

Homologies of digraphs

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June 2012

Abstract

We construct the chain and cochain complexes on a finite directed graph and study the corresponding homologies and cohomologies. We obtain new methods of computation of homology groups and relations between homologies of various types of graphs, including the product of graphs and a subgraph of the given graph. A number of explicit examples is given for computations of homology groups of graphs.

Contents

1	Introduction	2
2	Differential forms on a finite set	4
2.1	Exterior derivative	4
2.2	Concatenation and product rule	5
2.3	Spaces of paths and the boundary operator	6
2.4	The path product	8
2.5	Regular forms	9
2.6	Regular paths	10
3	Elements of homological algebra	11
3.1	Cochain complexes	11
3.2	Chain complexes	13
3.3	Sub-complexes and quotient complexes	14
3.4	Zigzag Lemma	15
3.5	Reduced cohomologies and homologies	16

*Partially supported by SFB 701 of German Research Council and a Visiting Grant of MSC, Tsinghua University

†Partially supported by the CONACyT Grant 98697 and SFB 701 of German Research Council

4	<i>p</i>-forms on digraphs	16
4.1	Allowed paths	17
4.2	The space <i>p</i> -forms on a digraph	17
4.3	Computation of $\dim H^0$	18
4.4	Some condition for $\dim \Omega^p = 0$	18
4.5	Disjoint union	19
5	∂-invariant paths	19
5.1	The space of ∂ -invariant paths	19
5.2	Semi-edges and ∂ -invariant paths	20
5.3	Triangles, squares and $\dim \Omega_p$	21
5.4	Star-like graphs and cones	24
5.5	An example of direct computation of $\dim H_p$	25
5.6	Cycle-graphs	26
5.7	Sierpinski triangles	28
5.8	Snakes	29
5.9	Triangulation as a closed path	30
5.10	Lemma of Sperner revisited	31
6	Cartesian product of digraphs and paths	34
6.1	Cross product of regular paths	34
6.2	Some property of the cross product	36
6.3	The product rule	38
6.4	Cartesian product of digraphs	40
6.5	Cylinders and hypercubes	42
6.6	Bi-cylinders	43
6.7	∂ -invariant paths on Cartesian product	47
6.8	Special paths on the product	48
6.9	Homologies of Cartesian product	50
7	Homologies of subgraphs	53
7.1	Chain complex of a subgraph	53
7.2	Removing a vertex of degree 1	54
7.3	Removing a vertex of degree <i>n</i>	55
7.4	Removing a vertex of degree $1 + 1$	57
7.5	Suspension and spheres	63
8	Minimal paths and hole detection	68
	References	70

1 Introduction

The subject of this paper is a theory of homologies and cohomologies on finite directed graphs that is based on two fundamental notions: *p*-forms and *p*-paths, where *p* is a non-negative integer.

The notion of a *p*-form is somewhat similar to the notion of a differential form of order *p* on a smooth manifold. In particular, there is an exterior derivative *d* acting from *p*-forms to

$(p + 1)$ -forms and satisfying the identity $d^2 = 0$. The latter allows us to introduce the notion of cohomology of digraphs similarly to de Rham cohomology for manifolds.

The notion of a p -path is somewhat similar to the notion of a face of a simplicial complex. There is a boundary operator ∂ acting from p -paths to $(p - 1)$ -paths and satisfying $\partial^2 = 0$, which allows us to define the notion of homology of digraphs similarly to homology of simplicial complexes.

The spaces of p -paths and p -forms are dual, and the operators d and ∂ are also dual, which implies the duality of homologies and cohomologies.

The notion of a differential form on a digraph was introduced by Dimakis and Müller-Hoissen [1] using the universal algebra of differential forms on the space of functions on a finite set V . Then the differential calculus on a digraph (V, E) with the set of vertices V and the set of directed edges E is obtained by taking an appropriate quotient reflecting the graph structure. The notion of cohomology of (V, E) was also introduced in [1], but no substantial development followed.

In the present paper we provide a simpler alternative approach to the notion of a p -form. We define a p -form on a set V as a scalar function on V^{p+1} , and then pass to equivalence classes using the graph structure. This approach allows to determine easily the dual notion of p -paths that are formal linear combinations of sequences of $p + 1$ vertices, where some of the sequences are identifies with 0 according to the set of edges.

One might have expected that the graph structure alone is too poor in order to have non-trivial homologies. Indeed, a graph as a simplicial complex has only faces of dimension 0 and 1. However, due to a correct definition, the p -paths see “higher dimensional” subgraphs such as simplexes and hypercubes of all dimensions. Consequently, the homologies of all orders can be non-trivial. Non-zero homology classes can under certain restriction be associated with subgraphs that leads to a detection of holes of arbitrary dimensions by means of algorithmic computation.

Some similarity between the homology theory on graphs and the classical homology theories should not create a false impression that our theory is a particular case of a theory of some simplicial objects. The homology theory on graphs does not admit natural geometric realization as homology theory (or even generalized homology theory) on some kind of topological objects (only for some specific graphs such presentation exists). Furthermore, there exists an opposite inclusions of the theories, i.e. for any finite simplicial complex there exists a naturally defined digraph with the homology groups that are isomorphic to the simplicial homology groups of the complex. This construction will be presented in a sequel paper.

In the present paper we obtain new methods of computation of homology groups and relations between the homologies of various types of graphs, including the Cartesian product of graphs and a subgraph of the given graph, and give a number of explicit examples of computations.

The paper is organized as follows. In Section 2, we define the notions of p -forms and p -paths on a finite set V , as well as the operators d and ∂ .

In Section 3, we recall the basic facts from homological algebra in the form which we need in the subsequent sections. All these results can be found in the standard books on homological algebra (see, for example, [3], [2]).

In Section 4, we define the cochain complex of p -forms on digraphs and prove several facts about the cohomology groups.

In Section 5, we define the chain complex of p -paths on digraph and describe relations between the chain and cochain complexes. We compute also homology groups of several classes of graphs, such as star-likes graphs, cones, cycle-graphs, snakes, Sierpinski triangles, and so on.

In Section 6, we investigate the homological properties of the direct product of the graphs

and obtain very explicit results for some cases. We describe the space of p -forms and compute homologies on cylinders and bi-cylinders, in particular, on binary hypercubes.

In Section 7, we investigate relations between homology groups of a graph and its subgraph. We obtain general relations that are given by an exact sequence and then apply them to obtain explicit relations for several classes of graphs. From example, we obtain the triviality of the reduced homology groups for trees. Also, we introduce the notion of a suspension of a graph and compute the homologies of the suspension. The operation of suspension allows to define graphs whose homological properties are identical to n -sphere.

In Section 8 we sketch an approach to hole detection on graphs.

2 Differential forms on a finite set

Let V be a non-empty finite set. Denote by $\Lambda^0 = \Lambda^0(V)$ the linear space of all \mathbb{K} -valued functions on V , where \mathbb{K} is a fixed scalar field, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. More generally, for any integer $p \geq 0$, denote by $\Lambda^p = \Lambda^p(V)$ the linear space of all \mathbb{K} -valued functions on $V^{p+1} = \underbrace{V \times \dots \times V}_{p+1}$.

Clearly, $\dim \Lambda^p = |V|^{p+1}$.

Definition 2.1 Elements of Λ^p are referred to as p -forms on V .

The value of a p -form ω at a point $(i_0, i_1, \dots, i_p) \in V^{p+1}$ will be denoted by $\omega_{i_0 i_1 \dots i_p}$. In particular, the value of a function $f \in \Lambda^0(V)$ at $i \in V$ will be denoted by f_i .

Denote by $e^{j_0 \dots j_p}$ a p -form that takes value $1 \in \mathbb{K}$ at the point (j_0, j_1, \dots, j_p) and 0 at all other points. For example, e^j is a function on V that is equal to 1 at j and 0 away from j . Let us refer to $e^{j_0 \dots j_p}$ as an *elementary* p -form. Clearly, the family $\{e^{j_0 \dots j_p}\}$ of all elementary p -forms is a basis in the linear space Λ^p and, for any $\omega \in \Lambda^p$,

$$\omega = \sum_{j_0, \dots, j_p \in V} \omega_{j_0 \dots j_p} e^{j_0 \dots j_p}.$$

2.1 Exterior derivative

Definition 2.2 Define the exterior derivative $d: \Lambda^p \rightarrow \Lambda^{p+1}$ by

$$(d\omega)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \dots \widehat{i}_q \dots i_{p+1}}, \quad (2.1)$$

for any $\omega \in \Lambda^p$, where the hat \widehat{i}_q means omission of the index i_q .

For example, for a function $f \in \Lambda^0$ we have

$$(df)_{ij} = f_j - f_i,$$

for 1-form ω

$$(d\omega)_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij},$$

for a 2-form ω

$$(d\omega)_{ijkl} = \omega_{jkl} - \omega_{ikl} + \omega_{ijl} - \omega_{ijk}.$$

It follows from (2.1) that

$$\begin{aligned}
(de^{j_0 \dots j_p})_{i_0 \dots i_{p+1}} &= \sum_{q=0}^{p+1} (-1)^q e^{j_0 \dots j_p}_{i_0 \dots \widehat{i}_q \dots i_{p+1}} \\
&= \sum_{q=0}^{p+1} (-1)^q e^{j_0 \dots j_{q-1} i_q j_q \dots j_p}_{i_0 \dots \widehat{i}_q \dots i_{p+1}} \\
&= \sum_{q=0}^{p+1} (-1)^q \sum_i e^{j_0 \dots j_{q-1} i j_q \dots j_p}_{i_0 \dots \widehat{i}_q \dots i_{p+1}}
\end{aligned}$$

whence

$$de^{j_0 \dots j_p} = \sum_i \sum_{q=0}^{p+1} (-1)^q e^{j_0 j_1 \dots j_{q-1} i j_q \dots j_p}. \quad (2.2)$$

For example,

$$\begin{aligned}
de^j &= \sum_i (e^{ij} - e^{ji}), \\
de^{jk} &= \sum_i (e^{ijk} - e^{jik} + e^{kji}).
\end{aligned}$$

Lemma 2.3 For any $p \geq 0$ and all $\omega \in \Lambda^p$,

$$d^2\omega = 0. \quad (2.3)$$

Proof. We have

$$\begin{aligned}
(d^2\omega)_{i_0 \dots i_{p+2}} &= \sum_{q=0}^{p+2} (-1)^q (d\omega)_{i_0 \dots \widehat{i}_q \dots i_{p+2}} \\
&= \sum_{q=0}^{p+2} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r \omega_{i_0 \dots \widehat{i}_r \dots \widehat{i}_q \dots i_{p+2}} + \sum_{r=q+1}^{p+2} (-1)^{r-1} \omega_{i_0 \dots \widehat{i}_q \dots \widehat{i}_r \dots i_{p+2}} \right) \\
&= \sum_{0 \leq r < q \leq p+2} (-1)^{q+r} \omega_{i_0 \dots \widehat{i}_r \dots \widehat{i}_q \dots i_{p+2}} \\
&\quad - \sum_{0 \leq q < r \leq p+2} (-1)^{q+r} \omega_{i_0 \dots \widehat{i}_q \dots \widehat{i}_r \dots i_{p+2}}.
\end{aligned}$$

After switching q and r in the second sum we see that it is equal to the first one, whence $d^2\omega = 0$ follows. ■

2.2 Concatenation and product rule

Definition 2.4 For forms $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$ denote by $\varphi\psi$ a form from Λ^{p+q} that is defined by

$$(\varphi\psi)_{i_0 \dots i_{p+q}} = \varphi_{i_0 \dots i_p} \psi_{i_{p+1} \dots i_{p+q}}. \quad (2.4)$$

The form $\varphi\psi$ is called the *concatenation* of φ and ψ .

Clearly, $\varphi\psi$ is a bilinear operation with respect to φ, ψ . For example, if φ is a function, that is, $p = 0$, then $\varphi\psi \in \Lambda^q$ and

$$(\varphi\psi)_{i_0\dots i_q} = \varphi_{i_0}\psi_{i_0\dots i_q}.$$

Also $\psi\varphi \in \Lambda^q$ and

$$(\psi\varphi)_{i_0\dots i_q} = \psi_{i_0\dots i_q}\varphi_{i_q}.$$

For the elementary forms $e^{i_0\dots i_p}$ and $e^{j_0\dots j_q}$ we have

$$e^{i_0\dots i_p}e^{j_0\dots j_q} = \begin{cases} 0, & i_p \neq j_0, \\ e^{i_0\dots i_p j_1\dots j_q}, & i_p = j_0. \end{cases} \quad (2.5)$$

Lemma 2.5 For all $\varphi \in \Lambda^p$ and $\psi \in \Lambda^q$, we have

$$d(\varphi\psi) = (d\varphi)\psi + (-1)^p \varphi d\psi. \quad (2.6)$$

Proof. Denoting $\omega = \varphi\psi$, we have

$$\begin{aligned} (d\omega)_{i_0\dots i_{p+q+1}} &= \sum_{r=0}^{p+q+1} (-1)^r \omega_{i_0\dots \widehat{i}_r\dots i_{p+q+1}} \\ &= \sum_{r=0}^p (-1)^r \omega_{i_0\dots \widehat{i}_r\dots i_{p+1}\dots i_{p+q+1}} + \sum_{r=p+1}^{p+q+1} (-1)^r \omega_{i_0\dots i_p\dots \widehat{i}_r\dots i_{p+q+1}} \\ &= \sum_{r=0}^p (-1)^r \varphi_{i_0\dots \widehat{i}_r\dots i_{p+1}} \psi_{i_{p+1}\dots i_{p+q+1}} + \sum_{r=p+1}^{p+q+1} (-1)^r \varphi_{i_0\dots i_p} \psi_{i_p\dots \widehat{i}_r\dots i_{p+q+1}}. \end{aligned}$$

Noticing that

$$(d\varphi)_{i_0\dots i_{p+1}} = \sum_{r=0}^{p+1} (-1)^r \varphi_{i_0\dots \widehat{i}_r\dots i_{p+1}}$$

and

$$(d\psi)_{i_p\dots i_{p+q+1}} = \sum_{r=p}^{p+q+1} (-1)^{r-p} \psi_{i_p\dots \widehat{i}_r\dots i_{p+q+1}},$$

we obtain

$$\begin{aligned} (d\omega)_{i_0\dots i_{p+q+1}} &= \left[(d\varphi)_{i_0\dots i_{p+1}} - (-1)^{p+1} \varphi_{i_0\dots i_p} \right] \psi_{i_{p+1}\dots i_{p+q+1}} \\ &\quad + (-1)^p \varphi_{i_0\dots i_p} \left[(d\psi)_{i_p\dots i_{p+q+1}} - \psi_{i_{p+1}\dots i_{p+q+1}} \right] \\ &= ((d\varphi)\psi)_{i_0\dots i_{p+q+1}} + (-1)^p (\varphi d\psi)_{i_0\dots i_{p+q+1}} \end{aligned}$$

which was to be proved. ■

2.3 Spaces of paths and the boundary operator

An *elementary p-path* is any (ordered) sequence i_0, \dots, i_p of $p+1$ vertices of V that will be denoted simply by $i_0\dots i_p$ or by $e_{i_0\dots i_p}$. We use the notation $e_{i_0\dots i_p}$ when we consider the elementary path as an element of a linear space $\Lambda_p = \Lambda_p(V)$ that consists of all formal linear combinations of all elementary p -paths. The elements of Λ_p are called *p-paths*. Each p -path has a form

$$v = \sum_{i_0 i_1 \dots i_p} v^{i_0 i_1 \dots i_p} e_{i_0 i_1 \dots i_p},$$

with arbitrary scalars $v^{i_0 i_1 \dots i_p}$, that are called the coefficients of v .

We have a natural pairing of p -forms and p -paths as follows:

$$(\omega, v) := \sum_{i_0 \dots i_p} \omega_{i_0 \dots i_p} v^{i_0 \dots i_p}$$

for all $\omega \in \Lambda^p$ and $v \in \Lambda_p$. It follows that the spaces Λ^p and Λ_p are dual.

Definition 2.6 Define the *boundary operator* $\partial : \Lambda_{p+1} \rightarrow \Lambda_p$ by

$$(\partial v)^{i_0 \dots i_p} = \sum_k \sum_{q=0}^{p+1} (-1)^q v^{i_0 \dots i_{q-1} k i_q \dots i_p} \quad (2.7)$$

where the index k is inserted so that it is preceded by q indices.

This definition is valid for $p \geq 0$. Sometimes we need also the operator $\partial : \Lambda_0 \rightarrow \Lambda_{-1}$ where we set $\Lambda_{-1} = \{0\}$ and $\partial v = 0$ for all $v \in \Lambda_0$.

If v is an 1-path, then ∂v is given by

$$(\partial v)^i = \sum_k (v^{ki} - v^{ik}).$$

If v is a 2-path then

$$(\partial v)^{ij} = \sum_k (v^{kij} - v^{ikj} + v^{ijk}).$$

It follows from (2.7) that

$$\begin{aligned} (\partial e_{j_0 \dots j_{p+1}})^{i_0 \dots i_p} &= \sum_k \sum_{q=0}^{p+1} (-1)^q e_{j_0 \dots j_{p+1}}^{i_0 \dots i_{q-1} k i_q \dots i_p} \\ &= \sum_{q=0}^{p+1} \sum_k (-1)^q e_{j_0 \dots j_{p+1}}^{i_0 \dots i_{q-1} k i_q \dots i_p} \\ &= \sum_{q=0}^{p+1} (-1)^q e_{j_0 \dots j_{q-1} j_{q+1} \dots j_{p+1}}^{i_0 \dots i_{q-1} i_q \dots i_p} \end{aligned}$$

whence

$$\partial e_{j_0 \dots j_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{j_0 \dots \widehat{j}_q \dots j_{p+1}}. \quad (2.8)$$

For example, we have

$$\partial e_{ij} = e_j - e_i \quad (2.9)$$

and

$$\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}.$$

Lemma 2.7 For any p -form ω and any $(p+1)$ -path v the following identity holds

$$(d\omega, v) = (\omega, \partial v).$$

Hence, the operators $d : \Lambda^p \rightarrow \Lambda^{p+1}$ and $\partial : \Lambda_{p+1} \rightarrow \Lambda_p$ are dual.

Proof. It suffices to prove this for $v = e_{i_0 \dots i_{p+1}}$. We have

$$(d\omega, v) = (d\omega)_{i_0 \dots i_{p+1}} = (d\omega)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \dots \widehat{i}_q \dots i_{p+1}}$$

while

$$(\omega, \partial v) = \left(\omega, \sum_{q=0}^{p+1} (-1)^q e_{i_0 \dots \widehat{i}_q \dots i_{p+1}} \right) = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \dots \widehat{i}_q \dots i_{p+1}},$$

whence the identity of the two expressions follows. ■

Corollary 2.8 *For any $v \in \Lambda_p$, we have $\partial^2 v = 0$.*

2.4 The path product

For any two paths $u \in \Lambda_p$ and $v \in \Lambda_q$ define the *path product* $uv \in \Lambda_{p+q+1}$ as follows:

$$(uv)^{i_0 \dots i_p j_0 \dots j_q} = u^{i_0 \dots i_p} v^{j_0 \dots j_q}. \quad (2.10)$$

For example, if $u = e_{i_0 \dots i_p}$ and $v = e_{j_0 \dots j_q}$, then

$$e_{i_0 \dots i_p} e_{j_0 \dots j_q} = e_{i_0 \dots i_p j_0 \dots j_q}. \quad (2.11)$$

This definition is valid for all $p, q \geq 0$.

Note that path product is essentially different from concatenation of forms, which can be seen from comparison of (2.10) and (2.4): in the former the index i_p is used twice in the right hand side whereas in the latter – only once. The both definitions are designed to satisfy the product rules with respect to the operators d and ∂ , respectively.

To state a product rule for $\partial(uv)$ we need also the notion of a product also for $p = -1$ or $q = -1$. For that consider instead of $\Lambda_{-1} = \{\emptyset\}$ a modified space $\tilde{\Lambda}_{-1} \equiv \mathbb{K}$ so that any $u \in \tilde{\Lambda}_{-1}$ is just a scalar. Then (2.10) can be used again to define the product uv for $u \in \tilde{\Lambda}_{-1}$ (or $v \in \tilde{\Lambda}_{-1}$) because the right hand side of (2.10) amounts to multiplying by the scalar u (resp. v). That is, if $p = -1$ then uv is just the multiple of v with the coefficient u .

We need then a modified version of ∂ when acting from Λ_0 to $\tilde{\Lambda}_{-1}$. Define the operator

$$\tilde{\partial} : \Lambda_p \rightarrow \Lambda_{p-1}$$

as follows. If $p > 0$ then $\tilde{\partial} \equiv \partial$, and for $p = 0$ define $\tilde{\partial} : \Lambda_0 \rightarrow \mathbb{K}$ by setting $\tilde{\partial} e_i = 1$ and extending to all $v \in \Lambda_0$ by linearity. In other word, for $v \in \Lambda_0$ we have

$$\tilde{\partial} v = (1, v) = \sum_{i \in V} v_i.$$

This definition of $\tilde{\partial}$ is the same as the one used in the extended chain complex (3.23). It is easy to see that $\tilde{\partial}^2 = 0$.

Lemma 2.9 *For any paths $u \in \Lambda_p$ and $v \in \Lambda_q$ with $p, q \geq 0$, we have*

$$\tilde{\partial}(uv) = (\tilde{\partial}u)v + (-1)^{p+1} u\tilde{\partial}v. \quad (2.12)$$

Note that under the conditions of Lemma 2.9 $uv \in \Lambda_{p+q+1}$ so that $\tilde{\partial}(uv) = \partial(uv)$.

Proof. By bilinearity it suffices to prove (2.12) for $u = e_{i_0 \dots i_p}$ and $v = e_{j_0 \dots j_q}$. Consider first the case $p = q = 0$. Then $u = e_i$, $v = e_j$ and $\tilde{\partial}u = \tilde{\partial}v = 1$ and

$$\tilde{\partial}(uv) = \partial(e_{ij}) = e_j - e_i = (\tilde{\partial}u)v - u(\tilde{\partial}v),$$

which proves (2.12) in this case.

If $p = 0$ and $q \geq 1$ then $u = e_i$ and $v = e_{j_0 \dots j_q}$, whence

$$\begin{aligned} \tilde{\partial}(uv) &= \partial e_{ij_0 \dots j_q} = e_{j_0 \dots j_q} - e_{ij_1 j_2 \dots j_q} + e_{ij_0 j_2 \dots j_q} - \dots \\ &= v - e_i(\partial v) = (\tilde{\partial}u)v - u(\tilde{\partial}v) \end{aligned}$$

which proves (2.12) in this case.

If $p \geq 1$ and $q = 0$ then $u = u_{i_0 \dots i_p}$, $v = e_j$, whence

$$\begin{aligned} \tilde{\partial}(uv) &= \partial e_{i_0 \dots i_p j} = e_{i_1 \dots i_p j} - e_{i_0 i_2 \dots i_p j} - \dots + (-1)^{p+1} e_{i_0 \dots i_p} \\ &= (\partial u)e_j + (-1)^{p+1} u = (\tilde{\partial}u)v + (-1)^{p+1} u\tilde{\partial}v. \end{aligned}$$

Finally, if $p \geq 1$ and $q \geq 1$ then

$$\begin{aligned} \tilde{\partial}(uv) &= \partial e_{i_0 \dots i_p j_0 \dots j_q} = e_{i_1 \dots i_p j_0 \dots j_q} - e_{i_0 i_2 \dots i_p j_0 \dots j_q} + \dots \\ &\quad + (-1)^{p+1} (e_{i_0 \dots i_p j_1 \dots j_q} - e_{i_0 \dots i_p j_0 j_2 \dots j_q} + \dots) \\ &= (\partial u)v + (-1)^{p+1} u(\partial v), \end{aligned}$$

which finishes the proof. ■

2.5 Regular forms

We say that a path $i_0 \dots i_p$ is *regular* if $i_k \neq i_{k+1}$ for all $k = 0, \dots, p-1$, and irregular otherwise. Consider the following subspace of Λ^p :

$$\begin{aligned} \mathcal{R}^p &= \text{span} \{ e^{i_0 \dots i_p} : i_0 \dots i_p \text{ is regular} \} \\ &= \{ \omega \in \Lambda^p : \omega_{i_0 \dots i_p} = 0 \text{ if } i_0 \dots i_p \text{ is irregular} \}, \end{aligned}$$

where $\text{span } A$ denotes the linear space spanned by the set A .

The elements of \mathcal{R}^p are called regular p -forms. For example, $\omega \in \mathcal{R}^1$ if $\omega_{ii} \equiv 0$ and $\omega \in \mathcal{R}^2$ if $\omega_{iij} \equiv \omega_{jii} \equiv 0$. The condition $f \in \mathcal{R}^0$ has no additional restriction so that $\mathcal{R}^0 = \Lambda^0$.

The operations of exterior derivative and concatenation can be restricted to regular forms.

Lemma 2.10 *If $\omega \in \mathcal{R}^p$ then $d\omega \in \mathcal{R}^{p+1}$. If $\varphi \in \mathcal{R}^p$ and $\psi \in \mathcal{R}^q$ then $\varphi\psi \in \mathcal{R}^{p+q}$.*

Proof. Let $\omega \in \mathcal{R}^p$. To prove that $d\omega \in \mathcal{R}^{p+1}$, we must show that

$$(d\omega)_{i_0 \dots i_{p+1}} = 0 \tag{2.13}$$

whenever $i_0 \dots i_{p+1}$ is irregular, say $i_k = i_{k+1}$. We have by (2.1)

$$(d\omega)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \dots \hat{i}_q \dots i_{p+1}}.$$

If $q \neq k, k+1$ then both i_k, i_{k+1} are present in $\omega_{i_0 \dots \widehat{i_q} \dots i_{p+1}}$ which makes this term equal to 0 since ω is regular. In the remaining two cases $q = k$ and $q = k+1$ the term $\omega_{i_0 \dots \widehat{i_q} \dots i_{p+1}}$ has the same values (because the sequences $i_0 \dots \widehat{i_q} \dots i_{p+1}$ are the same) but the signs $(-1)^q$ are opposite. Hence, they cancel out, which proves (2.13).

Let us prove that $\varphi\psi$ is regular provided so are φ and ψ . By (2.4), we have

$$(\varphi\psi)_{i_0 \dots i_{p+q}} = \varphi_{i_0 \dots i_p} \psi_{i_p \dots i_{p+q}}.$$

If the sequence $i_0 \dots i_{p+q}$ is irregular, say $i_k = i_{k+1}$ then the both indices i_k, i_{k+1} are present either in the sequence $i_0 \dots i_p$ or in $i_p \dots i_{p+q}$, which implies that one of the terms $\varphi_{i_0 \dots i_p}, \psi_{i_p \dots i_{p+q}}$ vanishes and, hence, $(\varphi\psi)_{i_0 \dots i_{p+q}} = 0$. ■

2.6 Regular paths

We say that an elementary p -path $e_{i_0 \dots i_p}$ is regular (or irregular) if the path $i_0 \dots i_p$ is regular (resp. irregular). We would like to define the boundary operator ∂ on the subspace of Λ_p spanned by regular elementary paths. Just restriction of ∂ to the subspace does not work as ∂ is not invariant on this subspace, so that we have to consider a quotient space instead.

