# THE ARITHMETIC OF DISTRIBUTIONS IN FREE PROBABILITY THEORY

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Abstract. We give an analytical approach to the definition of additive and multiplicative free convolutions which is based on the theory of Nevanlinna and of Schur functions. We consider the set of probability distributions as a semigroup $\mathbf{M}$ equipped with the operation of free convolution and prove a Khintchine type theorem for the factorization of elements of this semigroup. An element of $\mathbf{M}$ contains either indecomposable ("prime") factors or it belongs to a class, say $I_0$, of distributions without indecomposable factors. In contrast to the classical convolution semigroup in the free additive and multiplicative convolution semigroups the class $I_0$ consists of units (i.e. Dirac measures) only. Furthermore we show that the set of indecomposable elements is dense in $\mathbf{M}$.

## 1. Introduction

In recent years a larger number of papers has been devoted to applications and extensions of the definition of free convolution of measures introduced by D. Voiculescu. The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for non-commutative random variables. As in the classical probability where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures on the real line, called free convolution. As one might expect there are many classical results for sums of independent random variables having a counterpart in this theory, such as the law of large numbers, the central limit theorem, the Lévy-Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [48] (1992) and to Hiai and Petz [28] (2000) for an introduction to these topics. One of the main problems in dealing with free convolution is that its definition is rather indirect. In the first part of this paper we propose an analytical approach to the definition of additive and

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multiplicative free convolution. This approach which develops an analytic method due to Maassen [36] is based on the classical theory of Nevanlinna and Schur functions and allows us to give a direct definition of free convolutions (see Theorem 2.1, Theorem 2.4, Theorem 2.7) by purely analytic methods. This approach shows the equivalence of the “characteristic function” approach and the probabilistic approach for additive and multiplicative free convolutions. Note that Bercovici and Belinschi [9] gave a related approach using other analytic methods.

In the second part of the paper we study the arithmetic structure of the Voiculescu semigroups $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$, and $(\mathcal{M}_*, \boxtimes)$ of probability measures on $\mathbb{R}$ with additive free convolution ($\boxplus$), on $\mathbb{R}_+$ and on $\mathbb{T}$ with multiplicative free convolution ($\boxtimes$), respectively. This subject had its origin in the work of Khintchine on the convolution semigroup $(\mathcal{M}, *)$ of probability measures on the real line. He derived for this semigroup the three basic theorems listed in Section 2. Kendall [29] (1967), [30] (1968) introduced the so called Delphic semigroups, which are commutative topological semigroups satisfying the central limit theorem for triangular arrays. Their arithmetic is similar to the convolution arithmetic of probability measures on $\mathbb{R}$. A characteristic feature of all these semigroups is the presence of *infinitely divisible* (i.d.) elements, which for every positive integer $n$ may be represented as $n$-th power of some element of the semigroup.

In any Delphic semigroup there are three classes of its elements:

- the indecomposable or *simple* elements, which have no factors besides themselves and the identity, (a set we shall denote by “$S$”);
- the elements which are decomposable and have an indecomposable factor, (a set we shall denote by “$D$”);
- the infinitely divisible elements which have no indecomposable factors, (a set we shall denote by “$I_0$”).

It turns out that a lot of important semigroups, in particular $(\mathcal{M}, *)$, are Delphic or almost Delphic.

It is convenient to formulate the Delphic hypothesis rather restrictively, and then to show that semigroups like $(\mathcal{M}, *)$ are 'almost' Delphic, that means they satisfy these hypothesis with nonessential modifications. In this context Davidson [24]–[26] (1968), (1969) introduced the concept of an hereditary sub-semigroup to verify that semigroups are almost (or properly) Delphic. Using a multivariate analytic description of free convolutions we show that the Voiculescu semigroups $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$, and $(\mathcal{M}_*, \boxtimes)$ are close by the structure to almost Delphic semigroups (but are not almost Delphic semigroups). Using some ideas of Khintchine, Kendall and Davidson, we deduce the three basic Delphic theorems for these semigroups. One of them states that each element of a Delphic semigroup may be written as a product of a countable number of indecomposable elements and an element of $I_0$, so a knowledge of $I_0$ and the set of indecomposable elements is essential for the arithmetic of the semigroup. As a consequence we show that in the Voiculescu semigroups the class $I_0$ consists of

Dirac measures $\delta_a$ only and for each semigroup there is a dense set of indecomposable elements.

As another consequence of this approach we obtain an analogue of Khintchine's limit theorem in Voiculescu's semigroups.

The paper is organized as follows. In Section 2 we discuss the results of the paper. In Section 3 we collect auxiliary results from complex analysis, free probability. In Sections 4 and 5 we prove the necessary analytical results for our approach to free convolutions. In Section 6 we prove basic Delphic theorems for the Voiculescu semigroups, in Section 7 we describe the class $I_0$, and in Section 8 we describe some dense classes of indecomposable elements in these semigroups.

Note that the contents of this paper is close to the contents of an earlier version [21], (2005) with some changes. We changed the text of Introduction, added the assertion (3) in Proposition 3.13, added some details in the proof of Lemma 4.2. We corrected the text of Section 6. Here we omitted some propositions about the reduction to the theory of Delphic semigroups which were not correct. In additional arguments in Section 6, still based on the ideas of [21] (2005), we now only use the hereditary property and the well-known methods of Delphic semigroups for the proof of Theorems 2.10–2.12. Note that independently a corrected proof of part of these results has been published by J. Williams [52]. In Section 8 we give some new free indecomposability conditions for probability measures using the arguments of [21] (2005).

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## 2. Results

Denote by $\mathcal{M}$ the family of all Borel probability measures (p-measures for short) defined on the real line $\mathbb{R}$. On the set $\mathcal{M}$ there are defined two associative composition laws denoted $*$ and $\boxplus$. Let $\mu_1, \mu_2 \in \mathcal{M}$. The measure $\mu_1 * \mu_2$ is the classical convolution of $\mu_1$ and $\mu_2$. In probabilistic terms, $\mu_1 * \mu_2$ is the probability distribution of $X + Y$, where $X$ and $Y$ are (commuting) independent random variables with distributions $\mu_1$ and $\mu_2$ respectively. The measure $\mu_1 \boxplus \mu_2$ denotes the free (additive) convolution of $\mu_1$ and $\mu_2$ introduced by Voiculescu [46] (1986) for compactly supported measures. The notion of free convolution was extended by Maassen [36] (1992) to measures with finite variance and by Bercovici and Voiculescu [12] (1993) to all measures in $\mathcal{M}$. Here, $\mu_1 \boxplus \mu_2$ may be considered as the probability distribution of $X+Y$, where $X$ and $Y$ are free random variables with distributions $\mu_1$ and $\mu_2$, respectively. For positive random variables and for random variables with values on $\mathbb{T}$ we consider multiplicative convolutions as well and their free analogues of multiplicative convolutions which were introduced by Voiculescu [47] (1987).

In this section we give an analytical approach to the definition of $\mu_1 \boxplus \mu_2$ which extends Maassen's definition. Furthermore, we shall present an analytical approach to the definition of multiplicative free convolution $\boxtimes$.

Let $\mathbb{C}^+$ ($\mathbb{C}^-$) denote the open upper (lower) half of the complex plane. If $\mu \in \mathcal{M}$, then its Cauchy transform is

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}, \qquad z \in \mathbb{C}^+. \tag{2.1}$$

Following Maassen [36] and Bercovici and Voiculescu [12], in the sequel we will consider the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{2.2}$$

Let $\mathcal{F}$ denote the corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$. This class admits the following simple description.

The class $\mathcal{F} \subset \mathcal{N}$ of reciprocal Cauchy transforms of p-measures introduced above coincides with the subclass of Nevanlinna functions $F \in \mathcal{N}$ such that $F(z)/z \to 1$ as $z \to \infty$ nontangentially to $\mathbb{R}$ (i.e., such that $|\operatorname{Re} z|/\operatorname{Im} z$ stays bounded). See Section 3. Recall that the class of all analytic functions $F : \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$, say $\mathcal{N}$, is called *Nevanlinna class*.

This implies that $F_\mu$ has certain invertibility properties. To be precise, for two numbers $\alpha > 0, \beta > 0$ we set

$$\Gamma_\alpha = \{z = x + iy \in \mathbb{C}^+ : |x| < \alpha y\} \quad \text{and} \quad \Gamma_{\alpha,\beta} = \{z = x + iy \in \Gamma_\alpha : y > \beta\}.$$

Then, by the relation (3.4) of Section 3, for every $\alpha > 0$ there exists $\beta = \beta(\mu, \alpha)$ such that $F_\mu$ has a right inverse $F_\mu^{(-1)}$ defined on $\Gamma_{\alpha,\beta}$. The function

$$\varphi_\mu(z) = F_\mu^{(-1)}(z) - z$$

is called the Voiculescu transform of $\mu$. It is not hard to show that $\operatorname{Im} \varphi_\mu(z) \leqslant 0$ for $z \in \Gamma_{\alpha,\beta}$ where $\varphi_\mu$ is defined. We also have $\varphi_\mu(z) = o(z)$ as $|z| \to \infty$, $z \in \Gamma_\alpha$.

If $\mu$ is the point measure $\delta_a$ at $a$, then $F(z) = z - a$ whereas $F(z) = z + ib$, $b \in \mathbb{R}$, corresponds to the Cauchy distribution with density $x \mapsto b/(\pi(x^2 + b^2))$ which has infinite variance.

## Additive free convolution

Let $\mu_1$ and $\mu_2$ be p-measures in $\mathcal{M}$ and let $F_{\mu_1}(z)$ and $F_{\mu_2}(z)$ denote their reciprocal Cauchy transforms respectively. We shall define the free convolution $\mu_1 \boxplus \mu_2$, based on $F_{\mu_1}(z)$ and $F_{\mu_2}(z)$ using the following result.

**Theorem 2.1.** *There exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,*

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \textit{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{2.3}$$

The function $F_{\mu_1}(Z_1(z))$ is in $\mathcal{F}$ again, hence there exists some p-measure $\mu$ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ denotes the Cauchy transform (2.1) of $\mu$.

Since the p-measure $\mu$ depends on $\mu_1$ and $\mu_2$ only, we define $\mu_1 \boxplus \mu_2 := \mu$.

Thus, we defined the free additive convolution by purely complex analytic methods (see Chistyakov and Götze (2005) [20], [21] for a previous version of this paper). A related approach has been suggested later by Belinschi (2006) in [4] and Belinschi and Bercovici (2007) in [9] .

The symmetry of the relation (2.3) obviously implies that this operation is commutative. Furthermore, choosing $\mu_2 = \delta_a$ in (2.3), where $\delta_a$ denotes a Dirac measure concentrated at the point $a$, we get $\mu_1 \boxplus \delta_a = \mu_1 * \delta_a$. The definition (2.3) does not restrict the class of p-measures and allows an obvious extension to the case of multiplicative convolutions, described below.

Moreover, on any set $\Gamma_{\alpha,\beta}$, where the functions $\varphi_{\mu_1}(z)$, $\varphi_{\mu_2}(z)$ and $\varphi_{\mu_1\boxplus\mu_2}(z)$ are defined, we obtain immediately from (2.3) that

$$\varphi_{\mu_1\boxplus\mu_2}(z) = \varphi_{\mu_1}(z) + \varphi_{\mu_2}(z). \tag{2.4}$$

The relation (2.4) implies at once that the operation $\boxplus$ is associative.

The equation (2.4) for the distribution $\mu_1 \boxplus \mu_2$ of $X + Y$, where $X$ and $Y$ are free random variables is due to Voiculescu [46]. He considered p-measures $\mu$ with compact support. The result was extended by Maassen [36] to p-measures with finite variance; the general case was proved by Bercovici and Voiculescu [12]. Note here that Voiculescu's and Bercovici's approach to the definition of $\mu_1 \boxplus \mu_2$ based on the operator algebras. Maassen's analytic approach to the definition is closer to the one presented here.

We see from (2.4) that our definition of $\mu_1 \boxplus \mu_2$ coincides with the Voiculescu, Bercovici, Maassen definition.

Since one can investigate free convolutions of several *different* p-measures $\mu_1, \dots,$ $\mu_n$ using the multivariate approach (2.3) with the help of Nevanlinna functions which are defined on the whole half-plane $\mathbb{C}^+$ (see Corollary 2.2 below), limit laws can be sucessfully described as in classical probability theory (see [22], [23]).

Voiculescu [49] showed for compactly supported p-measures that there exist unique functions $F_1, F_2 \in \mathcal{F}$ such that $G_{\mu_1\boxplus\mu_2}(z) = G_{\mu_1}(F_1(z)) = G_{\mu_2}(F_2(z))$ for all $z \in \mathbb{C}^+$. Using Speicher's combinatorial approach [44] (1998) to freeness, Biane [19] (1998) proved this result in the general case. It follows from Theorem 2.1 that $F_1(z)$ and $F_2(z)$ are $Z_1(z)$ and $Z_2(z)$ in (2.3), respectively. About the existence and uniqueness of the subordinating functions $Z_1$ and $Z_2$ see also Voiculescu [50], [51].

Pastur and Vasilchuk [42] (2000) studied the normalized eigenvalue counting measure of the sum of two $n \times n$ unitary matrices rotated independently by random unitary Haar distributed measures. They established the convergence in probability as $n \to \infty$ to a limiting nonrandom measure. They derive functional equations for the Cauchy transforms of limiting distributions assuming the existence of the mean of the limiting measures. It follows from Theorem 2.1 that their equations are equivalent to (2.3).

**Corollary 2.2.** *Let $\mu_1, \dots, \mu_n \in \mathcal{M}$. There exist unique functions $Z_1(z), \dots, Z_n(z)$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,*

$$z = Z_1(z)+\dots+Z_n(z)-(n-1)F_{\mu_1}(Z_1(z)), \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = \dots = F_{\mu_n}(Z_n(z)).$$

*Moreover, $F_{\mu_1 \boxplus \dots \boxplus \mu_n}(z) = F_{\mu_1}(Z_1(z))$ for all $z \in \mathbb{C}^+$.*

Specializing to $\mu_1 = \mu_2 = \dots = \mu_n = \mu$ write $\mu_1 \boxplus \dots \boxplus \mu_n = \mu^{n\boxplus}$. Then we get

**Corollary 2.3.** *Let $\mu \in \mathcal{M}$. There exists an unique function $Z \in \mathcal{F}$ such that*

$$z = nZ(z) - (n-1)F_\mu(Z(z)), \quad z \in \mathbb{C}^+, \tag{2.5}$$

*and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$, $z \in \mathbb{C}^+$.*

By (2.5), we see that $(Z^{(-1)}(z) - z)/(n-1) = z - F_\mu(z)$ for $z$ from some domain $\Gamma_{\alpha,\beta}$. It follows from this that $Z^{(-1)}(z) - z$ has an analytic continuation to $\mathbb{C}^+$ with values in $\mathbb{C}^- \cup \mathbb{R}$. Since $Z \in \mathcal{F}$, it is easy to see, that $(Z^{(-1)}(iy) - iy)/y \to 0$ as $y \to +\infty$. Hence $Z(z) = F_\nu(z)$, $z \in \mathbb{C}^+$, where $\nu \in \mathcal{M}$ is infinitely divisible relative to the free additive convolution (the definition and the characterization of the $\boxplus$-infinitely divisibility see in this section below). Note that the relation (2.5) holds if the integers $n$ are replaced by real numbers $t \geqslant 1$. This shows that there is a semigroup $\nu_t \in \mathcal{M}$, $t \geqslant 1$, such that $t\varphi_\nu = \varphi_{\nu_t}$. Thus we give an analytical approach to the existence of the semigroup $\nu_t \in \mathcal{M}$, $t \geqslant 1$, with the described property.

This fact was shown in [13], [39], [5] by other methods.

Having defined the Voiculescu semigroup $(\mathcal{M}, \boxplus)$, based on the properties of the functions of the subclass $\mathcal{F}$ of the class $\mathcal{N}$ of Nevanlinna functions, we shall proceed by studying multiplicative free convolutions.

## Multiplicative free convolution on $\mathbb{R}_+$

Let $\mathcal{M}_+$ be the set of p-measures $\mu$ on $\mathbb{R}_+ = [0, +\infty)$ such that $\mu(\{0\}) < 1$. Define, following Voiculescu [47], the $\psi_\mu$-function of a p-measure $\mu \in \mathcal{M}_+$, by

$$\psi_\mu(z) = \int\limits_{\mathbb{R}_+} \frac{z\xi}{1 - z\xi}\, \mu(d\xi) \tag{2.6}$$

for $z \in \mathbb{C} \setminus \mathbb{R}_+$. The measure $\mu$ is completely determined by $\psi_\mu$ because $z(\psi_\mu(z) + 1) = G_\mu(1/z)$. Note that $\psi_\mu : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}$ is an analytic function such that $\psi_\mu(\bar{z}) = \overline{\psi_\mu(z)}$, and $z(\psi_\mu(z) + 1) \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$. Consider the function

$$K_\mu(z) := \psi_\mu(z)/(1 + \psi_\mu(z)), \quad z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{2.7}$$

It is easy to see that $K_\mu(z) \in \mathcal{N}$ and $K_\mu(z)$ is analytic and nonpositive on the negative real axis $(-\infty, 0)$. In addition, for $x > 0$, $K_\mu(-x) \to 0$ as $x \to 0$.

Denote by $\mathcal{K}$ the subclass of $\mathcal{N}$ of functions $f$ such that $f(z)$ is analytic and nonpositive on the negative real axis, and, for $x > 0$, $f(-x) \to 0$ as $x \to 0$.

By Krein's results (see Section 3), the function $K_\mu(z)$, being analytic and nonpositive on $(-\infty,0)$, and $\lim_{x\to 0,x>0} K_\mu(-x)=0$, belongs to $\mathcal{K}$ and admits by Corollary 3.3 (2) (like *all* functions in $\mathcal{K}$) the following representation

$$\frac{K_\mu(z)}{z}=a+\int\limits_{(0,\infty)}\frac{\tau(dt)}{t-z},\quad 0<\arg z<2\pi, \tag{2.8}$$

where $a\geqslant 0$ and $\tau$ is a nonnegative measure such that

$$\int_{(0,\infty)}\frac{\tau(dt)}{1+t}<\infty. \tag{2.9}$$

In view of Proposition 3.4, see Section 3, a function $K_\mu$ has the inverse function $\tilde{\chi}_\mu$ on the image $K_\mu(i\mathbb{C}^+)$. We define the $\Sigma$-transform of $\mu$ as the function

$$\Sigma_\mu(z):=\tilde{\chi}_\mu(z)/z,\quad z\in K_\mu(i\mathbb{C}^+).$$

Note that $K_\mu(i\mathbb{C}^+)\supseteq\Gamma^+_{\alpha,\beta,\Delta}:=\{z\in\mathbb{C}:\beta<|z|<\Delta,\alpha<\arg z<2\pi-\alpha\}$ for some $0<\beta<\Delta$ and $\alpha\in(0,\pi)$. In addition we conclude from (2.8) that $\arg K_\mu(z)\geqslant\arg z$ for $z\in\mathbb{C}^+$ and $\arg K_\mu(z)=\pi$ for $z\in(-\infty,0)$. Therefore $\arg\Sigma_\mu(z)\leqslant 0$, $\operatorname{Im} z\geqslant 0$, and $\arg\Sigma_\mu(z)=0$, $z\in(-\infty,0)$, where $\Sigma_\mu(z)$ is defined.

Let $\mu_1$ and $\mu_2$ denote $p$-measures in $\mathcal{M}_+$ with corresponding transforms $K_{\mu_1}$ and $K_{\mu_2}$ defined in (2.7), which are in the klass $\mathcal{K}$.

We shall define the free multiplicative convolution using the transforms $K_{\mu_1}$ and $K_{\mu_2}$ by means of the following characterization which (after exchanging addition with multiplication) is identical to the characterization (2.3) for the additive convolution.

**Theorem 2.4.** *There exist two uniquely determined functions $Z_1(z)$ and $Z_2(z)$ in the Krein class $\mathcal{K}$ such that*

$$Z_1(z)Z_2(z)=zK_{\mu_1}(Z_1(z))\quad\textit{and}\quad K_{\mu_1}(Z_1(z))=K_{\mu_2}(Z_2(z)),\quad z\in\mathbb{C}^+. \tag{2.10}$$

Introduce

$$K(z):=K_{\mu_1}(Z_1(z))\quad\text{and}\quad\psi(z):=K(z)/(1-K(z)),\quad z\in\mathbb{C}^+.$$

Then, by (2.8) for $K_{\mu_1}$ and $Z_1$, we note that $K(z)$ and $K(z)/z$ are functions in the class $\mathcal{N}$ and $K(-x)\to 0$ as $x\to 0$ for $x>0$. Hence, by Corollary 3.3 (1), $K(z)\in\mathcal{K}$. Using this assertion we easily see that $\psi(z)\in\mathcal{N}$, $\psi(z)/z\in\mathcal{N}$, and $\lim_{x\to 0,x>0}\psi(-x)=0$. Moreover, by the representation (2.8) for $K(z)$, we have $\lim_{x\to-\infty}\psi(x)/x=0$. Hence the function $\psi(z)/z$ admits the representation (2.8) with $a=0$, i.e.,

$$\frac{\psi(z)}{z}=\int\limits_{(0,\infty)}\frac{\tau_\psi(dt)}{t-z}=\int\limits_{(0,\infty)}\frac{u}{1-uz}\mu_\psi(du),\quad z\in\mathbb{C}^+, \tag{2.11}$$

where $\tau_\psi$ is a nonnegative measure satisfying the condition (2.9) and $\mu_\psi$ is a nonnegative finite measure.

Note that $\lim_{x\to-\infty}\psi(x)=-1$ if and only if in the representation (2.8) for $K(z)$ either $a>0$ or $\tau((0,\infty))=\infty$. In this case, by (2.11), we may represent $\psi(z)\in\mathcal{K}$ as

$\psi = \psi_\mu$, see (2.6), with some p-measure $\mu \in \mathcal{M}_+$ such that $\mu(\{0\}) = 0$. In addition $K \in \mathcal{K}$ may be represented as $K(z) = K_\mu(z)$, $z \in \mathbb{C}^+$. Therefore $\psi_{\mu_1}(Z_1(z)) = \psi_\mu(z)$.

Let in (2.8) for $K(z)$ $a = 0$ and $\tau((0,\infty)) < \infty$. Then, as it is easy to see, $\lim_{x\to-\infty} \psi(x) = -p = -\tau((0,\infty))/(1+\tau((0,\infty)))$. Again, by (2.11), we get for $\psi(z)$ the representation (2.6) with some p-measure $\mu \in \mathcal{M}_+$ and $\mu(\{0\}) = 1-p$. Thus, $\psi(z) = \psi_\mu(z)$ and $K(z) = K_\mu(z)$, $z \in \mathbb{C}^+$. Hence $\psi_{\mu_1}(Z_1(z)) = \psi_\mu(z)$.

The p-measure $\mu$ is determined uniquely by the p-measures $\mu_1$ and $\mu_2$.

We define $\mu := \mu_1 \boxtimes \mu_2$.

This defines the free multiplicative convolution on the non-negative half-line by purely complex analytic methods as above on p. 4.

Since $K_{\mu_1}(Z_1(z)) = K_{\mu_2}(Z_2(z))$ for $z \in \mathbb{C}^+$, we have $\mu_1 \boxtimes \mu_2 = \mu_2 \boxtimes \mu_1$ and it is easily verified that this convolution is associative as well.

From Theorem 2.4 we conclude that the relation

$$\Sigma_{\mu_1}(z)\Sigma_{\mu_2}(z) = \Sigma_\mu(z) \tag{2.12}$$

holds for $z$, where $\Sigma_{\mu_1}(z)$, $\Sigma_{\mu_2}(z)$ and $\Sigma_\mu(z)$ are defined. This relation is due to Voiculescu [47] and Bercovici and Voiculescu [12].

Using Speicher's combinatorial approach [44] (1998) to freeness, Biane [19] (1998) showed that there exist unique functions $R_1, R_2 \in \mathcal{K}$ such that $\psi_{\mu_1\boxtimes\mu_2}(z) = \psi_{\mu_1}(R_1(z)) = \psi_{\mu_2}(R_2(z))$ for all $z \in \mathbb{C}^+$. It follows from Theorem 2.4 that $R_1(z)$ and $R_2(z)$ are $Z_1(z)$ and $Z_2(z)$ in (2.10), respectively.

**Corollary 2.5.** *Let $\mu_1, \dots, \mu_n \in \mathcal{M}_+$. There exist unique functions $Z_1(z), \dots, Z_n(z)$ in the class $\mathcal{K}$ such that, for $z \in \mathbb{C}^+$,*

$$Z_1(z)\dots Z_n(z) = z\big(K_{\mu_1}(Z_1(z))\big)^{n-1}, \quad \textit{and} \quad K_{\mu_1}(Z_1(z)) = \dots = K_{\mu_n}(Z_n(z)).$$

*Moreover, $K_{\mu_1\boxtimes\dots\boxplus\mu_n}(z) = K_{\mu_1}(Z_1(z))$ for all $z \in \mathbb{C}^+$.*

Let $\mu_1 = \mu_1 = \dots = \mu_n = \mu$. Denote $\mu_1 \boxtimes \dots \boxtimes \mu_n = \mu^{n\boxtimes}$.

**Corollary 2.6.** *Let $\mu \in \mathcal{M}_+$. There exists an unique function $Z \in \mathcal{K}$ such that*

$$(Z(z))^n = z\big(K_\mu(Z(z))\big)^{n-1}, \quad z \in \mathbb{C}^+, \tag{2.13}$$

*and $K_{\mu^{n\boxtimes}}(z) = K_\mu(Z(z))$, $z \in \mathbb{C}^+$.*

Again the relation (2.13) holds if we replace integers $n$ by real numbers $t \geqslant 1$. This shows that there is a semigroup $\nu_t \in \mathcal{M}_+$, $t \geqslant 1$, such that $(\Sigma_\nu(z))^t = \Sigma_{\nu_t}(z)$.

Thus we have defined the Voiculescu semigroup $(\mathcal{M}_+, \boxtimes)$, based on properties of functions of the Krein subclass $\mathcal{K}$ of the class $\mathcal{N}$ of Nevanlinna functions. In the following we consider the case of spectral p-measures on the unit circle $\mathbb{T}$.

Multiplicative free convolution on the unit circle

Let $\mu$ be a p-measure on $\mathbb{T}$. Following Voiculescu [47], we define a transform of the p-measure $\mu$ on $\mathbb{T}$, by

$$\psi_\mu(z)=\int\limits_{\mathbb{T}}\frac{z\xi}{1-z\xi}\,\mu(d\xi).$$

The function $\psi$ has a convergent power series representation in $\mathbb{D}:=\{z\in\mathbb{C}:\,|z|<1\}$, the open unit disk of $\mathbb{C}$, such that $\psi_\mu(0)=0$.

Let $\mathcal{M}_*$ denote the set of p-measures on $\mathbb{T}$ such that $\int_{\mathbb{T}}\xi\,\mu(d\xi)\ne 0$.

If $\mu\in\mathcal{M}_*$, it follows that the function

$$Q_\mu:=\psi_\mu/(1+\psi_\mu)$$

has a right inverse $Q_\mu^{(-1)}$, defined in a neighborhood of 0 denoted by $\mathbb{D}_\alpha:=\{z\in\mathbb{C}:|z|<\alpha\}$ with some $0<\alpha\leqslant 1$, such that $Q_\mu^{(-1)}(0)=0$. Let

$$\Sigma_\mu(z)=Q_\mu^{(-1)}(z)/z$$

denote the so called $\Sigma$-transform of $\mu$.

Denote by $\mathcal{C}$ the class of analytic functions $H(z)$ on $\mathbb{D}\to -i(\mathbb{C}^+\cup\mathbb{R})$ introduced by Carathéodory.

Note that

$$Q_\mu(z)=\frac{\psi_\mu(z)}{1+\psi_\mu(z)}=\frac{H(z)-1}{H(z)+1} \tag{2.14}$$

where $H(z):=1+2\psi_\mu(z)$ is a function of Carathéodory's class $\mathcal{C}$. Such functions $H(z)$, by (3.1) and $H(0)=1$, see Section 3, have the form

$$H(z)=\int\limits_{\mathbb{T}}\frac{\xi+z}{\xi-z}\,\sigma(d\xi),$$

where $\sigma$ is a p-measure.

Define $\mathcal{S}$ to be so called *Schur class* of analytic functions $\mathbb{D}\to\overline{\mathbb{D}}$ (see Section 3), where $\overline{\mathbb{D}}$ is the closure of $\mathbb{D}$.

We see from (2.14), that $Q_\mu\in\mathcal{S}$ and since $\psi_\mu(0)=0$ and $\mu\in\mathcal{M}_*$, $Q_\mu(0)=0$, $Q'_\mu(0)\ne 0$.

In the sequel we denote by $\mathcal{S}_*$ the subclass of $\mathcal{S}$ which consists of Schur functions $Q$ with properties $Q(0)=0$ and $Q'(0)\ne 0$.

Since $Q_\mu\in\mathcal{S}_*$, by the Schwarz lemma, we have $|Q_\mu(z)/z|\leqslant 1$ for $z\in\mathbb{D}$. Hence $|Q_\mu^{(-1)}(z)/z|\geqslant 1$ in a neighborhood of 0. Moreover, both (3.2) induces a one-to-one correspondence between the classes $\mathcal{C}$ and $\mathcal{S}$, and (2.14) induces a one-to-one correspondence between functions $H\in\mathcal{C}$ such that $H(0)=1$ and $H'(0)\ne 0$, and functions $Q_\mu$ of the class $\mathcal{S}_*$.

Let $\mu_1$ and $\mu_2$ denote p-measures in $\mathcal{M}_*$ and let $Q_{\mu_1}$ and $Q_{\mu_2}$ be Schur functions which correspond to these measures, by (2.14). We now define the free multiplicative free convolution $\mu_1\boxtimes\mu_2$ based on $Q_{\mu_1}$ and $Q_{\mu_2}$ using the following characterization.

**Theorem 2.7.** *There exist two functions $Z_1(z)$ and $Z_2(z)$ in the set $\mathcal{S}_*$ such that*

$$Z_1(z)Z_2(z) = zQ_{\mu_1}(Z_1(z)) \quad \text{and} \quad Q_{\mu_1}(Z_1(z)) = Q_{\mu_2}(Z_2(z)), \quad z \in \mathbb{D}. \tag{2.15}$$

*The functions $Z_1(z)$ and $Z_2(z)$ are unique solutions of* (2.15) *in the class $\mathcal{S}_*$.*

Consider the function $Q_{\mu_1}(Z_1(z))$. It is easy to see that it belongs to Schur's class $\mathcal{S}$ and $Q_{\mu_1}(Z_1(0)) = 0$, $Q'_{\mu_1}(0)Z'_1(0) \neq 0$. Therefore $Q_{\mu_1}(Z_1) \in \mathcal{S}_*$ and $Q_{\mu_1}(Z_1(z)) = Q_\mu(z)$ for $z \in \mathbb{D}$, where $Q_\mu(z)$ has form (2.14) for some p-measure $\mu \in \mathcal{M}_*$. This measure is determined uniquely by the p-measures $\mu_1$ and $\mu_2$. Define $\mu := \mu_1 \boxtimes \mu_2$.

This defines the free multiplicative convolution on the unit circle by purely complex analytic methods as on p. 4.

Since $Q_{\mu_1}(Z_1(z)) = Q_{\mu_2}(Z_2(z))$ for $z \in \mathbb{D}$, we get $\mu_1 \boxtimes \mu_2 = \mu_2 \boxtimes \mu_1$. It is easy to verify that the operation $\boxtimes$ is associative. Using the relation (2.14) between the function $Q_\mu(z) \in \mathcal{S}_*$ and the function $\psi_\mu(z)$, we conclude that $\psi_\mu(z) = \psi_{\mu_1}(Z_1(z))$ for $z \in \mathbb{D}$. In addition we have in some neighborhood of 0

$$\frac{Q_{\mu_1}^{(-1)}(z)}{z}\frac{Q_{\mu_2}^{(-1)}(z)}{z} = \frac{Q_\mu^{(-1)}(z)}{z} \quad \text{or} \quad \Sigma_{\mu_1}(z)\Sigma_{\mu_2}(z) = \Sigma_\mu(z). \tag{2.16}$$

This formula is due to Voiculescu [47], [11] (1992).

Biane [19] (1998) showed that there exist unique functions $Q_1, Q_2 \in \mathcal{S}_*$ such that $\psi_{\mu_1 \boxtimes \mu_2}(z) = \psi_{\mu_1}(Q_1(z)) = \psi_{\mu_2}(Q_2(z))$ for all $z \in \mathbb{D}$. It follows from Theorem 2.7 that $Q_1(z)$ and $Q_2(z)$ are $Z_1(z)$ and $Z_2(z)$ in (2.15), respectively.

Note that Vasilchuk [45] (2001) studied the normalized eigenvalue counting measure of the product of two $n \times n$ unitary matrices and the measure of product of three $n \times n$ Hermitian positive matrices rotated independently by random unitary Haar distributed measures. He established the convergence in probability as $n \to \infty$ to a limiting nonrandom measure and derived functional equations for the Herglotz and Cauchy transforms of limiting distributions under some restriction on counting measures. From Theorem 2.4 and Theorem 2.7 it follows that his equations are equivalent to (2.10) and (2.15).

For the multiplicative free convolution the analogues of Corrolary 2.2 and Corollary 2.3 hold.

