

RATIONAL HOMOLOGICAL STABILITY FOR GROUPS OF PARTIALLY SYMMETRIC AUTOMORPHISMS OF FREE GROUPS

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ABSTRACT. Let F_{n+m} be the free group of rank $n + m$, with generating set $\{x_1, \dots, x_{n+m}\}$. An automorphism ϕ of F_{n+m} is called partially symmetric if for each $1 \leq i \leq m$, $\phi(x_i)$ is conjugate to x_j or x_j^{-1} for some $1 \leq j \leq m$. Let ΣAut_n^m be the group of partially symmetric automorphisms. We prove that for any $m \geq 0$ the inclusion $\Sigma\text{Aut}_n^m \rightarrow \Sigma\text{Aut}_{n+1}^m$ induces an isomorphism in rational homology for $n \geq (3(i+1) + m)/2$, with a similar statement for the groups $P\Sigma\text{Aut}_n^m$ of pure partially symmetric automorphisms. We also prove that for any $n \geq 0$ the inclusion $\Sigma\text{Aut}_n^m \rightarrow \Sigma\text{Aut}_n^{m+1}$ induces an isomorphism in rational homology for $m > 3(i+1)/2$.

1. INTRODUCTION

Let $\text{Aut}(F_{n+m})$ be the group of automorphisms of the free group F_{n+m} . If $S = \{x_1, \dots, x_{n+m}\}$ is a fixed basis of F_{n+m} , an automorphism ϕ of F_{n+m} is called *partially symmetric* if for each $1 \leq i \leq m$, $\phi(x_i)$ is conjugate to x_j or x_j^{-1} for some $1 \leq j \leq m$. If ϕ is an automorphism such that each $\phi(x_i)$ is even conjugate to x_i we call ϕ *pure partially symmetric*. Call these first m generators *distinguished*. Let ΣAut_n^m be the group of symmetric automorphisms of F_{n+m} , and $P\Sigma\text{Aut}_n^m$ the group of pure symmetric automorphisms.

To keep the notation straight, we mention that in [BCV09] the “outer” version of the group we are calling $P\Sigma\text{Aut}_n^m$ is denoted $P\Sigma(n, k)$, where n is the rank and k the number of distinguished generators. In [JW04] the same group is denoted A_n^k , where n and k are the number of non-distinguished and distinguished generators, respectively. In [JW04] certain other groups denoted $A_{n,k}$ are considered, which are central extensions of A_n^k , but these are not the same as the groups ΣAut_n^m considered here.

In [HV98a] it is shown that the homology of $\text{Aut}(F_n) = \Sigma\text{Aut}_n^0$ is stable with respect to n , meaning that it is independent of n for n large enough. In [Gal11] the stable rational homology is even shown to be trivial, namely, $H_i(\text{Aut}(F_n); \mathbb{Q}) = 0$ for all $n > 2i+1$ [Gal11, Corollary 1.2]. At the other end of the spectrum, in [HW10] it is shown that the group of *symmetric automorphisms* $\Sigma\text{Aut}(F_m) = \Sigma\text{Aut}_0^m$ is homologically stable in m , and it turns out the rational homology actually vanishes in every dimension; see [Gri, Wil]. In contrast, the rational homology of $P\Sigma\text{Aut}_0^m$ is not stable in m [JW04]. In the present work we analyze ΣAut_n^m , using a version of a space constructed in [BCV09] and using methods inspired by [MZ].

We actually obtain stability results for a range of families of subgroups of ΣAut_n^m , which includes the groups $P\Sigma\text{Aut}_n^m$. Consider any family of groups G_n^m such that $P\Sigma\text{Aut}_n^m \leq G_n^m \leq \Sigma\text{Aut}_n^m$ for each n and m , and such that the inclusion $\Sigma\text{Aut}_n^m \hookrightarrow \Sigma\text{Aut}_{n+1}^m$, given by extending $\phi \in \Sigma\text{Aut}_n^m$ to F_{m+n+1} via $\phi(x_{n+m+1}) = x_{n+m+1}$,

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restricts to an inclusion $G_n^m \hookrightarrow G_{n+1}^m$. Of course $P\Sigma\text{Aut}_n^m$ and ΣAut_n^m themselves are examples of such families of groups. Our main result for these groups is the following theorem.

Theorem 1.1. *For any $m \geq 0$ and any family of groups G_n^m satisfying the above conditions, the map $H_i(G_n^m; \mathbb{Q}) \rightarrow H_i(G_{n+1}^m; \mathbb{Q})$ induced by inclusion is an isomorphism for $n \geq (3(i+1) + m)/2$.*

Corollary. *The rational homology of ΣAut_n^m is stable in n , as is the rational homology of $P\Sigma\text{Aut}_n^m$. \square*

We also consider stability in the other parameter, m . Renumber the elements of S as $\{x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}\}$, so an automorphism ϕ is partially symmetric if for all $1 \leq i \leq m$, $\phi(x_{n+i})$ is conjugate to x_{n+j} or x_{n+j}^{-1} for some $1 \leq j \leq m$. We now have a natural inclusion map $\Sigma\text{Aut}_n^m \hookrightarrow \Sigma\text{Aut}_n^{m+1}$, given by extending $\phi \in \Sigma\text{Aut}_n^m$ to F_{m+n+1} via $\phi(x_{n+m+1}) = x_{n+m+1}$.

Theorem 1.2. *For any $n \geq 0$, the map $H_i(\Sigma\text{Aut}_n^m; \mathbb{Q}) \rightarrow H_i(\Sigma\text{Aut}_n^{m+1}; \mathbb{Q})$ induced by inclusion is an isomorphism for $m > 3(i+1)/2$.*

In Section 2 we provide some background on the spine of Auter space K_{n+m} , and describe a family of subcomplexes ΔK_n^m that admit a nice ΣAut_n^m action. In Section 3 we define a height function on ΔK_n^m , and in Section 4 we use this height function to show that the sublevel sets of ΔK_n^m with respect to *weighted degree* are highly connected. Lastly, in Section 5 we prove Theorems 1.1 and 1.2.

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2. AUTER SPACE AND GRAPHS WITH DISJOINT DISTINGUISHED CYCLES

We will analyze the homology of ΣAut_n^m by considering its action on a certain topological space. Our starting point is the well-studied *spine of Auter space* K_n introduced in [HV98a]. Let R_n be the rose with n edges, i.e., the graph with a single vertex p_0 and n edges. Here by a *graph* we mean a connected one-dimensional CW-complex, with the usual notions of vertices and edges. We identify F_n with $\pi_1(R_n)$. If Γ is a graph with basepoint vertex p , a homotopy equivalence $\rho: R_n \rightarrow \Gamma$ is called a *marking* on Γ if ρ takes p_0 to p . We will consider two markings to be equivalent if there is a basepoint-preserving homotopy between them. Also, we only consider graphs such that p is at least bivalent and all other vertices are at least trivalent. Note however that we are allowing separating edges.

For graphs Γ_1 and Γ_2 , a basepoint-preserving homotopy equivalence $d: \Gamma_1 \rightarrow \Gamma_2$ is called a *forest collapse* or a *blow-down* if it amounts to collapsing a subforest of Γ_1 . The reverse of a blow-down is, naturally, called a *blow-up*. This gives us a partial ordering on equivalence classes of triples (Γ, p, ρ) , namely $(\Gamma', p, \rho') \leq (\Gamma, p, \rho)$ if there is a forest collapse $d: \Gamma \rightarrow \Gamma'$ such that ρ' is equivalent to $d \circ \rho$. The spine K_n of Auter space is then the geometric realization of the poset of equivalence classes of triples (Γ, p, ρ) with Γ a rank- n graph, with this partial ordering. In particular the vertices of K_n are equivalence classes of such triples.

Since we are identifying F_n with $\pi_1(R_n)$, we can also identify $\text{Aut}(F_n)$ with the group of basepoint-preserving homotopy equivalences of R_n , up to homotopy. This is of course the same as the group of markings of R_n , so we can denote markings on R_n by elements of $\text{Aut}(F_n)$. There is a (right) action of $\text{Aut}(F_n)$ on K_n in the following way: given a vertex (Γ, p, ρ) in K_n and $\phi \in \text{Aut}(F_n)$, we have

$\phi(\Gamma, p, \rho) = (\Gamma, p, \rho \circ \phi)$. In particular this action only affects markings, and it is easy to see that $\text{Aut}(F_n)$ permutes markings arbitrarily.

To analyze the groups ΣAut_n^m we will work with a certain (full) subcomplex ΔK_n^m of K_{n+m} . The vertices of ΔK_n^m are marked basepointed graphs (Γ, p, ρ) such that Γ is a *viable graph* and ρ is an *admissible marking*. A viable graph is a basepointed graph (Γ, p) such that there exist m reduced cycles C_1, \dots, C_m in Γ that are pairwise disjoint. See Figure 1 for an example. A marking ρ is called admissible if there is a maximal tree T in Γ such that for $\pi : \Gamma \rightarrow \Gamma/T = R_{n+m}$, $\pi \circ \rho \in \Sigma\text{Aut}_n^m$ (recall our identification of $\text{Aut}(F_{n+m})$ with markings on R_{n+m}), and the reduced cycles C_i obtained by reducing $\rho(x_i)$ for $1 \leq i \leq m$ are pairwise disjoint. Also see [BCV09] for a more complete description. For brevity we will just define a *viable marked graph* to be a viable graph with an admissible marking. The cycles C_i for $1 \leq i \leq m$ are called *distinguished cycles*, and we similarly refer to vertices, edges, and half-edges as *distinguished* if they are contained in some C_i . A forest F in a viable marked graph Γ is called *admissible* if Γ/F is again viable and the induced marking is again admissible. The characterizing property of admissible forests is that any admissible tree T can intersect at most one distinguished cycle C , and if $T \cap C$ is nonempty then it must either be a single vertex or a connected edge path in C . An example of an admissible and a non-admissible forest (for some marking ρ) are shown in gray in Figure 1.