Let I_p be the subspace of Λ_p that is spanned by irregular $e_{i_0 \dots i_p}$. Consider the quotient spaces

$$\mathcal{R}_p := \Lambda_p / I_p.$$

The elements of \mathcal{R}_p are the equivalence classes $v \bmod I_p$ where $v \in \Lambda_p$, and they are called *regularized p -paths*. The next lemma shows that the boundary operator ∂ , the product and the pairing are well-defined for regularized paths. We write $u = v \bmod W$ if $u - v \in W$, assuming that u, v are elements of some linear space and W is its subspace.

Lemma 2.11 (a) *If $v_1, v_2 \in \Lambda_p$ and $v_1 = v_2 \bmod I_p$ then $\partial v_1 = \partial v_2 \bmod I_{p-1}$.*

(b) *If $\omega \in \mathcal{R}^p$, $v_1, v_2 \in \Lambda_p$ and $v_1 = v_2 \bmod I_p$ then $(\omega, v_1) = (\omega, v_2)$.*

(b) *Let $u_1, u_2 \in \Lambda_p$ and $v_1, v_2 \in \Lambda_q$. If $u_1 = u_2 \bmod I_p$ and $v_1 = v_2 \bmod I_q$ then $u_1 v_1 = u_2 v_2 \bmod I_{p+q+1}$.*

Proof. (a) It suffices to prove that if $v = 0 \bmod I_p$ then $\partial v = 0 \bmod I_{p-1}$. Since v is a linear combination of irregular paths, it suffices to prove that $\partial e_{i_0 \dots i_p}$ is irregular provided $e_{i_0 \dots i_p}$ is irregular. If $e_{i_0 \dots i_p}$ is irregular then there exists an index k such that $i_k = i_{k+1}$. Then we have

$$\begin{aligned} \partial e_{i_0 \dots i_p} &= e_{i_1 \dots i_p} - e_{i_0 i_2 \dots i_p} + \dots \\ &\quad + (-1)^k e_{i_0 \dots i_{k-1} i_{k+1} i_{k+2} \dots i_p} + (-1)^{k+1} e_{i_0 \dots i_{k-1} i_k i_{k+2} \dots i_p} \\ &\quad + \dots + (-1)^p e_{i_0 \dots i_{p-1}}. \end{aligned} \tag{2.14}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (2.14) cancel out, whereas all other terms are irregular, whence, $\partial e_{i_0 \dots i_p} \in I_{p-1}$.

(b) Indeed, $v_1 - v_2 \in I_p$ is a linear combination of irregular paths $e_{i_0 \dots i_p}$. Since $(\omega, e_{i_0 \dots i_p}) = 0$ for irregular paths, it follows that $(\omega, v_1 - v_2) = 0$ and $(\omega, v_1) = (\omega, v_2)$.

(c) Observe first that if $u \in \Lambda_p$, $v \in \Lambda_q$ then $uv = 0 \bmod I_{p+q+1}$ provided $u = 0 \bmod I_p$ or $v = 0 \bmod I_q$. Indeed, if for example $u = 0 \bmod I_p$ then u is a linear combination of irregular paths $e_{i_0 \dots i_p}$, and the product of an irregular path with any path is irregular. Since

$$u_1 v_1 - u_2 v_2 = (u_1 - u_2) v_1 + u_2 (v_1 - v_2)$$

and $u_1 - u_2 = 0 \bmod I_p$, $v_1 - v_2 = 0 \bmod I_q$, we conclude that

$$u_1 v_1 = u_2 v_2 \bmod I_{p+q+1}.$$

■

It follows from Lemma 2.7 that, for all $\omega \in \mathcal{R}^p$ and $v \in \mathcal{R}_{p+1}$,

$$(d\omega, v) = (\omega, \partial v). \quad (2.15)$$

By Lemma 2.9, we obtain that, for all $u \in \mathcal{R}_p$ and $v \in \mathcal{R}_q$,

$$\partial(uv) = (\tilde{\partial}u)v + (-1)^{p+1} u\tilde{\partial}v. \quad (2.16)$$

Clearly, \mathcal{R}_p is linearly isomorphic to the space of regular p -paths:

$$\text{span} \{ e_{i_0 \dots i_p} : i_0 \dots i_p \text{ is regular} \}.$$

For simplicity of notation, we will identify \mathcal{R}_p with this space, by setting all irregular p -paths to be equal to 0. Hence, when applying the formulas for ∂ and for the product in the spaces \mathcal{R}_p , one should make the following adjustments: all elementary irregular paths $e_{i_0 \dots i_p}$ are equal to zero, and the components $v^{i_0 \dots i_p}$ for irregular paths $i_0 \dots i_p$ vanish by definition. In particular, the formula (2.7) for the component $(\partial v)^{i_0 \dots i_p}$ is valid only for regular $i_0 \dots i_p$, whereas the formula (2.8) for $\partial e_{j_0 \dots j_{p+1}}$ remains valid for all $j_0 \dots j_{p+1}$.

Let V' be a subset of V . Clearly, every elementary regular p -path $e_{i_0 \dots i_p}$ on V' is also a regular p -path on V , so that we have a natural inclusion

$$\mathcal{R}_p(V') \subset \mathcal{R}_p(V). \quad (2.17)$$

By (2.8), $\partial e_{i_0 \dots i_p}$ has the same expression in the both spaces $\mathcal{R}_p(V')$, $\mathcal{R}_p(V)$ so that ∂ commutes with the inclusion (2.17).

Note for comparison that for p -forms the inclusion $\mathcal{R}^p(V') \subset \mathcal{R}^p(V)$ is also valid, but the operator d does not commute with it. For example, in the formula

$$de^i = \sum_j (e^{ji} - e^{ij})$$

the summation index j on the right hand side runs over all vertices, and the result depends on the set of vertices.

3 Elements of homological algebra

3.1 Cochain complexes

A *cochain complex* X is a sequence

$$0 \rightarrow X^0 \xrightarrow{d} X^1 \xrightarrow{d} \dots \xrightarrow{d} X^{p-1} \xrightarrow{d} X^p \xrightarrow{d} \dots \quad (3.1)$$

of vector spaces $\{X^p\}_{p=0}^\infty$ over a field \mathbb{K} and linear mappings $d : X^p \rightarrow X^{p+1}$ with the property that $d^2 = 0$ at each level. To distinguish the operators d on different spaces, we will denote by $d|_{X^p}$ the operator $d : X^p \rightarrow X^{p+1}$. The condition $d^2 = 0$ means that

$$\text{Im } d|_{X^{p-1}} \subset \ker d|_{X^p}.$$

This allows to define the de Rham cohomologies of the complex X by

$$H^p(X) = \ker d|_{X^p} / \operatorname{Im} d|_{X^{p-1}}$$

where $X^{-1} := \{0\}$. The sequence (3.1) is called exact if $H^p(X) = \{0\}$ for all $p \geq 0$.

We always assume that the spaces X^p are finitely dimensional.

Lemma 3.1 *We have for any $p \geq 0$*

$$\dim H^p(X) = \dim X^p - \dim dX^p - \dim dX^{p-1} \quad (3.2)$$

$$= \dim \ker d|_{X^p} + \dim \ker d|_{X^{p-1}} - \dim X^{p-1}. \quad (3.3)$$

Proof. By definition, we have

$$\dim H^p(X) = \dim \ker d|_{X^p} - \dim \operatorname{Im} d|_{X^{p-1}}. \quad (3.4)$$

Applying the nullity-rank theorem to the mapping $d : X^p \rightarrow X^{p+1}$, we obtain

$$\dim \ker d|_{X^p} = \dim X^p - \dim \operatorname{Im} d|_{X^p}.$$

Substituting into (3.4), we obtain (3.2).

In the same way, substituting into (3.4) the identity

$$\dim \operatorname{Im} d|_{X^{p-1}} = \dim X^{p-1} - \dim \ker d|_{X^{p-1}},$$

we obtain (3.3). ■

Lemma 3.2 *For a finite cochain complex*

$$0 \rightarrow X^0 \xrightarrow{d} X^1 \xrightarrow{d} \dots \xrightarrow{d} X^{n-1} \xrightarrow{d} X^n \xrightarrow{d} 0, \quad (3.5)$$

the following identity is satisfied

$$\sum_{k=0}^n (-1)^k \dim H^k(X) = \sum_{k=0}^n (-1)^k \dim X^k. \quad (3.6)$$

In particular, if the sequence (3.5) is exact, then

$$\sum_{k=0}^n (-1)^k \dim X^k = 0. \quad (3.7)$$

Proof. We have by (3.2)

$$\begin{aligned} \sum_{k=0}^n (-1)^k \dim H^k(X) &= \sum_{k=0}^n (-1)^k \dim X^k - \sum_{k=0}^n (-1)^k \dim dX^k - \sum_{k=0}^n (-1)^k \dim dX^{k-1} \\ &= \sum_{k=0}^n (-1)^k \dim X^k - \sum_{k=0}^{n-1} (-1)^k \dim dX^k - \sum_{j=0}^{n-1} (-1)^{j+1} \dim dX^j \\ &= \sum_{k=0}^n (-1)^k \dim X^k, \end{aligned}$$

whence (3.6) follows. If in addition the sequence (3.5) is exact then the left hand side of (3.6) vanishes, whence (3.7) follows. ■

For any finite cochain complex (3.5), define its Euler characteristic by

$$\chi(X) = \sum_{p=0}^n (-1)^p \dim X^p.$$

Then (3.6) implies

$$\chi(X) = \sum_{k=0}^n (-1)^k \dim H^k(X).$$

3.2 Chain complexes

Given a cochain complex (3.1) with finite-dimensional spaces X^p , denote by X_p the dual space to X^p and by ∂ the dual operator to d . Then we obtain a chain complex

$$0 \leftarrow X_0 \xleftarrow{\partial} X_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} X_{p-1} \xleftarrow{\partial} X_p \xleftarrow{\partial} \dots \quad (3.8)$$

Denoting by (\cdot, \cdot) the natural pairing of dual spaces.

For $\omega \in X^p$ and $v \in X_p$ we write $\omega \perp v$ if $(\omega, v) = 0$. If S is a subset of X^p then S^\perp denotes the annihilator in the dual space X_p , that is,

$$S^\perp = \{v \in X_p : \omega \perp v \quad \forall \omega \in S\}.$$

Clearly, S^\perp is a linear subspace of X_p . In the same way one defines the annihilator of subsets of X_p .

By definition we have

$$(d\omega, v) = (\omega, \partial v)$$

for all $\omega \in X^p$ and $v \in X_{p+1}$. Since $d^2 = 0$, it follows that also $\partial^2 = 0$. Hence, one can define the *homologies* of the chain complex (3.8) by

$$H_p(X) = \ker \partial|_{X_p} / \text{Im } \partial|_{X_{p+1}}.$$

By duality we have

$$\ker \partial|_{X_p} = (\text{Im } d|_{X^{p-1}})^\perp, \quad \ker d|_{X^p} = (\text{Im } \partial|_{X_{p+1}})^\perp. \quad (3.9)$$

Lemma 3.3 *The spaces $H^p(X)$ and $H_p(X)$ are dual. In particular, $\dim H^p(X) = \dim H_p(X)$.*

Proof. Let $\omega \in X^p$ be a representative of an element of $H^p(X)$ and $v \in X_p$ be a representative of an element of $H_p(X)$. Let us show that the pairing (ω, v) of ω and v in the dual spaces X^p and X_p is also well-defined for the elements of $H^p(X)$ and $H_p(X)$. Indeed, v is defined mod $\text{Im } \partial|_{X_{p+1}}$, that is, v and $v + \partial u$ represent the same element of $H_p(X)$ for any $u \in X_{p+1}$. Since $\omega \in \ker d|_{X^p}$, we obtain

$$(\omega, v) = (\omega, v + \partial u) = (\omega, v) + (d\omega, u) = (\omega, v).$$

In the same way, (ω, v) does not change when adding $d\varphi$ to ω .

If $(\omega, v) = 0$ for all $v \in \ker \partial|_{X_p}$ then $\omega \perp \ker \partial|_{X_p}$ whence by (3.9) $\omega \in \text{Im } d|_{X^{p-1}}$, that is, ω represent the zero element of $H^p(X)$. In the same way, if $(\omega, v) = 0$ for all $\omega \in \ker d|_{X^p}$ then v represents the zero element of $H_p(X)$. Hence, (ω, v) is a pairing between $H^p(X)$ and $H_p(X)$, whence the duality of these spaces follows. ■

Lemma 3.4 *We have for any $p \geq 0$*

$$\begin{aligned} \dim H_p(X) &= \dim X_p - \dim \partial X_p - \dim \partial X_{p+1} \\ &= \dim \ker \partial|_{X_p} + \dim \ker \partial|_{X_{p+1}} - \dim X_{p+1}. \end{aligned} \quad (3.10)$$

The proof is similar to Lemma 3.1.

3.3 Sub-complexes and quotient complexes

Let X be a cochain complex as in (3.1), and assume that each X^p has a subspace J^p so that d is invariant on $\{J^p\}$, that is, $dJ^p \subset J^{p+1}$. Then we have a cochain *sub-complex* J as follows:

$$0 \rightarrow J^0 \xrightarrow{d} J^1 \xrightarrow{d} \dots \xrightarrow{d} J^{p-1} \xrightarrow{d} J^p \xrightarrow{d} \dots \quad (3.11)$$

Since the operator d is well defined also on the quotient spaces X^p/J^p , we obtain also a cochain *quotient complex* X/J :

$$0 \rightarrow X^0/J^0 \xrightarrow{d} X^1/J^1 \xrightarrow{d} \dots \xrightarrow{d} X^{p-1}/J^{p-1} \xrightarrow{d} X^p/J^p \xrightarrow{d} \dots \quad (3.12)$$

Consider the annihilator of J^p , that is the space

$$(J^p)^\perp = \{v \in X_p : v \perp J^p\}.$$

Lemma 3.5 *The dual operator ∂ of d is invariant on $\{(J^p)^\perp\}$, and the chain sub-complex*

$$0 \leftarrow (J^0)^\perp \xleftarrow{\partial} (J^1)^\perp \xleftarrow{\partial} \dots \xleftarrow{\partial} (J^{p-1})^\perp \xleftarrow{\partial} (J^p)^\perp \xleftarrow{\partial} \dots \quad (3.13)$$

is dual to the cochain quotient complex (3.12).

Proof. If $v \in (J^p)^\perp$ then, for any $\omega \in J^{p-1}$, we have $d\omega \in J^p$ and, hence,

$$(\omega, \partial v) = (d\omega, v) = 0,$$

which implies $\partial v \in (J^{p-1})^\perp$. Hence, ∂ maps $(J^p)^\perp$ to $(J^{p-1})^\perp$, so that the complex (3.13) is well-defined.

To prove the duality of (3.12) and (3.13), observe that $(J^p)^\perp$ is naturally isomorphic to the dual space $(X^p/J^p)'$. Indeed, each $v \in (J^p)^\perp$ defines a linear functional on X^p/J^p simply by $\omega \mapsto (\omega, v)$ where $\omega \in X^p$ is a representative of an element of X^p/J^p . If $\omega_1 = \omega_2 \pmod{J^p}$ then $\omega_1 - \omega_2 \in J^p$ whence $(\omega_1 - \omega_2, v) = 0$ and $(\omega_1, v) = (\omega_2, v)$. Clearly, the mapping $(J^p)^\perp \rightarrow (X^p/J^p)'$ is injective and, hence, surjective because of the identity of the dimensions of the two spaces. Finally, the duality of the operators d and ∂ on the complexes (3.12) and (3.13) is a trivial consequence of their duality on the complexes X^\bullet and X_\bullet . ■

Let us describe a specific method of constructing of d -invariant subspaces.

Lemma 3.6 *Given any subspace S^p of X^p , set*

$$J^p = S^p + dS^{p-1}. \quad (3.14)$$

Then d is invariant on $\{J^p\}$. Besides, we have the following identity

$$(J^p)^\perp = \left\{ v \in (S^p)^\perp : \partial v \in (S^{p-1})^\perp \right\}. \quad (3.15)$$

Proof. The first claim follows from $d^2 = 0$ since

$$dJ^p \subset dS^p + d^2S^{p-1} = dS^p \subset J^{p+1}.$$

The condition $v \in (J^p)^\perp$ means that

$$v \perp S^p \quad \text{and} \quad v \perp dS^{p-1}. \quad (3.16)$$

Clearly, the first condition here is equivalent to $v \in (S^p)^\perp$, while the second condition is equivalent to

$$(d\omega, v) = 0 \quad \forall \omega \in S^{p-1} \Leftrightarrow (\omega, \partial v) = 0 \quad \forall \omega \in S^{p-1} \Leftrightarrow \partial v \perp S^{p-1} \Leftrightarrow \partial v \in (S^{p-1})^\perp,$$

which proves (3.15). ■

3.4 Zigzag Lemma

Consider now three cochain complexes X, Y, Z connected by vertical linear mappings as on the diagram:

$$\begin{array}{ccccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & Y^0 & \xrightarrow{d} & Y^1 & \xrightarrow{d} & Y^2 & \rightarrow & \dots \\
\downarrow & & \downarrow^\alpha & & \downarrow^\alpha & & \downarrow^\alpha & & \\
0 & \rightarrow & X^0 & \xrightarrow{d} & X^1 & \xrightarrow{d} & X^2 & \rightarrow & \dots \\
\downarrow & & \downarrow^\alpha & & \downarrow^\alpha & & \downarrow^\alpha & & \\
0 & \rightarrow & Z^0 & \xrightarrow{d} & Z^1 & \xrightarrow{d} & Z^2 & \rightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \dots
\end{array} \quad (3.17)$$

Each horizontal mapping is denoted by d and each vertical mapping is denoted by α . We assume that the diagram is commutative. Let us also assume that each column in (3.17) is an exact cochain complex, that is, the mapping $\alpha : Y^p \rightarrow X^p$ is an injection, and $\alpha : X^p \rightarrow Z^p$ a surjection with the kernel X^p . In this case we can identify Y^p with a subspace of X^p and Z^p with the quotient X^p/Y^p .

Proposition 3.7 (Zigzag Lemma) *Under the above conditions the sequence*

$$0 \rightarrow H^0(Y) \rightarrow H^0(X) \rightarrow H^0(Z) \rightarrow \dots \rightarrow H^p(Y) \rightarrow H^p(X) \rightarrow H^p(Z) \rightarrow H^{p+1}(Y) \rightarrow \dots \quad (3.18)$$

is exact.

The sequence (3.18) is called a *long exact sequence in cohomology*. A similar result holds for homologies of chain complexes.

The meaning of the statement is that the mappings denoted in (3.18) by arrows, can be defined so that this sequence is exact. For example, the mappings

$$H^p(Y) \rightarrow H^p(X) \rightarrow H^p(Z)$$

are obvious extensions of the mapping α in (3.17), whereas the mapping $H^p(Z) \rightarrow H^{p+1}(Y)$ is defined in a more tricky way. The details of the proof can be found in [3].

We will normally apply Proposition 3.7 in the following form: if X is a cochain complex (3.1) and J is its sub-complex (3.11), then the following long sequence is exact:

$$0 \rightarrow \dots \rightarrow H^p(J) \rightarrow H^p(X) \rightarrow H^p(X/J) \rightarrow H^{p+1}(J) \rightarrow \dots \quad (3.19)$$

Similarly, if X is a chain complex (3.8) and J its sub-complex, then the following long sequence is exact:

$$0 \leftarrow \dots \leftarrow H_p(X/J) \leftarrow H_p(X) \leftarrow H_p(J) \leftarrow H_{p+1}(X/J) \leftarrow \dots \quad (3.20)$$

3.5 Reduced cohomologies and homologies

In the cochain and chain complexes (3.1) and (3.8) one naturally defines the spaces X^{-1} and X_{-1} as $\{0\}$. In a number of situations there is a need in another choice of X^{-1} and X_{-1} .

Assume that there is a injection $\tilde{d} : \mathbb{K} \rightarrow X^0$ that satisfies the relation $d\tilde{d} = 0$. Setting $\tilde{X}^{-1} \equiv \mathbb{K}$, we obtain an extended cochain complex

$$0 \rightarrow \tilde{X}^{-1} \xrightarrow{\tilde{d}} X^0 \xrightarrow{d} X^1 \xrightarrow{d} \dots \xrightarrow{d} X^{p-1} \xrightarrow{d} X^p \xrightarrow{d} \dots \quad (3.21)$$

The cohomologies of the complex (3.21) are denoted by $\tilde{H}^p(X)$ and are called the *reduced cohomologies*. Obviously, we have

$$\tilde{H}^p(X) \cong \begin{cases} H^p(X), & p \geq 1, \\ H^0(X) / \text{const}, & p = 0. \end{cases} \quad (3.22)$$

The dual space \tilde{X}_{-1} is also \mathbb{K} , and the dual operator $\tilde{\partial} : X_0 \rightarrow \mathbb{K}$ of \tilde{d} is given by $\tilde{\partial}v = (\tilde{d}1, v)$ for any $v \in X_0$. Hence, we obtain an extended chain complex

$$0 \leftarrow \tilde{X}_{-1} \xleftarrow{\tilde{\partial}} X_0 \xleftarrow{\partial} X_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} X_{p-1} \xleftarrow{\partial} X_p \xleftarrow{\partial} \dots \quad (3.23)$$

and the reduced homologies $\tilde{H}_p(X)$.

For example, let X^0 be a space of \mathbb{K} -valued functions over a finite set V and assume that $d \text{const} = 0$. Define a mapping $\tilde{d} : \mathbb{K} \rightarrow X^0$ as follows: for any $c \in \mathbb{K}$, $\tilde{d}c$ is the constant function on V taking the value c . It follows that $d\tilde{d} = 0$ so that the reduced cohomologies are well-defined. In this case we have, for any $v \in X_0$,

$$\tilde{\partial}v = (1, \tilde{\partial}v) = (\tilde{d}1, v) = (1, v),$$

where 1 is regarded as a constant function on V . It follows that

$$\tilde{H}_0(X) \cong H_0(X) / \{v \in X_0 : (1, v) = 0\}. \quad (3.24)$$

4 p -forms on digraphs

A digraph is a pair (V, E) where V is an arbitrary set and E is a subset of $V \times V \setminus \{\text{diag}\}$. The elements of V are called *vertices* and the elements of E are called (*directed*) *edges*. The set V will be always assumed non-empty and finite.

4.1 Allowed paths

Let $i_0 \dots i_p$ be an elementary regular p -path on V . It is called *allowed* if $i_k i_{k+1} \in E$ for any $k = 0, \dots, p-1$, and *non-allowed* otherwise. The set of all allowed elementary p -paths will be denoted by E_p , and non-allowed – by N_p . For example, $E_0 = V$ and $E_1 = E$.

Denote by $\mathcal{A}_p = \mathcal{A}_p(V, E)$ the subspace of \mathcal{R}_p spanned by the allowed elementary p -paths, that is,

$$\mathcal{A}_p = \text{span} \{e_{i_0 \dots i_p} : i_0 \dots i_p \in E_p\} = \{v \in \mathcal{R}_p : v^{i_0 \dots i_p} = 0 \ \forall i_0 \dots i_p \in N_p\}. \quad (4.1)$$

The elements of \mathcal{A}_p are called *allowed p -paths*.

Similarly, denote by \mathcal{N}^p the subspace of \mathcal{R}^p , spanned by the non-allowed elementary p -forms, that is,

$$\mathcal{N}^p = \text{span} \{e^{i_0 \dots i_p} : i_0 \dots i_p \in N_p\} = \{\omega \in \mathcal{R}^p : \omega_{i_0 \dots i_p} = 0 \ \forall i_0 \dots i_p \in E_p\}.$$

Clearly, we have $\mathcal{A}_p = (\mathcal{N}^p)^\perp$ where \perp refers to the annihilator subspace with respect to the couple $(\mathcal{R}^p, \mathcal{R}_p)$ of dual spaces.

4.2 The space p -forms on a digraph

We would like to reduce the space \mathcal{R}^p of regular p -forms so that the non-allowed forms can be treated as zeros. Consider the following subspaces of spaces \mathcal{R}^p

$$\mathcal{J}^p \equiv \mathcal{J}^p(V, E) := \mathcal{N}^p + d\mathcal{N}^{p-1}, \quad (4.2)$$

that are d -invariant by Lemma 3.6, and define the space Ω^p of p -forms on the digraph (V, E) by

$$\Omega^p \equiv \Omega^p(V, E) := \mathcal{R}^p / \mathcal{J}^p. \quad (4.3)$$

Then d is well-defined on Ω^p and we obtain a cochain complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \dots \quad (4.4)$$

Shortly we write $\Omega^\bullet = \mathcal{R}^\bullet / \mathcal{J}^\bullet$ where Ω^\bullet is the complex (4.4) and \mathcal{R}^\bullet and \mathcal{J}^\bullet refer to the corresponding cochain complexes.

If the digraph (V, E) is complete, that is, $E = V \times V \setminus \{\text{diag}\}$ then the spaces \mathcal{N}^p and \mathcal{J}^p are trivial, and $\Omega^p = \mathcal{R}^p$.

Let us show that the concatenation is also well-defined on the spaces Ω^p .

Lemma 4.1 *Let $\varphi \in \mathcal{R}^p$ and $\psi \in \mathcal{R}^q$. If $\varphi \in \mathcal{J}^p$ or $\psi \in \mathcal{J}^q$ then $\varphi\psi \in \mathcal{J}^{p+q}$, that is, $\{\mathcal{J}^p\}$ is a graded ideal for the concatenation. Consequently, the concatenation of two forms is well-defined on the spaces \mathcal{J}^p as well as on Ω^p , and it satisfies the product rule (2.6).*

Proof. Observe first that if $\varphi \in \mathcal{N}^p$ then $\varphi\psi \in \mathcal{N}^{p+q}$. Indeed, it suffices to prove this for elementary forms $\varphi = e^{i_0 \dots i_p}$ and $\psi = e^{j_0 \dots j_q}$ where the claim is obvious: if the p -path $i_0 \dots i_p$ is non-allowed then so is the concatenated $(p+q)$ -path $i_0 \dots i_p j_1 \dots j_q$.

If $\varphi \in \mathcal{J}^p$ then $\varphi = \varphi_0 + d\varphi_1$ where $\varphi_0 \in \mathcal{N}^p$ and $\varphi_1 \in \mathcal{N}^{p-1}$. Then we have

$$\begin{aligned} \varphi\psi &= \varphi_0\psi + (d\varphi_1)\psi \\ &= \varphi_0\psi + d(\varphi_1\psi) - (-1)^{p-1} \varphi_1 d\psi. \end{aligned}$$

By the above observation, all the forms $\varphi_0\psi, \varphi_1\psi, \varphi_1 d\psi$ are in \mathcal{N}^\bullet . It follows that $d(\varphi_1\psi) \in \mathcal{J}^{p+q}$ and, hence, $\varphi\psi \in \mathcal{J}^{p+q}$. In the same way one handles the case $\psi \in \mathcal{J}^q$.