**Corollary 2.8.** *Let $\mu_1, \dots, \mu_n \in \mathcal{M}_*$. There exist uniquely determined functions $Z_1(z), \dots, Z_n(z)$ in the class $\mathcal{S}_*$ such that, for $z \in \mathbb{D}$,*

$$Z_1(z) \dots Z_n(z) = z\big(Q_{\mu_1}(Z_1(z))\big)^{n-1}, \quad \text{and} \quad Q_{\mu_1}(Z_1(z)) = \dots = Q_{\mu_n}(Z_n(z)).$$

*Moreover, $Q_{\mu_1 \boxtimes \dots \boxtimes \mu_n}(z) = Q_{\mu_1}(Z_1(z))$ for all $z \in \mathbb{D}$.*

**Corollary 2.9.** *Let $\mu \in \mathcal{M}_*$. There exists an unique function $Z \in \mathcal{S}_*$ such that*

$$(Z(z))^n = z\big(Q_\mu(Z(z))\big)^{n-1}, \quad z \in \mathbb{D}, \tag{2.17}$$

*and $Q_{\mu^{n\boxtimes}}(z) = Q_\mu(Z(z))$, $z \in \mathbb{D}$.*

Rewrite (2.17) in the form

$$\tilde{Z}(z) = \big(\tilde{Q}_\mu(Z(z))\big)^{n-1}, \quad z \in \mathbb{D}, \tag{2.18}$$

where $\tilde{Z}(z) := Z(z)/z$ and $\tilde{Q}_\mu(z) := Q_\mu(z)/z$, $z \in \mathbb{D}$, are functions of the class $\mathcal{S}$ and $\tilde{Z}(0) \neq 0$ and $\tilde{Q}_\mu(0) \neq 0$. From (2.18) it follows that $\tilde{Z}(z) \neq 0$ and $\tilde{Q}_\mu(Z(z)) \neq 0$ for $z \in \mathbb{D}$.

Note that the relation (2.18) holds if we replace integers $n$ by real $t \geqslant 1$ for the functions $\tilde{Q}_{\mu\boxtimes\mu}(z)$ and for $\tilde{Q}_\mu(z)$ if $Q_\mu(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$. This shows that there is a semigroup $\nu_t \in \mathcal{M}_*$, $t \geqslant 2$, such that $(\Sigma_\nu(z))^t = \Sigma_{\nu_t}(z)$ in the general case and there is a semigroup $\nu_t \in \mathcal{M}_*$, $t \geqslant 1$, such that $(\Sigma_\nu(z))^t = \Sigma_{\nu_t}(z)$ in the case $Q_\nu(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$.

Thus we have defined the Voiculescu semigroup $(\mathcal{M}_*, \boxtimes)$, based on properties of functions of the subclass $\mathcal{S}_*$ of the class $\mathcal{S}$ of Schur functions.

The relations (2.3), (2.10), and (2.14) were used to good effect in the papers of Belinshi [3], [5], Belinshi and Bercovici [6], [7], Bercovici and Voiculescu [14], Biane [18]. In our paper we use the relations (2.3), (2.10), and (2.14) to study the arithmetic of p-measures in Voiculescu's semigroups.

## Arithmetic of p-measures in Voiculescu's semigroups

Now we consider the problem of the decomposition of measures $\mu$ of the commutative semigroups $(\mathcal{M}, \boxplus)$, $(\mathcal{M}_+, \boxtimes)$, and $(\mathcal{M}_*, \boxtimes)$. In the sequel we shall denote these semigroups by a symbol $(\mathbf{M}, \circ)$, where $\mathbf{M}$ means $\mathcal{M}, \mathcal{M}_+, \mathcal{M}_*$ and $\circ$ means the operations $\boxplus, \boxtimes$. The following notions are analogues of the classical ones for $(\mathcal{M}, *)$.

We shall say that $\mu_1 \in \mathbf{M}$ is a *free factor* or just *factor* of $\mu \in \mathbf{M}$ if there exists $\mu_2 \in \mathbf{M}$ such that $\mu = \mu_1 \circ \mu_2$. Every $\mu \in \mathbf{M}$ has factors. Indeed, we have $\mu = \delta_a \circ (\mu \circ \delta_b)$, where $b = -a, a \in \mathbb{R}$, in the case of the semigroup $(\mathcal{M}, \boxplus)$, $b = 1/a, a > 0$, in the case of $(\mathcal{M}_+, \boxtimes)$, and $b = 1/a, a \in \mathbb{T}$, in the case of $(\mathcal{M}_*, \boxtimes)$. Hence $\delta_a$ and $\mu \circ \delta_b$ are factors of $\mu$. Such factors are called *improper*. A p-measure $\mu$ which is not a Dirac measure is called *indecomposable* if it has improper factors only. Such p-measures may be regarded as simple elements of this semigroup. If $\mu$ in $(\mathbf{M}, \circ)$ is not indecomposable it is called *decomposable*. Two measures $\mu_1$ and $\mu_2$ are called *equivalent*, $\mu_1 \sim \mu_2$, if $\mu_1 = \mu_2 \circ \delta_a$, where $a \in \mathbb{R}$ in the case of $(\mathcal{M}, \boxplus)$, $a > 0$ in the case of $(\mathcal{M}_+, \boxtimes)$, and $a \in \mathbb{T}$ in the case of $(\mathcal{M}_*, \boxtimes)$.

As in the classical theory of convolutions, a measure $\mu$ is called $\circ$-*infinitely divisible* (or i.d. for short) if, for every natural number $n$, $\mu$ can be written as $\mu = \nu_n \circ \nu_n \circ \cdots \circ \nu_n$ ($n$ times) with $\nu_n \in \mathbf{M}$. The measure $\delta_a$ is necessarily infinitely divisible. Note that all i.d. measures are decomposable and Dirac measures have this property. As mentioned in the introduction a measure $\mu \in \mathbf{M}$ is to the class $I_0$ relative $\circ$ if $\mu$ is $\circ$-infinitely divisible and has no $\circ$-indecomposable factors.

Khintchine [31] (1937) was the first who studied the arithmetic of the semigroup $(\mathcal{M}, *)$ of distribution functions on $\mathbb{R}$ with respect to the convolution $*$. He derived for this semigroup the three basic results.

1. **Limit of triangular arrays.** The limit of a convergent infinitesimal triangular array is an i.d. element $\mu$ of $(\mathcal{M}, *)$.

2. **Classification.** Any element $\mu$ of $(\mathcal{M}, *)$ belongs to one of the following classes. Either

(1) $\mu$ is indecomposable,
(2) $\mu$ is decomposable (possibly i.d.) and has an indecomposable factor,
(3) $\mu$ is i.d. and has no indecomposable factors. (This class of p-measures is denoted by $I_0$.)

(Compare the description of Delphic semigroups in the introduction.)

3. **Representation.** For each $\mu$ in $\mathcal{M}$ one may decompose

$$\mu = \nu * \mu_1 * \mu_2 * \dots$$

in at least one way, where $\nu$ is i.d. and has no indecomposable sub-factor, and each $\mu_j$ is indecomposable. The convolution product is at most countable and may be finite or void.

Note that Gaussian distributions (Cramér (1936)), Poisson distributions (Raikov (1937)) and the convolution of Gaussian and Poisson distributions (Linnik (1957)) belong to the class $I_0$. Hence in the semigroup $(\mathcal{M}, *)$ the class $I_0$ has nontrivial elements (besides the trivial units $\delta_a$).

A number of papers have been devoted to the study of the arithmetic of semigroups of p-measures. We refer the reader to the monograph of Linnik and Ostrovskii [34] (1977), the surveys of Livshic, Ostrovskii and Chistyakov [35] (1975), Ostrovskii [40] (1977), [41] (1986).

We shall consider the semigroups $(\mathbf{M}, \circ)$ introduced above and we study their arithmetic using the theory of Delphic semigroups, (see Kendall [30], Davidson [24] –[26]). Kendall and Davidson developed Khintchine's theory for a wide class of semigroups (Delphic) where Khintchine's basic theorems remain valid.

In the semigroup $(\mathcal{M}, \boxplus)$ i.d. p-measures were first considered in Voiculescu [46], where compactly supported $\boxplus$-i.d. measures were characterized. P-measures with a finite variance were considered in Maassen [36] and Bercovici, Voiculescu [12] gave a characterization of general i.d. p-measures $\mu \in \mathcal{M}$. There is an analogue of the Lévy-Khintchine formula, (see [48], [11], [12]) which states that a p-measure $\mu$, on $\mathbb{R}$, is i.d. if and only if the function $\varphi_\mu(z)$ has an analytic continuation to the whole of $\mathbb{C}^+$, with values in $\mathbb{C}^- \cup \mathbb{R}$, and one has

$$\lim_{y \to +\infty} \frac{\varphi_\mu(iy)}{y} = 0. \tag{2.19}$$

By the Nevanlinna representation for such function, we know that there exist a real number $\alpha$, and a finite nonnegative measure $\nu$, on $\mathbb{R}$, such that

$$\varphi_\mu(z) = \alpha + \int\limits_{\mathbb{R}} \frac{1 + uz}{z - u}\, \nu(du). \tag{2.20}$$

The formula (2.20) is an analogue of the well-known Lévy-Khintchine formula for the logarithm of characteristic functions $\varphi(t;\mu) := \int_{\mathbb{R}} e^{itu}\,\mu(du)$, $t \in \mathbb{R}$, of $*$-i.d. measures $\mu \in \mathcal{M}$. A measure $\mu \in \mathcal{M}$ is $*$-i.d. if and only if there exist a finite nonnegative Borel measure $\nu$ on $\mathbb{R}$, and a real number $\alpha$ such that

$$\log\varphi(t;\mu) = f_\mu(t) := \exp\Big\{i\alpha t + \int\limits_{\mathbb{R}} \Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big)\frac{1+u^2}{u^2}\,\nu(du)\Big\}, \quad t \in \mathbb{R}, \tag{2.21}$$

where $(e^{itu} - 1 - itu/(1+u^2))(1+u^2)/u^2$ will be interpreted as $-t^2/2$ for $u = 0$.

In the classical case the precise formulation of the Khintchine limit theorem for $(\mathcal{M}, *)$ is as follows:

Let $\{\mu_{nk} : n \geqslant 1,\, 1 \leqslant k \leqslant n\}$ be an array of infinitesimal measures in $\mathcal{M}$, i.e.,

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant n}\mu_{nk}(\{u : |u| > \varepsilon\}) = 0 \tag{2.22}$$

for every $\varepsilon > 0$. In order that $\mu \in \mathcal{M}$ be the limit in the weak topology of distributions $\mu^{(n)} = \delta_{a_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nn} \to \mu$ for some suitably chosen constants $a_n$, it is necessary and sufficient that $\mu$ be i.d.

The Khintchine limit theorem in free probability theory has the same form for the measures $\mu^{(n)} = \delta_{a_n} \boxplus \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nn}$.

This theorem was early proved by Bercovici and Pata [16] (2000). We give another proof of this result, using arguments of the theory of Delphic semigroups.

The i.d. measures in $(\mathcal{M}_+, \boxtimes)$ have been characterized by Voiculescu [47], Bercovici and Voiculescu [11], [12]. There is an analogue of the Lévy-Khintchine formula which states that a measure $\mu \in \mathcal{M}_+$ is $\boxtimes$-i.d. if and only if there exist a finite nonnegative measure $\nu$ on $(0, \infty)$ and real numbers $a$ and $b \geqslant 0$ such that

$$\Sigma_\mu(z) = \exp\Big\{a - bz + \int\limits_{\mathbb{R}_+} \frac{1+uz}{z-u}\,\nu(du)\Big\}, \tag{2.23}$$

for $z$, where $\Sigma_\mu(z)$ is defined.

In other words, a measure $\mu \in \mathcal{M}_+$ is $\boxtimes$-i.d. if and only if

$$\Sigma_\mu(z) = \exp\{-u(z)\}, \tag{2.24}$$

where $u(z) \in \mathcal{N}$ and $u(z)$ is analytic and real valued on the negative half-line $(-\infty, 0)$.

Khintchine's limit problem for multiplicative free convolution may be formulated as follows.

Let $\{\mu_{nk} : n \geqslant 1,\, 1 \leqslant k \leqslant n\}$ be an array of measures in $\mathcal{M}_+$ such that

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant n}\mu_{nk}(\{u :\, |u-1| > \varepsilon\}) = 0 \tag{2.25}$$

for every $\varepsilon > 0$. The measures $\mu_{nk} \in \mathcal{M}_+$ are called *infinitesimal*.

We shall characterize in Theorem 2.10 the class of p-measures $\mu \in \mathcal{M}_+$ such that $\mu^{(n)} = \delta_{a_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \cdots \boxtimes \mu_{nn} \to \mu$ in the weak topology for some suitably chosen positive constants $a_n$.

The i.d. measures of the semigroup $(\mathcal{M}_*, \boxtimes)$ were characterized in [47], [11]. There is an analogue of the Lévy-Khintchine formula which states that a measure $\mu \in \mathcal{M}_*$ is $\boxtimes$-i.d. if and only if there exist a finite nonnegative measure $\nu$ on $\mathbb{T}$ and a real number $a$ such that

$$\Sigma_\mu(z) = \exp\Big\{ia + \int\limits_{\mathbb{T}} \frac{1+z\xi}{1-z\xi}\,\nu(d\xi)\Big\}, \tag{2.26}$$

for $z$, where $\Sigma_\mu(z)$ is defined.

In other words, a measure $\mu \in \mathcal{M}_*$ is $\boxtimes$-i.d. if and only if

$$\Sigma_\mu(z) = \exp\{v(z)\}, \tag{2.27}$$

where $v(z) \in \mathcal{C}$.

Let $\{\mu_{nk} : n \geqslant 1, 1 \leqslant k \leqslant n\}$ be an array of measures in $\mathcal{M}_*$. We shall call the measures $\mu_{nk}$ *infinitesimal* if

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant n} \mu_{nk}(\{\xi : |\arg\xi| > \varepsilon\}) = 0. \tag{2.28}$$

The Khintchine limit problem for multiplicative free convolution for measures $\mu \in \mathcal{M}_*$ has the same form as in the case of $\mu \in \mathcal{M}_+$ with constants $a_n \in \mathbb{T}$.

We give the solution of the Khintchine limit problem, proving the following result.

**Theorem 2.10.** *Let $\{\mu_{nk} : n \geqslant 1, 1 \leqslant k \leqslant n\}$ be infinitesimal probability measures in the semigroup $(\mathbf{M}, \circ)$. The family of limit measures of sequences $\mu^{(n)} = \delta_{a_n} \circ \mu_{n1} \circ \mu_{n2} \circ \cdots \circ \mu_{nn}$ for some suitably chosen constants $a_n$ coincides with the family of $\circ$-infinitely divisible p-measures.*

We give the proof of this theorem, using arguments of the theory of Delphic semigroups. About the Khintchine limit problem for multiplicative free convolution see Belinschi–Bercovici [8] as well.

The arithmetic of the semigroups $(\mathbf{M}, \circ)$ is described in the following results.

**Theorem 2.11.** *The element $\mu$ of $(\mathbf{M}, \circ)$ can be classified as follows. Either*

(1) *$\mu$ is indecomposable,*
(2) *$\mu$ is decomposable (possibly infinitely divisible) and has an indecomposable factor,*
(3) *$\mu$ is infinitely divisible and has no indecomposable factors. (This class will be denoted by $I_0$.)*

P-measures may be decomposed as follows

**Theorem 2.12.** *Every probability measure $\mu$, which has indecomposable factors, can be expressed in the form*

$$\mu = \mu_0 \circ \mu_1 \circ \mu_2 \circ \ldots, \tag{2.29}$$

*where $\mu_0 \in I_0$ and the probability measures $\mu_1, \mu_2, \ldots$ are indecomposable (its number may be finite or denumerable).*

We will show that the representation (2.29) is not unique.

The representation (2.29) has been proved independently by Williams in [52].

The class $I_0$ in $(\mathbf{M}, \circ)$, mentioned above may be described as follows.

**Theorem 2.13.** *In Voiculescu's semigroup* $(\mathbf{M}, \circ)$ *the class* $I_0$ *is trivial, that is* $I_0$ *is the class of Dirac measures.*

In Section 8 we describe wide classes of indecomposable elements in $(\mathcal{M}, \circ)$ (see Theorems 8.1–8.3) from which it follows the following result.

**Theorem 2.14.** *The probability measures with support consisting of a finite number of points are indecomposable in* $(\mathbf{M}, \circ)$.

**Corollary 2.15.** *The class of indecomposable elements of* $(\mathbf{M}, \circ)$ *is dense in* $(\mathbf{M}, \circ)$ *in the weak topology.*

Theorem 2.14 follows from Belinschi's [5] and Bercovici–Wang [17] results. But, as we note in Section 8, Theorems 8.1–8.3 do not follow from these results.

Theorem 2.11 describes the class $I_0$ as the class of i.d. elements of $(\mathbf{M}, \circ)$ which have i.d. components only. Bercovici and Voiculescu [13] proved that a semicircular measure does not belong to the class $I_0$ in the semigroup $(\mathcal{M}, \boxplus)$. Benaych-Georges [10] proved a similar result for the free Poisson measure. These results follow from Theorem 2.13.

Speicher and Woroudi [43] (1997) introduced a further convolution operation on $\mathcal{M}$ denoted $\uplus$. Let $\mu \in \mathcal{M}$. Denote as before by $G_\mu$ the Cauchy transform of $\mu$ and by $F_\mu = 1/G_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ its reciprocal. We have $\operatorname{Im} z \leqslant \operatorname{Im} F_\mu(z)$ so that the function $E_\mu(z) := z - F_\mu(z)$ maps $\mathbb{C}^+$ to $\mathbb{C}^- \cup \mathbb{R}$, and, in addition, $E_\mu(z)/z \to 0$ as $z \to \infty$, $z \in \Gamma_\alpha$ for any fixed $\alpha > 0$. Conversely, if $E : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ is an analytic function so that $E(z)/z \to 0$ as $z \to \infty$, $z \in \Gamma_\alpha$ for any fixed $\alpha$, then there exists $\mu \in \mathcal{M}$ such that $E_\mu = E$. This observation leads to the formal definition of the Boolean convolution. Given $\mu, \nu \in \mathcal{M}$, there exists $\rho \in \mathcal{M}$ such that

$$E_\rho = E_\mu + E_\nu.$$

The measure $\rho$ is called the *Boolean convolution* of $\mu$ and $\nu$, and it denoted $\mu \uplus \nu$. Boolean convolution is an associative and commutative law, with $\delta_0$ as the zero element. We have $\delta_s \uplus \delta_t = \delta_{s+t}$, but generally $\delta_t \uplus \mu$ is not a translate of $\mu$. Speicher and Woroudi [43] treated the central and Poisson limit theorems, characterized i.d. and stable distributions and proved analogues of the classical theorems of Cramér, Marcinkiewicz, Kac and Loève. Bercovici and Pata [15] (1999) established limit laws for Boolean convolutions.

Speicher and Woroudi [43] noted that all p-measures $\mu$ are i.d. It is easy to see that the class of indecomposable elements in the semigroup $(\mathcal{M}, \uplus)$ is empty. Moreover, the class of i.d. elements coincides with the class $I_0$. Therefore Theorem 2.12 obviously holds for $(\mathcal{M}, \uplus)$ .

## 3. Auxiliary results

In the following we state some results about classes of analytic functions (see Akhiezer [1] (1965), Section 3, Akhiezer and Glazman [2] (1963), Section 6, §59, and Krein and Nudelman [32] (1977), Appendix).

By $\mathcal{C}$ we denote C. Carathéodory's class of analytic functions $F(z) : \mathbb{D} \to \{z : \operatorname{Re} z \geqslant 0\}$. A function $F$ is in $\mathcal{C}$ if and only if it admits the following representation (Herglotz, G., Riesz, F.)

$$F(z) = ia + \int\limits_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, \sigma(d\xi), \tag{3.1}$$

where $a = \operatorname{Im} F(0)$, $\mathbb{T}$ is the unit circle, and $\sigma$ is finite nonnegative measure. The number $a$ and the measure $\sigma$ are uniquely determined by $F$.

By $\mathcal{S}$ we denote J. Schur's class of analytic functions $\varphi(z) : \mathbb{D} \to \overline{\mathbb{D}}$. The classes $\mathcal{C}$ and $\mathcal{S}$ are connected via

$$\varphi(z) = \frac{1}{z} \frac{F(z) - F(0)}{F(z) + \overline{F(0)}}, \tag{3.2}$$

which induces a one-to-one correspondence between $\mathcal{C}$ and $\mathcal{S}$.

Finally we denote by $\mathcal{N}$ R. Nevanlinna's class of analytic functions $f(z) : \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}$. A function $f$ is in $\mathcal{N}$ if and only if it admits an integral representation

$$f(z) = a + bz + \int\limits_{\mathbb{R}} \frac{1 + uz}{u - z}\, \tau(du) = a + bz + \int\limits_{\mathbb{R}} \Big(\frac{1}{u - z} - \frac{u}{1 + u^2}\Big)(1 + u^2)\, \tau(du), \tag{3.3}$$

where $b \geqslant 0$, $a \in \mathbb{R}$, and $\tau$ is finite nonnegative measure. Here $a, b$ and $\tau$ are uniquely determined by $f$. More precisely we have $a = \operatorname{Re} f(i)$ and $\tau(\mathbb{R}) = \operatorname{Im} f(i) - b$. From this formula it follows that

$$f(z) = (b + o(1))z \tag{3.4}$$

for $z \in \mathbb{C}^+$ such that $|\operatorname{Re} z|/\operatorname{Im} z$ stays bounded as $|z|$ tends to infinity. Hence if $b \neq 0$, then $f$ has a right inverse $f^{(-1)}$ defined on the region $\Gamma_{\alpha,\beta}$ defined (in Section 2) for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$.

A function $f \in \mathcal{N}$ admits the representation

$$f(z) = \int\limits_{\mathbb{R}} \frac{\sigma(du)}{u - z}, \quad z \in \mathbb{C}^+, \tag{3.5}$$

where $\sigma$ is a finite nonnegative measure, if and only if $\sup_{y \geqslant 1} y|f(iy)| < \infty$.

Note that the class $\mathcal{F}$ coincides with the subclass of Nevanlinna functions for which $f(z)/z \to 1$ as $z \to \infty$ nontangentially. Indeed, reciprocal Cauchy transforms of p-measures have obviously this property. Let $f \in \mathcal{N}$ and $f(z)/z \to 1$ as $z \to \infty$ nontangentially. Then, by (3.4), $f$ admits the representation (3.3), where $b = 1$. By

(3.4) and (3.5), we see that $-1/f(z)$ admits the representation (3.5), where $\sigma \in \mathcal{M}$. It follows from (3.3) with $b = 1$ that a function $f \in \mathcal{F}$ satisfies the inequality

$$\operatorname{Im} f(z) \geqslant \operatorname{Im} z, \qquad z \in \mathbb{C}^+. \tag{3.6}$$

The Stieltjes-Perron inversion formula for a function $f \in \mathcal{N}$ has the following form. Let $\psi(u) := \int_0^u (1+t^2)\,\tau(dt)$. Then

$$\psi(u_2) - \psi(u_1) = \lim_{\eta \to 0} \frac{1}{\pi} \int_{u_1}^{u_2} \operatorname{Im} f(\xi + i\eta)\, d\xi, \tag{3.7}$$

where $u_1 < u_2$ are continuity points of the function $\psi(u)$.

The following two results are due to Krein, M.

**Theorem 3.1.** *The function $f(z)$ admits the representation*

$$f(z) = a + \int_{\mathbb{R}_+} \frac{\tau(du)}{u - z}, \quad 0 < \arg z < 2\pi, \tag{3.8}$$

*where $a \geqslant 0$ and $\tau$ is a nonnegative measure such that*

$$\int_{\mathbb{R}_+} \frac{\tau(du)}{1+u} < \infty, \tag{3.9}$$

*if and only if $f(z) \in \mathcal{N}$ and $f(z)$ is analytic and nonnegative on $(-\infty, 0)$.*

**Theorem 3.2.** *A function $f(z) \in \mathcal{N}$ is analytic and nonnegative on $(-\infty, 0)$ if and only if $zf(z)$ is in $\mathcal{N}$.*

**Corollary 3.3.** (1) *A function $f(z) \in \mathcal{N}$ is analytic and nonpositive on $(-\infty, 0)$ if and only if $f(z)/z$ is in $\mathcal{N}$.*
(2) *A function $f(z)/z$ admits a representation* (3.8) *with a nonnegative measure $\tau$ satisfying assumptions* (3.9) *and $\tau(\{0\}) = 0$ if and only if $f(z) \in \mathcal{K}$.*

*Proof.* At first we shall prove the assertion (1). Let the function $f(z) \in \mathcal{N}$ be analytic and nonpositive on $(-\infty, 0)$. Then the function $-1/f(z) \in \mathcal{N}$ and satisfies the assumptions of Theorem 3.2. By this theorem, $-z/f(z) \in \mathcal{N}$ and therefore $f(z)/z \in \mathcal{N}$. The converse assertion follows by repeating the previous arguments.

Let us now prove the assertion (2). It easy to see that $f(z) \in \mathcal{K}$ if the function $f(z)/z$ admits a representation (3.8) with a nonnegative measure $\tau$ which satisfies the assumption (3.9) and $\tau(\{0\}) = 0$. The converse assertion follows from the assertion (1) of Corollary 3.3 and Theorem 3.1. □

**Theorem 3.4.** *Any function $f(z)$ of class $\mathcal{N}$ not identically zero admits a unique multiplicative representation*

$$f(z) = C \exp\Big\{ \int_{-\infty}^{\infty} \Big( \frac{1}{t - z} - \frac{t}{1+t^2} \Big) p(t)\, dt \Big\},$$

*where* $C > 0$, $p(t)$ *is a summable function such that* $0 \leqslant p(t) \leqslant 1$ *almost everywhere and*

$$\int_{-\infty}^{\infty} \frac{p(t)}{1+t^2}\,dt < \infty.$$

*If* $f(z)$ *is analytic and nonnegative on the negative real axis* $(-\infty, 0)$, *then* $p(t) = 0$ *for* $t < 0$.

**Proposition 3.5.** *Let* $f(z) \in \mathcal{N}$ *be analytic and nonpositive on* $(-\infty, 0)$. *Then* $f(z)$ *admits an analytic continuation to* $\mathbb{C} \setminus \mathbb{R}_+$ *and is univalent in the left half-plane* $i\mathbb{C}^+$.

*Proof.* The first assertion of the proposition follows immediately from the representation (3.8) for $f(z)/z$. Since the function $f(z)/z$ admits the representation (3.8), we see that

$$f(z_1) - f(z_2) = (z_1 - z_2)\Big(a + \int_{(0,\infty)} \frac{u}{(u-z_1)(u-z_2)}\,\tau(du)\Big)$$

for all $z_1, z_2 \in \mathbb{C}^+$. Since for $z_1 \neq z_2$ and $z_1, z_2 \in \mathbb{C}^+ \cap i\mathbb{C}^+$

$$\operatorname{Im} \int_{(0,\infty)} \frac{u}{(u-z_1)(u-z_2)}\,\tau(du)$$
$$= \int_{(0,\infty)} \frac{-(\operatorname{Re} z_1 \operatorname{Im} z_2 + \operatorname{Re} z_2 \operatorname{Im} z_1) + u(\operatorname{Im} z_1 + \operatorname{Im} z_2)}{|u-z_1|^2 |u-z_2|^2}\, u\tau(du) \neq 0,$$

we conclude from the preceding formula that $f(z_1) \neq f(z_2)$ for the considered $z_1$ and $z_2$. Hence $f(z)$ is univalent in $\mathbb{C}^+ \cap i\mathbb{C}^+$. Since $f(\mathbb{C}^+) \subset \mathbb{C}^+$ and $f$ is strictly increasing on $(-\infty, 0)$, the univalence of $f$ on $i\mathbb{C}^+$ follows from the identity $f(\bar{z}) = \overline{f(z)}$. $\square$

We need as well the following well-known result about Schur functions $Q_\mu \in \mathcal{S}_*$ (the definition of $Q_\mu$ see in (2.14)).

**Proposition 3.6.** *Let* $\mu \in \mathcal{S}_*$, *then*

$$Q_\mu(z) = z\frac{Q'_\mu(0) + z\varphi_1(z)}{1 + \overline{Q'_\mu(0)}z\varphi_1(z)},$$

*where* $\varphi_1(z) \in \mathcal{S}$.

We need the following result about the behavior of nondecreasing functions (see [34], Ch. 4, §16).

**Proposition 3.7.** *Let* $\nu$ *be a nonnegative finite measure. If* $\nu((a, b)) > 0$ *and* $\nu([a, x))$, $a < x < b$, *is a continuous function, then there exists* $[\alpha, \beta] \subset (a, b)$ *such that* $\nu([\alpha, \alpha + h]) > ch$ *and* $\nu([\beta - h, \beta]) > ch$ *for all* $0 < h \leqslant h_0$, *where* $c > 0$ *and* $h_0 > 0$ *depend on the measure* $\nu$.

We also need some results of the theory of real and complex variable (see [27] (1969), [37] (1965)).

**Theorem 3.8.** (*Weierstrass' preparation theorem*) *Let $F(z,w)$ be a function of two complex variables which is analytic in neighborhood $|z-z_0|<r$, $|w-w_0|<\rho$ of the point $(z_0,w_0)$, and suppose that*

$$F(z_0,w_0)=0, \qquad F(z_0,w)\not\equiv 0. \tag{3.10}$$

*Then there is a neighborhood $|z-z_0|<r'<r$, $|w-w_0|<\rho'<\rho$ where $F(z,w)$ may be written as*

$$F(z,w)=\big(A_0(z)+A_1(z)w+\dots+A_{k-1}(z)w^{k-1}+w^k\big)G(z,w), \tag{3.11}$$

*$k$ is determined by*

$$\frac{\partial F(z_0,w_0)}{\partial w}=\dots\frac{\partial^{k-1}F(z_0,w_0)}{\partial w^{k-1}}=0, \qquad \frac{\partial^k F(z_0,w_0)}{\partial w^k}\neq 0,$$

*the functions $A_0(z),A_1(z),\dots,A_{k-1}(z)$ are analytic for $|z-z_0|<r'$, and the function $G(z,w)$ is analytic and nonzero on the set $|z-z_0|<r'$, $|w-w_0|<\rho'$.*

**Theorem 3.9.** *Let $\{u_n(z)\}_{n=1}^\infty$ denote a sequence of functions which are harmonic on a domain $B$. If $\{u_n(z)\}_{n=1}^\infty$ is uniformly bounded in the interior of $B$, it contains a subsequence that converges uniformly in the interior of $B$ to a harmonic function on $B$.*

**Theorem 3.10.** *Let $\{u_n(z)\}_{n=1}^\infty$ denote a sequence of functions which are harmonic on a domain $B$ such that $u_n(z)\leqslant u_{n+1}(z)$ for all $z\in B$. Suppose that the sequence converges at some point $z_0\in B$. Then the sequence converges uniformly in the interior of $B$ to a harmonic function on $B$.*

**Theorem 3.11.** *Given a domain $B$ and a sequence $\{f_n(z)\}_{n=1}^\infty$ of analytic functions on $B$, suppose the sequence $\{u_n(z)\}_{n=1}^\infty=\{\operatorname{Re} f_n(z)\}_{n=1}^\infty$ converges uniformly on every compact subset of $B$, and suppose $\{f_n(z)\}_{n=1}^\infty$ converges at some point $z_0\in B$. Then $\{f_n(z)\}_{n=1}^\infty$ converges uniformly on every compact subset of $B$ to an analytic function.*

**Theorem 3.12.** (*Vitali*) *Let $\{f_n(z)\}_{n=1}^\infty$ denote a sequence function that are analytic on $B$. Suppose that $\{f_n(z)\}_{n=1}^\infty$ is uniformly bounded in the interior of $B$ and converges on a set of points $z_k\in B$, $k=1,2,\dots$, that has a cluster point in the interior of $B$. Then, the sequence $\{f_n(z)\}_{n=1}^\infty$ converges uniformly in the interior of $B$.*

In the sequel we shall need the following result.

**Proposition 3.13.** *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of p-measures on $\mathbb{R}$. The following assertions are equivalent.*

(1) *The sequence $\{\mu_n\}_{n=1}^\infty$ converges weakly to a p-measure $\mu$.*
(2) *There exist $\alpha,\beta>0$ such that the sequence $\{\varphi_{\mu_n}\}_{n=1}^\infty$ converges uniformly on the compact subsets of $\Gamma_{\alpha,\beta}$ to a function $\varphi$, and $\varphi_{\mu_n}(z)=o(|z|)$ uniformly in $n$ as $z\to\infty$, $z\in\Gamma_{\alpha,\beta}$.*

(3) *There exist* $\alpha', \beta' > 0$ *such that the sequence* $\{\varphi_{\mu_n}\}_{n=1}^{\infty}$ *converges uniformly on the compact subsets of* $\Gamma_{\alpha',\beta'}$ *to a function* $\varphi$*, and* $\operatorname{Im}\varphi_{\mu_n}(iy) = o(y)$ *uniformly in* $n$ *as* $y \to +\infty$.

*Moreover, if* (1) *and* (2) *are satisfied, we have* $\varphi = \varphi_\mu$ *in* $\Gamma_{\alpha,\beta}$.

This result without the assertion (3) was proved by Bercovici and Voiculescu [12]. We should prove that the assertion (3) implies (1) only.