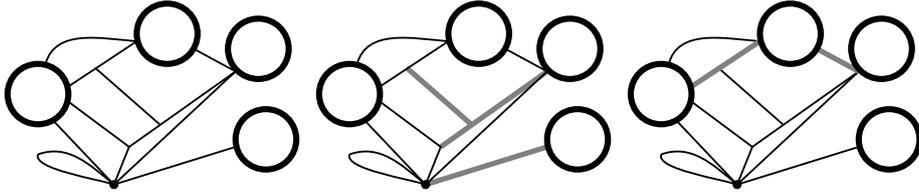


FIGURE 1. From left to right: A viable graph, an admissible forest and a non-admissible forest.

The action of ΣAut_n^m on K_{n+m} only affects markings, and takes admissible markings to admissible markings, so we can consider the action of ΣAut_n^m on ΔK_n^m . Let $\Delta Q_n^m := \Delta K_n^m / \Sigma\text{Aut}_n^m$ be the orbit space.

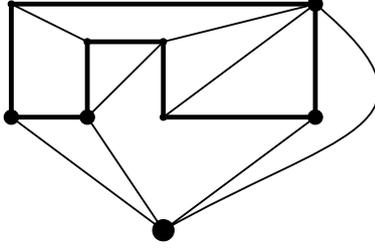
Proposition 2.1. [BCV09, Section 3] ΣAut_n^m acts on ΔK_n^m with finite stabilizers and finite quotient ΔQ_n^m , and ΔK_n^m is contractible.

It is also clear that if an element of ΣAut_n^m stabilizes a simplex then it fixes it pointwise, since the vertices of any simplex correspond to pairwise non-isomorphic graphs. The upshot of this that ΔQ_n^m and ΣAut_n^m have the same rational homology; see for example Exercise 2 on page 174 in [Bro82].

It is difficult to analyze ΔQ_n^m directly, and so we will work with a certain filtration. For a vertex (Γ, p, ρ) in ΔK_n^m , define the *weighted valency* $val_w(v)$ of a vertex v to be the number of non-distinguished half-edges at v , plus half the number of distinguished half-edges. Define the *weighted degree* $d_w(\Gamma)$ to be $2n + m - val_w(p)$. It is clear that $1 \leq val_w(p) \leq 2n + m$, and so $0 \leq d_w(\Gamma) \leq 2n + m - 1$. For $k \in \mathbb{N}_0$ let $\Delta K_{n,k}^m$ be the subspace of ΔK_n^m spanned by marked basepointed graphs of weighted degree less than or equal to k . In particular for $k \geq 2n + m - 1$, $\Delta K_{n,k}^m = \Delta K_n^m$. We will also make use of the notion of *degree* from [HV98a], which we define to be $d_0(\Gamma) := 2n + 2m - val(p)$. If c denotes the number of distinguished cycles not containing p , it is clear that $d_w = d_0 - c$.

3. A HEIGHT FUNCTION

We now define a height function h on ΔK_n^m . This height function is related to the one defined in [MZ] on the space K_n . Let (Γ, p, ρ) be a basepointed viable marked graph. For vertices v, v' in Γ , let $d(v, v')$ be the number of edges in a minimal-length edge path from v to v' . Also, for a subforest F of Γ , define the *level* $D(F)$ from F to p to be the smallest i such that F has a vertex at distance i from p . Let $\Lambda_i(\Gamma) := \{v \in \Gamma \mid d(p, v) = i\}$ be the i *th level* of Γ , so for example $\Lambda_0(\Gamma) = \{p\}$. A reduced edge path γ in Γ will be called an *arc* if both of its endpoints lie in $\Lambda_{D(\gamma)}$. If v is a vertex that is in a distinguished cycle C , and $d(p, v) \leq d(p, v')$ for any other vertex v' in C , then we will say that v is a *base vertex* for C , and call $i_C := d(p, v)$ the *base height* of C . If v is a base vertex for some C , call v a base vertex. Note that the basepoint p is a base vertex if and only if it is distinguished, if and only if $c = m - 1$. In Figure 2 the distinguished cycle C is indicated by thick edges, the base vertices are given by the larger dots, and the basepoint is the largest dot at the bottom.

FIGURE 2. Distinguished cycle C with $i_C = 1$.

For each $i \geq 0$ let $m_i(\Gamma)$ be the number of base vertices in $\Lambda_i(\Gamma)$, define $n_i(\Gamma) := -|\Lambda_i(\Gamma)|$ and let $d_i(\Gamma) := \sum_{v \notin \Lambda_i} (\text{val}(v) - 2)$. Note that $m_0 = m - c$, n_0 is constant -1 and $d_0 = 2n + 2m - \text{val}(p)$ is the degree. Now define $h_i(\Gamma) := (m_i(\Gamma), n_i(\Gamma), d_i(\Gamma))$, and set

$$h(\Gamma) = (h_0(\Gamma), h_1(\Gamma), h_2(\Gamma), \dots)$$

with the lexicographic order. We remark that in [MZ] the height function used was $(d_0, n_1, d_1, n_2, d_2, \dots)$, which encodes the same information as our h when $m = 0$. Extend h to the vertices of ΔK_n^m via $h(\Gamma, p, \rho) = h(\Gamma)$. In general we will just write Γ to denote vertices of ΔK_n^m , with the basepoint and marking understood. It is worth emphasizing that h is *not* a refinement of the coarse height function d_w , indeed d_w does not even appear as a factor in h . This will be important in Section 4.3.

Note that for any admissible forest F , blowing down F either increases or decreases $h_{D(F)}$. For example, if $n_{D(F)}$ does not change, then $d_{D(F)}$ must decrease. Also, blowing down F does not change any h_i for $i < D(F)$, since this is clearly true for m_i and n_i , and is easy to check for d_i . In general, of all the terms changed by blowing down F , there is one that is lexicographically first, which we will call the *essential* term of F . Similarly, any blow-up has an essential term. We remark that a blow-down at level i cannot decrease n_i , and a blow-up at level i cannot decrease d_i , though blow-downs and blow-ups can both increase or decrease m_i .

For a vertex Γ in ΔK_n^m , the *descending star* $\text{st}\downarrow(\Gamma)$ with respect to h is the set of simplices in the star of Γ whose vertices all have strictly lower height than Γ . The *descending link* $\text{lk}\downarrow(\Gamma)$ consists of the faces of simplices in $\text{st}\downarrow(\Gamma)$ that do not themselves contain Γ . There are two types of vertex in $\text{lk}\downarrow(\Gamma)$: those obtained

from Γ by a descending blow-up, and those obtained by a descending blow-down. Here we say that a blow-up or blow-down is *descending* if the resulting graph has a lower height than the starting graph. Call the subcomplex of $\text{lk}\downarrow(\Gamma)$ spanned by vertices of the first type the *up-link*, and the subcomplex spanned by vertices of the second type the *down-link*. Any vertex in the up-link is related to every vertex in the down-link by a blow-down, so $\text{lk}\downarrow(\Gamma)$ is the simplicial join of the up- and down-links. (This is exactly the kind of decomposition of $\text{lk}\downarrow(\Gamma)$ that occurs in [MZ].) We remark that we only consider admissible blow-downs, and on the other hand observe that any blow-up of a viable graph is again viable. If a forest blow-down is descending we call the forest itself *descending*, and similarly we refer to *ascending* forests. As remarked above, any forest blow-down either increases or decreases h . Since adjacent vertices of ΔK_n^m are related via forest blow-downs, this means that adjacent vertices have different heights, so the descending stars of two vertices with the same height must be disjoint.

For an edge ϵ in Γ with vertices v and v' , we call ϵ *horizontal* if $d(p, v) = d(p, v')$. Otherwise we call ϵ *vertical*. If $v = v'$ then we say ϵ is a *loop*, and if some distinguished cycle C has only one edge we call it a *distinguished loop*. For a vertical edge ϵ with vertices v_1 and v_2 , by comparing $d(v_1, p)$ and $d(v_2, p)$ we get a natural notion of the *top* vertex and *bottom* vertex of ϵ . A half-edge may also have either a top or a bottom. If a vertex v has only one incident vertical edge ϵ with v as its top, we call that edge *decisive at v* . In other words, if every minimal-length path from v to p begins with ϵ , then ϵ is decisive at v . If an edge ϵ in Γ is decisive at its top vertex we call it a decisive edge. Given a vertex v with a decisive edge ϵ , if the only other edge incident to v is a distinguished loop ℓ then the subgraph consisting of ℓ , v and ϵ will be called a *lollipop*, with *stick* ϵ .

It will be important to have a somewhat explicit description of which forests are descending.

Lemma 3.1. *Let F be an admissible forest in Γ with $i := D(F)$.*

- (1) *If $m_i(\Gamma/F) < m_i(\Gamma)$, then F is descending.*
- (2) *If $m_i(\Gamma/F) > m_i(\Gamma)$, then F is ascending.*
- (3) *If $m_i(\Gamma/F) = m_i(\Gamma)$ and F connects vertices in Λ_i , then F is ascending.*
- (4) *If $m_i(\Gamma/F) = m_i(\Gamma)$ and F does not connect vertices in Λ_i , then F is descending.*

Proof. The essential term of F occurs in h_i , so the first two claims are immediate. Suppose $m_i(\Gamma/F) = m_i(\Gamma)$. If F connects vertices in Λ_i , then blowing down F increases n_i and so is ascending. If F does not connect vertices in Λ_i , then blowing down F does not change n_i , but decreases d_i , so F is descending. \square

3.1. An idea of the proof of connectivity of $\Delta K_{n,k}^m$. The height function h is quite cumbersome, but the idea of how it will be used is not too complicated. The goal is to prove Corollary 4.15, that $\Delta K_{n,k}^m$ is $(k-1)$ -connected. Since ΔK_n^m is contractible, it suffices to show that we can remove vertices from ΔK_n^m along their relative links, in such a way that we never create a non-trivial homotopy sphere of dimension less than k , and so that we eventually end up with $\Delta K_{n,k}^m$. The function h tells us in which order to remove vertices, so the relative link of Γ is $\text{lk}\downarrow(\Gamma)$, and the problem (almost) reduces to showing that $\text{lk}\downarrow(\Gamma)$ is $(d_w(\Gamma) - 2)$ -connected for any Γ . Since h does not refine d_w , things are a little more complicated, but this issue is dealt with in Section 4.3. (This last issue aside, this is all standard discrete Morse theory.)

