The Gorenstein projective modules for the Nakayama algebras.

Claus Michael Ringel

Abstract: The aim of this note is to outline the structure of the category of the Gorenstein projective Λ-modules, where Λ is a Nakayama algebra. We are going to introduce the resolution quiver of Λ, which provides a fast algorithm in order to obtain the Gorenstein projective Λ-modules and to decide whether Λ is a Gorenstein algebra or not, and whether it is CM-free or not.

Throughout the paper, Λ will be a connected Nakayama algebra with s simple modules. The minimal length of an indecomposable projective module will be denoted by p and we usually will assume that $p > s$ (thus that the endomorphism ring of any indecomposable projective module has a non-zero radical).

Let $\operatorname{mod}\Lambda$ be the category of all Λ-modules of finite length. We denote by $\operatorname{gp}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ given by the Gorenstein projective modules, and $\operatorname{gp}_0\Lambda$ denotes its full subcategory of all Gorenstein projective modules without any indecomposable projective direct summand. Recall that Λ is said to be CM-free provided all Gorenstein projective modules are projective.

Theorem 1. *Let Λ be a Nakayama algebra. Let $\mathcal{C}$ be the additive subcategory of* $\operatorname{mod}\Lambda$ *whose indecomposable objects are the indecomposable non-projective Gorenstein projective Λ-modules as well as their projective covers. Then $\mathcal{C}$ is a full exact abelian subcategory of* $\operatorname{mod}\Lambda$ *closed under extensions, projective covers and minimal left Λ-approximations such that*

$$\operatorname{gp}_0\Lambda \ \subseteq\ \mathcal{C}\ \subseteq\ \operatorname{gp}\Lambda.$$

The category $\mathcal{C}$ is equivalent to $\operatorname{mod}\Lambda'$, *where Λ' is a connected self-injective Nakayama algebra and the inclusion functor $\mathcal{C}\to\operatorname{gp}\Lambda$ induces an equivalence $\underline{\operatorname{mod}}\ \Lambda' \to \underline{\operatorname{gp}}\ \Lambda$.*

The subcategory $\mathcal{C}$ may be called the *Gorenstein core* of $\operatorname{mod}\Lambda$. In order to determine $\mathcal{C}$ it is sufficient to decide which indecomposable projective modules belong to $\mathcal{C}$. An arbitrary module M belongs to $\mathcal{C}$ if and only if the modules P, P' in a minimal projective presentation $P' \to P \to M \to 0$ belong to $\mathcal{C}$.

In section 3, we will introduce the "resolution quiver" R of Λ. The vertices of R are the simple Λ-modules and such a vertex S is said to be black provided the projective dimension of S is at least 2, otherwise it is said to be red. As we will see any connected component

2010 *Mathematics Subject Classification.* Primary 16G10, 16G50. Secondary: 16D90, 16E05, 16G70, 18G25.

of R contains precisely one cycle. The indecomposable projective module P belongs to $\mathcal{C}$ if and only if top P belongs to a cycle of black vertices.

Theorem 2. *The algebra Λ is a Gorenstein algebra if and only if any cycle in R contains only back vertices. The algebra Λ is* CM*-free if and only if any cycle in R contains at least one red vertex.*

Acknowledgment. This investigation is part of an ongoing project with Zhang Pu devoted to determine explicitly the Gorenstein projective modules for suitable classes of rings. The author is grateful to Zhang Pu and his collaborators at SJTU, but also to Xiao-Wu Chen for advice and comments.

1. Notation.

We denote by $Q = Q(\Lambda)$ the quiver of Λ; its vertices are the (isomorphism classes of the) simple Λ-modules and there is an arrow $S \to S'$ provided there is a length 2 module with top S and socle S'. Since we assume that no simple module is projective, the quiver Q is just a cycle. Let τ be the Auslander-Reiten translation. Since Λ is a Nakayama algebra without simple projective modules, an arrow $S \to S'$ in Q corresponds to the assertion $\tau(S) = S'$. Often, we will use the vertices x of the quiver Q in order to index corresponding modules: thus, we write $S(x)$ for the simple module itself, $P(x)$ for the projective cover of $S(x)$ and $H(x)$ for the factor module of $P(x)$ of length s.

In general, given a module M, we denote by $P(M)$ a projective cover, and by $\Omega(M)$ the first syzygy module. Inductively, let $P_0(M) = P(M)$, $\Omega_0(M) = M$, and for $n \geq 1$, let $P_n(M) = P(\Omega_n(M))$ and $\Omega_n(M) = \Omega(\Omega_{n-1}(M))$. Thus, there are exact sequences

$$0 \to \Omega_{n+1}(M) \to P_n(M) \to \Omega_n(M) \to 0$$

for all $n \geq 0$. A minimal projective resolution of M has the form

$$\cdots P_n(M) \to P_{n-1}(M) \to \cdots \to P_1(M) \to P_0(M) \to M \to 0,$$

with $\Omega_n(M)$ being the image of $P_n(M) \to P_{n-1}(M)$.

Recall that a complex $P_\bullet = (P_i, \delta_i)$ with maps $\delta : P_i \to P_{i-1}$ is called a *complete projective resolution* provided it is an exact complex of projective modules P_i such that also the complex $\operatorname{Hom}(P_\bullet, \Lambda)$ is exact. The latter condition is equivalent to the requirement that the inclusion map $\operatorname{Im}(\delta_i) \subseteq P_{i-1}$ is a left Λ-approximation, for all i. A module M is said to be *Gorenstein projective* provided there is a complete projective resolution (P_i, δ_i) such that $M = \operatorname{Im}(\delta_0)$.