*Proof.* Let (3) hold. We should prove that there exists $\mu \in \mathcal{M}$ such that $\varphi(z) = \varphi_\mu(z)$ and $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to $\mu$. Since $\operatorname{Im}\varphi_{\mu_n}(iy) = o(y)$ iniformly in $n$ as $y \to +\infty$, we obtain the relation

$$F_{\mu_n}(iy + \varphi_{\mu_n}(iy)) = iy \tag{3.12}$$

for sufficiently large $y \geqslant y_0 > 0$ and $n \geqslant 1$. The functions $F_{\mu_n}(z) \in \mathcal{F}$ therefore

$$F_{\mu_n}(z) = z + a_n + \int\limits_{\mathbb{R}} \frac{1+uz}{u-z}\,\sigma_n(du), \quad z \in \mathbb{C}^+, \tag{3.13}$$

where $a_n \in \mathbb{R}$ and $\sigma_n$ are finite nonnegative measures. Rewrite (3.12) in the form

$$\varphi_{\mu_n}(iy) + a_n + \int\limits_{\mathbb{R}} \frac{1+u(iy+\varphi_{\mu_n}(iy))}{u-(iy+\varphi_{\mu_n}(iy))}\,\sigma_n(du) = 0, \quad y \geqslant y_0. \tag{3.14}$$

From this relation it follows, for $y \geqslant y_0$,

$$-\frac{\operatorname{Im}\varphi_{\mu_n}(iy)}{y+\operatorname{Im}\varphi_{\mu_n}(iy)} = \int\limits_{\mathbb{R}} \frac{1+u^2}{(u-\operatorname{Re}\varphi_{\mu_n}(iy))^2 + (y+\operatorname{Im}\varphi_{\mu_n}(iy))^2}\,\sigma_n(du). \tag{3.15}$$

Since $y + \operatorname{Im}\varphi_{\mu_n}(iy) = y(1+o(1))$ uniformly in $n$ for sufficiently large $y$ and $\varphi_{\mu_n}(iy) \to \varphi(iy)$ as $n \to \infty$, we obtain from (3.15) for sufficiently large $y$: $\sigma_n(\mathbb{R}) \leqslant c_1(y,\varphi)$, $n \geqslant 1$. Then we deduce from (3.14) that $|a_n| \leqslant c_2(y,\varphi)$, $n \geqslant 1$. Here $c_j(y,\varphi)$, $j = 1, 2$, are positive constants depended on $y, \varphi$ only. By the vague compactness theorem (see [33], p.179), there exists a subsequence $\{n'\}$ such that

$$a_{n'} \to a, \quad \sigma_{n'}(\mathbb{R}) \to b,$$

where $a \in \mathbb{R}, b \in \mathbb{R}_+$ and $\{\sigma_{n'}\}$ converges in the vague topologue to some nonnegative mesure $\sigma$ such that $\sigma(\mathbb{R}) \leqslant b$. Using the Helly-Brey lemma ([33], p.181) we obtain from (3.13)

$$\begin{aligned} F_{\mu_{n'}}(z) \to F(z) &:= z + a + \int\limits_{\mathbb{R}} \Big(\frac{1+uz}{u-z} - z\Big)\,\sigma(du) + bz \\ &= z + a + \int\limits_{\mathbb{R}} \frac{1+uz}{u-z}\,\sigma(du) + (b - \sigma(\mathbb{R}))z, \quad n' \to \infty, \end{aligned} \tag{3.16}$$

uniformly on every compact set in $\mathbb{C}^+$. It is easy to see from (3.14) that

$$\varphi(iy)+a+\int\limits_{\mathbb{R}}\frac{1+u(iy+\varphi(iy))}{u-(iy+\varphi(iy))}\,\sigma(du)+(b-\sigma(\mathbb{R}))(iy+\varphi(iy))=0,\quad y\geqslant y_0. \tag{3.17}$$

We deduce from this equality that $(b-\sigma(\mathbb{R}))y+(1+b-\sigma(\mathbb{R}))\,\mathrm{Im}\,\varphi(iy))\leqslant 0$ for $y\geqslant y_0$. This implies $\sigma(\mathbb{R})=b=\lim_{n'\to\infty}\sigma_{n'}(\mathbb{R})$. Hence $\{\sigma_{n'}\}$ converges in the weak topologue to the mesure $\sigma$. In addition there exists $\mu\in\mathcal{M}$ such that $F(z)=F_\mu(z)$, $z\in\mathbb{C}^+$, and we have

$$F_{\mu_{n'}}(z)=z(1+o(1)),\quad F_\mu(z)=z(1+o(1))\quad \text{as}\quad z\to\infty,\ z\in\Gamma_{\alpha,\beta},$$

uniformly in $n'$ for some $\alpha>0,\beta>0$. Therefore $\varphi_{\mu_{n'}}(z)=o(z)$ uniformly in $n'$ as $z\to\infty$, $z\in\Gamma_{\alpha,\beta}$. Hence assumptions of the assertion (2) hold and, by the equivalence of (1) and (2), $\{\mu_{n'}\}$ converges weakly to $\mu$. In other words we proved that under assumptions of the assertion (3) we can choose a subsequence $\{n'\}$ such that $\{\mu_{n'}\}$ converges weakly to some p-measure $\mu$. It remains to show that $\{\mu_n\}$ converges weakly to $\mu$. Let to the contrary $\{\mu_n\}$ does not converge weakly to $\mu$. This means that there exists $\{n''\}$ such that $\{\mu_{n''}\}$ converges weakly to $\nu\in\mathcal{M}$ such that $\nu\not\equiv\mu$. On the other hand, as it is follows from (3.12), $F_\mu(z)$ and $F_\nu(z)$ satisfy the equation

$$F_\mu(iy+\varphi(iy))=iy,\quad F_\nu(iy+\varphi(iy))=iy,\quad y\geqslant y_0,$$

and we have $F_\mu(iy+\varphi(iy))=F_\nu(iy+\varphi(iy))$ for $y\geqslant y_0$. This implies $F_\mu(z)=F_\nu(z)$, $z\in\mathbb{C}^+$, and $\mu\equiv\nu$, a contradiction. The proposition is proved. □

We shall need the following result of Bercovici and Voiculescu [12] as well.

**Proposition 3.14.** *Let $\{\mu_n\}_{n=1}^\infty$ be a tight sequence of p-measures on $\mathbb{R}_+$ such that $\delta_0$ is not in the weak closure of $\{\mu_n\}_{n=1}^\infty$. The following assertions are equivalent.*

(1) *The sequence $\{\mu_n\}_{n=1}^\infty$ converges in the weak topology to a measure $\mu$.*
(2) *There exist numbers $\alpha\in(0,\pi)$ and $0<\beta<\Delta$ such that the sequence $\{\Sigma_{\mu_n}\}_{n=1}^\infty$ converges uniformly on $\Gamma^+_{\alpha,\beta,\Delta}$ to a function $\Sigma$.*

*Moreover, if (1) and (2) are satisfied, we have $\Sigma=\Sigma_\mu$ in $\Gamma^+_{\alpha,\beta,\Delta}$.*

We need the following result of Bercovici and Voiculescu [11] as well.
For positive number $\alpha$ denote $\mathbb{D}_\alpha:=\{z\in\mathbb{C}:|z|<\alpha\}$.

**Proposition 3.15.** *Consider a measure $\mu\in\mathcal{M}_*$ and a sequence $\mu_j\in\mathcal{M}_*$, $j=1,\dots$. The following assertions are equivalent.*

(1) *The sequence $\{\mu_n\}_{n=1}^\infty$ converges in the weak topology to a measure $\mu\in\mathcal{M}_*$.*
(2) *There exists a positive number $\alpha$ such that the sequence $\{\Sigma_{\mu_j}\}_{j=1}^\infty$ converges uniformly on $\mathbb{D}_\alpha$ to a function $\Sigma_\mu$.*

*Moreover, if (1) and (2) are satisfied, we have $\Sigma=\Sigma_\mu$ in $\mathbb{D}_\alpha$.*

## 4. Additive free convolution

In this section we prove Theorem 2.1 and its consequences. We need the following auxiliary results.

**Lemma 4.1.** *Let $g : \mathbb{C}^+ \to \mathbb{C}^-$ be analytic with*

$$\liminf_{y\to+\infty} \frac{|g(iy)|}{y} = 0. \tag{4.1}$$

*Then the function $f : \mathbb{C}^+ \to \mathbb{C}$ defined via $z \mapsto z+g(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.*

This lemma generalizes a result of Maassen [36] (see Lemma 2.3). Maassen proved Lemma 4.1 under the additional restriction $|g(z)| \leqslant c/\operatorname{Im} z$ for $z \in \mathbb{C}^+$, where $c$ is a constant.

*Proof.* Since $g : \mathbb{C}^+ \to \mathbb{C}^-$ is analytic, it can be written in Nevanlinna's integral form (see (3.3) in Section 3)

$$g(z) = a - bz - \int_{\mathbb{R}} \frac{1+uz}{u-z}\,\sigma(du), \tag{4.2}$$

where $a, b \in \mathbb{R}$, $b \geqslant 0$ and $\sigma$ is a finite nonnegative measure. By (4.1), $b = 0$. Denote $\alpha_c := \sigma(\{|u| > c\})$, where $c > 0$ is chosen such that $\alpha_c < 1$. We may decompose $1+uz$ in the integral (4.2) as $1+u^2+u(z-u)$ for $|u| \leqslant c$ and as $1+z^2+z(u-z)$ for $|u| > c$. Hence we get $f(z) = a_c + f_1(z) + f_2(z)$ for $z \in \mathbb{C}^+$, where $a_c := a + \int_{[-c,c]} u\,\sigma(du)$ and

$$f_1(z) := (1-\alpha_c)z - \int_{[-c,c]} \frac{1+u^2}{u-z}\,\sigma(du), \quad f_2(z) := -(1+z^2)\int_{|u|>c} \frac{\sigma(du)}{u-z}.$$

Let $w \in \mathbb{C}^+$, denote $w_1 := w - a_c$. For every fixed $w \in \mathbb{C}^+$ we consider a closed rectifiable curve $\gamma_1 = \gamma_1(w)$ (see Figure 1) consisting of some smooth curve $\gamma_{1,1}$ connecting $w_1 - R$ to $w_1 + R$ inside the strip $0 < \operatorname{Im} z < \operatorname{Im} w$, the arc $\gamma_{1,2} : 0 < \arg(z - w_1) \leqslant \eta$ on the circle $|z - w_1| = R$ connecting $w_1 + R$ to $w_1 + Re^{i\eta}$, the arc $\gamma_{1,3} : \eta < \arg(z-w_1) \leqslant \pi - \eta$ on the circle $|z - w_1| = R$ connecting $w_1 + Re^{i\eta}$ to $w_1 - Re^{-i\eta}$, and the arc $\gamma_{1,4} : \pi - \eta < \arg(z - w_1) \leqslant \pi$ on the circle $|z - w_1| = R$ connecting $w_1 - Re^{-i\eta}$ to $w_1 - R$. Here $\eta$ is given by $\eta := 10^{-2}\min\{\arg w_1, -\arg \overline{w}_1\}$. We also assume that $R > 0$ is sufficiently large.

Note by (4.2) with $b = 0$ that $\max_{z\in\gamma_{1,3}} |g(z)|/|z| \to 0$ as $R \to \infty$. We also see that on $\gamma_{1,3}$ $\operatorname{Im} z \geqslant \operatorname{Im} w + R\sin\eta$. Since $-\operatorname{Im} g(z) = o(R)$, $z \in \gamma_{1,3}$, we have $\operatorname{Im} f(z) \geqslant \operatorname{Im} w + R\sin\eta - o(R) > \operatorname{Im} w$, $z \in \gamma_{1,3}$. Therefore, if $z$ runs through $\gamma_{1,3}$ the image $f(z)$ lies in the half-plane $\operatorname{Im} z > \operatorname{Im} w$.

For $z \in \gamma_{1,1}$, $\operatorname{Im}(z+g(z)) < \operatorname{Im} z \leqslant \operatorname{Im} w$. Therefore, if $z$ runs through $\gamma_{1,1}$ the image $f(z)$ lies in the half-plane $\operatorname{Im} z < \operatorname{Im} w$.

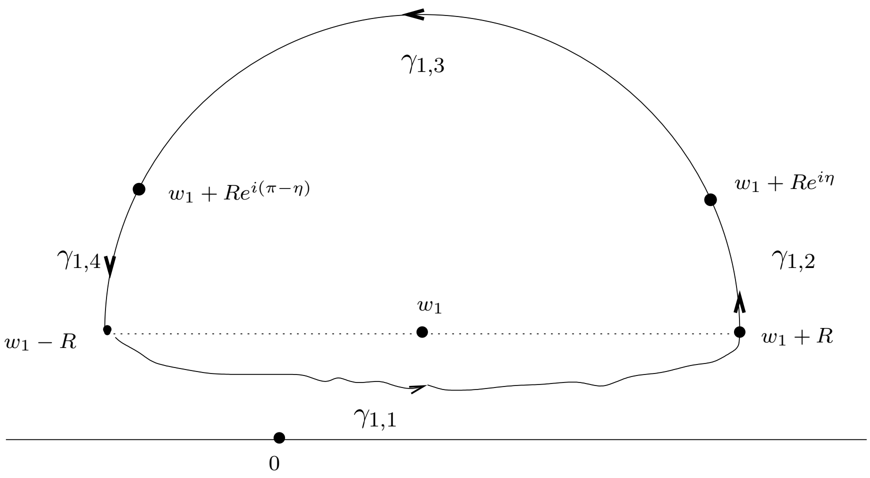


FIGURE 1

Let $z \in \gamma_{1,2}$. It is easy to see that, for $|z| > c$,

$$f_1(z) = (1-\alpha_c)z\Big(1 + \frac{1}{1-\alpha_c}\sum_{k=1}^{\infty}\frac{1}{z^{k+1}}\int\limits_{[-c,c]} u^{k-1}(1+u^2)\,\sigma(du)\Big)$$

Therefore we obtain the following formula, for $|z| > 2(c^2+1)(\sigma(\mathbb{R})+1)/(1-\alpha_c)$,

$$\log f_1(z) = \log(1-\alpha_c) + \log z + \sum_{m=1}^{\infty}\frac{a_m}{z^{m+1}},$$

where $a_m$ are real coefficients such that $|a_m| \leqslant K^m$, $m = 1,\dots,$ with some positive constant $K$. Here and in the sequel we choose the principle branch of the logarithm. Hence, for $|z| > 2K$,

$$\arg f_1(z) = \arg z - \sum_{m=1}^{\infty}\frac{a_m \sin\big((m+1)\arg z\big)}{|z|^{m+1}} \tag{4.3}$$
$$= \Big(1 + \Theta\sum_{m=1}^{\infty}\frac{K^m(m+1)}{|z|^{m+1}}\Big)\arg z = \Big(1 + 6\Theta\frac{K}{|z|^2}\Big)\arg z,$$

where $\Theta$ denotes a real-valued quantity such that $|\Theta| \leqslant 1$. On the other hand we easily obtain, for $|z| > 2$,

$$\arg(1+z^2) = 2\Big(1 + \frac{2\Theta}{|z|^2}\Big)\arg z.$$

Therefore we conclude from the definition of $f_2(z)$, taking into account that $-\int_{|u|>c}\frac{\sigma(du)}{u-z} \in \mathbb{C}^-$,

$$-\pi + 2\Big(1+\frac{2\Theta}{|z|^2}\Big)\arg z \leqslant \arg f_2(z) \leqslant 2\Big(1+\frac{2\Theta}{|z|^2}\Big)\arg z, \quad z \in \gamma_{1,2}. \tag{4.4}$$

From (4.3) and (4.4) it follows that, for $z \in \gamma_{1,2}$, $|\arg f_1(z) - \arg f_2(z)| < \pi$ and therefore

$$-\pi + 2\Big(1 + \frac{2\Theta}{|z|^2}\Big)\arg z \leqslant \arg(f_1(z) + f_2(z)) \leqslant 2\Big(1 + \frac{2\Theta}{|z|^2}\Big)\arg z, \quad z \in \gamma_{1,2}. \tag{4.5}$$

We conclude from (4.5) that the image $\zeta = f(z)$ lies in the domain $D_1 := \{\zeta \in \mathbb{C} : -\pi < \arg \zeta < 3\eta\}$ when $z$ runs through $\gamma_{1,2}$. In addition the point $w$ does not lie in $D_1$ by the choice of the parameter $\eta$. In the same way we deduce that the image $\zeta = f(z)$ lies in $D_2 := \{\zeta \in \mathbb{C} : \pi - 3\eta < \arg \zeta < 2\pi\}$ and $w \notin D_2$ when $z$ runs through $\gamma_{1,4}$.

Hence $f(z)$ winds around $w$ once, and it follows from the argument principle that inside the curve $\gamma_1$ there is a unique point $z_0$ such that $f(z_0) = w$. Since this relation holds for all sufficiently large $R > 0$ and all curves $\gamma_{1,1}$, we deduce that the point $z_0$ is unique in $\mathbb{C}^+$.

Hence the inverse function $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ exists and is analytic in $\mathbb{C}^+$. By the condition (4.1), $\lim_{y\to+\infty}(iy/f^{(-1)}(iy)) = 1$ and therefore $f^{(-1)} \in \mathcal{F}$. This proves the lemma. □

Let $z_1 \in \mathbb{C}^+$ and $z_2 \in \mathbb{C}^+$, and introduce the functions

$$w_1(z_1, z_2) := z_1 + z_2 - F_{\mu_2}(z_2), \qquad w_2(z_1, z_2) := z_1 + z_2 - F_{\mu_1}(z_1).$$

**Lemma 4.2.** *For every $z \in \mathbb{C}^+$ there exist unique points $z_1 \in \mathbb{C}^+$ and $z_2 \in \mathbb{C}^+$ such that*

$$z = w_1(z_1, z_2) \quad \textit{and} \quad z = w_2(z_1, z_2). \tag{4.6}$$

*Proof.* Let us fix $z \in \mathbb{C}^+$. For every $z_2 \in \mathbb{C}^+$ we define $z_1 := z - (z_2 - F_{\mu_2}(z_2))$. Recall that $F_{\mu_j} \in \mathcal{F}$, $j = 1, 2$. Since, by (3.6), $(z_2 - F_{\mu_2}(z_2)) \in \mathbb{C}^- \cup \mathbb{R}$, it follows that $z_1 \in \mathbb{C}^+$. Hence it suffices to prove that the equation

$$F_{\mu_2}(z_2) = F_{\mu_1}\big(z - (z_2 - F_{\mu_2}(z_2))\big) \tag{4.7}$$

has a unique solution $z_2 \in \mathbb{C}^+$. Rewrite (4.7) in the form

$$z = z_2 - g_{\mu_1}\big(z - (z_2 - F_{\mu_2}(z_2))\big),$$

where $g_{\mu_1}(w) := F_{\mu_1}(w) - w$ for $w \in \mathbb{C}^+$.

By (3.3), the functions $g_{\mu_j}(w)$, $j = 1, 2$, admit the representation

$$g_{\mu_j}(w) = a_j + \int_{\mathbb{R}} \frac{1 + uw}{u - w}\,\tau_j(du), \quad w \in \mathbb{C}^+,$$

where $a_j \in \mathbb{R}$ and $\tau_j$ are finite nonnegative measures. Therefore

$$\operatorname{Im} g_{\mu_j}(iy) = y \int_{\mathbb{R}} \frac{1 + u^2}{u^2 + y^2}\,\tau_j(du)$$

and

$$\operatorname{Re} g_{\mu_j}(iy) = a_j + \int_{\mathbb{R}} \frac{u(1 - y^2)}{u^2 + y^2}\,\tau_j(du).$$

For $y \geqslant 1$, we obtain the following estimates

$$\operatorname{Im} g_{\mu_j}(iy) \geqslant \frac{1}{2y} \int\limits_{[-y,y]} (1+u^2)\,\tau_j(du) + \frac{y}{2} \int\limits_{|u|>y} \tau_j(du)$$

and

$$|\operatorname{Re} g_{\mu_j}(iy)| \leqslant |a_j| + \int\limits_{[-y,y]} |u|\,\tau_j(du) + y^2 \int\limits_{|u|>y} \frac{\tau_j(du)}{u}.$$

We conclude from the last two inequalities and Lyapunov's inequality that

$$|\operatorname{Re} g_{\mu_j}(iy)| \leqslant c\big(1 + (y \operatorname{Im} g_{\mu_j}(iy))^{1/2} + \operatorname{Im} g_{\mu_j}(iy)\big), \quad y \geqslant 1, \tag{4.8}$$

where $c$ is a positive constant.

We shall prove that

$$|g_{\mu_1}\big(z - (i \operatorname{Im} z_2 - F_{\mu_2}(i \operatorname{Im} z_2))\big)|/\operatorname{Im} z_2 \to 0 \quad \text{as} \quad \operatorname{Im} z_2 \to +\infty. \tag{4.9}$$

Let to the contrary

$$|g_{\mu_1}\big(z - (iy_k - F_{\mu_2}(iy_k)\big)|/y_k \geqslant c > 0 \tag{4.10}$$

for some sequence $\{y_k\}_{k=1}^\infty$ such that $y_k \to +\infty$ and for a positive constant $c$.

If $\liminf_{y_k\to+\infty} |iy_k - F_{\mu_2}(iy_k)| < \infty$, then there exists a subsequence $\{y_k'\}_{k=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $\lim_{y_k'\to+\infty} |iy_k' - F_{\mu_2}(iy_k')| < \infty$. It is easy to see, $\operatorname{Im}(z - (iy_k' - F_{\mu_2}(iy_k')) \geqslant \operatorname{Im} z$ and $|z - (iy_k' - F_{\mu_2}(iy_k')| \leqslant |z| + c_1$ for all $y_k$ and for some constant $c_1 > 0$. Hence in this case we have $|g_{\mu_1}\big(z - (iy_k' - F_{\mu_2}(iy_k'))\big)| \leqslant c_2$ for all $y_k'$ and for some constant $c_2 > 0$. This estimate contradicts to (4.10).

Let $\liminf_{y_k\to+\infty} |iy_k - F_{\mu_2}(iy_k)| = \infty$ and $\liminf_{y_k\to+\infty} |\operatorname{Re}(iy_k - F_{\mu_2}(iy_k))| = \infty$. We see that

$$|g_{\mu_1}(z + g_{\mu_2}(iy_k))| \leqslant |a_1| + \frac{\tau_1(\mathbb{R})}{\operatorname{Im} z} + |z + g_{\mu_2}(iy_k)| \int\limits_{\mathbb{R}} \frac{|u|\,\tau_1(du)}{|u - z - g_{\mu_2}(iy_k)|}$$

$$\leqslant |a_1| + \frac{\tau_1(\mathbb{R})}{\operatorname{Im} z} + |z + g_{\mu_2}(iy_k)|\big(I_1(iy_k) + I_2(iy_k) + I_3(iy_k)\big),$$

where

$$I_1(iy_k) := \int\limits_{|u|\leqslant x_k/2} \frac{|u|\,\tau_1(du)}{|u - z - g_{\mu_2}(iy_k)|} \leqslant c\frac{|\operatorname{Re} g_{\mu_2}(iy_k)|}{|g_{\mu_2}(iy_k)|} \leqslant c,$$

$$I_2(iy_k) := \int\limits_{x_k/2<|u|\leqslant 2x_k} \frac{|u|\,\tau_1(du)}{|u - z - g_{\mu_2}(iy_k)|} \leqslant c\frac{|\operatorname{Re} g_{\mu_2}(iy_k)|}{\operatorname{Im} z + \operatorname{Im} g_{\mu_2}(iy_k)}\tau_1(\{|u| > x_k/2\}),$$

and

$$I_3(iy_k) := \int\limits_{|u|>2x_k} \frac{|u|\,\tau_1(du)}{|u - z - g_{\mu_2}(iy_k)|} \leqslant c$$

with some positive constant and $x_k := |\operatorname{Re} g_{\mu_2}(iy_k)|$. Using (4.8), we finally obtain

$$|g_{\mu_1}(z + g_{\mu_2}(iy_k))| \leqslant |a_1| + \frac{\tau_1(\mathbb{R})}{\operatorname{Im} z} + cy_k\tau_1(\{|u| > x_k/2\}) + c|z + g_{\mu_2}(iy_k)|.$$

From this estimate it follows immediately that

$$\frac{|g_{\mu_1}(z - (iy_k - F_{\mu_2}(iy_k)))|}{y_k} \to 0 \quad \text{as} \quad y_k \to +\infty,$$

a contradiction with (4.10).

Let $\liminf_{y_k \to +\infty} |iy_k - F_{\mu_2}(iy_k)| = \infty$ and $\liminf_{y_k \to +\infty} |\operatorname{Re}(iy_k - F_{\mu_2}(iy_k))| < \infty$. Without loss of generality we can assume that $\limsup_{y_k \to +\infty} |\operatorname{Re}(iy_k - F_{\mu_2}(iy_k))| < \infty$.

Since $|g_{\mu_1}(z + z_2)|/\operatorname{Im} z_2 \to 0$ as $z_2 \to +\infty$ nontangentially to $\mathbb{R}$ and $|i \operatorname{Im} z_2 - F_{\mu_2}(i \operatorname{Im} z_2)|/\operatorname{Im} z_2 \to 0$ as $\operatorname{Im} z_2 \to +\infty$, we see that

$$\frac{|g_{\mu_1}(z - (iy_k - F_{\mu_2}(iy_k)))|}{|z - (iy_k - F_{\mu_2}(iy_k))|} \frac{|z - (iy_k - F_{\mu_2}(iy_k))|}{y_k} \to 0 \quad \text{as} \quad y_k \to +\infty,$$

a contradiction with (4.10). Hence (4.9) is proved.

Consider the function $\tilde{f}(z_2) := z_2 - \tilde{g}(z_2)$, where $\tilde{g}(z_2) := g_{\mu_1}\big(z - (z_2 - F_{\mu_2}(z_2))\big)$, $z_2 \in \mathbb{C}^+$. By the definition of the function $g_{\mu_1}$, we see that $-\tilde{g} : \mathbb{C}^+ \to \mathbb{C}^-$. By (4.9), the $-\tilde{g}$ satisfies the condition (4.1). Applying Lemma 4.1 to the function $\tilde{f}(z_2)$, we obtain that (4.7) has a unique solution $z_2 \in \mathbb{C}^+$ for every fixed $z \in \mathbb{C}^+$, thus proving the lemma. □

*Proof of Theorem* 2.1. Consider the function $F(z, z_2) := F_{\mu_2}(z_2) - F_{\mu_1}\big(z - (z_2 - F_{\mu_2}(z_2))\big)$ as a function of the two complex variables $z$ and $z_2$. It is analytic on $\mathbb{C}^+ \times \mathbb{C}^+$. By Lemma 4.2, for every fixed $z = z^0 \in \mathbb{C}^+$ the equation (4.7) has an unique solution $z_2^0 \in \mathbb{C}^+$. Hence $F(z^0, z_2^0) = 0$. We shall verify that $F(z^0, z_2) \not\equiv 0$ for $z_2 \in \mathbb{C}^+$. Assume that $F(z^0, z_2) \equiv 0$ holds in $z_2 \in \mathbb{C}^+$. Since $\operatorname{Im} F_{\mu_2}(iy)/y \to 1$ and $\operatorname{Im} F_{\mu_1}(z - (iy - F_{\mu_2}(iy)))/y \to 0$ as $y \to +\infty$, by arguments as in Lemma 4.2, we arrive at a contradiction. Therefore the function $F(z, z_2)$ satisfies the assumption (3.10) of Theorem 3.8 (Weierstrass' preparation theorem) at the point $(z^0, z_2^0)$. Moreover, by this theorem and Lemma 4.2, the function $F(z, z_2)$ admits the representation (3.11) in a neighborhood $|z - z^0| < r'$, $|z_2 - z_2^0| < \rho'$ with the positive integer $k = 1$. Let us show that there exists $0 < r'' \leqslant r'$ such that the equation $F(z, z_2) = 0$ has a unique root $z_2$ in $|z_2 - z_2^0| < \rho'$ for any given $z$ with $|z - z^0| < r''$. Since $F(z, z_2) = 0$, (3.11) implies that

$$P(z, z_2) := A_0(z) + z_2 = 0$$

for $z, z_2$ from the above neighborhood. Here the functions $A_0(z)$ is analytic in the domain $|z - z^0| < r''$ and $A_0(z^0) = -z_2^0$. For every $z$ with $|z - z^0| < r''$ and a sufficiently small $r'' = r''(z^0) \in (0, r')$, this equation has a root $z_2(z; z_2^0) := -A_0(z)$ in $|z_2 - z_2^0| < \rho'$.

Thus, we have proved that for every given point $z^0 \in \mathbb{C}^+$ there exists a neighborhood $|z - z^0| < r''(z^0)$ such that (4.7) has an unique regular solution $z_2 = z_2(z; z^0)$ with values in $\mathbb{C}^+$. Note that for points $z' \in \mathbb{C}^+$ and $z'' \in \mathbb{C}^+$, $z' \neq z''$, we have $z_2(z; z') = z_2(z; z'')$ for all $z$ in $\{|z - z'| < r''(z')\} \cap \{|z - z''| < r''(z'')\}$. By the monodromy theorem (see

[37], v. 3, p. 269, [38], p. 217), there exists a regular function $Z_2(z)$, $z \in \mathbb{C}^+$, such that, for every point $z^0 \in \mathbb{C}^+$, $Z_2(z) = z_2(z; z^0)$ for $|z - z^0| < r''(z^0)$. Therefore $Z_2(z) \in \mathcal{N}$ and it is an unique solution of (4.7).

It is easy to see from (4.7) that $Z_2(z)$ is in the class $\mathcal{F}$. Indeed, we note that the function $F_{\mu_2}(Z_2(z)) - Z_2(z) \in \mathcal{N}$ and, by (3.4), $F_{\mu_2}(Z_2(z)) - Z_2(z) = (b + o(1))z$, where $b \geqslant 0$ is some constant, for $z \in \mathbb{C}^+$ such that $|\operatorname{Re} z| / \operatorname{Im} z$ stays bounded as $|z|$ tends to infinity. Hence the function $z + F_{\mu_2}(Z_2(z)) - Z_2(z) \in \mathcal{N}$ and $|z + F_{\mu_2}(Z_2(z)) - Z_2(z)| \to \infty$ for the same $z$. In addition $|\operatorname{Re}(z + F_{\mu_2}(Z_2(z)) - Z_2(z))| / \operatorname{Im}(z + F_{\mu_2}(Z_2(z)) - Z_2(z))$ remains bounded as $|z|$ tends to infinity. Therefore, by (3.4),

$$F_{\mu_1}(z + F_{\mu_2}(Z_2(z)) - Z_2(z)) = (z + F_{\mu_2}(Z_2(z)) - Z_2(z)(1 + o(1))$$

for the considered $z$. Using this relation, we conclude from (4.7) and (3.4) that

$$Z_2(z) = (1 + o(1))z + o(1)(F_{\mu_2}(Z_2(z)) - Z_2(z)) = (1 + o(1))z$$

for the same $z$. Thus $Z_2 \in \mathcal{F}$. The desired result is proved.

Choosing $Z_1(z) := z - Z_2(z) + F_{\mu_2}(Z_2(z))$, $z \in \mathbb{C}^+$, we see that $Z_1$ and $Z_2$ are unique solutions of (4.6) in the class $\mathcal{F}$. Hence Theorem 2.1 is proved. □

*Proof of Corollary* 2.2. For simplicity we shall prove this corollary in the case $n = 3$. The general case follows by induction.

Denote $\mu_{2,3} := \mu_2 \boxplus \mu_3$. We have, by associativity, $\mu_1 \boxplus \mu_2 \boxplus \mu_3 = \mu_1 \boxplus \mu_{2,3}$. By Theorem 2.1, there exist unique functions $W_1$ and $W_{2,3}$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = W_1(z) + W_{2,3}(z) - F_{\mu_1}(W_1(z)) \quad \text{and} \quad F_{\mu_1}(W_1(z)) = F_{\mu_{2,3}}(W_{2,3}(z)). \quad (4.11)$$

On the other hand, again by Theorem 2.1, there exist unique functions $W_2 \in \mathcal{F}$ and $W_3 \in \mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = W_2(z) + W_3(z) - F_{\mu_2}(W_2(z)) \quad \text{and} \quad F_{\mu_2}(W_2(z)) = F_{\mu_3}(W_3(z)).$$

Hence, replacing $z$ by $W_{2,3}(z)$ in the last equation we get, for all $z \in \mathbb{C}^+$,

$$W_{2,3}(z) = W_2(W_{2,3}(z)) + W_3(W_{2,3}(z)) - F_{\mu_2}(W_2(W_{2,3}(z))) \quad (4.12)$$

and

$$F_{\mu_2}(W_2(W_{2,3}(z))) = F_{\mu_3}(W_3(W_{2,3}(z))). \quad (4.13)$$

Comparing (4.11) and (4.12), (4.13), we obtain the assertion of Corollary 2.2 with $Z_1(z) = W_1(z)$, $Z_2(z) = W_2(W_{2,3}(z))$ and $Z_3(z) = W_3(W_{2,3}(z))$. □

Corollary 2.3 is an obvious consequence of Corollary 2.2. Note that the continuous semigroup version of Corollary 2.3 with $t \geqslant 1$ replacing $n$ is proved using Lemma 4.1 and Lemma 4.2, and repeating the arguments of Theorem 2.1.

## 5. Multiplicative free convolution

1. Consider the case of multiplicative convolution for p-measures of class $\mathcal{M}_+$. In order to prove Theorem 2.4 we need the following two auxiliary results.

Let $\mu_1, \mu_2 \in \mathcal{M}_+$ and $w \in \mathbb{C}^+$. Introduce the functions

$$f_1(z) = w\frac{K_{\mu_1}(z)}{z}, \quad f_2(z) = \frac{K_{\mu_2}(f_1(z))}{f_1(z)}, \quad \text{and} \quad f_3(z) = \frac{z}{f_2(z)}, \quad z \in \mathbb{C}^+.$$

**Lemma 5.1.** *The function $f_3 : \mathbb{C}^+ \to \mathbb{C}$ takes the value $w \in \mathbb{C}^+$ precisely once. Moreover, $f_3$ takes this value in $D_w := \{z \in \mathbb{C} : \arg w \leqslant \arg z < \pi\}$.*

*Proof.* We shall fix $w \in \mathbb{C}^+$. Let $\alpha \in (0, \arg w)$. Denote by $\gamma_2 = \gamma_2(\alpha)$ (see Figure 2) the closed rectifiable curve consisting of the line segment $\gamma_{2,1} : te^{i\alpha}$, $1/R \leqslant t \leqslant R$, connecting $e^{i\alpha}/R$ to $Re^{i\alpha}$, the arc $\gamma_{2,2} : \alpha < \arg z < \pi$ on the circle $|z| = R$ connecting $Re^{i\alpha}$ to $-R$, the line segment $\gamma_{2,3} : -R \leqslant t \leqslant -1/R$ connecting $-R$ to $-1/R$, and the arc $\gamma_{2,4} : \alpha < \arg z < \pi$ on the circle $|z| = 1/R$ connecting $-1/R$ to $e^{i\alpha}/R$. Here the parameter $R > 0$ and will be chosen later sufficiently large. Let $z$ run through $\gamma_2$ in the counter clockwise direction.