It may also be helpful to think of the Morse function induced by h on ΔK_n^m . As indicated in the results of the next section, the critical points with respect to this Morse function, i.e., the points where the homotopy type changes, occur at

very specific graphs Γ . Namely, no non-base vertices of Γ should be the top of a decisive edge, and every base vertex in Γ should have as small a valency as possible. In this situation, $\text{lk}\downarrow(\Gamma)$ is a wedge of spheres of dimension $d_w(\Gamma) - 1$, as seen in Corollary 4.12.

4. CONNECTIVITY

In this section we analyze the descending links of vertices in ΔK_n^m , with the goal of showing that $\Delta K_{n,k}^m$ is $(k - 1)$ -connected. Since $\text{lk}\downarrow(\Gamma)$ is the join of the up-link and down-link, it is immediate that if either of these is contractible, so is $\text{lk}\downarrow(\Gamma)$. In the next two subsections we will show that if this is not the case, then $\text{lk}\downarrow(\Gamma)$ is highly spherical. The main results of the section are Corollaries 4.12 and 4.15. In reading these subsections, the reader may find it helpful to refer to the corresponding sections in [MZ], which cover what amounts here to the $m = 0$ case.

4.1. Connectivity of the descending down-link. In this section we analyze the down-link of Γ . In order to get an induction to run, we will need to lift the restriction on the valency of vertices. Our height function h does not work well with such graphs though, for instance the trivalency of non-basepoint vertices is crucial to the fact that blowing down F either increases $n_{D(F)}$ or decreases $d_{D(F)}$. Thanks to Lemma 3.1 though, we have a condition on forests that is equivalent to being descending for graphs $\Gamma \in \Delta K_n^m$, and does not refer to the functions n_i or d_i . For the rest of this subsection, Γ is a connected graph with basepoint p and m disjoint distinguished cycles, with no restriction on the valency of vertices. The definitions of Λ_i and m_i remain valid, and are as given above.

Definition 4.1. Let F be an admissible forest in Γ . Define

$$\Delta m_i(\Gamma, F) := m_i(\Gamma/F) - m_i(\Gamma)$$

for any i . Now let $i := D(F)$. If $\Delta m_i(\Gamma, F) < 0$ call F *base-decreasing*, if $\Delta m_i(\Gamma, F) > 0$ call F *base-increasing* and if $\Delta m_i(\Gamma, F) = 0$ call F *base-preserving*. If F connects vertices in Λ_i , or equivalently if F contains an arc γ with $D(\gamma) = D(F)$, call F *arc-ed*. If F does not connect vertices in Λ_i , call F *arc-free*. Finally, if F is base-decreasing, or if it is base-preserving and arc-free, call F *good*. A forest is *bad* if it is not good.

Lemma 3.1 says that for any $\Gamma \in \Delta K_n^m$, a forest F in Γ is descending if and only if it is good.

Remark 4.2. There are a few important observations about single-edge forests that we collect here. A vertical edge is arc-free and cannot be base-decreasing, and a distinguished vertical edge cannot be base-increasing, so must be base-preserving and arc-free, hence good. A horizontal edge is arc-ed and cannot be base-increasing, and a base-decreasing horizontal edge must be distinguished. Hence a horizontal edge is good if and only if it is distinguished and base-decreasing, i.e., connects two base vertices.

It is also easy to see whether an edge path γ in a distinguished cycle C is good or bad. Such a γ cannot be base-increasing, so if γ is arc-free then it is automatically good. If γ is arc-ed and $D(\gamma) = i_C$, then γ contains an arc connecting base vertices and so is base-decreasing, hence good. If γ is arc-ed and $D(\gamma) > i_C$ then it is base-preserving, hence bad. To summarize, γ is bad if it is arc-ed and $D(\gamma) > i_C$, and otherwise is good. See Figure 3 for some examples.

Let $P(\Gamma)$ be the poset of good admissible forests in Γ . For $\Gamma \in \Delta K_n^m$, the down-link of Γ is the geometric realization $|P(\Gamma)|$ of $P(\Gamma)$, so the goal of this section is to calculate the homotopy type of $|P(\Gamma)|$. For the rest of this section we will omit

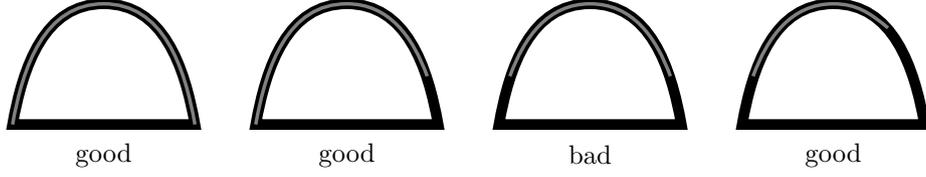


FIGURE 3. Good and bad distinguished edge paths.

the vertical bars, and just refer to $P(\Gamma)$ as having a homotopy type. For each edge ϵ of Γ , let $P_1(\Gamma, \epsilon)$ be the poset of all good admissible forests except the forest just consisting of ϵ , and let $P_0(\Gamma, \epsilon) \subseteq P_1(\Gamma, \epsilon)$ be the poset of good admissible forests that do not contain ϵ . Whenever Γ and ϵ are understood from context we will just write P , P_1 and P_0 .

Lemma 4.3. *For any admissible distinguished edge ϵ , $P_0(\Gamma, \epsilon)$ is contractible.*

Proof. Let C be the distinguished cycle containing ϵ , and let ϕ be the forest consisting of all edges of C other than ϵ . Since $D(\phi) = i_C$, ϕ is good by Remark 4.2. Let $f : P_0 \rightarrow P_0$ be given by $F \mapsto F \cup \phi$. We claim that for $F \in P_0$, $F \cup \phi$ is an admissible good forest, so f is well defined. Since $\epsilon \notin F$, and F is admissible, it is clear that $F \cup \phi$ is an admissible forest. Let ϕ' be the image of ϕ in Γ/F , so $\Gamma/F \cup \phi = (\Gamma/F)/\phi'$. By Remark 4.2, ϕ' is not base-increasing, which tells us that if F is base-decreasing then so is $F \cup \phi$, and so the claim follows. The other way F can be good is if it is base-preserving and arc-free. Then by the same argument, $F \cup \phi$ is not base-increasing, so it suffices to show that if $F \cup \phi$ is arced, then it is base-decreasing. If ϕ itself is arced then it must be base-decreasing, which implies $F \cup \phi$ is base-decreasing. Suppose instead that ϕ is arc-free (and recall that F is too). For $F \cup \phi$ to be arced then, we need that $D(F) = D(\phi) =: i$ and that ϕ' is arced. But by Remark 4.2, if ϕ' is arced then it is base-decreasing, in which case $F \cup \phi$ is base-decreasing, so the claim follows in this case as well.

We conclude that f is well defined, and so it follows immediately from [Qui78, Section 1.5] that P_0 is contractible. \square

For an admissible edge ϵ with endpoints v_1 and v_2 , call ϵ *maximally distant* if among all admissible edges, ϵ maximizes the quantity $\min\{d(p, v_1), d(p, v_2)\}$. Note that if ϵ is maximally distant then $D(\epsilon)$ is maximized, though the converse is only necessarily true if ϵ is vertical, or if *every* maximally distant edge is horizontal. In general, all maximally distant edges have the same orientation, i.e., horizontal or vertical. We also need one more definition. If a maximally distant edge ϵ maximizes the quantity $\Delta m_{D(\epsilon)}(\Gamma, \epsilon)$ among all maximally distant edges, call ϵ *optimal*. Note that if there exists a good optimal edge, then either every maximally distant edge is vertical and good, or else every maximally distant edge is horizontal and connects base vertices (and so is good).

Proposition 4.4. *Let ϵ be an optimal maximally distant edge. Then $P_1(\Gamma, \epsilon)$ is homotopy equivalent to $P_0(\Gamma, \epsilon)$.*

Proof. Let $P_{\frac{1}{2}} = P_{\frac{1}{2}}(\Gamma, \epsilon)$ be the subcomplex of P spanned by good admissible forests F for which $F \setminus \{\epsilon\}$ is again a (non-empty) good admissible forest. Clearly $P_0 \subseteq P_{\frac{1}{2}} \subseteq P_1$. Let $f : P_{\frac{1}{2}} \rightarrow P_{\frac{1}{2}}$ be given by $F \mapsto F \setminus \{\epsilon\}$. This is a well defined poset map that is the identity on its image P_0 , and so induces a homotopy equivalence between $P_{\frac{1}{2}}$ and P_0 ; see [Qui78, Section 1.3]. Now suppose that ϵ is non-distinguished, and we claim that $P_{\frac{1}{2}} = P_1$. Let $F \in P_1$ and let $i := D(F)$. We want to show that $F \setminus \{\epsilon\}$ is good. We may assume ϵ is (properly) contained in F ,

which since ϵ is maximally distant tells us that $D(F \setminus \{\epsilon\}) = i$. If ϵ' is the image of ϵ in $\Gamma/(F \setminus \{\epsilon\})$ then ϵ' is non-distinguished, and so cannot be base-decreasing. Hence

$$\Delta m_i(\Gamma, F) = \Delta m_i(\Gamma/(F \setminus \{\epsilon\}), \epsilon') + \Delta m_i(\Gamma, F \setminus \{\epsilon\}) \geq \Delta m_i(\Gamma, F \setminus \{\epsilon\}).$$

Clearly if F is arc-free then $F \setminus \{\epsilon\}$ is too. From this fact and the above equation, we conclude that if F is good then so is $F \setminus \{\epsilon\}$. We remark that so far we have not used the hypothesis that ϵ is optimal.