An artin algebra Λ is said to be a *Gorenstein algebra* provided the injective dimension of ${}_\Lambda\Lambda$ as well as of Λ_Λ is finite. If this is the case, these dimensions are equal and called the *Gorenstein dimension* (or also the *virtual dimension*) of Λ; we will denote it by $v(\Lambda)$. Note that Λ is a Gorenstein algebra with Gorenstein dimension at most v if and only if $\Omega_v(M)$ is a Gorenstein projective module, for every module M.

An artin algebra is said to be CM-free provided every Gorenstein projective module is projective.

2. Projective resolutions of indecomposable Λ-modules.

An indecomposable projective module P is said to be *minimal* provided its radical is non-projective (thus provided $P = P(S)$ for some simple module S of projective dimension at least 2). Note that a proper non-zero submodule of a minimal projective module is not projective (this explains the name).

Lemma 1. *Let M be an indecomposable non-projective module. An embedding $M \to P$ is a minimal left Λ-approximation provided P is a minimal projective module.*

Proof. If there exists an embedding $M \to P$, with P projective, then there is such an embedding $M \to P_0$ with P_0 indecomposable. Since we assume that M is not projective, there is such an embedding $M \to P_0$, where P_0 is in addition minimal projective.

It remains to be shown that any embedding $u\colon M \to P_0$ with P_0 minimal projective is a left Λ-approximation. We have to show that any non-zero map $f\colon M \to P_1$ with P_1 indecomposable projective factors through u. Here, we can assume that also P_1 is minimal projective (namely, since M is not projective, it will map into $\operatorname{rad} P_1$, thus, if $\operatorname{rad} P_1$ is projective, we may replace P_1 by $\operatorname{rad} P_1$).

There are several ways to show that f factors through u. One possibility is to look at the Auslander-Reiten quiver of Λ and consider various right almost split maps. Here is a proof which uses the fact that M is projective and injective in the category of Λ-modules of Loewy length at most $|M|$ and that any automorphism of a submodule of an indecomposable module X can be extended to an automorphism of X.

We use induction on the length of $\operatorname{Ker}(f)$. If f is a monomorphism, then P_0 has to be a submodule of P_1 and the image of f coincides with the image of u. It follows that f factors through u. Now assume that $\operatorname{Ker}(f) \neq 0$. Let S be the socle of the image of f and take a composition series

$$\operatorname{Ker}(f) = K_0 \subset K_1 \supset \cdots \supset K_s = 0$$

of $\operatorname{Ker}(f)$. Note that $K_{i-1}/K_i = \tau^i S$ for $1 \le i \le s$.

Since M is not projective, f maps into $N = \operatorname{rad} P$, and by assumption N is not projective. Let $q\colon P = P(N) \to N$ be a projective cover of N, say with kernel U. Let

$$U = U_0 \subset U_1 \supset \cdots \supset U_u = 0$$

be a composition series of U. Then also $U_{i-1}/U_i = \tau^i S$ for $1 \le i \le u$. Note that for $1 \le i \le u-1$, the module P/U_i is injective and not projective, thus the modules $\tau^i S = U_{i-1}/U_i$ for $1 \le i \le u-1$ are not torsionless.

Now assume that $s < u$. Then $1 \le s \le u-1$ and therefore $K_{s-1}/K_s = \tau^s S$ is not torsionless. But this is the socle of the module M and by assumption there is the embedding $u\colon M \to P_1$. This contradiction shows that we must have $u \le s$.

Consider the restriction of $q\colon P(N) \to N$ to $q^{-1}(\operatorname{Im}(f))$. The length of $q^{-1}(\operatorname{Im}(f))$ is

$$|q^{-1}(\operatorname{Im}(f))| = |\operatorname{Im}(f)| + |\operatorname{Ker}(q)| = |\operatorname{Im}(f)| + u \le |\operatorname{Im}(f)| + s = |M|,$$

and therefore we can lift the map $f\colon M \to \operatorname{Im}(f)$ to $q^{-1}(\operatorname{Im}(f))$. Thus, there is a (surjective) map $f'\colon M \to q^{-1}(\operatorname{Im}(f))$ such that $qf' = f$. Note that the kernel of f' (as well as the kernel of the composition $u'f'$ of f' with the embedding $u'\colon q^{-1}(\operatorname{Im}(f)) \to P(N)$) has length $s - u < s$), thus by induction $u'f'$ factors through u, therefore f factors through u.

Lemma 2. *Let M be an indecomposable non-projective module. The following assertions are equivalent:*
(1) *The module M is Gorenstein projective.*
(2) *All the projective modules occurring in a minimal projective resolution of M are minimal projective.*
(3) *There is an exact sequence*

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that all the modules P_i are minimal projective modules.

Proof. (1) $\implies$ (2). This follows from Lemma 1.

(2) $\implies$ (3). We assume that we deal with the projective resolution

$$\to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Now all the images $\Omega_n(M)$ are indecomposable, thus there are natural numbers $0 \le n' < n$ such that $\Omega_n(M) = \Omega_{n'}(M)$. Choose n minimal and assume that $n' \ge 1$. Now $\Omega_n(M)$ is a submodule of P_{n-1}, and $\Omega_{n'}(M)$ is a submodule of $P_{n'-1}$. Since $\Omega_n(M) = \Omega_{n'}(M)$, the minimality of P_{n-1} and $P_{n'-1}$ implies that $P_{n-1} = P'_{n'-1}$ and therefore $\Omega_{n-1}(M) = \Omega_{n'-1}(M)$. But this contradicts the minimality of n. Thus $n' = 0$ and $\Omega_n(M) = \Omega_0(M) = M$.

(3) $\implies$ (1). We assume that there is given an exact sequence

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

such that all the modules P_i are minimal projective modules. Concatenation of countably many such sequences yields a complete resolution with M as one of the images. This shows that M is Gorenstein projective.

