Free Infinitely Divisible Approximations of n-fold Free Convolutions.

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Abstract Based on the method of subordinating functions we prove a free analog of error bounds in classical Probability Theory for the approximation of n-fold convolutions of probability measures by infinitely divisible distributions.

1 Introduction

In recent years a number of papers are investigating limit theorems for the free convolution of probability measures defined by D. Voiculescu. The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for noncommutative random variables. As in the classical probability where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures on the real line, the free convolution. Classical results for the convolution of probability measures have their counterpart in this new theory, such as the law of large numbers, the central limit theorem, the Lévy-Khinchine formula and others. We refer to Voiculescu, Dykema and Nica [26] and Hiai and Petz [17] for introduction to these topics. Bercovici and Pata [10] established the distributional behavior of sums of free identically distributed random variables and described explicitly the correspondence between limits laws for free and classical additive convolution. Chistyakov and Götze [14] generalized the results of Bercovici and Pata to the case of free non-identically distributed random variables. They showed that the parallelism found by Bercovici and Pata holds in the general case of free non-identically distributed random variables. Using the method of subordination functions they proved the semi-circle ap-

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proximation theorem (an analog of the Berry-Esseen inequality). See Kargin's paper [18] as well.

In the classical probability Doeblin [15] showed that it is possible to construct independent identically distributed random variables $X_1, X_2, \dots$ such that the distribution of the centered and normalized sum $b_{n_k}^{-1}(X_1 + \dots + X_{n_k} - a_{n_k})$ does not converge to any nondegenerate distribution, whatever the choice of the constants a_n and b_n and of the sequence $n_1 < n_2 < \dots$. Kolmogorov [19] initiated the study of approximations of sequences $\{\mu^{n*}\}_{n=1}^{\infty}$ of convolutions of some distribution μ by elements of the class of infinitely divisible distributions in some metric as $n \to \infty$. Prokhorov [23] and Kolmogorov [20] studied this problem which subsequently led to seminal results by Arak and Zaitsev in their monograph [3] on this problem.

Due to the Bercovici–Pata parallelism between limits laws for free and classical additive convolution results like those of Doeblin should hold for free random variables as well, which we discuss in Section 2. Thus Kolmogorov's approach would be natural in free Probability Theory as well but has not been done yet and we would like to start research in this direction. In particular in this paper we study the problem of approximating n-fold additive free convolutions of probability measures by additive free infinitely divisible probability measures.

The paper is organized as follows. In Section 2 we formulate and discuss the main results of the paper. In Section 3 we formulate auxiliary results. Section 4 contains a upper bound in the approximation problem.

2 Results

Denote by $\mathscr{M}$ the family of all Borel probability measures defined on the real line $\mathbb{R}$. On $\mathscr{M}$ define the associative composition laws denoted $*$ and $\boxplus$ as follows. For $\mu_1, \mu_2 \in \mathscr{M}$ let a probability measure $\mu_1 * \mu_2$ denote the classical convolution of μ_1 and μ_2. In probabilistic terms, $\mu_1 * \mu_2$ is the probability distribution of $X + Y$, where X and Y are (commuting) independent random variables with distributions μ_1 and μ_2, respectively. A measure $\mu_1 \boxplus \mu_2$ on the other hand denotes the free (additive) convolution of μ_1 and μ_2 introduced by Voiculescu [25] for compactly supported measures. Free convolution was extended by Maassen [21] to measures with finite variance and by Bercovici and Voiculescu [9] to the class $\mathscr{M}$. Thus, $\mu_1 \boxplus \mu_2$ is the probability distribution of $X + Y$, where X and Y are free random variables with distributions μ_1 and μ_2, respectively.

Let $\rho(\mu, \nu)$ be the Kolmogorov distance between probability measures μ and ν, i.e.,

$$\rho(\mu, \nu) = \sup_{x \in \mathbb{R}} |\mu((-\infty, x)) - \nu((-\infty, x))|.$$