Now assume ϵ is distinguished, so we know $\Delta m_{D(\epsilon)}(F, \epsilon) \leq 0$. We have to do a bit more work in this case. Define a height function e on P_1 as follows. For $F \in P_1$, if $F \in P_{\frac{1}{2}}$ set $e(F) = 0$ and otherwise let $e(F)$ be the number of edges in F . Since adjacent vertices (forests) in $P_1 \setminus P_{\frac{1}{2}}$ have different e values, we can build up from $P_{\frac{1}{2}}$ to P_1 by gluing in vertices along their descending links. We claim these descending links are contractible, so by [Bux99, Lemma 4] the homotopy type does not change, and the result follows. The descending link of F is the join of two subcomplexes, which we will call the *out-link* and the *in-link*. The out-link is spanned by forests in $P_{\frac{1}{2}}$ containing F , and the in-link by forests in P_1 properly contained in F . It suffices to show that the in-link is contractible.

A forest F in P_1 but not in $P_{\frac{1}{2}}$ is characterized by F being good and $F \setminus \{\epsilon\}$ being bad. This is a relatively specific situation, and we will be able to restrict the possibilities quite a bit. First of all, $\epsilon \subseteq F$, and ϵ is maximally distant so $D(F \setminus \{\epsilon\}) = i := D(F)$. Consider again the equation $\Delta m_i(\Gamma, F) = \Delta m_i(\Gamma/(F \setminus \{\epsilon\}), \epsilon') + \Delta m_i(\Gamma, F \setminus \{\epsilon\})$, where ϵ' is the image of ϵ in $\Gamma/(F \setminus \{\epsilon\})$. Since F is good and $F \setminus \{\epsilon\}$ is bad, and since if F is arc-free then so is $F \setminus \{\epsilon\}$, it is clear that $\Delta m_i(\Gamma/(F \setminus \{\epsilon\}), \epsilon')$ cannot be 0 or 1. The only other option is that it equals -1 . This implies that ϵ' is horizontal, and so in particular F must be arced, with an arc containing ϵ . Since F is good it therefore must be base-decreasing, and so we conclude that $\Delta m_i(\Gamma, F) = -1$, $\Delta m_i(\Gamma/(F \setminus \{\epsilon\}), \epsilon') = -1$ and $\Delta m_i(\Gamma, F \setminus \{\epsilon\}) = 0$. Then since $F \setminus \{\epsilon\}$ is bad, it must be arced.

We are now in a position to analyze the in-link. Let C be the distinguished cycle containing ϵ . Since $\epsilon \subseteq F$ and F is admissible, we know $F \cap C$ is a forest. Let γ' be the connected edge path in $F \cap C$ containing ϵ . By the previous paragraph, we see that γ' must contain an arc at level $D(F)$ that in turn contains ϵ . Let γ be the shortest arc in γ' containing ϵ with $D(\gamma) = D(F)$. If $\gamma = \epsilon$ then $D(F) = D(\epsilon)$, and ϵ being an arc implies that it, and so every edge of F , is horizontal and connects base vertices. Hence $F \setminus \{\epsilon\}$ is base-decreasing, which we know is not the case. We can therefore assume γ properly contains ϵ . According to Remark 4.2, γ is base-decreasing, hence good, and it is easy to see that $\gamma \setminus \{\epsilon\}$ is arc-free and non-base-increasing, so also good. Since $F \setminus \{\epsilon\}$ is bad, this means γ does not equal F , so γ is really in the in-link. See Figure 4 for an idea of γ' and γ .

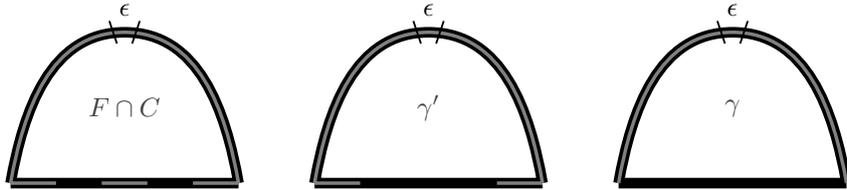


FIGURE 4. $F \cap C$, γ' and γ .

The idea now is to retract the in-link to the relative star of γ . We claim that for any F' in the in-link, $F' \cup \gamma$ is also in the in-link. It is clear that $F' \cup \gamma$ is

admissible, since it is contained in F . If $\gamma \subseteq F'$ there is nothing to show, so we can assume rather that the image of γ in Γ/F' is an arc, which necessarily connects base vertices and so is base-decreasing. Since F' is good we conclude that $F' \cup \gamma$ is base-decreasing, and so is also good.

It remains only to show that $F' \cup \gamma \neq F$. We claim that for any $\emptyset \neq \delta \subseteq \gamma$, $F \setminus \delta$ is bad. Since F' is good, this will imply that $F' \cup \gamma \neq F$. Note that if $\epsilon \notin \delta$ and $F \setminus \delta$ is good, the connected component of $(F \setminus \delta) \cap C$ containing ϵ is arc-free, so by the previous paragraphs $F \setminus \delta \in P_{\frac{1}{2}}$, i.e., $(F \setminus \delta) \setminus \{\epsilon\}$ is good. In particular if $F \setminus (\delta \cup \{\epsilon\})$ is bad then so is $F \setminus \delta$, so we can assume without loss of generality that $\epsilon \subseteq \delta$. Since $F \setminus \{\epsilon\}$ is arced we have $D(F \setminus \gamma) = i$, and so $D(F \setminus \delta) = i$. It is clear that $\Delta m_i(\Gamma, F \setminus \delta) \geq \Delta m_i(\Gamma, F \setminus \{\epsilon\}) = 0$, so to show $F \setminus \delta$ is bad, it suffices to assume it is arc-free and prove it is base-increasing. For $F \setminus \{\epsilon\}$ to be arced and $F \setminus \delta$ to be arc-free, there must exist an arc in $F \setminus \{\epsilon\}$ containing an edge of $\delta \setminus \{\epsilon\}$. In particular, the image of $\delta \setminus \{\epsilon\}$ in $\Gamma/(F \setminus \delta)$ is an arced forest consisting of distinguished edges, with an arc connecting base vertices. This must be base-decreasing, which tells us that $\Delta m_i(\Gamma, F \setminus \delta) > \Delta m_i(\Gamma, F \setminus \{\epsilon\})$, and we are done.

The claim now follows, and so $F' \cup \gamma$ is in the in-link. In particular the in-link is contractible by [Qui78, Section 1.5]. \square

In general if ϵ is any admissible good edge, then we have $P(\Gamma) = P_1(\Gamma, \epsilon) \cup \text{st}(\epsilon)$ and $P_1(\Gamma, \epsilon) \cap \text{st}(\epsilon) = \text{lk}(\epsilon)$, where star and link here are taken in $P(\Gamma)$. The previous results provide tools to analyze $P_1(\Gamma, \epsilon)$, and the next lemma tells us something about $\text{lk}(\epsilon)$.

Lemma 4.5. *Let ϵ be an optimal edge in Γ such that $\epsilon \in P(\Gamma)$, i.e., ϵ is good. Let F be an admissible forest properly containing ϵ . Then $F \in P(\Gamma)$ if and only if $F/\epsilon \in P(\Gamma/\epsilon)$. Also, $\text{lk}(\epsilon) \cong P(\Gamma/\epsilon)$.*

Proof. Let $i := D(F) = D(F/\epsilon)$. Since ϵ is good, $\Delta m_i(\Gamma, \epsilon) \in \{-1, 0\}$. First suppose that $\Delta m_i(\Gamma, \epsilon) = 0$, for example if $D(\epsilon) > i$. It is clear that F is arced if and only if F/ϵ is arced. Also, $\Delta m_i(\Gamma, F) = \Delta m_i(\Gamma/\epsilon, F/\epsilon) + \Delta m_i(\Gamma, \epsilon)$, so $\Delta m_i(\Gamma, F) = \Delta m_i(\Gamma/\epsilon, F/\epsilon)$. Hence, F is base-decreasing if and only if F/ϵ is, and F is base-preserving and arced if and only if F/ϵ is, which implies that $F \in P(\Gamma)$ if and only if $F/\epsilon \in P(\Gamma/\epsilon)$.

Next suppose $\Delta m_i(\Gamma, \epsilon) = -1$, so $D(\epsilon) = i$. We claim that in fact F and F/ϵ must both be base-decreasing, and hence good. We know that ϵ , and indeed every maximally distant edge, is horizontal and connects base vertices. In particular since $D(\epsilon) = i$, every edge of F must be maximally distant, and so connects base vertices. Since F has more than one edge, it is clear that $\Delta m_i(\Gamma, F) \leq -2$, so F is base-decreasing. Also,

$$\Delta m_i(\Gamma/\epsilon, F/\epsilon) = \Delta m_i(\Gamma, F) - \Delta m_i(\Gamma, \epsilon) \leq -2 + 1 = -1$$

so F/ϵ is base-decreasing.

Now consider the map $f : \text{lk}(\epsilon) \rightarrow P(\Gamma/\epsilon)$ sending F to F/ϵ . This is well-defined by the previous paragraphs, and is clearly injective. We claim that f is bijective. Let $\Phi \in P(\Gamma/\epsilon)$. There are two forests in Γ that map to Φ under blowing down ϵ , one that contains ϵ and one that does not. Let Φ' be the one that does, so $\Phi' \in \text{lk}(\epsilon)$ and $f(\Phi') = \Phi$. If Φ was admissible then Φ' is too. Also, if Φ was good then so is Φ' , again by the previous paragraphs. So f is an isomorphism. \square

Let V be the number of vertices in Γ and E_{ad} the number of admissible edges. The next two results are generalizations of Proposition 3.2 and Lemma 3.3 from [MZ]. Recall that $c = m - m_0$ is the number of distinguished cycles not at p .

Proposition 4.6. $P(\Gamma)$ is homotopy equivalent to a (possibly empty) wedge of spheres of dimension $V - c - 2$.

Proof. The proof is similar to the proof of Proposition 2.2 in [Vog90] and Proposition 3.2 in [MZ]. We induct on the number of admissible edges E_{ad} . Since non-distinguished loops do not affect $P(\Gamma)$, V or c , we may assume there are none. The base case is $E_{ad} = 0$, for which clearly $P(\Gamma)$ is empty, i.e., S^{-1} . When $m > 0$, if there are no admissible edges then $V = m$ and $c = m - 1$. If $m = 0$ and there are no admissible edges then $V = 1$ and $c = 0$. In both cases, $-1 = V - c - 2$, which finishes the base case.