3. The resolution quiver of Λ.

Let $\gamma S = \tau \operatorname{soc} P(S)$. The importance of the map γ stems from two observations, see Lemmas 3 and 4. As Xiao-Wu Chen has pointed out, the map γ has been considered already by W. H. Gustafson [G].

Lemma 3. *Let M be an indecomposable module. Then either the projective dimension of M is at most 1 and $\Omega_2(M) = 0$, or else* $\operatorname{top} \Omega_2(M) = \gamma \operatorname{top} M$.

Proof: Write $M = P(M)/U$ for some submodule U of $P(M)$. We can assume that U is a proper submodule of $P(M)$. There are exact sequences of the following form

$$0 \to \Omega_2(M) \to P_1(M) \to \Omega_1(M) \to 0$$
$$0 \to \Omega_1(M) \to P(M) \to M \to 0.$$

Clearly, $\operatorname{soc} \Omega_1(M) = \operatorname{soc} P(M)$ and the first exact sequence shows that either $\Omega_2(M) = 0$ (thus the projective dimension of M is at most 1) or else $\operatorname{top} \Omega_2(M) = \tau \operatorname{soc} \Omega_1(M)$. In the latter case, $\operatorname{top} \Omega_2(M) = \gamma \operatorname{top} M$.

Inductively, we see:

Corollary. *Let M be an indecomposable module and $m \in \mathbb{N}$. Then either $\Omega_{2m}(M) = 0$ or else $\operatorname{top}\Omega_{2m}(M) = \gamma^m \operatorname{top} M$.*

Lemma 4. *Let S be a simple module. Then*

$$\Omega_2 H(S) = H(\gamma S).$$

Proof. Since $H(S)$ has length s, its projective presentation is

$$P(S) \to P(S) \to H(S) \to 0,$$

and therefore $\Omega_2 H(S)$ has length s. On the other hand, according to Lemma 3, we know that $\Omega_2 H(S)$ is a factor module of $P(\gamma S)$.

Corollary. *Let S be a simple module and m a natural number. Then*

$$\Omega_{2m} H(S) = H(\gamma^m S).$$

We say that a vertex x or the corresponding simple or projective modules $S(x)$ and $P(x)$ are *black* provided $P(x)$ is minimal projective, otherwise x (and $S(x)$ and $P(x)$) will be said to be *red.* Note that x is black if and only if the projective dimension of $S(x)$ is equal to 1, and red if and only if the projective dimension of $S(x)$ is greater or equal 2.

The *resolution quiver* R has as vertices the (isomorphism classes of the) simple Λ-modules, and for any simple module S, there is an arrow $S \to \gamma S$.

Note that *any connected component of the resolution quiver has a unique cycle.* Proof: This follows from the fact that at any vertex x precisely one arrow starts; thus given any connected component, the number of arrows in the component is equal to the number of vertices in the component.

A cycle in R will be said to be *black* provided all the vertices occurring in the cycle are black, otherwise it is said to be *non-black.* A vertex x is said to be *cyclically black* provided x belongs to a black cycle.

Proposition 1. *The resolution quiver has no black cycles if and only if Λ is* CM*-free.*

Proof: Assume that there are no black cycles. Let M be an indecomposable non-projective module. We show that there exists n such that $P_n(M)$ is red, thus M cannot be Gorenstein projective.

First, assume that M has finite projective dimension, say assume that M has finite global dimension, thus $\Omega_n(M)$ is indecomposable projective for some $n \geq 0$. Then $\Omega_{n+1}(M) \to P_n(M)$ is a proper inclusion, thus P_n is not minimal projective (thus red).

Second, assume that M has infinite projective dimension. Then $P_{2m}(M)$ belongs to a non-black cycle for $m \gg 0$, thus there is such an m with $P_{2m}(M)$ red.

Conversely, let us assume that there is a black cycle $x_1 \to x_2 \to \cdots \to x_n \to x_1$. Then we concatenate the exact sequences

$$0 \to H(x_{i+1}) \to P(x_i) \to P(x_i) \to H(x_i) \to 0$$

in order to obtain a complete resolution of $H(x_1)$. This shows that $H(x_1)$ is Gorenstein projective.

The resolution quiver has the following property: *any vertex y is end point of at most one arrow $x \to y$ with x black.* Namely, if S, S' are black simple modules with $\gamma S = \gamma S'$, then $\tau \operatorname{soc} P(S) = \tau \operatorname{soc} P(S')$, thus $\operatorname{soc} P(S) = \operatorname{soc} P(S')$ and therefore $P(S) \subseteq P(S')$ or $P(S') \subseteq P(S')$. However, since both $P(S)$ and $P(S')$ are minimal projective modules, it follows that $P(S) = P(S')$ and thus $S = S'$.

As a consequence we obtain:

Red entrance Lemma. *If $x_0 \to x_1 \to \cdots \to x_a \to y$ is a path such that y is cyclically black, whereas x_a is not cyclically black, then x_a is red.*

Proof. Since y is cyclically black, there is an arrow $x' \to y$ such that also x' is cyclically black. Since x_a is not cyclically black, we have $x_a \neq x'$, and it follows that x_a cannot be black, thus it is red.

Lemma 5. *Let M be indecomposable and not projective. Then the following conditions are equivalent:*
(1) *M is Gorenstein projective.*
(2) *Both* $\operatorname{top} M$ *and* $\operatorname{top} \Omega(M)$ *are cyclically black.*
(3) *Both* $\operatorname{top} M$ *and* $\tau \operatorname{soc} M$ *are cyclically black.*

Proof: The assertions (2) and (3) are equivalent, since $\tau \operatorname{soc} M = \operatorname{top} \Omega(M)$.

