On the noise-induced passage through an unstable periodic orbit II: General case

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Abstract

Consider a dynamical system given by a planar differential equation, which exhibits an unstable periodic orbit surrounding a stable periodic orbit. It is known that under random perturbations, the distribution of locations where the system's first exit from the interior of the unstable orbit occurs, typically displays the phenomenon of cycling: The distribution of first-exit locations is translated along the unstable periodic orbit proportionally to the logarithm of the noise intensity as the noise intensity goes to zero. We show that for a large class of such systems, the cycling profile is given, up to a model-dependent change of coordinates, by a universal function given by a periodicised Gumbel distribution. Our techniques combine action-functional or large-deviation results with properties of random Poincaré maps described by continuous-space discrete-time Markov chains.

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1 Introduction

Many interesting effects of noise on deterministic dynamical systems can be expressed as a stochastic exit problem. Given a subset $\mathcal{D}$ of phase space, usually assumed to be positively invariant under the deterministic flow, the stochastic exit problem consists in determining when and where the noise causes solutions to leave $\mathcal{D}$.

If the deterministic flow points inward $\mathcal{D}$ on the boundary $\partial\mathcal{D}$, then the theory of large deviations provides useful answers to the exit problem in the limit of small noise intensity [FW98]. Typically, the exit locations are concentrated in one or several points, in which the so-called quasipotential is minimal. The mean exit time is exponentially long as a function of the noise intensity, and the distribution of exit times is asymptotically exponential [Day83].

The situation is more complicated when $\partial\mathcal{D}$, or some part of it, is invariant under the deterministic flow. Then the theory of large deviations does not suffice to characterise

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the distribution of exit locations. An important particular case is the one of a two-dimensional deterministic ordinary differential equation (ODE), admitting an unstable periodic orbit. Let $\mathcal{D}$ be the part of the plane inside the periodic orbit. Day [Day90a, Day90b] discovered a striking phenomenon called *cycling*: As the noise intensity σ goes to zero, the exit distribution rotates around the boundary $\partial\mathcal{D}$, by an angle proportional to $|\log\sigma|$. Thus the exit distribution does not converge as $\sigma \to 0$. The phenomenon of cycling has been further analysed in several works by Day [Day92, Day94, Day96], by Maier and Stein [MS96, MS97], and by Getfert and Reimann [GR09, GR10].

The noise-induced exit through an unstable periodic orbit has many important applications. For instance, in synchronisation it determines the distribution of noise-induced phase slips [PRK01]. The first-exit distribution also determines the residence-time distribution in stochastic resonance [GHJM98, MS01, BG05]. In neuroscience, the interspike interval statistics of spiking neurons is described by a stochastic exit problem [Tuc75, Tuc89, BG09, BL12]. In certain cases, as for the Morris–Lecar model [ML81] for a region of parameter values, the spiking mechanism involves the passage through an unstable periodic orbit (see, e.g. [RE89, TP04, TKY$^+$06, DG12]). In all these cases, it is important to know the distribution of first-exit locations as precisely as possible.

In [BG04], we introduced a simplified model, consisting of two linearised systems patched together by a switching mechanism, for which we obtained an explicit expression for the exit distribution. In appropriate coordinates, the distribution has the form of a periodicised Gumbel distribution, which is common in extreme-value theory. Note that the standard Gumbel distribution also occurs in the description of reaction paths for overdamped Langevin dynamics [CGLM12]. The aim of the present work is to generalise the results of [BG04] to a larger class of more realistic systems. Two important ingredients of the analysis are large-deviation estimates near the unstable periodic orbit, and the theory of continuous-space Markov chains describing random Poincaré maps.

The remainder of this paper is organised as follows. In Section 2, we define the system under study, discuss the heuristics of its behaviour, and state the main result (Theorem 2.3). Subsequent sections are devoted to the proof of this result. Section 3 describes a coordinate transformation to polar-type coordinates used throughout the analysis. Section 4 contains the large-deviation estimates for the dynamics near the unstable orbit. Section 5 states results on Markov chains and random Poincaré maps, while Section 6 contains estimates on the sample-path behaviour needed to apply the results on Markov chains. Finally, in Section 7 we complete the proof of Theorem 2.3.

2 Results

2.1 Stochastic differential equations with an unstable periodic orbit

Consider the two-dimensional deterministic ODE

$$\dot{z} = f(z) \,, \tag{2.1}$$

where $f \in \mathcal{C}^2(\mathcal{D}_0, \mathbb{R}^2)$ for some open, connected set $\mathcal{D}_0 \subset \mathbb{R}^2$. We assume that this system admits two distinct periodic orbits, that is, there are periodic functions $\gamma_\pm : \mathbb{R} \to \mathbb{R}^2$, of respective periods $T_\pm$, such that

$$\dot{\gamma}_\pm(t) = f(\gamma_\pm(t)) \qquad\qquad \forall t \in \mathbb{R} \ . \tag{2.2}$$

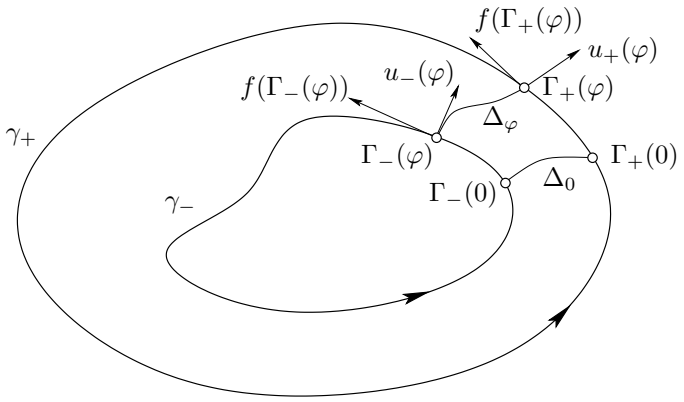

FIGURE 1. Geometry of the periodic orbits. The stable orbit γ_- is located inside the unstable orbit γ_+. $\Gamma_\pm(\varphi)$ denote parametrisations of the orbits, and $u_\pm(\varphi)$ are eigenvectors of the monodromy matrix used to construct a set of polar-type coordinates.

We set $\Gamma_\pm(\varphi) = \gamma_\pm(T_\pm\varphi)$, so that $\varphi \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ gives an equal-time parametrisation of the orbits. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\Gamma_\pm(\varphi) = T_\pm f(\Gamma_\pm(\varphi)) \,, \tag{2.3}$$

and thus $\dot\varphi = 1/T_\pm$ is constant on the periodic orbits.

Concerning the geometry, we will assume that the orbit Γ_- is contained in the interior of Γ_+, and that the annulus-shaped region $\mathcal{S}$ between the two orbits contains no invariant proper subset. This implies in particular that the orbit through any point in $\mathcal{S}$ approaches one of the orbits $\Gamma_\pm$ as $t \to \infty$ and $t \to -\infty$.

Let $A_\pm(\varphi) = \partial_z f(\Gamma_\pm(\varphi))$ denote the Jacobian matrices of f at $\Gamma_\pm(\varphi)$. The principal solutions associated with the linearisation around the periodic orbits are defined by

$$\partial_\varphi U_\pm(\varphi,\varphi_0) = T_\pm A_\pm(\varphi) U_\pm(\varphi,\varphi_0) \,, \qquad\qquad U_\pm(\varphi_0,\varphi_0) = \mathbb{1} \,. \tag{2.4}$$

In particular, the monodromy matrices $U_\pm(\varphi+1,\varphi)$ satisfy

$$\det U_\pm(\varphi+1,\varphi) = \exp\biggl\{ T_\pm \int_\varphi^{\varphi+1} \mathrm{Tr}\, A_\pm(\varphi')\, \mathrm{d}\varphi' \biggr\} \,, \tag{2.5}$$

with $\mathrm{Tr}\, A_\pm(\varphi') = \mathrm{div}\, f(\Gamma_\pm(\varphi'))$. Taking the derivative of (2.3) shows that each monodromy matrix $U_\pm(\varphi+1,\varphi)$ admits $f(\Gamma_\pm(\varphi))$ as eigenvector with eigenvalue 1. The other eigenvalue is thus also independent of φ, and we denote it $\mathrm{e}^{\pm\lambda_\pm T_\pm}$, where

$$\pm\lambda_\pm = \int_0^1 \mathrm{div}\, f(\Gamma_\pm(\varphi))\, \mathrm{d}\varphi \tag{2.6}$$

are the Lyapunov exponents of the orbits. We assume that λ_+ and λ_- are both positive, which implies that Γ_- is stable and Γ_+ is unstable.

Consider now the stochastic differential equation (SDE)

$$\mathrm{d}z_t = f(z_t)\, \mathrm{d}t + \sigma g(z_t)\, \mathrm{d}W_t \,, \tag{2.7}$$

where f satisfies the same assumptions as before, $\{W_t\}_t$ is a k-dimensional standard Brownian motion, $k \geqslant 2$, and $g \in \mathcal{C}^1(\mathcal{D}_0, \mathbb{R}^{2\times k})$ satisfies the uniform ellipticity condition

$$c_1\|\xi^2\| \leqslant \langle \xi, g(z)g(z)^T\xi\rangle \leqslant c_2\|\xi^2\| \qquad \forall z \in \mathcal{D}_0\ \forall \xi \in \mathbb{R}^2 \tag{2.8}$$

with $c_2 \geqslant c_1 > 0$.

Proposition 2.1 (Polar-type coordinates)**.** *There exist $L > 1$ and a set of coordinates $(r, \varphi) \in (-L, L) \times \mathbb{R}$, in which the SDE (2.7) takes the form*

$$\begin{aligned} \mathrm{d}r_t &= f_r(r_t, \varphi_t; \sigma)\ \mathrm{d}t + \sigma g_r(r_t, \varphi_t)\, \mathrm{d}W_t\ , \\ \mathrm{d}\varphi_t &= f_\varphi(r_t, \varphi_t; \sigma)\ \mathrm{d}t + \sigma g_\varphi(r_t, \varphi_t)\, \mathrm{d}W_t\ . \end{aligned} \tag{2.9}$$

The functions f_r, f_φ, g_r and g_φ are periodic with period 1 in φ, and g_r, g_φ satisfy a uniform ellipticity condition similar to (2.8). The unstable orbit lies in $r = 1 + \mathcal{O}(\sigma^2)$, and

$$\begin{aligned} f_r(r, \varphi) &= \lambda_+(r-1) + \mathcal{O}((r-1)^2)\ , \\ f_\varphi(r, \varphi) &= \frac{1}{T_+} + \mathcal{O}((r-1)^2) \end{aligned} \tag{2.10}$$

as $r \to 1$. The stable orbit lies in $r = -1 + \mathcal{O}(\sigma^2)$, and

$$\begin{aligned} f_r(r, \varphi) &= -\lambda_-(r+1) + \mathcal{O}((r+1)^2)\ , \\ f_\varphi(r, \varphi) &= \frac{1}{T_-} + \mathcal{O}((r+1)^2) \end{aligned} \tag{2.11}$$

as $r \to -1$. Furthermore, f_φ is strictly larger than a positive constant for all $(r, \varphi) \in (-L, L) \times \mathbb{R}$, and f_r is negative for $-1 < r < 1$.

We give the proof in Section 3. We emphasize that after performing this change of coordinates, the stable and unstable orbit are not located exactly in $r = \pm 1$, but are slightly shifted by an amount of order σ^2, owing to second-order terms in Itô's formula.

The question we are interested in is the following: Assume the system starts with some initial condition $(r_0, \varphi_0) = (r_0, 0)$ close to the stable periodic orbit. What is the distribution of the first-hitting location of the unstable orbit? We define the first-hitting time of (a σ^2-neighbourhood of) the unstable orbit by

$$\tau = \inf\bigl\{t > 0 \colon r_t = 1\bigr\}\ , \tag{2.12}$$

so that the random variable φ_τ gives the first-exit location. Note that we consider φ as belonging to $\mathbb{R}_+$ instead of the circle $\mathbb{R}/\mathbb{Z}$, which means that we keep track of the number of rotations around the periodic orbits.

2.2 Heuristics 1: Large deviations

A first key ingredient to the understanding of the distribution of exit locations is the theory of large deviations, which has been developed in the context of SDEs by Freidlin and Wentzell [FW98]. The theory tells us that for a set Γ of paths $\gamma : [0, T] \to \mathbb{R}^2$, one has

$$-\inf_{\Gamma^\circ} I \leqslant \liminf_{\sigma \to 0} \sigma^2 \log \mathbb{P}\bigl\{(z_t)_{t\in[0,T]} \in \Gamma\bigr\} \leqslant \limsup_{\sigma \to 0} \sigma^2 \log \mathbb{P}\bigl\{(z_t)_{t\in[0,T]} \in \Gamma\bigr\} \leqslant -\inf_{\overline{\Gamma}} I\ , \tag{2.13}$$

where the *rate function* $I = I_{[0,T]} : \mathcal{C}^0([0, T], \mathbb{R}^2) \to \mathbb{R}_+$ is given by

$$I(\gamma) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^T (\dot\gamma_s - f(\gamma_s))^{\mathrm{T}} D(\gamma_s)^{-1} (\dot\gamma_s - f(\gamma_s))\ \mathrm{d}s & \text{if } \gamma \in H^1, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.14}$$

with $D(z) = g(z)g(z)^{\mathrm{T}}$ (the *diffusion matrix*). Roughly speaking, Equation (2.13) tells us that

$$\mathbb{P}\big\{(z_t)_{t\in[0,T]} \in \Gamma\big\} \simeq \mathrm{e}^{-\inf_\Gamma I/\sigma^2} \qquad \text{or, symbolically,} \quad \mathbb{P}\big\{(z_t)_{t\in[0,T]} = \gamma\big\} \simeq \mathrm{e}^{-I(\gamma)/\sigma^2} \ . \tag{2.15}$$

For deterministic solutions, we have $\dot\gamma = f(\gamma)$ and $I(\gamma) = 0$, so that (2.15) does not yield useful information. However, for paths γ with $I(\gamma) > 0$, (2.15) tells us how unlikely γ is.

The minimisers of I obey Euler–Lagrange equations, which are equivalent to Hamilton equations generated by the Hamiltonian

$$H(\gamma,\psi) = \frac{1}{2}\psi^{\mathrm{T}} D(\gamma)\psi + f(\gamma)^{\mathrm{T}}\psi \ , \tag{2.16}$$

where $\psi = D(\gamma)^{-1}(\dot\gamma - f(\gamma))$ is the moment conjugated to γ. The rate function thus takes the form

$$I(\gamma) = \frac{1}{2}\int_0^T \psi_s^{\mathrm{T}} D(\gamma_s)\psi_s \ \mathrm{d}s \ . \tag{2.17}$$

Writing $\psi^T = (p_r, p_\varphi)$, the Hamilton equations associated with (2.16) read

$$\begin{aligned}
\dot r &= f_r(r,\varphi) + D_{rr}(r,\varphi)p_r + D_{r\varphi}(r,\varphi)p_\varphi \ , \\
\dot\varphi &= f_\varphi(r,\varphi) + D_{r\varphi}(r,\varphi)p_r + D_{\varphi\varphi}(r,\varphi)p_\varphi \ , \\
\dot p_r &= -\partial_r f_r(r,\varphi)p_r - \partial_r f_\varphi(r,\varphi)p_\varphi - \frac{1}{2}\sum_{ij\in\{r,\varphi\}} \partial_r D_{ij}(r,\varphi)p_i p_j \ , \\
\dot p_\varphi &= -\partial_\varphi f_r(r,\varphi)p_r - \partial_\varphi f_\varphi(r,\varphi)p_\varphi - \frac{1}{2}\sum_{ij\in\{r,\varphi\}} \partial_\varphi D_{ij}(r,\varphi)p_i p_j \ ,
\end{aligned} \tag{2.18}$$

We can immediately note the following points:

- the plane $p_r = p_\varphi = 0$ is invariant, it corresponds to the deterministic dynamics;
- there are two periodic orbits, given by $p_r = p_\varphi = 0$ and $r = \pm 1$, which are, of course, the original periodic orbits of the deterministic system;
- $\dot\varphi$ is positive, bounded away from zero, in a neighbourhood of the deterministic manifold.

The Hamiltonian being a constant of the motion, the four-dimensional phase space is foliated in three-dimensional invariant manifolds, which can be labelled by the value of H. Since $\partial_{p_r} H = \dot r$ is positive near the deterministic manifold, one can express p_φ as a function of H, r, φ and p_r, and thus describe the dynamics on each invariant manifold by an effective three-dimensional equation for (r, φ, p_r). It is furthermore possible to use φ as new time, which yields a two-dimensional, non-autonomous equation.[1]

The linearisation of the system around the periodic orbits is given by

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\begin{pmatrix} r \\ p_r \end{pmatrix} = \begin{pmatrix} \pm\lambda_\pm T_\pm & D_{rr}(\pm 1,\varphi) \\ 0 & \mp\lambda_\pm T_\pm \end{pmatrix}\begin{pmatrix} r \\ p_r \end{pmatrix} \ . \tag{2.19}$$

The characteristic exponents of the periodic orbit in $r = 1$ are thus $\pm T_+\lambda_+$, and those of the periodic orbit in $r = -1$ are $\pm T_-\lambda_-$. The Poincaré section at $\varphi = 0$ will thus have hyperbolic fixed points at $(r, p_r) = (\pm 1, 0)$.

[1] The associated Hamiltonian is the function $P_\varphi(r, p_r, H, \varphi)$ obtained by expressing p_φ as a function of the other variables.