In 1955 Prokhorov [23] proved that

$$\rho(\mu^{n*}, \mathbf{D}^*) := \inf_{\nu \in \mathbf{D}^*} \rho(\mu^{n*}, \nu) \to 0, \quad n \to \infty, \tag{1}$$

for any $\mu \in \mathscr{M}$, where μ^{n*} denotes the n-fold convolution of the probability measure μ and $\mathbf{D}^*$ denotes the set of infinitely divisible probability measures (with respect to classical convolution). Kolmogorov [20] noted that the convergence in (1) is uniform with respect to μ throughout the class $\mathscr{M}$. Work by a number of researchers (a detailed history of the problem may be found in [3]) eventually proved upper and lower bounds for the function $\psi(n) := \sup_{\mu \in \mathscr{M}} \rho(\mu^{n*}, \mathbf{D}^*)$. A final answer was given by Arak [4], [5], who proved the following bound:

$$c_1 n^{-2/3} \leq \psi(n) \leq c_2 n^{-2/3},$$

where c_1 and c_2 are absolute positive constants. Chistyakov [12] returned to Prohorov's result [23] and studied the problem of determining the possible rate of convergence of $\rho(\mu^{n*}, \mathbf{D}^*)$ to zero as $n \to \infty$ for probability measures $\mu \notin \mathbf{D}^*$.

Define the distance in variation between two signed measures μ and ν by

$$\rho_{var}(\mu, \nu) := \sup_{S \in \mathscr{B}} |\mu(S) - \nu(S)|,$$

where $\mathscr{B}$ denotes the σ-algebra of Borel subsets of $\mathbb{R}$. If μ, ν are probability measures, then $\rho_{var}(\mu, \nu) := var(\mu - \nu)/2$. It is natural to consider the analogous problems for the distance in variation. Prokhorov's paper [23] states that the quantity

$$\rho_{var}(\mu^{n*}, \mathbf{D}^*) := \inf_{\nu \in \mathbf{D}^*} \rho_{var}(\mu^{n*}, \nu) \tag{2}$$

tends to zero as $n \to \infty$ if $\mu \in \mathscr{M}$ is discrete or it has a nondegenerate absolutely continuous component.

Zaitsev [28] proved that there exist probability measures μ whose set of n-fold convolutions is uniformly separated from the set of infinitely divisible measures in the sense of the variation distance.

Let $\mu \in \mathscr{M}$, denote $\mu^{n\boxplus} := \mu \boxplus \cdots \boxplus \mu$ (n times). Recall that $\nu \in \mathscr{M}$ is $\boxplus$-infinitely divisible if, for every $n \in \mathbb{N}$, there exists $\nu_n \in \mathscr{M}$ such that $\nu = \nu_n^{n\boxplus}$. In the sequel we will write in this case that $\nu \in \mathbf{D}^{\boxplus}$.

Fix now $\mu, \nu \in \mathscr{M}$. We will say that μ belongs to the partial $*$-domain of attraction (resp., partial $\boxplus$-domain of attraction) of ν if there exist measures $\mu_1, \mu_2, \ldots$ equivalent to μ, and natural numbers $k_1 < k_2 < \ldots$ such that

$$\mu_n * \mu_n * \cdots * \mu_n, \quad (k_n \text{ times}) \quad \Big(\text{resp.,} \quad \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n, \quad (k_n \text{ times})\Big)$$

converges weakly to ν as $n \to \infty$. Recall that μ_j and μ are equivalent if there exist real numbers a, b with $a > 0$, such that $\mu_j(S) = \mu(aS + b)$ for every $S \in \mathscr{B}$. Denote by $\mathscr{P}_*(\nu)$ (resp., $\mathscr{P}_{\boxplus}(\nu)$) the partial $*$-domain of attraction (resp., partial $\boxplus$-domain of attraction) of ν. Khinchine proved the following result for the classical convolution (for free convolution it was proved by Pata [22]).

A measure $\nu \in \mathscr{M}$ is $*$-infinitely divisible (resp., $\boxplus$-infinitely divisible) if and only if $\mathscr{P}_*(\nu)$ (resp., $\mathscr{P}_{\boxplus}(\nu)$) is not empty.

The next result is due to Bercovici and Pata [10] and is known as the Bercovici-Pata bijection.

There exists a bijection $\nu \leftrightarrow \nu'$ between $*$-infinitely divisible measures ν and $\boxplus$-infinitely divisible measures ν' such that $\mathscr{P}_*(\nu) = \mathscr{P}_{\boxplus}(\nu')$. More precisely, let $\mu_n \in \mathscr{M}$, let $k_1 < k_2 < \dots$ be positive integers, and set

$$\nu_n = \mu_n * \mu_n * \cdots * \mu_n \quad (k_n \text{ times}), \qquad \nu'_n = \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n \quad (k_n \text{ times}).$$

Then ν_n converges weakly to ν if and only if ν'_n converges weakly to ν'.