(2) $\implies$ (1). If $\operatorname{top} M$ is cyclically black, then all the modules $P_{2m}(M)$ are black, for $m \geq 0$. If $\operatorname{top} \Omega(M)$ is cyclically black, then all the modules $P_{2m+1}(M)$ are black, for $m \geq 0$. Thus, if both conditions are satisfied, then all the projective modules occurring in the minimal resolution of M are minimal, thus M is Gorenstein projective, according to Lemma 2.

(1) $\implies$ (2). Assume that $\operatorname{top} M$ is not cyclically black. Then there is some $m \geq 0$ such that $\gamma^m \operatorname{top} M$ is red, but $P_{2m}(M) = P(\gamma^m \operatorname{top} M)$. Similarly, if $\operatorname{top} \Omega M$ is not cyclically black, then there is some $m \geq 0$ such that $\gamma^m \Omega \operatorname{top} M$ is red, and $P_{2m+1}(M) = P(\gamma^m \operatorname{top} \Omega M)$. Thus, M cannot be Gorenstein projective, according to Lemma 2.

Remark. It seems to be of interest to identify the sources of the resolution quiver R: *The simple module S is a source of R if and only if $\tau^- S$ cannot be embedded into Λ* (thus if and only if $I(\tau^- S)$ is not projective). Namely, if S is in the image of γ, then there is a simple module S' with $S = \gamma S' = \tau \operatorname{soc} P(S')$, thus $\tau^- S = \operatorname{soc} P(S') \subseteq \operatorname{soc} \Lambda$. And conversely, if $\tau^- S$ can be embedded into Λ, then it can be embedded into some indecomposable projective module $P(S')$, and then $\gamma S' = \tau \operatorname{soc} P(S') = \tau\tau^- S = S$, thus S is in the image of γ.

As a consequence, *the number of sources of R is equal to $s - t$,* where t is the number of indecomposable modules which are both projective and injective (and this is also the number of minimal projective modules).

The same argument shows that in general *the number of arrows ending in S is equal to the number of projective modules with socle $\tau^- S$.*

4. The elementary Gorenstein projective modules.

Lemma 6. *Let S be a simple module which is cyclically black. Denote by $G(S)$ the non-zero factor module of $P(S)$ which is Gorenstein projective and of smallest possible length. Then the length of $G(S)$ is at most s.*

The modules $G(S)$ will be called the *elementary* Gorenstein projective modules.

Proof. The module $H(S)$ is Gorenstein projective, since $\operatorname{top}\Omega H(S) = \operatorname{top} H(S) = S$. Thus $G(S)$ has to be a factor module of $H(S)$, and therefore $|G(S)| \leq s$.

Proposition 2. *Assume that there is at least one cyclically black vertex, say x. Let $x = x_1, \dots x_g$ be the cyclically black vertices. Then the modules $G(x_1), \dots, G(x_g)$ are pairwise orthogonal bricks and*

$$\sum\nolimits_{i=1}^{g} |G(x_i)| = s.$$

Proof: Write $x_i = \tau^{n(i)} x$ with $0 \leq n(i) < s$ and we may reorder the indices so that $0 = n(1) < n(2) < \cdots < n(g)$. Then $H(x)$ has a filtration

$$H(x) = M_0 \supset M_1 \supset \cdots \supset M_{g-1} \supset M_g = 0$$

such that $M_{i-1}/M_i = G(x_i)$ for $1 \leq i \leq g$.

Thus we see: The elementary Gorenstein projective modules are thin, and have pairwise disjoint support. If there is at least one non-projective Gorenstein projective module, then the union of the support of the elementary modules is the set of all simple modules.

Proposition 3. *Let $\mathcal{C}$ be the set of Λ-modules with a filtration with factors in $\mathcal{G} = \{G(x_1), \dots, G(x_g)\}$. Then*

(a) *The category $\mathcal{C}$ is an abelian exact subcategory with simple objects $G(x_1), \dots, G(x_g)$ and projective objects $P(x_1), \dots, P(x_g)$.*

(b) *We have*

$$\mathrm{gp}_0 \subseteq \mathcal{C} \subseteq \mathrm{gp}\,.$$

and $\mathcal{C}$ is closed under extensions, projective covers and minimal left Λ-approximations.

(c) *The category $\mathcal{C}$ is equivalent to the module category of a self-injective Nakayama algebra Λ' with g simple modules.*

(d) *The embedding* $\operatorname{mod}\Lambda' \to \operatorname{mod}\Lambda$ *induces an embedding* $\underline{\operatorname{mod}}\Lambda' \to \underline{\operatorname{mod}}\Lambda$ *with image just* $\underline{\mathrm{gp}}$.

Proof: (a) and (b). All the modules $G(x)$ are Gorenstein projective, thus $\mathcal{C} \subseteq \mathrm{gp}$. On the other hand, if X is in gp_0, let X' be a non-zero Gorenstein projective factor module of X with smallest possible length. Consider the projective cover $P(x)$ of X, then x has to be cyclically black and $X' = G(x)$. Also, the kernel X'' of $X \to X'$ is Gorenstein projective and by induction belongs to $\mathcal{C}$. Since both X'' and X' belong to $\mathcal{C}$, also X belongs to $\mathcal{C}$. This shows that $\mathrm{gp}_0 \subseteq \mathcal{C}$.

The process of simplification (see [R]) asserts that the category $\mathcal{C}$ is abelian with simple objects $G(x_1), \dots, G(x_g)$.

If x is cyclically black, then there is the surjection $P(x) \to G(x)$. Since $G(x)$ is Gorenstein projective, also the kernel of $P(x) \to G(x)$ is Gorenstein projective ....

(c) The category $\mathcal{C}$ is equivalent to the module category of an artin algebra Λ'. Since the indecomposable objects in $\mathcal{C}$ have a unique composition series as Λ-modules, they also have a unique composition series as objects in $\mathcal{C}$, thus Λ' is a Nakayama algebra. Of course, the modules $P(x)$ with x cyclically black, are projective as objects of $\mathcal{C}$, thus these are the indecomposable projective Λ'-modules.