Now assume $E_{ad} > 0$, so in particular there exists a maximally distant edge. Let ϵ be an optimal (maximally distant) edge. First suppose that ϵ is distinguished. By Lemma 4.3 and Proposition 4.4, $P_1(\Gamma, \epsilon)$ is contractible. If ϵ is bad then $P(\Gamma) = P_1(\Gamma, \epsilon)$ and we are done, so assume ϵ is good. Then $\text{lk}(\epsilon) \cong P(\Gamma/\epsilon)$ by Lemma 4.5, and admissible blow-downs necessarily decrease E_{ad} , so by induction $\text{lk}(\epsilon)$ is $(V - c - 3)$ -spherical. Since $P(\Gamma) = P_1(\Gamma, \epsilon) \cup \text{st}(\epsilon)$ and $P_1(\Gamma, \epsilon) \cap \text{st}(\epsilon) = \text{lk}(\epsilon)$, we conclude that $P(\Gamma)$ is $(V - c - 2)$ -spherical.

Next suppose that ϵ is not distinguished, and is not a separating edge. By the same argument as above, if ϵ is good then $\text{lk}(\epsilon)$ is $(V - c - 3)$ -spherical, so we just need to inspect $P_1(\Gamma, \epsilon)$, which by Proposition 4.4 is homotopy equivalent to $P_0(\Gamma, \epsilon)$. Since ϵ is not a separating edge, we can remove it from Γ and we still have a connected graph with m distinguished cycles and V vertices, and strictly fewer admissible edges. By induction then, $P(\Gamma \setminus \epsilon)$ is $(V - c - 2)$ -spherical (since c did not change either). Consider the map $g : P(\Gamma \setminus \epsilon) \rightarrow P_0(\Gamma, \epsilon)$ induced by $\Gamma \setminus \epsilon \hookrightarrow \Gamma$. Adding ϵ to the graph cannot affect whether a forest F in $\Gamma \setminus \epsilon$ is admissible or not. Also, since ϵ is maximally distant, ϵ cannot be decisive, so adding ϵ to the graph does not change the levels Λ_i . In particular adding ϵ cannot affect whether a forest F in $\Gamma \setminus \epsilon$ is good or bad, so g is an isomorphism. We conclude that $P_0(\Gamma, \epsilon)$ is $(V - c - 2)$ -spherical, and hence so is $P(\Gamma)$. Of course if ϵ is bad then $P(\Gamma) = P_1(\Gamma, \epsilon)$, and again we get the result.

Lastly suppose ϵ is not distinguished, but is an (admissible) separating edge. If ϵ is good then for any $F \in P(\Gamma)$ it is clear that $F \cup \epsilon$ is again an admissible good forest. In this case $P(\Gamma)$ is contractible by [Qui78, Section 1.5]. Incidentally, this completely finishes the $m = 0$ case. If ϵ is bad then its top must be a base vertex. Since ϵ is maximally distant, and Γ has no non-distinguished loops, ϵ is the stick of a lollipop ℓ . The graph $\Gamma \setminus \ell$ has $V - 1$ vertices and $c - 1$ distinguished cycles not at p , and has fewer admissible edges than Γ . By induction then, $P(\Gamma) = P_1(\Gamma, \epsilon) \simeq P_0(\Gamma, \epsilon) = P(\Gamma \setminus \ell)$ is $(V - 1) - (c - 1) - 2 = (V - c - 2)$ -spherical. \square

Lemma 4.7. If Γ has a non-base vertex with an admissible decisive edge then $P(\Gamma)$ is contractible.

Proof. The proof is essentially the same as the previous lemma. Induct on E_{ad} . In the base case, there are no admissible edges, much less admissible decisive edges, so the claim is vacuously true. Now assume $E_{ad} > 0$. Let ϵ be an optimal maximally distant edge, so $P_1(\Gamma, \epsilon)$ and $P_0(\Gamma, \epsilon)$ are homotopy equivalent. If ϵ is a separating edge, and good, then $P(\Gamma)$ is already contractible with cone point ϵ . If ϵ is a separating edge, and bad, then its top is a base vertex. The only way a maximally distant edge can be decisive is if it is separating, and so we can assume there is a decisive edge $\eta \neq \epsilon$ with top a non-base vertex.

First suppose that ϵ is distinguished. Then $P_1(\Gamma, \epsilon)$ is contractible, so if ϵ is bad we are done. If ϵ is good, we still have that $\text{lk}(\epsilon) \cong P(\Gamma/\epsilon)$ as in the previous proof. By Lemma 3.1, ϵ is either vertical, or is horizontal and connects base vertices. In either case, η maps to a decisive edge in Γ/ϵ , with a non-base vertex for a top,

and so $\text{lk}(\epsilon)$ is contractible by induction. Therefore $P(\Gamma)$ is contractible. Now suppose ϵ is not distinguished. Again, $\text{lk}(\epsilon)$ is contractible if ϵ is good, so we just need to inspect $P_0(\Gamma, \epsilon)$. If ϵ is not a separating edge we may remove it as in the previous proof and get that $P_0(\Gamma, \epsilon) \cong P(\Gamma \setminus \epsilon)$ is contractible by induction. The only case remaining is when ϵ is a separating edge whose top is a distinguished vertex, so it is the stick of a lollipop ℓ . Obviously η is still a decisive edge in $\Gamma \setminus \ell$, so $P(\Gamma) = P_0(\Gamma, \epsilon) = P(\Gamma \setminus \ell)$ is contractible by induction. \square

4.2. Connectivity of the descending up-link. Now consider the up-link. We return to only considering graphs coming from ΔK_n^m , so all vertices $v \neq p$ are at least trivalent and p is at least bivalent. Let $\text{BU}(v)$ be the poset of all blow-ups at the vertex v , and let $\text{BU}\downarrow(v)$ be the poset of descending blow-ups at v . We will use the combinatorial framework for graph blow-ups described in [CV86] and [Vog90], so we think of $\text{BU}(v)$ as the poset of *compatible partitions* of the set of incident half-edges. Let $[n] := \{1, \dots, n\}$, and consider partitions of $[n]$ into two blocks. Denote such a partition by $\alpha = \{a, \bar{a}\}$, where $1 \in a$. Distinct partitions $\{a, \bar{a}\}$ and $\{b, \bar{b}\}$ are called *compatible* if either $a \subset b$ or $b \subset a$. Let $\Sigma(v)$ be the simplicial complex of partitions $\alpha = \{a, \bar{a}\}$ of $[\text{val}(v)]$ into blocks a and \bar{a} such that a and \bar{a} each have at least two elements. (If v is the basepoint p , then one block may have size one, since p is allowed to be bivalent.) That is, the vertices of $\Sigma(v)$ are partitions, and a j -simplex is given by a collection of $j + 1$ distinct, pairwise compatible partitions. Also let $\Sigma\downarrow(v)$ be the subcomplex of $\Sigma(v)$ spanned by descending partitions, i.e., partitions that correspond to descending single-edge blow-ups.

For $v \neq p$, the geometric realization $|\text{BU}(v)|$ of $\text{BU}(v)$ is isomorphic to the barycentric subdivision of $\Sigma(v)$. The idea is that a partition describes an *ideal edge*, i.e., an edge blow-up at a vertex, and the blocks a and \bar{a} indicate which half-edges attach to which endpoints of the new edge. See [CV86] and [Vog90] for more details. It is also clear that the geometric realization $|\text{BU}\downarrow(v)|$ contains the barycentric subdivision of $\Sigma\downarrow(v)$ as a subcomplex, and that any simplex in $|\text{BU}\downarrow(v)|$ has at least one vertex in $\Sigma\downarrow(v)$. Hence there is a map $|\text{BU}\downarrow(v)| \rightarrow |\text{BU}\downarrow(v)|$ sending each simplex to its face spanned by vertices in $\Sigma\downarrow(v)$, which induces a deformation retraction from $|\text{BU}\downarrow(v)|$ to $\Sigma\downarrow(v)$.

The next lemma relates the up-link of Γ to these complexes $\Sigma\downarrow(v)$. The proof is very similar to the proof of [MZ, Proposition 4.5].

Lemma 4.8. *Let $\text{BU}\downarrow(\Gamma) := \ast_{v \in \Gamma} \text{BU}\downarrow(v)$. Then $|\text{BU}\downarrow(\Gamma)|$ is homotopy equivalent to the up-link of Γ .*

Proof. For a poset P , let \underline{P} be $P \sqcup \{\perp\}$, where \perp is a formal minimal element. Then we have that $P \ast Q \simeq \underline{P} \times \underline{Q} - \{(\perp, \perp)\}$. Let

$$U := \{f \in \prod_v \underline{\text{BU}}(v) - \{(\perp)_v\} \mid f \text{ is descending}\},$$

so the geometric realization $|U|$ is the up-link. Define a poset map $r : U \rightarrow U$ via

$$(f_v)_v \mapsto \left(\begin{cases} f_v & \text{for } f_v \in \text{BU}\downarrow(v) \\ \perp & \text{for } f_v \notin \text{BU}\downarrow(v) \end{cases} \right)_v$$

where f_v is a blow-up at v in the tuple f . This map is well defined since if f is descending then f_v must be descending for some v . It is easy to see that r is the identity when restricted to $\text{BU}\downarrow(\Gamma)$. Also, $r(f) \leq f$ for all $f \in U$, and so by [Qui78, 1.3] this induces a homotopy equivalence between $|U|$ and $|\text{BU}\downarrow(\Gamma)|$. \square

In particular the up-link is homotopy equivalent to $\ast_{v \in \Gamma} \Sigma\downarrow(v)$, so we can analyze the up-link by looking at the complexes $\Sigma\downarrow(v)$. In light of Lemma 4.7, one important situation is when v is a non-base vertex with no decisive edges.