To prove that concatenation is well defined on Ω^p , we need to verify that if $\varphi = \varphi' \bmod \mathcal{J}^p$ and $\psi = \psi' \bmod \mathcal{J}^q$ then $\varphi\psi = \varphi'\psi' \bmod \mathcal{J}^{p+q}$. Indeed, we have

$$\varphi\psi - \varphi'\psi' = \varphi(\psi - \psi') + (\varphi - \varphi')\psi',$$

and each of the terms in the right hand side belong to \mathcal{J}^{p+q} by the first part. Finally, the Leibniz formula for equivalence classes follows from that for their representatives. ■

Frequently it will be convenient to use the following notation. For p -forms $\omega', \omega'' \in \mathcal{R}^p$ we write

$$\omega' \simeq \omega'' \quad \text{if} \quad \omega' = \omega'' \bmod \mathcal{J}^p.$$

Then the equivalence classes of \simeq are exactly the elements of Ω^p .

As it follows from Lemmas 3.6 and 4.1, $\omega \simeq 0$ implies $d\omega \simeq 0$, and if $\varphi \simeq 0$ or $\psi \simeq 0$ then $\varphi\psi \simeq 0$.

Let us define the cohomologies of the digraph (V, E) by

$$H^p(V, E) := H^p(\Omega) = \ker d|_{\Omega^p} / \text{Im } d|_{\Omega^{p-1}}.$$

Here and in what follows we use the notation $H^p(\Omega)$ for the cohomology of the complex Ω^\bullet . Later on we will introduce a dual complex Ω_\bullet and its homologies will be denoted by $H_p(\Omega)$, which will never lead to any ambiguity in the notation.

4.3 Computation of $\dim H^0$

Proposition 4.2 *We have*

$$\dim H^0(V, E) = C, \tag{4.5}$$

where C is the number of (undirected) connected components of the digraph (V, E) .

Proof. By definition,

$$H^0(\Omega) = \ker d|_{\Omega^0} = \{f \in \Omega^0 : df \simeq 0\}.$$

The condition $df \simeq 0$ means that $(df)_{ij} = 0$ for all $ij \in E$, that is, $f_i = f_j$ for all edges ij . The latter is equivalent to the fact that $f = \text{const}$ on any connected component of (V, E) , and the dimension of this space of functions is clearly C . ■

4.4 Some condition for $\dim \Omega^p = 0$

Proposition 4.3 *If $\dim \Omega^n \leq 1$ then $\dim \Omega^p = 0$ for all $p > n$.*

Proof. Assume $\dim \Omega^n = 0$. Any regular p -form $e^{i_0 \dots i_p}$ with $p > n$ is a concatenation of an n -form and a $(p - n)$ -form:

$$e^{i_0 \dots i_p} = e^{i_0 \dots i_n} e^{i_n \dots i_p}.$$

Since $e^{i_0 \dots i_n} \simeq 0$ by hypothesis, it follows by Lemma 4.1 that also $e^{i_0 \dots i_p} \simeq 0$, whence $\dim \Omega^p = 0$.

Let now $\dim \Omega^n = 1$. We have

$$e^{i_0 \dots i_p} = e^{i_0 \dots i_n} e^{i_n \dots i_p} = e^{i_0 i_1} e^{i_1 \dots i_{n+1}} e^{i_{n+1} \dots i_p}$$

We claim that

$$\text{either } e^{i_0 \dots i_n} \simeq 0 \quad \text{or} \quad e^{i_1 \dots i_{n+1}} \simeq 0, \tag{4.6}$$

which would imply that $e^{i_0 \dots i_p} \simeq 0$ and $\dim \Omega^p = 0$. Indeed, if (4.6) fails then both forms $e^{i_0 \dots i_n}$ and $e^{i_1 \dots i_{n+1}}$ belong to non-zero equivalence classes modulo \mathcal{J}^n . Since the latter has dimension 1, it follows that

$$e^{i_0 \dots i_n} = \text{const } e^{i_1 \dots i_{n+1}} \pmod{\mathcal{J}^n}.$$

Clearly, this identity is only possible if $e^{i_0 \dots i_n} = e^{i_1 \dots i_{n+1}}$ whence $i_0 = i_1 \dots = i_{n+1}$, which contradicts the regularity. ■

4.5 Disjoint union

Proposition 4.4 *Let (V', E') and (V'', E'') be two digraphs. Define a digraph (V, E) as a disjoint union of (V', E') and (V'', E'') , that is,*

$$V = V' \sqcup V'' \quad \text{and} \quad E = E' \sqcup E''.$$

Then, for any $p \geq 0$,

$$H^p(V, E) \cong H^p(V', E') \oplus H^p(V'', E'').$$

Proof. Indeed, every allowed elementary path on (V, E) is a path either on (V', E') or on (V'', E'') , which implies the identity

$$\Omega^p(V, E) = \Omega^p(V', E') \oplus \Omega^p(V'', E'').$$

It is obvious that d on $\Omega^p(V, E)$ is the direct sum of the operators d on $\Omega^p(V', E')$ and $\Omega^p(V'', E'')$, whence the claim follows. ■

5 ∂ -invariant paths

5.1 The space of ∂ -invariant paths

Note that the spaces \mathcal{A}_p of allowed paths are in general *not* invariant for ∂ . Consider the following subspaces of \mathcal{A}_p

$$\Omega_p \equiv \Omega_p(V, E) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\} \tag{5.1}$$

that are ∂ -invariant. Indeed, $v \in \Omega_p \Rightarrow \partial v \in \mathcal{A}_{p-1} \subset \Omega_{p-1}$. The elements of Ω_p are called *∂ -invariant p -paths*.

A p -path $v \in \mathcal{A}_p$ is called *closed* if $\partial v = 0$. It follows from (5.1) that a closed path is automatically ∂ -invariant. Similarly, v is called *exact* if $v = \partial u$ for some $u \in \Omega_{p+1}$. Clearly, any exact path is also closed.

We obtain a chain complex Ω_\bullet

$$0 \leftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \dots$$

that, in fact, is dual to Ω^\bullet . Indeed, by Lemma 3.5, the dual to the cochain complex $\Omega^\bullet = \mathcal{R}^\bullet / \mathcal{J}^\bullet$ is

$$0 \leftarrow (\mathcal{J}^0)^\perp \xleftarrow{\partial} (\mathcal{J}^1)^\perp \xleftarrow{\partial} \dots \xleftarrow{\partial} (\mathcal{J}^{p-1})^\perp \xleftarrow{\partial} (\mathcal{J}^p)^\perp \xleftarrow{\partial} \dots$$

while by Lemma 3.6 we have

$$\begin{aligned} (\mathcal{J}^p)^\perp &= \left\{ v \in (\mathcal{N}^p)^\perp : \partial v \in (\mathcal{N}^{p-1})^\perp \right\} \\ &= \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\} = \Omega_p. \end{aligned}$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$ so that

$$\dim \Omega_0 = |V| \quad \text{and} \quad \dim \Omega_1 = |E|,$$

while in general $\Omega_p \subset \mathcal{A}_p$.

Note that, unlike the operation of concatenation of forms, the operation of path product is *not* invariant in the spaces \mathcal{A}_p or Ω_p .

Let us define the homologies of the digraph (V, E) by

$$H_p(V, E) := H_p(\Omega) = \ker \partial|_{\Omega_p} / \text{Im } \partial|_{\Omega_{p+1}}.$$

As it was already mentioned above, in the notation $H_p(\Omega)$ we use the letter Ω instead of Ω_\bullet in order to simplify the notation.

Recall that by Lemma 3.3 the spaces $H^p(V, E)$ and $H_p(V, E)$ are dual and, hence, their dimensions are the same. The values of $\dim H_p(V, E)$ can be regarded as invariants of the digraph (V, E) .

The elements of $H_p(V, E)$ are called the *homology classes*. Two closed p -paths u and v on V are said to be homological if u and v represent the same homology class, that is, if $u - v$ is exact.

The Euler characteristic of the digraph (V, E) is defined by

$$\chi(V, E) = \sum_{p=0}^n (-1)^p \dim H_p(\Omega) \tag{5.2}$$

provided n is so big that

$$\dim H_p(\Omega) = 0 \quad \text{for all } p > n. \tag{5.3}$$

Note that the condition (5.3) is not always fulfilled. If it is not fulfilled then $\chi(V, E)$ is not defined.

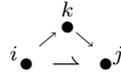
If $\dim \Omega_p = 0$ for $p > n$, then by Lemma 3.2

$$\chi(V, E) = \sum_{p=0}^n (-1)^p \dim \Omega_p. \tag{5.4}$$

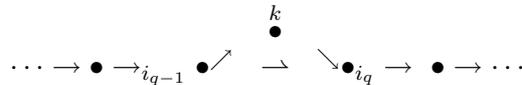
The definition (5.2) has an advantage that it may work even when $\dim \Omega_p > 0$ for all p .

5.2 Semi-edges and ∂ -invariant paths

Let us describe more explicitly the ∂ -invariant paths. Let us say that a pair ij of vertices is a *semi-edge* if it is not an edge but there is a vertex k (not necessarily unique) such that ik and kj are edges. The 2-path ikj is called a *bridge* of the semi-edge ij . The semi-edge ij will be denoted by $i \rightharpoonup j$ as on the diagram:



Let us say that an elementary path $i_0 \dots i_p$ is *semi-allowed* if among the pairs $i_{q-1}i_q$, $q = 1, \dots, p$, there is exactly one semi-edge, while all others are edges, as on the diagram:



A path $i_0 \dots i_{q-1} k i_q \dots i_p$ that is obtained by replacing in $i_0 \dots i_{q-1} i_q \dots i_p$ the semi-edge $i_{q-1} i_q$ by the bridge $i_{q-1} k i_q$, is obviously allowed and will be called an allowed extension of $i_0 \dots i_p$.

Let us use the following notation: if $i_0 \dots i_p$ is semi-allowed with the semi-edge $i_{q-1} i_q$ then, for any p -path v , define its *deficiency* $[v]^{i_0 \dots i_p}$ along the path $i_0 \dots i_p$ by

$$[v]^{i_0 \dots i_p} := \sum_{k \in V} v^{i_0 \dots i_{q-1} k i_q \dots i_p}. \quad (5.5)$$

Clearly, it suffices to restrict the summation to those k forming a bridge $i_{q-1} k i_q$. Alternatively, one can say that the summation in (5.5) is performed across all allowed extensions of the path $i_0 \dots i_p$.

Lemma 5.1 *Let $p \geq 1$. A path $v \in \mathcal{A}_{p+1}$ belongs to Ω_{p+1} if and only if for all semi-allowed paths $i_0 \dots i_p$,*

$$[v]^{i_0 \dots i_p} = 0.$$

Proof. The condition $v \in \Omega_{p+1}$ is equivalent to $\partial v \in \mathcal{A}_p$, while the latter is equivalent to

$$(\partial v)^{i_0 \dots i_p} = 0 \quad (5.6)$$

for all non-allowed regular paths $i_0 \dots i_{p-1}$. By (2.7) we have

$$(\partial v)^{i_0 \dots i_p} = \sum_{q=0}^{p+1} \sum_k (-1)^q v^{i_0 \dots i_{q-1} k i_q \dots i_p}. \quad (5.7)$$

If $i_0 \dots i_p$ is not semi-allowed then all the paths $i_0 \dots i_{q-1} k i_q \dots i_p$ are not allowed, because by inserting k one can eliminate only one non-edge. Hence, for such $i_0 \dots i_p$ the condition (5.6) is satisfied automatically, so that (5.6) is non-void only for semi-allowed paths. If the only semi-edge in $i_0 \dots i_p$ is $i_{q-1} i_q$ then (5.6) amounts to

$$\sum_k v^{i_0 \dots i_{q-1} k i_q \dots i_p} = 0,$$

which was to be proved. ■

5.3 Triangles, squares and $\dim \Omega_p$

Recall that $\dim \Omega_0 = \dim \mathcal{A}_0 = |V|$ and $\dim \Omega_1 = \dim \mathcal{A}_1 = |E|$. Here we compute $\dim \Omega_2$. Denote by \mathcal{S} the set of all semi-edges of a digraph (V, E) (see Section 5.2 for the definition of a semi-edge).

Proposition 5.2 *We have*

$$\dim \Omega_2 = \dim \mathcal{A}_2 - |\mathcal{S}| = |E_2| - |\mathcal{S}|. \quad (5.8)$$

Proof. Recall that

$$\mathcal{A}_2 = \text{span} \{e_{abc} : abc \text{ is allowed}\}, \quad \dim \mathcal{A}_2 = |E_2|,$$

and

$$\Omega_2 = \{v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1\} = \{v \in \mathcal{A}_2 : \partial v = 0 \text{ mod } \mathcal{A}_1\}.$$

If abc is allowed then ab and bc are edges, whence

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} = -e_{ac} \bmod \mathcal{A}_1.$$

If ac is an edge then $e_{ac} = 0 \bmod \mathcal{A}_1$. If ac is not an edge then ac is a semi-edge, and in this case

$$\partial e_{abc} \neq 0 \bmod \mathcal{A}_1.$$

For any $v \in \Omega_2$, we have

$$v = \sum_{\{abc \text{ is allowed}\}} v^{abc} e_{abc}$$

hence it follows that

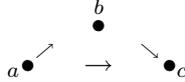
$$\partial v = - \sum_{\{abc: ac \text{ is semi-edge}\}} v^{abc} e_{ac} \bmod \mathcal{A}_1.$$

The condition $\partial v = 0 \bmod \mathcal{A}_1$ is equivalent to

$$\sum_{\{abc: ac \text{ is semi-edge}\}} v^{abc} e_{ac} = 0 \bmod \mathcal{A}_1,$$

which is equivalent to $\sum_b v^{abc} = 0$ for all semi-edges ac . The number of these conditions is exactly $|\mathcal{S}|$, and they all are independent for different semi-edges, because a triple abc determines at most one semi-edge. Hence, Ω_2 is obtained from \mathcal{A}_2 by imposing $|\mathcal{S}|$ linearly independent conditions, which implies (5.8). ■

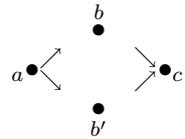
Let us call by a *triangle* a sequence of three distinct vertices $a, b, c \in V$ such that ab, bc, ac are edges:



Note that a triangle determines a 2-path $e_{abc} \in \Omega_2$ as $e_{abc} \in \mathcal{A}_2$ and

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1.$$

Let us called by a *square* a sequence of four distinct vertices $a, b, b', c \in V$ such that $ab, bc, ab', b'c$ are edges:



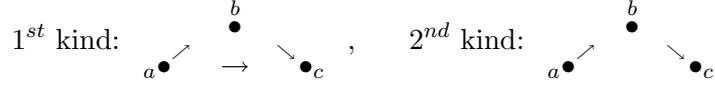
Note that a square determines a 2-path $v := e_{abc} - e_{ab'c} \in \Omega_2$ as $v \in \mathcal{A}_2$ and

$$\begin{aligned} \partial v &= (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1. \end{aligned}$$

Theorem 5.3 *Assume that a digraph (V, E) contains no squares (as subgraphs). Then $\dim \Omega_2$ is equal to the number of distinct triangles in (V, E) , and $\dim \Omega_p = 0$ for all $p > 2$.*

In particular, if (V, E) contains neither triangles nor squares then $\dim \Omega_p = 0$ for all $p \geq 2$. Consequently, $\dim H_p(\Omega) = 0$ for all $p \geq 2$.

Proof. Let us split the family E_2 of allowed 2-paths into two subsets: an allowed path abc is of the first kind if ac is an edge and of the second kind otherwise:

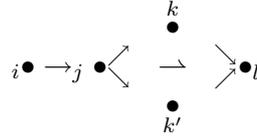


Clearly, the paths of the first kind are in one-to-one correspondence with triangles. Each path abc of the second kind determines a semi-edge ac . The mapping of $abc \mapsto ac$ from the paths of second kind to semi-edges is also one-to-one: if $abc \mapsto ac$ and $ab'c \mapsto ac$ then we obtain a square a, b, b', c which contradicts the hypotheses. Hence, the number of the path of the second kind is equal to $|\mathcal{S}|$, which implies that the number of the paths of the first kind is equal to $|E_2| - |\mathcal{S}|$, and so is the number of triangles. Comparing with (5.8) we obtain that $\dim \Omega_2$ is equal to the number of triangles.

Let us prove that $\Omega_3 = \{0\}$, that is, any $v \in \Omega_3$ is identical 0. It suffices to prove that $v^{ijkl} = 0$ for any allowed path $ijkl$ on (V, E) . Fix an allowed path $ijkl$ and assume first jl is a semi-edge. Then ijl is semi-allowed, and by Lemma 5.1 we obtain $[v]^{ijl} = 0$, that is,

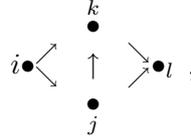
$$\sum_{k'} v^{ijk'l} = 0.$$

However, the only allowed path of the form $ijk'l$ is $ijkl$ because of the absence of squares:



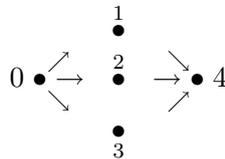
We conclude that $v^{ijkl} = 0$, provided jl is a semi-edge. In the same way $v^{ijkl} = 0$ provided ik is a semi-edge.

Now we claim that, for any allowed path $ijkl$, either ik or jl is a semi-edge. Indeed, if neither of them is a semi-edge then both ik and jl must be edges, which implies that the sequence i, j, k, l forms a square:



which contradicts the hypothesis. It follows that $v^{ijkl} = 0$ for any allowed path $ijkl$, which proves that $\Omega_3 = \{0\}$. By Proposition 4.3 we conclude that $\Omega_p = \{0\}$ for all $p \geq 3$. ■

In the presence of squares one cannot relate directly $\dim \Omega_2$ to the number of squares and triangles as there may be a linear dependence between them. Indeed, in the following digraph



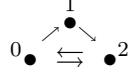
there are three squares $0, i, j, 4$, where $i, j = 1, 2, 3$, which determine three paths

$$e_{014} - e_{024}, \quad e_{024} - e_{034}, \quad e_{034} - e_{014}$$

that are linearly dependent (the sum is equal to 0). In fact, $\dim \Omega_2 = 2$ as $|E_2| = 3$ and there is only one semi-edge 04 .

Also, in the presence of squares one may have non-trivial Ω_p for arbitrary p as one can see from examples in the rest of the paper.

5.4 Star-like graphs and cones

We say that a digraph (V, E) is *star-like* if there is a vertex $a \in V$ (called a star center) such that $ai \in E$ for all $i \neq a$. For example, a digraph  is star-like with the star center 0.

Of course, any complete digraph is star-like.

Theorem 5.4 *If (V, E) is a star-like digraph, then $\dim H_p(V, E) = 0$ for any $p \geq 1$ and $\dim H_0(V, E) = 1$. Consequently, $\chi(V, E) = 1$.*

Proof. To prove that $H_p(\Omega) = \{0\}$ for $p \geq 1$, we need to show that if $v \in \Omega_p$ and $\partial v = 0$ then $v = \partial\omega$ for some $\omega \in \Omega_{p+1}$. Set $\omega = e_a v$. We claim that $\omega \in \mathcal{A}_{p+1}$. Since v is a linear combination of allowed paths $e_{i_0 \dots i_p}$, it suffices to show that $e_{ai_0 \dots i_p} \in \mathcal{A}_{p+1}$ for any allowed path $e_{i_0 \dots i_p}$. Indeed, if $i_0 = a$ then $e_{ai_0 \dots i_p} = 0 \in \mathcal{A}_{p+1}$. If $i_0 \neq a$ then $e_{ai_0 \dots i_p}$ is allowed by the star condition. Hence, we have $\omega \in \mathcal{A}_{p+1}$.

By the product rule (2.16) we have

$$\partial\omega = \partial(e_a v) = v - e_a \partial v = v,$$

where we have used $\partial v = 0$. It follows that $\partial\omega \in \mathcal{A}_p$ and, hence, $\omega \in \Omega_{p+1}$, which finishes the proof of $H_p(V, E) = \{0\}$.

Since the graph (V, E) is connected, we have by Proposition 4.2 $\dim H_0(\Omega) = \dim H^0(\Omega) = 1$. One can see this also independently as follows. All 0-paths e_i are closed, whereas $e_i - e_a = \partial e_{ai}$ is exact, so that e_i and e_a determine the same homology class. The class of e_a is non-zero because $(e_a, 1) = 1$ while for any 1-form φ we have $(\partial\varphi, 1) = (\varphi, d1) = 0$, whence $e_a \neq \partial\varphi$. Hence, $H_0(\Omega)$ contains only one non-trivial homology class $\text{const } e_a$, which proves $\dim H_0(\Omega) = 1$.

It follows that $\chi = 1$. ■

Remark 5.5 In a similar manner one can handle the *inverse* star-like graphs, that is, when the requirement $ai \in E$ in the definition of a star property is replaced by $ia \in E$. Using the right multiplication with e_a , one proves in the same way that the statement of Theorem 5.4 remains true for inverse star-like graph.

Definition 5.6 Given a digraph $G = (V, E)$, by adding one more vertex a and the edges ia for all $i \in V$, we obtain a new digraph that is called a *cone* over G and is denoted by $\text{Cone } G$.

Since the cone is clearly inverse star-like, we obtain by Theorem 5.4 that the homologies $H_p(\text{Cone } G)$ are trivial for all $p \geq 1$.

However, the idea of the proof of Theorem 5.4 allows to say more about spaces $\Omega_p(\text{Cone } G)$. Indeed, for any ∂ -invariant p -path v on G consider, its *lifting* ve_a is a ∂ -invariant $(p+1)$ -path on $\text{Cone } G$, because if v and ∂v are allowed in G then ve_a and $\partial(ve_a) = (\partial v)e_a + (-1)^{p+1}v$ are allowed in $\text{Cone } G$. Hence, the lifting operator $v \mapsto ve_a$ is a mapping from $\Omega_p(G)$ to $\Omega_{p+1}(\text{Cone } G)$, which allows to construct easily examples of digraphs with non-trivial Ω_p for a given p .

Example 5.7 Let us define inductively a sequence $\{\text{Sm}_n\}_{n=0}^\infty$ of *simplex-digraphs* as follows: the digraph Sm_0 consists of a single vertex 0 and $\text{Sm}_n = \text{Cone } \text{Sm}_{n-1}$ for $n = 1, 2, \dots$. For example,

$$\text{Sm}_1 = 0 \bullet \rightarrow \bullet^1, \quad \text{Sm}_2 = 0 \bullet \begin{array}{c} \nearrow \bullet^2 \\ \rightarrow \bullet^1 \end{array}$$

and Sm_3 is shown on Fig. 1.

By Theorem 5.4, $H_p(\text{Sm}_n) = \{0\}$ for any $p \geq 1$. Since $\Omega_0(\text{Sm}_0)$ contains a non-trivial 0-path e_0 , we obtain by induction that $\Omega_n(\text{Sm}_n)$ contains a non-trivial n -path $e_{01 \dots n}$.

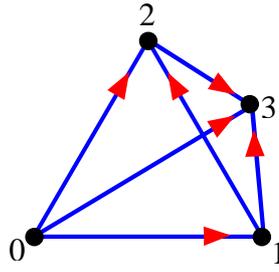
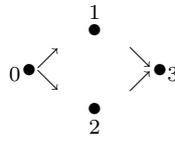


Figure 1: A 3-simplex graph Sm_3

Example 5.8 Let G be a square graph



Then $\text{Cone } G$ is a *pyramid* shown on Fig. 2. $\Omega_2(G)$ contains a 2-path $e_{013} - e_{023}$, we obtain

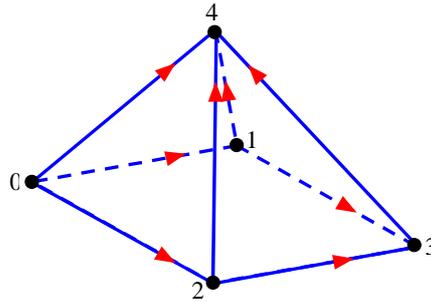


Figure 2: A pyramid graph

that $\Omega_3(\text{Cone } G)$ contains a 3-path $e_{0134} - e_{0234}$.

5.5 An example of direct computation of $\dim H_p$

Consider the digraph (V, E) with $V = \{0, 1, 2, 3, 5\}$ and $E = \{01, 02, 13, 14, 23, 24, 53, 54\}$, see Fig. 3.

Let us compute the spaces Ω_p and the homologies $H_p(\Omega)$. We have

$$\begin{aligned} \Omega_0 &= \mathcal{A}_0 = \text{span}\{e_0, e_1, e_2, e_3, e_4, e_5\}, & \dim \Omega_0 &= 6 \\ \Omega_1 &= \mathcal{A}_1 = \text{span}\{e_{01}, e_{02}, e_{13}, e_{14}, e_{23}, e_{24}, e_{53}, e_{54}\}, & \dim \Omega_1 &= 8 \\ \mathcal{A}_2 &= \text{span}\{e_{013}, e_{014}, e_{023}, e_{024}\}, & \dim \mathcal{A}_2 &= 4. \end{aligned}$$

The set of semi-edges is $\mathcal{S} = \{e_{03}, e_{04}\}$ so that $\dim \Omega_2 = \dim \mathcal{A}_2 - |\mathcal{S}| = 2$. The basis in Ω_2 can be easily spotted as each of two squares $0, 1, 2, 3$ and $0, 1, 2, 4$ determine a ∂ -invariant 2-paths, whence

$$\Omega_2 = \text{span}\{e_{013} - e_{023}, e_{014} - e_{024}\}.$$

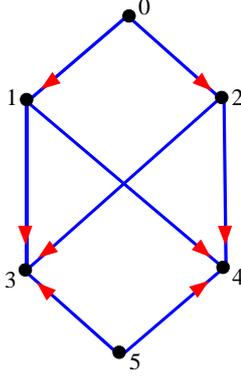


Figure 3: A digraph with 6 vertices and 8 edges

Since there are no allowed 3-paths, we see that $\mathcal{A}_3 = \Omega_3 = \{0\}$. It follows that

$$\chi = \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 = 6 - 8 + 2 = 0.$$

Let us compute $\dim H_2$ by (3.10):

$$\dim H_2 = \dim \Omega_2 - \dim \partial\Omega_2 - \dim \partial\Omega_3 = 2 - \dim \partial\Omega_2.$$

The image $\partial\Omega_2$ is spanned by two 1-paths

$$\begin{aligned} \partial(e_{013} - e_{023}) &= e_{13} - e_{03} + e_{01} - (e_{23} - e_{03} + e_{02}) = e_{13} + e_{01} - e_{23} - e_{02} \\ \partial(e_{014} - e_{024}) &= e_{14} - e_{04} + e_{01} - (e_{24} - e_{04} + e_{02}) = e_{14} + e_{01} - e_{24} - e_{02} \end{aligned}$$

that are clearly linearly independent. Hence, $\dim \partial\Omega_2 = 2$ whence $\dim H_2 = 0$. The dimension of H_1 can be computed similarly, but we can do easier using the Euler characteristic:

$$\dim H_0 - \dim H_1 + \dim H_2 = \chi = 0$$

whence $\dim H_1 = 1$.