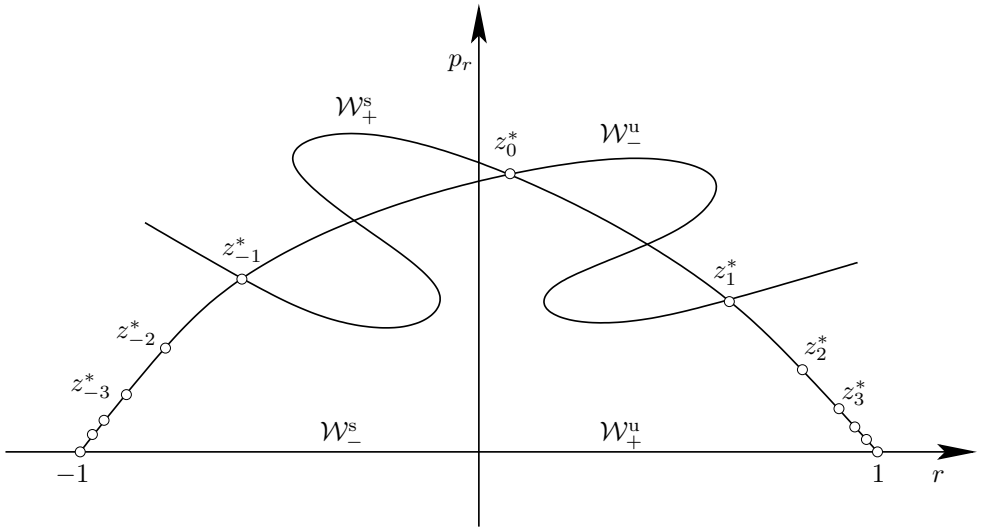

FIGURE 2. Poincaré section of the Hamiltonian flow associated with the large-deviation rate function. The stable periodic orbit is located in $(-1,0)$, the unstable one in $(1,0)$. We assume that the unstable manifold $\mathcal{W}^{\mathrm{u}}_-$ of $(-1,0)$ intersects the stable manifold $\mathcal{W}^{\mathrm{s}}_+$ of $(1,0)$ transversally. The intersections of both manifolds define a heteroclinic orbit $\{z^*_k\}_{-\infty<k<\infty}$ which corresponds to the minimiser of the rate function.

Consider now the event Γ that the stochastic system, starting on the stable orbit at $\varphi = 0$, hits the unstable orbit for the first time near $\varphi = s$. The probability of Γ will be determined by the infimum of the rate function I over all paths connecting $(r,\varphi) = (-1,0)$ to $(r,\varphi) = (1,s)$. Note however that if $t > s$, we can connect $(1,s)$ to $(1,t)$ for free in terms of the rate function I by following the deterministic dynamics along the unstable orbit. We conclude that *on the level of large deviations, all exit points on the unstable orbit are equally likely.*

This does not mean, however, that all paths connecting the stable and unstable orbits are optimal. In fact, it turns out that the infimum of the rate function is reached on a heteroclinic orbit connecting the orbits in infinite time. It is possible to connect the orbits in finite time, at the cost of increasing the rate function. In what follows, we will make the following simplifying assumption.

Assumption 2.2. In the Poincaré section for $H = 0$, the unstable manifold $\mathcal{W}^{\mathrm{u}}_-$ of $(-1,0)$ intersects the stable manifold $\mathcal{W}^{\mathrm{s}}_+$ of $(1,0)$ transversally (Figure 2). Let γ_∞ denote the heteroclinic orbit meeting the Poincaré section at the set $\{z^*_k\}_{-\infty<k<\infty}$ of intersections of the manifolds. Then γ_∞ minimises the rate function over all paths connecting the two periodic orbits, and this minimiser is unique (up to translations $\varphi \mapsto \varphi + 1$).

This assumption obviously fails to hold if the system is perfectly rotation symmetric, because then the two manifolds do not intersect transversally but are in fact identical. The assumption is likely to be true *generically* for small-amplitude perturbations of φ-independent systems (cf. Melnikov's method), for large periods $T_\pm$ (adiabatic limit) and for small periods (averaging regime), but may not hold in general. See in particular [GT84, GT85, MS97] for discussions of possible complications.

It will turn out in our analysis that the probability of crossing the unstable orbit near a sufficiently large finite value of φ will be determined by a finite number $n = \lfloor \varphi \rfloor$ of translates of the minimising orbit.

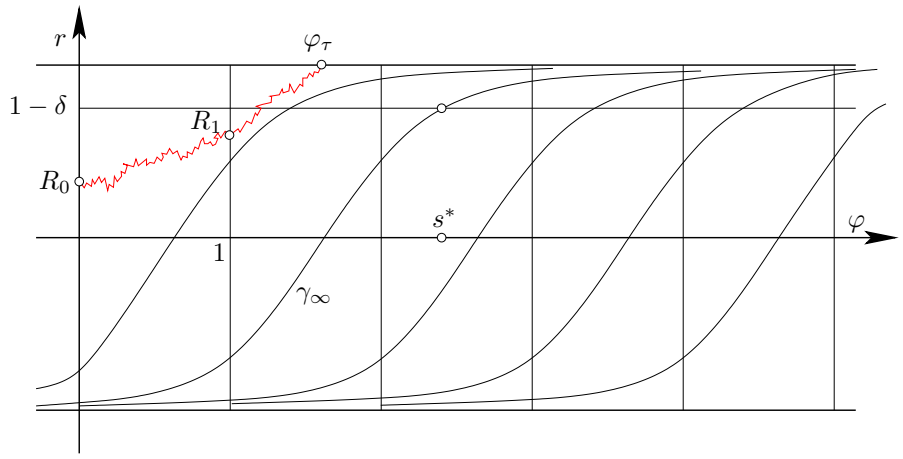

FIGURE 3. The optimal path γ_∞ minimising the rate function, and its translates. We define a random Poincaré map, giving the location R_1 of the first crossing of the line $\varphi = 1$ of a path starting in $r = R_0$ and $\varphi = 0$.

2.3 Heuristics 2: Random Poincaré maps

The second key ingredient of our analysis are Markov chains describing Poincaré maps of the stochastic system. Choose an initial condition $(R_0, 0)$, and consider the value $R_1 = r_{\tau_1}$ of r at the time

$$\tau_1 = \inf\{t > 0 \colon \varphi_t = 1\} \,, \tag{2.20}$$

when the sample path first reaches $\varphi = 1$ (Figure 3). Since we are interested in the first-passage time through the unstable orbit, we declare that whenever the sample path (r_t, φ_t) reaches $r = 1$ before $\varphi = 1$, then R_1 has reached a cemetery state ∂, which it never leaves again. Successively, we define $R_n = r_{\tau_n}$, where $\tau_n = \inf\{t > 0 \colon \varphi_t = n\}$, $n \in \mathbb{N}$.

By periodicity of the system in φ and the strong Markov property, the sequence $(R_0, R_1, \dots)$ forms a Markov chain, with kernel K, that is, for a Borel set A,

$$\mathbb{P}^{R_n}\big\{R_{n+1} \in A\big\} = K(R_n, A) = \int_A K(R_n, \mathrm{d}y) \qquad \text{for all } n \geqslant 0 \,, \tag{2.21}$$

where $\mathbb{P}^x\big\{R_n \in A\big\}$ denotes the probability that the Markov chain, starting in x, is in A at time n.

Results on harmonic measures [BAKS84] imply that $K(x, \mathrm{d}y)$ actually has a density $k(x, y)$ with respect to Lebesgue measure (see also [Dah77, JK82, CZ87] for related results). Thus the density of R_n evolves according to an integral operator with kernel k. Such operators have been studied, among others, by Fredholm [Fre03], Jentzsch [Jen12] and Birkhoff [Bir57]. In particular, we know that k has a discrete set of eigenvalues $\lambda_0, \lambda_1, \dots$ of finite multiplicity, where λ_0 is simple, real, positive, and larger than the modules of all other eigenvalues. It is called the *principal eigenvalue* of the Markov chain. In our case, we have $\lambda_0 < 1$ due to the killing at the unstable orbit.

Fredholm theory yields a decomposition[2]

$$k(x, y) = \lambda_0 h_0(x) h_0^*(y) + \lambda_1 h_1(x) h_1^*(y) + \dots \tag{2.22}$$

[2]If λ_1 has multiplicity $m > 1$, the second term in (2.22) has to be replaced by a sum with m terms.

where the h_i and h_i^* are right and left orthonormal eigenfunctions of the integral operator. It follows that

$$\mathbb{P}^{R_0}\big\{R_n \in A\big\} =: K_n(R_0, A) = \lambda_0^n h_0(R_0) \int_A h_0^*(y)\,\mathrm{d}y \Big[1 + \mathcal{O}\Big(\Big(\frac{|\lambda_1|}{\lambda_0}\Big)^n\Big)\Big] \,. \tag{2.23}$$

Thus the *spectral gap* $\lambda_0 - |\lambda_1|$ plays an important role in the convergence of the distribution of R_n. For times n satisfying $n \gg (\log(\lambda_0/|\lambda_1|))^{-1}$, the distribution of R_n will have a density proportional to h_0^*. More precisely, if

$$\pi_0(\mathrm{d}x) = \frac{h_0^*(x)\,\mathrm{d}x}{\int h_0^*(y)\,\mathrm{d}y} \tag{2.24}$$

is the so-called *quasistationary distribution*, then the asymptotic distribution of the process R_n, conditioned on survival, will be π_0, while the survival probability decays like λ_0^n.

Furthermore, the (sub-)probability density of the first-exit location φ_τ at $n + s$, with $n \in \mathbb{N}$ and $s \in [0, 1)$, can be written as

$$\int K_n(R_0, \mathrm{d}y)\mathbb{P}^y\big\{\varphi_\tau \in \mathrm{d}s\big\} = \lambda_0^n h_0(R_0) \int h_0^*(y)\mathbb{P}^y\big\{\varphi_\tau \in \mathrm{d}s\big\}\,\mathrm{d}y \Big[1 + \mathcal{O}\Big(\Big(\frac{|\lambda_1|}{\lambda_0}\Big)^n\Big)\Big] \,. \tag{2.25}$$

This shows that the distribution of the exit location is asymptotically equal to a periodically modulated exponential distribution. Note that the integral appearing in (2.25) is proportional to the expectation of φ_τ when starting in the quasistationary distribution.

In order to combine the ideas based on Markov chains and on large deviations, we will rely on the approach first used in [BG04], and decompose the dynamics into two subchains, the first one representing the dynamics away from the unstable orbit, and the second one representing the dynamics near the unstable orbit. We consider:

1. A chain for the process killed upon reaching, at time τ_-, a level $1 - \delta$ below the unstable periodic orbit. We denote its kernel K^{s}. By Assumption 2.2, the first-hitting location φ_{τ_-} will be concentrated near places $s^* + n$ where a translate $\gamma_\infty(\cdot + n)$ of the minimiser γ_∞ crosses the level $1 - \delta$. We will establish a spectral-gap estimate for K^{s} (see Theorem 6.12), showing that φ_{τ_-} indeed follows a periodically modulated exponential of the form

$$\mathbb{P}^0\big\{\varphi_{\tau_-} \in [\varphi_1, \varphi_1 + \Delta]\big\} \simeq (\lambda_0^{\mathrm{s}})^{\varphi_1}\, \mathrm{e}^{-J(\varphi_1)/\sigma^2} \,, \tag{2.26}$$

   where J is periodic and minimal in points of the form $s^* + n$.
2. A chain for the process killed upon reaching either the unstable periodic orbit at $r = 1$, or a level $1 - 2\delta$, with kernel K^{u}. We show in Theorem 6.7 that its principal eigenvalue is of the form

$$\lambda_0^{\mathrm{u}} = \mathrm{e}^{-2\lambda_+ T_+}(1 + \mathcal{O}(\delta)) \,. \tag{2.27}$$

   Together with a large-deviation estimate, this yields a rather precise description of the distribution of φ_τ, given the value of φ_{τ_-}, of the form

$$\mathbb{P}^{\varphi_{\tau_-}}\big\{\varphi_\tau \in [\varphi, \varphi + \Delta]\big\} \simeq \mathrm{e}^{-2\lambda_+ T_+(\varphi - \varphi_{\tau_-})} \exp\Big\{-\frac{1}{\sigma^2}\Big[I_\infty + \mathcal{O}(\mathrm{e}^{-2\lambda_+ T_+(\varphi - \varphi_{\tau_-})})\Big]\Big\} \,, \tag{2.28}$$

   where I_∞ is again related to the rate function, and the term $\mathcal{O}(\mathrm{e}^{-2\lambda_+ T_+(\varphi - \varphi_{\tau_-})})$ can be computed explicitly to leading order. The double-exponential dependence of (2.28) on $2\lambda_+ T_+(\varphi - \varphi_{\tau_-})$ is in fact what characterises the Gumbel distribution.

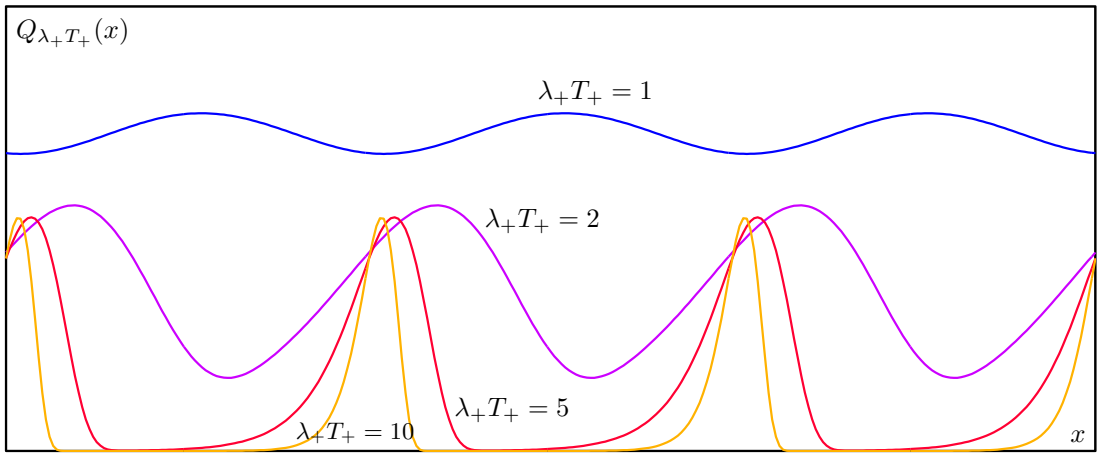

FIGURE 4. The cycling profile $x \mapsto Q_{\lambda_+ T_+}(x)$ for different values of the parameter $\lambda_+ T_+$, shown for $x \in [0, 3]$.

By combining the two above steps, we obtain that the first-exit distribution is given by a sum of shifted Gumbel distributions, in which each term is associated with a translate of the optimal path γ_∞.

2.4 Main result: Cycling

In order to formulate the main result, we introduce the notation

$$h^{\mathrm{per}}(\varphi) = \frac{\mathrm{e}^{2\lambda_+ T_+ \varphi}}{1 - \mathrm{e}^{-2\lambda_+ T_+}} \int_\varphi^{\varphi+1} \mathrm{e}^{-2\lambda_+ T_+ u} \, D_{rr}(1, u) \, \mathrm{d}u \tag{2.29}$$

for the periodic solution of the equation $\mathrm{d}h/\,\mathrm{d}\varphi = 2\lambda_+ T_+ h - D_{rr}(1, \varphi)$, where

$$D_{rr}(1, \varphi) = g_r(1, \varphi) g_r(1, \varphi)^T \tag{2.30}$$

measures the strength of diffusion in the direction orthogonal to the periodic orbit.

Theorem 2.3 (Main result)**.** *For any sufficiently small $\delta, \Delta > 0$, there exists a $\sigma_0 > 0$ such that the following holds: For any r_0 sufficiently close to -1 and $\sigma < \sigma_0$,*

$$\begin{aligned} \mathbb{P}^{r_0, 0}\big\{\tau \in [\varphi, \varphi + \Delta]\big\} = C_0(\sigma) (\lambda_0)^\varphi \chi_\Delta(\varphi) Q_{\lambda_+ T_+}\left(\frac{|\log \sigma| - \theta(\varphi) + \mathcal{O}(\delta)}{\lambda_+ T_+} \right) \\ \times \left[1 + \mathcal{O}\Big(\mathrm{e}^{-c\varphi/|\log \sigma|} \Big) + \mathcal{O}(\delta |\log \delta|) \right], \end{aligned} \tag{2.31}$$

where $c > 0$ and we use the following notations:

- *$Q_{\lambda_+ T_+}(x)$ is periodic with period 1 and given be the periodicised Gumbel distribution*

$$Q_{\lambda_+ T_+}(x) = \sum_{n=-\infty}^{\infty} A\big(\lambda_+ T_+ (n - x)\big) \qquad \text{where} \qquad A(x) = \frac{1}{2} \exp\Big\{ -2x - \frac{1}{2} \mathrm{e}^{-2x} \Big\} . \tag{2.32}$$

- *$\theta(\varphi)$ satisfies $\theta(\varphi + 1) = \theta(\varphi) + \lambda_+ T_+$ and is given by*

$$\theta(\varphi) = \lambda_+ T_+ \varphi - \frac{1}{2} \log \left[\frac{1}{2} \delta^2 \frac{h^{\mathrm{per}}(\varphi)}{h^{\mathrm{per}}(s^*)^2} \right], \tag{2.33}$$

where s^* denotes the value of φ where the optimal path γ_∞ crosses the level $1 - \delta$.