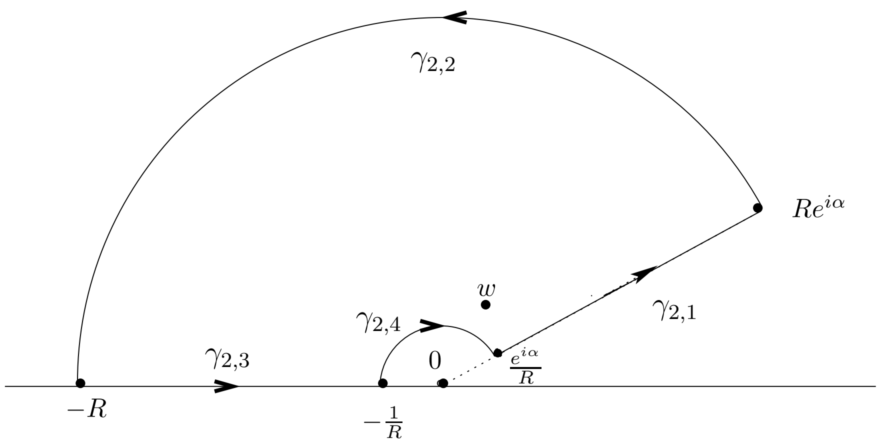


FIGURE 2

Since the function $K_{\mu_1}$ is in the Krein class $\mathcal{K}$, we note that $\arg w \leqslant \arg f_1(z) < \pi$ for all $z \in \mathbb{C}$ with $\arg w \leqslant \arg z \leqslant \pi$. Let $z \in \mathbb{C}$ such that $\arg w < \arg z < \pi + \arg w$. Since $K_{\mu_1}(\bar{z}) = \overline{K_{\mu_1}(z)}$ and $0 \leqslant \arg(K_{\mu_1}(z)/z) < \pi - \arg z$ for $z \in \mathbb{C}^+$, we conclude that $\operatorname{Im} f_1(z) > 0$ on the angular domain $\{z \in \mathbb{C} : \arg w < \arg z < \pi + \arg w\}$. Therefore the function $f_2(ze^{i \arg w})$ is in the class $\mathcal{N}$.

On the other hand we see that $\arg w \leqslant \arg f_1(z) < \pi + \arg w$ for all $z \in \mathbb{C}^+$. Since $K_{\mu_2}(\bar{z}) = \overline{K_{\mu_2}(z)}$ and $0 \leqslant \arg(K_{\mu_2}(z)/z) < \pi - \arg z$ for $z \in \mathbb{C}^+$, we deduce that $e^{i \arg w} f_2(z)$ is in the class $\mathcal{N}$.

Using the representation (3.3) for functions $f \in \mathcal{N}$, we note that if $|f(re^{i\alpha})|/r \to 0$ as $r \to \infty$ for any fixed $\alpha \in (0, \pi)$, then $|f(z)|/r \to 0$ as $r \to \infty$ uniformly in the angle $\delta \leqslant \arg z \leqslant \pi - \delta$ with any fixed $\delta \in (0, \pi)$. From this and from the relations $f_2(ze^{i \arg w}) \in \mathcal{N}$ and $e^{i \arg w} f_2(z) \in \mathcal{N}$ we easily obtain that the estimate

$$\max_{z \in \gamma_{2,2}} \frac{|f_2(z)|}{R} \to 0, \quad R \to \infty, \tag{5.1}$$

follows from the estimate

$$\frac{|f_2(-R)|}{R} \to 0, \quad R \to \infty. \tag{5.2}$$

Let us prove (5.2). At the first step we shall consider the behavior of the function $f_1(-R)$ for $R \geqslant 1$. The functions $K_{\mu_j} \in \mathcal{K}$, $j = 1, 2$, admit (see (2.8)) the representation

$$K_{\mu_j}(z) = a_j z + z g_j(-z) := a_j z + z \int\limits_{(0,\infty)} \frac{\tau_j(du)}{u - z}, \quad z \in \mathbb{C}^+, \tag{5.3}$$

where $a_j \geqslant 0$ and the nonnegative measures $\tau_j$ satisfy the condition (2.9). Hence we have

$$f_1(-R) = w\Big(a_1 + \int\limits_{(0,\infty)} \frac{\tau_1(du)}{u + R}\Big) = w(a_1 + g_1(R)).$$

Note that $g_1(R) \to 0$ as $R \to \infty$ and, if $\tau_1 \not\equiv 0$, then

$$g_1(R) \geqslant \frac{\tau_1((0, R))}{R} \geqslant \frac{c}{R} \tag{5.4}$$

for sufficiently large $R \geqslant 1$. Here and in the sequel we shall denote by $c$ positive constants which do not depend on $R$. In view of (5.3), we obtain the formula

$$f_2(-R) = a_2 + \int\limits_{(0,\infty)} \frac{\tau_2(du)}{u - w(a_1 + g_1(R))} = a_2 + g_2(-w(a_1 + g_1(R))). \tag{5.5}$$

If in (5.5) $a_1 > 0$, then, it is easy to see, that $|f_2(-R)| \leqslant c$ and we arrive at (5.2). Let $a_1 = 0$. In this case we need an upper bound for the function $|g_2(-wg_1(R))|$ for large $R \geqslant 1$.

Since $a_1 = 0$, we conclude $\tau_1 \not\equiv 0$ and write, using (5.4),

$$\begin{aligned} \operatorname{Im} g_2(-wg_1(R)) &= \int\limits_{(0,\infty)} \frac{g_1(R) \operatorname{Im} w}{(u - g_1(R) \operatorname{Re} w)^2 + (g_1(R) \operatorname{Im} w)^2} \tau_2(du) \\ &\leqslant \frac{\tau_2((0, -g_1(R) \log g_1(R)))}{g_1(R) \operatorname{Im} w} + c g_1(R) \int\limits_{[-g_1(R) \log g_1(R),\infty)} \frac{\tau_2(du)}{u^2} \\ &\leqslant c\Big(\frac{\tau_2((0, -g_1(R) \log g_1(R)))}{g_1(R)} + \frac{1}{g_1(R)(\log g_1(R))^2}\Big) = o\Big(\frac{1}{g_1(R)}\Big) = o(R) \end{aligned} \tag{5.6}$$

as $R\to\infty$. In addition we have

$$|\operatorname{Re} g_2(-wg_1(R))| = \Big|\int\limits_{(0,\infty)} \frac{(u-g_1(R)\operatorname{Re} w)}{(u-g_1(R)\operatorname{Re} w)^2+(g_1(R)\operatorname{Im} w)^2}\,\tau_2(du)\Big| \tag{5.7}$$

$$\leqslant c\Big(\frac{\tau_2((0,2|w|g_1(R)))}{g_1(R)} + \int\limits_{[2|w|g_1(R),-g_1(R)\log g_1(R))} \frac{\tau_2(du)}{u}$$

$$+ \int\limits_{[-g_1(R)\log g_1(R),\infty)} \frac{\tau_2(du)}{u}\Big) = o\Big(\frac{1}{g_1(R)}\Big) = o(R), \quad R\to\infty.$$

The estimate (5.2) and, hence (5.1), follows immediately from (5.6) and (5.7).

Now we shall prove that

$$R\max_{z\in\gamma_{2,4}}|f_2(z)|\to\infty, \quad R\to\infty. \tag{5.8}$$

We shall write $f_1(z)$ in the form

$$\begin{aligned} f_1(z) &= w(\operatorname{Re}(K_{\mu_1}(z)/z) + i\operatorname{Im}(K_{\mu_1}(z)/z)) \\ &= w\Big(a_1 + \int\limits_{(0,\infty)} \frac{(u-\operatorname{Re} z)}{(u-\operatorname{Re} z)^2+(\operatorname{Im} z)^2}\,\tau_1(du) \\ &\quad + i\int\limits_{(0,\infty)} \frac{\operatorname{Im} z}{(u-\operatorname{Re} z)^2+(\operatorname{Im} z)^2}\,\tau_1(du)\Big). \end{aligned} \tag{5.9}$$

Let $|z|=1/R$ and $\eta|\operatorname{Re} z|\leqslant \operatorname{Im} z$, where $\eta:=\min\{\tan\alpha, 1/10\}$. For these $z$, using (5.9), we obtain for sufficiently large $R\geqslant 1$ the inequality

$$\Big|\operatorname{Re}\frac{K_{\mu_1}(z)}{z}\Big| \leqslant a_1 + \frac{\tau_1((0,2\operatorname{Im} z/\eta))}{\operatorname{Im} z} + 2\int\limits_{[2\operatorname{Im} z/\eta,\infty)} \frac{\tau_1(du)}{u} = o(R). \tag{5.10}$$

On the other hand the following lower bound holds

$$\Big|\operatorname{Re}\frac{K_{\mu_1}(z)}{z}\Big| \geqslant a_1 + \frac{1}{2}\int\limits_{[2\operatorname{Im} z/\eta,\infty)} \frac{\tau_1(du)}{u} - \frac{\tau_1((0,2\operatorname{Im} z/\eta))}{\operatorname{Im} z}. \tag{5.11}$$

In addition, using (5.9), we deduce, for the same $z$,

$$\frac{\eta^2}{10}q_1(z) \leqslant \operatorname{Im}\frac{K_{\mu_1}(z)}{z} \leqslant 4q_1(z), \tag{5.12}$$

where

$$q_1(z) := \frac{\tau_1((0,2\operatorname{Im} z/\eta))}{\operatorname{Im} z} + \operatorname{Im} z\int\limits_{[2\operatorname{Im} z/\eta,\infty)} \frac{\tau_1(du)}{u^2} = o(R). \tag{5.13}$$

Comparing (5.11) and the left-hand side of the inequality (5.12), we conclude that $|K_{\mu_1}(z)/z| \geqslant c$ for some positive constant $c$ and the $z$ considered above. From (5.10) and the right-hand side of the inequality (5.12) we see that $|f_1(z)| = o(R)$ for those $z$. Moreover by (5.3) and by the definition of $f_1(z)$, it follows that $\arg w \leqslant \arg f_1(z) \leqslant \pi + \arg w$ for $z \in \mathbb{C}^+$. Hence, we get, for $|z| = 1/R, \eta|\operatorname{Re} z| \leqslant \operatorname{Im} z$,

$$c \leqslant |f_1(z)| = o(R) \quad \text{and} \quad \arg w \leqslant \arg f_1(z) \leqslant \pi + \arg w, \tag{5.14}$$

with some positive constant $c$.

Let $|z| = 1/R$ and $\operatorname{Re} z \leqslant 0, -\eta \operatorname{Re} z > \operatorname{Im} z$. Repeating the previous arguments we obtain the estimate (5.14) for such $z$. Thus, (5.14) holds for all $z \in \gamma_{2,4}$.

Using (5.3), it is not difficult to conclude that $|K_{\mu_2}(z)| \geqslant c(\delta_1, \delta_2)$ for $|z| \geqslant \delta_1$ and $\delta_2 \leqslant \arg z \leqslant 2\pi - \delta_2$, where $\delta_1 > 0$, $\delta_2 \in (0, \pi)$ are constants and $c(\delta_1, \delta_2)$ is a positive constant depending on $\delta_1$ and $\delta_2$. Using this estimate, we arrive at the lower bound

$$\Big|\frac{K_{\mu_2}(f_1(z))}{f_1(z)}\Big| \geqslant \frac{c}{|f_1(z)|} \geqslant \frac{N(R)}{R}, \quad z \in \gamma_{2,4},$$

where $N(R) \to \infty$ as $R \to \infty$. The relation (5.8) follows from this bound.

Now we let $z$ run through $\gamma_2$ in the counter clockwise direction.

We see from the arguments at the beginning of the proof of the lemma that $\alpha - \arg w < \arg f_2(z) < \pi - \arg w$ for $z \in \mathbb{C}$ such that $\alpha \leqslant \arg z < \pi$. If $z$ traverses $\gamma_{2,1}$, the image $\zeta = f_3(z)$ lies in the angular region $\arg w + \alpha - \pi < \arg \zeta < \arg w$ in the $\zeta$-plane.

If $z$ traverses $\gamma_{2,2}$, by (5.1), the image $\zeta = f_3(z)$ lies in the domain $|\zeta| > N_1(R)$, $\arg w + \alpha - \pi \leqslant \arg \zeta \leqslant \arg w + \pi - \alpha$, where $N_1(R) \to \infty$ as $R \to \infty$.

If $z$ traverses $\gamma_{2,3}$, the image $\zeta = f_3(z)$ lies in the angular domain $\arg w < \arg \zeta \leqslant \arg w + \pi - \alpha$.

Finally, by (5.8), when $z$ moves in $\gamma_{2,4}$, the image $\zeta = f_3(z)$ lies in the disk $|\zeta| < 1/N_2(R)$, where $N_2(R) \to \infty$ as $R \to \infty$.

In view of these results about the image $\zeta = f_3(z)$, we see that the winding number of $f_3(z)$ around $w$, when $z$ traverses $\gamma_2$ in the counter clockwise direction, is equal to one. Hence, by the argument principle, the function $f_3(z)$ takes the value $w$ precisely once inside $\gamma_2$. Since this assertion holds for all sufficiently large $R > 1$ and all $0 < \alpha < \arg w$, $f_3(z)$ takes the value $w$ precisely once in $\mathbb{C}^+$. In addition, it takes this value in the domain $\arg w \leqslant \arg z < \pi$. This proves the lemma. □

**Lemma 5.2.** *For every $z \in \mathbb{C}^+$ there exist unique points $z_1 \in \mathbb{C}^+$ and $z_2 \in \mathbb{C}^+$ such that $z_1 z^{-1} \in \mathbb{C}^+ \cup (0, \infty)$ and $z_2 z^{-1} \in \mathbb{C}^+ \cup (0, \infty)$, and*

$$z_1 z_2 = z K_{\mu_1}(z_1) \quad \text{and} \quad z_1 z_2 = z K_{\mu_2}(z_2). \tag{5.15}$$

*Proof.* Fix $z \in \mathbb{C}^+$. In view of the second relation of (5.15), we have $z_1 = z K_{\mu_2}(z_2)/z_2$. Since $K_{\mu_2}(z_2)$ belongs to the class $\mathcal{K}$, we obtain $z_1 \in \mathbb{C}^+$ and $z_1 z^{-1} \in \mathbb{C}^+ \cup (0, \infty)$ provided that $z_2 \in \mathbb{C}^+$ and $z_2 z^{-1} \in \mathbb{C}^+ \cup (0, \infty)$. It remains to solve the functional equation

$$K_{\mu_2}(z_2) = K_{\mu_1}\big(z K_{\mu_2}(z_2)/z_2\big). \tag{5.16}$$

Rewrite it in the form $z = z_2/K(z_2;z)$, where

$$K(z_2;z) := K_{\mu_1}\big(zK_{\mu_2}(z_2)/z_2\big)/(zK_{\mu_2}(z_2)/z_2) = K_{\mu_1}(z_1)/z_1. \tag{5.17}$$

We see that, for fixed $z \in \mathbb{C}^+$, the function $\tilde{f}_3(z_2) := z_2/K(z_2;z)$ has the same type as the function $f_3$ in Lemma 5.1. Applying Lemma 5.1 to the function $\tilde{f}_3$, we obtain the assertion of the lemma. □

We need as well the following auxiliary lemma which is an analogue of Lemma 4.1. Denote $\mathbb{S}_\pi := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$.

**Lemma 5.3.** *Assume that $f_4$ has the representation*

$$f_4(z) := -a_1 z + \frac{a_2}{z} + \int\limits_{(0,\infty)} \frac{1+uz}{z-u}\,\sigma(du), \quad 0 < \arg z < 2\pi, \tag{5.18}$$

*where $a_j \geqslant 0$, $j = 1,2$, and $\sigma$ ($\sigma \not\equiv 0$) is a finite nonnegative measure. Then the function $f_5(z) := \log z + f_4(z)$, $f_5 : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{S}_\pi$ precisely once. The inverse function $f_5^{(-1)} : \mathbb{S}_\pi \to \mathbb{C}^+$ thus defined has the property that $f_5^{(-1)}(\log z)$ belongs to the class $\mathcal{K}$.*

*Proof.* We shall prove that for every $w \in \mathbb{S}_\pi$ there exists an unique $z \in \mathbb{C}^+$ such that

$$w = f_5(z) = \log z + f_4(z). \tag{5.19}$$

Recall that we take the principle branch of the logarithm only. Let $\alpha \in (0, \operatorname{Im} w)$ and let $\gamma_2$ be the curve defined in the proof of Lemma 5.1.

Let $z$ traverse $\gamma_{2,3}$. The image $\zeta = f_5(z)$ lies on the line $\operatorname{Im}\zeta = \pi$. Furthermore, we have $\log(-R) + f_4(-R) = \log R + f_4(-R) + i\pi$ with $\log R + f_4(-R) \geqslant (\log R)/2$, and $\log(-1/R) + f_4(-1/R) = -\log R + f_4(-1/R) + i\pi$ with $-\log R + f_4(-1/R) \leqslant -(\log R)/2$.

Let $z$ move in $\gamma_{2,4}$. Since

$$\int\limits_{(0,\infty)} \frac{u|z|}{|z-u|}\,\sigma(du) \leqslant \sigma((0,\infty))|z|d(z), \quad z \in \mathbb{C}^+,$$

where $d(z) := 1$ for $\operatorname{Re} z \leqslant 0$ and $d(z) := |z|/\operatorname{Im} z$, for $\operatorname{Re} z > 0$, we easily conclude, for $z \in \gamma_{2,4}$, $\operatorname{Re} z \leqslant 0$,

$$\operatorname{Re}\Big(f_4(z) + a_1 z - \frac{a_2}{z}\Big) \leqslant \frac{c}{R}, \qquad R \to \infty,$$

with some positive constant $c$, and, for $z \in \gamma_{2,4}$, $\operatorname{Re} z > 0$,

$$\arg\Big(f_4(z)+a_1z-\frac{a_2}{z}\Big) = \arg\Big(-\int\limits_{(0,\infty)} \frac{u\sigma(du)}{|z-u|^2} + \bar{z}\int\limits_{(0,\infty)} \frac{\sigma(du)}{|z-u|^2} + O(1/R)\Big) \leqslant 2\pi - \frac{\alpha}{2}.$$

Hence the image $\zeta = f_5(z)$ either lies in the domain $\{\zeta \in \mathbb{C} : \alpha < \operatorname{Im}\zeta < \pi,\ |\zeta| \geqslant c(\alpha)\log R\}$, where $c(\alpha)$ is a positive constant, or in the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \leqslant \alpha\}$.

Let $z$ traverse $\gamma_{2,1}$. Note that the image $\zeta = f_5(z)$ lies in the half-plane $\operatorname{Im}\zeta \leqslant \alpha$.

Let finally $z$ traverse $\gamma_{2,2}$. Since $\operatorname{Im} f_4(z) \leqslant 0$, we see that the image $\zeta = f_5(z)$ lies in the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \leqslant \alpha\}$ or in the domain $\{\zeta \in \mathbb{C} : \alpha < \operatorname{Im}\zeta < \pi\}$.

Assume that $\zeta = f_5(z)$ lies in $\{\zeta \in \mathbb{C} : \alpha < \operatorname{Im}\zeta < \pi\}$. In this case we easily see that $-\operatorname{Im} f_4(z) \leqslant \pi - \alpha$ and we obtain the inequality

$$-\operatorname{Im} f_4(z) = a_1 \operatorname{Im} z + a_2 \frac{\operatorname{Im} z}{|z|^2} + \operatorname{Im} z \int\limits_{(0,\infty)} \frac{1+u^2}{|z-u|^2}\,\sigma(du) \leqslant \pi - \alpha, \qquad (5.20)$$

On the other hand note that, for $z \in \gamma_{2,2}$,

$$\operatorname{Re} f_4(z) = -a_1 \operatorname{Re} z + \int\limits_{(0,\infty)} \frac{u(|z|^2 - u \operatorname{Re} z)}{|z-u|^2}\,\sigma(du) + O(1), \quad R \to \infty. \qquad (5.21)$$

Since, for $z \in \gamma_{2,2}$, $\big|f_4(z) + a_1 z - a_2/z\big| = o(R)$, we deduce from (5.21) in the case $a_1 \neq 0$ the bound $|\operatorname{Re} f_4(z)| \geqslant cR$. This means that $\zeta = f_5(z)$ lies in the domain $\{\zeta \in \mathbb{C} : \alpha < \operatorname{Im}\zeta < \pi,\ |\zeta| > \frac{1}{2}\log R\}$. If $a_1 = 0$, then we have from (5.20) and (5.21)

$$\begin{aligned}\operatorname{Re} f_4(z) &\geqslant \int\limits_{(0,\infty)} \frac{uR^2}{|z-u|^2}\,\sigma(du) - (1 + sign(\operatorname{Re} z))\frac{\operatorname{Re} z}{\operatorname{Im} z}(\pi - \alpha) + O(1) \\ &\geqslant R^2 \int\limits_{(0,\infty)} \frac{u\sigma(du)}{|z-u|^2} + O(1), \quad R \to \infty.\end{aligned}$$

We conclude again that $\zeta = f_5(z)$ lies in the domain $\{\zeta \in \mathbb{C} : \alpha < \operatorname{Im}\zeta < \pi,\ |\zeta| > \frac{1}{2}\log R\}$.

Hence the image $\zeta = f_5(z)$ for $z \in \gamma_{2,2}$ lies in the half-plane $\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta \leqslant \alpha\}$ or in the domain $\{\zeta \in \mathbb{C} : \alpha < \operatorname{Im}\zeta < \pi,\ |\zeta| > \frac{1}{2}\log R\}$.

Therefore we conclude that the image $\zeta = f_5(z)$ winds around $w$ once when $z$ runs through $\gamma_2$. By the argument principle, there is an unique point $z$ inside $\gamma_2$ such that (5.19) holds. This relation is valid for all sufficiently large $R > 1$ and sufficiently small $\alpha > 0$. Thus for every fixed $w \in \mathbb{S}_\pi$ there is an unique point $z \in \mathbb{C}^+$ such that (5.19) holds. This implies that the inverse function $q = f_5^{(-1)} : \mathbb{S}_\pi \to \mathbb{C}^+$ exists and is analytic on $\mathbb{S}_\pi$.

Let us show that $q(z)$ admits an analytic continuation on the half-line $\gamma_- : \operatorname{Im} z = \pi, \operatorname{Re} z < 0$, and that its value on this half-line is negative. It is easy to see that

$$f_5'(x) = \frac{1}{x} - a_1 - \frac{a_2}{x^2} - \int\limits_{(0,\infty)} \frac{(1+u^2)\sigma(du)}{(x-u)^2} < 0, \quad x < 0.$$

Since $f_5(z)$ is analytic on $(-\infty, 0)$, we conclude that $f_5^{(-1)}$ exists and is analytic on $\gamma_-$ as well. Since, as shown above, for every fixed $w \in \mathbb{S}_\pi$ there is an unique point $z \in \mathbb{C}^+$ such that (5.19) holds, this function coincides for $z \in \mathbb{C}^+$ with the function $q(z)$ obtained early. Introduce the function $f_6(z) := q(\log z)$, $z \in \mathbb{C}^+$. Note that $f_6 \in \mathcal{N}$ and

$f_6^{(-1)}(z) = z\exp\{f_4(z)\}$, on the domain $\mathbb{C}^+$, where $f_6^{(-1)}$ exists. Moreover, the function $f_6(z)$ admits an analytic continuation on $(-\infty, 0)$.

From the definition of $f_6(z)$ it follows that $f_6(x) < 0$ for $x < 0$ and $f_6(x) \to 0$ as $x \uparrow 0$. By Corollary 3.3, $f_6$ belongs to the class $\mathcal{K}$ and we obtain the assertion of the lemma. $\square$

*Proof of Theorem* 2.4. Consider the function

$$F(z, z_2) := K_{\mu_2}(z_2) - K_{\mu_1}\big(zK_{\mu_2}(z_2)/z_2\big)$$

which, by (2.8), is analytic on $\mathbb{C}^+ \times \mathbb{C}^+$. By Lemma 5.2, for every fixed $z = z^0 \in \mathbb{C}^+$ the equation (5.16) has an unique solution $z_2 = z_2^0 \in \{z_2 \in \mathbb{C} : \arg z^0 \leqslant \arg z_2 < \pi\}$. Let us show that the function $F(z, z_2)$ satifies (3.10) at the point $(z^0, z_2^0)$. Note that if $K_{\mu_2}(z_2) = K_{\mu_1}(z^0 K_{\mu_2}(z_2)/z_2)$ holds for $z_2 \in \mathbb{C}^+$ such that $|z_2 - z_2^0| < r'$ with $r' > 0$, then this equality holds for all $z_2 \in \mathbb{C}^+$. By (5.2), the function $K(-r; z^0)$ introduced in (5.17) has the property $K(-r; z^0)/r \to 0$ as $r \to \infty$. Since the relation $z^0 = -r/R(-r; z^0)$ holds for $r > 0$, we arrive at a contradiction. Hence the point $(z^0, z_2^0)$ satisfies the assumptions (3.10) of Theorem 3.7. Now repeating almost word for word the arguments of the proof of Theorem 2.1, using Lemma 5.2 instead of Lemma 4.2, and Theorem 3.1, Corollary 3.3, we obtain the assertion of the theorem. $\square$

We prove Corollaries 2.5 and 2.6 in the same way as Corollaries 2.2 and 2.3.

Rewrite the relation (2.13) with $n - 1 = t \in [0, \infty)$ in the form

$$Z(z) = z\Big(\frac{K_\mu(Z(z))}{Z(z)}\Big)^t, \quad z \in \mathbb{C}^+, \quad t \geqslant 0. \tag{5.22}$$

Consider the function $F(z) := K_\mu(z)/z$, $z \in \mathbb{C}^+$. This function belongs to the class $\mathcal{N}$, is analytic and nonnegative on the negative real axis $(-\infty, 0)$. By Theorem 3.4, $F(z)$ admits the representation (5.18) with $a_1 = a_2 = 0$. By Lemma 5.3, there exists a function $Z(z) \in \mathcal{K}$ such that (5.22) holds.

The relation (5.22) implies that there exists a semigroup $\mu_t \in \mathcal{M}_+$, $t \geqslant 1$, such that $(\Sigma_\mu(z))^t = \Sigma_{\mu_t}(z)$, $t \geqslant 1$.

2. Consider the case of multiplicative convolution for p-measures of the class $\mathcal{M}_*$. In order to prove Theorem 1.7 we need some auxiliary results.

**Lemma 5.4.** *Let* $Q \in \mathcal{S}$ *and* $Q(0) \neq 0$. *Then the function* $g : \mathbb{D}\backslash\{0\} \to \mathbb{C} : z \mapsto Q(z)/z$ *takes every value in* $\mathbb{C} \setminus \overline{\mathbb{D}}$ *precisely once.*

*Proof.* Without loss of generality we assume that $Q(0) > 0$. Let $w \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and let $r \in (0, 1)$. We assume that $1 - r$ is sufficiently small. It is easy to see that there exist $\theta = \theta(w, r) \in [0, 2\pi)$ and $\theta \neq \arg w$ such that $Q(z)/z \neq w$ for $z = \rho e^{i\theta}$, $1 - r \leqslant \rho \leqslant r$. Let $\gamma_3 = \gamma_3(w, \theta)$ (see Figure 3) be the closed rectifiable curve consisting of the two circles $\gamma_{3,2} : |z| = 1 - r$ and $\gamma_{3,4} : |z| = r$, and the segments $\gamma_{3,1}, \gamma_{3,3}$ which coincide with the segment $\gamma : te^{i\theta}, 1 - r \leqslant t \leqslant r$, traversed twice in opposite directions.

We note that the number of loops around $w$ by the map $z \mapsto Q(z)/z$ as $z$ traverses the curve $\gamma_3$ once in the counter clockwise direction is equal to 1. Indeed, if $z$ traverses

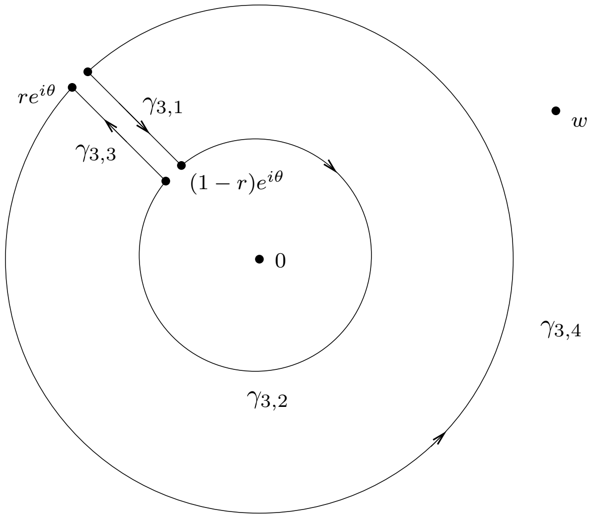


FIGURE 3

$\gamma_{3,1}$ the number of windings around $w$ is equal to some number $\alpha$. If $z$ traverses $\gamma_{3,4}$ the number of windings around $w$ is equal to 0. Finally, moving along $\gamma_{3,3}$, the number of windings around $w$ is equal to $-\alpha$. Hence traversing $\gamma_{3,1}$, $\gamma_{3,4}$ and $\gamma_{3,3}$ in the counter clockwise direction the number of windings around $w$ is equal to 0. Traversing $\gamma_{3,2}$, we wind around $w$ once. Hence, by the argument principle, the function $Q(z)/z$ takes the value $w$ in the domain $\{z : 1-r < |z| < r\}$ precisely once. Since this assertion holds for all $0 < r < 1$, $Q(z)/z$ takes every value $w \in \mathbb{C} \setminus \overline{\mathbb{D}}$ in $\mathbb{D}$ precisely once. □

Let $\mu_1$ and $\mu_2$ belong to the class $\mathcal{M}_*$. Denote $a_j := \int_{\mathbb{T}} \xi\, \mu_j(d\xi)$, $j = 1, 2$. Recall that $Q_{\mu_j}(0) = 0$, $a_j \neq 0$, $j = 1, 2$.

**Lemma 5.5.** *For every $z \in \mathbb{D} \setminus \{0\}$, there exist unique points $z_1 \in \mathbb{D}$ and $z_2 \in \mathbb{D}$ such that $(z_1, z_2) \neq (0, 0)$ and*

$$z_2 = z\frac{Q_{\mu_1}(z_1)}{z_1} \quad \textit{and} \quad z_1 = z\frac{Q_{\mu_2}(z_2)}{z_2}. \tag{5.23}$$

*Proof.* Fix $z \in \mathbb{D} \setminus \{0\}$. Assume for definiteness $|a_2| \leqslant |a_1|$. It follows from Proposition 3.6 that $Q_{\mu_1}(a_2 z) \neq 0$ for every $z \in \mathbb{D}$.

By Schwarz's lemma, $\big|Q_{\mu_2}(z_2)/z_2\big| \leqslant 1$, $z_2 \in \mathbb{D}$. The second relation of (5.23) implies that if $z_2 \in \mathbb{D}$, then $z_1 \in \mathbb{D}$ as well. Hence, by the first relation of (5.23), we need to solve the functional equation

$$Q_{\mu_2}(z_2) = Q_{\mu_1}\big(zQ_{\mu_2}(z_2)/z_2\big), \quad z_2 \in \mathbb{D}. \tag{5.24}$$

Rewrite this relation in the form $z_2/z = Q(z_2)$, where

$$Q(z_2) = Q_{\mu_1}\big(zQ_{\mu_2}(z_2)/z_2\big)/(zQ_{\mu_2}(z_2)/z_2).$$

The function $Q$ satisfies $Q(0) = Q_{\mu_1}(a_2 z)/(a_2 z) \neq 0$ and $Q \in \mathcal{S}$. By Lemma 5.4, we obtain the assertion of the lemma for all $z \in \mathbb{D} \setminus \{0\}$. □

We need as well the following auxiliary lemma which is an analogue of Lemma 4.1 for the semigroup $(\mathcal{M}_*, \boxtimes)$.

**Lemma 5.6.** *Define*

$$Q_1(z) := \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\,\sigma(du), \quad z \in \mathbb{D}, \tag{5.25}$$

*where $\sigma$ ($\sigma \not\equiv 0$) is finite nonnegative measure. Then the function $Q_2 : \mathbb{D} \to \mathbb{C}$, $Q_2(z) := z\exp\{Q_1(z)\}$, takes every value in $\mathbb{D}$ precisely once. The inverse $Q_2^{(-1)} : \mathbb{D} \to \mathbb{D}$ thus defined is in the class $\mathcal{S}_*$.*

*Proof.* Let $w \in \mathbb{D}$. Introduce the curve $\gamma_4 : |z| = r$, where $r \in (0,1)$ and $1 - r$ is sufficiently small. Note that $\operatorname{Im} Q_1(0) = 0$. Then, by the minimum and maximum principles for harmonic functions, for every $r \in (0,1)$ there exist points $\theta'$ and $\theta'' \in [0, 2\pi)$ such that $\operatorname{Im} Q_1(re^{i\theta'}) < 0$ and $\operatorname{Im} Q_1(re^{i\theta''}) > 0$. Since for every fixed $r \in (0,1)$ $\theta \mapsto \operatorname{Im} Q_1(re^{i\theta})$ is analytic on the segment $[0, 2\pi]$, there are only a finite number of points $\{re^{i\theta_l}\}_{l=1}^k$ with $0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_k < 2\pi$, such that $\operatorname{Im} Q_1(re^{i\theta_l}) = 0$, $l = 1, 2, \dots, k$. Consider the arcs $\gamma_{4,l} : \theta_l < \arg z \leqslant \theta_{l+1}$, $|z| = r$, $l = 1, \dots, k$, with $\theta_{k+1} = \theta_1 + 2\pi$.