In fact, a non-trivial element of H_1 is determined by 1-path

$$v = e_{13} - e_{14} - e_{53} + e_{54}.$$

Indeed, by a direct computation $\partial v = 0$, so that $v \in \ker \partial|_{\Omega_1}$ while for v to be in $\text{Im } \partial|_{\Omega_2}$ it should be a linear combination of $\partial(e_{013} - e_{023})$ and $\partial(e_{014} - e_{024})$, which is not possible since they do not have the term e_{54} .

5.6 Cycle-graphs

We say that a digraph (V, E) is a *cycle-graph* if it is connected (as undirected) and every vertex had the degree 2 (see Fig. 4).

For a cycle-graph we have $\dim H_0 = 1$ and

$$\dim \Omega_0 = |V| = |E| = \dim \Omega_1. \quad (5.9)$$

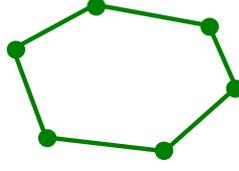


Figure 4: A cycle graph (no orientation is shown)

Proposition 5.9 *Let (V, E) be a cycle-graph. Then*

$$\begin{aligned} \dim \Omega_p &= 0 \text{ for all } p \geq 3 \\ \dim H_p(\Omega) &= 0 \text{ for all } p \geq 2. \end{aligned}$$

If (V, E) is a triangle or a square then

$$\dim \Omega_2 = 1, \quad \dim H_1(\Omega) = 0, \quad \chi = 1$$

whereas otherwise

$$\dim \Omega_2 = 0, \quad \dim H_1(\Omega) = 1, \quad \chi = 0.$$

Proof. Observe first that $\dim \Omega_2 \leq 1$ will imply $\dim \Omega_p = 0$ for all $p \geq 3$ by Proposition 4.3, whence $\dim H_p = 0$ for $p \geq 3$. Hence, we need only to handle the cases $p = 1, 2$.

Using two equivalent definition of the Euler characteristic, we have

$$\begin{aligned} \chi &= \dim H_0 - \dim H_1 + \dim H_2 \\ &= \dim \Omega_0 - \dim \Omega_1 + \dim \Omega_2 \end{aligned}$$

whence

$$\chi = \dim \Omega_2 = 1 - \dim H_1 + \dim H_2. \quad (5.10)$$

Assume first that (V, E) is neither a triangle nor a square. Then (V, E) contains neither a triangle nor a square. By Theorem 5.3 $\dim \Omega_2 = 0$ whence $\dim H_2 = 0$ and by (5.10) $\chi = 0$ and $\dim H_1 = 1$.

Let us construct an 1-path spanning H_1 . For that let us identify V with \mathbb{Z}_N where $N = |V|$ so that in the unoriented graph based on (V, E) , the edges are $i(i+1)$. Hence, in the digraph (V, E) either $i(i+1)$ or $(i+1)i$ is an edge. Consider an allowed 1-path σ with components

$$\sigma^{i(i+1)} = \begin{cases} 1, & \text{if } i(i+1) \text{ is an edge} \\ -1, & \text{if } (i+1)i \text{ is an edge,} \end{cases} \quad (5.11)$$

and all other components of σ vanish (see Fig. 5).

Since $\sigma \neq 0$, σ is not in $\text{Im } \partial|_{\Omega_2}$. However, $\sigma \in \ker \partial|_{\Omega_1}$ because by construction $\sigma^{i(i+1)} - \sigma^{(i+1)i} \equiv 1$ whence for any i

$$(\partial\sigma)^i = \sum_{j \in V} (\sigma^{ji} - \sigma^{ij}) = \sigma^{(i-1)i} + \sigma^{(i+1)i} - \sigma^{i(i-1)} - \sigma^{i(i+1)} = 1 - 1 = 0.$$

Let (V, E) be a triangle, say, with vertices a, b, c then $\dim \mathcal{A}_2 = 1$, $\mathcal{S} = \emptyset$ whence $\dim \Omega_2 = 1$ and $\chi = 1$. Note that in this case $\Omega_2 = \text{span } \{e_{abc}\}$. Since a triangle is star-like, we have by Theorem 5.4 $\dim H_p = 0$ for all $p \geq 1$.

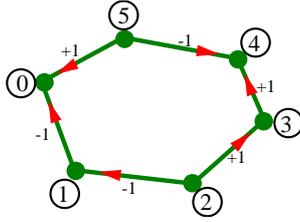
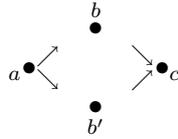


Figure 5: The 1-path $\sigma = -e_{01} - e_{12} + e_{23} + e_{34} - e_{45} + e_{50}$ spans H_1 .

Let (V, E) be a square, say a, b, b', c :



Then

$$\mathcal{A}_2 = \text{span} \{e_{abc}, e_{ab'c}\}, \quad \mathcal{S} = \{ac\}$$

whence $\dim \Omega_2 = 2 - 1 = 1$ and $\chi = 1$. Note that in this case

$$\Omega_2 = \text{span} \{e_{abc} - e_{ab'c}\}.$$

For $v = e_{abc} - e_{ab'c}$ we have $\partial v = e_{bc} - e_{b'c} + e_{ab} - e_{ab'} \neq 0$ so that $\ker \partial|_{\Omega_2} = 0$. It follows that $\dim H_2 = 0$. Then by (5.10) $\dim H_1 = 0$. ■

5.7 Sierpinski triangles

On Fig. 6 we have shown the process of constructing of Sierpinski gasket.

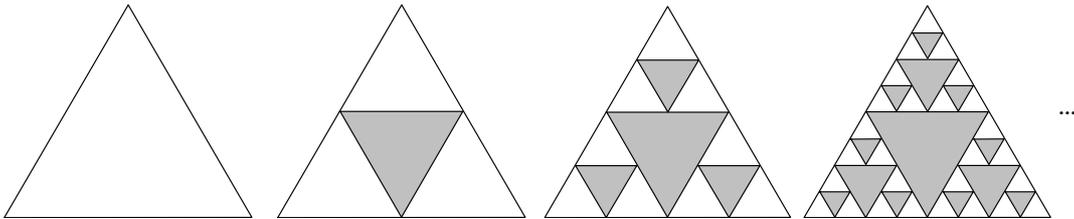


Figure 6: Construction of Sierpinski gasket

Starting with a flat triangle, one removes its middle quarter, then the same procedure is applied to each of the three remaining triangles, and so on. The set that remains after infinitely many steps is called Sierpinski gasket. Clearly, each of the shapes on Fig. 6 can easily be turned onto a graph that can be regarded as a graph approximation of the Sierpinski gasket. We go further and turn them into digraphs by choosing orientation as on Fig. 7.

The rules for choosing orientation are as follows:

1. Any white triangle gets orientation  so that it becomes a triangle as a digraph.

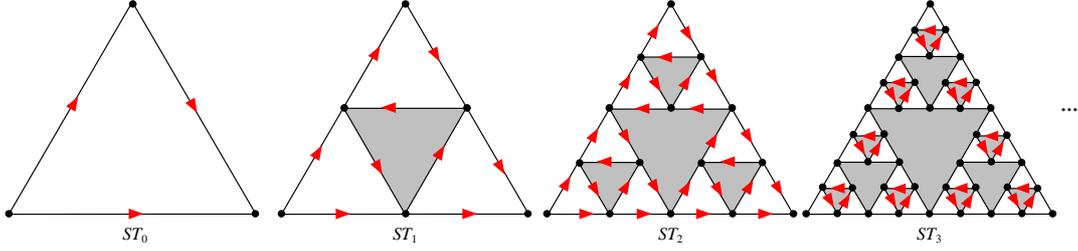


Figure 7: Sierpinski triangles ST_n (orientation on ST_3 is shown only partially)

2. Any shaded triangle gets orientation $\begin{array}{ccc} \bullet & \leftarrow & \bullet \\ & \searrow & \nearrow \\ & \bullet & \end{array}$ so that it becomes a directed cycle.

The digraph at step n is denoted by ST_n . Let us compute the dimensions of the spaces $\Omega_p(n)$ and $H_p(n)$ of the graph ST_n . Denote by v_n, e_n, t_n, c_n respectively the number of vertices, edges, white triangles and shaded triangles (cycles) on ST_n . It is easy to see that $v_0 = 3, e_0 = 3, t_0 = 1, c_0 = 0$ and

$$v_{n+1} = v_n + 3t_n, \quad e_{n+1} = e_n + 6t_n, \quad t_{n+1} = 3t_n, \quad c_{n+1} = c_n + t_n. \quad (5.12)$$

In particular, it follows that $t_n = 3^n$ and $c_n = \frac{3^n - 1}{2}$.

We have $\dim \Omega_0(n) = v_n$ and $\dim \Omega_1(n) = e_n$. Since ST_n contains no squares and t_n triangles, we obtain by Theorem 5.3 that $\dim \Omega_2(n) = t_n$ and $\dim \Omega_p(n) = 0$ for $p > 2$. Consequently, $\dim H_p(n) = 0$ for all $p \geq 2$. Denoting by χ_n the Euler characteristic of ST_n , we obtain

$$\chi_n = v_n - e_n + t_n,$$

whence by (5.12) $\chi_{n+1} = \chi_n - t_n$. The sequence $1 - c_n$ satisfies by (5.12) the same recurrence equation as χ_n and the same initial condition at $n = 0$ as $\chi_0 = 1$, which implies that $\chi_n = 1 - c_n$. Finally, using

$$\chi_n = \dim H_0(n) - \dim H_1(n) = 1 - \dim H_1(n),$$

we obtain that $\dim H_1(n) = c_n = \frac{3^n - 1}{2}$. The generating elements of $H_1(n)$ are obvious: all directed cycles, which are determined by the shaded triangles.

5.8 Snakes

A *snake* of length p is a subgraph of $p + 1$ vertices, say, $0, 1, \dots, p$ such that $i(i + 1)$ and $i(i + 2)$ are edges (see Fig. 8), which is equivalent to say that any triple $i(i + 1)(i + 2)$ is a triangle.

Any snake gives rise to a ∂ -invariant p -path $v = e_{01\dots p}$. This path is obviously allowed, its boundary

$$\partial v = \sum_{k=0}^p (-1)^k e_{0\dots\widehat{k}\dots p}$$

is also allowed (because $(k - 1)(k + 1)$ is an edge), whence $v \in \Omega_p$.

A simplex of dimension p is a subgraph of $p + 1$ vertices, say $0, 1, \dots, p$ so that any pair ij with $i < j$ is an edge – cf. Section 5.4 and Fig. 1. Since a simplex contains a snake as a subgraph, the p -path $v = e_{01\dots p}$ is ∂ -invariant also for a simplex, as we have already seen that in Section 5.4.

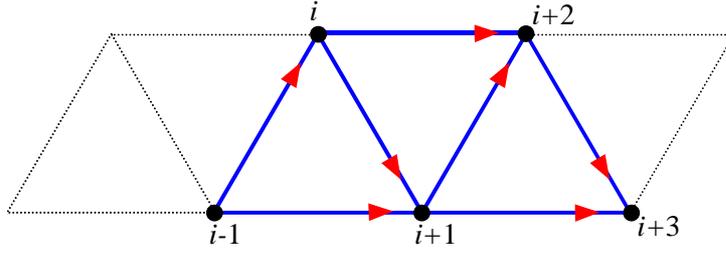


Figure 8: A snake

5.9 Triangulation as a closed path

Given a closed oriented n -dimensional manifold M , let T be its triangulation, that is, a partition into n -dimensional simplexes. Denote by V the set of all vertices of simplexes from T and by E - the set of all edges, so that (V, E) is a graph embedded on M . We would like to make (V, E) into a digraph and define on that digraph a closed n -path as a certain alternating sum of elementary n -paths arising from the simplexes from T .

Let us enumerate the set of vertices V by distinct integers. For any simplex from T with the vertices $i_0 \dots i_n$ define the quantity $\sigma^{i_0 \dots i_n}$ to be equal to 1 if the orientation of the simplex $i_0 \dots i_n$ matches the orientation of the manifold M , and -1 otherwise. Note that each simplex gives rise to $n!$ different ordered sequences of its vertices, each of them defining the quantity $\sigma^{i_0 \dots i_n}$.

Let us introduce the orientation on the set of edges E by choosing on each edge the direction from the vertex with a smaller number to the vertex with a larger number. Then each simplex from T becomes a digraph simplex as defined in Example 5.7. Denote by \vec{T} the set of the digraph simplexes constructed in this way. That is, $i_0 \dots i_n \in \vec{T}$ if $i_0 \dots i_n$ determines a simplex from T and the sequence i_0, \dots, i_n is monotone increasing.

Then consider the following n -path on the digraph (V, E) :

$$\sigma = \sum_{i_0 \dots i_n \in \vec{T}} \sigma^{i_0 \dots i_n} e_{i_0 \dots i_n}. \quad (5.13)$$

This path is allowed on (V, E) by the definition of the orientation of the edges.

We claim that the path σ is closed, that is, $\partial\sigma = 0$, which, in particular, implies that σ is ∂ -invariant. Observe that $\partial\sigma$ is the a linear combination with coefficients ± 1 of the terms $e_{j_0 \dots j_{n-1}}$ where the sequence j_0, \dots, j_{n-1} is monotone increasing and it forms an $(n-1)$ -dimensional face of one of the n -simplexes from T . In fact, every $(n-1)$ -face arises from *two* n -simplexes, say $A = j_0 \dots j_{k-1} a j_k \dots j_{n-1}$ and $B = j_0 \dots j_{l-1} b j_l \dots j_{n-1}$ (cf. Fig. 9).

We have by (2.8)

$$\partial e_{j_0 \dots j_{k-1} a j_k \dots j_{n-1}} = \dots + (-1)^k e_{j_0 \dots j_{k-1} j_k \dots j_{n-1}} + \dots$$

Since interchanging the order of two neighboring vertices in an n -simplex changes its orientation, we have

$$\sigma^{j_0 \dots j_{k-1} a j_k \dots j_{n-1}} = (-1)^k \sigma^{a j_0 \dots j_{k-1} j_k \dots j_{n-1}}.$$

Multiplying the above lines, we obtain

$$\partial (\sigma^A e_A) = \dots + \sigma^{a j_0 \dots j_{n-1}} e_{j_0 \dots j_{n-1}} + \dots,$$

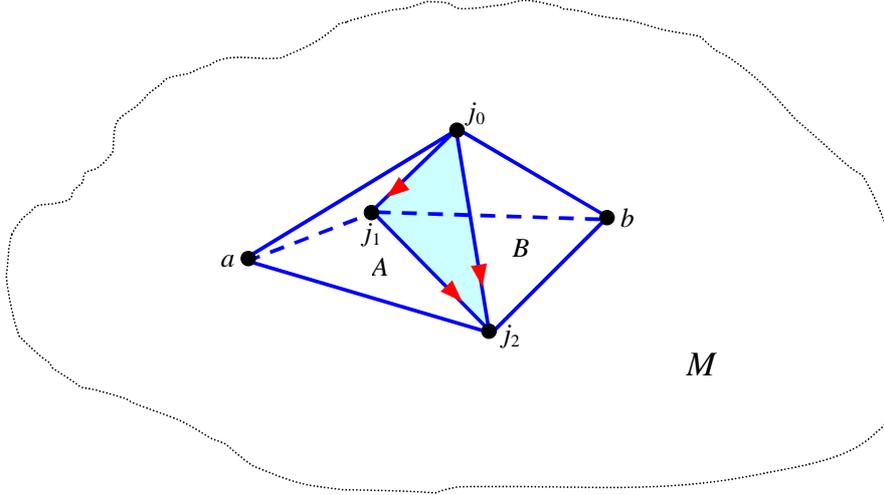


Figure 9: Two n -simplices A, B having a common $(n - 1)$ -dimensional face $j_0 \dots j_{n-1}$

and in the same way

$$\partial(\sigma^B e_B) = \dots + \sigma^{bj_0 \dots j_{n-1}} e_{j_0 \dots j_{n-1}} + \dots$$

However, the vertices a and b are located on the opposite sides of the face $j_0 \dots j_{n-1}$, which implies that the simplices $aj_0 \dots j_{n-1}$ and $bj_0 \dots j_{n-1}$ have the opposite orientations. Hence,

$$\sigma^{aj_0 \dots j_{n-1}} + \sigma^{bj_0 \dots j_{n-1}} = 0,$$

which means that the term $e_{j_0 \dots j_{n-1}}$ cancels out in the sum $\partial(\sigma^A e_A + \sigma^B e_B)$ and, hence, in $\partial\sigma$. This proves that $\partial\sigma = 0$.

The closed paths σ defined by (5.13) is called a *surface path* of M (or T).

There is a number of triangulations when a surface path σ happens to be exact, that is, $\sigma = \partial v$ for some $(n + 1)$ -path v . If this is the case then v is called a *solid path* as in this case v represents a “solid” shape whose boundary is given by M (or T).

Example 5.10 If $M = \mathbb{S}^1$ then T is a cycle graph, and a surface path σ in this case was constructed in the proof of Proposition 5.9. We have seen there that σ is exact if the digraph (V, E) is a triangle or square, and non-exact otherwise. In the former case a solid paths v represents a triangle or a square, respectively, in the latter case a solid path does not exist.

Example 5.11 Let $M = \mathbb{S}^n$ and let the faces of a triangulation T of M form a $(n + 1)$ -simplex, so that the digraph (V, E) is a $(n + 1)$ -simplex digraph. Denoting the vertices by $0, 1, \dots, n + 1$, we obtain $\partial e_{0 \dots n+1} = \sigma$ so that $e_{i_0 \dots i_{n+1}}$ is a solid paths representing a solid $(n + 1)$ -simplex.

There are also higher dimensional examples when a surface path is not exact, see Example 7.13 below. Further examples of surface and solid paths will be given in Section 6.

5.10 Lemma of Sperner revisited

Consider a triangle ABC on the plane \mathbb{R}^2 and its triangulation T . The set of vertices of T is colored with three colors $1, 2, 3$ in such a way that

- the vertices A, B, C are colored with 1, 2, 3 respectively;
- each vertex on any edge of ABC is colored with one of the two colors of the endpoints of the edge (see Fig. 10).

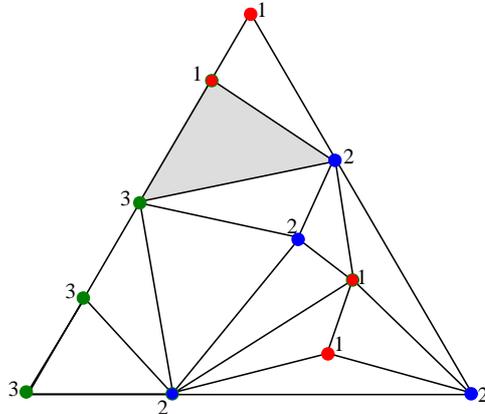


Figure 10: A Sperner coloring

The classical lemma of Sperner says that then there exists in T a 3-color triangle, that is, a triangle, whose vertices are colored with the three different colors. Moreover, the number of such triangles is odd.

We give here a new proof using the boundary operator ∂ for 1-paths. Although this proof is no shorter than the classical proof based on a double counting argument, it still provides a new insight into the subject, that 3-color triangles appear as sources and sinks of some “vector field” on a digraph.

Let us first do some reduction. Firstly, let us modify the triangulation T so that there are no vertices on the edges AB, AC, BC except for A, B, C . Indeed, if X is a vertex on AB then we move X a bit inside the triangle ABC . This gives rise to a new triangle in the triangulation T that is formed by X and its former neighbors, say Y and Z , on the edge AB (while keeping all old triangles). However, since all X, Y, Z are colored with two colors, no 3-color triangle emerges after that move. By induction, we remove all the vertices from the edges of ABC .

Secondly, we project the triangle ABC and the triangulation T onto the sphere \mathbb{S}^2 and add to the set T the triangle ABC itself from the other side of the sphere. Then we obtain a triangulation of \mathbb{S}^2 , denote it again by T , and we need to prove that the number of 3-color triangles is *even*. Indeed, since we know that one of the triangles, namely, ABC is 3-color, this would imply that the number of 3-color triangles in the original triangulation was odd.

Let us regard T as a graph on \mathbb{S}^2 and construct a dual graph V . Choose at each face of T a point and regard them as vertices of the dual graph V . The vertices in V are connected if the corresponding triangles in T have a common edge (see Fig. 11). Then the faces of V are in one-to-one correspondence to the vertices of T .

Hence, given a graph V on \mathbb{S}^2 such that each vertex has degree 3 and each face is colored with one of the colors 1, 2, 3, prove that the number of 3-color vertices (that is, the vertices, whose adjacent faces have all three colors) is even.

Let us make V into a digraph as follows. Each edge ξ in V has two adjacent faces. Choose the orientation on ξ so that the color from the left hand side and that from the right hand side

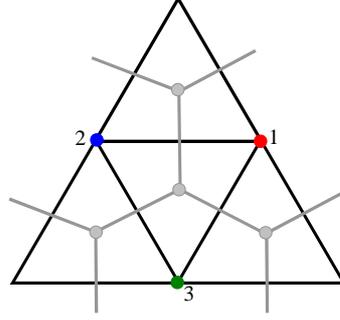


Figure 11: Construction of a dual graph

of ξ form one of the following pairs: $(1, 2)$, $(2, 3)$, $(3, 1)$ (see Fig. 12), while if the colors are the same then allow both orientations of ξ .



Figure 12: The orientation of an edge depends on the colors of adjacent faces

Examples of such orientations are shown on Fig. 13.

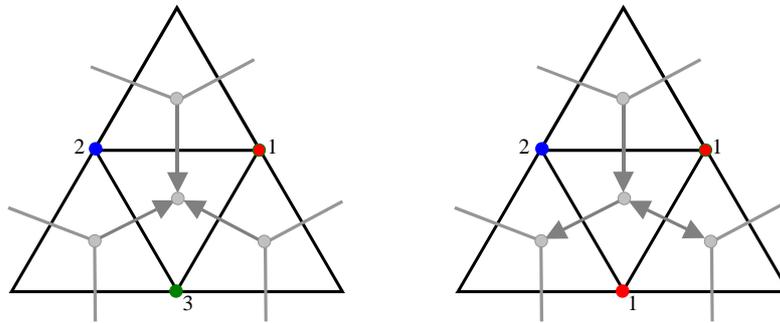


Figure 13: Oriented edges in the dual graph (V, E)

Denote by E the set of the oriented edges and set $v = \sum_{\{ab \in E\}} e_{ab}$. We have for any $a \in V$

$$(\partial v)_a = \sum_b v^{ba} - \sum_c v^{ac} = \#\{\text{incoming edges}\} - \#\{\text{outcoming edges}\},$$

where $\#A$ denotes the number of elements in the set A . If a is 3-color, then either all three edges at a are incoming or all are outcoming, whence $(\partial v)_a = 3$ or -3 , respectively. If a is not 3-color then $(\partial v)_a = 0$ (cf. Fig. 13). Denoting by n_1 the number of 3-color edges with incoming orientation and by n_2 that with outcoming orientation, we obtain that $(\partial v, 1) = 3(n_1 - n_2)$. On the other hand, $(\partial v, 1) = (v, d1) = 0$ whence we conclude that $n_1 = n_2$. In particular, the total number of 3-color vertices is $2n_1$, that is, even, which was to be proved. In fact, we have

proved a bit more: in a triangulation of a sphere, the numbers of 3-color triangles of the opposite orientations are the same.

6 Cartesian product of digraphs and paths

Let us fix some notation to be used in this section. For a finite set V denote by $R_p(V)$ the set of regular elementary p -paths on V , where p is a non-negative integer. Denote also by $R_\bullet(V)$ the (disjoint) union of the sets $R_p(V)$ for all $p \geq 0$. That is, $R_\bullet(V)$ is the set of all regular elementary paths on V .

By definition, the space $\mathcal{R}_p(V)$ of regular p -paths can be identified with the set of formal linear combinations of paths from $R_p(V)$ with the coefficients from the field \mathbb{K} . Let us define the direct sum of all spaces $\mathcal{R}_p(V)$:

$$\mathcal{R}_\bullet(V) = \bigoplus_{p \geq 0} \mathcal{R}_p(V).$$

Alternatively, $\mathcal{R}_\bullet(V)$ is the space of all formal finite linear combinations of the elements of $R_\bullet(V)$. The boundary operator ∂ extends by linearity to an operator in $\mathcal{R}_\bullet(V)$.

Let now V be a digraph. We denote the vertex set of V by the same letter V , while the edge set will be denoted by $E(V)$. For any non-negative integer p denote by $E_p(V)$ the set of all allowed elementary p -paths on V . For example, $E_0(V)$ is the set of vertices and $E_1(V)$ is the set of edges. Denote by $E_\bullet(V)$ the (disjoint) union of all the sets $E_p(V)$ for all $p \geq 0$.

Recall that the space $\mathcal{A}_p(V)$ of allowed p -paths is spanned by $E_p(V)$. Let $\mathcal{A}_\bullet(V)$ be the direct sum of all $\mathcal{A}_p(V)$. Finally, $\Omega_p(V)$ is the space of all ∂ -invariant p -paths on V and $\Omega_\bullet(V)$ is the direct sum of all $\Omega_p(V)$. We have the inclusions of subspaces

$$\Omega_\bullet(V) \subset \mathcal{A}_\bullet(V) \subset \mathcal{R}_\bullet(V),$$

and the subspace $\Omega_\bullet(V)$ is ∂ -invariant.

6.1 Cross product of regular paths

Given two finite sets X, Y , consider their Cartesian product

$$Z = X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

Let z be a regular elementary r -path on Z :

$$z = z_0 z_1 \dots z_r$$

where $z_k = (x_k, y_k)$ with $x_k \in X$ and $y_k \in Y$. We say that z is *stair-like* if, for any $k = 1, \dots, r$, either $x_{k-1} = x_k$ or $y_{k-1} = y_k$ is satisfied (in fact, exactly one of these conditions hold as z is regular). In other words, any couple $z_{k-1} z_k$ of consecutive vertices is either vertical (when $x_{k-1} = x_k$) or horizontal (when $y_{k-1} = y_k$):

$$\text{vertical couple } \begin{array}{c} \bullet \\ \uparrow \\ \bullet \end{array} \text{ and horizontal couple } \begin{array}{c} z_{k-1} \\ \bullet \end{array} \longrightarrow \begin{array}{c} z_k \\ \bullet \end{array}.$$

Any stair-like path z on Z determines the elementary paths x on X and y on Y by projection. More precisely, x is obtained from z by removing the Y -components in all the vertices of z and

then by collapsing in the resulting sequence any subsequence of repeated vertices to one vertex. The same rule applies to y . By construction, the projections x and y are *regular* elementary paths on X and Y , respectively.