Lemma 4.9. *Suppose v is a non-base vertex with no decisive edge. Then $\Sigma\downarrow(v) \simeq \bigvee S^{val(v)-4}$.*

Proof. We know that among the half-edges at v , at least two correspond to vertical edges with top v . Since v is a non-base vertex, a blow-up at v is descending if and only if it separates some of these half-edges with top v . (Here the essential term will be $n_{d(p,v)}$.) Thus $\Sigma\downarrow(v)$ is isomorphic to the complex denoted $SBU(v)$ in [MZ], and the result is immediate from Lemma 4.1 and Proposition 4.3 in [MZ]. \square

Next we describe one important case for which the up-link, and hence $\text{lk}\downarrow(\Gamma)$ is contractible. If a vertex $v \neq p$ has valency 3, or if $v = p$ and $val(v) = 2$, we say v has *minimal valency*. Otherwise we naturally say it has *non-minimal valency*.

Lemma 4.10. *If Γ has a base vertex with non-minimal valency, then the up-link is contractible, and so $\text{lk}\downarrow(\Gamma)$ is contractible.*

Proof. Let v be a base vertex with non-minimal valency. By Lemma 4.8 it suffices to show that $\Sigma\downarrow(v)$ is contractible. Label the distinguished half-edges at v by c_1 and c_2 , and label the non-distinguished half-edges by b_1, \dots, b_q . By hypothesis, $q > 1$, unless $v = p$ in which case $q > 0$. Let α_0 be the ideal edge at v that separates c_1, c_2 from all the other half-edges. See Figure 5 for an example. This is clearly a descending blow-up, with essential term $m_{d(p,v)}$. Also, any partition of $\{c_1, c_2, b_1, \dots, b_q\}$ that separates c_1 and c_2 is ascending, so indeed $\Sigma\downarrow(v)$ is contractible with cone point α_0 . \square

We remark that in this proof, blowing up α_0 does not change the weighted valency of v . In particular if $v = p$ then it does not change the weighted degree; this will be important in the next section.

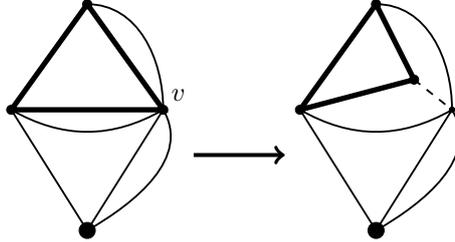


FIGURE 5. The blow-up at v given by α_0 . Here m_1 goes from 2 to 1.

We may now assume every base vertex has minimal valency, and so $\Sigma\downarrow(v)$ is empty for all base vertices v . Let V be the number of vertices of Γ , and recall that here $d_0 = d_0(\Gamma)$ is the degree of Γ , i.e., $d_0 = 2n + 2m - val(p)$.

Lemma 4.11. *Suppose Γ has no non-base vertices with an admissible decisive edge. Moreover suppose every base vertex has minimal valency. Then the up-link of Γ is homotopy equivalent to $\bigvee S^{d_0-V}$.*

Proof. By Lemma 4.8, the up-link is homotopy equivalent to $*_{v \in \Gamma} \Sigma\downarrow(v)$. It is clear that $\Sigma\downarrow(p) = \emptyset$, so this is the same as $*_{v \neq p} \Sigma\downarrow(v)$. Also, each base vertex $u \neq p$ has valency 3, so $\Sigma\downarrow(u) = \emptyset = S^{val(u)-4}$. Therefore by Lemma 4.9 the up-link is homotopy equivalent to

$$*_{v \neq p} (\bigvee S^{val(v)-4}),$$

which is a wedge of spheres of dimension $(V-2) + \sum_{v \neq p} (val(v)-4)$. Observe that

$$\sum_{v \neq p} (val(v)-2) = d_0, \text{ so this dimension equals } (V-2) + d_0 - 2(V-1) = d_0 - V. \quad \square$$

We can now prove our main result of this section. Here d_w is the weighted degree, which recall equals $d_0 - c$.

Corollary 4.12. *The descending link $\text{lk}\downarrow(\Gamma)$ is either contractible or a wedge of spheres of dimension $d_w - 1$.*

Proof. Assume that neither the up-link nor down-link is contractible. Then every base vertex has minimal valency, and no non-base vertex of Γ has a decisive edge. By Proposition 4.6, $P(\Gamma) \simeq \bigvee S^{V-c-2}$, and by Lemma 4.11 the up-link is homotopy equivalent to $\bigvee S^{d_0-V}$. Hence $\text{lk}\downarrow(\Gamma)$ is homotopy equivalent to

$$\left(\bigvee S^{V-c-2}\right) * \left(\bigvee S^{d_0-V}\right) \simeq \bigvee S^{V-c-2+d_0-V+1} = \bigvee S^{d_0-c-1} = \bigvee S^{d_w-1}.$$

□

4.3. Connectivity of sublevel sets. We can use Corollary 4.12 to argue that the sublevel sets $\Delta K_{n,k}^m$ are highly connected. To simplify the notation we make the following definitions. Let $X := \Delta K_n^m$ and $X_k := \Delta K_{n,k}^m$, and let X^0 be the subcomplex of X whose graphs Γ satisfy $m_0(\Gamma) = 0$, i.e., graphs for which p is not distinguished. The weighted degree of a graph can never exceed $2n + m - 1$, and m_0 is either 0 or 1, so if $N := 2n + m - 1$ and $k \geq N$ then $X_k = X$ is already contractible. For the remainder of the section we assume $k < N$.

Consider the following diagram, in which every map is inclusion.

$$\begin{array}{ccccc} X_k \cap X^0 & \longrightarrow & X_k & & \\ \downarrow & & \downarrow & & \\ X^0 & \longrightarrow & X_k \cup X^0 & \longrightarrow & X \end{array}$$

We claim that all the spaces involved (except X_k) are closed under taking descending links, that is, the descending link of a point in one of the spaces is completely contained in that space.

Lemma 4.13. *The spaces X^0 , $X_k \cap X^0$ and $X_k \cup X^0$ are closed under taking descending links.*

Proof. It is clear that X^0 is closed under taking descending links, since any vertex with a distinguished cycle at p is higher than any vertex with no distinguished cycle at p . Now let $x \in X_k \cap X^0$ and take y in $\text{lk}\downarrow(x)$, so y is obtained either by a forest blow-down or forest blow-up from x , and $h(y) < h(x)$. Suppose $y \notin X_k$, so $d_w(y) > k$. Since blow-downs cannot increase d_w , y is obtained from x by a blow-up involving blowing up at p . But $m_0(x) = 0$, so no blow-up at p is descending. We conclude that $y \in X_k \cap X^0$. Lastly let $x \in X_k \cup X^0$ and $y \in \text{lk}\downarrow(x)$. Suppose that $y \notin X_k$, so in particular either y has higher weighted degree than x , or $x \in X^0$. In the latter case $y \in X^0$ by the above argument. In the former case y must be obtained from x by a blow-up that involves blowing up at p . The only way this can be descending is if it reduces m_0 from 1 to 0, so in this case we also get $y \in X^0$. □

Proposition 4.14. *$X_k \cup X^0$, X^0 and $X_k \cap X^0$ are all $(k - 1)$ -connected.*

Proof. First, since $X_k \cup X^0$ is closed under taking descending links, we can build up from $X_k \cup X^0$ to X by attaching vertices in order of increasing height by gluing in their descending stars along their descending links. For vertices with the same height, the descending stars are disjoint so we may attach them in any order. Any vertex in X but not $X_k \cup X^0$ has weighted degree at least $k+1$, and so the descending link is $(k - 1)$ -connected by Corollary 4.12. Since X is contractible, this implies that $X_k \cup X^0$ is $(k - 1)$ -connected.

Next build up from X^0 to $X_k \cup X^0$ in the same way. Any vertex in $X_k \cup X^0$ but not X^0 has a distinguished cycle at the basepoint. Also, the weighted degree is less than or equal to k , which is less than N , so p has non-minimal valency. By Lemma 4.10 the descending link of such a vertex is contractible, so by [Bux99, Lemma 4], X^0 and $X_k \cup X^0$ are homotopy equivalent. In particular X^0 is $(k-1)$ -connected.

Lastly build up from $X_k \cap X^0$ to X^0 in the same way. As before, any vertex in X^0 but not in $X_k \cap X^0$ has weighted degree at least $k+1$, and so the descending link is $(k-1)$ -connected. We conclude that $X_k \cap X^0$ is $(k-1)$ -connected. \square

Corollary 4.15. *For each k , $\Delta K_{n,k}^m$ is $(k-1)$ -connected.*

Proof. Build up from $X_k \cap X^0$ to $X_k = \Delta K_{n,k}^m$ as in the previous proof. Since X_k is not closed under taking descending links, we have to attach each new vertex x along $\text{lk}\downarrow(x) \cap X_k$. Any vertex x in X_k but not in $X_k \cap X^0$ has a distinguished cycle at the basepoint, and since $k < N$ the basepoint has non-minimal valency. If α_0 is the cone point of $\text{lk}\downarrow(x)$ from the proof of Lemma 4.10, as remarked, blowing up α_0 does not change the weighted degree. In particular α_0 is still a cone point for $\text{lk}\downarrow(x) \cap X_k$. Hence X_k is homotopy equivalent to $X_k \cap X^0$ and so is $(k-1)$ -connected. \square

5. HOMOLOGICAL STABILITY

We now use the high connectivity of $\Delta K_{n,k}^m$ to obtain stability results for families of groups. As in Section 1, let G_n^m be any family of groups such that $P\Sigma\text{Aut}_n^m \leq G_n^m \leq \Sigma\text{Aut}_n^m$ for each n , and such that the inclusion $\Sigma\text{Aut}_n^m \hookrightarrow \Sigma\text{Aut}_{n+1}^m$ restricts to an inclusion $G_n^m \hookrightarrow G_{n+1}^m$. Consider the action of ΣAut_n^m on ΔK_n^m , and the restricted action of G_n^m on ΔK_n^m . For each k , $\Delta K_{n,k}^m$ is clearly stabilized by G_n^m , so it makes sense to consider the quotient $\Delta K_{n,k}^m / G_n^m$. Since this quotient is finite when $G_n^m = P\Sigma\text{Aut}_n^m$, it is finite for any G_n^m . Hence, as explained in [HV98a], since ΔK_n^m is contractible and each $\Delta K_{n,k}^m$ is $(k-1)$ -connected, $\Delta K_{n,k}^m / G_n^m$ has the same rational homology as G_n^m in dimensions less than k . To be precise, we have the following

