1) = \Sigma(n)$, we recover the fact that $\Sigma(n)$ is $(n - 4)$ -spherical, as shown in [Vog90, Theorem 2.4]. Coupling Proposition 4.3 with Lemma 4.1 we see that if there are least two half-edges with top v , then $|\text{SBU}(v)| \simeq \bigvee S^{\text{val}(v)-4}$.

Now let $A := *_{v \neq p} \text{SBU}(v)$, where the join is taken over all vertices $v \neq p$ in Γ . Recall that V is the number of vertices in Γ .

Corollary 4.4. If Γ has no decisive edges then $|A| \simeq \bigvee S^{d_0(\Gamma)-V}$.

Proof. Since there are no decisive edges, for any $v \neq p$ we know that there are at least two half-edges at v with top v . Hence $|\text{SBU}(v)| \simeq \bigvee S^{\text{val}(v)-4}$, and so

$$|A| \simeq *_{v \neq p} \bigvee S^{(\text{val}(v)-2)-2} = \bigvee S^{(d_0(\Gamma)-2(V-1))+(V-2)} = \bigvee S^{d_0(\Gamma)-V}.$$

\square

Proposition 4.5. If Γ has no decisive edges then the up-link is homotopy equivalent to $|A|$, and hence to $\bigvee S^{d_0(\Gamma)-V}$.

Proof. For a poset P , define \underline{P} to be $P \sqcup \{\perp\}$, with \perp a formal minimum element. Then $P * Q \cong \underline{P} \times \underline{Q} \setminus \{(\perp, \perp)\}$ for posets P and Q . The relevant example is that $A = *_{v \neq p} \text{SBU}(v) \cong \prod_{v \neq p} \underline{\text{SBU}(v)} - \{(\perp)_v\} =: Y$.

Define $X := \left\{ f \in \prod_{v \neq p} \underline{\text{BU}(v)} \mid \exists v \in \Lambda_{D(f)} \text{ with } f_v \in \text{SBU}(v) \right\}$. Here f_v is the

blow-up at vertex v in the tuple f , and $D(f)$ is the minimal level such that $f_v \neq \perp$ for some $v \in \Lambda_{D(f)}$. Note that $Y \subseteq X$. Define a map $r : X \rightarrow Y$ by

$$(f_v)_v \mapsto \left(\begin{array}{l} f_v \text{ for } f_v \in \text{SBU}(v) \\ \perp \text{ for } f_v \notin \text{SBU}(v) \end{array} \right)_v$$

Note that r is a poset map that is the identity on its image Y . Also, $r(f) \leq f$ for all $f \in X$, so r induces a homotopy equivalence between $|X|$ and $|Y|$ by [Qui78, Section 1.3]. But $|X|$ is precisely the up-link of Γ , so the up-link is homotopy equivalent to $\bigvee S^{d_0(\Gamma)-V}$ by Corollary 4.4. \square

Corollary 4.6. For any vertex Γ in L_n , $\text{lk}\downarrow(\Gamma)$ is either contractible or homotopy equivalent to $\bigvee S^{d_0(\Gamma)-1}$.

Proof. If the down-link of Γ is contractible, then so is $\text{lk}\downarrow(\Gamma)$. If the down-link is not contractible, then Γ has no decisive edges. Hence joining the up-link and down-link yields $\left(\bigvee S^{d_0(\Gamma)-V}\right) * \left(\bigvee S^{V-2}\right) \simeq \bigvee S^{d_0(\Gamma)-1}$. \square

Theorem 4.7. $L_{n,k}$ is $(k-1)$ -connected.

Proof. We build up from $L_{n,k}$ to the contractible space L_n by gluing in vertices of $L_n \setminus L_{n,k}$ along their descending links, in order of increasing height. If two vertices have the same height then their descending stars are disjoint, so we can choose any order to attach them. For any vertex Γ in $L_n \setminus L_{n,k}$ we have $d_0(\Gamma) > k$, so by the previous corollary $\text{lk}\downarrow(\Gamma)$ is at least $(k-1)$ -connected. We conclude that $L_{n,k}$ is $(k-1)$ -connected. \square

REFERENCES

- [CV86] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. **84** (1986), no. 1, 91–119.
- [HV98] A. Hatcher and K. Vogtmann, *Cerf theory for graphs*, J. London Math. Soc. (2) **58** (1998), no. 3, 633–655.
- [Qui78] D. Quillen, *Homotopy properties of the poset of nontrivial p -subgroups of a group*, Adv. in Math. **28** (1978), no. 2, 101–128.
- [Vog90] K. Vogtmann, *Local structure of some $\text{Out}(F_n)$ -complexes*, Proc. Edinburgh Math. Soc. (2) **33** (1990), no. 3, 367–379.
- [Zar] M. C. B. Zaremsky, *Rational homological stability for groups of partially symmetric automorphisms of free groups*. arXiv:1203.4845.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904
E-mail address: ram7s@virginia.edu

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, BIELEFELD, GERMANY 33615
E-mail address: zaremsky@math.uni-bielefeld.de