- λ_0 *is the principal eigenvalue of the Markov chain, and satisfies*

$$\lambda_0 = 1 - \mathrm{e}^{-H/\sigma^2} \tag{2.34}$$

  where $H > 0$ is close to the value of the rate function $I(\gamma_\infty)$.
- $\chi_\Delta(\varphi)$ *is periodic in φ with period* 1*. It is proportional to $F(\varphi + \Delta) - F(\varphi)$ for a probability distribution function F.*
- *The normalising constant $C_0(\sigma)$ is of order e^{-H/σ^2}.*

The proof is given in Section 7. We now comment the different terms in the expression (2.31) in more detail.

- **Cycling profile:** The function $Q_{\lambda_+ T_+}$ is the announced universal cycling profile. Relation (2.31) shows that the profile is translated along the unstable orbit proportionally to $|\log \sigma|$. The intuition is that this is the time needed for the optimal path γ_∞ to reach a σ-neighbourhood of the unstable orbit where escape becomes likely. For small values of $\lambda_+ T_+$, the cycling profile is rather flat, while it becomes more and more sharply peaked as $\lambda_+ T_+$ increases (Figure 4).
- **Periodic prefactor:** The periodic prefactor $\chi_\Delta(\varphi)$ is related to the subexponential behaviour of the first-exit distribution when starting near the unstable periodic orbit. Its expression contains a factor $F(\varphi + \Delta) - F(\varphi) = \mathbb{P}^{\pi_0^{\mathrm{u}}}\{\varphi_\tau \in [\varphi, \varphi + \Delta]\}$, where the initial distribution π_0^{u} is the quasistationary distribution near the unstable periodic orbit. In the linear case with switching [BG04], it can be computed explicitly by using the reflection principle, with the result

$$\chi_\Delta(\varphi) \approx \theta'(\varphi)\Delta = \frac{D_{rr}(1,\varphi)}{2h^{\mathrm{per}}(\varphi)}\Delta \qquad \text{for small } \Delta\,. \tag{2.35}$$

  We conjecture that this is still the case in the nonlinear situation (up to a multiplicative error of the form $1 + \mathcal{O}(\delta)$). This would imply that $\theta(\varphi)$ is a natural parametrisation of the unstable orbit, in which the first-exit distribution is given by the cycling profile.
- **Principal eigenvalue:** The principal eigenvalue λ_0 determines the slow exponential decay of the first-exit distribution. Writing $(\lambda_0)^\varphi = \mathrm{e}^{-\varphi|\log \lambda_0|}$, we see that the expected first-exit location is of order $1/|\log \lambda_0| \simeq \mathrm{e}^{H/\sigma^2}$. This time plays the same rôle as Kramers's time for gradient systems (see [Eyr35, Kra40] and e.g. [Ber11] for a recent review of Kramers' law). One may obtain sharper bounds on λ_0 using, for instance, the Donsker–Varadhan inequality [DV76].
- **Normalisation:** The prefactor $C_0(\sigma)$ can be estimated using the fact that the first-exit distribution is normalised to 1. It is of the order $|\log \lambda_0| \simeq \mathrm{e}^{-H/\sigma^2}$.
- **Transient behaviour:** The error term $\mathcal{O}(\mathrm{e}^{-c\varphi/|\log \sigma|})$ describes the transient behaviour when not starting in the quasistationary distribution. If the initial condition is concentrated near the stable periodic orbit, we expect the first-exit distribution to be bounded above by the leading term in (2.31) during the transient phase.
- **Dependence on a level δ:** While the left-hand side of (2.31) does not depend on δ and one would like to take the limit $\delta \to 0$ on the right-hand side, this would require also to pass to the limit $\sigma \to 0$ since the maximal value σ_0 depends on δ (as it does depend on Δ).

We conclude with some remarks on applications and possible improvements and extensions of Theorem 2.3.

- **Spectral decomposition:** In the proof presented here, we rely partly on large-deviation estimates, and partly on spectral properties of random Poincaré maps. By obtaining more precise information on the eigenfunctions and eigenvalues of the Markov chain K^{u}, one might be able to obtain the same result without using large deviations. This is the case for the linearised system (see Proposition 6.1), for which one can check that the eigenfunctions are similar to those of the quantum harmonic oscillator (Gaussians multiplied by Hermite polynomials).
- **Residence-time distribution:** Consider the situation where there is a stable periodic orbit surrounding the unstable one. Then sample paths of the system switch back and forth between the two stable orbits, in a way strongly influenced by noise intensity and period of the orbits. The residence-time distribution near each orbit is related to the above first-exit distribution [BG05], and has applications in the quantification of the phenomenon of stochastic resonance.
- **More general geometries:** In a similar spirit, one may ask what happens if the stable periodic orbit is replaced by a stable equilibrium point, or some other attractor. We expect the result to be similar in such a situation, because the presence of the periodic orbit is only felt inasmuch hitting points of the level $1-\delta$ are concentrated within each period.

3 Coordinate systems

3.1 Deterministic system

We start by constructing polar-like coordinates for the deterministic ODE (2.1).

Proposition 3.1. *There is an open subset $\mathcal{D}_1 = (-L, L) \times \mathbb{S}^1$ of the cylinder, with $L > 1$, and a $\mathcal{C}^2$-diffeomorphism $h : \mathcal{D}_1 \to \mathcal{D}_0$ such that (2.1) is equivalent, by the transformation $z = h(r, \varphi)$, to the system*

$$\begin{aligned} \dot{r} &= f_r(r, \varphi) \;, \\ \dot{\varphi} &= f_\varphi(r, \varphi) \;, \end{aligned} \tag{3.1}$$

where $f_r, f_\varphi : \mathcal{D}_1 \to \mathbb{R}$ satisfy

$$\begin{aligned} f_r(r, \varphi) &= \lambda_+(r-1) + \mathcal{O}((r-1)^2) \;, \\ f_\varphi(r, \varphi) &= \frac{1}{T_+} + \mathcal{O}((r-1)^2) \end{aligned} \tag{3.2}$$

as $r \to 1$, and

$$\begin{aligned} f_r(r, \varphi) &= -\lambda_-(r+1) + \mathcal{O}((r+1)^2) \;, \\ f_\varphi(r, \varphi) &= \frac{1}{T_-} + \mathcal{O}((r+1)^2) \end{aligned} \tag{3.3}$$

as $r \to -1$. Furthermore, $f_\varphi(r, \varphi)$ is positive, bounded away from 0, while $f_r(r, \varphi)$ is negative for $|r| < 1$ and positive for $|r| > 1$.

Proof: The construction of h proceeds in several steps. We start by defining h in a neighbourhood of $r = 1$, before extending it to all of $\mathcal{D}_1$.

1. We set $h(1, \varphi) = \Gamma_+(\varphi)$. Hence $f(\Gamma_+(\varphi)) = \partial_r h(1, \varphi)\dot{r} + \Gamma'_+(\varphi)\dot{\varphi}$, so that $\dot{\varphi} = 1/T_+$ and $\dot{r} = 0$ whenever $r = 1$.

2. Let $u_+(0)$ be an eigenvector of the monodromy matrix $U_+(1,0)$ with eigenvalue $\mathrm{e}^{\lambda_+ T_+}$. Then it is easy to check that

$$u_+(\varphi) = \mathrm{e}^{-\lambda_+ T_+ \varphi}\, U_+(\varphi,0) u_+(0) \tag{3.4}$$

is an eigenvector of the monodromy matrix $U_+(\varphi+1,\varphi)$ with same eigenvalue $\mathrm{e}^{\lambda_+ T_+}$, and that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} u_+(\varphi) = T_+ \big[A_+(\varphi) u_+(\varphi) - \lambda_+ u_+(\varphi) \big] \,. \tag{3.5}$$

We now impose that

$$h(r,\varphi) = \Gamma_+(\varphi) + (r-1) u_+(\varphi) + \mathcal{O}((r-1)^2) \tag{3.6}$$

as $r \to 1$. This implies that

$$f(h(r,\varphi)) = f(\Gamma_+(\varphi)) + (r-1) A_+(\varphi) u_+(\varphi) + \mathcal{O}((r-1)^2) \,, \tag{3.7}$$

which must be equal to

$$\begin{aligned} \dot z &= \big[\Gamma_+'(\varphi) + u_+'(\varphi)(r-1) + \mathcal{O}((r-1)^2) \big] \dot\varphi + \big[u_+(\varphi) + \mathcal{O}((r-1)) \big] \dot r \\ &= T_+ \big[f(\Gamma_+(\varphi)) + \big(A_+(\varphi) u_+(\varphi) - \lambda_+ u_+(\varphi) \big)(r-1) + \mathcal{O}((r-1)^2) \big] \dot\varphi \\ &\quad + \big[u_+(\varphi) + \mathcal{O}((r-1)) \big] \dot r \,. \end{aligned} \tag{3.8}$$

Comparing with (3.7) and, in a first step, projecting on a vector normal to $u_+(\varphi)$ shows that $\dot\varphi = 1/T_+ + \mathcal{O}((r-1)^2) + \mathcal{O}(\dot r(r-1))$. Then, in a second step, projecting on a vector perpendicular to $f(\Gamma_+(\varphi))$ shows that $\dot r = \lambda_+(r-1) + \mathcal{O}((r-1)^2)$, which also implies $\dot\varphi = 1/T_+ + \mathcal{O}((r-1)^2)$.

3. In order to extend $h(r,\varphi)$ to all of $\mathcal{D}_1$, we start by constructing a curve segment Δ_0, connecting $\Gamma_+(0)$ to some point $\Gamma_-(\varphi^\star)$ on the stable orbit, which is crossed by all orbits of the vector field in the same direction (see Figure 1). Reparametrising Γ_- if necessary, we may assume that $\varphi^\star = 0$. The curve Δ_0 can be chosen to be tangent to $u_+(0)$ in $\Gamma_+(0)$, and to the similarly defined vector $u_-(0)$ in $\Gamma_-(0)$. We set

$$h(r,\varphi) = \Gamma_-(\varphi) + (r+1) u_-(\varphi) + \mathcal{O}((r+1)^2) \tag{3.9}$$

as $r \to -1$, which implies in particular the relations (3.3).
The curve segment Δ_0 can be parametrised by a function $r \mapsto h(r,0)$ which is compatible with (3.7) and (3.9), that is, $\partial_r h(\pm 1, 0) = u_\pm(0)$. We proceed similarly with each element of a smooth deformation $\{\Delta_\varphi\}_{\varphi \in \mathbb{S}^1}$ of Δ_0, where Δ_φ connects $\Gamma_+(\varphi)$ to $\Gamma_-(\varphi)$ and is tangent to $u_\pm(\varphi)$. The parametrisation $r \mapsto h(r,\varphi)$ of Δ_φ can be chosen in such a way that whenever $\varphi < \varphi'$, the orbit starting in $h(r,\varphi) \in \mathcal{S}$ first hits $\Delta_{\varphi'}$ at a point $h(r',\varphi')$ with $r' < r$. This guarantees that

$$\begin{aligned} \dot r &= f_r(r,\varphi) \,, \\ \dot\varphi &= f_\varphi(r,\varphi) \end{aligned} \tag{3.10}$$

with $f_r(r,\varphi) < 0$ for $(r,\varphi) \in \mathcal{S}$.

4. We can always assume that $f_\varphi(r,\varphi) > 0$, replacing, if necessary, φ by $\varphi + \delta(r)$ for some function δ vanishing in $r = 1$. □

Remark 3.2. One can always use φ as new time variable, and rewrite (3.1) as the one-dimensional, non-autonomous equations

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{f_r(r,\varphi)}{f_\varphi(r,\varphi)} =: F(r,\varphi) \; . \tag{3.11}$$

Note, in particular, that

$$F(\pm 1,\varphi) = T_\pm \lambda_\pm (r \mp 1)) + \mathcal{O}\big((r \mp 1)^2\big) \; . \tag{3.12}$$

3.2 Stochastic system

We now turn to the SDE (2.7) which is equivalent, via the transformation $z = h(r,\varphi)$ of Proposition 3.1, to a system of the form

$$\begin{aligned} \mathrm{d}r_t &= f_r(r_t,\varphi_t;\sigma)\ \mathrm{d}t + \sigma g_r(r_t,\varphi_t)\,\mathrm{d}W_t \; , \\ \mathrm{d}\varphi_t &= f_\varphi(r_t,\varphi_t;\sigma)\ \mathrm{d}t + \sigma g_\varphi(r_t,\varphi_t)\,\mathrm{d}W_t \; . \end{aligned} \tag{3.13}$$

In fact, Itô's formula shows that $f = (f_r, f_\varphi)^{\mathrm{T}}$ and $g = (g_r, g_\varphi)^{\mathrm{T}}$, where g_r and g_φ are $(1 \times k)$-matrices, satisfy

$$\begin{aligned} g(h(r,\varphi)) &= \partial_r h(r,\varphi) g_r(r,\varphi) + \partial_\varphi h(r,\varphi) g_\varphi(r,\varphi) \\ f(h(r,\varphi)) &= \partial_r h(r,\varphi) f_r(r,\varphi) + \partial_\varphi h(r,\varphi) f_\varphi(r,\varphi) \\ &\quad + \frac{1}{2}\sigma^2 \big[\partial_{rr} h(r,\varphi) g_r g_r^{\mathrm{T}}(r,\varphi) + 2\partial_{r\varphi} h(r,\varphi) g_r g_\varphi^{\mathrm{T}}(r,\varphi) + \partial_{\varphi\varphi} h(r,\varphi) g_\varphi g_\varphi^{\mathrm{T}}(r,\varphi)\big] \; . \end{aligned} \tag{3.14}$$

The first equation allows to determine g_r and g_φ, by projection on $\partial_r h$ and $\partial_\varphi h$. The second one shows that

$$\begin{aligned} f_r(r,\varphi;\sigma) &= f_r^0(r,\varphi) + \sigma^2 f_r^1(r,\varphi) \; , \\ f_\varphi(r,\varphi;\sigma) &= f_\varphi^0(r,\varphi) + \sigma^2 f_\varphi^1(r,\varphi) \; , \end{aligned} \tag{3.15}$$

where f_r^0 and f_φ^0 are the functions of Proposition 3.1.

A drawback of the system (3.13) is that the drift term f_r in general no longer vanishes in $r = \pm 1$. This can be seen as an effect induced by the curvature of the orbit, since $f_r^1(\pm 1,\varphi)$ depends on $\Gamma'_\pm(\varphi)$ and $u'_\pm(\varphi)$. This problem can, however, be solved by a further change of variables.

Proposition 3.3. *There exists a change of variables of the form $y = r + \mathcal{O}(\sigma^2)$, leaving φ unchanged, such that the drift term for $\mathrm{d}y_t$ vanishes in $y = \pm 1$.*

Proof: We shall look for a change of variables of the form

$$y = Y(r,\varphi) = r - \sigma^2 \big[\Delta_-(\varphi)(r-1) + \Delta_+(\varphi)(r+1)\big] \; , \tag{3.16}$$

where $\Delta_\pm(\varphi)$ are periodic functions, representing the shift of variables near the two periodic orbits. Note that $y = 1$ for

$$r = 1 + 2\sigma^2 \Delta_+(\varphi) + \mathcal{O}(\sigma^4) \; . \tag{3.17}$$

Using Itô's formula, one obtains a drift term for $\mathrm{d}y_t$ satisfying

$$\begin{aligned} f_y(1,\varphi) = f_r(1 + 2\sigma^2\Delta_+(\varphi) + \mathcal{O}(\sigma^4), \varphi)\big[1 - \sigma^2[\Delta_-(\varphi) + \Delta_+(\varphi)]\big] \\ - \sigma^2\big[2\Delta'_+(\varphi) f_\varphi(1,\varphi) + \mathcal{O}(\sigma^2)\big] \\ - \frac{1}{2}\sigma^4\big[4\Delta'_+(\varphi) g_r(1,\varphi) g_\varphi(1,\varphi)^{\mathrm{T}} + 2\Delta''_+(\varphi) g_\varphi(1,\varphi) g_\varphi(1,\varphi)^{\mathrm{T}} + \mathcal{O}(\sigma^2)\big] \,, \end{aligned} \tag{3.18}$$

where the terms $\mathcal{O}(\sigma^2)$ depend on $\Delta_\pm$ and $\Delta'_\pm$. Using (3.2), we see that, in order that $f_y(1,\varphi)$ vanishes, $\Delta_+(\varphi)$ thus has to satisfy an equation of the form

$$\lambda_+\Delta_+(\varphi) - \frac{1}{T_+}\Delta'_+(\varphi) + r(\varphi, \Delta_\pm(\varphi), \Delta'_\pm(\varphi)) - \sigma^2 b(\varphi, \Delta_\pm(\varphi))\Delta''_+(\varphi) = 0 \,, \tag{3.19}$$

where $r(\varphi, \Delta_\pm, \Delta'_\pm) = f_r^1(1,\varphi) + \mathcal{O}(\sigma^2)$. Note that $b(\varphi, \Delta) > 0$ is bounded away from zero for small σ by our ellipticity assumption on g. A similar equation is obtained for $\Delta_-(\varphi)$. If $\Delta = (\Delta_+, \Delta_-)$ and $\Xi = (\Delta'_+, \Delta'_-)$, we arrive at a system of the form

$$\begin{aligned} \sigma^2 B(\varphi,\Delta)\Xi' &= -D\Xi + \Lambda\Delta + R(\varphi,\Delta,\Xi) \,, \\ \Delta' &= \Xi \,, \\ \varphi' &= 1 \,. \end{aligned} \tag{3.20}$$

Here D denotes a diagonal matrix with entries $1/T_+$ and $1/T_-$, and Λ denotes a diagonal matrix with entries $\lambda_\pm$. The system (3.20) is a slow–fast ODE, in which Ξ plays the rôle of the fast variable, and (Δ, φ) are the slow variables. The fast vector field vanishes on a uniformly hyperbolic slow manifold of the form $\Xi = \Xi^*(\Delta, \varphi)$, where

$$\Xi^*_\pm(\Delta,\varphi) = T_\pm\big[\lambda_\pm\Delta_\pm(\varphi) + f_r^1(\pm 1, \varphi)\big] + \mathcal{O}(\sigma^2) \,. \tag{3.21}$$

By Fenichel's theorem [Fen79], there exists an invariant manifold $\Xi = \overline{\Xi}(\varphi, \Delta)$ in a σ^2-neighbourhood of the slow manifold. The reduced equation on this invariant manifold takes the form

$$\Delta'_\pm = T_\pm\big[\lambda_\pm\Delta_\pm(\varphi) + f_r^1(\pm 1, \varphi)\big] + \mathcal{O}(\sigma^2) \,. \tag{3.22}$$

The limiting equation obtained by setting σ to zero admits an explicit periodic solution. Using standard arguments of regular perturbation theory, one then concludes that the full equation (3.22) also admits a periodic solution. □

4 Large deviations

In this section, we consider the dynamics near the unstable periodic orbit on the level of large deviations. We want to estimate the infimum I_φ of the rate function for the event $\Gamma(\delta)$ that a sample path, starting at sufficiently small distance δ from the unstable orbit, reaches the unstable orbit at the moment when the angular variable has increased by φ.