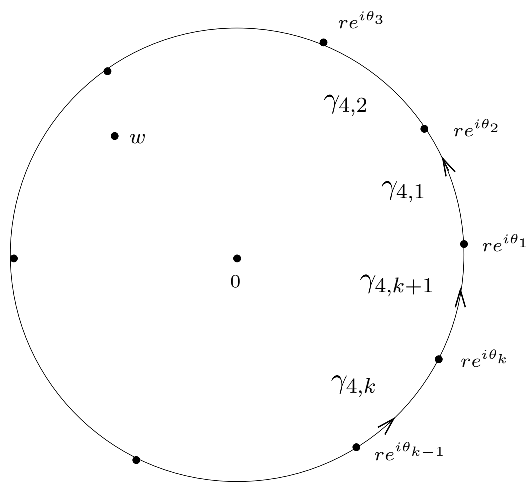


Figure 4

Let $z$ run transverse $\gamma_{4,l}$ in the counter clockwise direction. Since $|\exp\{Q_1(z)\}| \geqslant 1$, we note that the change in $Arg(Q_2(z) - w)$ is equals to

$$(\arg(re^{i\theta_{l+1}} - w) - \arg(re^{i\theta_l} - w))/(2\pi).$$

Hence the image $\zeta = Q_2(z)$ winds around $w$ once when $z$ transverses $\gamma_4$ once in the counter clockwise direction. By the argument principle, the function $Q_2 : \mathbb{D} \to \mathbb{C}$ takes the value $w \in \mathbb{D}$ in $\mathbb{D}$ precisely once. The inverse function $Q_2^{(-1)} : \mathbb{D} \to \mathbb{D}$ thus defined is analytic on $\mathbb{D}$ and obviously belongs to the class $\mathcal{S}_*$. The lemma is proved. $\square$

*Proof of Theorem* 2.7. We assume for definiteness that $|a_2| \leqslant |a_1|$. Consider the function $F(z, z_2) := Q_{\mu_2}(z_2) - Q_{\mu_1}\big(zQ_{\mu_2}(z_2)/z_2\big)$ which is analytic on $\mathbb{D} \times \mathbb{D}$. For a fixed $z = z^0 \in \mathbb{D}$ the equation (5.24) has, by Lemma 5.5, a unique solution $z_2 = z_2^0 \in \mathbb{D}$. Moreover, if $F(z^0, z_2) = 0$ for $z_2$ in some neighborhood of the point $z_2^0$, then $F(z^0, z_2) = 0$ for all $z_2 \in \mathbb{D}$. This relation is equivalent to $z_2/z^0 = Q(z_2)$, $z_2 \in \mathbb{D}$, where $Q \in \mathcal{S}$. Choosing here $|z_2| > |z^0|$ we arrive at a contradiction. Hence the function $F(z, z_2)$ satisfies the assumptions (3.10) of Theorem 3.7 at the point $(z^0, z_2^0)$. Repeating almost word for word the arguments of the proof of Theorem 2.1, using Lemma 5.5 instead of Lemma 4.2, we obtain that there exists a neighborhood $|z - z^0| < r(z^0)$ such that the equation $F(z, z_2) = 0$ has a unique analytic solution $z_2 = z_2(z; z^0)$ with values in $\mathbb{D}$. Choosing $z_1(z; z^0) := zQ_{\mu_2}(z_2(z; z^0))/z_2(z; z^0)$, there are two analytic functions $z_1(z; z^0)$ and $z_2(z; z^0)$ with values in $\mathbb{D}$ which satisfy the equations

$$z_1(z; z^0)z_2(z; z^0) = zQ_{\mu_1}(z_1(z; z^0)) \quad \text{and} \quad z_1(z; z^0)z_2(z; z^0) = zQ_{\mu_2}(z_2(z; z^0)).$$

By Lemma 5.5, for the points $z' \in \mathbb{D}$ and $z'' \in \mathbb{D}$, $z' \neq z''$, we obtain the identities $z_1(z; z') = z_1(z; z'')$, $z_2(z; z') = z_2(z; z'')$ for $z$ of the domain $\{|z - z'| < r(z')\} \cap \{|z - z''| < r(z'')\}$. By the monodromy theorem, there exist analytic functions $Z_1(z)$ and $Z_2(z)$, $z \in \mathbb{D}$, such that, for every point $z^0 \in \mathbb{D}$, $Z_j(z) = z_j(z; z^0)$, $j = 1, 2$, for $|z - z^0| < r(z^0)$. These functions belong to the class $\mathcal{S}$ and are unique solutions of equations (2.15) for $\mu_j$, $j = 1, 2$. Moreover, it is easy to see that $Z_1, Z_2 \in \mathcal{S}_*$. This proves the theorem. □

We prove Corollaries 2.8 and 2.9 in the same way as Corollaries 2.2 and 2.3.

Now we prove the relation (2.18) with $n = t \in [1, \infty)$. In the first step we note from (2.18) that $Q_{\mu\boxtimes\mu}(z)/z \neq 0$, $z \in \mathbb{D}$. Since this function belongs to the class $\mathcal{S}$, it has the form

$$\frac{Q_{\mu\boxtimes\mu}(z)}{z} = \exp\Big\{ai - \int\limits_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\,\sigma(d\zeta)\Big\}, \quad z \in \mathbb{D},$$

where $a \in \mathbb{R}$ and $\sigma$ is a finite nonnegative measure. By Lemma 5.6, we conclude that there exists $Z(z) \in \mathcal{S}_*$ such that

$$Z(z) = z\Big(\frac{Q_{\mu\boxtimes\mu}(Z(z))}{Z(z)}\Big)^t, \quad z \in \mathbb{D}. \quad t \geqslant 0, \tag{5.26}$$

We obtain from (5.26) the existence a semigroup $\mu_u \in \mathcal{M}_*$, $u \geqslant 2$, such that

$$\big(\Sigma_\mu(z)\big)^u = \Sigma_{\mu_u}(z), \quad u \geqslant 2. \tag{5.27}$$

If $Q_\mu(z)/z \neq 0$, $z \in \mathbb{D}$, then, as before, the following relation holds

$$Z(z) = z\Big(\frac{Q_\mu(Z(z))}{Z(z)}\Big)^t, \quad z \in \mathbb{D}, \quad t \geqslant 0, \tag{5.28}$$

and (5.27) is true for $u \geqslant 1$.

## 6. Basic arithmetic and Khintchine's limit theorems for the semigroups $(\mathbf{M}, \circ)$

In this section we shall prove Theorems 2.10, 2.11 and 2.12.

Introduce sets $\mathbf{M}^\gamma$ of measures of $\mathbf{M}$ in the following way.

Consider the semigroup $(\mathcal{M}, \boxplus)$. Let $\overline{\Gamma}_{\alpha,\beta}$ denote the closure of $\Gamma_{\alpha,\beta}$. Denote by $\mathcal{M}^{(\alpha,\beta)}$ the set of those $\mu \in \mathcal{M}$ such that $F_\mu(z)$ is univalent on $\Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^+$ is a domain, for which $F_\mu(\Omega_\mu) \supset \overline{\Gamma}_{\alpha,\beta}$. The function $F_\mu(z)$ has an inverse $F_\mu^{(-1)}(z)$ defined on $\overline{\Gamma}_{\alpha,\beta}$ such that $\operatorname{Im} \varphi_\mu(z) \leqslant 0$. We denote $\mathbf{M}^\gamma := \mathcal{M}^{(\alpha,\beta)}$, where $\gamma := (\alpha, \beta) \in \mathcal{G} := \{(\alpha, \beta) : \alpha > 0, \beta > 0\}$, a subset of the semigroup $(\mathcal{M}, \boxplus)$.

Consider $(\mathcal{M}_+, \boxtimes)$ and let $\overline{\Gamma}^+_{\alpha,\beta,\Delta}$ denote the closure of $\Gamma^+_{\alpha,\beta,\Delta}$. Recall that $\Gamma^+_{\alpha,\beta,\Delta} := \{z \in \mathbb{C} : \beta < |z| < \Delta,\ \alpha < \arg z < 2\pi - \alpha\}$ for some $0 < \beta < \Delta$ and $\alpha \in (0, \pi)$. Denote by $\mathcal{M}_+^{(\alpha,\beta,\Delta)}$ a subset of the set of those $\mu \in \mathcal{M}_+$ such that $K_\mu(z)$ is univalent on $\Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C} \setminus [0, +\infty)$ is a domain, with $K_\mu(\Omega_\mu) \supset \overline{\Gamma}^+_{\alpha,\beta,\Delta}$. The function $K_\mu(z)$ has an inverse $K_\mu^{(-1)}(z)$ defined on $\overline{\Gamma}^+_{\alpha,\beta,\Delta}$ such that $\arg \Sigma_\mu(z) \leqslant 0$ for $z \in \overline{\Gamma}^+_{\alpha,\beta,\Delta} \cap \mathbb{C}^+$ and $\arg \Sigma_\mu(z) \geqslant 0$ for $z \in \overline{\Gamma}^+_{\alpha,\beta} \cap \mathbb{C}^-$. As above we write $\mathbf{M}^\gamma := \mathcal{M}_+^{(\alpha,\beta,\Delta)}$, where $\gamma := (\alpha, \beta, \Delta) \in \mathcal{G} := \{(\alpha, \beta, \Delta) : \alpha \in (0, \pi), 0 < \beta < \Delta < \infty\}$, for a subset of the semigroup $(\mathcal{M}_+, \boxtimes)$.

Consider $(\mathcal{M}_*, \boxtimes)$ and let $\overline{\mathbb{D}}_\alpha$ denote the closure of $\mathbb{D}_\alpha := \{z \in \mathbb{C} : |z| < \alpha\}$. Denote by $\mathcal{M}_*^\alpha$ the set of those $\mu \in \mathcal{M}_*$ such that $Q_\mu(z)$ is univalent on $\Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{D}$ is a domain, with $Q_\mu(\Omega_\mu) \supset \overline{\mathbb{D}}_\alpha$. The function $Q_\mu(z)$ has an inverse $Q_\mu^{(-1)}(z)$ defined on $\overline{\mathbb{D}}_\alpha$ such that $|\Sigma_\mu(z)| \geqslant 1$. Introduce $\mathbf{M}^\gamma := \mathcal{M}_*^\alpha$, where $\gamma := \alpha \in \mathcal{G} := \{\alpha : 0 < \alpha < 1\}$, for a subset of the semigroup $(\mathcal{M}_*, \boxtimes)$. Note that $\mathbf{M} = \cup_{\gamma \in \mathcal{G}} \mathbf{M}^\gamma$.

A next proposition shows that the set $\mathbf{M}^\gamma$ is hereditary in $(\mathbf{M}, \circ)$.

**Proposition 6.1.** *Given $\gamma \in \mathcal{G}$, let $\mu \in \mathbf{M}^\gamma$ and $\mu = \mu_1 \circ \mu_2$, where $\mu_j \in \mathbf{M}$, $j = 1, 2$. Then $\mu_j \in \mathbf{M}^\gamma$, $j = 1, 2$.*

*Proof.* At first we shall prove the assertion of the proposition for $(\mathcal{M}, \boxplus)$. Let $\mu \in \mathcal{M}^{(\alpha,\beta)}$ and $\mu = \mu_1 \boxplus \mu_2$, where $\mu_1, \mu_2 \in \mathcal{M}$. We should show that $\mu_1, \mu_2 \in \mathcal{M}^{(\alpha,\beta)}$. By Theorem 2.1, there exist functions $Z_1, Z_2 \in \mathcal{F}$ such that

$$F_\mu(z) = F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)), \quad z \in \mathbb{C}^+. \tag{6.1}$$

By the definition of $\mathcal{M}^{(\alpha,\beta)}$, there exists a domain $\Omega_\mu$ such that $F_{\mu_j}(Z_j(z))$, $j = 1, 2$, are univalent on $\Omega_\mu$ and $F_{\mu_j}(Z_j(\Omega_\mu)) \supset \overline{\Gamma}_{\alpha,\beta}$, respectively. Consider the domains $\Omega_{\mu_j} := Z_j(\Omega_\mu)$, $j = 1, 2$. It is clear that $Z_j(z)$, $j = 1, 2$, are univalent on $\Omega_\mu$ and $F_{\mu_j}(z)$, $j = 1, 2$, are univalent on $\Omega_{\mu_j}$, $j = 1, 2$, respectively, thus proving the assertion.

Consider $(\mathcal{M}_+, \boxtimes)$. Let $\mu \in \mathcal{M}_+^{(\alpha,\beta,\Delta)}$ and $\mu = \mu_1 \boxtimes \mu_2$, where $\mu_j \in \mathcal{M}_+$, $j = 1, 2$. Let us show that $\mu_1, \mu_2 \in \mathcal{M}_+^{(\alpha,\beta,\Delta)}$. By Theorem 1.4, there exist functions $Z_1, Z_2 \in \mathcal{K}$ such that

$$K_\mu(z) = K_{\mu_1}(Z_1(z)) = K_{\mu_2}(Z_2(z)), \quad z \in \mathbb{C}^+. \tag{6.2}$$

By the definition of $\mathcal{M}^{(\alpha,\beta,\Delta)}$, there exists a domain $\Omega_\mu$ such that $K_{\mu_j}(Z_j(z))$ are univalent on $\Omega_\mu$ and $K_{\mu_j}(Z_j(\Omega_\mu)) \supset \overline{\Gamma}_{\alpha,\beta,\Delta}$, $j = 1, 2$, respectively. Denote $\Omega_{\mu_j} := Z_j(\Omega_\mu)$, $j = 1, 2$. We see that $Z_j(z)$, $j = 1, 2$, are univalent on $\Omega_\mu$ and $K_{\mu_j}(z)$, $j = 1, 2$, are univalent on $\Omega_{\mu_j}$, respectively, thus proving the assertion.

Repeating the previous arguments, using Theorem 2.7 instead of Theorem 2.4, we obtain the assertion of the proposition for $\mathcal{M}_*^\alpha$ as well which completes the proof of Proposition 6.1. □

Define an equivalence relation $E$ on $\mathbf{M}$ via $\mu E \nu$ if $\mu = \nu \circ \delta_a$, where $\delta_a$ is Dirac measure with the support at the point $a$. Here $a$ denotes a real number for $(\mathcal{M}, \boxplus)$, is a positive number for $(\mathcal{M}_+, \boxtimes)$ and $a \in \mathbb{T}$ for $(\mathcal{M}_*, \boxtimes)$. It is clear that $E$ is an equivalence relation on the set $\mathbf{M}^\gamma$ as well. We shall denote the equivalence class of $\mu \in \mathbf{M}$ as $E\mu$, and the element of $\mathbf{M}/E$ by $\mu^*, \nu^*$ etc. Note that $\mathbf{M}/E$ is a semigroup under $\circ$ multiplication of representatives. This semigroup has an identity element $e := \delta_a^*$, where $a = 0, 1, 1$ for the semigroups $(\mathcal{M}, \boxplus)$, $\mathcal{M}_+, \boxtimes)$, $(\mathcal{M}_*, \boxtimes)$, respectively. In the semigroup $(\mathbf{M}/E, \circ)$ an element $\mu^*$ is indecomposable if $\mu^* = E\mu$, where $\mu$ is indecomposable in $(\mathbf{M}, \circ)$.

For $\mu^*, \nu^* \in \mathbf{M}/E$ define a distance $L^*(\mu^*, \nu^*)$ via

$$L^*(\mu^*, \nu^*) = inf\{L(\mu, \nu) : E\mu = \mu^*,\ E\nu = \nu^*\},$$

where $L(\mu, \nu)$ denotes the Lévy metric. It is easy to check that $L^*(\mu^*, \nu^*)$ is a metric, and that $\{\mu_n^*\}$ converges to $\mu^* \in \mathbf{M}/E$ with respect to $L^*$ if and only if there is $\{\nu_n\}$ with $\nu_n \in \mathbf{M}$ and $\mu_n^* = E\nu_n$ for each $n$, and $\nu \in \mathbf{M}$ with $\mu^* = E\nu$, such that $\{\nu_n\}$ converges to $\nu$ with respect to $L$.

By $\mathbf{S}$ we denote the set of sequences of elements of $\mathbf{M}$ with weak limits in $\mathbf{M}$, and $l$ denotes the operation of taking the weak limit. We denote by $\mathbf{S}/E$ the set of sequences of elements of $\mathbf{M}/E$ that admit limits with respect to $L^*$ and by $l$ again the operation of taking the limit.

Define a homomorphism $D^\gamma$ of $\mathbf{M}^\gamma/E$ to non-negative real numbers with addition in the following way.

For the semigroup $(\mathcal{M}, \boxplus)$:
$D^\gamma(\mu^*) := -\operatorname{Im} \varphi_\nu(i(\beta + 1))$, where $\nu = E\mu$ and $\nu \in \mathcal{M}^{(\alpha,\beta)}$.
For the semigroup $(\mathcal{M}_+, \boxtimes)$:
$D^\gamma(\mu^*) := -\operatorname{Im} \log \Sigma_\nu((\beta + \Delta)e^{i(\pi+\alpha)/2}/2)$, where $\nu = E\mu$ and $\nu \in \mathcal{M}_+^{(\alpha,\beta,\Delta)}$.
For the semigroup $(\mathcal{M}_*, \boxtimes)$:
$D^\gamma(\mu^*) := \operatorname{Re} \log \Sigma_\nu(\alpha/2)$, where $\nu = E\mu$ and $\nu \in \mathcal{M}_*^\alpha$.
Note that $D^\gamma$ is finite-valued and does not depend on the particular $\nu$.

**Proposition 6.2.** *Let $\mu \in \mathbf{M}^\gamma$. If the relation $D^\gamma(\mu^*) = 0$ holds, then $\mu^* = e$.*

*Proof.* We shall prove the proposition for the semigroup $(\mathcal{M}, \boxplus)$. Let $D^{(\alpha,\beta)}(\mu^*) = 0$. Then $= -\operatorname{Im} \varphi_\mu(i(\beta + 1)) = 0$, where $\mu^* = E\mu$. Since the function $-\operatorname{Im} \varphi_\mu(z)$ is harmonic and nonnegative in $\Gamma_{\alpha,\beta}$ and the point $i(\beta + 1)$ lies in $\Gamma_{\alpha,\beta}$, we conclude, by minimum principle for harmonic functions, that $\operatorname{Im} \varphi_\mu(z) = 0$ as $z \in \Gamma_{\alpha,\beta}$. From this it follows that there exists a real number $a$ such that $\varphi_\mu(z) = a$ as $z \in \Gamma_{\alpha,\beta}$ and we have $F_\mu(z) = z - a$, $z \in \mathbb{C}^+$. Hence $\mu = \delta_a$ and the assertion is proved.

For the other semigroups $(\mathcal{M}_+, \boxtimes)$ and $(\mathcal{M}_*, \boxtimes)$ the proof of the proposition is similar. $\square$

**Proposition 6.3.** *Let $\mu$ and $\mu_n$, $n = 1, \dots,$ belong to the set $\mathbf{M}^\gamma$ for a fixed $\gamma \in \mathcal{G}$. Let $\{\mu_n^*\} \in \mathbf{S}/E$ and $l\{\mu_n^*\} = \mu^*$. Then $D^\gamma(\mu^*) = \lim_{n\to\infty} D^\gamma(\mu_n^*)$.*

We omit the proof of this simple proposition.

**Proposition 6.4.** *The operation $\circ$ is continuous with respect to Lévy's metric.*

This result is due to Bercovici and Voiculescu [11], [12]. For convenience of the reader we include a proof.

*Proof of Proposition* 6.4. Let $\{\mu_n\}$ and $\{\nu_n\} \in \mathbf{S}$. Let us prove that $\{\mu_n \boxplus \nu_n\} \in \mathbf{S}$ and $l\{\mu_n \boxplus \nu_n\} = l\{\mu_n\} \boxplus l\{\nu_n\}$.

At first consider the semigroup $(\mathcal{M}, \boxplus)$. By Proposition 3.13, the sequences $\{\varphi_{\mu_n}\}$ and $\{\varphi_{\nu_n}\}$ converge uniformly on compact subsets of some $\Gamma_{\alpha,\beta}$ to the functions $\varphi_\mu$ and $\varphi_\nu$, and $\varphi_{\mu_n}(iy) = o(y)$, $\varphi_{\nu_n}(iy) = o(y)$ uniformly in $n$ as $y \to +\infty$. Since $\varphi_{\mu_n \boxplus \nu_n}(z) = \varphi_{\mu_n}(z) + \varphi_{\nu_n}(z)$ and $\varphi_{\mu \boxplus \nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$ for $z \in \Gamma_{\alpha,\beta}$, note that $\{\varphi_{\mu_n \boxplus \nu_n}\}$ converges uniformly on compact subsets of some $\Gamma_{\alpha,\beta}$ to the function $\varphi_{\mu \boxplus \nu}$. Furthemore, $\varphi_{\mu_n \boxplus \nu_n}(iy) = o(y)$ uniformly in $n$ as $y \to +\infty$. The assertion for $(\mathcal{M}, \boxplus)$ now follows immediately from Proposition 3.13.

Consider the semigroup $(\mathcal{M}_+, \boxtimes)$. Assume that the sequences $\{\mu_n\}$ and $\{\nu_n\}$ converge in the weak topology to measures $\mu \neq \delta_0$ and $\nu \neq \delta_0$, respectively. Since $\Sigma_{\mu_n \boxtimes \nu_n} = \Sigma_{\mu_n}\Sigma_{\nu_n}$ on the set, where these functions are defined, we easily conclude from Proposition 3.14 that the sequence $\{\mu_n \boxtimes \nu_n\}$ converges as well to a p-measure which is not the Dirac measure $\delta_0$. Indeed, by Proposition 3.14, there exist numbers $\alpha \in (0, \pi)$ and $0 < \beta < \Delta$ such that the sequences $\{\Sigma_{\mu_n}\}$ and $\{\Sigma_{\nu_n}\}$ converge uniformly on $\Gamma^+_{\alpha,\beta,\Delta}$ to the functions $\Sigma_\mu$ and $\Sigma_\nu$, respectively. Hence, $\{\Sigma_{\mu_n}\Sigma_{\nu_n}\}$ converges uniformly on $\Gamma^+_{\alpha,\beta,\Delta}$ to the function $\Sigma_\mu\Sigma_\nu$ and, by Proposition 3.14, we conclude that $\{\mu_n \boxtimes \nu_n\}$ converges weakly to $\mu \boxtimes \nu \neq \delta_0$, which is the desired result.

It remains to consider the semigroup $(\mathcal{M}_*, \boxtimes)$. By Proposition 3.15, the sequences $\{\Sigma_{\mu_n}\}$ and $\{\Sigma_{\nu_n}\}$ converge uniformly in some neighborhood of zero, say $\mathbb{D}_\alpha$, $\alpha \in (0, 1]$, to the functions $\Sigma_\mu$ and $\Sigma_\nu$, respectively. Since $\Sigma_{\mu_n \boxtimes \nu_n} = \Sigma_{\mu_n}\Sigma_{\nu_n}$ in $\mathbb{D}_\alpha$, we conclude that $\{\Sigma_{\mu_n \boxtimes \nu_n}\}$ converge uniformly to $\Sigma_\mu\Sigma_\nu$. Applying Proposition 3.15, we arrive at the assertion of the proposition for $(\mathcal{M}_*, \boxtimes)$. $\square$

In the sequel we denote

$$a(\mu) := -\operatorname{Re}\varphi_\mu(i(\beta+1)) \quad \text{for} \quad \mu \in \mathcal{M}^{(\alpha,\beta)},$$

$$a(\mu) := 1/|\Sigma_\mu((\beta+\Delta)e^{i(\pi+\alpha)/2}/2)| \quad \text{for} \quad \mu \in \mathcal{M}_+^{(\alpha,\beta,\Delta)},$$

and

$$a(\mu) := \exp\{-i \arg \Sigma_\mu(\alpha/2)\} \quad \text{for} \quad \mu \in \mathcal{M}_*^\alpha.$$

**Proposition 6.5.** *If* $\{\mu_n^*\}$ *is any element of* $\mathbf{S}/E$, *and if, for each* $n \in \mathbb{N}$, $\nu_n^*$ *is a factor of* $\mu_n^*$, *then there is a subsequence* $\{n'\}$ *of* $\{n\}$ *such that* $\{\nu_{n'}^*\} \in \mathbf{S}/E$ *and* $l\{\nu_{n'}^*\}$ *is a factor of* $l\{\mu_n^*\}$.

*Proof.* Let $\{\mu_n^*\}$ converges to $\mu^* \in \mathbf{M}/E$ in the metric $L^*$, and let $\nu_n^*$ be a factor of $\mu_n^*$ for each $n$. Then for each $n$ there exist probability measures $\mu_n$ and $\nu_n$, such that $\mu_n^* = E\mu_n$ and $\nu_n^* = E\nu_n$, $\nu_n$ is a $\mathbf{M}$-factor of $\mu_n$, and $\{\mu_n\}$ converges to $\mu$ in the metric $L$, where $\mu \in \mathbf{M}$ and $\mu^* = E\mu$.

At first let us prove the assertion of the proposition for $(\mathcal{M}, \boxplus)$. By Proposition 3.13, the sequence $\{\varphi_{\mu_n}\}$ converges uniformly on compact subsets of some set $\Gamma_{\alpha,\beta}$ to the function $\varphi_\mu$, and $\varphi_{\mu_n}(iy) = o(y)$ uniformly in $n$ as $y \to +\infty$. Since, for $\mu \in \mathcal{M}^{(\alpha,\beta)}$, $-\operatorname{Im}\varphi_\mu(z) \geqslant 0$, $z \in \Gamma_{\alpha,\beta}$, we have $-\operatorname{Im}\varphi_{\nu_n}(z) \leqslant -\operatorname{Im}\varphi_{\mu_n}(z)$, $z \in \Gamma_{\alpha,\beta}$. By Theorem 3.9, there exists a subsequence $\{n'\}$ of $\{n\}$ such that $\{-\operatorname{Im}\varphi_{\nu_{n'}}(z)\}$ converges uniformly on compact subsets of $\Gamma_{\alpha,\beta}$ to a harmonic function $U(z) \geqslant 0$. Moreover $-\operatorname{Im}\varphi_{\nu_{n'}}(iy) = o(y)$ uniformly in $n$ as $y \to +\infty$. Consider the sequence of analytic functions $\{\varphi_{\nu_{n'}}(z) + a(\nu_{n'})\} = \{\varphi_{\nu_{n'}\boxplus\delta_{a(\nu_{n'})}}(z)\}$ in $\Gamma_{\alpha,\beta}$. This sequence converges at the point $i(\beta+1)$ and hence, by Theorem 3.11, $\{\varphi_{\nu_{n'}}(z) + a(\nu_{n'})\}$ converges uniformly on every compact subset of $\Gamma_{\alpha,\beta}$. Hence the assumptions of Proposition 3.13 hold for the sequence $\{\nu_{n'} \boxplus \delta_{a(\nu_{n'})}\}$. By this proposition, this sequence of p-measures converges in the weak topology to a p-measure $\nu$. Iterating this argument, we see that $\{\rho_{n'}\}$ converges weakly to $\rho$, say, where $\rho_{n'}$ is some cofactor of $\nu_{n'} \boxplus \delta_{a(\nu_{n'})}$ in $\mu_{n'}$. By continuity of the additive free convolution (Proposition 6.4), we have $\nu \boxplus \rho = \mu$, so $\nu$ is a factor of $\mu$. Thus Proposition 6.5 is proved for $(\mathcal{M}, \boxplus)$.

Now we shall prove the assertion for $(\mathcal{M}_+, \boxtimes)$. By Proposition 3.14, the sequence $\{\Sigma_{\mu_n}\}$ converges uniformly on compact subsets of some $\Gamma^+_{\alpha,\beta,\Delta}$ to the function $\Sigma_\mu$. Recall that for every $\mu \in \mathcal{M}_+^{(\alpha,\beta,\Delta)}$ the function $\log\Sigma_\mu(z)$ is analytic in $\Gamma^+_{\alpha,\beta,\Delta}$ and, in addition, $-\operatorname{Im}\log\Sigma_\mu(z) \geqslant 0$ for $z \in \Gamma^+_{\alpha,\beta,\Delta} \setminus \mathbb{C}^-$ and $\operatorname{Im}\log\Sigma_\mu(\bar z) = -\operatorname{Im}\log\Sigma_\mu(z)$ for $z \in \Gamma^+_{\alpha,\beta,\Delta}$. Since $-\operatorname{Im}\log\Sigma_{\nu_n}(z) \leqslant -\operatorname{Im}\log\Sigma_{\mu_n}(z)$ for $z \in \Gamma^+_{\alpha,\beta,\Delta} \setminus \mathbb{C}^-$, we may apply Theorem 3.9 to $\{-\operatorname{Im}\log\Sigma_{\nu_n}(z)\}$ and obtain that there exists a subsequence $\{n'\}$ of $\{n\}$ such that $\{\operatorname{Im}\log\Sigma_{\nu_{n'}}(z)\}$ converges uniformly on the compact subsets of $\Gamma^+_{\alpha,\beta,\Delta}$ to a harmonic function $U(z)$. Consider the sequence of analytic functions $\{\log(\Sigma_{\nu_{n'}}(z)a(\nu_{n'}))\} = \{\log\Sigma_{\nu_{n'}\boxtimes\delta_{a(\nu_{n'})}}(z)\}$ in $\Gamma^+_{\alpha,\beta,\Delta}$. This sequence converges at the point $(\beta+\Delta)e^{i(\pi+\alpha)/2}/2$ and then, by Theorem 3.11, $\{\Sigma_{\nu_{n'}\boxtimes\delta_{a(\nu_{n'})}}(z)\}$ converges uniformly on every compact subset of $\Gamma^+_{\alpha,\beta,\Delta}$. Hence the assumptions of Proposition 3.14 hold for the sequence $\{\nu_{n'} \boxtimes \delta_{a(\nu_{n'})}\}$ and the sequence of p-measures converges in the weak topology to a p-measure $\nu$. Iterating this argument, we see that $\{\rho_{n'}\}$ converges weakly to $\rho$, say, where $\rho_{n'}$ is some cofactor of $\nu_{n'} \boxtimes \delta_{a(\nu_{n'})}$ in $\mu_{n'}$. By continuity of the multiplicative free convolution (Proposition 6.4), we have $\nu \boxplus \rho = \mu$, so $\nu$ is a factor of $\mu$. Thus Proposition 6.5 is proved for $(\mathcal{M}_+, \boxtimes)$.

It remains to consider the case of the semigroup $(\mathcal{M}_*, \boxtimes)$. By Proposition 3.15, the sequence $\{\Sigma_{\mu_n}\}$ converges uniformly on compact subsets of some $\mathbb{D}_\alpha$ to the function $\Sigma_\mu$. Recall that for every $\mu \in \mathcal{M}_*^\alpha$ the function $\log\Sigma_\mu(z)$ is analytic in $\mathbb{D}_\alpha$ and

$\operatorname{Re}\log\Sigma_\mu(z) = \log|\Sigma_\mu(z)| \geqslant 0$ for $z \in \mathbb{D}_\alpha$. Since $\log|\Sigma_{\nu_n}(z)| \leqslant \log|\Sigma_{\mu_n}(z)|$ for $z \in \mathbb{D}_\alpha$, we may apply Theorem 3.9 to $\{\log|\Sigma_{\nu_n}(z)|\}$ and obtain that there exists a subsequence $\{n'\}$ of $\{n\}$ such that $\{\log|\Sigma_{\nu_{n'}}(z)|\}$ converges uniformly on compact subsets of $\mathbb{D}_\alpha$ to a harmonic function $U(z) \geqslant 0$. Consider the sequence of analytic functions in $\mathbb{D}_\alpha$ $\{\log(\Sigma_{\nu_{n'}}(z)a_{n'})\} = \{\log\Sigma_{\nu_{n'}\boxtimes\delta_{a(\nu_{n'})}}(z)\}$. This sequence converges at the point $\alpha/2$ and thus, by Theorem 3.11, the sequence $\{\Sigma_{\nu_{n'}\boxtimes\delta_{a(\nu_{n'})}}(z)\}$ converges uniformly on every compact subset of $\mathbb{D}_\alpha$. Hence the assumptions of Proposition 3.15 hold for the sequence $\{\nu_{n'} \boxtimes \delta_{a(\nu_{n'})}\}$. By this proposition, this sequence of p-measures converges in the weak topology to a p-measure $\nu$. Iterating the argument, we see that $\{\rho_{n'}\}$ converges weakly to $\rho$, say, where $\rho_{n'}$ is some cofactor of $\nu_{n'} \boxtimes \delta_{a(\nu_{n'})}$ in $\mu_{n'}$. By continuity of the multiplicative free convolution (Proposition 6.4), we have $\nu \boxplus \rho = \mu$, so $\nu$ is a factor of $\mu$, thus proving Proposition 6.5 for $(\mathcal{M}_*, \boxtimes)$.