Let the projections of z be $x = x_0 \dots x_p$ and $y = y_0 \dots y_q$. Then every vertex z_k of the path z has a form (x_i, y_j) , and the previous vertex z_{k-1} is either (x_i, y_{j-1}) or (x_{i-1}, y_j) as on the diagram:

$$\begin{array}{c} z_k = (x_i, y_j) \\ \bullet \\ \uparrow \\ z_{k-1} = (x_i, y_{j-1}) \\ \bullet \end{array} \quad \text{or} \quad \begin{array}{c} z_{k-1} = (x_{i-1}, y_j) \\ \bullet \\ \longrightarrow \\ z_k = (x_i, y_j) \\ \bullet \end{array} .$$

In particular, it follows that $z \in \mathcal{R}_{p+q}(Z)$.

Every vertex (x_i, y_j) of the path z can be represented as a point (i, j) of \mathbb{Z}^2 so that the whole path z is represented by a *staircase* $S(z)$ in \mathbb{Z}^2 connecting the points $(0, 0)$ and (p, q) . Define the *elevation* $L(z)$ of the path z as the number of cells in \mathbb{Z}_+^2 below the staircase $S(z)$ (a shaded area on Fig. 14).

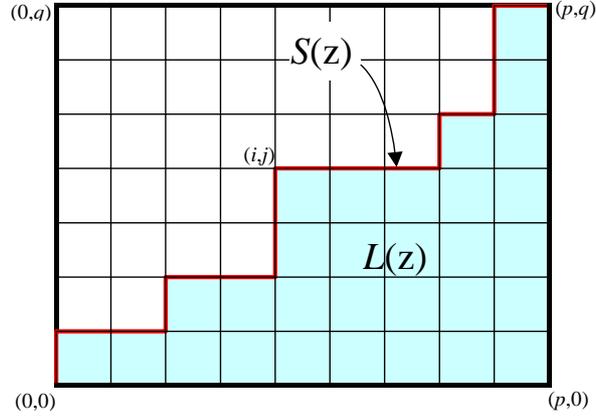


Figure 14: The staircase $S(z)$ and the elevation $L(z)$

Definition 6.1 Given paths $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ with some $p, q \geq 0$, define a path $u \times v$ on Z by the following rule: for any stair-like elementary $(p+q)$ -path z on Z , the component $(u \times v)^z$ is defined by

$$(u \times v)^z = (-1)^{L(z)} u^x v^y, \quad (6.1)$$

where x and y are the projections of z onto X and Y , respectively, and u^x and v^y are the corresponding components of u and v . For non-stair-like paths z set $(u \times v)^z = 0$.

The path $u \times v$ is called the (Cartesian) cross product of u and v . It follows that $u \times v \in \mathcal{R}_{p+q}(Z)$.

Given a stair-like $(p+q)$ -path z on Z , the projection of z onto X could be a p' -path with $p' \neq p$; in this case we set by definition $u^x \equiv 0$. The same rule applies to v^y . In other words, $(u \times v)^z$ may be non-zero only when the projections of z onto X and Y are, respectively, p -path x and q -path y with non-zero u^x and v^y .

For given paths $x \in R_p(X)$ and $y \in R_q(Y)$ with non-negative integers p, q , denote by $\Sigma_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are respectively x and y . It follows from (6.1) that

$$e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z. \quad (6.2)$$

The definition of the cross product $u \times v$ extends by linearity to all $u \in \mathcal{R}_\bullet(X)$ and $v \in \mathcal{R}_\bullet(Y)$, so that $u \times v \in \mathcal{R}_\bullet(Z)$.

Example 6.2 Let us denote the vertices on X by letters a, b, c etc and the vertices on Y by integers $1, 2, 3$, etc so that the vertices on Z can be denoted as $a1, b2$ etc as the fields on the chessboard. Then we have

$$e_a \times e_{12} = e_{a1a2}$$

$$e_{ab} \times e_1 = e_{a1b1}$$

$$e_{ab} \times e_{12} = e_{a1b1b2} - e_{a1a2b2}$$

$$e_{abc} \times e_{12} = e_{a1b1c1c2} - e_{a1b1b2c2} + e_{a1a2b2c2}$$

$$\begin{aligned} e_{abc} \times e_{123} &= e_{a1b1c1c2c3} - e_{a1b1b2c2c3} + e_{a1b1b2b3c3} \\ &\quad + e_{a1a2b2c2c3} - e_{a1a2b2b3c3} + e_{a1a2a3b3c3} \end{aligned}$$

etc (cf. Fig. 15).

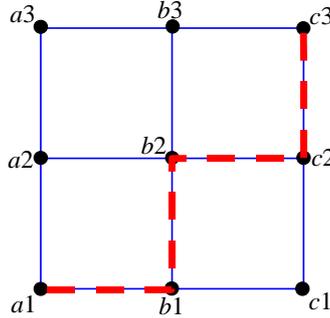


Figure 15: The path $a1b1b2c2c3$ has elevation 1. Hence, $e_{a1b1b2c2c3}$ enters the product $e_{abc} \times e_{123}$ with the negative sign.

6.2 Some property of the cross product

Let us prove the following relation between the cross product of paths and the product of paths on each constituent graph.

Lemma 6.3 *Let $u \in \mathcal{R}_{p-1}(X)$ and $v \in \mathcal{R}_{q-1}(Y)$ with some $p, q \geq 1$. Let $a \in X$ and $b \in Y$ be vertices. Then*

$$(e_a u) \times (e_b v) = e_c [u \times (e_b v) + (-1)^p (e_a u) \times v] \quad (6.3)$$

where $c = (a, b) \in X \times Y$.

Proof. It suffices to prove (6.3) when u and v are regular elementary paths, say $u = e_{i_1 \dots i_p}$ and $v = e_{j_1 \dots j_q}$. Denote also $i_0 = a$ and $j_0 = b$. If $a = i_1$ then $e_a e_{i_1 \dots i_p} = 0$ and the right hand side of (6.3) also vanishes. The same applies if $b = j_1$. Hence, we assume in the sequel that $a \neq i_1$ and $b \neq j_1$ so that $e_a u$ and $e_b v$ are regular paths.

Then we have by (6.2)

$$(e_a u) \times (e_b v) = e_{i_0 \dots i_p} \times e_{j_0 \dots j_q} = \sum_{z \in \Sigma_{\mathbf{i}, \mathbf{j}}} (-1)^{L(z)} e_z \quad (6.4)$$

where $\mathbf{i} = i_0 \dots i_p$ and $\mathbf{j} = j_0 \dots j_q$. Set $z = z_0 \dots z_{(p+q)}$ and note that

$$z_0 = (i_0, j_0) = c$$

while z_1 can be either (i_1, j_0) or (i_0, j_1) . The summation in (6.4) can be split into two parts according to the value of z_1 .

Set $z' = z_1 z_2 \dots z_{(p+q)}$. If $z_1 = (i_1, j_0)$ then z' projects onto $\mathbf{i}' = i_1 \dots i_p$ and $j_0 \dots j_q$; moreover, z' exhausts all such paths, and $L(z') = L(z)$ (cf. Fig. 16).

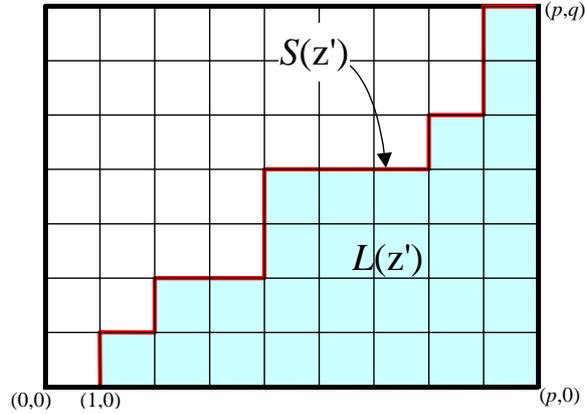


Figure 16: The staircase $S(z')$ starts at $(1, 0)$. In this case $L(z') = L(z)$.

Hence, we see that

$$\sum_{\{z \in \Sigma_{\mathbf{i}, \mathbf{j}}: z_1 = (i_1, j_0)\}} (-1)^{L(z)} e_z = \sum_{z' \in \Sigma_{\mathbf{i}', \mathbf{j}}} (-1)^{L(z')} e_{z_0} e_{z'} = e_c [u \times (e_b v)].$$

If $z_1 = (i_0, j_1)$ then z' projects onto $i_0 \dots i_p$ and $\mathbf{j}' = j_1 \dots j_q$. In this case, $L(z') = L(z) - p$, as one can see comparing Fig. 14 and 17.

It follows that

$$\sum_{\{z \in \Sigma_{\mathbf{i}, \mathbf{j}}: z_1 = (i_0, j_1)\}} (-1)^{L(z)} e_z = \sum_{z' \in \Sigma_{\mathbf{i}, \mathbf{j}'}} (-1)^{L(z') + p} e_{z_0} e_{z'} = (-1)^p e_c [e_a u \times v].$$

Combining the above identities, we obtain (6.3). ■

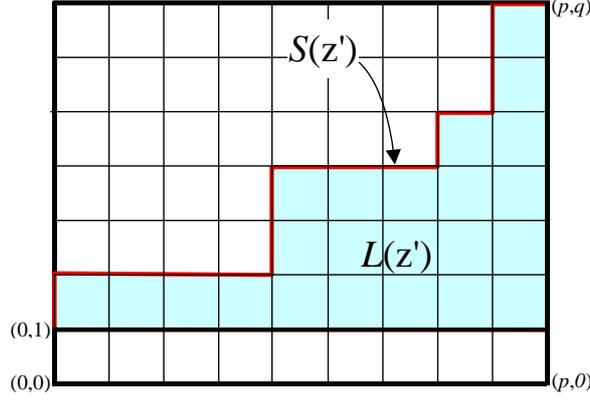


Figure 17: The staircase $S(z')$ starts at $(0,1)$. In this case $L(z') = L(z) - p$.

6.3 The product rule

Proposition 6.4 *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p, q \geq 0$, then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v) \quad (6.5)$$

Proof. It suffices to prove (6.5) for regular elementary paths u and v . The proof will be done by induction in $p+q$. If $p=0$ then $u = e_a$ for some vertex $a \in X$ and

$$\partial(u \times v) = \partial(e_a \times v) = e_a \times \partial v,$$

which implies (6.5) because $\partial u = 0$. In the same way one treats the case $q=0$.

Let now $p, q \geq 1$. Then we can write $u = e_a u'$ and $v = e_b v'$ where $u' \in \mathcal{R}_{p-1}(X)$ and $v' \in \mathcal{R}_{q-1}(Y)$. By Lemma 6.3 we have

$$u \times v = e_c (u' \times v + (-1)^p u \times v')$$

where $c = (a, b)$. Set

$$w = u' \times v + (-1)^p u \times v'$$

so that $w \in \mathcal{R}_{p+q-1}(Z)$. Applying the product rule (2.16) to $\partial(e_c w)$ and the inductive hypothesis to $\partial(u' \times v)$ and $\partial(u \times v')$ we obtain:

$$\begin{aligned} \partial(u \times v) &= w - e_c \partial w \\ &= u' \times v + (-1)^p u \times v' \\ &\quad - e_c \left[\partial u' \times v + (-1)^{p-1} u' \times \partial v + (-1)^p \partial u \times v' + u \times \partial v' \right]. \end{aligned}$$

Next, we have

$$u' \times \partial v = u' \times \partial(e_b v') = u' \times (v' - e_b \tilde{\partial} v') = u' \times v' - u' \times (e_b \tilde{\partial} v')$$

and in the same way

$$\partial u \times v' = u' \times v' - (e_a \tilde{\partial} u') \times v'.$$

We see that the two instances of the terms $u' \times v'$ enter the expression for $\partial(u \times v)$ with the opposite signs $(-1)^{p-1}$ and $(-1)^p$ so that they cancel out, and we obtain

$$\begin{aligned} \partial(u \times v) &= u' \times v + (-1)^p u \times v' \\ &\quad + e_c \left[-\partial u' \times v + (-1)^{p-1} u' \times (e_b \tilde{\partial} v') + (-1)^p (e_a \tilde{\partial} u') \times v' - u \times \partial v' \right]. \end{aligned}$$

On the other hand, we have

$$\partial u \times v = \partial(e_a u') \times v = (u' - e_a \tilde{\partial} u') \times v = u' \times v - (e_a \tilde{\partial} u') \times (e_b v')$$

and

$$u \times \partial v = u \times \partial(e_b v') = u \times v' - (e_a u') \times (e_b \tilde{\partial} v')$$

whence

$$\begin{aligned} &\partial(u \times v) - \partial u \times v - (-1)^p u \times \partial v \\ &= e_c \left[-\partial u' \times v + (-1)^{p-1} u' \times (e_b \tilde{\partial} v') + (-1)^p (e_a \tilde{\partial} u') \times v' - u \times \partial v' \right] \end{aligned} \quad (6.6)$$

$$+ (e_a \tilde{\partial} u') \times (e_b v') + (-1)^p (e_a u') \times (e_b \tilde{\partial} v') \quad (6.7)$$

We need to verify that the right hand side of (6.6)-(6.7) vanishes. Consider first the case when $p, q \geq 2$, so that $\tilde{\partial} u' = \partial u'$ and $\tilde{\partial} v' = \partial v'$. We have by (6.3)

$$(e_a \partial u') \times (e_b v') = e_c \left[\partial u' \times v + (-1)^{p-1} (e_a \partial u') \times v' \right]$$

and

$$(-1)^p (e_a u') \times (e_b \partial v') = e_c \left[(-1)^p u' \times (e_b \partial v') + u \times \partial v' \right].$$

Substituting into (6.6)-(6.7) we see that all the terms in the right hand side cancel out, which finishes the proof of (6.5) in this case.

If $p = 1$ then $u' = e_{a'}$ for some $a' \in X$ and $\partial u' = 0$, $\tilde{\partial} u' = 1$. In this case the right hand side of (6.6)-(6.7) is equal to

$$e_c [e_{a'} \times (e_b \tilde{\partial} v') - e_a \times v' - e_{aa'} \times \partial v'] + e_a \times (e_b v') - e_{aa'} \times (e_b \tilde{\partial} v'). \quad (6.8)$$

Clearly, $e_a \times (e_b v') = e_{(a,b)} (e_a \times v')$ so that this terms cancels out with $e_c [e_a \times v']$, and (6.8) becomes

$$e_c [e_{a'} \times (e_b \tilde{\partial} v') - e_{aa'} \times \partial v'] - e_{aa'} \times (e_b \tilde{\partial} v'). \quad (6.9)$$

To handle the rest terms, consider first the case $q = 1$ that is, $v' = e_{b'}$. Then the expression in (6.9) is equal to

$$e_c [e_{a'} \times e_b] - e_{aa'} \times e_b = e_{(a,b)(a',b)} - e_{(a,b)(a'b)} = 0.$$

If $q \geq 2$ then $\tilde{\partial} v' = \partial v'$, and we obtain by (6.3) that

$$e_{aa'} \times (e_b \partial v') = e_c [e_{a'} \times (e_b \partial v') - e_{aa'} \times \partial v'].$$

Substituting into (6.9) we see that all terms in (6.9) cancel out, so that we obtain again the identity (6.5).

The last case to consider is when $q = 1$ and $p \geq 2$. In this case $v' = e_{b'}$ for some $b' \in Y$ and $\partial v' = 0$, $\tilde{\partial} v' = 1$. The right hand side of (6.6)-(6.7) is equal to

$$e_c \left[-\partial u' \times e_{bb'} + (-1)^{p-1} u' \times e_b + (-1)^p (e_a \partial u') \times e_{b'} \right] + (e_a \partial u') \times e_{bb'} + (-1)^p (e_a u') \times e_b \quad (6.10)$$

We have

$$(-1)^p (e_a u') \times e_b = (-1)^p e_c [u' \times e_b]$$

and by (6.3)

$$(e_a \partial u') \times e_{bb'} = e_c [\partial u' \times e_{bb'} + (-1)^{p-1} (e_a \partial u') \times e_{b'}].$$

Substituting into (6.10) we see that all the terms cancel out, which finishes the proof. ■

6.4 Cartesian product of digraphs

In this section we denote a digraph and its set of vertices by the same letters to simplify the notation. Given two digraphs X and Y , define their Cartesian product digraph

$$Z = X \square Y$$

as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a, b) where $a \in X$ and $b \in Y$;
- the edges in Z are of two types: $(a, b) \rightarrow (a', b)$ where $a \rightarrow a'$ (a *horizontal* edge) and $(a, b) \rightarrow (a, b')$ where $b \rightarrow b'$ (a *vertical* edge):

$$\begin{array}{ccccccc}
 & & & & (a, b') & \longrightarrow & (a', b') & \dots \\
 & & b' \bullet & \dots & \bullet & & \bullet & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & \\
 & & b \bullet & \dots & \bullet & \longrightarrow & \bullet & \dots \\
 & & & & & & & \\
 Y & / & X & \dots & \bullet & \longrightarrow & \bullet & \dots \\
 & & & & a & & a' &
 \end{array}$$

It follows that any allowed elementary path in Z is stair-like. Moreover, if z is a stair-like path on Z then it is allowed if and only if its projections x and y on respectively X and Y are both allowed. In particular, if x and y are elementary allowed paths then all paths $z \in \Sigma_{x,y}$ are allowed. Then it follows from (6.2) that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \Rightarrow u \times v \in \mathcal{A}_{p+q}(Z).$$

Furthermore, the following is true.

Proposition 6.5 *If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then $u \times v \in \Omega_{p+q}(Z)$.*

Proof. Indeed, ∂u and ∂v are allowed, whence also $\partial u \times v$ and $u \times \partial v$ are allowed, whence $\partial(u \times v)$ is allowed by the product rule (6.5). It follows that $u \times v \in \Omega_{p+q}(Z)$.

Let us give another, direct proof of that, without product rule. Using Lemma 5.1, it suffices to prove that, for any semi-allowed path $z = i_1 \dots i_{p+q}$ on Z ,

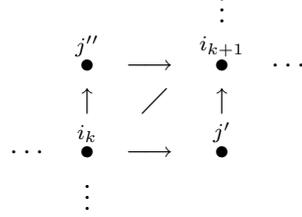
$$[u \times v]^z = 0, \tag{6.11}$$

where the deficiency $[\cdot]$ is defined by (5.5). Recall that the path $i_1 \dots i_{p+q}$ is semi-allowed if among the consecutive pairs $i_k i_{k+1}$ there is exactly one semi-edge and all others are edges. Let $i_k i_{k+1}$ be a semi-edge.

Observe first that semi-edges on Z are of the following three kinds:

1. $(a, b) \rightarrow (a', b)$ where $a \rightarrow a'$ (a horizontal semi-edge)
2. $(a, b) \rightarrow (a, b')$ where $b \rightarrow b'$ (a vertical semi-edge)
3. $(a, b) \rightarrow (a', b')$ where $a \rightarrow a'$ and $b \rightarrow b'$ (a diagonal semi-edge)

Assume first that the semi-edge $i_k i_{k+1}$ is diagonal. In this case we prove that (6.11) holds for all $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{A}_q(Y)$. The diagonal semi-edge $i_k i_{k+1}$ has only two bridges $i_k j' i_{k+1}$ and $i_k j'' i_{k+1}$ as on the diagram:



Consider the allowed paths

$$z' = i_1 \dots i_k j' i_{k+1} \dots i_{p+q} \quad \text{and} \quad z'' = i_1 \dots i_k j'' i_{k+1} \dots i_{p+q}$$

and recall that by (5.5)

$$[u \times v]^z = (u \times v)^{z'} + (u \times v)^{z''}.$$

Observe that the paths z' and z'' have the same projections, say x and y , while for their elevations we have

$$L(z') - L(z'') = \pm 1. \tag{6.12}$$

By (6.1), we have

$$(u \times v)^{z'} = (-1)^{L(z')} u^x v^y \quad \text{and} \quad (u \times v)^{z''} = (-1)^{L(z'')} u^x v^y.$$

Adding up these identities and using (6.12), we obtain (6.11).

Assume now that the semi-edge $i_k i_{k+1}$ is horizontal. In this case we prove that (6.11) holds for all $u \in \Omega_p(X)$ and $v \in \mathcal{A}_q(Y)$. Set $z = i_1 \dots i_{p+q}$ and denote by z' an arbitrary allowed extension of the semi-allowed path z . Note that the bridges $i_k j i_{k+1}$ of the semi-edge $i_k i_{k+1}$ have only horizontal edges, which implies that the value of the elevation $L(z')$ is the same for all possible z' ; denote it simply by L .

Let x be the projection of the path z onto X and y – that onto Y . Then x is semi-allowed and y is allowed. Moreover, the projection x' of z' onto X is an allowed extension of x , and the projection of z' onto Y is again y . Hence, we obtain

$$[u \times v]^z = \sum_{z'} (u \times v)^{z'} = \sum_{z'} (-1)^{L(z')} u^{x'} v^y = (-1)^L v^y \sum_{x'} u^{x'} = (-1)^L v^y [u]^x.$$

Since $u \in \Omega_p(X)$, we have $[u]^x = 0$ whence (6.11) follows.

Finally, the case when the semi-edge $i_k i_{k+1}$ is vertical is treated in the same way. ■

6.5 Cylinders and hypercubes

For any digraph X , the cylinder over X is the digraph

$$\text{Cyl } X := X \times \{\bullet^0 \rightarrow \bullet^1\}.$$

By Theorem 6.11,

$$H_p(\text{Cyl } X) = H_p(X) \quad \text{for all } p \geq 0.$$

It will be convenient to identify the set of vertices of $\text{Cyl } X$ with $X \sqcup X'$ where X' is a copy of X , and use the following notation for the vertices of $\text{Cyl } X$: $(x, 0) \equiv x$ and $(x, 1) \equiv x'$.

Define the operation of *lifting* paths from X to $\text{Cyl } X$ as follows: for any regular path v on X , the lifted path is denoted by \widehat{v} and is defined by

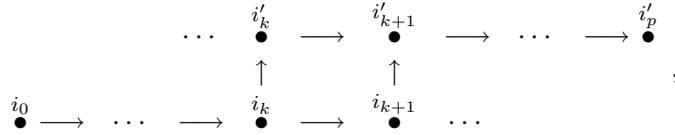
$$\widehat{v} = v \times e_{01}.$$

By the results of Section 6.5, if $v \in \mathcal{A}_p(X)$ then $\widehat{v} \in \mathcal{A}_{p+1}(\text{Cyl } X)$, and if $v \in \Omega_p(X)$ then $\widehat{v} \in \Omega_{p+1}(\text{Cyl } X)$.

For example, if $v = e_{i_0 \dots i_p}$ then

$$\widehat{v} = e_{i_0 \dots i_p} \times e_{01} = \sum_{k=0}^p (-1)^{p-k} e_{i_0 \dots i_k i'_k \dots i'_p}. \quad (6.13)$$

where the path $i_0 \dots i_k i'_k \dots i'_p$ has the elevation $p - k$ as can be seen on the diagram:



Example 6.6 The cylinder over a triangle $X = \begin{array}{ccc} & \bullet^1 & \\ & \nearrow & \searrow \\ \bullet^0 & & \bullet^2 \end{array}$ is shown on Fig. 18, where 0, 1, 2 are the vertices on X , and 3 = 0', 4 = 1', 5 = 2' are the vertices on X' .

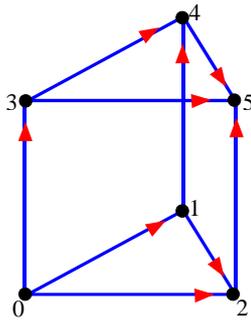


Figure 18: A cylinder over a triangle

Since 2-path e_{012} is ∂ -invariant on the triangle, lifting it to the cylinder, we obtain a ∂ -invariant 3-path $e_{00'1'2'} - e_{011'2'} + e_{0122'}$, that can be written in the form

$$e_{0345} - e_{0145} + e_{0125}.$$

Example 6.7 The cylinder over the graph $X = \{^0\bullet \rightarrow \bullet^1\}$ is a square

$$\begin{array}{ccc} 2\bullet & \longrightarrow & \bullet^3 \\ \uparrow & & \uparrow \\ 0\bullet & \longrightarrow & \bullet^1 \end{array} = \text{Cyl } \{^0\bullet \rightarrow \bullet^1\}$$

Lifting a ∂ -invariant 1-path e_{01} on X we obtain a ∂ -invariant 2-path on the square: $-e_{00'1'} + e_{011'}$, that can be rewritten in the form $e_{013} - e_{023}$.

The cylinder over a square is a 3-cube that is shown in Fig. 19.

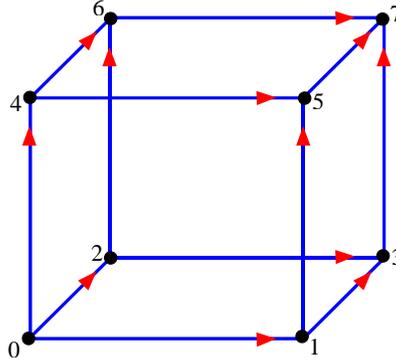


Figure 19: A 3-cube

Lifting the 2-path $e_{013} - e_{023}$ we obtain a ∂ -invariant 3-path

$$e_{00'1'3'} - e_{011'3'} + e_{0133'} - e_{00'2'3'} + e_{022'3'} - e_{0233'},$$

that we can rewrite in the form

$$e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}.$$

Defining further n -cube for any positive integer n by

$$\text{Cube}_n = \text{Cyl Cube}_{n-1},$$

we see that Cube_n determines a ∂ -invariant n -path that is a lifting of a ∂ -invariant $(n-1)$ -path from Cube_{n-1} and that is an alternating sum of $n!$ elementary terms. It is a curious observation that these terms correspond to partitioning of a solid n -cube into simplexes.

6.6 Bi-cylinders

The bi-cylinder over a digraph X is the digraph

$$\text{biCyl } X = X \times \{^0\bullet \rightleftharpoons \bullet^1\}.$$

By Theorem 6.11,

$$H_p(\text{biCyl } X) = H_p(X) \quad \text{for all } p \geq 0.$$

The set of vertices of $\text{biCyl } X$ can be identified with $X \sqcup X'$ where X' is a copy of X . As for cylinders, we use the following notation for vertices of $\text{biCyl } X$: $(x, 0) \equiv x$ and $(x, 1) = x'$.