Consider first the system linearised around the unstable orbit, given by

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\begin{pmatrix} r \\ p_r \end{pmatrix} = \begin{pmatrix} \lambda_+ T_+ & D_{rr}(0,\varphi) \\ 0 & -\lambda_+ T_+ \end{pmatrix}\begin{pmatrix} r \\ p_r \end{pmatrix} \,. \tag{4.1}$$

(We have redefined r so that the unstable orbit is in $r = 0$.) Its solution can be written in the form

$$\begin{pmatrix} r(\varphi) \\ p_r(\varphi) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\lambda_+ T_+(\varphi-\varphi_0)} & \mathrm{e}^{\lambda_+ T_+(\varphi-\varphi_0)} \int_{\varphi_0}^{\varphi} \mathrm{e}^{-2\lambda_+ T_+(u-\varphi_0)}\, D_{rr}(0,u)\,\mathrm{d}u \\ 0 & \mathrm{e}^{-\lambda_+ T_+(\varphi-\varphi_0)} \end{pmatrix} \begin{pmatrix} r(\varphi_0) \\ p_r(\varphi_0) \end{pmatrix} . \tag{4.2}$$

The off-diagonal term of the above fundamental matrix can also be expressed in the form

$$\mathrm{e}^{\lambda_+ T_+(\varphi-\varphi_0)}\, h^{\mathrm{per}}(\varphi_0) - \mathrm{e}^{-\lambda_+ T_+(\varphi-\varphi_0)}\, h^{\mathrm{per}}(\varphi) , \tag{4.3}$$

where

$$h^{\mathrm{per}}(\varphi) = \frac{\mathrm{e}^{2\lambda_+ T_+ \varphi}}{1-\mathrm{e}^{-2\lambda_+ T_+}} \int_{\varphi}^{\varphi+1} \mathrm{e}^{-2\lambda_+ T_+ u}\, D_{rr}(0,u)\,\mathrm{d}u \tag{4.4}$$

is the periodic solution of the equation $\mathrm{d}h/\,\mathrm{d}\varphi = 2\lambda_+ T_+ h - D_{rr}(0,\varphi)$. The expression (4.3) shows that for initial conditions satisfying $r(\varphi_0) = -h^{\mathrm{per}}(\varphi_0) p_r(\varphi_0)$, the orbit $(r(\varphi), p_r(\varphi))$ will converge to $(0,0)$. The stable manifold of the unstable orbit is thus given by the equation $r = -h^{\mathrm{per}}(\varphi) p_r$.

We consider now the following situation: Let $(r(0), p_r(0))$ belong to the stable manifold. The orbit starting in this point takes an infinite time to reach the unstable orbit, and gives rise to a value I_∞ of the rate function. We want to compare this value to the rate function I_φ of an orbit starting at the same $r(0)$, but reaching $r = 0$ in finite time φ.

Recall that the rate function has the expression

$$I(\gamma) = \frac{1}{2}\int_0^T \psi_s^{\mathrm{T}} D(\gamma_s)\psi_s \,\mathrm{d}s = \frac{1}{2}\int \Big[D_{rr}p_r^2 + 2D_{r\varphi}p_r p_\varphi + D_{\varphi\varphi}p_\varphi^2\Big]\,\mathrm{d}\varphi . \tag{4.5}$$

However, p_φ can be expressed in terms of r, φ and p_r using the Hamiltonian, and is of order $r^2 + p_r^2$. Thus the leading term in the rate function near the unstable orbit is $D_{rr}p_r^2$. As a first approximation we may thus consider

$$I^0(\gamma) = \frac{1}{2}\int D_{rr}p_r^2\,\mathrm{d}\varphi . \tag{4.6}$$

Proposition 4.1 (Comparison of rate functions in the linear case)**.** *Denote by I^0_∞ and I^0_φ the minimal value of the rate function I^0 for orbits starting in $r(0)$ and reaching the unstable orbit in infinite time or in time φ, respectively. We have*

$$I^0_\varphi - I^0_\infty = \frac{1}{2}\delta^2\,\mathrm{e}^{-2\lambda_+ T_+\varphi}\,\frac{h^{\mathrm{per}}(\varphi)}{h^{\mathrm{per}}(0)^2}\Big[1+\mathcal{O}\big(\mathrm{e}^{-2\lambda_+ T_+\varphi}\big)\Big] . \tag{4.7}$$

Proof: Let $(r^0, p_r^0)(u)$ be the orbit with initial condition $(r(0), p_r(0))$, and $(r^1, p_r^1)(u)$ the one with initial condition $(r(0), p_r(0)+q)$. Then we have by (4.2) with $\varphi_0 = 0$, (4.3) and the relation $r(0) = -h^{\mathrm{per}}(0)p_r(0)$

$$\begin{aligned} r^1(u) &= \mathrm{e}^{-\lambda_+ T_+ u}\,\frac{h^{\mathrm{per}}(u)}{h^{\mathrm{per}}(0)} r(0) + q\Big[\mathrm{e}^{\lambda_+ T_+ u}\, h^{\mathrm{per}}(0) - \mathrm{e}^{-\lambda_+ T_+ u}\, h^{\mathrm{per}}(u)\Big] , \\ p_r^1(u) &= \mathrm{e}^{-\lambda_+ T_+ u}(p_r(0)+q) . \end{aligned} \tag{4.8}$$

The requirement $r^1(\varphi) = 0$ implies

$$q = \mathrm{e}^{-2\lambda_+ T_+\varphi}\,\frac{h^{\mathrm{per}}(\varphi)}{h^{\mathrm{per}}(0)} p_r(0)\Big[1+\mathcal{O}\big(\mathrm{e}^{-2\lambda_+ T_+\varphi}\big)\Big] . \tag{4.9}$$

Since the solutions starting on the stable manifold satisfy $p_r^0(u) = \mathrm{e}^{-2\lambda_+ T_+ u}\, p_r(0)$, we get

$$p_r^1(u)^2 - p_r^0(u)^2 = 2qp_r^0(u)\,\mathrm{e}^{-\lambda_+ T_+ u} + q^2\,\mathrm{e}^{-2\lambda_+ T_+ u} \ . \tag{4.10}$$

The difference between the two rate functions is thus given by

$$2(I_\varphi^0 - I_\infty^0) = (2qp_r(0) + q^2)\int_0^\varphi \mathrm{e}^{-2\lambda_+ T_+ u}\, D_{rr}(0,u)\,\mathrm{d}u - p_r(0)^2 \int_\varphi^\infty \mathrm{e}^{-2\lambda_+ T_+ u}\, D_{rr}(0,u)\,\mathrm{d}u \ . \tag{4.11}$$

Using the relations (cf. (4.3))

$$\begin{aligned}
\int_0^\varphi \mathrm{e}^{-2\lambda_+ T_+ u}\, D_{rr}(0,u)\,\mathrm{d}u &= h^{\mathrm{per}}(0) - \mathrm{e}^{-2\lambda_+ T_+ \varphi}\, h^{\mathrm{per}}(\varphi) \ , \\
\int_0^\infty \mathrm{e}^{-2\lambda_+ T_+ u}\, D_{rr}(0,u)\,\mathrm{d}u &= h^{\mathrm{per}}(0) \ , \\
\int_\varphi^\infty \mathrm{e}^{-2\lambda_+ T_+ u}\, D_{rr}(0,u)\,\mathrm{d}u &= \mathrm{e}^{-2\lambda_+ T_+ \varphi}\, h^{\mathrm{per}}(\varphi)
\end{aligned} \tag{4.12}$$

and $h^{\mathrm{per}}(0)p_r(0) = -r(0) = -\delta$ yields the result. □

We can now draw on standard perturbation theory to obtain the following result for the nonlinear case.

Proposition 4.2 (Comparison of rate functions in the nonlinear case)**.** *For sufficiently small δ, the infimum I_φ of the rate function for the event $\Gamma(\delta)$ satisfies*

$$I_\varphi - I_\infty = \frac{1}{2}\delta^2\,\mathrm{e}^{-2\lambda_+ T_+ \varphi}\,\frac{h^{\mathrm{per}}(\varphi)}{h^{\mathrm{per}}(0)^2}\Big[1 + \mathcal{O}\big(\mathrm{e}^{-2\lambda_+ T_+ \varphi}\big) + \mathcal{O}(\delta)\Big] \ . \tag{4.13}$$

Proof: Writing $(r, \varphi, p_r, p_\varphi) = \delta(\bar r, \bar\varphi, \bar p_r, \bar p_\varphi)$, we can consider the nonlinear terms as a perturbation of order δ. Since the solutions we consider decay exponentially, the stable manifold theorem and a Gronwall argument allows to bound the effect of nonlinear terms by a multiplicative error of the form $1 + \mathcal{O}(\delta)$. □

5 Continuous-space Markov chains

5.1 Eigenvalues and eigenfunctions

Let $E \subset \mathbb{R}$ be an interval, equipped with the Borel σ-algebra. Consider a Markov kernel

$$K(x, \mathrm{d}y) = k(x,y)\,\mathrm{d}y \tag{5.1}$$

with density k with respect to Lebesgue measure. We assume k to be continuous and square-integrable. We allow for $K(x,E) < 1$, that is, the kernel may be substochastic. In that case, we add a cemetery state ∂ to E, so that K is stochastic on $E \cup \partial$. Given an initial condition X_0, the kernel K generates a Markov chain $(X_0, X_1, \dots)$ via

$$\mathbb{P}\{X_{n+1} \in A\} = \int_E \mathbb{P}\{X_n \in \mathrm{d}x\} K(x, A) \ . \tag{5.2}$$

We write the natural action of the kernel on bounded measurable functions f as

$$(Kf)(x) \coloneqq \mathbb{E}^x\big\{f(X_1)\big\} = \int_E k(x,y) f(y)\,\mathrm{d}y \ . \tag{5.3}$$

For a finite signed measure μ with density m, we set

$$(\mu K)(\mathrm{d}y) \coloneqq \mathbb{E}^{\mu}\{X_1 \in \mathrm{d}y\} = (mK)(y)\,\mathrm{d}y\ , \tag{5.4}$$

where

$$(mK)(y) = \int_E m(x)\,\mathrm{d}x\,k(x,y)\ . \tag{5.5}$$

We know by the work of Fredholm [Fre03] that the integral equation

$$(Kf)(x) - \lambda f(x) = g(x) \tag{5.6}$$

can be solved for any g, if and only if λ is not an eigenvalue, i.e., the eigenvalue equation

$$(Kh)(x) = \lambda h(x) \tag{5.7}$$

admits no nontrivial solution. All eigenvalues λ have finite multiplicity, and the properly normalised left and right eigenfunctions h_n^* and h_n form a complete orthonormal basis, that is,

$$\int_E h_n^*(x)h_m(x)\,\mathrm{d}x = \delta_{nm} \qquad \text{and} \qquad \sum_n h_n^*(x)h_n(y) = \delta(x-y)\ . \tag{5.8}$$

Jentzsch [Jen12] proved that if k is positive, there exists a simple eigenvalue $\lambda_0 \in (0,\infty)$, which is strictly larger in module than all other eigenvalues. It is called the *principal eigenvalue*. The associated eigenfunctions h_0 and h_0^* are positive. Birkhoff [Bir57] has obtained the same result under weaker assumptions on k. We call the probability measure π_0 given by

$$\pi_0(\mathrm{d}x) = \frac{h_0^*(x)\,\mathrm{d}x}{\int_E h_0^*(y)\,\mathrm{d}y} \tag{5.9}$$

the *quasistationary distribution* of the Markov chain. It describes the asymptotic distribution of the process conditioned on having survived.

Given a Borel set $A \subset E$, we introduce the stopping times

$$\begin{aligned} \tau_A &= \tau_A(x) = \inf\{t \geqslant 1\colon X_t \in A\}\ , \\ \sigma_A &= \sigma_A(x) = \inf\{t \geqslant 0\colon X_t \in A\}\ , \end{aligned} \tag{5.10}$$

where the optional argument x denotes the initial condition. Observe that $\tau_A(x) = \sigma_A(x)$ if $x \in E \setminus A$ while $\sigma_A(x) = 0 < 1 \leqslant \tau_A(x)$ if $x \in A$. The stopping times τ_A and σ_A may be infinite because the Markov chain can reach the cemetery state before hitting A (and, for the moment, we also don't assume that the chain conditioned to survive is recurrent).

Given $u \in \mathbb{C}$, we define the Laplace transforms

$$\begin{aligned} G_A^u(x) &= \mathbb{E}^x\big\{\mathrm{e}^{u\tau_A}\,1_{\{\tau_A<\infty\}}\big\}\ , \\ H_A^u(x) &= \mathbb{E}^x\big\{\mathrm{e}^{u\sigma_A}\,1_{\{\sigma_A<\infty\}}\big\}\ . \end{aligned} \tag{5.11}$$

Note that $G_A^u = H_A^u$ in $E \setminus A$ while $H_A^u = 1$ in A. The following result is easy to check by splitting the expectation defining G_A^u according to the location of X_1:

Lemma 5.1. *Let*

$$\gamma(A) = \sup_{x\in E\setminus A} K(x, E\setminus A) = \sup_{x\in E\setminus A} \mathbb{P}^x\big\{X_1 \in E\setminus A\big\}\ . \tag{5.12}$$

Then $G_A^u(x)$ is analytic in u for $\operatorname{Re} u < \log \gamma(A)^{-1}$, i.e., for $|\mathrm{e}^{-u}| > \gamma(A)$, and for these u it satisfies the bound

$$\sup_{x \in E \setminus A} \left| G_A^u(x) \right| \leqslant \frac{1}{|\mathrm{e}^{-u}| - \gamma(A)} \,. \tag{5.13}$$

The main interest of the Laplace transforms lies in the following result, which shows that H_A^u is "almost an eigenfunction", if G_A^u varies little in A.

Lemma 5.2. *For any $u \in \mathbb{C}$ such that G_A^u and K_A^u exist,*

$$K H_A^u = \mathrm{e}^{-u} \, G_A^u \,. \tag{5.14}$$

Proof: Splitting according to the location of X_1, we get

$$\begin{aligned}
(K H_A^u)(x) &= \mathbb{E}^x \Big\{ \mathbb{E}^{X_1} \big\{ \mathrm{e}^{u\sigma_A} \, 1_{\{\sigma_A < \infty\}} \big\} \Big\} \\
&= \mathbb{E}^x \Big\{ 1_{\{X_1 \in A\}} \mathbb{E}^{X_1} \big\{ \mathrm{e}^{u\sigma_A} \, 1_{\{\sigma_A < \infty\}} \big\} \Big\} + \mathbb{E}^x \Big\{ 1_{\{X_1 \in E \setminus A\}} \mathbb{E}^{X_1} \big\{ \mathrm{e}^{u\sigma_A} \, 1_{\{\sigma_A < \infty\}} \big\} \Big\} \\
&= \mathbb{E}^x \big\{ 1_{\{\tau_A = 1\}} \big\} + \mathbb{E}^x \big\{ \mathrm{e}^{u(\tau_A - 1)} \, 1_{\{1 < \tau_A < \infty\}} \big\} \\
&= \mathbb{E}^x \big\{ \mathrm{e}^{u(\tau_A - 1)} \, 1_{\{\tau_A < \infty\}} \big\} = \mathrm{e}^{-u} \, G_A^u(x) \,.
\end{aligned} \tag{5.15}$$

□

We have the following relation between an eigenfunction inside and outside A.

Proposition 5.3. *Let h be an eigenfunction of K with eigenvalue $\lambda = \mathrm{e}^{-u}$. Assume there is a set $A \subset E$ such that*

$$|\mathrm{e}^{-u}| > \gamma(A) = \sup_{x \in E \setminus A} \mathbb{P}^x \big\{ X_1 \in E \setminus A \big\} \,. \tag{5.16}$$

Then

$$h(x) = \mathbb{E}^x \big\{ \mathrm{e}^{u\tau_A} \, h(X_{\tau_A}) 1_{\{\tau_A < \infty\}} \big\} \tag{5.17}$$

for all $x \in E$.