The algebra Λ' is self-injective: The simple Λ'-modules are the modules $G(x)$ with x cyclically black. Now $\Omega_{\Lambda'}G(x) = \Omega_\Lambda G(x)$, thus $\Omega_{\Lambda'}G(x)$ is not projective. This shows that the Kupisch series $(p'_1, \dots, p'_g)$ of Λ' satisfies $p'_1 \le p'_2 \le \cdots \le p'_g \le p'_1$, thus it is constant.

(d) Let X, Y be indecomposable in gp_0, and $f\colon X \to Y$ a morphism which factors through a projective Λ-module, say P. Then the morphism $X \to P$ factors through P', where $X \to P'$ is the minimal left Λ-approximation of X. But P' belongs to $\mathcal{C}$ and is projective in $\mathcal{C}$, thus f is zero in $\underline{\mathrm{gp}}$.

Proposition 4. *If M belongs to $\mathcal{C}$, its length as object of $\mathcal{C}$ is the number of Λ-composition factors of M which are cyclically black.*

More information about Λ'. We know that the number of simple Λ'-modules is g. The length of the indecomposable projective Λ'-modules is given by the formula

$$p' = \tfrac{1}{s}\sum_{i=1}^{g} p(x_i)$$

(where $x_1, \dots, x_g$ are the cyclically black vertices and $p(x_i) = |P(x_i)|$).

Proof: We have

$$\begin{aligned}\sum_{i=1}^{g} p(x_i) &= \sum_{i=1}^{g} |P(x_i)| = \sum_{i=1}^{g}\sum_{i=1}^{p'} |G(x_i)| = \sum_{i=1}^{p'}\sum_{i=1}^{g} |G(x_i)| \\ &= \sum_{i=1}^{p'} |H(x_i)| = \sum_{i=1}^{p'} s = p's.\end{aligned}$$

Proposition 3 is essentially a refinement of Theorem 1. But the assertion of Theorem 1 allows that Λ is not connected (then we consider the connected components of Λ separately). Also, Theorem 1 includes the case that Λ (or a connected component of Λ) is CM-free: If Λ is CM-free, then $\mathcal{C}$ is the zero category.

Remark 1. In general, one may look for full exact abelian subcategories $\mathcal{C}'$ closed under projective covers in $\operatorname{mod}\Lambda$, such that

$$\mathrm{gp}_0 \subseteq \mathcal{C}' \subseteq \mathrm{gp}\,.$$

Such a subcategory $\mathcal{C}'$ is uniquely determined, if we assume that the endomorphism ring of any indecomposable projective module has a non-zero radical (our usual assumption

$p > s$): the objects of $\mathcal{C}'$ have to be just those of gp_0 as well as their projective covers. (Namely, assume that X is indecomposable and belongs to $\mathcal{C}'$, but not to gp_0, then X must be projective, say $X = P(S)$ for some simple module S. Since $\mathcal{C}'$ is closed under cokernels, $H(S)$ has to belong to $\mathcal{C}'$, thus must be Gorenstein projective. But this implies that S has to be cyclically black. Here we use that the modules $H(S)$ do exist and are not projective; this is just the assumption $p > s$.) In particular, in case Λ is CM-free, we must have $\mathcal{C}' = 0$.

On the other hand, if $p \leq s$ and $\mathrm{gp}_0 = \{0\}$, let P be an indecomposable projective module of length s. Then the full subcategory $\mathcal{C}'$ of all direct sums of copies of P is a full exact abelian subcategory of $\operatorname{mod}\Lambda$, closed under projective covers of $\operatorname{mod}\Lambda$ and satisfies $\mathrm{gp}_0 \subseteq \mathcal{C}' \subseteq \mathrm{gp}$.

Remark 2. The indecomposable objects in $\mathcal{C}$ are obtained from $\operatorname{mod}\Lambda$ by deleting some rays and some corays of the Auslander-Reiten quiver of Λ. Namely, we have to delete the corays ending in cyclically black simple modules S (these are the modules with top equal to S) and the rays starting in $\tau^- S$, where S is cyclically black (these are the modules with socle $\tau^- S$).

5. Gorenstein algebras.

Proposition 5. *The Nakayama algebra Λ is a Gorenstein algebra if and only if all cycles in the resolution quiver of Λ are black. In this case, the Gorenstein dimension of Λ is equal to $2d$, where d is the maximal distance between vertices and the black cycles.*

We need the following lemmas.

Lemma 7. *Let $x = x_0 \to \cdots \to x_d$ be a path in the resolution quiver such that x_d is cyclically black, whereas x_{d-1} is not. Then $G\text{-}\dim H(x) = 2d$.*

Proof. We show that $\Omega_{2d-1}H(x)$ is not Gorenstein projective, whereas $\Omega_{2d}H(x)$ is Gorenstein projective. According to Lemma 4 we know that $\Omega_{2d}H(S) = H(\gamma^d S) = H(x_d)$ and this is a Gorenstein projective module. On the other hand, there is the following exact sequence

$$0 \to H(x_d) \to P(x_{d-1}) \xrightarrow{f} P(x_{d-1}) \to H(x_{d-1}) \to 0$$

and $\Omega_{2d-1}H(x)$ is the image of f. In particular, we see that the top of $\Omega_{2d-1}H(x)$ is $S(x_{d-1})$ and this is a red vertex, according to the red entrance lemma. Since the top of $\Omega_{2d-1}H(x)$ is not even black, $\Omega_{2d-1}H(x)$ cannot be Gorenstein projective.