For any regular path v on X , the lifted path on $\text{biCyl } X$ is given by

$$\widehat{v} = v \times (e_{01} + e_{10})$$

that is ∂ -invariant, provided v is ∂ -invariant. For example, if $v = e_{i_0 \dots i_p}$ then

$$\begin{aligned} \widehat{v} &= e_{i_0 \dots i_p} \times (e_{01} + e_{10}) \\ &= \sum_{k=0}^p (-1)^{p-k} e_{i_0 \dots i_k i'_k \dots i'_p} + \sum_{k=0}^p (-1)^{p-k} e_{i'_0 \dots i'_k i_k \dots i_p}. \end{aligned}$$

Since $\partial(e_{01} + e_{10}) = 0$, we obtain by the product rule that \widehat{v} is closed if and only if v is closed.

For example, consider the graph $X = \{0 \bullet \leftrightarrow \bullet 1\}$ whose bi-cylinder is shown on Fig. 20. Fig 21 the same graph is shown embedded on the torus \mathbb{T}^2 .

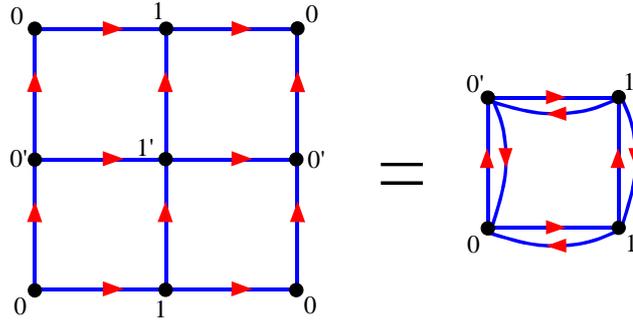


Figure 20: $\text{biCyl } \{0 \bullet \leftrightarrow \bullet 1\}$

For the closed path $v = e_{01} + e_{10}$ on X , the lifted path \widehat{v} is closed and is given by

$$\widehat{v} = -e_{00'1'} + e_{011'} - e_{0'01} + e_{0'1'1} - e_{11'0'} + e_{100'} - e_{1'00} + e_{1'0'0}.$$

Adding diagonals to the graph on Fig. 20, we obtain a triangulation T of the torus \mathbb{T}^2 as on Fig. 22. The path \widehat{v} is then the surface path of the triangulation T .

Consider another a triangle digraph

$$X = \begin{array}{c} \bullet 1 \\ \swarrow \quad \searrow \\ 0 \bullet \quad \bullet 2 \\ \rightarrow \end{array},$$

whose bi-cylinder is shown on Fig. 23.

The paths $v = e_{012}$ is ∂ -invariant on X , so that the lifted path

$$\widehat{v} = e_{00'1'2'} - e_{011'2'} + e_{0122'} + e_{0'012} - e_{0'1'12} + e_{0'1'2'2}$$

is ∂ -invariant on $\text{biCyl } X$. Using $\partial \widehat{v} = \widehat{\partial v}$ that is a consequence of the product rule, and

$$\partial v = e_{01} - e_{02} + e_{12},$$

we obtain

$$\begin{aligned} \partial \widehat{v} &= -e_{00'1'} + e_{011'} - e_{0'01} + e_{0'1'1} \\ &\quad + e_{00'2'} - e_{022'} + e_{0'02} - e_{0'2'2} \\ &\quad - e_{11'2'} + e_{122'} - e_{1'12} + e_{1'2'2}. \end{aligned}$$

Adding diagonals to the digraph on Fig. 23, we obtain a triangulation T of the torus \mathbb{T}^2 as on Fig. 24. The path $\partial \widehat{v}$ coincides with the surface path of this triangulation. Hence, \widehat{v} can be regarded as a solid path of the above torus.

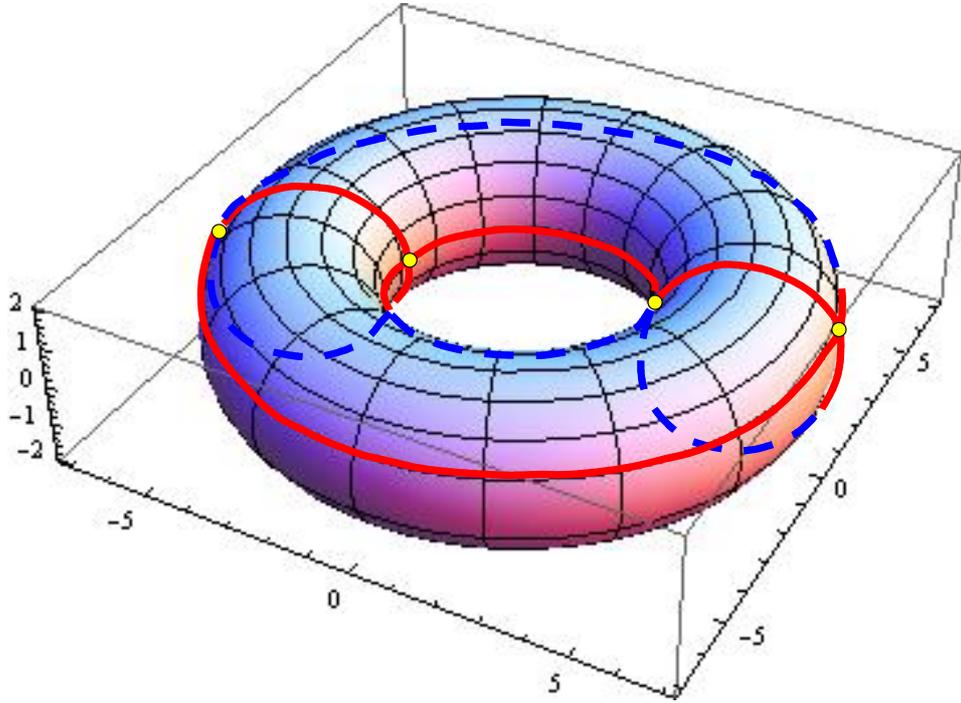


Figure 21: Graph $\text{biCyl } X$ is embedded on \mathbb{T}^2

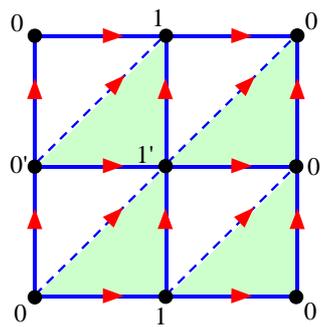


Figure 22: A triangulation T of torus \mathbb{T}^2 into 8 triangles. The shaded triangles are those with positive orientation

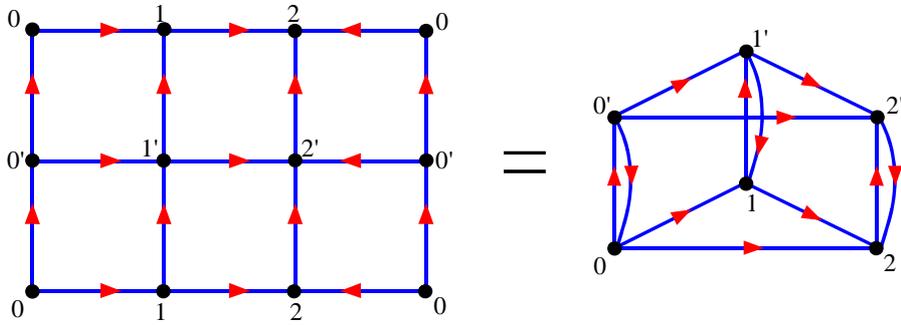


Figure 23: $\text{biCyl} \left\{ \begin{array}{c} \bullet \\ \nearrow \quad \searrow \\ 0 \bullet \quad \bullet 2 \\ \rightarrow \end{array} \right\}$

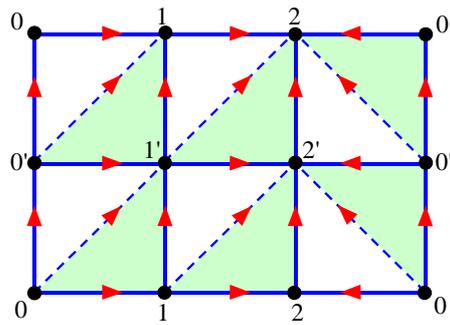


Figure 24: A triangulation T of torus \mathbb{T}^2 into 12 triangles. The shaded triangles are those with positive orientation.

6.7 ∂ -invariant paths on Cartesian product

We start here some preparations for Theorem 6.11 below. The following statement is a partial converse of Proposition 6.5.

Proposition 6.8 *Any path $w \in \Omega_\bullet(Z)$ admits a representation*

$$w = \sum_{x \in E_\bullet(X), y \in E_\bullet(Y)} c^{xy} (e_x \times e_y) \quad (6.14)$$

with some scalar coefficients c^{xy} (only finitely many coefficients are non-vanishing).

Proof. It suffices to prove (6.14) for $w \in \Omega_r$ with a fixed $r \geq 0$. If $x \in E_p(X)$ and $y \in E_q(Y)$ then the coefficient c^{xy} may be non-zero only if $p+q = r$. For a given $x \in E_p(X)$ and $y \in E_q(Y)$ with $p+q = r$, set

$$c^{xy} = (-1)^{L(z)} w^z \quad (6.15)$$

for any $z \in \Sigma_{x,y}$. Let us show that the value of c^{xy} is independent of the choice of z . Let $z = i_0 \dots i_{p+q}$. Let k be an index such that the edge $i_{k-1}i_k$ is horizontal, while $i_k i_{k+1}$ is vertical. Denote by i'_k the vertex on Z such that the edge $i_{k-1}i'_k$ is vertical and $i'_k i_{k+1}$ is horizontal:

$$\begin{array}{ccccc} & & & \vdots & \\ & & & i_{k+1} & \dots \\ & & & \bullet & \\ & & & \uparrow & \\ & & & i_k & \dots \\ & & & \bullet & \\ & & & \vdots & \\ \dots & & i_{k-1} & \longrightarrow & i_k & \dots \\ & & \bullet & & \bullet & \\ & & \uparrow & & \uparrow & \\ & & i'_k & \longrightarrow & i_{k+1} & \dots \\ & & \bullet & & \bullet & \\ & & \vdots & & \vdots & \end{array}$$

and set $z' = i_0 \dots i_{k-1} i'_k i_{k+1} \dots i_p$. Clearly, the projections of z' on X and Y are again x and y , respectively, and $z' \in \Sigma_{x,y}$. Since $i_0 \dots i_{k-1} i_{k+1} \dots i_p$ is a semi-allowed path, we obtain by Lemma 5.1 that $[w]^{i_0 \dots i_{k-1} i_{k+1} \dots i_p} = 0$ which amounts to

$$w^{z'} + w^z = 0.$$

Since $L(z') = L(z) + 1$, it follows that

$$(-1)^{L(z')} w^{z'} = (-1)^{L(z)} w^z. \quad (6.16)$$

In the same way the identity (6.16) holds if $i_{k-1}i_k$ is vertical, while $i_k i_{k+1}$ is horizontal. The transformation $z \mapsto z'$ described above, allows to obtain from a given $z \in \Sigma_{x,y}$ any other path in $\Sigma_{x,y}$ in a finite number of steps. Since the quantity $(-1)^{L(z)} w^z$ does not change under this transformation, it follows that it does not depend on a particular choice of $z \in \Sigma_{x,y}$, which was claimed. Hence, the coefficients c^{xy} in (6.15) are well-defined.

Let us show that the identity (6.14) holds with these coefficients. By (6.2) we have

$$e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z.$$

Using (6.15) we obtain

$$\begin{aligned}
\sum_{x \in E_{\bullet}(X), y \in E_{\bullet}(Y)} c^{xy} (e_x \times e_y) &= \sum_{x \in E_{\bullet}(X), y \in E_{\bullet}(Y)} c^{xy} \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z \\
&= \sum_{x \in E_{\bullet}(X), y \in E_{\bullet}(Y)} \sum_{z \in \Sigma_{x,y}} w^z e_z \\
&= \sum_{z \in E_{\bullet}(Z)} w^z e_z = w.
\end{aligned}$$

■

6.8 Special paths on the product

Let X, Y be finite sets. Denote by $S(X, Y)$ the space of all paths $w \in \mathcal{R}_{\bullet}(Z)$ that can be represented as a finite sum of the form

$$w = \sum_{x \in R_{\bullet}(X), y \in R_{\bullet}(Y)} c^{xy} e_x \times e_y, \quad (6.17)$$

where $c^{xy} \in \mathbb{K}$. Let us call such paths *special*.

Note that the coefficients c^{xy} in (6.17) are uniquely defined. Indeed, by (6.2), $e_x \times e_y$ is an alternating sum of the terms e_z where $z \in \Sigma_{x,y}$, which implies the representation

$$w = \sum_z w^z e_z, \quad (6.18)$$

where the summation runs over all stair-like paths z on Z . Clearly, the expansion (6.18) is unique. Each term e_z in (6.18) comes from $e_x \times e_y$ where x and y are projections on z onto X and Y , respectively, which implies $c^{xy} = w^z$. Hence, c^{xy} is uniquely determined.

The introduction of the space $S(X, Y)$ is obviously motivated by Proposition 6.8, although here we do not assume a digraph structure.

Given any vertex $a \in X$, define the following operator A on $S(X, Y)$:

$$Aw = \sum_{x \in R_{\bullet}(X), y \in R_{\bullet}(Y)} c^{xy} e_{ax} \times e_y.$$

Lemma 6.9 *For any $w \in S(X, Y)$ also $\partial w \in S(X, Y)$, and the following identity is true:*

$$w = e_a \times \left(\sum_{x \in E_0(X), y \in R_{\bullet}(Y)} c^{xy} e_y \right) + \partial(Aw) + A(\partial w). \quad (6.19)$$

Proof. For all $x, x' \in R_{\bullet}(X)$ define the coefficients $\varepsilon_x^{x'}$ by

$$\partial e_x = \sum_{x' \in R_{\bullet}(X)} \varepsilon_x^{x'} e_{x'}$$

and in the same way define $\varepsilon_y^{y'}$ for regular paths y, y' on Y . By (2.8) $\varepsilon_x^{x'}$ takes the values 0 and ± 1 .

Denote by $|x|$ the length of the path $x \in R_\bullet(X)$. In the next lines, x, x' vary in $R_\bullet(X)$ and y, y' vary in $R_\bullet(Y)$, unless otherwise specified. Using the product rule, we obtain from (6.17)

$$\begin{aligned}
\partial w &= \sum_{x,y} \left[c^{xy} (\partial e_x \times e_y) + c^{xy} (-1)^{|x|} e_x \times \partial e_y \right] \\
&= \sum_{x,y} \left[c^{xy} \left(\sum_{x'} \varepsilon_x^{x'} e_{x'} \times e_y \right) + c^{xy} (-1)^{|x|} e_x \times \sum_{y'} \varepsilon_y^{y'} e_{y'} \right] \\
&= \sum_{x,y,x'} c^{xy} \varepsilon_x^{x'} e_{x'} \times e_y + \sum_{x,y,y'} c^{xy} \varepsilon_y^{y'} (-1)^{|x|} e_x \times e_{y'} \\
&= \sum_{x,y,x'} c^{x'y} \varepsilon_x^{x'} e_{x'} \times e_y + \sum_{x,y,y'} c^{xy'} \varepsilon_y^{y'} (-1)^{|x|} e_x \times e_{y'} \\
&= \sum_{x,y} \left[\sum_{x'} c^{x'y} \varepsilon_x^{x'} + \sum_{y'} c^{xy'} \varepsilon_y^{y'} (-1)^{|x|} \right] e_x \times e_y
\end{aligned}$$

that is

$$\partial w = \sum_{x,y} b^{xy} e_x \times e_y,$$

where

$$b^{xy} = \sum_{x'} c^{x'y} \varepsilon_x^{x'} + \sum_{y'} (-1)^{|x|} c^{xy'} \varepsilon_y^{y'}. \quad (6.20)$$

Hence, $\partial w \in S(X, Y)$.

Let us prove (6.19). If $|x| \geq 1$ then

$$\partial e_{ax} = e_x - e_a \partial e_x$$

and

$$\begin{aligned}
\partial(e_{ax} \times e_y) &= \partial e_{ax} \times e_y + (-1)^{|x|+1} e_{ax} \times \partial e_y \\
&= e_x \times e_y - (e_a \partial e_x) \times e_y + (-1)^{|x|+1} e_{ax} \times \partial e_y \\
&= e_x \times e_y - \sum_{x'} \varepsilon_x^{x'} e_{ax'} \times e_y - (-1)^{|x|} \sum_{y'} \varepsilon_y^{y'} e_{ax} \times e_{y'}.
\end{aligned}$$

If $|x| = 0$ then we use $\partial e_{ax} = e_x - e_a$ and obtain instead

$$\begin{aligned}
\partial(e_{ax} \times e_y) &= \partial e_{ax} \times e_y + (-1)^{|x|+1} e_{ax} \times \partial e_y \\
&= e_x \times e_y - e_a \times e_y + (-1)^{|x|+1} e_{ax} \times \partial e_y \\
&= e_x \times e_y - e_a \times e_y - \sum_{x'} \varepsilon_x^{x'} e_{ax'} \times e_y - (-1)^{|x|} \sum_{y'} \varepsilon_y^{y'} e_{ax} \times e_{y'},
\end{aligned}$$

where we have used that in this case $\partial e_x = 0$ and, hence, $\varepsilon_x^{x'} = 0$. Separating the term $e_a \times e_y$

that appears only for $|x| = 0$, we obtain

$$\begin{aligned}
\partial(Aw) &= \sum_{x,y} c^{xy} \partial(e_{ax} \times e_y) \\
&= \sum_{x,y} c^{xy} e_x \times e_y - e_a \times \sum_{x \in E_0(X)} \sum_y c^{xy} e_y \\
&\quad - \sum_{x,y,x'} c^{xy} \varepsilon_x^{x'} e_{ax'} \times e_y - \sum_{x,y,y'} (-1)^{|x|} c^{xy} \varepsilon_y^{y'} e_{ax} \times e_{y'} \\
&= w - e_a \times \sum_{x \in E_0(X)} \sum_y c^{xy} e_y \\
&\quad - \sum_{x,y,x'} c^{x'y} \varepsilon_x^x e_{ax} \times e_y - \sum_{x,y,y'} (-1)^{|x|} c^{xy'} \varepsilon_{y'}^y e_{ax} \times e_y \\
&= w - e_a \times \sum_{x \in E_0(X)} \sum_y c^{xy} e_y - \sum_{x,y} b^{xy} e_{ax} \times e_y \\
&= w - e_a \times \sum_{x \in E_0(X)} \sum_y c^{xy} e_y - A(\partial w),
\end{aligned}$$

which proves (6.19). ■

6.9 Homologies of Cartesian product

Let X, Y be digraphs. Consider again the product path $u \times v$ with $u \in \mathcal{A}_p(X)$ and $v \in \mathcal{A}_q(Y)$. Since by (6.1) the product $u \times v$ is clearly linear in each argument and satisfies the identity

$$(cu) \times v = u \times (cv) = c(u \times v)$$

for any scalar $c \in \mathbb{K}$, the cross product extends to a linear mapping \mathcal{X} from the tensor product space $\mathcal{A}_p(X) \otimes \mathcal{A}_q(Y)$ to $\mathcal{A}_{p+q}(Z)$. Namely, the elements of the space $\mathcal{A}_p(X) \otimes \mathcal{A}_q(Y)$ have the form

$$\sum_{x \in E_p(X), y \in E_q(Y)} a^{xy} (e_x \otimes e_y)$$

with arbitrary scalar coefficients a^{xy} , where $\{e_x \otimes e_y\}_{x \in E_p(X), y \in E_q(Y)}$ is its basis, and the aforementioned mapping is given by

$$\mathcal{X}(e_x \otimes e_y) = e_x \times e_y.$$

Furthermore, by linearity one extends \mathcal{X} to a mapping

$$\mathcal{X} : \bigoplus_{p+q=r} \mathcal{A}_p(X) \otimes \mathcal{A}_q(Y) \rightarrow \mathcal{A}_r(Z)$$

where \bigoplus is the direct sum of the vector spaces over all couples of non-negative integers p, q such that $p + q = r$.

Lemma 6.10 *The mapping \mathcal{X} is well-defined on the homology spaces as below:*

$$\mathcal{X} : \bigoplus_{p+q=r} H_p(X) \otimes H_q(Y) \rightarrow H_r(Z). \quad (6.21)$$

Proof. Let us first show that if u and v are closed paths on X and Y , respectively, then $\mathcal{X}(u \otimes v) = u \times v$ is a closed path on Z . That u is a closed path is equivalent to $u \in \mathcal{A}_p(X)$ for some p and $\partial u = 0$. Let $v \in \mathcal{A}_q(Y)$ and $\partial v = 0$. Then by definition $u \times v \in \mathcal{A}_{p+q}(Z)$ and by Proposition 6.4

$$\partial(u \times v) = 0$$

whence it follows $u \times v \in \Omega_{p+q}(Z)$ and $u \times v$ is closed.

Let us show that if u, u' are homological closed paths on X and v, v' are homological closed paths on Y , then $u \times v$ and $u' \times v'$ are homological on Z . Indeed, we have

$$u' = u + \partial\varphi \quad \text{and} \quad v' = v + \partial\psi$$

for some $\varphi \in \Omega_{p+1}(X)$ and $\psi \in \Omega_{q+1}(Y)$, whence

$$u' \times v' = u \times v + u \times \partial\psi + \partial\varphi \times v' + \partial\varphi \times \partial\psi.$$

Observe that the path $u \times \partial\psi$ is exact as

$$\partial(u \times \psi) = \partial u \times \psi + (-1)^p u \times \partial\psi = (-1)^p u \times \partial\psi,$$

and in the same way $\partial\varphi \times v'$ and $\partial\varphi \times \partial\psi$ are exact. It follows that $u' \times v'$ and $u \times v$ represent the same homology class, which was to be proved. ■

The identity (6.21) raises the question if the mapping \mathcal{X} is bijective. If it is then (6.21) implies the Künneth formula

$$H_r(Z) \cong \bigoplus_{p+q=r} H_p(X) \otimes H_q(Y)$$

and, hence,

$$\dim H_r(Z) = \sum_{p+q=r} \dim H_p(X) \dim H_q(Y).$$

The validity of this formula will be the subject of a separate paper. Here we treat a particular case that will allow us to consider some interesting examples.

Theorem 6.11 *Assume that X is a star-like digraph. Then, for any digraph Y and for all $p \geq 0$,*

$$H_p(X \times Y) \cong H_p(Y). \quad (6.22)$$

Proof. Let $a \in X$ be a star center. Set as before $Z = X \times Y$ and consider the linear mapping

$$\Omega_p(Y) \ni u \mapsto e_a \times u \in \Omega_p(Z) \quad (6.23)$$

where $e_a \times u \in \Omega_p(Z)$ holds by Proposition 6.5.

If the path u is closed then by the product rule so is $e_a \times u$ (as e_a is closed). If $u \sim u'$ then $u - u' = \partial\varphi$ for some $\varphi \in \Omega_{p+1}(Y)$, whence

$$e_a \times u - e_a \times u' = e_a \times \partial\varphi = -\partial(e_a \times \varphi)$$

and $e_a \times u \sim e_a \times u'$. Hence, the mapping (6.23) extends to the mapping

$$H_p(Y) \ni u \mapsto e_a \times u \in H_p(Z). \quad (6.24)$$

We will prove that the mapping (6.24) is bijective, which will imply (6.22).

Let us prove that the mapping (6.24) is injective, that is, if $e_a \times u \sim 0$ for $u \in \Omega_\bullet(Y)$ then $u \sim 0$. Indeed, let

$$e_a \times u = \partial w \quad (6.25)$$

for some $w \in \Omega_\bullet(Z)$. By Proposition 6.8 w can be represented in the form (6.14). It follows from (6.25) that $A(\partial w) = 0$, and we obtain from (6.19)

$$w = e_a \times \varphi + \partial(Aw)$$

where

$$\varphi = \sum_{x \in E_0(X), y \in E_\bullet(Y)} c^{xy} e_y \in \mathcal{A}_\bullet(Y).$$

It follows that

$$e_a \times u = \partial w = \partial(e_a \times \varphi) = -e_a \times \partial\varphi,$$

which implies $u = -\partial\varphi$. Hence, $\partial\varphi \in \Omega_\bullet(Y)$, whence $\varphi \in \Omega_\bullet(Y)$ and, hence, $u \sim 0$.

Let us prove that the mapping (6.24) is surjective, that is, for any closed path $w \in \Omega_p(Z)$ there exists a closed path $u \in \Omega_p(Y)$ such that $w \sim e_a \times u$, that is,

$$w = e_a \times u + \partial\psi \quad (6.26)$$

for some $\psi \in \Omega_{p+1}(Z)$. In fact, it suffices to have (6.26) with some $u \in \mathcal{A}_p(Y)$ with $\partial u = 0$ and $\psi \in \mathcal{A}_{p+1}(Z)$. Indeed, $\partial u = 0$ implies then that $u \in \Omega_p(Y)$, whence also $e_a \times u \in \Omega_p(Z)$. Then by (6.26) $\partial\psi \in \Omega_p(Z)$ and, hence, $\psi \in \Omega_{p+1}(Z)$.

By Proposition 6.8 any path $w \in \Omega_p(Z)$ admits the representation

$$w = \sum_{x \in E_\bullet(X), y \in E_\bullet(Y)} c^{xy} e_x \times e_y. \quad (6.27)$$

We extend the summation in (6.27) to all $x \in R_\bullet(X)$ and $y \in R_\bullet(Y)$ by setting $c^{xy} = 0$ if either $x \notin E_\bullet(X)$ or $y \notin E_\bullet(Y)$. The identity (6.19) and $\partial w = 0$ imply then (6.26) with

$$u = \sum_{x \in E_0(X), y \in R_\bullet(Y)} c^{xy} e_y = \sum_{x \in E_0(X), y \in E_\bullet(Y)} c^{xy} e_y$$

and

$$\psi = Aw = \sum_{x \in R_\bullet(X), y \in R_\bullet(Y)} c^{x,y} e_{ax} \times e_y = \sum_{x \in E_\bullet(X), y \in E_\bullet(Y)} c^{x,y} e_{ax} \times e_y.$$

By construction we have $u \in \mathcal{A}_p(X)$. Since a is a star center, the path e_{ax} is also allowed, whence $\psi \in \mathcal{A}_{p+1}(Z)$.