We return to Doeblin's result [15]. Using this result and the two last results about $\mathscr{P}_*(\nu)$ and $\mathscr{P}_{\boxplus}(\nu)$ we see that there exist free identically distributed random variables $X_1, X_2, \dots$ such that the distribution of the centered and normalized sum $b_{n_k}^{-1}(X_1 + \cdots + X_{n_k} - a_{n_k})$ does not converge weakly to any nondegenerate distribution, whatever the choice of the constants a_n and b_n and of the sequence $n_1 < n_2 < \dots$.

Introduce the quantity

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) := \inf_{\nu \in \mathbf{D}^{\boxplus}} \rho_{var}(\mu^{n\boxplus}, \nu)$$

and raise the question of the behavior of this quantity when $n \to \infty$.

In the sequel we denote by $c(\mu), c_1(\mu)$ positive constants depending on μ only, while $c(\mu)$ is used to denote either generic constants for cases where we are not interested in particular values.

In order to formulate our main result we introduce the following notation

$$c_1(\mu) := \Im\Big(1 \Big/ \int\limits_{\mathbb{R}} \frac{\mu(dt)}{i-t}\Big) - 1.$$

It is easy to see that $c_1(\mu) > 0$ if and only if $\mu \neq \delta_b$ with $b \in \mathbb{R}$, where δ_b denotes the Dirac measure concentrated at the point b.

Theorem 1. *Let $\mu \in \mathscr{M}$ and $c_1(\mu) > 0$. Then*

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) \le c(\mu)\Big(\frac{1}{\sqrt{n}} \int\limits_{[-N_n/8, N_n/8]} |u|\,\mu(du) + \mu(\mathbb{R} \setminus [-N_n/8, N_n/8])\Big), \quad n \in \mathbb{N}, \tag{3}$$

where $N_n := \sqrt{c_1(\mu)(n-1)}$.

It was proved in [6] that $\mu^{n\boxplus}$ is Lebesgue absolutely continuous when n is sufficiently large, provided that $\mu \neq \delta_b$ for any b. Therefore we immediately obtain from Theorem 1 a free analog of Prokhorov's result (2).

Corollary 1. *For $\mu \in \mathscr{M}$,*

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) \to 0, \quad \text{as} \quad n \to \infty.$$

In addition this corollary shows that in contrast to the classical case the approximation of n-fold free additive convolutions by free infinitely divisible probability measures can be shown in variation distance for all $\mu \in \mathscr{M}$.

Denote by $\mathscr{M}_d$, $d \geq 0$, the set of probability measures such that

$$\beta_d(\mu) := \int_{\mathbb{R}} |x|^d \, \mu(dx) < \infty.$$

We easily obtain from Theorem 1 the following upper bound.

Corollary 2. *Let $\mu \in \mathscr{M}_d$ with some $d > 0$. Then*

$$\rho_{var}(\mu^{n\boxplus}, \mathbf{D}^{\boxplus}) \leq c(\mu, d) n^{-\min\{d/2, 1/2\}}, \quad n \in \mathbb{N}, \tag{4}$$

where $c(\mu, d)$ is a constant depending on μ and d only.

From this corollary it follows that for all $\mu \in \mathscr{M}_1$ the order of approximation of $\mu^{n\boxplus}$ by free infinitely divisible measures is of order $n^{-1/2}$ in variation distance.

In the classical case there exists results with the rate of approximation in Kolmogorov's metric depending on the existence of moments. See, for example, a paper of Zaitsev [29].

Proof. Let $d_0 := \min\{1, d\}$ and $c_1(\mu) > 0$. We have

$$\int_{[-N_n/8, N_n/8]} |u| \, \mu(du) \leq \Big(\frac{N_n}{8}\Big)^{1-d_0} \int_{[-N_n/8, N_n/8]} |u|^{d_0} \, \mu(du) \leq \beta_{d_0}(\mu) \Big(\frac{c_1(\mu) n}{64}\Big)^{\frac{1-d_0}{2}}. \tag{5}$$

In addition

$$\mu(\mathbb{R} \setminus [-N_n/8, N_n/8]) \leq \Big(\frac{8}{N_n}\Big)^{d_0} \int_{|u| > N_n/8} |u|^{d_0} \, \mu(du) \leq \beta_{d_0}(\mu) \Big(\frac{64}{c_1(\mu)(n-1)}\Big)^{\frac{d_0}{2}}. \tag{6}$$

Now we see that (4) follows immediately from (3) and (5), (6). □

3 Auxiliary results

We shall need some results about some classes of analytic functions (see [1], Section 3, and [2], Section 6, §59).