Lemma 8. *Assume that for any path $x_0 \to \cdots \to x_d$ in the resolution quiver, the vertex x_d is cyclically black. Then the G-dimension of any Λ-module is at most $2d$.*

Proof. We show that $\Omega_{2d}(M)$ is Gorenstein projective, for any module M. According to Lemma 5. we have to show that the modules $\operatorname{top}\Omega_{2d}(M)$ and $\operatorname{top}\Omega_{2d+1}(M)$ are zero or cyclically black. But $\operatorname{top}\Omega_{2d}(M)$ is zero or equal to $\gamma^d \operatorname{top} M$, and $\operatorname{top}\Omega_{2d+1}(M)$ is zero or equal to $\gamma^d \operatorname{top}\Omega M$. The assumption of the lemma can be rephrased by saying that $\gamma^d S$ is cyclically black for all simple modules S. This completes the proof.

Proof of proposition 5: First, assume that there is a non-black cycle in the resolution quiver, say involving the red vertex x. Then a minimal projective resolution of $H(x)$

involves infinite many copies of $P(x)$ and $P(x)$ is not minimal projective, thus $H(x)$ has infinite G-dimension. This shows that Λ is not a Gorenstein algebra.

On the other hand, if all the cycles in the resolution quiver are black, then all indecomposable modules have finite G-dimension, according to lemma 8. Thus Λ is a Gorenstein algebra.

Proposition 6. *If Λ is a Gorenstein algebra, then $v(\Lambda) \leq 2s - 2$.*

Proof. Any path $x = x_0 \to \cdots \to x_d$ in R such that x_{d-1} does not belong to a cycle involves pairwise different vertices, thus $d \leq s - 1$. Thus, Lemma 8 asserts that the G-dimension of any Λ-module is at most $2s - 2$.

As Xiao-Wu Chen has pointed out, this result (and its proof) corresponds to Gustafson's bound for the finitistic dimension of a Nakayama algebra.

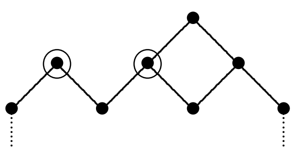
6. Calculation of resolution quivers.

Recall that $(p_1, \ldots, p_s)$ is said to be a *Kupisch series* for Λ, provided we have labeled the indecomposable projective modules $P_1, \ldots, P_s$ such that $\operatorname{rad} P_i$ is a factor module of P_{i+1} (thus provided there is an arrow $S_i \to S_{i+1}$) and $p_i = |P_i|$. The Kupisch series for Λ are obtained from any one by cyclic permutation.

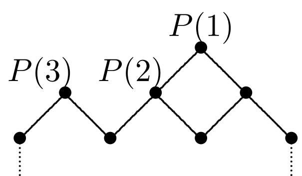
It is an easy exercise to draw the resolution quiver of Λ is a Kupisch series is known. Namely, if $(p_1, \ldots, p_s)$ is a Kupisch series for Λ, then $\gamma i = i + p_i$ (modulo s).

Two Nakayama algebras Λ, Λ' are said to have *the same roof* provided they have Kupisch series $(p_1, \ldots, p_s)$ and $(p'_1, \ldots, p'_s)$ such that $p_i - p'_i = c$, for $1 \leq i \leq s$, where $c = c(\Lambda, \Lambda')$ is a constant. *If $c(\Lambda, \Lambda')$ is a multiple of s, then Λ and Λ' have the same resolution quiver.*

In order to visualize roofs, it seems to be convenient to draw the Auslander-Reiten quivers for such a Nakayama algebras with $p_s = p = 2$. For example, here is the roof of the Nakayama algebra with Kupisch series $(m + 3, m + 2, m + 2)$:

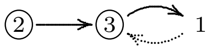


Let us stress the conventions which we use: if the Kupisch series is $(m + 3, m + 3, m + 2)$, then $p = m + 2$, thus we exhibit just the Auslander-Reiten quiver of the algebra with Kupisch series $(3, 3, 2)$. Always, the direction of the arrows will be deleted: all arrows are supposed to be directed from left to right, and the vertex far right has to be identified with the vertex far left (so that the Auslander-Reiten quiver lives on a cylinder). The projective vertices should be labeled **going from right to left** as $P(1), P(2), P(3)$, see



In the picture above, two of the projective vertices, namely $P(2)$ and $P(3)$ were encircled (we usually encircle the minimal projective modules, they correspond to the black vertices).

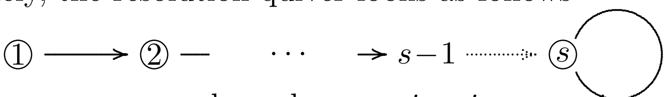
In order to draw the resolution quiver, we use as vertices the numbers $1, 2, \ldots, s$ and we draw an arrow $i \to j$ provided $j \equiv i + p_i \mod s$. For the convenience of the reader, the black vertices will be encircled and the arrows which start at a red vertex will be dotted. Thus, the resolution quiver for the algebra with Kupisch series $(5, 4, 4)$ or $(8, 7, 7)$ (and so on) looks as follows:



The bound $2s - 2$ in Proposition 6 is optimal as the algebras with Kupisch series

$$(ms + 1, ms + 1, \ldots, ms + 1, ms)$$

and $m \in \mathbb{N}_1$ show. Namely, the resolution quiver looks as follows

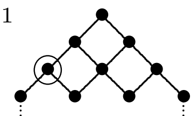
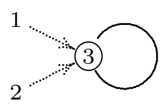
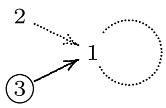
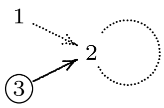
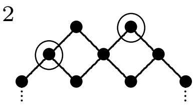
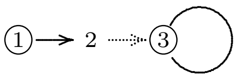
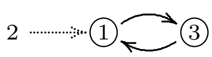
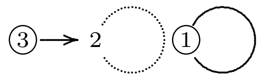
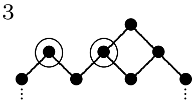
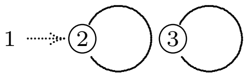
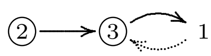
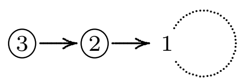


(since $\gamma s = s + p_s = s + ms \equiv s \mod s$, whereas $\gamma i = i + p_i = i + ms + 1 \equiv i + 1$ for $1 \le i \le s - 1$). The vertex $s - 1$ is red, all others are black. In particular, the loop at the vertex s is black. Thus, according to Lemma 7, the path $1 \to 2 \to \cdots \to s-1 \to s$ shows that the G-dimension of $H(s)$ is equal to $2s - 2$.