We are left to verify that $\partial u = 0$. We have

$$\begin{aligned} \partial u &= \sum_{x \in E_0(X), y, y' \in R_\bullet(Y)} c^{xy} \varepsilon_y^{y'} e_{y'} = \sum_{x \in E_0(X), y, y' \in R_\bullet(Y)} c^{xy'} \varepsilon_{y'}^y e_y \\ &= \sum_{y \in R_\bullet(Y)} \left(\sum_{x \in E_0(X), y' \in R_\bullet(Y)} c^{xy'} \varepsilon_{y'}^y \right) e_y. \end{aligned} \quad (6.28)$$

By (6.20) and $\partial w = 0$ we have, for all $x \in R_\bullet(X)$ and $y \in R_\bullet(Y)$, that

$$\sum_{y' \in R_\bullet(Y)} c^{xy'} \varepsilon_{y'}^y = -(-1)^{|x|} \sum_{x' \in R_\bullet(X)} c^{x'y} \varepsilon_{x'}^x.$$

Adding up these identities for all $x \in E_0(X)$ and using $|x| = 0$, we obtain

$$\sum_{x \in E_0(X), y' \in R_\bullet(Y)} c^{xy'} \varepsilon_{y'}^y = - \sum_{x' \in R_\bullet(X)} c^{x'y} \left(\sum_{x \in E_0(X)} \varepsilon_{x'}^x \right). \quad (6.29)$$

Let us show that, for all $x' \in R_\bullet(X)$,

$$\sum_{x \in E_0(X)} \varepsilon_{x'}^x = 0. \quad (6.30)$$

If $|x'| \neq 1$ then all the terms $\varepsilon_{x'}^x$ vanish. If $|x'| = 1$, that is, x' is an edge ab , then

$$\partial e_{x'} = e_b - e_a$$

so that $\varepsilon_{x'}^b = 1$, $\varepsilon_{x'}^a = -1$ and $\varepsilon_{x'}^x = 0$ for $x \neq a, b$, whence (6.30) follows. Substituting into (6.29) and (6.28), we obtain $\partial u = 0$, which finishes the proof. ■

7 Homologies of subgraphs

7.1 Chain complex of a subgraph

Let (V', E') be a subgraph of (V, E) in the sense that $V' \subset V$ and $E' \subset E$. Let us mark by the dash "'/" all the notation related to the graph (V', E') rather than to (V, E) .

As it was already observed (cf. (2.17)), $\mathcal{R}'_p \subset \mathcal{R}_p$ and ∂ commutes with this inclusion. It is also obvious that if $e_{i_0 \dots i_p}$ is an allowed path in (V', E') then it is also allowed in (V, E) , whence $\mathcal{A}'_p \subset \mathcal{A}_p$. Moreover, this argument shows that

$$\mathcal{A}'_p = \mathcal{A}_p \cap \mathcal{R}'_p. \quad (7.1)$$

By the definition (5.1) of Ω_p , we obtain that $\Omega'_p \subset \Omega_p$ and ∂ commutes with this inclusion. Consequently, the chain complex

$$0 \leftarrow \Omega'_0 \xleftarrow{\partial} \Omega'_1 \xleftarrow{\partial} \Omega'_2 \xleftarrow{\partial} \Omega'_3 \xleftarrow{\partial} \dots$$

is a sub-complex of

$$0 \leftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \Omega_2 \xleftarrow{\partial} \Omega_3 \xleftarrow{\partial} \dots$$

By Proposition 3.7 (cf. (3.20)) we obtain that the following long sequence is exact:

$$0 \leftarrow H_0(\Omega/\Omega') \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow \dots \leftarrow H_p(\Omega/\Omega') \leftarrow H_p(\Omega) \leftarrow H_p(\Omega') \leftarrow H_{p+1}(\Omega/\Omega') \leftarrow \dots \quad (7.2)$$

It is also worth mentioning that

$$\Omega'_p = \Omega_p \cap \mathcal{A}'_p = \Omega_p \cap \mathcal{R}'_p. \quad (7.3)$$

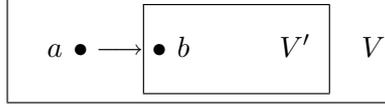
Indeed, the inclusions $\Omega'_p \subset \Omega_p \cap \mathcal{A}'_p \subset \Omega_p \cap \mathcal{R}'_p$ are obvious. To prove the opposite inclusions, observe that $v \in \Omega_p \cap \mathcal{R}'_p$ implies by (5.1) and (7.1)

$$v \in \mathcal{A}_p \cap \mathcal{R}'_p = \mathcal{A}'_p \quad \text{and} \quad \partial v \in \mathcal{A}_{p-1} \cap \mathcal{R}'_{p-1} = \mathcal{A}'_{p-1},$$

whence $v \in \Omega'_p$.

7.2 Removing a vertex of degree 1

Theorem 7.1 *Suppose that a graph (V, E) has a vertex a such that there is only one outgoing edge ab from a and no incoming edges to a . Let $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab\}$.*



Then $H_p(V, E) \cong H_p(V', E')$ for all $p \geq 0$.

Remark 7.2 The same is true if the edge ab in the statement is replaced by ba .

Proof. Let us first prove that¹

$$\Omega_p = \Omega'_p \text{ for all } p \geq 2, \quad (7.4)$$

which will imply that, for all $p \geq 2$,

$$\dim H_p(\Omega') = \dim H_p(\Omega). \quad (7.5)$$

In the view of (7.3), to prove (7.4) it suffices to show that, for all $p \geq 2$,

$$\Omega_p \subset \mathcal{A}'_p, \quad (7.6)$$

that is

$$v \in \mathcal{A}_p \text{ and } \partial v \in \mathcal{A}_{p-1} \Rightarrow v \in \mathcal{A}'_p.$$

Every elementary allowed p -path on (V, E) either is allowed on (V', E') or starts with ab , which implies that v can be represented in the form

$$v = e_{ab}u + v',$$

where $v' \in \mathcal{A}'_p$, while $u \in \mathcal{A}'_{p-2}$ is a linear combination of the paths $e_{i_0 \dots i_{p-2}} \in \mathcal{A}'_{p-2}$ with $i_0 \neq b$. It follows that

$$\partial v = (e_b - e_a)u + e_{ab}\partial u + \partial v'. \quad (7.7)$$

Note that $e_a u$ is a linear combination of the elementary paths $e_{ai_0 \dots i_{p-2}}$ where $i_0, \dots, i_{p-2} \in V'$ and $i_0 \neq b$. Since ai_0 is not an edge, those elementary paths are not allowed in (V, E) . No other terms in the right hand side of (7.7) has $e_{ai_0 \dots i_{p-2}}$ -component. Since ∂v is allowed in (V, E) , its $e_{ai_0 \dots i_{p-2}}$ -component is 0, which is only possible if $e_a u = 0$, that is, $u = 0$. It follows that $v = v' \in \mathcal{A}'_p$, which finishes the proof of (7.6).

Hence, we have the identity (7.5) for $p \geq 2$. For $p = 0$ this identity also true as the number of connected components of (V, E) and (V', E') is the same.

We are left to treat the case $p = 1$. Observe that

$$\Omega_0 = \Omega'_0 + \text{span}\{e_a\} \quad \text{and} \quad \Omega_1 = \Omega'_1 + \text{span}\{e_{ab}\}. \quad (7.8)$$

By (7.4) and (7.8) the cochain complex Ω/Ω' has the form

$$0 \longleftarrow \text{span}\{e_a\} \xleftarrow{\partial} \text{span}\{e_{ab}\} \longleftarrow 0 = \Omega_2/\Omega'_2.$$

¹Theorem 7.1 is a particular case of a more general Theorem 7.4 from the next section. We give here an independent proof based on the identity (7.4) that may be interesting on its own right.

Since

$$\partial e_{ab} = e_b - e_a = -e_a \pmod{\Omega'_0},$$

it follows that $\text{Im } \partial|_{\Omega_1/\Omega'_1} = \text{span } \{e_a\}$, while $\ker \partial|_{\Omega_1/\Omega'_1} = 0$, whence

$$\dim H_0(\Omega/\Omega') = \dim H_1(\Omega/\Omega') = 0.$$

By (7.2) we have a long exact sequence

$$H_0(\Omega/\Omega') = 0 \longleftarrow H_1(\Omega) \longleftarrow H_1(\Omega') \longleftarrow 0 = H_1(\Omega/\Omega')$$

which implies that

$$\dim H_1(\Omega) = \dim H_1(\Omega'),$$

thus finishing the proof. ■

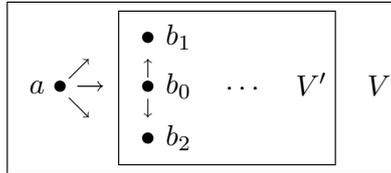
Corollary 7.3 *Let a digraph (V, E) be a tree (that is, the underlying undirected graph is a tree). Then $H_p(V, E) = 0$ for all $p \geq 1$.*

Proof. Induction in the number of edges $|E|$. If $|E| = 0$ then the claim is obvious. If $|E| > 0$ then there is a vertex $a \in V$ of degree 1 (indeed, if this is not the case then moving along undirected edges allows to produce a cycle). Removing this vertex and the adjacent edge, we obtain a tree (V', E') with $|E'| < |E|$. By the inductive hypothesis $H_p(V', E') = 0$ for $p \geq 1$, whence by Theorem 7.1 also $H_p(V, E) = 0$.

Note that $H_p(V, E) = 0$ for $p \geq 2$ follows also from Theorem 5.3. ■

7.3 Removing a vertex of degree n

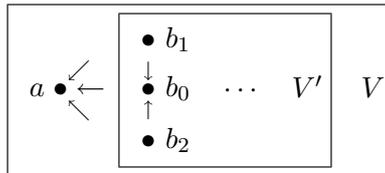
Theorem 7.4 *Suppose that a digraph (V, E) has a vertex a with n outgoing edges $ab_0, ab_1, \dots, ab_{n-1}$ and no incoming edges. Assume also that b_0b_i is an edge for all $i \geq 1$:*



Denote by (V', E') the digraph that is obtained from (V, E) by removing a vertex a with all adjacent edges, that is, $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab_i\}_{i=0}^{n-1}$. Then, for any $p \geq 0$,

$$\dim H_p(V, E) = \dim H_p(V', E'). \quad (7.9)$$

The same is true if a vertex a has n incoming edges $b_0a, b_1a, \dots, b_{n-1}a$ and no outgoing edges, while b_ib_0 is an edge for all $i \geq 1$:



Theorem 7.4 can be regarded as a discrete analogous of the classical result of homotopy invariance of homologies on manifolds.

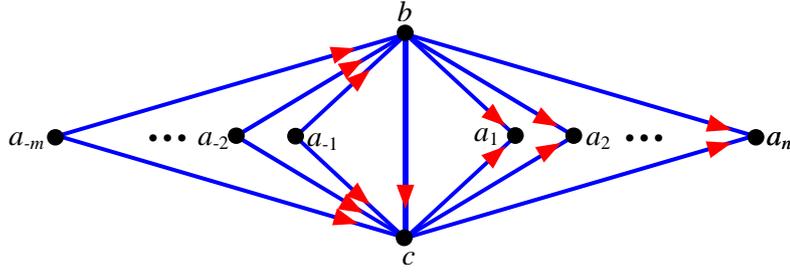


Figure 25: A digraph with many triangles and squares

Example 7.5 Consider a digraph (V, E) as shown in Fig. 25.

Each of the vertices a_k satisfies the hypotheses of Theorem 7.4 with $n = 2$ (either with incoming or outgoing edges). Removing successively the vertices a_k , we see that all the homologies of (V, E) are the same as those of the remaining graph $b \bullet \rightarrow \bullet c$. Since it is a star-like graph, we obtain $\dim H_0 = 1$ and $\dim H_p = 0$ for all $p \geq 1$. In particular, $\chi = 1$.

Note that for this digraph

$$\dim \Omega_0 = m + n + 2, \quad \dim \Omega_1 = 2m + 2n + 1.$$

Using Proposition 5.2 and observing that the number of semi-edges is mn , we obtain

$$\dim \Omega_2 = m + n + mn,$$

where the basis in Ω_2 is given by the triangles $e_{a_{-k}bc}$, e_{bca_l} and squares $e_{a_{-k}ba_l} - e_{a_{-k}ca_l}$. Finally, we have

$$\dim \Omega_3 = \dim \mathcal{A}_3 = mn$$

where the basis in Ω_3 is determined by the snakes $e_{a_{-k}bca_l}$, and $\dim \Omega_p = 0$ for all $p > 3$.

Proof of Theorem 7.4. Since the number of connected components of the graphs (V, E) and (V', E') is obviously the same, the identity (7.9) for $p = 0$ follows from Proposition 4.2.

For $p \geq 1$ consider the long exact sequence (7.2), that is,

$$\dots \leftarrow H_p(\Omega/\Omega') \leftarrow H_p(\Omega) \leftarrow H_p(\Omega') \leftarrow H_{p+1}(\Omega/\Omega') \leftarrow \dots,$$

that implies the identity

$$\dim H_p(\Omega) = \dim H_p(\Omega') \quad \text{for } p \geq 1,$$

if we prove that

$$\dim H_p(\Omega/\Omega') = 0 \quad \text{for } p \geq 1. \quad (7.10)$$

The condition (7.10) means that

$$\ker \partial|_{\Omega_p/\Omega'_p} \subset \text{Im } \partial|_{\Omega_{p+1}/\Omega'_{p+1}}$$

that is, if

$$v \in \Omega_p \quad \text{and} \quad \partial v = 0 \text{ mod } \Omega'_{p-1} \quad (7.11)$$

then there exists $\omega \in \Omega_{p+1}$ such that

$$\partial \omega = v \text{ mod } \Omega'_p. \quad (7.12)$$

In fact, it suffices to prove the existence of $\omega \in \mathcal{A}_{p+1}$ such that

$$\partial\omega = v \text{ mod } \mathcal{A}'_p. \quad (7.13)$$

Indeed, (7.13) implies $\partial\omega \in \mathcal{A}_p$ and, hence, $\omega \in \Omega_{p+1}$. Since $\partial\omega - v \in \mathcal{A}'_p$ and

$$\partial(\partial\omega - v) = -\partial v \in \mathcal{A}'_{p-1},$$

it follows that $\partial\omega - v \in \Omega'_p$ which proves (7.12).

To prove the existence of ω as above, observe that v can be represented in the form

$$v = e_a u \text{ mod } \mathcal{A}'_p, \quad (7.14)$$

where $u \in \mathcal{A}'_{p-1}$ and $e_a u \in \mathcal{A}_p$. We have then

$$\partial v = u - e_a \tilde{\partial} u \text{ mod } \mathcal{R}'_{p-1}.$$

Since $\partial v, u \in \mathcal{A}'_{p-1}$, it follows that

$$e_a \tilde{\partial} u = 0 \text{ mod } \mathcal{R}'_{p-1}. \quad (7.15)$$

However, all components of the path $e_a \tilde{\partial} u$ start with $e_{a\dots}$, whereas the condition (7.15) means that the path $e_a \tilde{\partial} u$ has no such component. Hence, (7.15) is only possible if $\tilde{\partial} u = 0$.

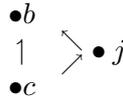
Since $e_a u \in \mathcal{A}_p$, any component of u has the form $e_{b_i\dots}$, which, together with the hypothesis that $b_0 b_i$ is an edge, implies that $e_{b_0} u \in \mathcal{A}'_p$ and $e_{ab_0} u \in \mathcal{A}_{p+1}$. Using $\tilde{\partial} u = 0$ and (7.14), we obtain

$$\partial(e_{ab_0} u) = (e_{b_0} - e_a) u + e_{ab_0} \tilde{\partial} u = e_{b_0} u - e_a u = -v \text{ mod } \mathcal{A}'_p,$$

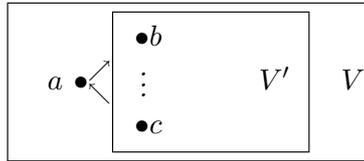
so that (7.13) holds with $\omega = -e_{ab_0} u$. ■

7.4 Removing a vertex of degree 1 + 1

Recall that a pair cb of distinct vertices on a graph is a *semi-edge* if cb is not an edge but there is a vertex j such that cjb is an edge:



Theorem 7.6 *Suppose that a graph (V, E) has a vertex a such that there is only one outgoing edge ab from a and only one incoming edge ca , where $b \neq c$. Let $V' = V \setminus \{a\}$ and $E' = E \setminus \{ab, ca\}$.*



Then the following is true.

(a) For any $p \geq 2$,

$$\dim H_p(V, E) = \dim H_p(V', E'). \quad (7.16)$$

- (b) If cb is an edge or a semi-edge in (V', E') then (7.16) is satisfied also for $p = 0, 1$, that is, for all $p \geq 0$.
- (c) If cb is neither edge nor semi-edge in (V', E') , but b, c belong to the same connected component of (V', E') then

$$\dim H_1(V, E) = \dim H_1(V', E') + 1$$

and $\dim H_0(V, E) = \dim H_0(V', E')$.

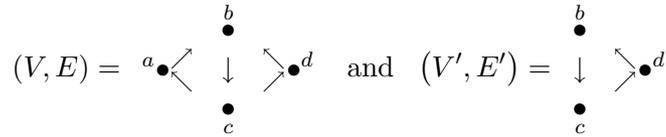
- (d) If b, c belong to different connected components of (V', E') then

$$\dim H_1(V, E) = \dim H_1(V', E')$$

and $\dim H_0(V, E) = \dim H_0(V', E') - 1$.

Consequently, in the case (b), $\chi(V, E) = \chi(V', E')$, whereas in the cases (c) and (d), $\chi(V, E) = \chi(V', E') - 1$.

Example 7.7 Consider the graphs



Since cb is semi-edge in (V', E') we have case (b) so that all homologies of (V, E) and (V', E') are the same. Removing further vertex d we obtain a digraph $b \bullet \rightarrow \bullet c$ that will be denoted by (V'', E'') . It is a star-like graph with all $\dim H_p(V'', E'') = 0$ for $p \geq 1$. Since cb is neither edge nor semi-edge in (V'', E'') , but the graph is connected, we conclude by case (c) that

$$H_p(V', E') = H_p(V'', E'') \quad \text{for } p \geq 2,$$

and

$$\dim H_1(V', E') = \dim H_1(V'', E'') + 1 = 1.$$

It follows that $\dim H_p(V, E) = 0$ for $p \geq 2$ and $\dim H_1(V, E) = 1$.

Example 7.8 Consider a digraph as on Fig. 26 (a kind of anti-snake).

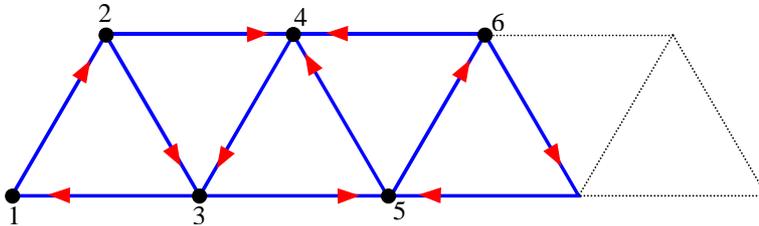


Figure 26: An anti-snake

We start building this graph with $1 \rightarrow 2$. Since 21 is neither edge nor semi-edge, adding a path $2 \rightarrow 3 \rightarrow 1$ increases $\dim H_1$ by 1 and preserves other homologies. Since 23 is an edge,

adding a path $2 \rightarrow 4 \rightarrow 3$ preserves all homologies. Since 34 is neither edge nor semi-edge, adding a path $3 \rightarrow 5 \rightarrow 4$ increases $\dim H_1$ by 1 and preserves other homologies. Similarly, adding a path $5 \rightarrow 6 \rightarrow 4$ preserves all homologies.

One can repeat this pattern arbitrarily many times. By doing so we construct a digraph with a prescribed value of $\dim H_1$ while keeping $\dim H_p = 0$ for all $p \geq 2$. Consequently, the Euler characteristic χ can take arbitrary negative values.

Proof of Theorem 7.6. *Proof of (a).* The identity (7.16) for $p \geq 2$ will follow if we prove that

$$\dim H_p(\Omega/\Omega') = 0 \text{ for } p \geq 2. \quad (7.17)$$

In order to prove (7.17) it suffices to show that

$$\ker \partial|_{\Omega_p/\Omega'_p} = 0,$$

which is equivalent to

$$v \in \Omega_p, \quad \partial v = 0 \text{ mod } \Omega'_{p-1} \Rightarrow v = 0 \text{ mod } \Omega'_p. \quad (7.18)$$

By the definition (5.1) of Ω_p , (7.18) is equivalent to

$$v \in \mathcal{A}_p \text{ and } \partial v \in \mathcal{A}'_{p-1} \Rightarrow v \in \mathcal{A}'_p. \quad (7.19)$$

Hence, let us prove (7.19) for all $p \geq 2$.

Every elementary allowed p -path on (V, E) either contains one of the edges ab , ca or is allowed in (V', E') . Let us show that, for any v as in (7.19), its components $v^{\dots ab \dots}$ and $v^{\dots ca \dots}$ vanish, which will imply that $v \in \mathcal{A}'_p$. Any such component can be written in the form $v^{\alpha ab \beta}$ or $v^{\gamma ca \beta}$ where α, β, γ are some paths. Consider the following cases. For further applications, in the Cases 1,2 we assume only that $v \in \Omega_p$ (whereas in the Case 3 v is as in (7.19)).

Case 1. Let us consider first the component $v^{\alpha ab \beta}$ where β is non-empty. If $\alpha ab \beta$ is not allowed in (V, E) then $v^{\alpha ab \beta} = 0$ by definition. Let $\alpha ab \beta$ be allowed in (V, E) . The path $\alpha a \beta$ is not allowed because the only outgoing edge from a is ab . Since $\partial v \in \mathcal{A}_{p-1}$, we have

$$(\partial v)^{\alpha a \beta} = 0.$$

Let us show that

$$(\partial v)^{\alpha a \beta} = \pm v^{\alpha ab \beta}, \quad (7.20)$$

which will imply $v^{\alpha ab \beta} = 0$. Indeed, by (2.7) $(\partial v)^{\alpha a \beta}$ is the sum of the terms $\pm v^\omega$ where ω is a p -path that is obtained from $\alpha a \beta$ by inserting one vertex. Since there is no edge from a to β , the only way ω can be allowed is when $\omega = \alpha ab \beta$. Since for any other ω we have $v^\omega = 0$, we obtain (7.20), which implies that $v^{\alpha ab \beta} = 0$.

Case 2. In the same way one proves that $v^{\gamma ca \beta} = 0$ provided γ is non-empty, using the fact that the only incoming edge in a is ca .

Case 3. Consider now an arbitrary component $v^{\alpha ab \beta}$. If β is non-empty then $v^{\alpha ab \beta} = 0$ by Case 1. Let β be empty. Then α must have the form $\alpha = \gamma c$ so that $v^{\alpha ab \beta} = v^{\gamma cab}$. If γ is non-empty then $v^{\gamma cab} = 0$ by Case 2. Finally, let γ be also empty so that $v^{\alpha ab \beta} = v^{cab}$ (which is only possible if $p = 2$). Since $\partial v \in \mathcal{A}'_1$, we have

$$(\partial v)^{ab} = 0.$$

On the other hand,

$$(\partial v)^{ab} = \sum_{i \in V} v^{iab} - v^{aib} + v^{abi}.$$

Here all the terms of the form v^{iab} vanish, except possibly for v^{cab} , because ia is not an edge unless $i = c$. All the terms v^{aib} vanish because ai is not an edge. All the terms v^{abi} vanish by Case 1. Hence, we obtain

$$(\partial v)^{ab} = v^{cab}$$

whence $v^{cab} = 0$ follows, thus finishing the proof of the part (a).

Proof of (b), (c), (d). If b, c belong to the same connected component of (V', E') then the number of connected components of (V, E) and that of (V', E') are the same, so that

$$\dim H_0(\Omega) = \dim H_0(\Omega'), \quad (7.21)$$

whereas if b, c belong to different connected components of (V', E') then after joining them by a the number of connected components reduces by 1, so that

$$\dim H_0(\Omega) = \dim H_0(\Omega') - 1. \quad (7.22)$$

To handle H_1 we use the long exact sequence (7.2) that by (7.17) has the form

$$0 \leftarrow H_0(\Omega/\Omega') \leftarrow H_0(\Omega) \leftarrow H_0(\Omega') \leftarrow H_1(\Omega/\Omega') \leftarrow H_1(\Omega) \leftarrow H_1(\Omega') \leftarrow 0. \quad (7.23)$$

Since we know already the relation between $H_0(\Omega')$ and $H_0(\Omega)$, to obtain the relation between $H_1(\Omega')$ and $H_1(\Omega)$ we need to compute $\dim H_0(\Omega/\Omega')$ and $\dim H_1(\Omega/\Omega')$ from the quotient complex Ω/Ω' . Observe that

$$\Omega_0 = \Omega'_0 + \text{span}\{e_a\}, \quad \Omega_1 = \Omega'_1 + \text{span}\{e_{ab}, e_{ca}\} \quad (7.24)$$

so that the quotient complex Ω/Ω' has the form

$$0 \longleftarrow \text{span}\{e_a\} \xleftarrow{\partial} \text{span}\{e_{ab}, e_{ca}\} \xleftarrow{\partial} \Omega_2/\Omega'_2 \xleftarrow{\partial} \dots$$

We need to determine $\text{Im } \partial|_{\Omega_1/\Omega'_1}$, $\ker \partial|_{\Omega_1/\Omega'_1}$, $\text{Im } \partial|_{\Omega_2/\Omega'_2}$. Since

$$\partial e_{ab} = e_b - e_a = -e_a \text{ mod } \Omega'_0,$$

it follows that

$$\text{Im } \partial|_{\Omega_1/\Omega'_1} = \Omega_0/\Omega'_0,$$

whence

$$\dim H_0(\Omega/\Omega') = 0. \quad (7.25)$$

For any scalars $k, l \in \mathbb{K}$, we have

$$\partial(ke_{ab} + le_{ca}) = (l - k)e_a \text{ mod } \Omega'_0,$$

so that $\partial(ke_{ab} + le_{ca}) = 0$ if and only if $k = l$, that is

$$\ker \partial|_{\Omega_1/\Omega'_1} = \text{span}(e_{ab} + e_{ca}) \text{ mod } \Omega'_1. \quad (7.26)$$

Let us now compute $\text{Im } \partial|_{\Omega_2/\Omega'_2}$. For any $v \in \Omega_2$ we have by the above Cases 1,2 that $v^{abi} = v^{jca} = 0$, which implies that v has the form

$$v = v' + v^{cab}e_{cab}, \quad (7.27)$$

where $v' \in \mathcal{A}'_2$. It follows that

$$\partial v = \partial v' + v^{cab} (e_{ab} - e_{cb} + e_{ca}). \quad (7.28)$$

Since all 1-paths ∂v , e_{ab} and e_{ca} belong to \mathcal{A}_1 , it follows that $\partial v' - v^{cab} e_{cb} \in \mathcal{A}_1$ whence also $\partial v' - v^{cab} e_{cb} \in \mathcal{A}'_1$. Therefore,

$$\partial v = v^{cab} (e_{ab} + e_{ca}) \bmod \Omega'_1. \quad (7.29)$$

Next consider two cases.

(i) Let Ω_2 contain an element v with $v^{cab} \neq 0$. Then by (7.29)

$$\text{Im } \partial|_{\Omega_2/\Omega'_2} = \text{span} (e_{ab} + e_{ca}) \bmod \Omega'_1, \quad (7.30)$$

which together with (7.26) implies

$$\dim H_1 (\Omega/\Omega') = 0. \quad (7.31)$$

Substituting (7.25) and (7.31) into the exact sequence (7.23), we obtain that the identity

$$\dim H_p (\Omega') = \dim H_p (\Omega)$$

holds for all $p \geq 0$.