Proposition 6.5 is completely proved. □

**Proposition 6.6.** *If $\{\mu_n^*\}$ is any element of $\mathbf{S}/E$, and if, for each $n \in \mathbb{N}$, $\mu_n^* = \nu_n^* \circ \rho_n^*$, where $\nu_n^*, \rho_n^* \in \mathbf{M}/E$ and $\nu_n^*$ is a factor of $\nu_{n+1}^*$, then $\{\nu_n^*\}$ and $\{\rho_n^*\}$ in $\mathbf{S}/E$ and $l\{\nu_n^*\}$ and $l\{\rho_n^*\}$ are factors of $l\{\mu_n^*\}$.*

*Proof.* We prove this proposition in the same way as Proposition 6.5. We keep all notations and demonstrate the difference between proofs of Proposition 6.5 and Proposition 6.6 in connection with the semigroup $(\mathcal{M}, \boxplus)$. We shall repeat all arguments accepting the following. By the assumptions of Proposition 6.6 we have

$$-\operatorname{Im}\varphi_{\nu_n}(z) \leqslant -\operatorname{Im}\varphi_{\nu_{n+1}}(z) \leqslant -\operatorname{Im}\varphi_{\mu_{n+1}}(z), \quad z \in \Gamma_{\alpha,\beta},$$

for all $n \geqslant 1$. Hence, by Theorem 3.10, $\{-\operatorname{Im}\varphi_{\nu_n}(z)\}$ converges uniformly on compact subsets of $\Gamma_{\alpha,\beta}$ to a harmonic function $U(z) \geqslant 0$. Then we repeat the arguments of Proposition 6.5 for $\{\varphi_{\nu_n}(z) + a(\nu_n)\}$ and obtain that $\{\nu_n \boxplus \delta_{a(\nu_n)}\}$ converges weakly to some p-measure $\nu$. Furthermore, as in the proof of Proposition 6.5 we see that $\{\rho_n\}$ converges weakly to $\rho$, say, where $\rho_n$ is the cofactor of $\nu_n \boxplus \delta_{a(\nu_n)}$ in $\mu_n$ such that $\rho_n^* = E\rho_n$, and $\nu \boxplus \rho = \mu$, so $\nu$ and $\rho$ are factors of $\mu$. Thus, Proposition 6.6 is proved for the semigroup $(\mathcal{M}, \boxplus)$.

The proof for the semigroups $(\mathcal{M}_+, \boxtimes)$ and $(\mathcal{M}_*, \boxtimes)$ is similar therefore we omit it. □

**Proposition 6.7.** *Let $\{\nu_n\}$ be a sequence of p-measures $\nu_n \in \mathbf{M}^\gamma$ and let $D^\gamma(\nu_n^*) \to 0$ as $n \to \infty$. For every fixed $\gamma' \in \mathcal{G}$ there exists $n(\gamma')$ such that $\nu_n \circ \delta_{a(\nu_n)} \in \mathbf{M}^{\gamma'}$ for $n \geqslant n(\gamma')$.*

*Proof.* Let us prove the assertion for the semigroup $(\mathcal{M}, \boxplus)$. Let $\{\nu_n\}$ denote a sequence of p-measures $\nu_n \in \mathcal{M}^{(\alpha,\beta)}$ and let $\operatorname{Im}\varphi_{\nu_n}(z_0) \to 0$ as $n \to \infty$, where $z_0 := i(\beta + 1)$. We shall prove that for every fixed $\alpha' > \alpha$ and $0 < \beta' < \beta$, there exists $n(\alpha', \beta')$ such that $\nu_n \boxplus \delta_{a(\nu_n)} \in \mathcal{M}^{(\alpha',\beta')}$ for $n \geqslant n(\alpha', \beta')$.

The functions $\operatorname{Im}\varphi_{\nu_n}(z)$ are harmonic non positive-valued functions in $\Gamma_{\alpha,\beta}$. Therefore, by the theorem of the mean for harmonic functions,

$$|\operatorname{Im}\varphi_{\nu_n}(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |\operatorname{Im}\varphi_{\nu_n}(z_0+\varepsilon e^{i\theta})|\,d\theta \tag{6.3}$$

for sufficiently small $\varepsilon > 0$. Since $\operatorname{Re}\varphi_{\nu_n\boxplus\delta_{a(\nu_n)}}(z_0) = 0$, we have, by Schwarz's formula (see Markushevich (1965), v. 2, p. 151),

$$\varphi_{\nu_n\boxplus\delta_{a(\nu_n)}}(z) = \frac{i}{2\pi}\int_0^{2\pi} \operatorname{Im}\varphi_{\nu_n}(z_0+\varepsilon e^{i\theta}/2)\,\frac{\varepsilon e^{i\theta}/2+(z-z_0)}{\varepsilon e^{i\theta}/2-(z-z_0)}\,d\theta$$

for $z \in \mathbb{C}$ such that $|z-z_0| < \varepsilon/2$. Using (6.3), we obtain from this formula, for $|z-z_0| \leqslant \varepsilon/4$,

$$\begin{aligned}|\varphi_{\nu_n\boxplus\delta_{a(\nu_n)}}(z)| &\leqslant \frac{1}{2\pi}\int_0^{2\pi} |\operatorname{Im}\varphi_{\nu_n}(z_0+\varepsilon e^{i\theta}/2)|\frac{\varepsilon/2+|z-z_0|}{\varepsilon/2-|z-z_0|}\,d\theta \\ &\leqslant \frac{3}{2\pi}\int_0^{2\pi} |\operatorname{Im}\varphi_{\nu_n}(z_0+\varepsilon e^{i\theta}/2)|\,d\theta = 3|\operatorname{Im}\varphi_{\nu_n}(z_0)|.\end{aligned} \tag{6.4}$$

Since $|\operatorname{Im}\varphi_{\nu_n}(z_0)| \to 0$ as $n \to \infty$, we conclude from (6.4) that $\varphi_{\nu_n\boxplus\delta_{a(\nu_n)}}(z) \to 0$ uniformly for $|z-z_0| \leqslant \varepsilon/4$ as $n \to \infty$. Recall that $F^{(-1)}_{\nu_n\boxplus\delta_{a(\nu_n)}}(z) = z + \varphi_{\nu_n\boxplus\delta_{a(\nu_n)}}(z)$. Therefore we easily conclude that $F_{\nu_n\boxplus\delta_{a(\nu_n)}} = z + o(1)$ and hence $G_{\nu_n\boxplus\delta_{a(\nu_n)}}(z) = 1/z + o(1)$ uniformly for $|z-z_0| \leqslant \varepsilon/8$ as $n \to \infty$. From the last relation we see that

$$\operatorname{Im}\Big(G_{\nu_n\boxplus\delta_{a(\nu_n)}}(z_0) - \frac{1}{z_0}\Big) = \frac{1}{\operatorname{Im} z_0}\int_{\mathbb{R}} \frac{u^2}{u^2+(\operatorname{Im} z_0)^2}\,\nu_n\boxplus\delta_{a(\nu_n)}(du) \to 0, \text{ as } n \to \infty. \tag{6.5}$$

From (6.5) it follows that $\nu_n\boxplus\delta_{a(\nu_n)}$ converges weakly to $\delta_0$ as $n \to \infty$. The last relation easily implies the assertion of Proposition 6.7 for $(\mathcal{M},\boxplus)$.

We shall consider the assertion for the semigroup $(\mathcal{M}_+,\boxtimes)$. Let $\{\nu_n\}$ denote a sequence of p-measures $\nu_n \in \mathcal{M}_+^{(\alpha,\beta,\Delta)}$ and assume that $\operatorname{Im}\log\Sigma_{\nu_n}(z_0) \to 0$ as $n \to \infty$, where $z_0 := (\beta+\Delta)e^{i(\pi+\alpha)/2}/2$. We shall prove that for every fixed $0 < \alpha' < \alpha$, $0 < \beta' < \beta$, and $\Delta' > \Delta$, there exists $n(\alpha',\beta',\Delta')$ such that $\nu_n \boxplus \delta_{a(\nu_n)} \in \mathcal{M}^{(\alpha',\beta',\Delta')}$ for $n \geqslant n(\alpha',\beta',\Delta')$.

The functions $\operatorname{Im}\log\Sigma_{\nu_n}(z)$ are harmonic non positive-valued functions in the domain $\Gamma^+_{\alpha,\beta,\Delta} \cap \mathbb{C}^+$ such that $\log|\Sigma_{\nu_n\boxtimes\delta_{a(\nu_n)}}(z_0)| = 0$. Repeating the previous arguments for the analytic functions $\log\Sigma_{\nu_n\boxtimes\delta_{a(\nu_n)}}(z)$ on $\Gamma^+_{\alpha,\beta,\Delta}$, we see that these functions $\log\Sigma_{\nu_n\boxtimes\delta_{a(\nu_n)}}(z) \to 0$ uniformly for $|z-z_0| \leqslant \varepsilon/4$ as $n \to \infty$, where as above $\varepsilon > 0$ is chosen sufficiently small. Hence $\Sigma_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z)$ converges to 1 uniformly for $|z-z_0| \leqslant \varepsilon/4$ as $n \to \infty$. Recalling the definition of $\Sigma_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z)$, we conclude from this relation that

$$K_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z) \to z \tag{6.6}$$

uniformly for $|z - z_0| \leqslant \varepsilon/8$ as $n \to \infty$. Taking into account the representation (2.8) for the functions $K_{\nu_n \boxtimes \delta_{a(\nu_n)}}$, we have

$$\frac{1}{z} K_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z) = a_n + \int\limits_{(0,\infty)} \frac{\tau_n(dt)}{t - z}, \qquad z \in \mathbb{C}^+, \tag{6.7}$$

where $a_n \geqslant 0$ and the nonnegative measures $\tau_n$ satisfy (2.9). We easily see from (6.6) and (6.7) that

$$a_n + \int\limits_{(0,\infty)} \frac{\tau_n(dt)}{t+1} \leqslant c(z_0) < \infty, \qquad n = 1, 2, \ldots,$$

where $c(z_0)$ is a positive constant depending on $z_0$. This estimate implies the inequality $|K_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z)/z| \leqslant c(\alpha', \beta', \Delta')$, $n = 1, 2, \ldots$, in every domain $\Gamma^+_{\alpha',\beta',\Delta'}$ with $0 < \alpha' < \alpha, 0 < \beta' < \beta$, and $\Delta' > \Delta$. Here $c(\alpha', \beta', \Delta')$ denotes a positive constant depending on $\alpha', \beta'$, and $\Delta'$. Hence, by (6.7), we conclude with the help of Vitali's theorem (see Theorem 3.12 of Section 3) that the relation (6.6) holds for $z \in \Gamma^+_{\alpha',\beta',\Delta'}$ with any fixed $\alpha', \beta', \Delta'$. From this relation we immediately obtain the assertion for $(\mathcal{M}_+, \boxtimes)$.

Let us prove the assertion for $(\mathcal{M}_*, \boxtimes)$. Let $\{\nu_n\}$ be a sequence of p-measures $\nu_n \in \mathcal{M}_*^\alpha$ such that let $\log |\Sigma_{\nu_n}(z_0)| \to 0$ as $n \to \infty$. Here $z_0 := \alpha/2$. We shall prove that for every fixed $\alpha' > \alpha$ there exists $n(\alpha')$ such that $\nu_n \boxplus \delta_{a(\nu_n)} \in \mathcal{M}^{\alpha'}$ for $n \geqslant n(\alpha')$.

The functions $\log |\Sigma_{\nu_n}(z)|$ are harmonic non negative-valued functions on $\mathbb{D}_\alpha$ such that $\operatorname{Im} \log \Sigma_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z_0) = 0$. Repeating the previous arguments for the analytic functions $\log \Sigma_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z)$ on $\mathbb{D}_\alpha$, we see that $\log \Sigma_{\nu_n \boxplus \delta_{a(\nu_n)}}(z) \to 0$ uniformly for $|z - z_0| \leqslant \varepsilon/4$ as $n \to \infty$. Hence $\Sigma_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z) \to 1$ uniformly for $|z - z_0| \leqslant \varepsilon/4$ as $n \to \infty$. Recalling the definition of $\Sigma_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z)$, we conclude from this relation that $Q_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z) \to z$ uniformly for $|z - z_0| \leqslant \varepsilon/8$ as $n \to \infty$. Since for $\mu \in \mathcal{S}_*$ the function $Q_\mu(z)/z$ is analytic and $|Q_\mu(z)/z| \leqslant 1$ for $z \in \mathbb{D}$, we conclude, by Vitali's theorem, that $Q_{\nu_n \boxtimes \delta_{a(\nu_n)}}(z) \to z$ uniformly for $z \in \mathbb{D}_{\alpha'}$ for any fixed $\alpha' \in (\alpha, 1)$. This implies at once the assertion for $(\mathcal{M}_*, \boxtimes)$.

Thus, Proposition 6.7 is proved. □

**Proposition 6.8.** *Let $\{\mu_{nk} : n \geqslant 1, 1 \leqslant k \leqslant n\}$ be an array of infinitesimal measures in $\mathbf{M}^\gamma$. Then, for every fixed $\gamma \in \mathcal{G}$, $\max_{1 \leqslant k \leqslant n} D^\gamma(\mu^*_{nk}) \to 0$ as $n \to \infty$.*

This proposition follows from obvious calculations which we omit.

**Proof of Theorem 2.10**. Let $\{\mu_{j,s} : 1 \leqslant s \leqslant j < \infty\}$ be a array of infinitesimal measures in $\mathbf{M}$ and let

$$l\{\mu^*_j\} = \mu^* \in \mathbf{M}/E, \quad \text{where} \quad \mu^*_j = \mu^*_{j,1} \circ \cdots \circ \mu^*_{j,j}.$$

It is clear that $\mu$ and $\mu_j$, $j = 1, \dots$, such that $\mu^* = E\mu$ and $\mu_j^* = E\mu_j$, $j = 1, \dots$, belong to the set $\mathbf{M}^\gamma$ with some $\gamma \in \mathcal{G}$ and, by Proposition 6.1 and 6.8,

$$\lim_{j\to\infty} \max_{1\leqslant s\leqslant j} D^\gamma(\mu_{j,s}^*) = 0.$$

Let $k \in \mathbb{N}$ be given. Since $\max_{1\leqslant s\leqslant j} D^\gamma(\mu_{j,s}^*) \to 0$ as $j \to \infty$ and, by Propositions 6.3, $D^\gamma(\mu_j^*) \to D^\gamma(\mu^*)$ as $j \to \infty$, given $n \in \mathbb{N}$ there exists $j(n)$ so large that we may group $\mu_{j(n),s}^*$, $1 \leqslant s \leqslant j(n)$, via free convolutions into $\nu_{k,l,n}^*$ say, $1 \leqslant l \leqslant k$, such that $\mu_{j(n)}^* = \nu_{k,1,n}^* \circ \dots \circ \nu_{k,k,n}^*$ and $|D^\gamma(\nu_{k,l,n}^*) - D^\gamma(\mu^*)/k| \leqslant 1/n$ for $1 \leqslant l \leqslant k$. By Proposition 6.5, there is a decomposition $\mu^* = \nu_{k,1}^* \circ \dots \circ \nu_{k,k}^*$ with $D^\gamma(\nu_{k,l}^*) = D^\gamma(\mu^*)/k$ for $1 \leqslant l \leqslant k$. Hence there is a decomposition $\mu = \nu_{k,1} \circ \dots \circ \nu_{k,k}$ for every $k \in \mathbb{N}$ such that $\nu_{k,l} \in \mathbf{M}^\gamma$ and $D^\gamma(\nu_{k,l}^*) = D^\gamma(\mu^*)/k$ for $1 \leqslant l \leqslant k$. Applying Proposition 6.7 to the sequence $\{\nu_{k,l}\}$ with $k = 1, \dots$ and $l = 1, \dots, k$, we see that, for any fixed $\gamma \in \mathcal{G}$, there exists $k(\gamma)$ such that $\nu_{k,l} \in \mathbf{M}^\gamma$ for $k \geqslant k(\gamma)$, $l = 1, \dots, k$.

From this relation we see in the case of $(\mathcal{M}, \boxplus)$ that $\varphi_\mu(z)$ admits an analytic continuation in $\mathbb{C}^+$ with values $\mathbb{C}^- \cup \mathbb{R}$ and hence, by the Bercovici and Voiculescu result (see Section 2), $\mu$ is infinitely divisible p-measure. This assertion follows from Lemma 4.1 as well.

In the case of $(\mathcal{M}_+, \boxtimes)$ we note that the function $\log \Sigma_\mu(z)$ admits an analytic continuation in $\mathbb{C}^+$ with values $\mathbb{C}^- \cup \mathbb{R}$. In addition this function is analytic and real-valued on the negative half-line. Again, by the Bercovici and Voiculescu result (see Section 2), $\mu$ is infinitely divisible p-measure. We can obtain this result with the help of Lemma 5.3 as well.

In the case of $(\mathcal{M}_*, \boxtimes)$ the function $\log \Sigma_\mu(z)$ admits an analytic continuation in $\mathbb{D}$ with values in $i(\mathbb{C}^+ \cup \mathbb{R})$. By the Bercovici and Voiculescu result (see Section 2), $\mu$ is infinitely divisible p-measure. This result follows from Lemma 5.6 as well.

The theorem is proved. □

**Proof of Theorem 2.11**. Let $\mu$ be not a Dirac measure and have no indecomposable factors. We shall show that $\mu$ is an infinitely divisible p-measure. Without loss of generality we assume that $\mu \in \mathbf{M}^\gamma$ with some $\gamma \in \mathcal{G}$. By Proposition 6.2, $D^\gamma(\mu^*) > 0$. Let us show that $\inf_{\nu\in S} D^\gamma(\nu^*) = 0$, where $S$ is the set of all factors $\nu$ of $\mu$ such that $\nu^* \neq e$. For suppose not: call the non-zero infimum $b$. By Proposition 6.1, Proposition 6.5, and Proposition 6.3, there is $\nu_0^* \in \mathbf{M}^\gamma/E$ such that $\nu_0^* = E\nu_0$ with $\nu_0 \in S$ and $D^\gamma(\nu_0^*) = b$. Since $\nu_0$ is a factor of $\mu$, it has factors from $S$, say $\nu_1$, on which $D^\gamma(\nu_1^*)$ is less then $b$, a contradiction.

Consider now all $n$-fold decompositions $\mathcal{D}:\ \mu = \nu_1 \circ \nu_2 \circ \dots \circ \nu_n$. By Proposition 6.1, $\nu_j \in \mathbf{M}^\gamma$, $j = 1, \dots, n$. Define $m(\mathcal{D}) := \min_{j=1,\dots,n} D^\gamma(\nu_j^*)$, so that $0 \leqslant m(\mathcal{D}) \leqslant (1/n)D^\gamma(\mu^*)$. Let $h := \sup_{\mathcal{D}} m(\mathcal{D})$. By Proposition 6.5 and Proposition 6.3, there is a decomposition $\mathcal{D}'$ such that $m(\mathcal{D}') = h$. Assume that $\nu_1', \nu_2', \dots, \nu_n'$ have been odered so that $D^\gamma((\nu_j')^*)$ increases with $j$; then $h = D^\gamma((\nu_1')^*)$. Now either all $D^\gamma$'s are equal or there is a least $l$ with $1 \leqslant l < n$ and $D^\gamma((\nu_l')^*) < D^\gamma((\nu_{l+1}')^*)$. By the first paragraph of this proof we can write

$$\nu_{l+1}' = \rho_1' \circ \rho_2' \circ \dots \circ \rho_l' \circ \nu_{l+1}'',$$

where $D^\gamma((\rho'_1)^*), \ldots, D^\gamma((\rho'_l)^*)$ are all arbitrary small but non-zero. If we choose them so that $D^\gamma((\nu''_{l+1})^*) > D^\gamma((\nu'_l)^*)$, we shall obtain a decomposition

$$\mathcal{D}'' : \mu = (\nu'_1 \circ \rho'_1) \circ \cdots \circ (\nu'_l \circ \rho'_l) \circ \nu''_{l+1} \circ \nu'_{l+2} \cdots \circ \nu'_n$$

which has $m(\mathcal{D}'') > h$, a contradiction. So in $\mathcal{D}'$ all the $D^\gamma$'s must be equal, entailing $D^\gamma((\nu'_j)^*) = (1/n)D^\gamma(\mu^*)$ for each $j$, $1 \leqslant j \leqslant n$.

We have a $\mathcal{D}'$ for every $n$: call it $\mathcal{D}'(n) : \mu = \nu'_1(n) \circ \nu'_2(n) \circ \cdots \circ \nu'_n(n)$. The $\mathcal{D}'(n)$ forms a triangular array whose row $\circ$-products are equal to $\mu$. Further, $\max_{j=1,\ldots,n} D^\gamma((\nu'_j(n))^*) = (1/n)D^\gamma(\mu^*)$. It follows from Theorem 2.10 that $\mu$ is infinitely divisible p-measure. The theorem is proved completely. $\square$

**Proof of Theorem 2.12**. It is clear that $\mu \in \mathbf{M}^\gamma$ with some $\gamma \in \mathcal{G}$. Hence the homomorphism $D^\gamma(\mu^*)$ is defined and finite. By Proposition 6.1, all factors of $\mu$ belong to $\mathbf{M}^\gamma$.

Denote by $S_1$ the set of all indecomposable factors of $\mu$. By the assumption of the theorem, this set is not empty. Denote $d_1 := \sup_{\nu \in S_1} D^\gamma(\nu^*)$. Let an indecomposable factor $\nu_1$ of $\mu$ have the property $D^\gamma(\nu_1^*) \geqslant d_1/2$. Since $\nu_1$ is a factor of $\mu$, we have $\mu = \nu_1 \circ \rho_1$, where $\rho_1 \in \mathbf{M}^\gamma$.

If $\rho_1$ is an indecomposable p-measure, the proof is complete. In the other case we repeat the previous arguments for $\rho_1$ and arrive at the relation $\mu = \nu_1 \circ \nu_2 \circ \rho_2$, where $\nu_2$ and $\rho_2$ belong to $\mathbf{M}^\gamma$, and $\nu_2$ is an indecomposable p-measure such that

$$D^\gamma(\nu_2^*) \geqslant d_2/2, \qquad d_2 := \sup_{\nu \in S_2} D^\gamma(\nu^*),$$

where $S_2\,(\neq \emptyset)$ is the set of all indecomposable factors of $\rho_1$.

Repeating next steps in the same way we obtain

$$\mu = \nu_1 \circ \nu_2 \circ \cdots \circ \nu_k \circ \rho_k, \tag{6.8}$$

where $\nu_1, \nu_2, \ldots, \nu_k$ are indecomposable factors of $\mu$ and $\rho_k$ is an indecomposable p-measure, then the proof is completed, or we have an infinite sequence of the relations (6.8). Recall that all p-measures $\nu_1, \nu_2, \ldots, \nu_k$ and $\rho_k$ belong to $\mathbf{M}^\gamma$. In addition we have

$$D^\gamma(\nu_k^*) \geqslant d_k/2, \qquad d_k := \sup_{\nu \in S_k} D^\gamma(\nu^*), \tag{6.9}$$

where $S_k\,(\neq \emptyset)$ is the set of all indecomposable factors of $\rho_{k-1}$ and, for $n > m$,

$$\rho_m = \nu_{m+1} \circ \nu_{m+2} \cdots \circ \nu_n \circ \rho_n. \tag{6.10}$$

By (6.8) and (6.9), we see that

$$\frac{1}{2}\sum_{k=1}^{\infty} d_k \leqslant \sum_{k=1}^{\infty} D^\gamma(\nu_k^*) \leqslant D^\gamma(\mu^*) < \infty. \tag{6.11}$$

Applying Proposition 6.6 to (6.8), we see that $\{\nu_1^* \circ \nu_2^* \circ \cdots \circ \nu_k^*\}$ and $\{\rho_k^*\}$ are elements of $\mathbf{S}/E$ and $l\{\nu_1^* \circ \nu_2^* \circ \cdots \circ \nu_k^*\}$ and $l\{\rho_k^*\}$ are factors of $\mu^*$.

It remains to show that $l\{\rho_k^*\}$ has no indecomposable factors. Applying Proposition 6.6 to (6.10), we obtain the relation

$$\rho_m^* = l\{\nu_{m+1}^* \circ \nu_2^* \circ \cdots \circ \nu_n^*\} \circ l\{\rho_n^*\}, \qquad m = 1, 2, \dots. \tag{6.12}$$

Let $l\{\rho_n^*\}$ have an indecomposable factor $\kappa^*$ such that $\kappa^* = E\kappa$ with $\kappa \in \mathbf{M}^\gamma$. We see from (6.12) that $\kappa$ is a factor of $\rho_m$ for every $m = 1, 2, \dots$ and hence belongs to all sets $S_m$, $m = 1, \dots$. By Proposition 6.2, $D^\gamma(\kappa^*) > 0$. On the other hand $d_k \geqslant D^\gamma(\kappa^*) > 0$ for all $k = 1, \dots$ and the series $\sum_{k=1}^\infty d_k$ should deverge. By (6.11), we arrive at a contradiction. Thus, $l\{\rho_n^*\}$ has no indecomposable factors and the theorem is proved completely. □

## 7. Description of the class $I_0$ in $(\mathbf{M}, \circ)$ and nonuniqueness of the representation (2.29)

Our next step is to prove Theorem 2.13 for the semigroups $(\mathbf{M}, \circ)$.

*Proof of Theorem* 2.13. Since $I_0$ is the set of i.d. elements all of whose factors are i.d., we should prove that any nontrivial i.d. element in $(\mathbf{M}, \circ)$ has a non i.d. factor.

**7.1**. Consider the case of $(\mathcal{M}, \boxplus)$. As we saw in Section 2 a measure $\mu \in \mathcal{M}$ is $\boxplus$-i.d. if and only if $\varphi_\mu$ admits a representation (2.20).

Let $a < b$ be real numbers such that $\nu([a, b]) > 0$. Consider the function

$$\phi(z) := \varepsilon_0 \int_{[a,b]} \frac{(1 + u^2)\nu(du)}{z - u}, \quad z \in \mathbb{C}^+, \tag{7.1}$$

with sufficiently small $\varepsilon_0 > 0$, depending on $\nu, a, b$ only, and introduce the functions $f_j(z) := 2\phi(z) + (-1)^j \varepsilon \phi^2(z)$, $j = 1, 2$. In order to prove the theorem we need the following lemma.

**Lemma 7.1.** *For sufficiently small $\varepsilon > 0$, there exist $\mu_j \in \mathcal{M}$, $j = 1, 2$, such that $\varphi_{\mu_j}(z) = f_j(z)$ for $z$, where $\varphi_{\mu_j}(z)$ is defined.*

*Proof.* Consider the function

$$F(z) := z + \phi(z), \quad z \in \mathbb{C}^+.$$

By Lemma 4.1, $F : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{C}^+$ precisely once. The inverse function $F^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.

Denote by $\mathbb{C}_\eta^+$ the half-plane $\operatorname{Im} z > \eta$ for any real $\eta$. Consider the domains $\Omega_\eta := F^{(-1)}(\mathbb{C}_\eta^+)$, $\eta \in (0, 1]$. Note that $\Omega_\eta$ are domains such that $\Omega_\eta \subset \mathbb{C}^+$.

The boundary of $\Omega_\eta$ is a curve $\gamma_5(\eta)$ characterized by the functional equation $\operatorname{Im} F(x + iy) = \eta$, $x \in \mathbb{R}$, $y > 0$. Using (7.1) rewrite this equation in the form

$$y\Big(1 - \varepsilon_0 \int_{[a,b]} \frac{(1 + u^2)\,\nu(du)}{(u - x)^2 + y^2}\Big) = \eta. \tag{7.2}$$

It is easy to see that (7.2) has an unique solution for every fixed $x \in \mathbb{R}$. Since $F'(z) \neq 0$ for $z \in \Omega_\eta$, $\eta > 0$, we conclude, by the implicit function theorem, that $\gamma_5(\eta)$, $\eta > 0$, are smooth Jordan curves.

In view of (7.2), we see that the curves $\gamma_5(\eta)$ lie in the union of domains $|z-(a+b)/2| < b-a+1$, $\operatorname{Im} z > 0$ and $0 < \operatorname{Im} z < 2\eta$. Denote by $\tilde{\gamma}_5(\eta)$ the part of the curve $\gamma_5(\eta)$ contained in the disk $|z-(a+b)/2| \leqslant b-a+1$. Let us show that

$$\sup_{0<\eta\leqslant 1} \sup_{z\in\tilde{\gamma}_5(\eta)} |\phi(z)| := A_1 < \infty. \tag{7.3}$$

Assume that $A_1 = \infty$. Then for any $N > 1$ there exist $\eta_0 \in (0,1]$ and $z_0 = z_0(\eta_0) \in \tilde{\gamma}_5(\eta_0)$ such that $\operatorname{Im} F(z_0) > 0$ and $|F(z_0)| > N$. Since $F(x)$ is real-valued for $x \in \mathbb{R} \setminus [a,b]$ and $\operatorname{Im} z \operatorname{Im} F(z) > 0$ for $|z| > |a|+|b|$, $\operatorname{Im} z \neq 0$, and $|\phi(z)| \to 0$ for $|z| = R \to \infty$, we easily see, by Rouché's theorem, that for $w \in \mathbb{C}^+$ with sufficiently large modulus the equation $w = F(z)$ has a solution $z = z(w) \in \mathbb{C}^+$ such that $z(w) = w + O(1)$. Recalling that $F(z) : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{C}^+$ precisely once, we conclude that $z_0 = F(z_0) + O(1)$, a contradiction for sufficiently large $N$.

Now let us show, for $j = 1,2$, that $F_j(z) := z + f_j(z) : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{C}^+$ precisely once in $\mathbb{C}^+$. By (7.3), we see that the inequality $1 + 2(-1)^j\varepsilon \operatorname{Re}\phi(z) > 0$ holds for $z \in \gamma_5(\eta)$, $\eta \in (0,1]$, and for sufficiently small $\varepsilon > 0$. Since $\operatorname{Im}\phi(z) \leqslant 0$ and $\operatorname{Im} F(z) = \eta$ for the same $z$, we conclude from the formula

$$\operatorname{Im} F_j(z) = \operatorname{Im} z + \operatorname{Im}\phi(z) + \operatorname{Im}\phi(z)\Big(1 + 2(-1)^j\varepsilon \operatorname{Re}\phi(z)\Big) \tag{7.4}$$

that $\operatorname{Im} F_j(z) \leqslant \eta$ for $z \in \gamma_5(\eta)$, $\eta \in (0,1]$.

Choose $R > 1$ to be determined later sufficiently large. Denote by $\tilde{a}$, $\operatorname{Re}\tilde{a} < 0$, and $\tilde{b}$, $\operatorname{Re}\tilde{b} > 0$, points of intersection of the curve $\gamma_5(\eta)$ with the circle $|z| = R$. Consider the closed rectifiable curve $\gamma_6 = \gamma_6(\eta)$, $\eta \in (0,1]$, consisting of $\gamma_{6,1}$: the part of $\gamma_5(\eta)$ lying in the disc $|z| \leqslant R$, connecting $\tilde{a}$ to $\tilde{b}$, and the arc $\gamma_{6,2} : Re^{i\theta}$, $\arg\tilde{b} < \theta < \arg\tilde{a}$, connecting $\tilde{b}$ to $\tilde{a}$.

Fix $w \in \mathbb{C}^+$. Assume that $\eta < \operatorname{Im} w$. If $z$ traverses $\gamma_{6,1}$, the image $\zeta = F_j(z)$ lies in the half-plane $\operatorname{Im}\zeta \leqslant \eta$. If $z$ traverses $\gamma_{6,2}$, the image $\zeta = F_j(z)$ is equal to $Re^{i\theta} + o(1)$, $\arg\tilde{b} < \theta < \arg\tilde{a}$, as $R \to \infty$. We conclude that the image $\zeta = F_j(z)$ winds around $w$ once when $z$ runs through $\gamma_6$ in the counter clockwise direction. By the argument principle, the function $F_j(z)$ takes the value $w$ precisely once inside $\gamma_6$. Since this assertion holds for all sufficiently large $R > 1$ and for all sufficiently small $\eta > 0$, we obtain the desired result.

Thus we conclude that $F_j^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$, $j = 1,2$, exist and are analytic functions. Hence $F_j^{(-1)} \in \mathcal{N}$. Moreover it is easy to see that $F_j^{(-1)} \in \mathcal{F}$, by proving the lemma. $\square$

We shall complete the proof of the theorem for the semigroup $(\mathcal{M}, \boxplus)$.

Since $\varphi_\mu(z)$ admits the representation (2.20), we may write $\mu$ in the form $\mu = \mu_1 \boxplus \mu_2 \boxplus \mu_3$, where p-measures $\mu_1$, $\mu_2$, and $\mu_3$ are determined as follows. The measures

$\mu_1, \mu_2$ are defined in Lemma 7.1. Since, by (7.1),

$$\varphi_{\mu_1}(z) + \varphi_{\mu_2}(z) = 4\phi(z) = 4\varepsilon_0 \int\limits_{[a,b]} \frac{1+uz}{z-u}\,\nu(du) - 4\varepsilon_0 \int\limits_{[a,b]} u\,\nu(du),$$

and $\varphi_\mu(z) = \varphi_{\mu_1}(z) + \varphi_{\mu_2}(z) + \varphi_{\mu_3}(z)$, we easily conclude that

$$\varphi_{\mu_3}(z) := \alpha + 4\varepsilon_0 \int\limits_{[a,b]} u\,\nu(du) + (1-4\varepsilon_0) \int\limits_{[a,b]} \frac{1+uz}{z-u}\,\nu(du) + \int\limits_{\mathbb{R}\setminus[a,b]} \frac{1+uz}{z-u}\,\nu(du).$$

We consider the functions $\varphi_\mu(z)$ and $\varphi_{\mu_j}(z)$, $j = 1, 2, 3$, for $z$, where they are defined. By the Bercovici and Voiculescu result [12] (see Section 2), the measure $\mu_3$ is i.d. for $\varepsilon_0 \leqslant 1/4$.