Examples. We present the different types of the connected Nakayama algebras with $s = 3$ and $s = 4$ (and $p > s$), but omit the self-injective ones. Note that for $s = 3$, this classification can be found already in the paper [CY] by Chen and Ye.

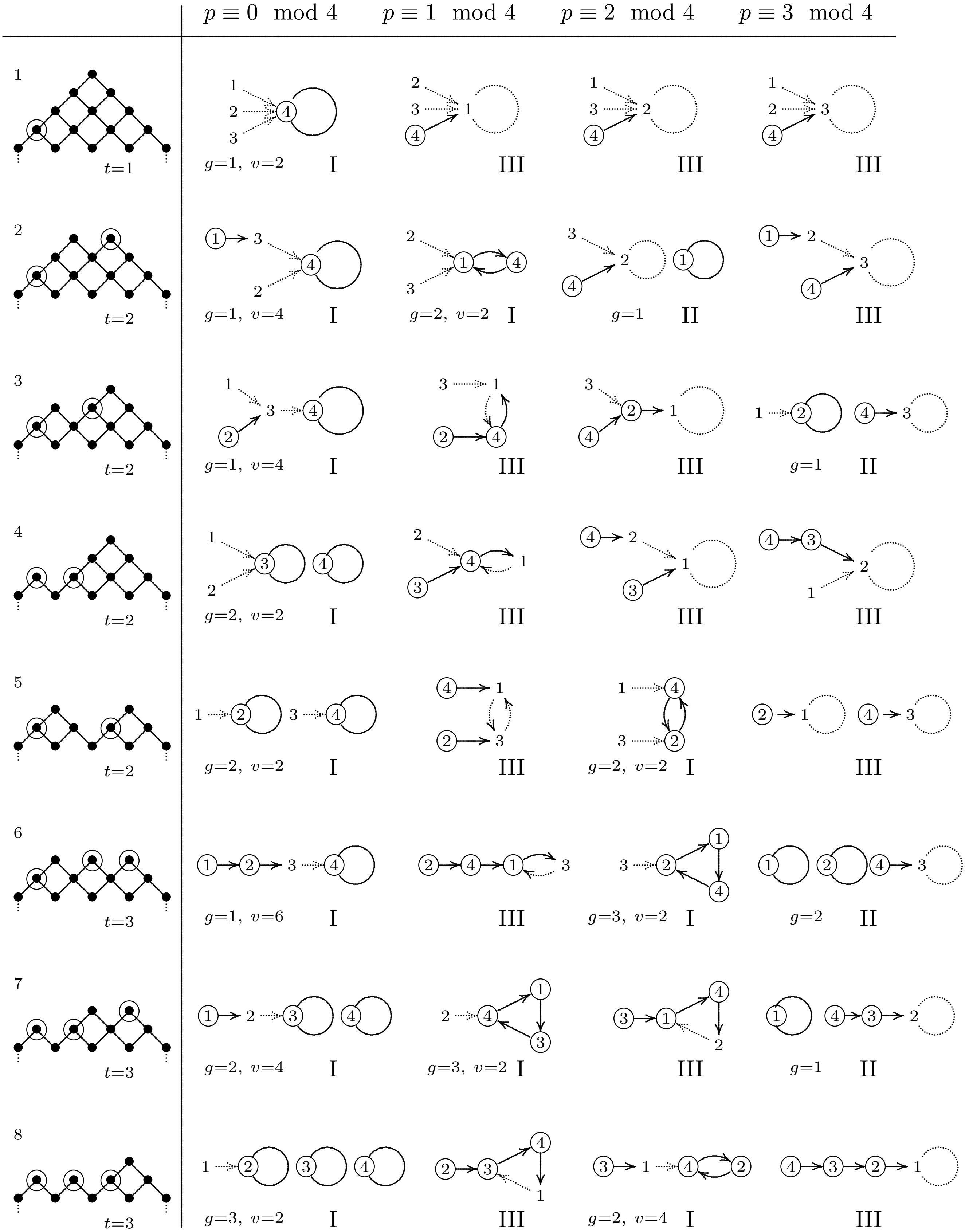
In both tables, we show on the left the different roofs. The remaining columns are indexed by the numbers a with $0 \le a \le s - 1$ and show the resolution quiver of the Nakayama algebras with given roof and $p \equiv a \mod s$.

The cases $s = 3$.

	$p \equiv 0 \mod 3$	$p \equiv 1 \mod 3$	$p \equiv 2 \mod 3$
1  $t{=}1$	 $g{=}1,\ v{=}2$ I	 III	 III
2  $t{=}2$	 $g{=}1,\ v{=}4$ I	 $g{=}2,\ v{=}2$ I	 $g{=}1$ II
3  $t{=}2$	 $g{=}2,\ v{=}2$ I	 III	 III

In addition of exhibiting the resolution quiver, we mention the type in question: type I means that we deal with a Gorenstein algebra (all cycles are black); the type III algebras are the CM-free algebras (no cycle is black); type II means that the algebra is not Gorenstein, but still there are non-projective Gorenstein projective modules (there are cycles which are black as well as cycles which are not black). In the left column, we mention the number t of minimal projective modules. For the algebras of type I and II, we add the number g of elementary Gorenstein projective modules, for those of type I (the Gorenstein algebras), the number v is the Gorenstein dimension (if d is the maximal distance between a vertex and a cyclically black vertex, then $v = 2d$).