Let $\mathbb{C}^+$ denote the open upper half of the complex plane. The class $\mathscr{N}$ (Nevanlinna, R.) denotes the class of analytic functions $f(z) : \mathbb{C}^+ \to \{z : \Im z \geq 0\}$. For such functions there is an integral representation

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1 + uz}{u - z} \, \tau(du) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{u - z} - \frac{u}{1 + u^2}\Big)(1 + u^2) \, \tau(du) \tag{7}$$

for $z \in \mathbb{C}^+$, where $b \geq 0$, $a \in \mathbb{R}$, and τ is a nonnegative finite measure. Moreover, $a = \Re f(i)$ and $\tau(\mathbb{R}) = \Im f(i) - b$. ¿From this formula it follows that

$$f(z) = (b + o(1))z \tag{8}$$

for $z \in \mathbb{C}^+$ such that $|\Re z|/\Im z$ stays bounded as $|z|$ tends to infinity (in other words $z \to \infty$ non-tangentially to $\mathbb{R}$). Hence if $b \neq 0$, then f has a right inverse $f^{(-1)}$ defined on the region

$$\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : |\Re z| < \alpha \Im z, \Im z > \beta\}$$

for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$.

A function $f \in \mathcal{N}$ admits the representation

$$f(z) = \int\limits_{\mathbb{R}} \frac{\sigma(du)}{u - z}, \quad z \in \mathbb{C}^+, \tag{9}$$

where σ is a finite nonnegative measure, if and only if $\sup_{y \geq 1} |y f(iy)| < \infty$ and $\sigma(\mathbb{R}) = \lim_{y \to \infty} y \Im f(iy)$.

For $\mu \in \mathcal{M}$, define its Cauchy transform by

$$G_\mu(z) = \int\limits_{-\infty}^{\infty} \frac{\mu(dt)}{z - t}, \qquad z \in \mathbf{C}^+. \tag{10}$$

The measure μ can be recovered from $G_\mu(z)$ as the weak limit of the measures

$$\mu_y(dx) = -\frac{1}{\pi} \Im G_\mu(x + iy)\,dx, \quad x \in \mathbb{R},\ y > 0,$$

as $y \downarrow 0$. If the function $\Im G_\mu(z)$ is continuous at $x \in \mathbb{R}$, then the probability distribution function $D_\mu(t) = \mu((-\infty, t))$ is differentiable at x and its derivative is given by

$$D'_\mu(x) = -\Im G_\mu(x)/\pi. \tag{11}$$

This inversion formula allows to extract the density function of the measure μ from its Cauchy transform.

Following Maassen [21] and Bercovici and Voiculescu [9], we shall consider in the following the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{12}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ will be denoted by $\mathcal{F}$. This class coincides with the subclass of Nevanlinna functions f for which $f(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Indeed, reciprocal Cauchy transforms of probability measures have obviously such property. Let $f \in \mathcal{N}$ and $f(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Then, by (8), f admits the representation (7) with $b = 1$. By (8) and (9), $-1/f(z)$ admits the representation (9) with $\sigma \in \mathscr{M}$.

The function $\phi_\mu(z) = F_\mu^{(-1)}(z) - z$ is called the Voiculescu transform of μ. It is not difficult to show that $\phi_\mu(z)$ is an analytic function on $\Gamma_{\alpha,\beta}$ and $\Im \phi_\mu(z) \le 0$ for $z \in \Gamma_{\alpha,\beta}$, where ϕ_μ is defined. Furthermore, note that $\phi_\mu(z) = o(z)$ as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$.

Voiculescu [27] showed that for compactly supported probability measures there exist unique functions $Z_1, Z_2 \in \mathscr{F}$ such that $G_{\mu_1 \boxplus \mu_2}(z) = G_{\mu_1}(Z_1(z)) = G_{\mu_2}(Z_2(z))$ for all $z \in \mathbb{C}^+$. Maassen [21] proved the similar result for probability measures with finite variance. Using Speicher's combinatorial approach [24] to freeness, Biane [11] proved this result in the general case.