If $\nu(\{a\}) > 0$, then we choose $b = a$ in (7.1). If $\nu(\{a\}) = 0$ for all $a \in \mathbb{R}$, then we choose the points $a < b$ such that $\nu([a, a+h]) > c_0 h$ and $\nu([b-h, b]) > c_0 h$ for all $0 < h \leqslant h_0$, where $c_0 > 0$ and $h_0 > 0$ depend on the measure $\nu$ only. Such points exist by Proposition 3.7. Let us show that under these assumptions the function $\phi(z)$ (see (7.1)), has the property

$$-\operatorname{Re}\phi(a-h+ih) \to +\infty, \quad h \downarrow 0, \quad \text{and} \quad \operatorname{Re}\phi(b+h+ih) \to +\infty, \quad h \downarrow 0. \quad (7.5)$$

This property is obvious in the case where $a = b$ and $\nu(\{a\}) > 0$. Consider the case where $\nu(\{a\}) = 0$ for all $a \in \mathbb{R}$. In this case we obtain for small $h > 0$ the estimates

$$\begin{aligned} -\operatorname{Re}\phi(a-h+ih) &= \varepsilon_0 \int\limits_{[a,b]} \frac{(u-a+h)(1+u^2)\nu(du)}{(u-a+h)^2+h^2} \geqslant \frac{\varepsilon_0}{2} \int\limits_{[a,b]} \frac{\nu(du)}{u-a+h} \\ &= \frac{\varepsilon_0}{2}\frac{\nu([a,b])}{b-a+h} + \frac{\varepsilon_0}{2}\int_a^b \nu([a,u))\frac{du}{(u-a+h)^2} \\ &\geqslant \frac{c_0\varepsilon_0}{2} \int_a^{\min\{a+h_0,b\}} (u-a)\frac{du}{(u-a+h)^2} \geqslant -\frac{c_0\varepsilon_0}{4}\log h. \end{aligned}$$

This lower bound implies the first assertion of (7.5). In the same way we obtain the second assertion of (7.5).

Note that the measures $\mu_j$, $j = 1, 2$, are not i.d. Indeed, the functions $\varphi_j(z)$, $j = 1, 2$, admit an analytic continuation in $\mathbb{C}^+$. We will denote again by $\varphi_j(z)$, $j = 1, 2$, respectively, these analytic continuations. Since

$$\operatorname{Im}\varphi_{\mu_j}(z) = 2\operatorname{Im}\phi(z)(1 + (-1)^j\varepsilon\operatorname{Re}\phi(z))$$

and $\operatorname{Im}\phi(z) < 0$, $z \in \mathbb{C}^+$, we see from (7.5) that there exist points in $\mathbb{C}^+$ where the functions $\operatorname{Im}\varphi_{\mu_1}(z)$ and $\operatorname{Im}\varphi_{\mu_2}(z)$ have positive values. By Bercovici and Voiculescu result [12] (see Section 2), the measures $\mu_j$, $j = 1, 2$, are not i.d.

Hence the measure $\mu$ has a non-i.d. factor and $\mu$ does not belong to the class $I_0$. $\square$

**7.2**. Let us consider the semigroup $(\mathcal{M}_+, \boxtimes)$. In order to prove Theorem 2.13 for this semigroup we need the following auxiliary lemmas.

**Lemma 7.2.** *For every $c > 0$ , there exist $\mu_j \in \mathcal{M}_+$, $j = 1, 2$, such that*

$$\Sigma_{\mu_j}(z) = \exp\Big\{ -\frac{cz}{2} + (-1)^j i\sqrt{cz}\Big\}$$

*for $z$, where $\Sigma_{\mu_j}(z)$ is defined.*

Here and in the sequel we shall choose the principle branch of $\sqrt{z}$ and $\log z$.

*Proof.* In the first step we consider the function

$$w = g_1(z) := \log z - \frac{cz}{2} - i\sqrt{cz}. \qquad z \in \mathbb{C}^+. \tag{7.6}$$

Let us show that for every fixed $w \in \mathbb{C}^+$ such that $0 < \operatorname{Im} w < \pi$ there exists a unique $z \in \mathbb{C}^+$ such that $w = g_1(z)$. To prove this assertion we consider the closed rectifiable curve $\gamma_7$, consisting of the interval $\gamma_{7,1} : t, -R \leqslant t \leqslant -1/R$, connecting $-R$ to $-1/R$, the arc $\gamma_{7,2} : e^{i\varphi}/R, 0 < \varphi < \pi$, connecting $-1/R$ to $1/R$, the interval $\gamma_{7,3} : t, 1/R \leqslant t \leqslant R$, connecting $1/R$ to $R$, and the arc $\gamma_{7,4} : Re^{i\varphi}, 0 < \varphi < \pi$, connecting $R$ to $-R$. The parameter $R \geqslant 1$ will be chosen later on sufficiently large.

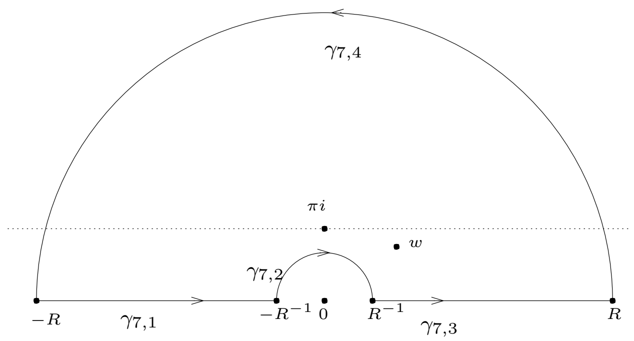


FIGURE 5

Let $z$ traverse $\gamma_7$ in the counter clockwise direction.

If $z$ traverses $\gamma_{7,1}$ the image $g_1(\gamma_{7,1})$ is contained in the line $\operatorname{Im}\zeta = \pi$, connecting $\log R + cR/2 + \sqrt{cR} + i\pi$ to $-\log R + c/(2R) + \sqrt{c/R} + i\pi$.

If $z$ traverses $\gamma_{7,2}$ the image $g_1(\gamma_{7,2})$ is contained in the half-plane $\operatorname{Re}\zeta \leqslant -\log R + 1$, connecting $-\log R + c/(2R) + \sqrt{c/R} + i\pi$ to $-\log R - c/(2R) - i\sqrt{c/R}$.

If $z$ traverses $\gamma_{7,3}$ the image $g_1(\gamma_{7,3})$ is contained in the half-plane $\operatorname{Im}\zeta < 0$, connecting $-\log R - c/(2R) - i\sqrt{c/R}$ to $\log R - cR/2 - i\sqrt{cR}$.

Finally, if $z$ traverses $\gamma_{7,4}$ the image $g_1(\gamma_{7,4})$ is contained in the domain $\{|\zeta| \geqslant cR/4, \operatorname{Im}\zeta \leqslant \pi\}$, connecting the points $\log R - cR/2 - i\sqrt{cR}$ to $\log R + cR/2 + \sqrt{cR} + i\pi$.

Therefore we can conclude that the image $\zeta = g_1(z), z \in \gamma_7$, winds around $w$ once when $z$ runs through $\gamma_7$. Hence there is a unique point $z$ inside the curve $\gamma_7$ such that $w = g_1(z)$. Thus we have proved that for every fixed $w \in \mathbb{S}_\pi = \{w \in \mathbb{C} : 0 < \operatorname{Im} w < \pi\}$

there exists a unique $z \in \mathbb{C}^+$ such that (7.6) holds. Thus the inverse function $g_1^{(-1)} : \mathbb{S}_\pi \to \mathbb{C}^+$ exists and is analytic in $\mathbb{S}_\pi$. Introduce the function $K_1(z) := g_1^{(-1)}(\log z)$, $z \in \mathbb{C}^+$. We note that $K_1 \in \mathcal{N}$ and $K_1^{(-1)}(z) = z \exp\{-cz/2 - i\sqrt{cz}\}$ in the domain of $\mathbb{C}^+$, where $K_1^{(-1)}$ is uniquely defined.

Let us show that $K_1$ admits an analytic continuation on $(-\infty, 0)$ and its value on $(-\infty, 0)$ is negative. Moreover $K_1(x) \to 0$ as $x \uparrow 0$. It is easy to see that $(e^{g_1(x)})' > 0$ for $x < 0$. Since $e^{g_1(z)}$ is analytic on $(-\infty, 0)$, we conclude that $(e^{g_1})^{(-1)}$ exists and is analytic on $(-\infty, 0)$ as well. This function coincides for $z \in \mathbb{C}^+$ with the function $K_1(z)$ defined earlier. From the definition of $e^{g_1(z)}$ it follows that $(e^{g_1})^{(-1)}(x) < 0$ for $x < 0$ and $(e^{g_1})^{(-1)}(x) \to 0$ as $x \uparrow 0$. By Corollary 3.3, $K_1(z)$ belongs to the class $\mathcal{K}$. Therefore there exists $\mu_1 \in \mathcal{M}_+$ such that $K_{\mu_1}(z) = K_1(z)$ for $z \in \mathbb{C}^+$ and we proved the assertion of the lemma in the case $j = 1$.

Now we shall consider the function

$$w = g_2(z) := \log z - \frac{cz}{2} + i\sqrt{cz}. \qquad z \in \mathbb{C}^+. \tag{7.7}$$

Let us show that for every fixed $w \in \mathbb{C}^+$ such that $-2\pi < \operatorname{Im} w < -\pi$ there exists a unique $z \in \mathbb{C}^+$ such that $w = g_2(z)$.

To prove this assertion we consider as before the closed rectifiable curve $\gamma_7$ and as before $z$ traverses $\gamma_7$ in the counter clockwise direction.

If $z$ traverses $\gamma_{7,1}$ the image $g_2(\gamma_{7,1})$ is contained in the line $\operatorname{Im}\zeta = \pi$, connecting $\log R + cR/2 - \sqrt{cR} + i\pi$ to $-\log R + c/(2R) - \sqrt{c/R} + i\pi$.

If $z$ traverses $\gamma_{7,2}$ the image $g_2(\gamma_{7,2})$ is contained in the half-plane $\operatorname{Re}\zeta \leqslant -\log R + 1$, connecting $-\log R + c/(2R) - \sqrt{c/R} + i\pi$ to $-\log R - c/(2R) + i\sqrt{c/R}$.

If $z$ traverses $\gamma_{7,3}$ the image $g_2(\gamma_{7,3})$ is contained in the half-plane $\operatorname{Im}\zeta > 0$, connecting $-\log R - c/(2R) + i\sqrt{c/R}$ to $\log R - cR/2 + i\sqrt{cR}$.

If $z$ traverses $\gamma_{7,4}$ the image $g_2(\gamma_{7,4})$ is contained in the domain $|\zeta| > cR/4$, $\operatorname{Im}\zeta < \pi$, $\operatorname{Re}\zeta \leqslant 0$, and $\operatorname{Im}\zeta < \sqrt{cR}$, $\operatorname{Re}\zeta > 0$, connecting $\log R - cR/2 + i\sqrt{cR}$ to $\log R + cR/2 - \sqrt{cR} + i\pi$.

Therefore we deduce that the image $\zeta = g_2(z)$ winds around $w$ once when $z$ runs through $\gamma_7$. Hence there is a unique point $z$ inside the curve $\gamma_7$ such that $w = g_2(z)$. This relation holds for all sufficiently large $R > 1$. Thus we have proved that for every fixed $w \in \mathbb{S}_{-\pi} := \{w \in \mathbb{C} : -2\pi < \operatorname{Im} w < -\pi\}$ there exists a unique $z \in \mathbb{C}^+$ such that (7.7) holds. Then we deduce that the inverse function $g_2^{(-1)} : \mathbb{S}_{-\pi} \to \mathbb{C}^+$ exists and is analytic in $\mathbb{S}_{-\pi}$. Introduce the function $K_2(z) := g_2^{(-1)}(\log z - 2\pi i)$, $z \in \mathbb{C}^+$. Obviously, $K_2^{(-1)}(z) = z \exp\{-cz/2 + i\sqrt{cz}\}$, where $K_2^{(-1)}$ is well defined.

As in the case of $j = 1$ we show show that $K_2(z)$ admits an analytic continuation on $(-\infty, 0)$ and its value on $(-\infty, 0)$ is negative. Moreover $K_2(x) \to 0$ as $x \uparrow 0$. Hence, by Corollary 3.3, $K_2(z)$ belongs to the class $\mathcal{K}$. Therefore there exists $\mu_2 \in \mathcal{M}_+$ such that $K_{\mu_2}(z) = K_2(z)$ for $z \in \mathbb{C}^+$ and we proved the assertion of the lemma in the case $j = 2$ as well. $\square$

**Lemma 7.3.** *For every $c > 0$, there exist $\mu_j \in \mathcal{M}_+$, $j = 1, 2$, such that*

$$\Sigma_{\mu_j}(z) = \exp\left\{\frac{c}{2z} + (-1)^j i\sqrt{\frac{c}{z}}\right\}$$

*for $z$, where $\Sigma_{\mu_j}(z)$ is defined.*

The proof of this lemma is similar to the proof of Lemma 7.2 therefore we omit it.

Let the functions $f_j(z)$, $j = 1, 2$, are defined as in Lemma 7.1. In addition we assume in (7.1) that $0 < a \leqslant b < \infty$.

**Lemma 7.4.** *For sufficiently small $\varepsilon > 0$, there exist $\mu_j \in \mathcal{M}_+$, $j = 1, 2$, such that*

$$\Sigma_{\mu_j}(z) = \exp\{f_j(z)\}$$

*for $z$, where $\Sigma_{\mu_j}(z)$ is defined.*

*Proof.* Consider the functions

$$G(z) := \log z + \phi(z), \quad z \in \mathbb{C}^+.$$

By Lemma 5.3, $G : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{S}_\pi$ precisely once. Recall that $\mathbb{S}_\pi := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}$. For any $\eta \in (0, 1/10)$ denote by $\mathbb{S}_\eta$ the strip $\{z \in \mathbb{C} : \eta < \operatorname{Im} z < \pi - \eta\}$. Consider the domains $\Omega_\eta := G^{(-1)}(\mathbb{S}_\eta)$, $\eta \in (0, 1/10)$. Note that $\Omega_\eta$ is a domain such that $\Omega_\eta \subset \mathbb{C}^+$.

The boundary of $\Omega_\eta$ is a curve $\gamma_8(\eta)$ consisting of curves $\gamma_{8,1}(\eta)$ and $\gamma_{8,2}(\eta)$ characterized by the equations $\operatorname{Im} G(x + iy) = \pi - \eta$, $x < 0, y > 0$, and $\operatorname{Im} G(x + iy) = \eta$, $x > 0, y > 0$, respectively. Since $G'(z) \neq 0$ for $z \in \Omega_\eta$, we conclude, by the implicit function theorem, that $\gamma_{8,1}(\eta)$ and $\gamma_{8,2}(\eta)$, $\eta \in (0, 1/10)$, are smooth Jordan curves.

We see, by the definition of $\gamma_{8,1}(\eta)$, that this curve, connecting $\infty$ to 0, lies in the angular region $\pi - 2\eta < \arg z < \pi$. By definition the curve $\gamma_{8,2}(\eta)$, connecting 0 to $\infty$, lies in the union of domains $|z - (a + b)/2| < b/2$, $\operatorname{Im} z > 0$, and $0 < \arg z < 2\eta$.

Denote by $\tilde{\gamma}_8(\eta)$ the part of the curve $\gamma_8(\eta)$, lying in the disk $|z - (a + b)/2| \leqslant b/2$. Let us show that

$$\sup_{0<\eta\leqslant\frac{1}{10}} \sup_{z\in\tilde{\gamma}_8(\eta)} |\phi(z)| := A_2 < \infty. \tag{7.8}$$

Assume that $A_2 = \infty$. Then for any $N > 1$ there exist $\eta_0 \in (0, 1/10]$ and $z_0 = z_0(\eta_0) \in \tilde{\gamma}_8(\eta_0)$ such that $0 < \operatorname{Im} G(z_0) < \pi$ and $|G(z_0)| > N$. Note that $G(x)$ is real-valued for $x > 0$ and $x \in \mathbb{R} \setminus [a, b]$. Let $\operatorname{Im} z \neq 0$, then $\operatorname{Im} z \operatorname{Im} G(z) > 0$ for $0 < \operatorname{Re} z \leqslant a/2$ and for $\operatorname{Re} z \geqslant b + a/2$. In addition $\operatorname{Im} G(x) = \pi$ for $x < 0$ and $\operatorname{Im} G(z) > \pi$ for $\operatorname{Re} z < 0$ and $\operatorname{Im} z < 0$. Moreover $|\phi(z)|/\log|z| \to 0$ as $|z| = R, 1/R$ and $R \to \infty$. Therefore we easily conclude, by Rouché's theorem, that for $w \in \mathbb{S}_\pi$ with sufficiently large modulus the equation $w = G(z)$ has a solution $z = z(w) \in \mathbb{S}_\pi$ such that $z(w) = \exp\{w(1+o(1))\}$. Recalling that $G(z) : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{S}_\pi$ precisely once, we conclude that $z_0 = \exp\{G(z_0)(1 + o(1))\}$, a contradiction for sufficiently large $N$.

Now let us show for $j = 1, 2$ that $G_j(z) := \log z + f_j(z) : \mathbb{C}^+ \to \mathbb{C}$ takes every value in $\mathbb{S}_\pi$ precisely once in $G^{(-1)}(\mathbb{S}_\pi)$. By the formula

$$\operatorname{Im} G_j(z) = \arg z + \operatorname{Im} \phi(z) + \operatorname{Im} \phi(z)(1 + 2(-1)^j \varepsilon \operatorname{Re} \phi(z)) \tag{7.9}$$

and by (7.8), we see that $\operatorname{Im} G_j(z) \leqslant \eta$, where $z \in \gamma_{8,2}(\eta)$, $\eta \in (0, 1/10)$, and $\varepsilon > 0$ is sufficiently small.

Using (7.1), we note that, for $z \in \gamma_{8,1}(\eta), \eta \in (0, 1/10)$,

$$-\operatorname{Im} \phi(z) = \varepsilon_0 \operatorname{Im} z \int\limits_{[a,b]} \frac{(1+u^2)\,\nu(du)}{(u-\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} \leqslant \varepsilon_0 \frac{1+b^2}{a} \nu([a,b]) \frac{\operatorname{Im} z}{|\operatorname{Re} z|}$$

$$\leqslant 2\varepsilon_0 \frac{1+b^2}{a} \nu([a,b])\eta.$$

Therefore we conclude from (7.9) that $\operatorname{Im} G_j(z) \geqslant \pi - 2\eta$ for $z \in \gamma_{8,1}(\eta), \eta \in (0, 1/10)$, and sufficiently small $\varepsilon_0, \varepsilon > 0$. In addition we see from (7.1) that for $|z| = 1/R$ and $|z| = R$ the inequality $|\phi(z)| \leqslant c$ holds, with a positive constant $c$, depending on $\phi$ only, where $R > 1$ is sufficiently large.

Let $R > 1$ be sufficiently large. Denote by $c'$ and $c''$ the intersection points of the curve $\gamma_{8,1}(\eta)$ with the circles $|z| = R$ and $|z| = 1/R$, respectively. We denote as well by $d''$ and $d'$ the intersection points of the curve $\gamma_{8,2}(\eta)$ with the circles $|z| = 1/R$ and $|z| = R$, respectively.

Consider the closed rectifiable curve $\gamma_9 = \gamma_9(\eta)$, $\eta \in (0, 1/10)$, consisting of $\gamma_{9,1}$: the part of $\gamma_{8,1}(\eta)$ lying in the disc $|z| \leqslant R$, connecting $c'$ to $c''$, the arc $\gamma_{9,2}$ : $e^{i\theta}/R$, $\arg d'' < \theta < \arg c''$, connecting $c''$ to $d''$, the curve $\gamma_{9,3}$: the part of $\gamma_{8,2}(\eta)$ lying in the disc $|z| \leqslant R$, connecting $d''$ to $d'$, and the arc $\gamma_{9,4}$ : $Re^{i\theta}$, $\arg d' < \theta < \arg c'$, connecting $d'$ to $c'$.

Assume that $w \in \mathbb{S}_\pi$ and $\eta < \frac{1}{2}\min\{\operatorname{Im} w, \pi - \operatorname{Im} w\}$. If $z$ traverses $\gamma_{9,1}$, the image $\zeta = G_j(z)$ lies in the half-plane $\operatorname{Im} \zeta \geqslant \pi - 4\eta$. If $z$ traverses $\gamma_{9,2}$, it is easy to see that the image $\zeta = G_j(z)$ lies in the half-plane $\operatorname{Re} \zeta < -\frac{1}{2}\log R$. If $z$ traverses $\gamma_{9,3}$, the image $\zeta = G_j(z)$ lies in the half-plane $\operatorname{Im} \zeta \leqslant \eta$. Finally, if $z$ traverses $\gamma_{9,4}$, the image $\zeta = G_j(z)$, lies in the half-plane $\operatorname{Re} \zeta > \frac{1}{2}\log R$.

Summarizing, it follows that the image $\zeta = G_j(z)$ winds around $w$ once when $z$ traverses $\gamma_9$. By the argument principle, the function $G_j(z)$ takes the value $w$ precisely once inside $\gamma_9$. Since this assertion holds for all sufficiently large $R > 1$ and for all sufficiently small $\eta > 0$, we obtain the desired result.

This result implies that the inverse functions $\rho_j = G_j^{(-1)} : \mathbb{S}_\pi \to \mathbb{C}^+$, $j = 1, 2$, exist and are analytic on $\mathbb{S}_\pi$.

Let us show that $\rho_j(z)$ admit an analytic continuation on the half-line $\gamma_- : \operatorname{Im} z = \pi, \operatorname{Re} z < 0$, and that their values on this half-line are negative. It is easy to see that

$$G_j'(x) = \frac{1}{x} + 2\phi'(x)(1 + (-1)^j \varepsilon\phi(x)) < 0, \quad x < 0,\ j = 1, 2.$$

Since $G_j(z)$ is analytic on $(-\infty, 0)$, we conclude that $G_j^{(-1)}$ exists and is analytic on $\gamma_-$ as well. Since, as shown above, for every fixed $w \in \mathbb{S}_\pi$ there is an unique point $z \in G^{(-1)}(\mathbb{C}^+)$ such that $w = \log z + f_j(z)$ holds, this function coincides for $z \in \mathbb{S}_\pi$ with the function $\rho_j(z)$ obtained early. Introduce the function $\tilde{\rho}_j(z) := \rho_j(\log z)$, $z \in \mathbb{C}^+$. Note that $\tilde{\rho}_j \in \mathcal{N}$ and $\tilde{\rho}_j^{(-1)}(z) = z\exp\{f_j(z)\}$, on the domain $\mathbb{C}^+$, where $\tilde{\rho}_j^{(-1)}$ is uniquely defined. Moreover, the function $\tilde{\rho}_j(z)$ admits an analytic continuation on $(-\infty, 0)$.

From the definition of $\tilde{\rho}_j(z)$ it follows that $\tilde{\rho}_j(x) < 0$ for $x < 0$ and $\tilde{\rho}_j(x) \to 0$ as $x \uparrow 0$. By Corollary 3.3, $\tilde{\rho}_j$, $j = 1, 2$, belong to the class $\mathcal{K}$ and we conclude the assertion of the lemma. $\square$

Now we complete the proof of Theorem 2.13 for the semigroup $(\mathcal{M}_+, \boxtimes)$.

At first note that the measure $\mu \in \mathcal{M}_+$ with $\Sigma_\mu(z) = \exp\{-cz\}$, where $c > 0$, does not belong to the class $I_0$. Indeed, by Lemma 7.2, there exist measures $\mu_1$ and $\mu_2$ of $\mathcal{M}_+$ such that

$$\Sigma_{\mu_j}(z) = \exp\{\tilde{f}_j(z)\}, \quad \text{where} \quad \tilde{f}_j(z) := -cz/2 + (-1)^j i\sqrt{cz}, \quad j = 1, 2,$$

for $z$, where $\Sigma_{\mu_j}(z)$ are defined. In addition $\tilde{f}_j(z)$ admit an analytic continuation in $\mathbb{C}^+$ and $\operatorname{Im} \tilde{f}_j(z)$ have positive values at some points of the half-plane $\mathbb{C}^+$. Hence, by Bercovici and Voiculescu result [11], [12], $\mu_j$, $j = 1, 2$, are not i.d. elements in the semigroup $(\mathcal{M}_+, \boxtimes)$. On the other hand $\Sigma_\mu(z) = \Sigma_{\mu_1}(z)\Sigma_{\mu_2}(z)$ and we see that $\mu$ has non-i.d. factors as was to be proved. In a similar way, using Lemma 7.3, we prove that the measure $\mu \in \mathcal{M}_+$ with $\Sigma_\mu(z) = \exp\{c/z\}$, where $c > 0$, does not belong to the class $I_0$.

Now we consider the case where $\Sigma_\mu(z)$ admits a representation (2.23) with $a = b = 0$, $\nu(\{0\}) = 0$ and $\nu((0, \infty)) > 0$.

For every fixed $0 < a \leqslant b$, we have the representation

$$\Sigma_\mu(z) = \Sigma_{\mu_1}(z)\Sigma_{\mu_2}(z)\Sigma_{\mu_3}(z),$$

where

$$\Sigma_{\mu_3}(z) := \exp\Big\{4\varepsilon_0 \int\limits_{[a,b]} u\,\nu(du) + (1 - 4\varepsilon_0) \int\limits_{[a,b]} \frac{1 + uz}{z - u}\,\nu(du)\Big\} + \int\limits_{\mathbb{R}\setminus[a,b]} \frac{1 + uz}{z - u}\,\nu(du)$$

and $\Sigma_{\mu_j}(z)$, $j = 1, 2$, are defined in Lemma 7.4. By Bercovici and Voiculescu result (see Section 2), the p-measure $\mu_3$ is i.d. for $\varepsilon_0 \leqslant 1/4$.

If $\nu(\{a\}) > 0$, then we assume in (7.1) $a = b$. If $\nu(\{a\}) = 0$ for all $a \in (0, \infty)$, then we choose the points $0 < a < b$ such that $\nu([a, a+h]) > ch$ and $\nu([b-h, b]) > ch$ for all $0 < h \leqslant h_0$, where $c > 0$ and $h_0 > 0$ depend on the measure $\nu$ only. Such points exist by Proposition 3.7. We showed in Section 7.1 that under these assumptions the function $\phi(z)$, see (7.1), has the limiting behaviour (7.5).

Note that the measures $\mu_j$, $j = 1, 2$, are not i.d. Indeed, by (7.5), there exist points in $\mathbb{C}^+$ at which the functions $\operatorname{Im} f_1(z)$ and $\operatorname{Im} f_2(z)$ have positive values and the desired assertion follows from Bercovici and Voiculescu result (see Section 2). Hence the measure $\mu$ has a non-i.d. factor and $\mu$ does not belong to the class $I_0$. $\square$

**7.3**. It remains to prove Theorem 2.13 for the semigroup $(\mathcal{M}_*, \boxtimes)$. We need the following auxiliary result.

**Lemma 7.5.** *For sufficiently small $\varepsilon > 0$, there exist $\mu_j \in \mathcal{M}_*$, $j = 1, 2$, such that, for $z$, where $\Sigma_{\mu_j}(z)$ is defined,*

$$\Sigma_{\mu_j}(z) = \exp\{q_j(z)\}.$$

*Here $q_j(z) := 2q(z) + (-1)^j i\varepsilon q^2(z)$, $j = 1, 2$, and, for $\xi_0 \in \mathbb{T}$ and $0 < \Delta < 1/100$,*

$$q(z) := \varepsilon_0 \int\limits_{\{\xi\in\mathbb{T}:\,|\xi-\xi_0|\leqslant\Delta\}} \frac{\xi + z}{\xi - z}\,\nu(d\xi), \quad z \in \mathbb{D}. \tag{7.10}$$

*In (7.10) $\varepsilon_0 > 0$ is a sufficiently small constant, depending on $\nu$ and $\Delta$ only, and $\nu(\{\xi \in \mathbb{T} : |\xi - \xi_0| \leqslant \Delta\}) > 0$.*

*Proof.* Let us prove the lemma in the case $j = 1$. One can prove the lemma in the case $j = 2$ in the same way. Without loss of generality we assume that in the definition of the function $q(z)$ the parameter $\xi_0$ is equal to 1.

Consider the function

$$Q(z) := \log z + q(z), \quad z \in \tilde{\mathbb{D}} := \mathbb{D} \setminus [-1, 0].$$

This function is analytic on $\tilde{\mathbb{D}}$.

Denote by $\gamma_{10,1}$, $\gamma_{10,2}$, and $\gamma_{10,3}$ the Jordan curves defined by the parametric equations $\zeta = \log(1-t)+i\pi+q(t-1)$, $0 \leqslant t < 1$, $\zeta = it+q(-1)$, $-\pi \leqslant t \leqslant \pi$, and $\zeta = \log t-i\pi+q(-t)$, $0 < t \leqslant 1$, respectively. Note that $\operatorname{Re} q(-1) = 0$. Define a curvilinear half-strip $S_-$ as a domain with boundary $\gamma_{10,1}, \gamma_{10,2}$, and $\gamma_{10,3}$. Let us show that $Q : \tilde{\mathbb{D}} \to \mathbb{C}$ takes every value in $S_-$ precisely once. For this we need to prove that for any $w \in S_-$ there exists an unique point $z \in \tilde{\mathbb{D}}$ such that $w = Q(z)$, provided that $\varepsilon_0 > 0$ is sufficiently small.

Consider the contour $\gamma_3$ with parameter $\theta = \pi$ (see Section 5). Choose $r \in (0, 1)$ such that $1 - r$ is small. If $z$ traverses $\gamma_{3,1}$ the image $\zeta = Q(z)$ lies on the curve $\gamma_{10,1}$. If $z$ traverses $\gamma_{3,2}$ the image $\zeta = Q(z)$ lies in the half-plane $\operatorname{Re}\zeta < \frac{1}{2}\log(1 - r)$. If $z$ traverses $\gamma_{3,3}$ the image $\zeta = Q(z)$ lies on the curve $\gamma_{10,3}$. Finally, by the inequality $\operatorname{Re} q(z) \geqslant 0$, $z \in \mathbb{D}$, if $z$ traverses $\gamma_{3,4}$ the image $\zeta = Q(z)$ lies in the half-plane $\operatorname{Re}\zeta > \log r$.

Hence, the image $\zeta = Q(z)$ winds around $w$ once when $z$ runs through $\gamma_3$ in the counter clockwise direction. By the argument principle, the function $Q(z)$ takes the value $w \in S_-$ precisely once inside $\gamma_3$ for sufficiently small $1 - r$ and the desired result is proved. Hence the inverse function $Q^{(-1)} : S_- \to \tilde{\mathbb{D}}$ is defined and is analytic function on $S_-$.

Let $\gamma_{10,2}(\eta)$ is the half-open interval of the vertical line $\operatorname{Re} z = -\eta, \eta \in (0, 1/100)$, lying between the curves $\gamma_{10,1}$ and $\gamma_{10,3}$. Denote by $\gamma_{11}(\eta)$ the closed rectifiable Jordan curve $Q^{(-1)}(\gamma_{10,2}(\eta))$. By the definition of $\gamma_{11}(\eta)$, we note that $\operatorname{Re} Q(z) = -\eta$ for $z \in \gamma_{11}(\eta)$, $\eta \in (0, 1/100)$. Since $Q : \tilde{\mathbb{D}} \to \mathbb{C}$ takes every value in $S_-$ precisely once, we note that if $z$ traverses the curve $\gamma_{11}(\eta)$ in the counter clockwise direction, then $\operatorname{Im} Q(z)$ is a monotone function such that $-\pi + \alpha(-\eta) \leqslant \operatorname{Im} Q(z) \leqslant \pi + \alpha(-\eta)$, where, as it is easy to see, $|\alpha(-\eta) - q(-1)| \leqslant 1/10$ for $\eta \in (0, 1/100)$.

Let us prove

$$\sup_{0<\eta\leqslant\frac{1}{100}}\ \sup_{z\in\gamma_{11}(\eta)}|q(z)|\leqslant 1/3. \tag{7.11}$$

We shall assume that $\eta\in(0,1/100)$. Note that $|q(z)|\leqslant 1/10$ for $z\in\gamma_{11}(\eta)\cap\{|z-1|\geqslant 2\Delta\}$. From the relation $\operatorname{Re}Q(z)=-\eta$, $z\in\gamma_{11}(\eta)$, and the inequality $\log|z|\geqslant\log(1-2\Delta)$ for $z\in\gamma_{11}(\eta)\cap\{|z-1|<2\Delta\}$ we see that the bound $|\operatorname{Re}q(z)|\leqslant 1/10$ holds for such $z$. In addition it is not difficult to see that the inequality $|Q(z')|\leqslant 1/10$ holds at points of intersection $z'$ of $\gamma_{11}(\eta)$ with the circle $|z-1|=2\Delta$. Since $\operatorname{Im}Q(z)$, $z\in\gamma_{11}(\eta)$, is a monotone function, we have $|\operatorname{Im}Q(z)|\leqslant 1/10$ and hence $|\operatorname{Im}q(z)|\leqslant-\log(1-2\Delta)+\arctan(2\Delta)+1/10\leqslant 1/5$ for $z\in\gamma_{11}(\eta)\cap\{|z-1|<2\Delta\}$. Thus (7.11) is proved.

Then, by the relation

$$\log|z|+\operatorname{Re}q_1(z)=\operatorname{Re}Q(z)+\operatorname{Re}q(z)(1+2\varepsilon\operatorname{Im}q(z))$$

and (7.11), we conclude that

$$\log|z|+\operatorname{Re}q_1(z)\geqslant-\eta,\ z\in\gamma_{11}(\eta),\ \eta\in(0,1/100). \tag{7.12}$$

Now we shall prove that $\tilde q_1(z):=ze^{q_1(z)}:\mathbb{D}\to\mathbb{C}$ takes every value $w\in\mathbb{D}$ precisely once in $Q^{(-1)}(S_-)\cup(-1,0]$.

Fix $w\in\mathbb{D}$. Let $\eta>0$ be sufficiently small. If $z$ traverses $\gamma_{11}(\eta)$ in the counter clockwise direction we see, by (7.11) and (7.12), that $|Arge^{q_1(z)}|\leqslant 1<\pi$ and $|e^{q_1(z)}|\geqslant e^{-\eta}$, respectively. By the argument principle, the function $ze^{q_1(z)}$ takes the value $w\in\mathbb{D}$ precisely once inside $\gamma_{11}(\eta)$ for sufficiently small $\eta>0$. We obtain the desired result.