The cases $s = 4$.



p ≡ 0 mod 4
p ≡ 1 mod 4
p ≡ 2 mod 4
p ≡ 3 mod 4
1
t=1
g=1, v=2 I
III
III
III
2
t=2
g=1, v=4 I
g=2, v=2 I
g=1 II
III
3
t=2
g=1, v=4 I
III
III
g=1 II
4
t=2
g=2, v=2 I
III
III
III
5
t=2
g=2, v=2 I
III
g=2, v=2 I
III
6
t=3
g=1, v=6 I
III
g=3, v=2 I
g=2 II
7
t=3
g=2, v=4 I
g=3, v=2 I
III
g=1 II
8
t=3
g=3, v=2 I
III
g=2, v=4 I
III

7. Some classes of Nakayama algebras.

Proposition 7. *Let $p \equiv 0 \mod s$. Then all the cycles in R are black loops. Thus Λ is a Gorenstein algebra and the indecomposable modules M in gp_0 are those with a minimal projective presentation $P' \to P \to M \to 0$ where both modules P, P' are of length p.*

Proof. We assume that $p_s = p$. Let i be a vertex. If $p_i = p$, then $\gamma i = i$, thus there is a black loop at the vertex i. Now assume that $p_i > p$. We claim that $i < \gamma i \le s$. Let U be the submodule of $P(i)$ of length p. Since p is a multiple of s, we see that U is an iterated self-extension of $H(\gamma i)$. In particular, the top of U is equal to the top of $H(\gamma i)$, thus equal to $S(\gamma i)$. On the other hand, looking at the roof of Λ, we see that the projective cover of U is equal to $P(j)$ with $i < j \le s$. Thus, we see that $i < \gamma j \le s$.

It follows that all paths in R are strictly increasing $i < \gamma i < \cdots < \gamma^s i$ until we reach a vertex $j = \gamma^s i$ with $p_j = p$, thus until we reach a black loop.

Thus we see that the cyclically black vertices are those simple modules S with $P(S)$ of length p.

Proposition 8. *If $p \equiv -1 \mod s$, then Λ is not Gorenstein.*

Proof. We show that the resolution quiver has a red loop. By assumption, the minimal possible length of an indecomposable projective module is $a \equiv -1 \mod s$. Since Λ is not self-injective, not all the indecomposable projective modules have the same length, thus we can assume that the Kupisch series ends with $(\dots, a+1, a)$. Thus $|P(s-1)| = a+1 \equiv 0 \mod s$. The inclusion $P(s) \subset P(s-1)$ shows that $P(s-1)$ is not minimal projective, thus $S(s-1)$ is a red vertex. On the other hand, since $|P(s)|$ is divisible by s, we have an exact sequence

$$0 \to H(s) \to P(s) \to P(s) \to H(s) \to 0.$$

It follows that the minimal projective resolution of $H(s)$ is of the form

$$\cdots \to P(s) \to \cdots \to P(s) \to P(s) \to H(s) \to 0,$$

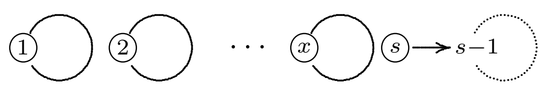
and, as we know, $P(s)$ is not minimal projective. Thus the G-dimension of $H(s)$ is infinite.

Proposition 7 asserts that there are Nakayama algebras with infinite global dimension, with $s \ge 2$ being arbitrarily large, which are not self-injective, but are Gorenstein algebras (type I algebras).

Similarly, all the Nakayama algebras with Kupisch series

$$(ms, ms, \dots, ms, ms-1)$$

and $m \ge 2$ are algebras of type II (neither Gorenstein algebras nor CM-free). Namely, the resolution quiver of such an algebra is of the form

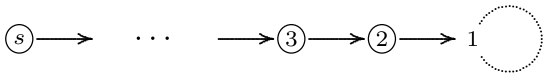


(with $x = s-2$); here we have $s-2$ black loops and one red loop.

Finally, the Nakayama algebras Λ with Kupisch series

$$(ms, ms-1, \ldots, ms-1)$$

and $m \geq 2$ are all of type III (they have infinite global dimension and are CM-free). Namely, they have resolution quivers of the form



There is just one cycle, namely a red loop.

8. References.

[ARS] M. Auslander, I. Reiten, S. Smalø. Representation Theory of Artin Algebras. Cambridge University Press (1997)

[C] X-W. Chen. Algebras with radical square zero are either self-injective of CM-free. Proc. Amer. Math. Soc. (to appear).

[CY] X.-W. Chen, Yu Ye. Retractions and Gorenstein homological properties. arXiv:1206.4415

[EJ] E. E. Enochs, O. M. G. Jenda. Relative homological algebra. De Gruyter Expo. Math. 30 (2000).

[G] W. H. Gustafson, Global dimension in serial rings, J. Algebra 97 (1985), 14-16.

[H] Hoshino, M.: Algebras of finite self-injective dimension. Proc. Amer. Math. Soc. 112 (1991), 619-622.

[R] Ringel: Representations of K-species and bimodules. J. Algebra 41 (1976), 269-302.

[RX] C. M. Ringel, B.-L. Xiong. On rings with radical square zero. arXiv:1112.1422

C. M. Ringel
Fakultät für Mathematik,
Universität Bielefeld
PO Box 100 131
D-33501 Bielefeld, Germany
e-mail: ringel@math.uni-bielefeld.de