Lemma 5.1. $H_i(\Delta K_{n,k}^m / G_n^m; \mathbb{Q}) \cong H_i(G_n^m; \mathbb{Q})$ for $i < k$, and $H_k(\Delta K_{n,k}^m / G_n^m; \mathbb{Q})$ surjects onto $H_k(G_n^m; \mathbb{Q})$. \square

To get homological stability for G_n^m we can now look for homological stability of $\Delta K_{n,k}^m / G_n^m$. Note that the vertices of $\Delta K_{n,k}^m / P\Sigma\text{Aut}_n^m$ are the homeomorphism types of basepointed graphs with m distinguished oriented cycles, disjoint and distinguishable from each other. In $\Delta K_{n,k}^m / \Sigma\text{Aut}_n^m$ the cycles become non-oriented and indistinguishable from each other, and in general $\Delta K_{n,k}^m / G_n^m$ interpolates between these two extremes. Embed $\Delta K_{n,k}^m$ into $\Delta K_{n+1,k}^m$ by sending (Γ, p, ρ) to $(\Gamma \vee S^1, p, \rho')$. Here ρ' is ρ extended to $\Gamma \vee S^1$ by sending the new generator to the new non-distinguished loop. This is the same embedding as described in [HV98a] for the K_n^0 case. This induces an embedding $\nu : \Delta K_{n,k}^m / G_n^m \hookrightarrow \Delta K_{n+1,k}^m / G_{n+1}^m$.

We also need an embedding $\mu : \Delta Q_{n,k}^m \hookrightarrow \Delta Q_{n,k}^{m+1}$, where $\Delta Q_{n,k}^m$ is the quotient $\Delta K_{n,k}^m / \Sigma\text{Aut}_n^m$. Start by embedding $\Delta K_{n,k}^m$ into $\Delta K_{n,k}^{m+1}$ by sending (Γ, p, ρ) to $(\Gamma \vee \ell, p, \rho')$, where ℓ is a lollipop and ρ' is now ρ extended to $\Gamma \vee \ell$ by sending the new generator to ℓ . Attaching a lollipop changes the degree, but does not change the weighted degree, so this is fine. This then induces an embedding $\mu : \Delta Q_{n,k}^m \hookrightarrow \Delta Q_{n,k}^{m+1}$. Colloquially, ν is the ‘‘attach a non-distinguished loop at the basepoint’’ map, and μ is the ‘‘attach a lollipop at the basepoint’’ map.

We now describe a way to ‘‘detect’’ the presence of certain subgraphs at the basepoint, as in [HV98a, Lemma 5.2]. If Γ has rank $n+m+1$ and has a non-distinguished loop at the basepoint p then (Γ, p, ρ) is in the image of ν for any ρ , so

we want to be able to detect loops. We similarly want to be able to detect lollipops, where we extend the definition of a lollipop to unmarked graphs with distinguished cycles. To get the precise stability bounds we want, we will also need to detect theta graph wedge summands at p , and a type of subgraph we will call a *double lollipop*. This is a graph obtained by taking a lollipop ℓ_1 based at p and wedging another lollipop ℓ_2 to ℓ_1 at any point. If the wedge point is p then this just makes two lollipops, but we allow any wedge point.

Lemma 5.2. *Let (Γ, p) be a basepointed graph with rank $n + m$, weighted degree d_w , and m pairwise disjoint distinguished cycles. The following hold:*

- (1) *If $n > 2d_w + m$ then Γ has a non-distinguished loop at the basepoint.*
- (2) *If $n > (3d_w + m)/2$ then Γ has either a non-distinguished loop at the basepoint or a theta graph wedge summand with no distinguished edges.*
- (3) *If $m > 2d_w + 1$ then Γ has a lollipop at the basepoint.*
- (4) *If $m > 3d_w/2 + 1$ then Γ has a lollipop or a double lollipop at the basepoint.*

Proof. First suppose $n > 2d_w + m$, so $\text{val}_w(p) = 2n + m - d_w > 3(n + m)/2$. If $m > 0$ then $\text{val}_w(p) > 1$, so there is at least one non-distinguished half-edge at p . If there is a distinguished cycle at p we can blow up an edge at p that pushes the cycle away from p and leaves all the other half-edges. This does not change n , d_w , or m , so without loss of generality there are no distinguished cycles at p , i.e., $c = m$ and $d_0 = d_w + m$. Since $n > 2d_w + m$, this tells us that $n > 2d_0 - m$, so $n + m > 2d_0$. By [HV98a, Lemma 5.2] there is a loop at p , and since $c = m$ this loop is not distinguished. Now suppose $n > (3d_w + m)/2$ and Γ has no non-distinguished loops at p . Then $\text{val}_w(p) = 2n + m - d_w > 4(n + m)/3$. If $m > 0$ then $\text{val}_w(p) > 1$, so as before without loss of generality there are no distinguished cycles at p , and $d_0 = d_w + m$. In particular there are no loops at p , and $n + m > 3d_0/2$, so by [HV98a, Lemma 5.2] there is a theta graph wedge summand at p , which obviously has no distinguished edges. As a remark, we cannot do better than detecting theta graph wedge summands, since higher-degree wedge summands may well have distinguished cycles away from p . Hence we do not have a general rule like in [HV98b, Lemma 5.4].

Next suppose that there are no lollipops at p , and we want to show that $m \leq 2d_w + 1$. We will induct on n . If $n = 0$ then every non-distinguished edge in Γ is a separating edge. Let Γ' be the graph obtained by blowing down every non-distinguished edge. Now Γ' is a *cactus graph* as in [Col89], i.e., every edge is contained in a unique reduced cycle. Note that Γ' is no longer in ΔK_0^m , since the distinguished cycles are not disjoint, but Γ' has the same weighted degree d_w as Γ . Let b' be the number of cycles in Γ' at p and $c' = m - b'$ the number of cycles not at p . Since Γ had no lollipops at p , Γ' has at most one loop at p . This tells us that $b' - 1 \leq c'$, and since $m = b' + c'$ we see that $m \leq 2c' + 1$. It is also clear that in Γ' , $c' = m - \text{val}(p)/2 = d_w$, so indeed $m \leq 2d_w + 1$. This finishes the base case, and we also note that if additionally Γ has no double lollipops then $(b' - 1) \leq c'/2$, so $m \leq 3c'/2 + 1 = 3d_w/2 + 1$.

Now assume $n > 0$. Then there exists a non-distinguished edge ϵ that is not a separating edge. Let Γ_1 be the graph obtained from Γ by removing ϵ , and then if any bivalent vertices $v \neq p$ arise, blowing down one of the edges containing v . Then Γ_1 is a connected graph with non-distinguished rank $n - 1$ and m distinguished cycles. Let $a \in \{0, 1, 2\}$ be such that the weighted degree $d_w(\Gamma_1)$ of Γ_1 is $d_w - a$. In particular $a = 0$ if and only if ϵ is a loop at p , and $a = 1$ if and only if p is an endpoint of ϵ and ϵ is not a loop. The graph Γ_1 has at most two lollipops at the basepoint, say there are b of them, so $b \in \{0, 1, 2\}$. Let Γ_2 be the graph obtained by removing all lollipops at p in Γ_1 . Then the weighted degree $d_w(\Gamma_2)$ of Γ_2 is the same as Γ_1 , the non-distinguished rank is $n - 1$, and there are $m - b$ distinguished

cycles. By induction, $m - b \leq 2(d_w - a) + 1$, so $m \leq 2d_w + 1 - (2a - b)$. It now suffices to show that $2a \geq b$. Clearly if $a = 0$ then $b = 0$, so suppose $a > 0$. Then the only case to check is when $b = 2$. But then p cannot be an endpoint of ϵ , so $a = 2$ and the result follows. We remark that the stronger statement $a \geq b$ even holds.

Lastly suppose that Γ has no lollipops or double lollipops at p . Let $b \in \{0, 1, 2\}$ be the number of lollipops in Γ_1 and $c \in \{0, 1, 2\}$ the number of double lollipops in Γ_1 , so $b + c \in \{0, 1, 2\}$. Let Γ_3 be the graph obtained by removing all lollipops and double lollipops at p in Γ_1 . Let $a \in \{0, 1, 2, 3, 4\}$ be such that Γ_3 has weighted degree $d_w - a$. Again, $a = 0$ if and only if ϵ is a loop at p . Also, if ϵ is not a loop but p is an endpoint of ϵ then $a = 1 + c$, and otherwise $a = 2 + c$. See Figure 6 for some examples. By the induction hypothesis $m - (b + 2c) \leq 3(d_w - a)/2 + 1$, so $m \leq 3d_w/2 + 1 - (3a/2 - (b + 2c))$. It now suffices to show that $3a \geq 2b + 4c$. If $a = 0$ then $b = c = 0$, so suppose $a > 0$. If p is an endpoint of ϵ then $b + c \leq 1$ and $a = 1 + c$, so $2b + 4c \leq 2 + 2c = 2a < 3a$. Now suppose p is not an endpoint of ϵ , so $b + c \leq 2$ and $c = a - 2$. Then $2b + 4c \leq 4 + 2c = 2a < 3a$ and we are done. Again, we find that a stronger statement holds, namely $a \geq b + 2c$. \square

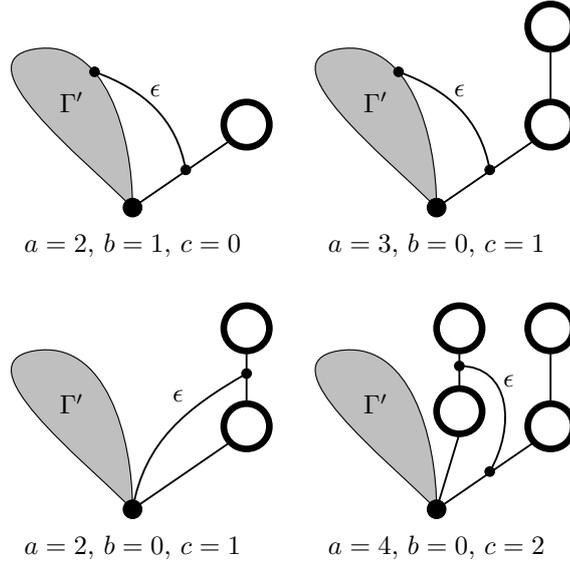


FIGURE 6.