(ii) Assume that $v^{cab} = 0$ for all $v \in \Omega_2$. Then by (7.29)

$$\text{Im } \partial|_{\Omega_2/\Omega'_2} = 0,$$

which together with (7.26) implies

$$\dim H_1 (\Omega/\Omega') = 1. \quad (7.32)$$

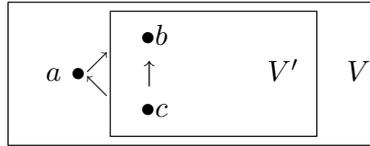
Using again the exact sequence (7.23), that is,

$$0 \leftarrow H_0 (\Omega) \leftarrow H_0 (\Omega') \leftarrow H_1 (\Omega/\Omega') \leftarrow H_1 (\Omega) \leftarrow H_1 (\Omega') \leftarrow 0,$$

we obtain by (3.7) and (7.32)

$$\dim H_1 (\Omega') - \dim H_1 (\Omega) + 1 - \dim H^0 (\Omega') + \dim H^0 (\Omega) = 0 \quad (7.33)$$

Let us now specify when (i) or (ii) occur. Assume first that cb is an edge:

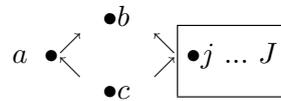


Then

$$\partial e_{cab} = e_{ab} - e_{cb} + e_{ca} \in \mathcal{A}_1,$$

whence it follows that $e_{cab} \in \Omega_2$. Hence, we have the case (i) with $v = e_{cab}$.

Assume now that cb is not an edge. Denote by J the set of vertices $j \in V'$ such that the 2-path cjb is allowed in (V', E') :



Assume first that J is non-empty, that is, cb is a semi-edge, and set

$$v = e_{cab} - \frac{1}{|J|} \sum_{j \in J} e_{cjb},$$

where $|J|$ is the number of elements in J . It is clear that $v \in \mathcal{A}_2$. We have

$$\begin{aligned} \partial v &= (e_{ab} - e_{cb} + e_{ca}) - \frac{1}{|J|} \sum_{j \in J} (e_{jb} - e_{cb} + e_{cj}) \\ &= (e_{ab} + e_{ca}) - \frac{1}{|J|} \sum_{j \in J} (e_{jb} + e_{cj}), \end{aligned} \tag{7.34}$$

where the term e_{cb} has cancelled out. It follows from (7.34) that $\partial v \in \mathcal{A}_1$ whence $v \in \Omega_2$, and we obtain again the case (i). This finishes the proof of (b).

Let us show that if $J = \emptyset$ (that is, when cb is neither edge nor semi-edge) then we have the case (ii). Any 2-path $v \in \Omega_2$ has the form (7.27) and ∂v is given by (7.28). It follows that

$$(\partial v)^{cb} = (\partial v')^{cb} - v^{cab}.$$

Since $\partial v \in \mathcal{A}_1$ and cb is not an edge, we have $(\partial v)^{cb} = 0$. We have by (2.7)

$$(\partial v')^{cb} = \sum_{j \in V'} (v')^{jcb} - (v')^{cjb} + (v')^{cbj},$$

which implies that $(\partial v')^{cb} = 0$ as no elementary 2-path of the form jcb, cjb, cbj is allowed in (V', E') , whereas $v' \in \mathcal{A}'_2$. It follows that $v^{cab} = 0$ so that we have the case (ii).

If in addition b, c belong to the same connected component of (V', E') then we have (7.21), that is,

$$\dim H^0(\Omega) = \dim H^0(\Omega').$$

Substituting into (7.33), we obtain

$$\dim H_1(\Omega) = \dim H_1(\Omega') + 1.$$

which proves part (c).

If b, c belong to different components of (V', E') then we have by (7.22)

$$\dim H^0(\Omega) = \dim H^0(\Omega') - 1,$$

whence by (7.33)

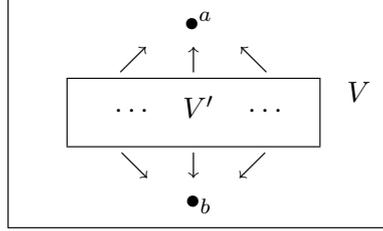
$$\dim H_1(\Omega) = \dim H_1(\Omega'),$$

which finishes the proof of part (d).

Finally, the identities for the Euler characteristic follows easily from the relations between $\dim H_p(\Omega)$ and $\dim H_p(\Omega')$. ■

7.5 Suspension and spheres

Let a digraph (V, E) have a subgraph (V', E') such that $V \setminus V' = \{a, b\}$ and $E \setminus E' = \{ia, ib, i \in V'\}$:



Definition 7.9 The digraph (V, E) is called a *suspension* of (V', E') and is denoted by $\text{Sus}(V', E')$.

Similarly, if a and b have outgoing edges then (V, E) is an *inverse suspension* of (V', E') . An example of a suspension graph is shown on Fig. 27.

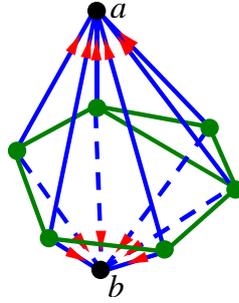


Figure 27: A suspension graph

The next theorem determines the homologies of a suspension.

Theorem 7.10 If (V, E) is a suspension (or inverse suspension) of (V', E') then, for any $p \geq 1$,

$$H_p(V, E) \cong \tilde{H}_{p-1}(V', E'). \quad (7.35)$$

Furthermore, the mapping $\tau v := v(e_a - e_b)$ that assigns to any $v \in \Omega'_{p-1}$ an element $\tau v \in \Omega_p$, extends to a bijective mapping from $\tilde{H}_{p-1}(V', E')$ onto $H_p(V, E)$.

Here \tilde{H}_p is a reduced homology: $\tilde{H}_p = H_p$ for $p \geq 1$ and

$$\tilde{H}_0 \cong H_0 / \{v \in \Omega_0 : (1, v) = 0\} \quad (7.36)$$

(cf. (3.24)). Denoting the digraph (V', E') by G , we can write the identity (7.35) in the functorial form as follows:

$$H_p(\text{Sus } G) = \tilde{H}_{p-1}(G).$$

Corollary 7.11 We have $\chi(\text{Sus } G) = 2 - \chi(G)$.

Proof. Denoting $\Gamma = \text{Sus } G$ we obtain

$$\begin{aligned}
\chi(\Gamma) &= 1 + \sum_{p \geq 1} (-1)^p \dim H_p(\Gamma) \\
&= 1 + \sum_{p \geq 1} (-1)^p \dim \tilde{H}_{p-1}(G) \\
&= 1 - \sum_{q \geq 0} (-1)^q \dim \tilde{H}_q(G) \\
&= 2 - \sum_{q \geq 0} (-1)^q \dim H_q(G) = 2 - \chi(G).
\end{aligned}$$

■

In particular, having examples of graphs G with arbitrary negative values of χ (cf. Example 7.8), we obtain examples of graphs $\text{Sus } G$ with arbitrary positive values of χ .

Example 7.12 Let S be any cycle-graph that is neither triangle nor square. We regard S as a circle. Define S_n inductively by $S_1 = S$ and $S_{n+1} = \text{Sus } S_n$. Then S_n can be regarded as n -dimensional sphere-graph. An example of S_2 is shown on Fig. 28.

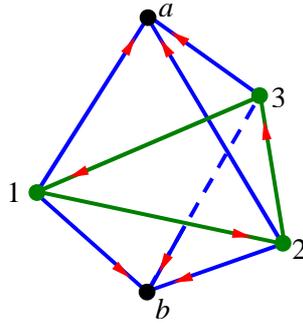


Figure 28: A graph S_2 based on a 3-vertex cycle S

Since $\chi(S) = 0$ by Proposition 5.9, it follows that $\chi(S_n) = 0$ if n is odd and $\chi(S_n) = 2$ if n is even. Theorem 7.10 also implies that $\dim H_n(S_n) = \dim H_1(S) = 1$, which gives an example of a non-trivial H_n with an arbitrary n .

Let v be an 1-path on S that spans $H_1(S)$ (see Section 5.6). If S_{n+1} is a suspension of S_n on the vertices a_n, b_n then we obtain by induction that the spanning element of $H_n(S_n)$ is

$$v(e_{a_1} - e_{b_1})(e_{a_2} - e_{b_2}) \dots (e_{a_{n-1}} - e_{b_{n-1}}).$$

For example, if S is a cycle-graph on Fig. 28 with $V = \{1, 2, 3\}$ and $E = \{12, 23, 31\}$, then $v = e_{12} + e_{23} + e_{31}$, whence the spanning element of $H_2(S_2)$ is

$$v(e_a - e_b) = (e_{12a} + e_{23a} + e_{31a}) - (e_{12b} + e_{23b} + e_{31b}).$$

Example 7.13 Another example of a 2-dimensional sphere-graph G is shown on Fig. 29.

Indeed, we have $G = \text{Sus } G'$ where G' is the subgraph with vertices $\{0, 1, 2, 3\}$ that is a cycle-graph. By Proposition 5.9 we have

$$\dim H_0(G') = 1, \quad \dim H_1(G') = 1, \quad \dim H_p(G') = 0 \text{ for } p \geq 2.$$

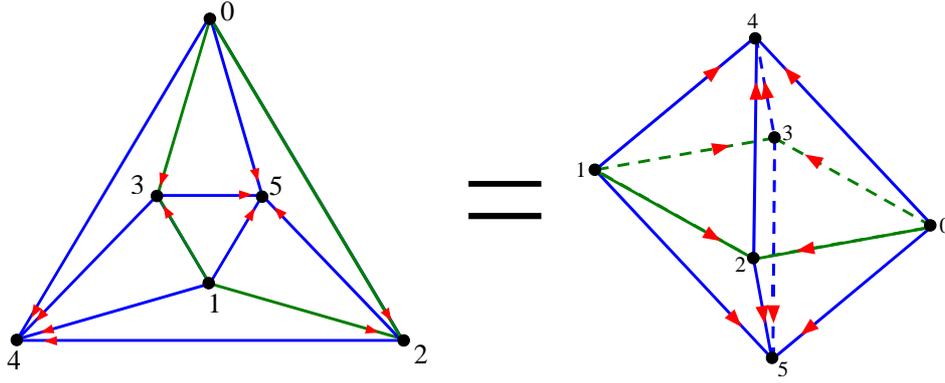


Figure 29: An octahedron G

The same follows from the fact that $G' = \text{Sus } G''$ where G'' is a subgraph with vertices $\{0, 1\}$. By Theorem 7.10 we obtain

$$\dim H_0(G) = 1, \dim H_1(G) = 0, \dim H_2(G) = 1, \dim H_p(G) = 0 \text{ for } p \geq 3. \quad (7.37)$$

Consequently, $\chi(G) = 2$.

In the digraph G we have

$$\dim \Omega_0 = |V| = 6 \text{ and } \dim \Omega_1 = |E| = 12$$

and

$$\mathcal{A}_2 = \text{span} \{e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135}\}.$$

The set of semi-edges is empty, whence by Proposition 5.2 $\dim \Omega_2 = \dim \mathcal{A}_2 = 8$ and, hence, $\Omega_2 = \mathcal{A}_2$. Alternatively, one can see that because all the 2-paths spanning \mathcal{A}_2 are triangles and there are no squares. Also, there are no allowed 3-paths, so that $\mathcal{A}_3 = \{0\}$ whence $\dim \Omega_p = 0$ for all $p \geq 3$.

Let us determine a spanning element of $H_2(G)$. Clearly, $H_0(G'') = \text{span} \{e_0, e_1\}$ whence by (7.36)

$$\tilde{H}_0(G'') = \text{span} \{e_0 - e_1\}.$$

By the second claim of Theorem 7.10, we have

$$\begin{aligned} H_1(G') &= \text{span} \{\tau(e_0 - e_1)\} = \text{span} \{(e_0 - e_1)(e_2 - e_3)\} \\ &= \text{span} \{e_{02} - e_{03} - e_{12} + e_{13}\}. \end{aligned}$$

Alternatively, the same spanning element can be obtained by (5.11).

Applying Theorem 7.10 again, we obtain

$$\begin{aligned} H_2(G) &= \text{span} \{\tau(e_{02} - e_{03} - e_{12} + e_{13})\} \\ &= \text{span} \{(e_{02} - e_{03} - e_{12} + e_{13})(e_4 - e_5)\} \\ &= \text{span} \{e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}\}. \end{aligned}$$

Note that the spanning element of $H_2(G)$ is exactly a surface path of a triangulation of \mathbb{S}^2 into an octahedron (cf. Section 5.9). In this case the surface path is not exact so that there is no solid path representing an octahedron.

Proof of Theorem 7.10. For any $p \geq 0$ consider a linear mapping

$$\tau : \mathcal{A}'_p \rightarrow \mathcal{A}_{p+1},$$

defined by

$$\tau v = v(e_a - e_b). \quad (7.38)$$

Since every vertex from V' is connected to a and b , the path τv is indeed allowed. By the product rule (2.16) we have

$$\partial(\tau v) = (\partial v)(e_b - e_a) + (-1)^{p+1} v \partial(e_a - e_b) = \tau \partial v$$

so that the operators ∂ and τ commute. It follows that

$$\tau(\Omega'_p) \subset \Omega_{p+1}.$$

Indeed, if $v \in \Omega'_p$ then

$$v \in \mathcal{A}'_p \quad \text{and} \quad \partial v \in \mathcal{A}'_{p-1}$$

whence

$$\tau v \in \mathcal{A}_{p+1} \quad \text{and} \quad \partial(\tau v) = \tau(\partial v) \in \mathcal{A}_p,$$

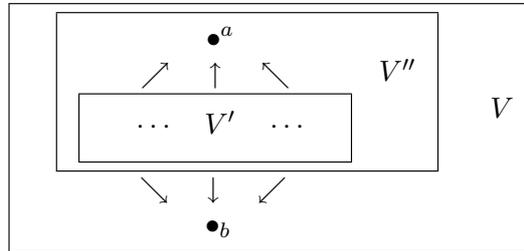
whence $\tau v \in \Omega_{p+1}$. Hence, the following diagram is commutative for all $p \geq 1$:

$$\begin{array}{ccc} \Omega'_{p-1} & \xleftarrow{\partial} & \Omega'_p \\ \downarrow \tau & & \downarrow \tau \\ \Omega_p & \xleftarrow{\partial} & \Omega_{p+1} \end{array} \quad (7.39)$$

Let us extend it to the case $p = 0$. Set $\Omega'_{-1} = \mathbb{K}$ as in the case of reduced homology. The operator $\tau : \mathbb{K} \rightarrow \Omega_0$ is also defined by (7.38), which now amounts to $\tau 1 = e_a - e_b$. The operator ∂ should be replaced by $\tilde{\partial} : \Omega'_0 \rightarrow \mathbb{K}$ where $\tilde{\partial} e_i = 1$ (this is the same operator $\tilde{\partial}$ that is used in the reduced homologies and in the product rule). The above argument, based on the product rule, remains valid. Hence, the diagram (7.39) remains commutative also for $p = 0$, where it takes the form

$$\begin{array}{ccc} \mathbb{K} & \xleftarrow{\tilde{\partial}} & \Omega'_0 \\ \downarrow \tau & & \downarrow \tau \\ \Omega_0 & \xleftarrow{\partial} & \Omega_1 \end{array}$$

Consider the digraph (V'', E'') that is obtained by adding to (V', E') the vertex a and all the edges ia with $i \in V'$, that is, (V'', E'') is a cone over (V', E') :



Let us mark by a double dash '' all the notation related to this digraph. For any $p \geq 0$, define a linear mapping $\rho : \mathcal{A}_p \rightarrow \mathcal{A}''_p$ by

$$\rho e_{i_0 \dots i_p} = \begin{cases} e_{i_0 \dots i_p}, & \text{if } i_p \neq b \\ e_{i_0 \dots i_{p-1} a}, & \text{if } i_p = b. \end{cases} \quad (7.40)$$

Clearly, ρ is surjective. Let us show that ρ commutes with ∂ . If $v = e_{i_0 \dots i_p}$ with $i_p \neq b$ then $\rho v = v$ and $\rho(\partial v) = \partial v$ so that $\rho(\partial v) = \partial(\rho v)$. If $i_p = b$ then, setting $u = e_{i_0 \dots i_{p-1}}$, we obtain $\rho v = u e_a$ and

$$\partial(\rho v) = (\partial u) e_a + (-1)^p u.$$

On the other hand, we have

$$\partial v = (\partial u) e_a + (-1)^p u$$

whence it follows that

$$\rho(\partial v) = (\partial u) e_b + (-1)^p u,$$

which proves that $\rho(\partial v) = \partial(\rho v)$.

It follows that ρ maps Ω_p to Ω''_p and the following diagram is commutative for any $p \geq 0$:

$$\begin{array}{ccc} \Omega_p & \xleftarrow{\partial} & \Omega_{p+1} \\ \downarrow \rho & & \downarrow \rho \\ \Omega''_p & \xleftarrow{\partial} & \Omega''_{p+1} \end{array} \quad (7.41)$$

We will merge the diagrams (7.41) and (7.39), and for that we need to verify that the following sequence is exact for all $p \geq -1$:

$$0 \longrightarrow \Omega'_p \xrightarrow{\tau} \Omega_{p+1} \xrightarrow{\rho} \Omega''_{p+1} \longrightarrow 0. \quad (7.42)$$

Since τ is injective and ρ is surjective, it suffices to show that $\text{Im } \tau = \ker \rho$. We have

$$\tau e_{i_0 \dots i_p} = e_{i_0 \dots i_p a} - e_{i_0 \dots i_p b}$$

so that $\text{Im } \tau$ consists of all p -paths of the form

$$\sum_{i_0, \dots, i_p \in V'} c^{i_0 \dots i_p} (e_{i_0 \dots i_p a} - e_{i_0 \dots i_p b}) \quad (7.43)$$

with arbitrary coefficients $c^{i_0 \dots i_p}$. Observe that, for any $u \in \Omega_{p+1}$,

$$\rho u = \sum_{i_0, \dots, i_{p+1} \in V'} u^{i_0 \dots i_{p+1}} e_{i_0 \dots i_p i_{p+1}} + \sum_{i_0, \dots, i_p \in V'} (u^{i_0 \dots i_p a} + u^{i_0 \dots i_p b}) e^{i_0 \dots i_p a}. \quad (7.44)$$

Then the equation $\rho u = 0$ that amounts to the system

$$\begin{cases} u^{i_0 \dots i_{p+1}} = 0, & \text{for all } i_0 \dots i_{p+1} \in V', \\ u^{i_0 \dots i_p a} + u^{i_0 \dots i_p b} = 0, & \text{for all } i_0 \dots i_p \in V', \end{cases} \quad (7.45)$$

that is, to the identity

$$u = \sum_{i_0, \dots, i_p \in V'} u^{i_0 \dots i_p a} (e_{i_0 \dots i_p a} - e_{i_0 \dots i_p b}). \quad (7.46)$$

Comparing with (7.43) we see that $\text{Im } \tau = \ker \rho$.

Hence, we have constructed the following commutative diagram where the rows are chain complexes and the columns are exact:

$$\begin{array}{cccccccccccc}
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dots \\
\downarrow & & \downarrow \\
0 & \longleftarrow & \Omega'_{-1} & \xleftarrow{\bar{\partial}} & \Omega'_0 & \xleftarrow{\partial} & \Omega'_1 & \xleftarrow{\partial} & \dots & \xleftarrow{\partial} & \Omega'_{p-1} & \xleftarrow{\partial} & \Omega'_p & \xleftarrow{\partial} & \dots \\
\downarrow & & \downarrow^\tau \\
0 & \longleftarrow & \Omega_0 & \xleftarrow{\partial} & \Omega_1 & \xleftarrow{\partial} & \Omega_2 & \xleftarrow{\partial} & \dots & \xleftarrow{\partial} & \Omega_p & \xleftarrow{\partial} & \Omega_{p+1} & \xleftarrow{\partial} & \dots \\
\downarrow & & \downarrow^\rho \\
0 & \longleftarrow & \Omega''_0 & \xleftarrow{\partial} & \Omega''_1 & \xleftarrow{\partial} & \Omega''_2 & \xleftarrow{\partial} & \dots & \xleftarrow{\partial} & \Omega''_p & \xleftarrow{\partial} & \Omega''_{p+1} & \xleftarrow{\partial} & \dots \\
\downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \dots
\end{array} \tag{7.47}$$

The homologies of the first chain complex in (7.47) are the reduced homologies $\tilde{H}_\bullet(\Omega')$, while the second and the third complexes yield the homologies $H_\bullet(\Omega)$ and $H_\bullet(\Omega'')$ respectively. By (3.20) we obtain a long exact sequence

$$0 \leftarrow \dots \leftarrow H_p(\Omega'') \leftarrow H_p(\Omega) \leftarrow \tilde{H}_{p-1}(\Omega') \leftarrow H_{p+1}(\Omega'') \leftarrow \dots \tag{7.48}$$

Since (V'', E'') is a star-like, we have by Theorem 5.4 $H_p(\Omega'') = \{0\}$ for any $p \geq 1$, whence it follows that

$$\dim H_p(\Omega) = \dim \tilde{H}_{p-1}(\Omega'),$$

which proves (7.35).

Finally, note that the mapping $\tilde{H}_{p-1}(\Omega') \rightarrow H_p(\Omega)$ in (7.48) is the extension of the vertical mapping $\tau : \Omega'_{p-1} \rightarrow \Omega_p$ on the diagram (7.47). The exactness of (7.48) and $H_p(\Omega'') = \{0\}$ imply that the mapping $\tau : \tilde{H}_{p-1}(\Omega') \rightarrow H_p(\Omega)$ is bijective, which finishes the proof. ■

8 Minimal paths and hole detection

The elements of $H_p(V, E)$ can be regarded as p -dimensional holes in the digraph (V, E) . To make this notion more geometric, we can work with representatives of the homologies classes, that are closed p -paths. Recall that two closed p -paths u and v are homological, that is, represent the same homology class, if $u - v$ is exact. We write in this case $u \sim v$.

For any p -path v define its length by

$$\ell(v) = \sum_{i_0, \dots, i_p \in V} |v^{i_0 \dots i_p}|.$$

Given a closed p -paths v_0 , consider the minimization problem

$$\ell(v) \mapsto \min \quad \text{for } v \sim v_0. \tag{8.1}$$

This problem has always a solution, although not necessarily unique. Any solution of (8.1) is called a *minimal* p -path. It is hoped that minimal p -paths (in a given homology class) match our geometric intuition of what holes in a graph should be. In this section we give some examples of minimal paths to support this claim.

Example 8.1 Consider the digraph (V, E) as on Fig. 30.

By Theorem 7.4, we can remove successively the vertices 9, B, C, A, D (and their adjacent edges) without changing the homologies. Then by Theorem 7.6 we can remove the vertices 7, 6, 8 equally without changing the homologies. We are left with the graph (V', E') formed by vertices $V' = \{0, 1, 2, 3, 4, 5\}$ and the edges $E' = \{01, 12, 23, 34, 45, 50\}$. By Proposition 5.9 we obtain $\dim H_1(V, E) = 1$, while $H_p(V, E) = \{0\}$ for all $p \geq 2$.

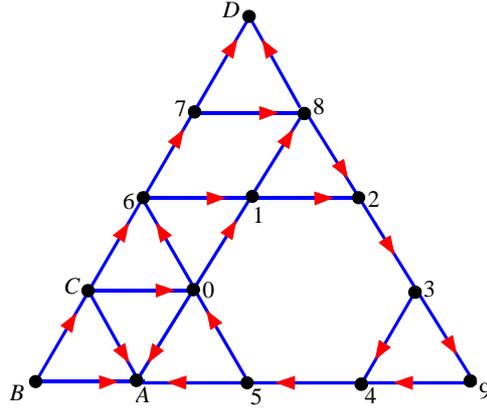


Figure 30: A digraph with 14 vertices and 23 edges

The following closed 1-path is a minimal path in the non-trivial homology class:

$$v = e_{01} + e_{12} + e_{23} + e_{34} + e_{45} + e_{50},$$

that is obviously associated with a hexagonal hole on Fig. 30.

Example 8.2 Consider a digraph on Fig. 31.

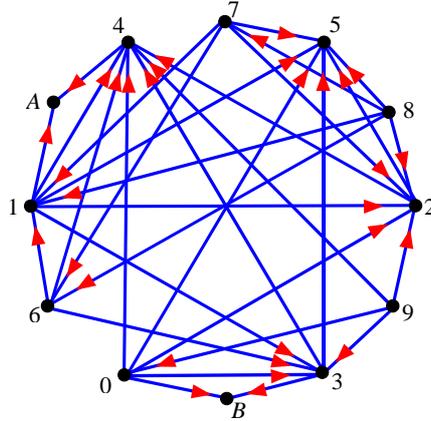


Figure 31: A digraph with 12 vertices and 32 edges.

Removing successively the vertices $A, B, 8, 9, 6, 7$ by Theorem 7.4, we obtain a digraph (V', E') with $V' = \{0, 1, 2, 3, 4, 5\}$ and $E' = \{02, 03, 04, 05, 12, 13, 14, 15, 24, 25, 34, 35\}$ that has the same homologies as (V, E) . The digraph (V', E') is shown in two ways on Fig. 32. Clearly, the second representation of this graph is reminiscent of an octahedron.

The digraph (V', E') is the same as the 2-dimensional sphere-graph of Example 7.13 (cf. Fig. 29). Hence, we obtain by (7.37) that $\dim H_2(V, E) = 1$ while $H_p(V, E) = \{0\}$ for $p = 1$ and $p > 2$.

The following closed 2-path is a minimal path in the non-trivial homology class:

$$v = e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135},$$

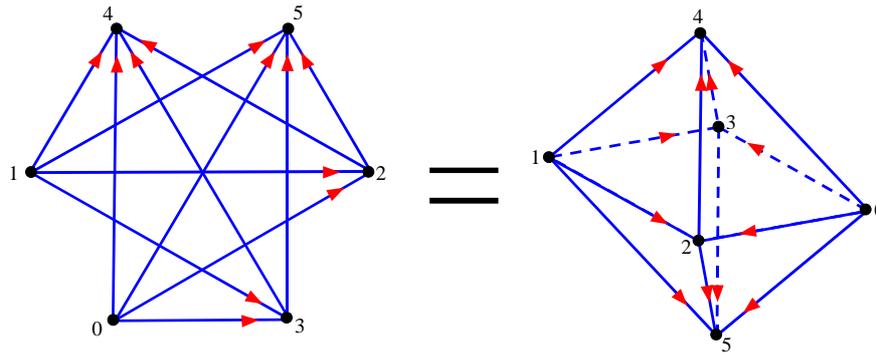


Figure 32: Two representations of the digraph (V', E')

that is a 2-path that determines a 2-dimensional hole in (V, E) given by the octahedron. Note that on Fig. 31 this octahedron is hardly visible, but it can be computed purely algebraically as shown above.

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