Chistyakov and Götze [13], Bercovici and Belinschi [7], Belinschi [8], proved, using methods from complex analysis, that there exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathscr{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{13}$$

The function $F_{\mu_1}(Z_1(z))$ belongs again to the class $\mathscr{F}$ and there exists a probability measure μ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ is the Cauchy transform as in (10).

Specializing to $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$ write $\mu_1 \boxplus \ldots \mu_n = \mu^{n\boxplus}$. The relation (13) admits the following consequence (see for example [13]).

Proposition 1. *Let $\mu \in \mathscr{M}$. There exists a unique function $Z_n(z) \in \mathscr{F}$ such that*

$$z = nZ_n(z) - (n-1)F_\mu(Z_n(z)), \quad z \in \mathbb{C}^+, \tag{14}$$

and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z_n(z))$.

Using the last proposition we now state and prove some auxiliary results about the behavior of the function $Z_n(z)$.

From (14) we obtain the formula

$$Z_n^{(-1)}(z) = nz - (n-1)F_\mu(z) \tag{15}$$

for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$. This equation provides an analytic continuation of the function $Z_n^{(-1)}(z)$ defined on $\mathbb{C}^+$. By (7), we have the following representation for the function $F_\mu(z)$

$$F_\mu(z) = c + z + \int_{\mathbb{R}} \frac{1+uz}{u-z}\, \tau(du), \quad z \in \mathbb{C}^+, \tag{16}$$

where $c \in \mathbb{R}$, and τ is a nonnegative finite measure. Moreover, $c = \Re F_\mu(i)$ and $\tau(\mathbb{R}) = \Im F_\mu(i) - 1 = \Im(1/G_\mu(i)) - 1 = c_1(\mu)$.

Bercovici and Voiculescu [9]) proved the following result.

Proposition 2. *A probability measure* μ *is* $\boxplus$*-infinitely divisible if and only if the function* $\phi_\mu(z)$ *has an analytic continuation defined on* $\mathbb{C}^+$*, with values in* $\mathbb{C}^- \cup \mathbb{R}$*, such that*

$$\lim_{y\to+\infty} \frac{\phi_\mu(iy)}{y} = 0. \tag{17}$$

It follows from Proposition 2 and (15), (16) that a probability measure ν_n such that $F_{\nu_n}(z) = Z_n(z), z \in \mathbb{C}^+$, is $\boxplus$-infinitely divisible.

The next lemma was proved in [13].

Lemma 1. *Let* $g : \mathbb{C}^+ \to \mathbb{C}^-$ *be analytic with*

$$\liminf_{y\to+\infty} \frac{|g(iy)|}{y} = 0. \tag{18}$$

Then the function $f : \mathbb{C}^+ \to \mathbb{C}$ *defined via* $z \mapsto z + g(z)$ *takes every value in* $\mathbb{C}^+$ *precisely once. The inverse* $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ *thus defined is in the class* $\mathscr{F}$*.*

This lemma generalizes a result of Maassen [21] (see Lemma 2.3). Maassen proved Lemma 1 under the additional restriction $|g(z)| \le c(g)/\Im z$ for $z \in \mathbb{C}^+$, where $c(g)$ is a constant depending on g.

Denote $z = x + iy$, where $x, y \in \mathbb{R}$. Using the representation (16) for $F_\mu(z)$ we see that, for $\Im z > 0$,

$$\Im\Big(nz - (n-1)F_\mu(z)\Big) = y\Big(1 - (n-1)I_\mu(x,y)\Big),$$

where

$$I_\mu(x,y) := \int\limits_{\mathbb{R}} \frac{(1+u^2)\,\tau(du)}{(u-x)^2 + y^2}.$$