Proof: The eigenvalue equation can be written in the form

$$\mathrm{e}^{-u} \, h(x) = (Kh)(x) = \mathbb{E}^x \big\{ h(X_1) 1_{\{X_1 \in A\}} \big\} + \mathbb{E}^x \big\{ h(X_1) 1_{\{X_1 \in E \setminus A\}} \big\} \,. \tag{5.18}$$

Consider first the case $x \in E \setminus A$. Define a linear operator $\mathcal{T}$ on the Banach space $\mathcal{X}$ of continuous functions $f : E \setminus A \to \mathbb{C}$ equipped with the supremum norm, by

$$(\mathcal{T} f)(x) = \mathbb{E}^x \big\{ \mathrm{e}^u \, h(X_1) 1_{\{X_1 \in A\}} \big\} + \mathbb{E}^x \big\{ \mathrm{e}^u \, f(X_1) 1_{\{X_1 \in E \setminus A\}} \big\} \,. \tag{5.19}$$

Is is straightforward to check that under Condition (5.16), $\mathcal{T}$ is a contraction. Thus it admits a unique fixed point in $\mathcal{X}$, which must coincide with h. Furthermore, let h_n be a sequence of functions in $\mathcal{X}$ defined by $h_0 = 0$ and $h_{n+1} = \mathcal{T} h_n$ for all n. Then one can show by induction that

$$h_n(x) = \mathbb{E}^x \big\{ \mathrm{e}^{u\tau_A} \, h(X_{\tau_A}) 1_{\{\tau_A \leqslant n\}} \big\} \,. \tag{5.20}$$

Since $\lim_{n \to ?\infty} h_n(x) = h(x)$ for all $x \in E \setminus A$, (5.17) holds for these x. It remains to show that (5.17) also holds for $x \in A$. This follows by a similar computation as in the proof of Lemma 5.2. □

The following result provides a simple way to estimate the principal eigenvalue λ_0.

Proposition 5.4. *For any $n \geqslant 1$, and any interval $A \subset E$ with positive Lebesgue measure, we have*

$$\left[\inf_{x\in A} K_n(x,A)\right]^{1/n} \leqslant \lambda_0 \leqslant \left[\sup_{x\in E} K_n(x,E)\right]^{1/n} . \tag{5.21}$$

PROOF: Since the principal eigenvalue of K_n is equal to λ_0^n, it suffices to prove the relation for $n = 1$. Let x^* be the point where $h_0(x)$ reaches its supremum. Then the eigenvalue equation yields

$$\lambda_0 = \int_E k(x^*,y)\frac{h_0(y)}{h_0(x^*)}\,\mathrm{d}y \leqslant K(x^*,E) , \tag{5.22}$$

which proves the upper bound. For the lower bound, we use

$$\lambda_0 \int_A h_0^*(y)\,\mathrm{d}y = \int_E h_0^*(x)K(x,A)\,\mathrm{d}x \geqslant \inf_{x\in A} K(x,A)\int_A h_0^*(y)\,\mathrm{d}y , \tag{5.23}$$

and the integral over A can be divided out since A has positive Lebesgue measure. □

The following result allows to bound the spectral gap, between λ_0 and the remaining eigenvalues.

Proposition 5.5. *Let A be an open subset of E. Assume there exists $m : A \to (0,\infty)$ such that*

$$m(y) \leqslant k(x,y) \leqslant Lm(y) \qquad \forall x,y \in A \tag{5.24}$$

holds with a constant L satisfying $\lambda_0 L > 1$. Then any eigenvalue $\lambda \neq \lambda_0$ of K satisfies

$$|\lambda| \leqslant \max\left\{2\bar\gamma(A)\ ,\ \lambda_0 L - 1 + p_{\text{kill}}(A) + \bar\gamma(A)\frac{\lambda_0 L}{\lambda_0 L - 1}\left[1 + \frac{1}{\lambda_0 - \bar\gamma(A)}\right]\right\} \tag{5.25}$$

where

$$\bar\gamma(A) = \sup_{x\in E} K(x, E\setminus A) \quad \textit{and} \quad p_{\text{kill}}(A) = \sup_{x\in A}[1 - K(x,E)] . \tag{5.26}$$

Remark 5.6. Proposition 5.4 shows that $\lambda_0 \geqslant 1 - p_{\text{kill}}(A) - \gamma(A)$. Thus, if A is chosen in such a way that $\bar\gamma(A)$ and $p_{\text{kill}}(A)$ are small, the bound (5.25) reads

$$|\lambda| \leqslant L - 1 + \mathcal{O}\big(\bar\gamma(A)\big) + \mathcal{O}\big(p_{\text{kill}}(A)\big) . \tag{5.27}$$

PROOF: The eigenvalue equation for λ and orthogonality of the eigenfunctions yield

$$\lambda h(x) = \int_E k(x,y)h(y)\,\mathrm{d}y , \tag{5.28}$$

$$0 = \int_E h_0^*(y)h(y)\,\mathrm{d}y . \tag{5.29}$$

For any $\kappa > 0$ we thus have

$$\lambda h(x) = \int_E \big[\kappa h_0^*(y) - k(x,y)\big]h(y)\,\mathrm{d}y . \tag{5.30}$$

Let x_0 be the point in A where $|h|$ reaches its supremum. Evaluating the last equation in x_0 we obtain

$$|\lambda| \leqslant \int_A \big|\kappa h_0^*(y) - k(x_0,y)\big|\,\mathrm{d}y + \int_{E\setminus A}\Big[\kappa h_0^*(y) + k(x_0,y)\Big]\frac{|h(y)|}{|h(x_0)|}\,\mathrm{d}y . \tag{5.31}$$

We start by estimating the first integral. Since for all $y \in A$,

$$\lambda_0 h_0^*(y) = \int_E h_0^*(x) k(x,y)\,\mathrm{d}x \geqslant m(y) \int_A h_0^*(x)\,\mathrm{d}x\ , \tag{5.32}$$

choosing $\kappa = \lambda_0 L (\int_A h_0^*(x)\,\mathrm{d}x)^{-1}$ allows to remove the absolute values so that

$$\int_A \bigl|\kappa h_0^*(y) - k(x_0,y)\bigr|\,\mathrm{d}y \leqslant \lambda_0 L - K(x_0,A) \leqslant \lambda_0 L - 1 + \bar\gamma(A) + p_{\mathrm{kill}}(A)\ . \tag{5.33}$$

From now on, we assume $|\lambda| \geqslant 2\bar\gamma(A)$, since otherwise there is nothing to show. In order to estimate the second integral in (5.31), we first use Proposition 5.3 with $\mathrm{e}^{-u} = \lambda$ and Lemma 5.1 to get for all $x \in E \setminus A$

$$|h(x)| \leqslant \mathbb{E}^x\Bigl\{\mathrm{e}^{(\mathrm{Re}\,u)\tau_A}\bigl|h(X_{\tau_A})\bigr| 1_{\{\tau_A<\infty\}}\Bigr\} \leqslant \frac{|h(x_0)|}{|\lambda| - \bar\gamma(A)}\ . \tag{5.34}$$

The second integral is thus bounded by

$$\frac{1}{|\lambda| - \bar\gamma(A)} \int_{E\setminus A} \Bigl[\kappa h_0^*(y) + k(x_0,y)\Bigr]\,\mathrm{d}y \leqslant \frac{1}{|\lambda| - \bar\gamma(A)} \left[\lambda_0 L \frac{\displaystyle\int_{E\setminus A} h_0^*(y)\,\mathrm{d}y}{\displaystyle\int_A h_0^*(y)\,\mathrm{d}y} + \bar\gamma(A)\right]\ . \tag{5.35}$$

Now the eigenvalue equation for λ_0 yields

$$\lambda_0 \int_{E\setminus A} h_0^*(y)\,\mathrm{d}y = \int_E h_0^*(x) K(x, E\setminus A)\,\mathrm{d}x \leqslant \bar\gamma(A) \int_E h_0^*(x)\,\mathrm{d}x\ . \tag{5.36}$$

Hence the second integral can be bounded by

$$\frac{1}{|\lambda| - \bar\gamma(A)} \left[\lambda_0 L \frac{\bar\gamma(A)}{\lambda_0 - \bar\gamma(A)} + \bar\gamma(A)\right]\ . \tag{5.37}$$

Substituting in (5.31), we thus get

$$|\lambda| \leqslant \lambda_0 L - 1 + \bar\gamma(A) + p_{\mathrm{kill}}(A) + \frac{\bar\gamma(A)}{|\lambda| - \bar\gamma(A)} \left[1 + \frac{\lambda_0 L}{\lambda_0 - \bar\gamma(A)}\right]\ . \tag{5.38}$$

Now it is easy to check the following fact: Let $|\lambda|, \alpha, \beta, \bar\gamma$ be positive numbers such that $\alpha, |\lambda| > \bar\gamma$. Then

$$|\lambda| \leqslant \alpha + \frac{\beta}{|\lambda| - \bar\gamma} \quad \Rightarrow \quad |\lambda| \leqslant \alpha + \frac{\beta}{\alpha - \bar\gamma}\ . \tag{5.39}$$

This yields the claimed result. □

5.2 Harmonic measures

Consider an SDE in $\mathbb{R}^2$ given by

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma g(x_t)\,\mathrm{d}W_t\ , \tag{5.40}$$

where $(W_t)_t$ is a standard k-dimensional Brownian motion, $k \geqslant 2$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by

$$\mathcal{L} = \sum_{i=1}^2 f_i \frac{\partial}{\partial x_i} + \frac{\sigma^2}{2} \sum_{i,j=1}^2 \bigl(gg^T\bigr)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{5.41}$$

the infinitesimal generator of the associated diffusion. Given a bounded open set $\mathcal{D} \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\mathcal{D}$, we are interested in properties of the first-exit location $x_\tau \in \partial\mathcal{D}$, where

$$\tau = \tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\} \tag{5.42}$$

is the first-exit time from $\mathcal{D}$. We will assume that f and g are uniformly bounded in $\overline{\mathcal{D}}$, and that g is uniformly elliptic in $\overline{\mathcal{D}}$. Dynkin's formula and Riesz's representation theorem imply the existence of a *harmonic measure* $H(x, \mathrm{d}y)$, such that

$$\mathbb{P}^x\big\{x_\tau \in B\big\} = \int_B H(x, \mathrm{d}y) \tag{5.43}$$

for all Borel sets $B \subset \partial\mathcal{D}$. Note that $x \mapsto H(x, \mathrm{d}y)$ is $\mathcal{L}$-harmonic, i.e., it satisfies $\mathcal{L}h = 0$ in $\mathcal{D}$. The uniform ellipticity assumption implies that for all $x \in \mathcal{D}$,

$$H(x, \mathrm{d}y) = h(x, y)\, \mathrm{d}y \tag{5.44}$$

admits a density h with respect to the arclength (one-dimensional surface measure) $\mathrm{d}y$, which is smooth wherever the boundary is smooth. This has been shown, e.g., in [BAKS84] (under a weaker hypoellipticity condition).

We now derive some bounds on the magnitude of oscillations of h, based on Harnack inequalities.

Lemma 5.7. *For any set $\mathcal{D}_0$ such that its closure satisfies $\overline{\mathcal{D}}_0 \subset \mathcal{D}$, there exists a constant C, independent of σ, such that*

$$\frac{\sup\limits_{x \in \mathcal{D}_0} h(x, y)}{\inf\limits_{x \in \mathcal{D}_0} h(x, y)} \leqslant \mathrm{e}^{C/\sigma^2} \tag{5.45}$$

holds for all $y \in \partial\mathcal{D}$.

Proof: Let $\mathcal{B}$ be a ball of radius $R = \sigma^2$ contained in $\mathcal{D}_0$. By [GT01, Corollary 9.25], we have for any $y \in \partial\mathcal{D}$

$$\sup_{x \in \mathcal{B}} h(x, y) \leqslant C_0 \inf_{x \in \mathcal{B}} h(x, y)\ , \tag{5.46}$$

where the constant $C_0 \geqslant 1$ depends only on the ellipticity constant of g and on νR^2, where the parameter ν is an upper bound on $(\|f\|/\sigma^2)^2$. Since $R = \sigma^2$, C_0 does not depend on σ. Consider now two points $x_1, x_2 \in \mathcal{D}$. They can be joined by a sequence of $N = \lceil \|x_2 - x_1\|/\sigma^2 \rceil$ overlapping balls of radius σ^2. Iterating the bound (5.46), we obtain

$$h(x_2, y) \leqslant C_0^N h(x_1, y) = \mathrm{e}^{(\log C_0)\lceil \|x_2 - x_1\|/\sigma^2 \rceil}\, h(x_1, y)\ , \tag{5.47}$$

which implies the result. □

Lemma 5.8. *Let $\mathcal{B}_r(x)$ denote the ball of radius r centred in x, and let $\mathcal{D}_0$ be such that its closure satisfies $\overline{\mathcal{D}}_0 \subset \mathcal{D}$. Then, for any $x_0 \in \mathcal{D}_0, y \in \partial\mathcal{D}$ and $\eta > 0$, one can find a constant $r = r(y, \eta)$, independent of σ, such that*

$$\sup_{x \in \mathcal{B}_{r\sigma^2}(x_0)} h(x, y) \leqslant (1 + \eta) \inf_{x \in \mathcal{B}_{r\sigma^2}(x_0)} h(x, y)\ . \tag{5.48}$$

PROOF: Let r_0 be such that $\mathcal{B}_{r_0\sigma^2}(x_0) \subset \mathcal{D}_0$, and write $R_0 = r_0\sigma^2$. Since h is harmonic and positive, we can apply the Harnack estimate [GT01, Corollary 9.24], showing that for any $R \leqslant R_0$,

$$\operatorname*{osc}_{\mathcal{B}_R(x_0)} h := \sup_{x\in\mathcal{B}_R(x_0)} h(x,y) - \inf_{x\in\mathcal{B}_R(x_0)} h(x,y) \leqslant C_1 \left(\frac{R}{R_0}\right)^{\alpha} \operatorname*{osc}_{\mathcal{B}_{R_0}(x_0)} h \;, \tag{5.49}$$

where, as in the previous proof, the constants $C_1 \geqslant 1$ and $\alpha > 0$ do not depend on σ. By [GT01, Corollary 9.25], we also have

$$\sup_{x\in\mathcal{B}_{R_0}(x_0)} h(x,y) \leqslant C_2 \inf_{x\in\mathcal{B}_{R_0}(x_0)} h(x,y) \;, \tag{5.50}$$

where again $C_2 > 1$ does not depend on σ. Combining the two estimates, we obtain

$$\frac{\sup\limits_{x\in\mathcal{B}_R(x_0)} h(x,y)}{\inf\limits_{x\in\mathcal{B}_R(x_0)} h(x,y)} - 1 \leqslant \frac{\operatorname*{osc}\limits_{\mathcal{B}_R(x_0)} h}{\inf\limits_{x\in\mathcal{B}_{R_0}(x_0)} h(x,y)} \leqslant C_1 \left(\frac{R}{R_0}\right)^{\alpha} (C_2 - 1) \;. \tag{5.51}$$

The result thus follows by taking $r = R/\sigma^2$, where $R = R_0[\eta/(C_1(C_2-1))]^{1/\alpha}$. □

5.3 Random Poincaré maps

Consider now an SDE of the form (5.40), where $x = (\varphi, r)$ and f and g are periodic in φ, with period 1. Consider the domain

$$\mathcal{D} = \big\{(\varphi, r)\colon\ -M < \varphi < 1\ ,\ a < r < b\big\} \;, \tag{5.52}$$

where $a < b$, and M is some (large) integer. We have in mind drift terms with a positive φ-component, so that sample paths are very unlikely to leave $\mathcal{D}$ through the segment $\varphi = -M$.

Given an initial condition $x_0 = (0, r_0) \in \mathcal{D}$, we can define

$$k(r_0, r_1) = h((0,r_0),(1,r_1)) \;, \tag{5.53}$$

where $h(x,y)\,\mathrm{d}y$ is the harmonic measure. Then by periodicity of f and g and the strong Markov property, k defines a Markov chain on $E = [a,b]$, keeping track of the value R_n of r_t whenever φ_t first reaches $n \in \mathbb{N}$. This Markov chain is substochastic because we only take into account paths reaching $\varphi = 1$ before hitting any other part of the boundary of $\mathcal{D}$. In other words, the Markov chain describes the process killed upon r_t reaching a or b (or φ_t reaching $-M$).

We denote by $K_n(x,y)$ the n-step transition kernel, and by $k_n(x,y)$ its density. Given an interval $A \subset E$, we write $K_n^A(x,y) = K_n(x,y)/K_n(x,A)$ for the n-step transition kernel for the Markov chain conditioned to stay in A, and $k_n^A(x,y)$ for the corresponding density.

Proposition 5.9. *Fix an interval $A \subset E$. For $x_1, x_2 \in A$ define the integer stopping time*

$$N = N(x_1,x_2) = \inf\big\{n \geqslant 1\colon |X_n^{x_2} - X_n^{x_1}| < r_\eta\sigma^2\big\} \;, \tag{5.54}$$

where $r_\eta = r(y,\eta)$ is the constant of Lemma 5.8 and $X_n^{x_0}$ denotes the Markov chain with transition kernel $K^A(x,y)$ and initial condition x_0. The two Markov chains $X_n^{x_1}$ and $X_n^{x_2}$

are coupled in the sense that their dynamics is derived from the same realization of the Brownian motion, cf. (5.40).