Remark 5.3. In the last two paragraphs of the proof, it is interesting that the induction would have run even with sharper bounds. In fact, whatever the best possible bound is for the $n = 0$ case automatically extends to all cases, as long as the slope is not less than 1. In particular, we can detect “triple lollipops,” “quadruple lollipops,” as so forth, with increasingly better bounds. Ultimately, we find that whenever $m > d_w + 1$, there is always some non-trivial wedge summand that is an iterated wedge of lollipops. However, since we currently do not have a way to make use of this fact to get better bounds for homological stability, we will content ourselves with just detecting lollipops and double lollipops.

Proposition 5.4. *The map $\nu : \Delta K_{n,k}^m / G_n^m \hookrightarrow \Delta K_{n+1,k}^m / G_{n+1}^m$ is a homeomorphism for $2k + m < n + 1$ and a homotopy equivalence for $(3k + m)/2 < n + 1$.*

Proof. The proof is very similar to the proof of Proposition 5.4 in [HV98a]. If $2k + m < n + 1$ then every Γ in $\Delta K_{n+1,k}^m / G_{n+1}^m$ has a non-distinguished loop at p , so ν is a homeomorphism. Now suppose $(3k + m)/2 < n + 1$, and let Γ be a vertex not in the image of ν . Then Γ has no non-distinguished loops at p but does have at least one theta graph wedge summand with no distinguished edges. Let Θ be the subgraph of Γ consisting of all such theta graphs at p , say there are $r \geq 1$ of them. Then $\Gamma = \Theta \vee \Gamma'$, for some Γ' with rank $n + m + 1 - 2r$. Now, the open star of Γ in $\Delta K_{n+1,k}^m / G_{n+1}^m$ is the product of open stars of Θ in $\Delta K_{2r,r}^0 / G_{2r}^0$ and Γ' in $\Delta K_{n+1-2r,k-r}^m / G_{n+1-2r}^m$. The former consists of a single simplex, since all non-loop edges in Θ are equivalent under automorphisms of Θ ; moreover, every other vertex of this star has lower weighted degree since blowing down any edge reduces d_w by 1. So, collapsing any non-loop edge of Θ gives a deformation retraction of the star of Γ into the image of ν . \square

As remarked in the proof of Lemma 5.2, we cannot necessarily detect non-distinguished wedge summands of (weighted) degree 2 or 3, and so we cannot immediately improve the bound to $(5k + m)/4 < n + 1$, as was done for the $m = 0$ case in [HV98b]. It seems likely that we could nonetheless improve this bound by directly inspecting examples with low weighted degree, in the spirit of [HV98b], but we leave this for future work.

Proposition 5.5. *The map $\mu : \Delta Q_{n,k}^m \hookrightarrow \Delta Q_{n,k}^{m+1}$ is a homeomorphism for $2k < m$ and a homotopy equivalence for $3k/2 < m$.*

Proof. If $2k < m$ then every Γ in $\Delta Q_{n,k}^{m+1}$ has a lollipop at p , so μ is a homeomorphism. Now suppose $3k/2 < m$, and let Γ be a vertex not in the image of μ . Then Γ has no lollipops at p but does have at least one double lollipop. Let $\Lambda\Lambda$ be the subgraph of Γ consisting of all double lollipops at p , say there are $r \geq 1$ of them. Then $\Gamma = \Lambda\Lambda \vee \Gamma'$, for some Γ' with rank $n + m + 1 - 2r$. The open star of Γ in $\Delta Q_{n,k}^{m+1}$ is the product of open stars of $\Lambda\Lambda$ in $\Delta Q_{0,r}^{2r}$ and Γ' in $\Delta Q_{n-2r,k-r}^{m+1}$. We claim that there is a retraction of the former that yields a retraction of the star of Γ into the image of μ , similar to the previous proof. Consider again the height function h from Section 3, thought of on $\Delta K_{0,r}^{2r}$, and note that since h only depends on ρ inasmuch as ρ determines which cycles are distinguished, h descends to a function \bar{h} on $\Delta Q_{0,r}^{2r}$. Since $\Delta Q_{0,r}^{2r}$ is not simplicial we think of \bar{h} as a height function in the sense of [Bux99]. It now suffices to show that the descending link $\overline{\text{lk}}_{\downarrow}(\Gamma)$ is contractible.

There are three homeomorphism types of double lollipops, depending on where the second lollipop is wedged to the first. If it is wedged to a point in the interior of the stem, call this Type 1. If it is wedged to a point on the distinguished cycle not in the stem, call this Type 2. If it is wedged to the top of the stem call this Type 3. See Figure 7. If $\Lambda\Lambda$ has a double lollipop of Type 1 then blowing down the edge connecting the wedge point to p is descending (with essential term d_0). Moreover, every simplex in $\overline{\text{lk}}_{\downarrow}(\Gamma)$ is compatible with this move since descending blow-ups cannot affect double lollipops of Type 1, so it is a cone point of $\overline{\text{lk}}_{\downarrow}(\Gamma)$. Next, if $\Lambda\Lambda$ has a double lollipop of Type 2, then blowing down either edge connecting the wedge point to the top of the stem is descending (with essential term d_0). These edges differ by a homeomorphism of Γ , so they actually correspond to the same blow-down. Again, every simplex in $\overline{\text{lk}}_{\downarrow}(\Gamma)$ is compatible with this move since descending blow-ups cannot affect double lollipops of Type 2, so it is a cone point of $\overline{\text{lk}}_{\downarrow}(\Gamma)$. Finally suppose $\Lambda\Lambda$ has a double lollipop of Type 3. Consider the blow-up that pushes the base of the first cycle away from the wedge point, creating a double lollipop of Type 1. This is descending, with essential term m_1 , and since descending (admissible) blow-downs cannot affect double lollipops of Type 3, it is a

cone point for $\overline{\text{lk}\downarrow}(\Gamma)$. We conclude that attaching Γ does not change the homotopy type, by [Bux99, Lemma 4], so the result follows. \square

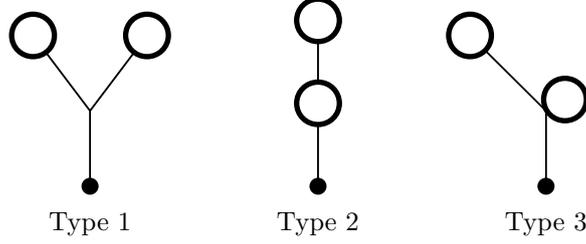


FIGURE 7. Types of double lollipops.

There is evidence to suggest that the descending links $\overline{\text{lk}\downarrow}(\Gamma)$ are always contractible whenever there is a non-trivial wedge summand that is an iterated wedge of lollipops. As indicated by Remark 5.3, this would imply that μ is a homotopy equivalence whenever $k < m$. From this we would also recover the fact that ΣAut_0^m has trivial rational homology. For now though, we will content ourselves with the double lollipop situation.

Since ν is natural with respect to $G_n^m \hookrightarrow G_{n+1}^m$ and μ is natural with respect to $\Sigma\text{Aut}_n^m \hookrightarrow \Sigma\text{Aut}_n^{m+1}$, we can now prove our main results.

Proof of Theorem 1.1. We know that when $(3k + m)/2 < n + 1$,

$$H_i(G_n^m; \mathbb{Q}) \rightarrow H_i(G_{n+1}^m; \mathbb{Q})$$

is an isomorphism for all $i < k$, by Lemma 5.1 and Proposition 5.4. If $n \geq (3(i + 1) + m)/2$ then we can use $k = i + 1$ and the result follows. \square

Proof of Theorem 1.2. We know that when $3k/2 < m$,

$$H_i(\Sigma\text{Aut}_n^m; \mathbb{Q}) \rightarrow H_i(\Sigma\text{Aut}_n^{m+1}; \mathbb{Q})$$

is an isomorphism for all $i < k$, by Lemma 5.1 and Proposition 5.5. If $m > 3(i + 1)/2$ then we can use $k = i + 1$ and the result follows. \square

When $m = 0$, so $G_n^0 = \text{Aut}(F_n)$, from the first theorem we recover the stability bound for $\text{Aut}(F_n)$ given in [HV98a], though not the improved one given in [HV98b]. When $n = 0$ the bound in the second theorem is slightly worse than the bound $m > 3(i - 1)/2$ found in [Zar]. The reason for this can be traced to the fact that weighted degree cannot “tell the difference” between a lollipop at p and a distinguished loop at p . The space of *cactus graphs* considered in [Zar] is thus slightly more natural in the $n = 0$ case, though does not generalize well to the $n > 0$ case for various reasons. However, in the $n = 0$ case the rational homology is known to always be trivial anyway.

Remark 5.6. We conclude with some questions that now naturally arise. First, the stable rational homology of ΣAut_n^0 in n is trivial, and the rational homology of ΣAut_0^m is trivial in every dimension, so it seems likely that the stable homology in m and n is always trivial; is this indeed the case? Some additional evidence for this is Theorem 7.4 in [JW04], which implies that $H_1(P\Sigma\text{Aut}_n^m; \mathbb{Q}) = 0$ for any $n > 2$ and any $m \geq 0$. Second, there exist examples where $H_i(\Sigma\text{Aut}_n^0; \mathbb{Q}) = \mathbb{Q}$, but when can non-trivial rational homology occur in general, e.g., if $m > 0$? This is an interesting question for outer automorphisms as well. Third, when $n = 0$ or $m = 0$, we have stable integral homology, so an obvious question is whether this holds in general.

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