For every real fixed x, consider the equation

$$y\Big(1 - (n-1)I_\mu(x,y)\Big) = 0, \quad y > 0. \tag{19}$$

Since in the case $\tau(\mathbb{R}) \ne 0$ $y \mapsto I_\mu(x,y)$, $y > 0$, is positive and monotone, and decreases to 0 as $y \to \infty$, it is clear that the equation (19) has at most one positive solution. If such a solution exists, denote it by $y_n(x)$. Note that (19) does not have a solution $y > 0$ for any given $x \in \mathbb{R}$ if and only if $I_\mu(x,0) \le 1/(n-1)$. Consider the set $S := \{x \in \mathbb{R} : I_\mu(x,0) \le 1/(n-1)\}$. We put $y_n(x) = 0$ for $x \in S$. By Fatou's lemma, $I_\mu(x_0,0) \le \liminf_{x\to x_0} I_\mu(x,0)$ for any given $x_0 \in \mathbb{R}$, hence the set S is closed. Therefore $\mathbb{R} \setminus S$ is the union of finitely or countably many intervals (x_k, x_{k+1}), $x_k < x_{k+1}$. The function $y_n(x)$ is continuous on the interval (x_k, x_{k+1}). Since the set $\{z \in \mathbb{C}^+ : n\Im z - (n-1)\Im F_\mu(z) > 0\}$ is open, we see that $y_n(x) \to 0$ if $x \downarrow x_k$ and $x \uparrow x_{k+1}$. Hence the curve γ_n given by the equation $z = x + iy_n(x)$, $x \in \mathbb{R}$, is a Jordan curve. In the case $\tau(\mathbb{R}) = 0$ we put $y_n(x) := 0$ for all $x \in \mathbb{R}$.

Consider the open domain $D_n := \{z = x + iy, x, y \in \mathbb{R} : y > y_n(x)\}$.

Lemma 2. *Let* $Z_n(z)$ *be the solution of the equation* (14). *The map* $Z_n(z) : \mathbb{C}^+ \mapsto D_n$ *is univalent. Moreover the function* $Z_n(z), z \in \mathbb{C}^+$, *is continuous up to the real axis and it maps the real axis bicontinuously onto the curve* γ_n.

Proof. Using the formula (15) for $z \in \Gamma_{\alpha,\beta}$ with some $\alpha, \beta > 0$, we see that the function $Z_n^{(-1)}(z)$ has an analytic continuation defined on $\mathbb{C}^+$. In view of the representation (16) for the function $F_\mu(z)$, we note that $Z_n^{(-1)}(z) = z + g(z)$, $z \in \mathbb{C}^+$, where $g(z)$ is analytic on $\mathbb{C}^+$ and satisfies the assumptions of Lemma 1. By Lemma 1, we conclude that the function $Z_n^{(-1)}(z)$ takes every value in $\mathbb{C}^+$ precisely once. Moreover, as it is easy to see, $Z_n^{(-1)}(D_n) = \mathbb{C}^+$. The inverse $Z_n(z)$ gives us a conformal mapping of $\mathbb{C}^+$ onto D_n. By well-known results of the theory of analytic functions (see [16]), $Z_n(z)$ is continuous up to the real axis and it maps the real axis bicontinuously onto the curve γ_n. □

Lemma 3. *Let* $c_1(\mu) > 0$ *and let* $Z_n(z)$ *be the solution of the equation* (14). *Then the following lower bound holds*

$$|Z_n(z)| \geq \frac{1}{4}\sqrt{c_1(\mu)(n-1)}, \quad z \in \mathbb{C}^+, \quad n \geq c(\mu). \tag{20}$$

Proof. We shall prove that, for real x such that $|x| \leq \frac{1}{4}N_n = \frac{1}{4}\sqrt{c_1(\mu)(n-1)}$, the lower bound $y_n(x) > \frac{1}{2}N_n$ holds. Indeed, for $|x| \leq \frac{1}{4}N_n$ and $|u| \leq \frac{1}{4}N_n$, the inequality $(u-x)^2 + y_n^2(x) \leq \frac{1}{4}c_1(\mu)(n-1) + y_n^2(x) = \frac{1}{4}\tau(\mathbb{R})(n-1) + y_n^2(x)$ is valid. Therefore, using (19), we deduce the following chain of inequalities

$$\frac{1}{n-1}\int\limits_{[-N_n/4,N_n/4]} \frac{\tau(du)}{\frac{1}{4}\tau(\mathbb{R}) + \frac{1}{n-1}y_n^2(x)} \leq \int\limits_{[-N_n/4,N_n/4]} \frac{\tau(du)}{(u-x)^2 + y_n^2(x)}$$
$$\leq \int\limits_{\mathbb{R}} \frac{(1+u^2)\,\tau(du)}{(u-x)^2 + y_n^2(x)} \leq \frac{1}{n-1}. \tag{21}$$

Assume that there exists an $x_0 \in [\mathbb{N}_n/4, N_n/4]$ such that $0 \leq y_n(x_0) \leq N_n/2$. Then it follows from (21) that

$$\frac{\tau([-N_n/4, N_n/4])}{\frac{1}{4}\tau(\mathbb{R}) + \frac{1}{4}\tau(\mathbb{R})} \leq 1. \tag{22}$$

Since, for all sufficiently large $n \geq c(\mu)$, the lower bound $\tau([-N_n/4, N_n/4]) \geq \frac{3}{4}\tau(\mathbb{R})$ holds, we arrive at contradiction.