Let

$$\rho_n = \sup_{x_1,x_2\in A} \mathbb{P}\big\{N(x_1,x_2) > n\big\} \,. \tag{5.55}$$

Then for any $n \geqslant 2$ *and* $\eta > 0$,

$$\frac{\sup\limits_{x\in A} k_n^A(x,y)}{\inf\limits_{x\in A} k_n^A(x,y)} \leqslant 1 + \eta + \rho_{n-1}\, \mathrm{e}^{C/\sigma^2} \tag{5.56}$$

holds for all $y \in A$, *where* C *does not depend on* σ.

Proof: We decompose

$$\begin{aligned}\mathbb{P}\big\{X_n^{x_1} \in \mathrm{d}y\big\} = \sum_{k=1}^{n-1} &\mathbb{P}\big\{X_n^{x_1} \in \mathrm{d}y \mid N = k\big\}\mathbb{P}\big\{N = k\big\} \\ &+ \mathbb{P}\big\{X_n^{x_1} \in \mathrm{d}y \mid N > n-1\big\}\mathbb{P}\big\{N > n-1\big\} \,.\end{aligned} \tag{5.57}$$

Let $k_n^{(2)}((x_1,x_2),(z_1,z_2) \mid N = k)$ denote the conditional joint density for a transition for $(X_l^{x_1}, X_l^{x_2})$ in n steps from (x_1,x_2) to (z_1,z_2), given $N = k$. Note that this density is concentrated on the set $\{|z_2 - z_1| < r_\eta \sigma^2\}$. For $k = 1,\dots,n-1$ and any measurable $B \subset E$, we use Lemma 5.8 to estimate

$$\begin{aligned}&\mathbb{P}\big\{X_n^{x_1} \in B \mid N = k\big\} \\ &\quad= \int_A\int_A \mathbb{P}\big\{X_{n-k}^{z_1} \in B\big\} k_n^{(2)}((x_1,x_2),(z_1,z_2) \mid N = k)\, \mathrm{d}z_2\, \mathrm{d}z_1 \\ &\quad\leqslant (1+\eta)\int_A\int_A \mathbb{P}\big\{X_{n-k}^{z_2} \in B\big\} k_n^{(2)}((x_1,x_2),(z_1,z_2) \mid N = k)\, \mathrm{d}z_2\, \mathrm{d}z_1 \\ &\quad= (1+\eta)\mathbb{P}\big\{X_n^{x_2} \in B \mid N = k\big\} \,.\end{aligned} \tag{5.58}$$

Writing $k_{n-1}(x_1, z_1 \mid N > n-1)$ for the conditional $(n-1)$-step transition density of $X_l^{x_1}$, the last term in (5.57) can be bounded by

$$\begin{aligned}\mathbb{P}\big\{X_n^{x_1} \in B \mid N > n-1\big\} &= \int_A \mathbb{P}\big\{X_1^{z_1} \in B\big\} k_{n-1}(x_1, z_1 \mid N > n-1)\, \mathrm{d}z_1 \\ &\leqslant \sup_{z_1\in A} \mathbb{P}\big\{X_1^{z_1} \in B\big\}\mathbb{P}\big\{X_{n-1}^{x_1} \in E \mid N > n-1\big\} \\ &\leqslant \sup_{z_1\in A} \mathbb{P}\big\{X_1^{z_1} \in B\big\} \,.\end{aligned} \tag{5.59}$$

We thus have

$$\mathbb{P}\big\{X_n^{x_1} \in \mathrm{d}y\big\} \leqslant (1+\eta)\mathbb{P}\big\{X_n^{x_2} \in \mathrm{d}y\big\} + \rho_{n-1} \sup_{z_1\in A} \mathbb{P}\big\{X_1^{z_1} \in \mathrm{d}y\big\} \,. \tag{5.60}$$

On the other hand, we have

$$\mathbb{P}\big\{X_n^{x_1} \in \mathrm{d}y\big\} \geqslant \mathbb{P}\big\{X_{n-1}^{x_1} \in A\big\} \inf_{z_1\in A} \mathbb{P}\big\{X_1^{z_1} \in \mathrm{d}y\big\} \,. \tag{5.61}$$

Combining the upper and lower bound, we get

$$\frac{\sup\limits_{x\in A} k_n^A(x,y)}{\inf\limits_{x\in A} k_n^A(x,y)} \leqslant 1 + \eta + \rho_{n-1}\, \frac{\sup\limits_{z\in A} k^A(z,y)}{\inf\limits_{z\in A} k^A(z,y)} \,. \tag{5.62}$$

Hence the result follows from Lemma 5.7. □

6 Sample-path estimates

6.1 The principal eigenvalue $\lambda_0^{\mathfrak{u}}$

We consider in this section the system

$$\begin{aligned} \mathrm{d}r_t &= \bigl[\lambda_+ r_t + b_r(r_t,\varphi_t)\bigr]\,\mathrm{d}t + \sigma g_r(r_t,\varphi_t)\,\mathrm{d}W_t\;, \\ \mathrm{d}\varphi_t &= \Bigl[\frac{1}{T_+} + b_\varphi(r_t,\varphi_t)\Bigr]\,\mathrm{d}t + \sigma g_\varphi(r_t,\varphi_t)\,\mathrm{d}W_t\;, \end{aligned} \tag{6.1}$$

describing the dynamics near the unstable orbit. We have redefined r in such a way that the unstable orbit is located in $r=0$, and that the stable orbit lies in the region $\{r>0\}$. Here $\{W_t\}_t$ is a k-dimensional standard Brownian motion, $k \geqslant 2$, and $g = (g_r^T, g_\varphi^T)$ satisfies a uniform ellipticity condition. The functions b_r, b_φ, g_r and g_φ are periodic in φ with period 1 and the nonlinear drift terms satisfy $|b_r(r,\varphi)|, |b_\varphi(r,\varphi)| \leqslant Mr^2$.

Note that in first approximation, φ_t is close to t/T_+. Therefore we start by considering the linear process r_t^0 defined by

$$\mathrm{d}r_t^0 = \lambda r_t^0\,\mathrm{d}t + \sigma g_0(t)\,\mathrm{d}W_t\;, \tag{6.2}$$

where $g_0(t) = g_r(0, t/T_+)$, and λ will be chosen close to λ_+.

Proposition 6.1 (Linear system)**.** *Choose a $T>0$ and fix a small constant $\delta>0$. Given $r_0 \in (0,\delta)$ and an interval $A \subset (0,\delta)$, define*

$$P(r_0, A) = \mathbb{P}^{r_0}\bigl\{0 < r_t^0 < \delta\ \forall t \in [0,T], r_T \in A\bigr\}\;, \tag{6.3}$$

and let

$$v_t = \int_0^t \mathrm{e}^{-2\lambda s}\, g_0(s) g_0(s)^T\,\mathrm{d}s \qquad \text{for } t \in [0,T]\;. \tag{6.4}$$

1. Upper bound: *For any $T>0$,*

$$P(r_0,(0,\delta)) \leqslant \frac{1}{\sqrt{2\pi}} \frac{\delta^2 r_0}{\sigma^3 v_T^{3/2}}\, \mathrm{e}^{-r_0^2/2\sigma^2 v_T}\, \mathrm{e}^{-2\lambda T}\biggl[1 + \mathcal{O}\Bigl(\frac{r_0^2\,\mathrm{e}^{-2\lambda T}}{\sigma^4 v_T^2}\Bigr)\biggr]\;. \tag{6.5}$$

2. Lower bound: *Assume $A = [\sigma a, \sigma b]$ for two constants $0<a<b$. Then there exist constants $C_0, C_1, c > 0$, depending only on a, b, λ, such that for any $r_0 \in A$ and $T \geqslant 1$,*

$$P(r_0, A) \geqslant \biggl(C_0 - C_1 T \frac{\mathrm{e}^{-c\delta^2/\sigma^2}}{\delta^2}\biggr)\,\mathrm{e}^{-2\lambda T}\;. \tag{6.6}$$

Proof: We shall work with the rescaled process $z_t = \mathrm{e}^{-\lambda t}\, r_t^0$, which satisfies

$$\mathrm{d}z_t = \mathrm{e}^{-\lambda t}\,\sigma g_0(t)\,\mathrm{d}W_t\;. \tag{6.7}$$

Note that z_t is Gaussian with variance $\sigma^2 v_t$. Using André's reflection principle, we get

$$\begin{aligned} P(r_0,(0,\delta)) &\leqslant \mathbb{P}^{r_0}\bigl\{z_t > 0\ \forall t \in [0,T], 0 < z_T < \delta\,\mathrm{e}^{-\lambda T}\bigr\} \\ &= \mathbb{P}^{r_0}\bigl\{0 < z_T < \delta\,\mathrm{e}^{-\lambda T}\bigr\} - \mathbb{P}^{-r_0}\bigl\{0 < z_T < \delta\,\mathrm{e}^{-\lambda T}\bigr\} \\ &= \frac{1}{\sqrt{2\pi\sigma^2 v_T}} \int_0^{\delta\,\mathrm{e}^{-\lambda T}} \Bigl[\mathrm{e}^{-(r_0-z)^2/2\sigma^2 v_T} - \mathrm{e}^{-(r_0+z)^2/2\sigma^2 v_T}\Bigr]\,\mathrm{d}z \\ &\leqslant \frac{2}{\sqrt{2\pi\sigma^2 v_T}}\, \mathrm{e}^{-r_0^2/2\sigma^2 v_T} \int_0^{\delta\,\mathrm{e}^{-\lambda T}} \sinh\Bigl(\frac{r_0 z}{\sigma^2 v_T}\Bigr)\,\mathrm{d}z\;, \end{aligned} \tag{6.8}$$

and the upper bound (6.5) follows by using $\cosh(u) - 1 = \frac{1}{2}u^2 + \mathcal{O}(u^4)$.

To prove the lower bound, we introduce the notations τ_0 and τ_δ for the first-hitting times of r_t of 0 and δ. Then we can write

$$P(r_0, A) = \mathbb{P}^{r_0}\big\{\tau_0 > T, r_T \in A\big\} - \mathbb{P}^{r_0}\big\{\tau_\delta < T < \tau_0 \ , \ r_T \in A\big\} \ . \tag{6.9}$$

The first term on the right-hand side can be bounded below by a similar computation as for the upper bound. Using that r_0 is of order σ, that v_T has order 1 for $T \geqslant 1$, and taking into account the different domain of integration, one obtains a lower bound $C_0 \, \mathrm{e}^{-2\lambda T}$. As for the second term on the right-hand side, it can be rewritten as

$$\mathbb{E}^{r_0}\Big\{1_{\{\tau_\delta < T \wedge \tau_0\}} \mathbb{P}^\delta\big\{\tau_0 > T - \tau_\delta \ , \ r_{T-\tau_\delta} \in A\big\}\Big\} \ . \tag{6.10}$$

By the upper bound (6.5), the probability inside the expectation is bounded by a constant times $\mathrm{e}^{-2\lambda(T-\tau_\delta)} \, \mathrm{e}^{-c\delta^2/\sigma^2}/\delta^2$. It remains to estimate $\mathbb{E}^{r_0}\big\{1_{\{\tau_\delta < T \wedge \tau_0\}} \, \mathrm{e}^{2\lambda\tau_\delta}\big\}$. Integration by parts and another application of (6.5) show that this term is bounded by a constant times T, and the lower bound is proved. □

Remark 6.2.

1. The upper bound (6.5) is maximal for $r_0 = \sigma\sqrt{v_T}$, with a value of order $(\delta^2/\sigma^2 v_T) \, \mathrm{e}^{-2\lambda T}$.
2. Applying the reflection principle at a level $-a$ instead of 0, one obtains

$$\mathbb{P}^{r_0}\big\{-a \, \mathrm{e}^{\lambda t} < r_t^0 < \delta \ \forall t \in [0, T]\big\} \leqslant C_0 \frac{(\delta \, \mathrm{e}^{-\lambda T} + a)^2}{\sigma^2 v_T} \tag{6.11}$$

for some constant C_0 (provided the higher-order error terms are small).

We will now extend these estimates to the general nonlinear system (6.1). We first show that φ_t does not differ much from t/T_+ on rather long timescales. To ease the notation, given $h, h_1 > 0$,we introduce two stopping times

$$\begin{aligned} \tau_h &= \inf\{t > 0 \colon r_t \geqslant h\} \ , \\ \tau_\varphi &= \inf\Big\{t > 0 \colon \Big|\varphi_t - \frac{t}{T_+}\Big| \geqslant M\big(h^2 t + h_1\big)\Big\} \ . \end{aligned} \tag{6.12}$$

Proposition 6.3 (Control of the diffusion along φ)**.** *There is a constant C_1, depending only on the ellipticity constants of the diffusion terms, such that*

$$\mathbb{P}^{(r_0,0)}\big\{\tau_\varphi < \tau_h \wedge T\big\} \leqslant \mathrm{e}^{-h_1^2/(C_1 h^2 \sigma^2 T)} \tag{6.13}$$

holds for all $T, \sigma > 0$ and all $h, h_1 > 0$.

Proof: Just note that $\eta_t = \varphi_t - t/T_+$ is given by

$$\eta_t = \int_0^t b_\varphi(r_s, \varphi_s) \, \mathrm{d}s + \sigma \int_0^t g_\varphi(r_s, \varphi_s) \, \mathrm{d}Ws \ . \tag{6.14}$$

For $t < \tau_h$, the first term is bounded by $Mh^2 t$, while the probability that the second one becomes large can be bounded by the Bernstein-type estimate Lemma A.1. □

In the following, we will set $h_1 = \sqrt{h^3 T}$. In that case, $h^2 t + h_1 \leqslant h(1 + 2hT)$, and the right-hand side of (6.13) is bounded by $\mathrm{e}^{-h/(C_1\sigma^2)}$. All results below hold for all σ sufficiently small, as indicated by the σ-dependent error terms.

Proposition 6.4 (Upper bound on the probability to stay near the unstable orbit)**.** *Let $h = \sigma^\gamma$ for some $\gamma \in (1/2, 1)$, and let $\mu > 0$ satisfy $(1 + 2\mu)/(2 + 2\mu) > \gamma$. Then for any $0 < r_0 < h$ and all $0 < T \leqslant 1/h$,*

$$\mathbb{P}^{r_0}\big\{0 < r_t < h \;\forall t \in [0, T \wedge \tau_\varphi]\big\} \leqslant \frac{1}{\sigma^{2\mu(1-\gamma)}} \exp\biggl\{-\lambda_+ T\biggl[\frac{2\mu}{1+\mu} - \mathcal{O}\Bigl(\frac{1}{|\log\sigma|}\Bigr)\biggr]\biggr\} \,. \tag{6.15}$$

Proof: The proof is very close in spirit to the proof of [BG06, Theorem 3.2.2], so that we will only give the main ideas. The principal difference is that we are interested in the exit from an asymmetric interval $(0, h)$, which yields an exponent close to 2λ instead of λ for a symmetric interval $(-h, h)$. To ease the notation, we will write λ instead of λ_+ throughout the proof.

We introduce a partition of $[0, T]$ into intervals of equal length Δ/λ, for a Δ to be chosen below. Then the Markov property implies that the probability (6.15) is bounded by

$$q(\Delta)^{-1} \exp\biggl\{-\lambda T \frac{\log(q(\Delta)^{-1})}{\Delta}\biggr\} \,, \tag{6.16}$$

where $q(\Delta)$ is an upper bound on the probability to leave $(0, h)$ on a time interval of length Δ/λ. We thus want to show that $\log(q(\Delta)^{-1})/\Delta$ is close to 2 for a suitable choice of Δ.