Hence the inverse function $\tilde q_1^{(-1)}:\mathbb{D}\to\mathbb{D}$ thus defined is analytic on $\mathbb{D}$ and, as it is easy to see, belongs to the class $\mathcal{S}_*$. The lemma is proved. □

Let us complete the proof of Theorem 2.13 for the semigroup $(\mathcal{M}_*,\boxtimes)$. Without loss of generality, we assume that $\nu(A)>0$, where $A:=\{\xi\in\mathbb{T},|\xi-1|\leqslant\Delta\}=\{\xi\in\mathbb{T},-\Delta_1\leqslant\arg\xi\leqslant\Delta_1\}$ and write

$$\Sigma_\mu(z)=\Sigma_{\mu_1}(z)\Sigma_{\mu_2}(z)\Sigma_{\mu_3}(z),$$

where

$$\Sigma_{\mu_3}(z):=(1-4\varepsilon_0)\int\limits_A\frac{\xi+z}{\xi-z}\,\nu(d\xi)+\int\limits_{\mathbb{T}\setminus A}\frac{\xi+z}{\xi-z}\,\nu(d\xi)$$

and $\Sigma_j(z)$, $j=1,2$, are defined in Lemma 7.5. If $\nu(\{1\}>0$, we assume $A:=\{1\}$. If $\nu(\{\xi\})=0$ for all $\xi\in\mathbb{T}$, we choose the point $\Delta_1>0$ so that $\nu(\{\xi\in\mathbb{T},\Delta_1-h\leqslant\arg\xi\leqslant\Delta_1\}>ch$ for all $0<h\leqslant h_0$, where the constant $c>0$ and $h_0>0$ depend on the measure $\nu$ only. By Proposition 3.6, such points exist. Repeating the arguments used in the proof of (7.5), we obtain

$$\operatorname{Im}q(e^{i(h+\Delta_1)}(1-h))\to+\infty,\qquad h\downarrow 0. \tag{7.13}$$

Note that the measures $\mu_j, j = 1, 2$ are not i.d. Indeed, $q_2(z)$ is an analytic function in $\mathbb{D}$ and, by (7.13), there exist points $z \in \mathbb{D}$ where $\operatorname{Re} q_2(z) < 0$. Hence, by Bercovici and Voiculescu result [11], the measure $\mu$ has a non-i.d. factor and therefore $\mu \notin I_0$. $\square$

Now we shall prove that there exists a measure $\mu \in \mathbf{M}$ for which the representation (2.29) is not unique. We establish this result for the semigroup $(\mathcal{M}, \boxplus)$. One can prove this fact for the semigroups $(\mathcal{M}_+, \boxtimes)$ and $(\mathcal{M}_*, \boxtimes)$ in the same way.

Assume to the contrary that every measure $\mu \in \mathcal{M}$, which has indecomposable factors, admits a unique representation

$$\mu = \mu_1 \boxplus \mu_2 \dots, \tag{7.14}$$

where $\mu_1, \mu_2, \dots$ are some indecomposable nondegenerate elements of the semigroup $(\mathcal{M}, \boxplus)$, with respect to the equivalence relation $\mu \sim \nu$ if $\mu = \nu \boxplus \delta_a$ for some $a \in \mathbb{R}$.

Let $\mu$ be an i.d. p-measure. Hence, for every $n \in \mathbb{N}$, $\mu = \nu_n \boxplus \dots \boxplus \nu_n$ ($n$ times), where $\nu_n \in \mathcal{M}$. Since the p-measure $\nu_n$ admits an unique representation of the form (7.14), we see that, for every $n \in \mathbb{N}$, $\mu = \mu_1^{n\boxplus} \boxplus \rho_n$, where $\mu_1$ denotes the measure from the representation (7.14) and $\rho_n \in \mathcal{M}$. We return to the notation of Section 6. Note that the measure $\mu$ belongs to the set $\mathcal{M}^{(\alpha,\beta)}$ for some $\alpha > 0$ and $\beta > 0$. Hence $0 < -\operatorname{Im} \varphi_\mu(i(\beta+1)) < \infty$. By Proposition 6.1, $\mu_1^{n\boxplus}$, $\rho_n \in \mathcal{M}^{(\alpha,\beta)}$ as well and $-\operatorname{Im} \varphi_{\mu_1^{n\boxplus}}(i(\beta+1)) \geqslant 0$ and $-\operatorname{Im} \varphi_{\rho_n}(i(\beta+1)) \geqslant 0$. Since

$$\begin{aligned}\operatorname{Im} \varphi_\mu(i(\beta+1)) &= \operatorname{Im} \varphi_{\mu_1^{n\boxplus}}(i(\beta+1))) + \operatorname{Im} \varphi_{\rho_n}(i(\beta+1)) \\ &= n \operatorname{Im} \varphi_{\mu_1}(i(\beta+1))) + \operatorname{Im} \varphi_{\rho_n}(i(\beta+1)),\end{aligned}$$

we have $\operatorname{Im} \varphi_{\mu_1}(i(\beta+1)) \to 0$ as $n \to \infty$ and we get $\operatorname{Im} \varphi_{\mu_1}(i(\beta+1)) = 0$. By Proposition 6.2, $\mu_1 = \delta_a$ for some $a \in \mathbb{R}$. We arrive at a contradiction which proves the assertion. $\square$

## 8. Dense classes of indecomposable elements in $(\mathbf{M}, \circ)$

In this section we describe a wide class of indecomposable elements in $(\mathbf{M}, \circ)$. Theorem 2.14 and Corollary 2.15 follow immediately from our results.

**8.1**. At first we shall formulate and prove our result for the semigroup $(\mathcal{M}, \boxplus)$.

**Theorem 8.1.** *Let $\mu \in \mathcal{M}$ such that $\mu(\{a\}) > 0$, $\mu([b,\infty)) > 0$ and $\mu(\{a\}) + \mu([b,\infty)) = 1$, where $a < b$ are real numbers and $\lim_{x \uparrow b} G_\mu(x) = 0$. Then $\mu$ is an indecomposable p-measure in $(\mathcal{M}, \boxplus)$.*

From Proposition 3.7 it follows that there exist p-measures $\mu$ with an unique atom $a$ satisfying conditions of Theorem 8.1 (see similar arguments in Section 7). Therefore the statement of this theorem does not follow from Belinschi [5] and Bercovici and Wang [17] results. This conclusion holds for Theorem 8.2 and 8.3 as well.

*Proof.* It follows from the assumptions of the theorem that $F_\mu(z)$ is an analytic function on $\mathbb{C} \setminus [b,\infty)$ and $F_\mu(x)$ is strictly monotone on $(-\infty, b)$. In addition $F_\mu(x) \downarrow -\infty$ as

$x \downarrow -\infty$ and $F_\mu(x) \uparrow +\infty$ as $x \uparrow b$. Using the Stieltjes-Perron inversion formula (3.7), we see that $F_\mu(z)$ admits the representation

$$F_\mu(z) = z + c + \int\limits_{[b,\infty)} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\sigma(dt), \quad z \in \mathbb{C}^+, \tag{8.1}$$

where $c \in \mathbb{R}$ and $\sigma$ is a nonnegative measure such that $\int_{[b,\infty)} \sigma(dt)/(1+t^2) < \infty$.

Let $\mu_j \in \mathcal{M}$, $j = 1, 2$, and $\mu = \mu_1 \boxplus \mu_2$. Let us prove that either $\mu_1 = \mu \boxplus \delta_\alpha$, $\mu_2 = \delta_{-\alpha}$ or $\mu_1 = \delta_{-\alpha}$, $\mu_2 = \mu \boxplus \delta_\alpha$ with $\alpha \in \mathbb{R}$. By Theorem 2.1, there exist functions $Z_j(z) \in \mathcal{F}$, $j = 1, 2$, such that (2.3) holds and $F_\mu(z) = F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z))$, $z \in \mathbb{C}^+$. Using the integral representation (3.3) for Nevanlinna functions, we rewrite the relation (8.1) in the form

$$\begin{aligned} z + c + \int\limits_{[b,\infty)} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\sigma(dt) &= b_j + Z_j(z) + W_j(z) \\ := b_j + Z_j(z) + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-Z_j(z)} - \frac{t}{1+t^2}\Big)\sigma_j(dt), &\quad z \in \mathbb{C}^+,\ j = 1, 2, \end{aligned} \tag{8.2}$$

where $b_j \in \mathbb{R}$ and $\sigma_j$ are nonnegative measures such that $\int_{\mathbb{R}} \sigma_j(dt)/(1+t^2) < \infty$. Since, again by the representation (3.3) for functions in $\mathcal{F}$, we have

$$Z_j(z) = z + c_j + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\nu_j(dt), \quad z \in \mathbb{C}^+,\ j = 1, 2,$$

where $c_j \in \mathbb{R}$ and $\nu_j$ are nonnegative measures such that $\int_{\mathbb{R}} \nu_j(dt)/(1+t^2) < \infty$. Moreover, by (3.4), $|Z_j(iy) - iy| = o(y)$, $j = 1, 2$, as $y \to +\infty$. We note that the functions $W_j(z) \in \mathcal{N}$, $j = 1, 2$, and, as it is easy to see, $|W_j(iy)|/y \to 0$, $j = 1, 2$, as $y \to +\infty$. Therefore, by (3.3) and (3.4), $W_j(z)$, $j = 1, 2$, admit the representation

$$W_j(z) = \tilde{c}_j + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\tilde{\sigma}_j(dt), \quad z \in \mathbb{C}^+,$$

where $\tilde{c}_j \in \mathbb{R}$ and $\tilde{\sigma}_j$ are nonnegative measures such that $\int_{\mathbb{R}} \tilde{\sigma}_j(dt)/(1+t^2) < \infty$. Hence, we finally obtain, for $z \in \mathbb{C}^+$ and $j = 1, 2$,

$$c + \int\limits_{[b,\infty)} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\sigma(dt) = b_j + c_j + \tilde{c}_j + \int\limits_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)(\nu_j + \tilde{\sigma}_j)(dt). \tag{8.3}$$

Applying the Stieltjes-Perron inversion formula (3.7) to (8.3), we see that the measures $\nu_j$, $j = 1, 2$, have supports which are contained on the set $[b, \infty)$. Hence either $Z_j(z) = z + c_j$ or

$$Z_j(z) = z + c_j + \int\limits_{[b,\infty)} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\,\nu_j(dt), \tag{8.4}$$

where $c_j \in \mathbb{R}$ and $\int_{[b,\infty)} \nu_j(dt)/(1+t^2) > 0$.

Let one of $Z_j(z)$, say $Z_1(z)$, be of the form $Z_1(z) = z + c_1$, then $F_{\mu_1}(z) = F_\mu(z - c_1)$ and we have $\mu_1 = \mu \boxplus \delta_{c_1}$ and $\varphi_{\mu_1}(z) = \varphi_\mu(z) + c_1$. From the relation (2.4) $\varphi_\mu(z) = \varphi_{\mu_1}(z) + \varphi_{\mu_2}(z)$, we obtain $\varphi_{\mu_2}(z) = -c_1$ which implies $\mu_2 = \delta_{-c_1}$.

It remains to consider the case where both $Z_1(z)$ and $Z_2(z)$ have the form (8.4). In this case $Z_1(z)$ and $Z_2(z)$ are analytic functions on $\mathbb{C} \setminus [b,\infty)$. Moreover $Z_1'(x) > 0$ and $Z_2'(x) > 0$ for $x \in (-\infty, b)$, and $Z_1(x) \downarrow -\infty$ and $Z_2(x) \downarrow -\infty$ for $x \downarrow -\infty$.

By the relation (2.3), we obtain

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) = Z_1(z) + Z_2(z) - F_\mu(z), \quad z \in \mathbb{C}^+.$$

Recalling that $F_\mu(x) \uparrow +\infty$ as $x \uparrow b$ we see from this formula that one of $Z_j(z)$, say $Z_1(z)$, has the property $Z_1(x) \uparrow +\infty$ as $x \uparrow b$.

Since $Z_1(z) \in \mathcal{F}$, the function $1/(t - Z_1(z))$ is in $\mathcal{N}$ and $z/(t - Z_1(z))$ converges to 1 as $z \to \infty$ for any fixed $t \in \mathbb{R}$. Moreover the function $1/(t - Z_1(z))$ is analytic on $\mathbb{C} \setminus [b,\infty)$ with the exception of a simple pole $\beta = \beta(t) \in (-\infty, b)$. Note that the function $\beta(t)$, as a function on the variable $t$, is a strictly increasing continuous function such that $\beta(t) \downarrow -\infty$ as $t \downarrow -\infty$ and $\beta(t) \uparrow b$ as $t \uparrow \infty$. In view of these properties we conclude that the Nevanlinna integral representation (3.3) for the functions $1/(t - Z_1(z))$, $z \in \mathbb{C}^+$ for every fixed $t \in \mathbb{R}$, has the form

$$\frac{1}{t - Z_1(z)} = \frac{\alpha(t)}{\beta(t) - z} + W(z,t) := \frac{\alpha(t)}{\beta(t) - z} + \int\limits_{[b,\infty)} \frac{\rho_t(du)}{u - z}, \tag{8.5}$$

where $\alpha(t) > 0$ and $\rho_t$ is a nonnegative finite measure. Moreover $\alpha(t)$ is a continuous function. We see from (8.5) that $W(z,t) \in \mathcal{N}$ and $W(z,t)$ is bounded by modulus for $z$ from every compact set in $\mathbb{C}^+$ and $t \in [-N, N]$ for every $N > 0$. From (8.2) with $j = 1$ and (8.5) we obtain the relation, for every $a > 0$,

$$z + c + \int\limits_{[b,\infty)} \Big(\frac{1}{t - z} - \frac{t}{1+t^2}\Big)\, \sigma(dt) = Z_1(z) + \int\limits_{[-N,N]} \frac{\alpha(t)\, \sigma_1(dt)}{\beta(t) - z} + T_N(z), \quad z \in \mathbb{C}^+, \tag{8.6}$$

where $T_N(z) \in \mathcal{N}$.

Since $\alpha(t) > 0$, $t \in \mathbb{R}$, and $\beta(t) < b$, $t \in \mathbb{R}$, we note that the second summand on the right hand-side of (8.6) can be written in the form

$$\int\limits_{[-N,N]} \frac{\alpha(t)\, \sigma_1(dt)}{\beta(t) - z} = \int\limits_{[t_1,t_2]} \frac{\alpha(\beta^{(-1)}(u))\, \tilde{\sigma}_1(du)}{u - z},$$

where $t_1 := \beta(-N) < b$, $t_2 := \beta(N) < b$, and $\tilde{\sigma}_1$ is a measure such that $\tilde{\sigma}_1(B) := \sigma_1(\beta^{(-1)}(B))$ for any Borel set $B$. Note that $\tilde{\sigma}_1([t_1, t_2]) = \sigma_1([-N, N])$. Applying to both sides of (8.6) the inversion formula (3.4), we obtain that $\sigma_1([-N, N]) = 0$ for every $N > 0$ and therefore $\sigma_1(\mathbb{R}) = 0$. Hence the relation (8.6) implies that $\sigma_1 \equiv 0$. Thus, $F_{\mu_1}(z) = z + b_1$ and $\mu_1 = \delta_{-b_1}$. Hence $\mu_2 = \mu \boxplus \delta_{b_1}$. This implies the statement of the theorem. $\square$

**8.2**. We shall now formulate and prove the result for the semigroup $(\mathcal{M}_+, \boxtimes)$.

**Theorem 8.2.** *Let* $\mu \in \mathcal{M}_+$ *such that* $\mu((0,a]) > 0$, $\mu(\{b\}) > 0$, *and* $\mu([0,a]) + \mu(\{b\}) = 1$, *where* $0 < a < b$, *and* $\lim_{x\uparrow 1/a} \psi_\mu(x) \geqslant -1$. *Then* $\mu$ *is an indecomposable p-measure in* $(\mathcal{M}_+, \boxtimes)$.

*Proof.* Denote $a_1 := 1/a$ and $b_1 := 1/b$. We have from the assumptions of the theorem that $\psi_\mu(z)$ is an analytic function on $\mathbb{C} \setminus [a_1, \infty)$ with the exception of the simple pole $b_1$. Moreover $\psi_\mu(x)$ is a strictly monotone function on $(-\infty, b_1)$ and on $(b_1, a_1)$, and $\psi_\mu(x) \downarrow -1 + \mu(\{0\})$ as $x \downarrow -\infty$, $\psi_\mu(x) \uparrow \infty$ as $x \uparrow b_1$, $\psi_\mu(x) \downarrow -\infty$ as $x \downarrow b_1$, and $\lim_{x\uparrow a_1} \psi_\mu(x) \geqslant -1$.

Since $K_\mu(z) := 1 - 1/(1 + \psi_\mu(z))$, we conclude that $K_\mu(z)$ is an analytic function on $\mathbb{C} \setminus [d, \infty)$, where $b_1 < d \leqslant a_1$ is a point such that $\lim_{x\uparrow d} \psi_\mu(x) = -1$. In addition $K_\mu(x)$ is a strictly monotone function on $(-\infty, d)$, and $K_\mu(0) = 0$ and $K_\mu(x) \uparrow \infty$ as $x \uparrow d$. Since $K_\mu(z) \in \mathcal{K}$ it admits the integral representation

$$K_\mu(z) = \gamma z + \int\limits_{[d,\infty)} \frac{z}{u - z}\, \tau(du), \tag{8.7}$$

where $\gamma \geqslant 0$ and $\tau$ is a nonnegative measure such that $\int_{[d,\infty)} \tau(du)/(1+u) < \infty$.

Let $\mu_1, \mu_2$ denote measures in $\mathcal{M}_+$ and assume that $\mu := \mu_1 \boxtimes \mu_2$. We have to prove that either $\mu_1 = \mu \boxtimes \delta_a$ and $\mu_2 = \delta_{1/a}$ or $\mu_1 = \delta_{1/a}$ and $\mu_2 = \mu \boxtimes \delta_a$, where $a \in (0, +\infty)$.

By Theorem 2.4, there exist functions $Z_j(z) \in \mathcal{K}$, $j = 1, 2$, such that $K_\mu(z) = K_{\mu_1}(Z_1(z)) = K_{\mu_2}(Z_2(z))$ for $z \in \mathbb{C}^+$ and

$$Z_1(z) Z_2(z) = z K_\mu(z), \quad z \in \mathbb{C}^+, \quad j = 1, 2. \tag{8.8}$$

Since $K_{\mu_j}(z) \in \mathcal{K}$, $j = 1, 2$, rewrite the first of these relations in the form

$$K_\mu(z) = d_j Z_j(z) + \int\limits_{(0,\infty)} \frac{Z_j(z)}{u - Z_j(z)}\, \tau_j(du), \quad z \in \mathbb{C}^+,\ j = 1, 2, \tag{8.9}$$

where $d_j \geqslant 0$ and $\tau_j$ are nonnegative measures such that $\int_{(0,\infty)} \tau_j(du)/(1+u) < \infty$.

Let us show that $Z_j(z)$, $j = 1, 2$, are analytic functions on $\mathbb{C} \setminus [d, \infty)$.

If $d_j \neq 0$ in (8.9), then, applying the Stieltjes-Perron inversion formula (3.7) to (8.9), we note that $Z_j(z)$ is an analytic function on $\mathbb{C} \setminus [d, \infty)$.

Let $d_j = 0$. Recalling the definition of the Krein class $\mathcal{K}$, note that the functions

$$K_j(z; u) = \frac{Z_j(z)}{u - Z_j(z)}, \qquad z \in \mathbb{C}^+, \quad j = 1, 2,$$

belong to $\mathcal{K}$ for every fixed $u > 0$. Therefore they admit the representation

$$K_j(z; u) = a_j(u) z + z \int\limits_{(0,\infty)} \frac{\tau_{u,j}(dt)}{t - z}, \quad z \in \mathbb{C}^+, \quad u > 0, \quad j = 1, 2, \tag{8.10}$$

where $a_j(u) \geqslant 0$ and $\tau_{u,j}$ are nonnegative measures, satisfying the condition

$$\int\limits_{(0,\infty)} \frac{\tau_{u,j}(dt)}{(1+t)} < \infty.$$

It is easy to see that $a_j(u) = 0$, $u > 0$, $j = 1, 2$. Moreover the functions $u \mapsto \tau_{u,j}(B)$, where $B$ is a Borel set on $(0, \infty)$, are measurable. Using (8.10), we easily deduce from (8.9) with $d_j = 0$ the relation

$$K_\mu(z) = \int\limits_{(0,\infty)} K_j(z; u)\, \tau_j(du) = z \int\limits_{(0,\infty)} \frac{\nu_j(dt)}{t - z}, \quad z \in \mathbb{C}^+, \qquad (8.11)$$

where $\nu_j(B) = \int_{(0,\infty)} \tau_{u,j}(B)\, \tau_j(du)$ for every Borel set on $\mathbb{R}_+$ and $\int_{(0,\infty)} \nu_j(dt)/(1+t) < \infty$.

By (8.7) and the Stieltjes-Perron inversion formula (3.7), we deduce from (8.11) that $\nu_j((0, d)) = 0$. Hence there exists a measurable set $B_j$ such that $\int_{B_j} \tau_j(dt)/(1 + t) > 0$ and $\tau_{u,j}((0, d)) = 0$ for $u \in B_j$. Thus we have from (8.10) the formula

$$K_j(z; u) = z \int\limits_{[d,\infty)} \frac{\tau_{u,j}(dt)}{t - z}, \qquad z \in \mathbb{C}^+, \quad u \in B_j, \quad j = 1, 2.$$

Since $Z_j(z) = uK_j(z; u)/(1+K_j(z; u))$ for $u \in B_j$, are analytic functions on $\mathbb{C} \setminus [d, \infty)$, we proved the desired assertion.

Hence $Z_j(z)$, $j = 1, 2$, are analytic functions on $\mathbb{C} \setminus [d, \infty)$ and $Z_j(0) = 0$. Moreover $Z_j(x)$, $j = 1, 2$, are strictly monotone functions on $(-\infty, d)$. By (8.8), we conclude that one of $Z_j(z)$, say $Z_1(z)$, has the following property: $Z_1(x) \uparrow \infty$ as $x \uparrow d$. Since $Z_1(z) \in \mathcal{K}$, the function $Z_1(z)/(u - Z_1(z))$ is in $\mathcal{K}$ for every fixed $u \in (0, \infty)$. Moreover the function $Z_1(z)/(u - Z_1(z))$ is analytic on $\mathbb{C} \setminus [d, \infty)$ with the exception of a simple pole $\beta = \beta(u) \in (0, d)$. Note that the function $\beta(u)$, as a function of the variable $u$, is strictly increasing continuous function such that $\beta(u) \downarrow 0$ as $u \downarrow 0$ and $\beta(u) \uparrow d$ as $u \uparrow \infty$. In view of these properties we conclude that the Nevanlinna integral representation (3.8) for the functions $Z_1(z)/(z(u - Z_1(z)))$, $z \in \mathbb{C}^+$, for every fixed $u \in (0, \infty)$, has the form

$$\frac{1}{z} \cdot \frac{Z_1(z)}{u - Z_1(z)} = \frac{\alpha(u)}{\beta(u) - z} + R(z, u) := \frac{\alpha(u)}{\beta(u) - z} + p + \int\limits_{[d,\infty)} \frac{\rho_u(ds)}{s - z}, \qquad (8.12)$$

where $p \geqslant 0$, $\alpha(u) > 0$ and $\rho_u(ds)$ is a nonnegative measure such that $\int_{[d,\infty)} \rho_u(ds)$ $/(1 + s) < \infty$. Moreover $\alpha(u)$ is a continuous function. We see from (8.12) that $R(z, u) \in \mathcal{N}$ and $R(z, u)$ is bounded by modulus for $z$ from every compact set in $\mathbb{C}^+$ and $u \in (0, N]$ for every $N > 0$. Recalling (8.7), (8.9) with $j = 1$ and (8.12), we obtain the relation, for every $N > 0$,

$$\gamma + \int\limits_{[d,\infty)} \frac{\tau(du)}{u-z} = d_1 \frac{Z_1(z)}{z} + \int\limits_{(0,N]} \frac{\alpha(u)\,\tau_1(du)}{\beta(u)-z} + T_N(z), \quad z \in \mathbb{C}^+, \tag{8.13}$$

where $T_N(z) \in \mathcal{N}$.

Since $\alpha(u) > 0$, $u \in (0,\infty)$, and $0 < \beta(u) < d$, $u \in (0,\infty)$, we note that the second summand on the right hand-side of (8.13) can be written in the form

$$\int\limits_{(0,N]} \frac{\alpha(u)\,\sigma_1(du)}{\beta(u)-z} = \int\limits_{(0,u_1]} \frac{\alpha(\beta^{(-1)}(u))\,\tilde{\tau}_1(du)}{u-z},$$

where $u_1 := \beta(N)$, and $\tilde{\tau}_1$ is a measure such that $\tilde{\tau}_1(B) := \tau_1(\beta^{(-1)}(B))$ for any Borel set $B$. Note that $\tilde{\tau}_1((0,u_1]) = \tau_1((0,N])$. Applying to both sides of (8.13) the inversion formula (3.4), we obtain that $\tau_1((0,N]) = 0$ for every $N > 0$ and therefore $\tau_1((0,\infty)) = 0$. Hence relation (8.13) implies that $\tau_1 \equiv 0$. Thus, $K_{\mu_1}(z) = d_1 z$ and $\mu_1 = \delta_{1/d_1}$. Hence $\mu_2 = \mu \boxplus \delta_{d_1}$. This implies the statement of the theorem. □

**8.3**. It remains to describe a wide class of indecomposable elements in the semigroup $(\mathcal{M}_*, \boxtimes)$. Our result in this case has the form.

Denote $\gamma_\alpha := \{\zeta \in \mathbb{T} : -\alpha < \arg\zeta < \alpha\}$, $0 < \alpha < \pi$, and $\zeta_\alpha = e^{i\alpha}$.

**Theorem 8.3.** *Let $\mu \in \mathcal{M}_*$ such that $\mu(\{1\}) > 0$, $\mu(\gamma_\alpha \setminus \{1\}) = 0$, $\mu(\mathbb{T} \setminus \gamma_\alpha) > 0$, and $\lim_{\theta\uparrow\alpha} \operatorname{Im}\psi_\mu(e^{i\theta}) = -\infty$, $\lim_{\theta\downarrow-\alpha} \operatorname{Im}\psi_\mu(e^{i\theta}) = \infty$. Then $\mu$ is an indecomposable p-measure in $(\mathcal{M}_*, \boxtimes)$.*

*Proof.* We have from the assumptions of the theorem that $\psi_\mu(z)$ is an analytic function on $\mathbb{C} \setminus (\mathbb{T} \setminus \gamma_\alpha)$ with the exception of the simple pole 1. Moreover $\operatorname{Im}\psi_\mu(e^{i\theta})$ is a strictly monotone function on $(-\alpha, 0)$ and on $(0,\alpha)$, and $\operatorname{Im}\psi_\mu(e^{i\theta}) \uparrow \infty$ as $\theta \downarrow -\alpha$, $\operatorname{Im}\psi_\mu(e^{i\theta}) \downarrow -\infty$ as $\theta \uparrow 0$, $\operatorname{Im}\psi_\mu(e^{i\theta}) \uparrow \infty$ as $\theta \downarrow 0$, and $\operatorname{Im}\psi_\mu(e^{i\theta}) \downarrow -\infty$ as $\theta \uparrow \alpha$. Introduce the function $H_\mu(z) = 1 + 2\psi_\mu(z)$ and consider the function

$$Q_\mu(z) = \frac{H_\mu(z) - 1}{H_\mu(z) + 1}, \quad z \in \mathbb{D}.$$

It is analytic on $\mathbb{C} \setminus (\mathbb{T} \setminus \gamma_\alpha)$ and $|Q_\mu(z)| = 1$ for $z \in \gamma_\alpha$. It is easy to verify that $\operatorname{Arg}Q_\mu(e^{i\theta})$, $\operatorname{Arg}Q_\mu(1) = 0$, is strictly monotone function on the interval $-\alpha < \theta < \alpha$ and $\operatorname{Arg}Q_\mu(e^{i(\alpha-0)}) - \operatorname{Arg}Q_\mu(e^{i(-\alpha+0)}) = 4\pi$.

Let $\mu_1, \mu_2, \in \mathcal{M}_*$ We have to prove that either $\mu_1 = \mu \boxtimes \delta_a$ and $\mu_2 = \delta_{1/a}$ or $\mu_1 = \delta_{1/a}$ and $\mu_2 = \mu \boxtimes \delta_a$, where $a \in \mathbb{T}$. By Theorem 2.7, there exists a function $Z_1(z) \in \mathcal{S}_*$ such that $\psi_\mu(z) = \psi_{\mu_1}(Z_1(z)) = \psi_{\mu_2}(Z_2(z))$ for $z \in \mathbb{D}$ and

$$Z_1(z)Z_2(z) = zQ_\mu(z) = zQ_{\mu_j}(Z_j(z)) = z\frac{\psi_{\mu_j}(Z_j(z))}{1 + \psi_{\mu_j}(Z_j(z))} = z\frac{H_{\mu_j}(Z_j(z)) - 1}{H_{\mu_j}(Z_j(z)) + 1} \tag{8.14}$$

for $z \in \mathbb{D}$, $\quad j=1,2$, where $H_{\mu_j}(z) := 2\psi_{\mu_j}(z)+1$. We obtain from (8.14) the relation

$$\int\limits_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\,\mu(d\zeta) = \int\limits_{\mathbb{T}} \frac{1+\zeta Z_j(z)}{1-\zeta Z_j(z)}\,\mu_j(d\zeta), \quad z \in \mathbb{D}, \quad j=1,2. \tag{8.15}$$

Since the functions

$$C_j(z;\xi) = \frac{1+Z_j(z)\xi}{1-Z_j(z)\xi}, \qquad z \in \mathbb{D}, \quad j=1,2, \tag{8.16}$$

belong to the Carathéodory class $\mathcal{C}$ for every fixed $\xi \in \mathbb{T}$, they admit the representation

$$C_j(z;\xi) = \int\limits_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\,\sigma_{\xi,j}(d\zeta), \qquad z \in \mathbb{D}, \quad j=1,2,$$

where $\sigma_{\xi,j}$ are p-measures. Moreover for $j=1,2$ the functions $\xi \mapsto \sigma_{\xi,j}(B)$, where $B$ is a Borel set on $\mathbb{T}$, are measurable. From (8.15) we deduce the relations

$$H_\mu(z) = \int\limits_{\mathbb{T}} C_j(z;\xi)\,\mu_j(d\xi) = \int\limits_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}\,\nu_j(d\zeta), \quad z \in \mathbb{D}, \tag{8.17}$$

where $\nu_j(B) := \int_{\mathbb{T}} \sigma_{\xi,j}(B)\mu_j(d\xi)$ for every Borel set $B$ on $\mathbb{T}$.

By (8.17) and the inversion formula, the p-measures $\nu_j$ have the property: $\nu_j(\gamma_\alpha \setminus \{1\}) = 0$, $j=1,2$. Hence, there exist measurable sets $B_j$, $j=1,2$, such that $\mu_j(B_j) = 1$ and $\sigma_{\xi,j}(\gamma_\alpha \setminus \{1\}) = 0$ for $\xi \in B_j$. Thus we obtain, for such points $\xi$,

$$\frac{1+\xi Z_j(z)}{1-\xi Z_j(z)} = C_j(z;\xi) = p_j(\xi)\frac{1+z}{1-z} + \int\limits_{\mathbb{T}\setminus\gamma_\alpha} \frac{1+\zeta z}{1-\zeta z}\,\sigma_{\xi,j}(d\zeta),\ z \in \mathbb{D},\ j=1,2, \tag{8.18}$$

where $0 \leqslant p_j(\xi) \leqslant 1$. Moreovere there exist points $\xi_j \in B_j$, $j=1,2$, such that $p_j(\xi_j) > 0$. Since

$$\xi_j Z_j(z) = \frac{C_j(z;\xi_j)-1}{C_j(z;\xi_j)+1}, \quad j=1,2,$$

we see from (8.18) that the functions $Z_j(z)$, $j=1,2$, are analytic on $\mathbb{C} \setminus (\mathbb{T} \setminus \gamma_\alpha)$ and $|Z_j(z)| = 1$ for $z \in \gamma_\alpha$. Hence $Z_j(e^{i\theta}) = e^{ig_j(\theta)}$, $j=1,2$, for $-\alpha < \theta < \alpha$, where $g_j(\theta)$ are continuous real-valued functions. In addition, as it is easy to see, $g_j(\theta)$, $j=1,2$, are strictly monotone functions on $(-\alpha,\alpha)$ and $Z_j(1) = 1/\xi_j$.

We note from (8.14) that

$$\begin{aligned} &g_1(\alpha-0) - g_1(-\alpha+0) + g_2(\alpha-0) - g_2(-\alpha+0) \\ &\qquad = 2\alpha + \arg Q_\mu(e^{i(\alpha-0)}) - \arg Q_\mu(e^{i(-\alpha+0)}) = 2\alpha + 4\pi. \end{aligned} \tag{8.19}$$

We conclude from (8.19) that one of $Z_j$, say $Z_1(z)$, has the property: $g_1(\alpha-0) - g_1(\alpha+0) > 2\pi$.

Return to the formula (8.17) with $j = 1$. We obtain from the definition (8.16) of the functions $C_1(z; e^{i\psi})$, $z \in \mathbb{D}$, $-\pi \leqslant \psi < \pi$, that they have the form

$$C_1(z; e^{i\psi}) = \alpha(\psi)\frac{1 + e^{i\beta(\psi)}z}{1 - e^{i\beta(\psi)}z} + M(z; \psi), \tag{8.20}$$

where $\alpha(\psi)$ is a positive continuous function, $\beta(\psi)$ is a strictly monotone continuous function on $[-\pi, \pi)$ such that $-\alpha < \beta(\psi) < \alpha$, $\beta(0) = -\arg\xi_1$, and, for every fixed $\psi \in [-\pi, \pi)$, $M(z; \psi)$ is an analytic function on $\mathbb{C} \setminus (\mathbb{T} \setminus \gamma_\alpha)$. Comparing (8.18) and (8.20) we arrive at a contradiction. Theorem 8.3 is proved completely.

□

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