Finally note that the assertion of the lemma follows from Lemma 2. □

4 A Upper Bound in the Approximation Theorem

Proof of Theorem 1. By Proposition 1 there exists a unique function $Z_n(z) \in \mathscr{F}$ such that (14) holds and $G_{\mu^{n\boxplus}}(z) = G_\mu(Z_n(z))$, $z \in \mathbb{C}^+$. We have shown in Section 3 that the function Z_n satisfies $1/Z_n(z) = G_{\nu_n}(z)$, $z \in \mathbb{C}^+$, where ν_n is an $\boxplus$-infinitely divisible probability measure. Our aim is to estimate $\rho_{var}(\mu^{n\boxplus}, \nu_n)$ for all $n \in \mathbb{N}$. Without loss of generality we assume that n is sufficiently large, i.e., $n \ge c(\mu)$. For any $z \in \mathbb{C}^+$, we may represent $G_{\mu^{n\boxplus}}(z)$ as

$$G_{\mu^{n\boxplus}}(z) = I_{n1}(z) + I_{n2}(z) := \Big(\int\limits_{[-N_n/8, N_n/8]} + \int\limits_{\mathbb{R}\setminus[-N_n/8, N_n/8]} \Big) \frac{\mu(du)}{Z_n(z) - u}. \qquad (23)$$

Since $Z_n(z) \in \mathscr{F}$, by (8), we have $Z_n(iy) = (1 + o(1))iy$ as $y \to \infty$. Therefore

$$-y\Im \frac{1}{Z_n(iy) - u} = y \frac{\Im Z_n(iy)}{|Z_n(iy) - u|^2} = 1 + o(1)$$

as $y \to \infty$ for all fixed $u \in \mathbb{R}$, and, by the inequality $\Im Z_n(iy) \ge y$, $y > 0$,

$$-y\Im \frac{1}{Z_n(iy) - u} \le \frac{y}{\Im Z_n(iy)} \le 1, \quad u \in \mathbb{R}, \quad y > 0.$$

By Lebesgue's theorem , we easily deduce the relations

$$\lim_{y\to\infty} (-y\Im I_{n1}(iy)) = \mu([-N_n/8, N_n/8])$$

and

$$\lim_{y\to\infty} (-y\Im I_{n2}(iy)) = \mu(\mathbb{R} \setminus [-N_n/8, N_n/8]).$$

Therefore, by (9),

$$I_{nj}(z) = \int\limits_{\mathbb{R}} \frac{\sigma_{nj}(dt)}{z - t}, \quad z \in \mathbb{C}^+, \quad j = 1, 2,$$

where σ_{nj}, $j = 1, 2$, denote nonnegative measures such that $\sigma_{n1}(\mathbb{R}) = \mu([-N_n/8, N_n/8])$ and $\sigma_{n2}(\mathbb{R}) = \mu(\mathbb{R} \setminus [-N_n/8, N_n/8])$.

By Lemma 2, the map $Z_n(z) : \mathbb{C}^+ \mapsto D_n$ is univalent. Moreover the function $Z_n(z)$ is continuous on $\mathbb{C}^+ \cup \mathbb{R}$ and it maps $\mathbb{R}$ bicontinuously onto the curve γ_n. The function $F_\mu(z)$ admits the representation (16), where, by the assumption of the theorem, $\tau(\mathbb{R}) = c_1(\mu) > 0$.