We write the equation for r_t in the form

$$\mathrm{d}r_t = \bigl[\lambda r_t + b(r_t, \varphi_t)\bigr]\,\mathrm{d}t + \sigma g_0(t)\,\mathrm{d}W_t + \sigma g_1(r_t, \varphi_t, t)\,\mathrm{d}W_t \,. \tag{6.17}$$

Note that for $|r_t| < h$ and $t < \tau_\varphi \wedge T$, we may assume that $g_1(r_t, \varphi_t, t)$ has order $h + h^2 T$, which has in fact order h since we assume $T \leqslant 1/h$. Introduce the Gaussian processes

$$r_t^\pm = r_0\,\mathrm{e}^{\lambda^\pm t} + \sigma\,\mathrm{e}^{\lambda^\pm t} \int_0^t \mathrm{e}^{-\lambda^\pm s}\, g_0(s)\,\mathrm{d}W_s \,, \tag{6.18}$$

where $\lambda^\pm = \lambda \pm Mh$. Applying the comparison principle to $r_t - r_t^+$, we have

$$r_t^- + \sigma\,\mathrm{e}^{\lambda^- t}\,\mathcal{M}_t^- \leqslant r_t \leqslant r_t^+ + \sigma\,\mathrm{e}^{\lambda^+ t}\,\mathcal{M}_t^+ \tag{6.19}$$

as long as $0 < r_t \leqslant h$, where $\mathcal{M}_t^\pm$ are the martingales

$$\mathcal{M}_t^\pm = \int_0^t \mathrm{e}^{-\lambda^\pm s}\, g_1(r_s, \varphi_s, s)\,\mathrm{d}W_s \,. \tag{6.20}$$

We also have the relation

$$r_t^+ = \mathrm{e}^{2Mht}\, r_t^- + \sigma\,\mathrm{e}^{\lambda^+ t}\,\mathcal{M}_t^0 \qquad \text{where} \quad \mathcal{M}_t^0 = \int_0^t \bigl[\mathrm{e}^{\lambda^+ s} - \mathrm{e}^{\lambda^- s}\bigr] g_0(s)\,\mathrm{d}W_s \,. \tag{6.21}$$

Using Itô's isometry, one obtains that $\mathcal{M}_t^0$ has a variance of order h^2. This, as well as Lemma A.1 in the case of $\mathcal{M}_t^\pm$, shows that

$$\mathbb{P}\Bigl\{\sup_{0<s<t} |\sigma\,\mathrm{e}^{\lambda^+ s}\,\mathcal{M}_s^{0,\pm}| > H\Bigr\} \leqslant \exp\biggl\{-\frac{H^2}{2C_1 h^2 \sigma^2\,\mathrm{e}^{2\lambda^+ t}}\biggr\} \tag{6.22}$$

for some constant C_1. Combining (6.19) and (6.21), we obtain that $0 < r_t < h$ implies

$$-\sigma\,\mathrm{e}^{\lambda^+ t}\,\mathcal{M}_t^+ < r_t^+ < \mathrm{e}^{2Mht}\big[h + \sigma\,\mathrm{e}^{\lambda^- t}\,\mathcal{M}_t^-\big] - \sigma\,\mathrm{e}^{\lambda^+ t}\,\mathcal{M}_t^0\,. \tag{6.23}$$

The probability we are looking for is thus bounded by

$$q(\Delta) = \mathbb{P}\Big\{-H < r_t^+ < \mathrm{e}^{2Mht}\big[h+H\big] + H\ \forall t \in [0,\Delta/\lambda]\Big\} + 3\exp\left\{-\frac{H^2}{2C_1h^2\sigma^2\,\mathrm{e}^{2\lambda^+\Delta/\lambda}}\right\}. \tag{6.24}$$

The first term on the right-hand side can be bounded using (6.11) with $a = H$, yielding

$$q(\Delta) \leqslant \frac{C_0}{\sigma^2}\left[(\mathrm{e}^{2Mh\Delta/\lambda}\big[h+H\big] + H)\,\mathrm{e}^{-\Delta} + H\right]^2 + 3\exp\left\{-\frac{H^2}{2C_1h^2\sigma^2\,\mathrm{e}^{2\lambda^+\Delta/\lambda}}\right\}. \tag{6.25}$$

We now make the choices

$$H = \mathrm{e}^{-\Delta}\,h \qquad \text{and} \qquad \Delta = \frac{1+\mu}{2}\log\left(1+\mu+\frac{h^2}{\sigma^2}\right). \tag{6.26}$$

Substituting in (6.25) and carrying out computations similar to those in [BG06, Theorem 3.2.2] yields $\log(q(\Delta)^{-1})/\Delta \geqslant 2\mu/(1+\mu) - \mathcal{O}(1/|\log\sigma|)$, and hence the result. □

The estimate (6.15) can be extended to the exit from a neighbourhood of order 1 of the unstable orbit, using exactly the same method as in [BGK12, Section D]:

Proposition 6.5. *Fix a small constant $\delta > 0$. Then for any $\kappa < 2$, there exist constants $\sigma_0, \alpha, C > 0$ and $0 < \nu < 2$ such that*

$$\mathbb{P}^{r_0}\big\{0 < r_t < \delta\ \forall t \in [0,T]\big\} \leqslant \frac{C}{\sigma^\alpha}\,\mathrm{e}^{-\kappa\lambda_+ T} \tag{6.27}$$

holds for all $r_0 \in (0,\delta)$, all $\sigma < \sigma_0$ and all $T \leqslant \sigma^{-\nu}$.

Proof: The proof follows along the lines of [BGK12, Sections D.2 and D.3]. The idea is to show that once sample paths have reached the level $h = \sigma^\gamma$, they are likely to reach level δ after a relatively short time, without returning below the level $h/2$. To control the effect of paths which switch once or several times between the levels h and $h/2$ before leaving $(0,\delta)$, one uses Laplace transforms.

Let τ_1 denote the first-exit time of r_t from $(0,h)$, where we set $\tau_1 = T$ if r_t remains in $(0,h)$ up to time T. Combining Proposition 6.3 with $h_1 = \sqrt{h^3 t}$ and Proposition 6.4, we obtain

$$\mathbb{P}^{r_0}\big\{\tau_1 > t\big\} \leqslant \frac{1}{\sigma^{2\mu(1-\gamma)}}\,\mathrm{e}^{-\kappa_1\lambda_+ t} + \mathrm{e}^{-1/(C_1\sigma^{2-\gamma})} \quad \forall t \in [0,T]\,, \tag{6.28}$$

where $\kappa_1 = 2\mu/(1+\mu) - \mathcal{O}(1/|\log\sigma|)$. The first term dominates the second one as long as $\nu < 2-\gamma$. Thus the Laplace transform $\mathbb{E}\{\mathrm{e}^{u(\tau_1\wedge T)}\}$ exists for all $u < 1/(\kappa_1\lambda_+)$.

Let τ_2 denote the first-exit time of r_t from $(h/2,\delta)$. As in [BGK12, Proposition D.4], using the fact that the drift term is bounded below by a constant times r, that $\{\tau_2 > t\} \subset \{r_t < \delta\}$, an endpoint estimate and the Markov property to restart the process at times which are multiples of $|\log\sigma|$, we obtain

$$\mathbb{P}^h\big\{\tau_2 > t\big\} \leqslant \exp\left\{-C_2\frac{t}{\sigma^{2(1-\gamma)}|\log\sigma|}\right\} \quad \forall t \in [0,T] \tag{6.29}$$

for some constant C_2. Therefore the Laplace transform $\mathbb{E}\{\mathrm{e}^{u(\tau_2\wedge T)}\}$ exists for all u of order $1/(\sigma^{2(1-\gamma)}|\log\sigma|)$. In addition, one can show that the probability that sample paths starting at level h reach $h/2$ before δ is exponentially small in $1/\sigma^{2(1-\gamma)}$.

We can now use [BGK12, Lemma D.5] to estimate the Laplace transform of $\tau = \tau_0 \wedge \tau_\delta \wedge T$, and thus the decay of $\mathbb{P}\{\tau > t\}$ via the Markov inequality. Given $\kappa = 2-\epsilon$, we first choose μ and σ_0 such that $\kappa_1 \leqslant 2 - \epsilon/2$. This allows to estimate $\mathbb{E}\{\mathrm{e}^{u\tau}\}$ for $u = \kappa_1 - \epsilon/2$ to get the desired decay, and determines α. The choice of μ also determines γ and thus ν. □

Proposition 6.6 (Lower bound on the probability to stay near the unstable orbit)**.** *Let $h = \sigma^\gamma$ for some $\gamma \in (1/2, 1)$, and let $A = [\sigma a, \sigma b]$ for constants $0 < a < b$. Then there exists a constant C such that*

$$\mathbb{P}^{r_0}\big\{0 < r_t < h \;\forall t \in [0,T]\;,\; r_T \in A\big\} \geqslant C\exp\biggl\{-2\lambda_+ T\biggl[1 + \mathcal{O}\Bigl(\frac{1}{|\log\sigma|}\Bigr)\biggr]\biggr\} \tag{6.30}$$

holds for all $r_0 \in A$ and all $T \leqslant 1/h$.

Proof: Consider again a partition of $[0,T]$ into intervals of length Δ/λ_+, and let $q(\Delta)$ be a lower bound on

$$\mathbb{P}^{r_0}\big\{0 < r_t < h \;\forall t \in [0,\Delta/\lambda_+]\;,\; r_{\Delta/\lambda_+} \in A\big\} \tag{6.31}$$

valid for all $r_0 \in A$. By comparing, as in the proof of Proposition 6.4, r_t with solutions of linear equations, and using the lower bound of Proposition 6.1, we obtain

$$q(\Delta) = C_1\,\mathrm{e}^{-2\Delta} - C_2\,\mathrm{e}^{-c/(\sigma^2\,\mathrm{e}^{4\Delta})} \tag{6.32}$$

for constants $C_1, C_2, c > 0$, where the second term bounds the probability that the martingales $\sigma\mathcal{M}_t^{0,\pm}$ exceed $H = \mathrm{e}^{-\Delta}\,h$ times an exponentially decreasing curve. By the Markov property, we can bound the probability we are interested in below by the expression (6.16). The result follows by choosing $\Delta = c_0|\log\sigma|$ for a constant c_0. □

We can now use the last two bounds to estimate the principal eigenvalue of the Markov chain on $E = [0, 2\delta]$ with kernel K^{u}, describing the process killed upon hitting either the unstable orbit at $r = 0$ or level $r = 2\delta$.

Theorem 6.7 (Bounds on the principal eigenvalue λ_0^{u})**.** *For any sufficiently small $\delta > 0$, there exist constants $\sigma_0, c > 0$ such that*

$$(1 - c\delta^2)\,\mathrm{e}^{-2\lambda_+ T_+} \leqslant \lambda_0^u \leqslant (1 + c\delta^2)\,\mathrm{e}^{-2\lambda_+ T_+} \tag{6.33}$$

holds for all $\sigma < \sigma_0$.

Proof: We will apply Proposition 5.4. In order to do so, we pick $n \in \mathbb{N}$ such that

$$T = \frac{nT_+}{1 + M\delta^2 T_+} \tag{6.34}$$

satisfies Proposition 6.5 and is of order $\sigma^{-\nu}$ with $\nu < 2$. Proposition 6.3 shows that with probability larger than $1 - \mathrm{e}^{-\delta/(C_1\sigma^{2-\gamma})}$,

$$\varphi_t \leqslant \frac{t}{T_+} + M\delta^2 T \qquad \text{for all } t \leqslant T \wedge \tau_h \tag{6.35}$$

for $h = \sigma^\gamma$ as before, with $\gamma > \nu$. In particular, we have $\varphi_T \leqslant n$. Together with Propositions 6.4 and 6.5 applied for $\kappa = 2 - \delta^2$, this shows that for any $r_0 \in E$

$$K_n^{\mathrm{u}}(r_0, E) \leqslant \frac{C}{\sigma^\alpha} \exp\biggl\{ -(2-\delta^2)\lambda_+ \frac{nT_+}{1 + M\delta^2 T_+} \biggr\} + \mathrm{e}^{-1/(C_1\sigma^{2-\gamma})} + \frac{1}{\sigma^{2\mu(1-\gamma)}} \exp\biggl\{ -\lambda_+ T \biggl[\frac{2\mu}{1+\mu} - \mathcal{O}\Bigl(\frac{1}{|\log\sigma|} \Bigr) \biggr] \biggr\} . \tag{6.36}$$

Using $\log(a+b+c) \leqslant \log 3 + \max\{\log a, \log b, \log c\}$ and the fact that $\nu < 2$, we obtain

$$\frac{1}{n} \log K_n^{\mathrm{u}}(r_0, E) \leqslant \max\biggl\{ -(2-\delta^2) \frac{\lambda_+ T_+}{1 + M\delta^2 T_+}, -\frac{2\mu}{1+\mu} \frac{\lambda_+ T_+}{1 + M\delta^2 T_+}, \frac{1}{n} \mathcal{O}\bigl(|\log\sigma| + \sigma^{-(2-\gamma)} \bigr) \biggr\} \tag{6.37}$$

Since n has order $\sigma^{-\nu}$, we can make σ small enough for all error terms to be of order δ^2. Choosing first μ, then the other parameters, proves the upper bound. The proof of the lower bound is similar. It is based on Proposition 6.6, a basic comparison between r_T and the value of r_t at the time t when φ_t reaches n, and the lower bound in Proposition 5.4. □

6.2 The principal eigenvalue λ_0^{s} and the spectral gap

We consider in this section the system

$$\begin{aligned} \mathrm{d}r_t &= \bigl[-\lambda_- r_t + b_r(r_t, \varphi_t) \bigr] \, \mathrm{d}t + \sigma g_r(r_t, \varphi_t) \, \mathrm{d}W_t \, , \\ \mathrm{d}\varphi_t &= \Bigl[\frac{1}{T_+} + b_\varphi(r_t, \varphi_t) \Bigr] \, \mathrm{d}t + \sigma g_\varphi(r_t, \varphi_t) \, \mathrm{d}W_t \, , \end{aligned} \tag{6.38}$$

describing the dynamics away from the unstable orbit. We have redefined r in such a way that the stable orbit is now located in $r = 0$, and that the unstable orbit is located in $r = 1$.

In what follows we consider the Markov chain of the process killed upon reaching level $1 - \delta$ where $\delta \in (1/2, 1)$, whose kernel we denote K^{s}. The corresponding state space is given by $E = [-L, 1-\delta]$ for some $L \geqslant 1$.

Proposition 6.8 (Lower bound on the principal eigenvalue λ_0^{s})**.** *There exists a constant $\kappa > 0$ such that*

$$\lambda_0^s \geqslant 1 - \mathrm{e}^{-\kappa/\sigma^2} \ . \tag{6.39}$$

Proof: Let $A = [-h, h]$ for some $h > 0$. If h is sufficiently small, the stability of the periodic orbit in $r = 0$ implies that any deterministic solution starting in $(r_0, 0)$ with $r_0 \in A$ satisfies $|r_1| \leqslant h_1 < h$ when it reaches the line $\varphi = 1$ at a point $(r_1, 1)$. In fact, by slightly enlarging h_1 we can ensure that $|r_t| \leqslant h_1 < h$ whenever φ_t is in a small neighbourhood of 1. Using, for instance, [BG06, Theorem 5.1.18],[3] one obtains that the random sample path with initial condition $(r_0, 0)$ stays, on timescales of order 1, in a ball around the deterministic solution with high probability. The probability of leaving the ball is exponentially small in $1/\sigma^2$. This shows that $K(x, A)$ is exponentially close to 1 for all $x \in A$ and proves the result, thanks to Proposition 5.4. □

[3] This might seem like slight overkill, but it works.

Note that in the preceding proof we showed that for $A = [-h, h]$, there exist constants $C_1, \kappa > 0$ such that

$$\sup_{x \in A} K(x, E \setminus A) + p_{\text{kill}}(A) \leqslant C_1 \, \mathrm{e}^{-\kappa/\sigma^2} \ . \tag{6.40}$$

The following proposition gives a similar estimate allowing for initial conditions $x \in E$.

Proposition 6.9 (Bound on the "contraction constant"). *Let $A = [-h, h]$. For any $h > 0$, there exist $n_1 \in \mathbb{N}$ and constants $C_1, \kappa > 0$ such that*

$$\gamma^{n_1}(A) \coloneqq \sup_{x \in E} K_{n_1}(x, E \setminus A) \leqslant C_1 \, \mathrm{e}^{-\kappa/\sigma^2} \ . \tag{6.41}$$

Proof: Consider a deterministic solution $x_t^{\text{det}} = (r_t^{\text{det}}, \varphi_t^{\text{det}})$ with initial condition $x_0 = (r_0, 0)$. The stability of the orbit in $r = 0$ implies that x_t^{det} will reach a neighbourhood of size $h/2$ of this orbit in a time T of order 1. By [BG06, Theorem 5.1.18], we have for all $t \geqslant 0$

$$\mathbb{P}^{x_0} \left\{ \sup_{0 \leqslant s \leqslant t} \| x_t - x_t^{\text{det}} \| > h_0 \right\} \leqslant C_0 (1 + t) \, \mathrm{e}^{-\kappa_0 h_0^2/\sigma^2} \tag{6.42}$$

for some constants $C_0, \kappa_0 > 0$. The estimate holds for all $h_0 \leqslant h_1/\chi(t)$, where h_1 is another constant, and $\chi(t)$ is related to the local Lyapunov exponent of x_t^{det}. Though $\chi(t)$ may grow exponentially at first, it will ultimately (that is after a time of order 1) grow at most linearly in time, because x_t^{det} is attracted by the stable orbit. Thus we have $\chi(T) \leqslant 1 + CT$ for some constant C. Applying (6.42) with $h_0 = h/2$, we find that any sample path which is not killed before time n_1 close to T will hit A with a high probability, which yields the result. □

Let $X_1^{x_1}$ and $X_1^{x_2}$ denote the values of the first component r_t of the solution of the SDE (6.38) with initial condition x_1 or x_2, respectively, at the random time at which φ_t first reaches the value 1. Note that both processes are driven by the same realization of the Brownian motion.

Proposition 6.10 (Bound on the difference of two orbits). *There exist constants $h_0, c > 0$ and $\rho < 1$ such that*

$$\mathbb{P}\big\{ |X_1^{x_2} - X_1^{x_1}| \geqslant \rho |x_2 - x_1| \big\} \leqslant \mathrm{e}^{-c/\sigma^2} \tag{6.43}$$

holds for all $x_1, x_2 \in A = [-h_0, h_0]$.

Proof: Let (ξ_t, η_t) denote the difference of the two sample paths started in $(x_1, 0)$ and $(x_2, 0)$, respectively. It satisfies a system of the form

$$\begin{aligned} \mathrm{d}\xi_t &= -\lambda_- \xi_t \, \mathrm{d}t + b_1(\xi_t, \eta_t) \, \mathrm{d}t + \sigma g_1(\xi_t, \eta_t) \, \mathrm{d}W_t \ , \\ \mathrm{d}\eta_t &= b_2(\xi_t, \eta_t) \, \mathrm{d}t + \sigma g_2(\xi_t, \eta_t) \, \mathrm{d}W_t \ , \end{aligned} \tag{6.44}$$

with initial condition $(\xi_0, 0)$, where we may assume that $\xi_0 = x_2 - x_1 > 0$. Here $|b_i(\xi, \eta)| \leqslant M(\xi^2 + \eta^2)$ and $|g_i(\xi, \eta)| \leqslant M(|\xi| + |\eta|)$ for $i = 1, 2$ (remember that both solutions are driven by the same Brownian motion). Consider the stopping times

$$\begin{aligned} \tau_\xi &= \inf\{ t > 0 \colon \xi_t > H \, \mathrm{e}^{-\lambda_- t} \} \ , \\ \tau_\eta &= \inf\{ t > 0 \colon |\eta_t| > h \} \ , \end{aligned} \tag{6.45}$$

where we set $H = \xi_0 + h = Lh$. Writing η_t in integral form and using Lemma A.1, we obtain

$$\mathbb{P}\big\{\tau_\eta < \tau_\xi \wedge 2\big\} \leqslant \mathrm{e}^{-c_1/\sigma^2} \tag{6.46}$$

for some constant $c_1 > 0$, provided h is smaller than some constant depending only on M and L. In a similar way, one obtains

$$\mathbb{P}\big\{\tau_\xi < \tau_\eta \wedge 2\big\} \leqslant \mathrm{e}^{-c_2/\sigma^2} \tag{6.47}$$

for some constant $c_2 > 0$, provided H is smaller than some constant depending only on M. It follows that

$$\mathbb{P}\big\{\tau_\xi \wedge \tau_\eta \leqslant 2\big\} \leqslant \mathrm{e}^{-c/\sigma^2} \tag{6.48}$$

for a $c > 0$. Together with the control on the diffusion along φ (cf. Proposition 6.3), we can thus guarantee that both sample paths have crossed $\varphi = 1$ before time 2, at a distance

$$|X_1^{x_2} - X_1^{x_1}| \leqslant \mathrm{e}^{-\lambda_-/2}(\xi_0 + h) \tag{6.49}$$

with probability exponentially close to 1. For any $\rho \in (\mathrm{e}^{-\lambda_-/2}, 1)$, we can find h such that the right-hand side is smaller than $\rho\xi_0 = \rho|x_2 - x_1|$. This yields the result. □

From (6.43), we immediately get

$$\mathbb{P}\big\{|X_n^{x_2} - X_n^{x_1}| \geqslant \rho^n|x_2 - x_1|\big\} \leqslant n\,\mathrm{e}^{-c/\sigma^2} \qquad \forall n \geqslant 1\,. \tag{6.50}$$

Fix $a > 0$ and let N be the integer stopping time

$$N = N(x_1, x_2) = \inf\big\{n \geqslant 1\colon |X_n^{x_2} - X_n^{x_1}| < a\sigma^2\big\}\,. \tag{6.51}$$

If n_0 is such that $\rho^{n_0}\mathrm{diam}(A) \leqslant a\sigma^2$, then (6.50) implies $\mathbb{P}\{N > n_0\} \leqslant n_0\,\mathrm{e}^{-c/\sigma^2}$ whenever $x_1, x_2 \in A$. Using Proposition 6.9 and the Markov property, we obtain the following improvement.

Proposition 6.11 (Bound on the hitting time of a small ball)**.** *There is a constant C_2 such that for any $k \geqslant 1$ and all $x_1, x_2 \in A$, we have*

$$\mathbb{P}\big\{N(x_1, x_2) > kn_0 \mid X_l^{x_1}, X_l^{x_2} \in A\,\forall l \leqslant kn_0\big\} \leqslant \big(C_2|\log(a\sigma^2)|\,\mathrm{e}^{-\kappa_2/\sigma^2}\big)^k\,, \tag{6.52}$$

where $\kappa_2 = c$.

Proof: By the definition of n_0 and (6.40), for any $x_1, x_2 \in A$ we have

$$\begin{aligned}\mathbb{P}\big\{N > n_0 \mid X_l^{x_1}, X_l^{x_2} \in A\,\forall l \leqslant n_0\big\} &\leqslant \frac{\mathbb{P}\{N > n_0\}}{\mathbb{P}\{X_l^{x_1}, X_l^{x_2} \in A\,\forall l \leqslant n_0\}} \\ &\leqslant \frac{n_0\,\mathrm{e}^{-c/\sigma^2}}{1 - n_0C_1\,\mathrm{e}^{-\kappa/\sigma^2}} \leqslant 2n_0\,\mathrm{e}^{-c/\sigma^2}\,.\end{aligned}$$

Thus the result follows by applying the Markov property at times which are multiples of n_0, and recalling that n_0 has order $|\log(a\sigma^2)|$. □

Combining the last estimates with Proposition 5.5, we finally obtain the following result.

Theorem 6.12 (Spectral gap estimate for K^{s})**.** *There exists a constant $c > 0$ such that for sufficiently small σ, the first eigenvalue of K^{s} satisfies*

$$|\lambda_1^s| \leqslant \mathrm{e}^{-c/|\log \sigma|} \tag{6.53}$$

Proof: We take $n = k(n_0 + n_1)$, where k will be chosen below. Fix $h > 0$ and set $A = (-h, h)$. We apply Proposition 5.5 for the Markov chain K_n^{s}, conditioned on not leaving A, with $m(y) = \inf_{x\in A} k_n(x, y)$, which yields

$$|\lambda_1|^n \leqslant \max\left\{2\gamma^n(A)\ ,\ (\lambda_0^{\mathrm{s}})^n L - 1 + \gamma^n(A)\frac{(\lambda_0^{\mathrm{s}})^n L}{(\lambda_0^{\mathrm{s}})^n L - 1}\left[1 + \frac{1}{(\lambda_0^{\mathrm{s}})^n - \gamma^n(A)}\right]\right\} . \tag{6.54}$$

Proposition 6.9 shows that $\gamma^n(A) \leqslant \gamma^n([-h/2, h/2])$ is exponentially small, since $n \geqslant n_1$. Proposition 6.8 shows that $(\lambda_0^{\mathrm{s}})^n$ is bounded below by $1 - n\,\mathrm{e}^{-\kappa/\sigma^2}$. It thus remains to estimate L. Proposition 5.9 shows that

$$L \leqslant \frac{1 + \eta + \sup_{x_1,x_2\in A} \mathbb{P}\{N(x_1, x_2) > n - 1\}\,\mathrm{e}^{C/\sigma^2}}{\inf_{x\in A} K_n(x, A)} , \tag{6.55}$$

where the parameter a in the definition of the stopping time N is determined by the choice of η. We thus fix, say, $\eta = 1/4$, and $k = \lceil C/\kappa_2\rceil + 1$. In this way, the numerator in (6.55) is exponentially close to $1 + \eta$. Since n has order $|\log \sigma|$, the denominator $K_{n-1}(x, A)$ is still exponentially close to 1, by the same argument as in Proposition 6.8. Making σ small enough, we can guarantee that $L - 1 \leqslant 3/8$ and $(\lambda_0^{\mathrm{s}})^n L > 1$, and thus $|\lambda_1|^n \leqslant 1/2$. The result thus follows from the fact that n has order $|\log \sigma|$. □

7 Distribution of exit locations

We can now complete the proof of Theorem 2.3, which is close in spirit to the proof of [BG04, Theorem 2.3]. We fix an initial condition $(r_0, 0)$ close to the stable periodic orbit and a small positive constant Δ. Let

$$P_\Delta(\varphi) = \mathbb{P}^{r_0,0}\big\{\varphi_{\tau_-} \in [\varphi_1, \varphi_1 + \Delta]\big\} \tag{7.1}$$

be the probability that the first hitting of level $1 - \delta$ occurs in the interval $[\varphi_1, \varphi_1 + \Delta]$. If we write $\varphi_1 = k + s$ with $k \in \mathbb{N}$ and $s \in [0, 1)$, we have by the same argument as the one given in (2.25),

$$P_\Delta(k + s) = C(r_0)(\lambda_0^{\mathrm{s}})^k \mathbb{P}^{\pi_0^{\mathrm{s}}}\big\{\varphi_{\tau_-} \in [s, s + \Delta]\big\}\left[1 + \mathcal{O}\left(\left(\frac{\lambda_1^{\mathrm{s}}}{\lambda_0^{\mathrm{s}}}\right)^k\right)\right] , \tag{7.2}$$

where $C(r_0)$ is a normalising constant, π_0^{s} is the quasistationary distribution for K^{s}, and

$$\mathbb{P}^{\pi_0^{\mathrm{s}}}\big\{\varphi_{\tau_-} \in [s, s + \Delta]\big\} = \int_{-L}^{1-\delta} \pi_0^{\mathrm{s}}(r)\mathbb{P}^r\big\{\varphi_{\tau_-} \in [s, s + \Delta]\big\}\,\mathrm{d}r . \tag{7.3}$$

Note that π_0^{s} is concentrated near the stable periodic orbit. By the large deviation principle and our assumption on the uniqueness of the minimal path γ_∞, $\mathbb{P}^r\{\varphi_{\tau_-} \in [s, s + \Delta]\}$ is maximal at the point s^* where γ_∞ crosses the level $1 - \delta$, and decays exponentially fast in $1/\sigma^2$ away from s^*. In addition, our spectral gap estimate Theorem 6.12 shows that the

error term in (7.2) has order δ as soon as k has order $|\log\sigma||\log\delta|$. For these k we thus have

$$P_\Delta(k+s) = C_1(\lambda_0^{\mathrm{s}})^k\,\mathrm{e}^{-J(s)/\sigma^2}\big[1+\mathcal{O}(\delta)\big]\;, \tag{7.4}$$

where $J(s)$ is periodic with a unique minimum per period in s^*. The minimal value $J(s^*)$ is close to the value of the rate function of the optimal path up to level $1-\delta$.

Now let us fix an initial condition $(1-\delta,\varphi_1)$, and consider the probability

$$Q_\Delta(\varphi_1,\varphi_2) = \mathbb{P}^{1-\delta,\varphi_1}\big\{\varphi_\tau\in[\varphi_2,\varphi_2+\Delta]\big\} \tag{7.5}$$

of first reaching the unstable orbit during the interval $[\varphi_2,\varphi_2+\Delta]$ when starting at level $1-\delta$. By the same argument as above,

$$Q_\Delta(\varphi_1,\ell+s) = C_2(\lambda_0^{\mathrm{u}})^\ell\mathbb{P}^{\pi_0^{\mathrm{u}}}\big\{\varphi_\tau\in[s,s+\Delta]\big\}\bigg[1+\mathcal{O}\bigg(\Big(\frac{\lambda_1^{\mathrm{u}}}{\lambda_0^{\mathrm{u}}}\Big)^\ell\bigg)\bigg]\;. \tag{7.6}$$

On the other hand, by the large-deviation principle (cf. Proposition 4.2), we have

$$Q_\Delta(\varphi_1,\ell+s) = D_\ell(s)\exp\bigg\{-\frac{1}{\sigma^2}\Big[I_\infty+\frac{c(s)}{2}\,\mathrm{e}^{-2\ell\lambda_+T_+}\big[1+\mathcal{O}(\mathrm{e}^{-2\ell\lambda_+T_+})+\mathcal{O}(\delta)\big]\Big]\bigg\}\;, \tag{7.7}$$

where

$$c(s) = \delta^2\,\mathrm{e}^{-2\lambda_+T_+s}\,\frac{h^{\mathrm{per}}(s)}{h^{\mathrm{per}}(\varphi_1)^2}\;, \tag{7.8}$$

and the σ-dependent prefactor satisfies $\lim_{\sigma\to0}\sigma^2\log D_\ell(s)=0$. In particular we have

$$-\lim_{\sigma\to0}\sigma^2\log Q_\Delta(\varphi_1,\ell+s) = I_\infty+\frac{c(s)}{2}\,\mathrm{e}^{-2\ell\lambda_+T_+}\big[1+\mathcal{O}(\mathrm{e}^{-2\ell\lambda_+T_+})+\mathcal{O}(\delta)\big]\;. \tag{7.9}$$

This implies in particular that the kernel K^{u} has a spectral gap. Indeed, assume by contradiction that $\lim_{\sigma\to0}(\lambda_1^{\mathrm{u}}/\lambda_0^{\mathrm{u}})=1$. Using this in (7.6), we obtain that $\sigma^2\log Q_\Delta(\varphi_1,\ell+s)$ converges to a quantity independent of ℓ, which contradicts (7.9). We thus conclude that $\lim_{\sigma\to0}(\lambda_1^{\mathrm{u}}/\lambda_0^{\mathrm{u}})=\rho<1$. In fact, (7.9) suggests that $\rho\simeq\mathrm{e}^{-2\lambda_+T_+}$, but we will not attempt to prove this here. The existence of a spectral gap shows that for all $\ell\gg|\log\delta|$, we have

$$D_\ell(s) = (\lambda_0^{\mathrm{u}})^\ell D^*(s)\big[1+\mathcal{O}(\delta)\big]\;, \tag{7.10}$$

where $D^*(s)=\lim_{\ell\to\infty}(\lambda_0^{\mathrm{u}})^{-\ell}D_\ell(s)$. Periodicity implies that $D_\ell(s+1)=D_{\ell+1}(s)$, so that necessarily

$$D^*(s+1) = \mathrm{e}^{-2\lambda_+T_+}\,D^*(s)\big[1+\mathcal{O}(\delta)\big]\;. \tag{7.11}$$

Let us now define

$$\theta(s) = -\frac{1}{2}\log c(s) = \lambda_+T_+s-\frac{1}{2}\log\bigg(\delta^2\frac{h^{\mathrm{per}}(s)}{h^{\mathrm{per}}(\varphi_1)^2}\bigg)\;. \tag{7.12}$$

Note that $\theta(s+1)=\theta(s)+\lambda_+T_+$. Since $\sigma^{-2}=\mathrm{e}^{2|\log\sigma|}$, we can rewrite (7.7) in the form

$$Q_\Delta(\varphi_1,\ell+s) = D_\ell(s)\,\mathrm{e}^{-I_\infty/\sigma^2}\exp\bigg\{-\frac{1}{2}\,\mathrm{e}^{-2\left[\theta(s)+\ell\lambda_+T_+-|\log\sigma|\right]}\big[1+\mathcal{O}(\mathrm{e}^{-2\ell\lambda_+T_+})+\mathcal{O}(\delta)\big]\bigg\}\;. \tag{7.13}$$

In a similar way, the prefactor $D_\ell(s)$ can be rewritten, for $\ell\gg|\log\delta|$, as

$$D_\ell(s) = \sigma^2\,\mathrm{e}^{2\theta(s)}\,D^*(s)\exp\bigg\{-2\Big[\theta(s)+\ell\lambda_+T_+[1+\mathcal{O}(\delta)]-|\log\sigma|\Big]\bigg\}\;. \tag{7.14}$$

Introducing the notation

$$A(x) = \frac{1}{2} \exp\Big\{-2x - \frac{1}{2}\,\mathrm{e}^{-2x}\Big\}\,, \tag{7.15}$$

we can thus write

$$Q_\Delta(\varphi_1, \ell + s) = 2\sigma^2\,\mathrm{e}^{-I_\infty/\sigma^2}\,\mathrm{e}^{2\theta(s)}\,D^*(s) A\Big(\theta(s) + \ell\lambda_+ T_+ - |\log\sigma| + \mathcal{O}(\delta\ell)\Big)\,. \tag{7.16}$$

Let us finally consider the probability

$$\mathbb{P}^{r_0,0}\big\{\varphi_\tau \in [n+s, n+s+\Delta]\big\}\,. \tag{7.17}$$

As in [BG04, Section 5], it can be written as an integral over φ_1, which can be approximated by the sum

$$S = \sum_{\ell=1}^{n-1} P_\Delta(n - \ell + s^*) Q_\Delta(s^*, \ell + s)\,. \tag{7.18}$$

Note that since K^{u} is defined by killing the process when it reaches a distance 2δ from the unstable periodic orbit, we have to show that the contribution of paths switching back and forth several times between distance δ and 2δ is negligible (cf. [BG04, Section 4.3]). This is the case here as well, in fact we have used the same argument in the proof of Proposition 6.5. From here on, we can proceed as in [BG04, Section 5.2], to obtain

$$S = C_1\sigma^2\,\mathrm{e}^{-(I_\infty + J(s^*))/\sigma^2} (\lambda_0^{\mathrm{s}})^n\,\mathrm{e}^{2\theta(s)}\,D^*(s) S_1\,, \tag{7.19}$$

where

$$S_1 = \sum_{\ell=-\infty}^{\infty} (\lambda_0^{\mathrm{s}})^{-\ell} A\Big(\theta(s) + \ell\lambda_+ T_+ - |\log\sigma| + \mathcal{O}(\delta)\Big)\big[1 + \mathcal{O}(\delta|\log\delta|)\big]\,. \tag{7.20}$$

The main point is to note that only indices ℓ in a window of order $|\log\delta|$ contribute to the sum, and that $A(x+\varepsilon) = A(x)[1 + \mathcal{O}(\varepsilon)]$ for $x \geqslant 0$. Also, since λ_0^{s} is exponentially close to 1, the factor $(\lambda_0^{\mathrm{s}})^{-\ell}$ can be replaced by 1, with an error which is negligible compared to $\delta|\log\delta|$. Extending the bounds to $\pm\infty$ only generates a small error. Now (7.19) and (7.20) yield the main result, recalling that $\chi_\Delta(s) = \mathrm{e}^{2\theta(s)}\,D^*(s)$ is periodic in s.

A A Bernstein–type estimate

Lemma A.1 ([RW00, Thm. 37.8])**.** *Consider the martingale*

$$M_t = \int_0^t g(X_s, s)\,\mathrm{d}W_s\,, \tag{A.1}$$

where X_s is adapted to the filtration generated by W_s. Assume

$$g(X_s, s) g(X_s, s)^T \leqslant G(s)^2 \tag{A.2}$$

almost surely and that

$$V(t) = \int_0^t G(s)^2\,\mathrm{d}s < \infty\,. \tag{A.3}$$

Then

$$\mathbb{P}\Big\{\sup_{0\leqslant s\leqslant t} M_s > L\Big\} \leqslant \mathrm{e}^{-L^2/2V(t)} \tag{A.4}$$

holds for all $L > 0$.

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