By (20), we have

$$|Z_n(x + i\varepsilon) - u| \ge |Z_n(x + i\varepsilon)| - |u| \ge \frac{1}{8} N_n \qquad (24)$$

for $x \in \mathbb{R}$, $\varepsilon \in (0, 1]$ and $u \in [-N_n/8, N_n/8]$. Therefore, by Lemma 2 and (24), $\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im(1/(u - Z_n(x + i\varepsilon))$ exists for every $x \in \mathbb{R}$, $u \in [-N_n/8, N_n/8]$, and this limit is a continuous probability density for every fixed $u \in [-N_n/8, N_n/8]$. By Lebesgue's theorem, the measure σ_{n1} is absolutely continuous and its density $p_1(x)$ has the form

$$p_1(x) = \frac{1}{\pi} \int\limits_{[-N_n/8, N_n/8]} \lim_{\varepsilon \downarrow 0} \Im \frac{1}{u - Z_n(x+i\varepsilon)} \, \mu(du).$$

The probability measure ν_n is absolutely continuous as well with a density

$$p_2(x) = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im \frac{1}{Z_n(x+i\varepsilon)}.$$

Since

$$\Im \Big(\frac{1}{Z_n(x+i\varepsilon) - u} - \frac{1}{Z_n(x+i\varepsilon)} \Big) = u \frac{2\Re Z_n(x+i\varepsilon) - u}{|Z_n(x+i\varepsilon) - u|^2} \Im \frac{1}{Z_n(x+i\varepsilon)},$$

we obtain, using (24),

$$\Big| \Im \Big(\frac{1}{Z_n(x+i\varepsilon) - u} - \frac{1}{Z_n(x+i\varepsilon)} \Big) \Big| \le -c(\mu) \frac{|u|}{\sqrt{n}} \Im \frac{1}{Z_n(x+i\varepsilon)}$$

for all $x \in \mathbb{R}$, $u \in [-N_n/8, N_n/8]$ and $\varepsilon \in (0,1]$. From this bound we conclude that

$$\begin{aligned}
&\int\limits_{\mathbb{R}} |p_1(x) - \mu([-N_n/8, N_n/8]) p_2(x)| \, dx \\
&\quad = \frac{1}{\pi} \int\limits_{\mathbb{R}} \Big| \int\limits_{[-N_n/8, N_n/8]} \lim_{\varepsilon \downarrow 0} \Im \Big(\frac{1}{u - Z_n(x+i\varepsilon)} + \frac{1}{Z_n(x+i\varepsilon)} \Big) \mu(du) \Big| \, dx \\
&\quad \le \frac{1}{\pi} \int\limits_{\mathbb{R}} \int\limits_{[-N_n/8, N_n/8]} \limsup_{\varepsilon \downarrow 0} \Big| \Im \Big(\frac{1}{u - Z_n(x+i\varepsilon)} + \frac{1}{Z_n(x+i\varepsilon)} \Big) \Big| \mu(du) \, dx \\
&\quad \le \frac{c(\mu)}{\sqrt{n}} \int\limits_{-N_n/8}^{N_n/8} |u| \, \mu(du) \int\limits_{\mathbb{R}} \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Big| \Im \frac{1}{Z_n(x+i\varepsilon)} \Big| \, dx \le \frac{c(\mu)}{\sqrt{n}} \int\limits_{-N_n/8}^{N_n/8} |u| \, \mu(du).
\end{aligned} \tag{25}$$

In view of the relation

$$\begin{aligned}
var(\mu^{n\boxplus} - \nu_n) &= var(\sigma_{n1} + \sigma_{n2} - \nu_n) \\
&\le var(\sigma_{n1} - \mu([-N_n/8, N_n/8]) \nu_n) + var\, \sigma_{n2} \\
&\qquad + \mu(\mathbb{R} \setminus [-N_n/8, N_n/8]) \, var\, \nu_n,
\end{aligned}$$

we have

$$\begin{aligned}\rho_{var}(\mu^{n\boxplus},\nu_n) &\le \frac{1}{2}\int\limits_{\mathbb{R}} |p_1(x)-\mu([-N_n/8,N_n/8])\,p_2(x)|\,dx \\ &\qquad\qquad + \frac{1}{2}\sigma_{n2}(\mathbb{R}) + \frac{1}{2}\mu(\mathbb{R}\setminus[-N_n/8,N_n/8]) \\ &\le \frac{1}{2}\int\limits_{\mathbb{R}} |p_1(x)-\mu([-N_n/8,N_n/8])\,p_2(x)|\,dx + \mu(\mathbb{R}\setminus[-N_n/8,N_n/8]).\end{aligned} \tag{26}$$

Note that the statement of the theorem now follows immediately from (25) and (26). □

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