Analysis and geometry of the measurable Riemannian structure on the Sierpiński gasket

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ABSTRACT. This expository article is devoted to a survey of existent results concerning the measurable Riemannian structure on the Sierpiński gasket and to a brief account of the author's recent result on Weyl's eigenvalue asymptotics of its associated Laplacian. In particular, properties of the Hausdorff measure with respect to the canonical geodesic metric are described in some detail as a key step to the proof of Weyl's asymptotics. A complete characterization of minimal geodesics is newly proved and applied to invalidity of Ricci curvature lower bound conditions such as the curvature-dimension condition and the measure contraction property. Possibility of and difficulty in extending the results to other self-similar fractals are also discussed.

CONTENTS

1. Introduction

The purpose of this expository article is to review known results concerning the measurable Riemannian structure on the *Sierpiński gasket* (Figure 1) and describe its connections to general theories of analysis and geometry on metric measure spaces. We also state the author's recent result on Weyl's eigenvalue asymptotics of its associated Laplacian and briefly explain the idea of its proof. In particular, we present various properties of the Hausdorff measure with respect to the canonical geodesic metric as the key facts for the proof of Weyl's asymptotics.

The notion of the *measurable Riemannian structure on the Sierpiński gasket* was first introduced by Kigami [52] on the basis of Kusuoka's construction in [63] of "weak gradients" for Dirichlet forms on fractals. In [52], Kigami proved that the Sierpiński gasket can be embedded in $\mathbb{R}^2$ by a certain harmonic map, whose image is now called the *harmonic Sierpiński gasket* (Figure 2), and that Kusuoka's "weak gradients" can be identified as the gradients with respect to the (measurable) "Riemannian structure" inherited from $\mathbb{R}^2$ through this embedding. (A related result is also found in Hino [35].) These results are reviewed in Section 3 after a brief account of the Sierpiński gasket and its standard Dirichlet form in Section 2.

Kigami further proved in [54] that the heat kernel associated with this "Riemannian structure" satisfies the two-sided *Gaussian* bound in terms of the natural geodesic metric, unlike typical fractal diffusions treated e.g. in [11, 60, 24, 7, 8] for whose transition densities (heat kernels) the two-sided *sub*-Gaussian bounds hold. Later in [44] the author proved some more detailed asymptotics of this heat kernel such as Varadhan's asymptotic relation, together with an analytic characterization of the geodesic metric and slight generalizations and improvements of the results in [54]. These results are reviewed in Section 5 following a summary of basic geometric properties of the measurable Riemannian structure in Section 4, where we also newly prove a complete characterization of minimal geodesics (Theorem 4.19).

Very recently, the author has also proved Weyl's Laplacian eigenvalue asymptotics for this case, which is to be treated in a forthcoming paper [47]. The proof of Weyl's asymptotics require some detailed properties of the Hausdorff measure with respect to the geodesic metric and this is reviewed in Section 6, along with the singularity of the Hausdorff measure to the energy measures. Then in Section 7, we give the statement of Weyl's asymptotics and sketch the idea of its proof.

Since the situation of the measurable Riemannian structure on the Sierpiński gasket looks similar to that of Riemannian manifolds, it is natural to expect close connections to general theories of analysis and geometry on metric measure spaces which are not applicable to the case of typical fractal diffusions. In fact, Koskela and Zhou [58, Section 4] recently proved that the theory of differential calculus on metric measure spaces, established by Cheeger [17] and developed further by e.g. Shanmugalingam [81] and Keith [48, 49, 50], is applicable to the measurable Riemannian structure on the Sierpiński gasket. To be more precise, they prove that in this case the $(1, 2)$-Sobolev space equipped with a natural $(1, 2)$-seminorm, due
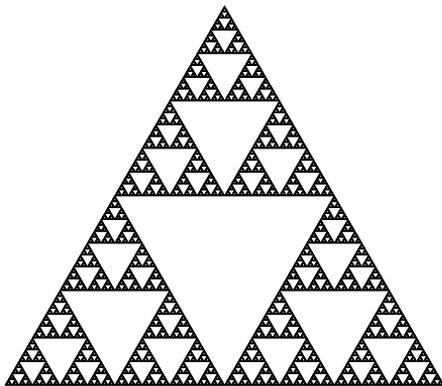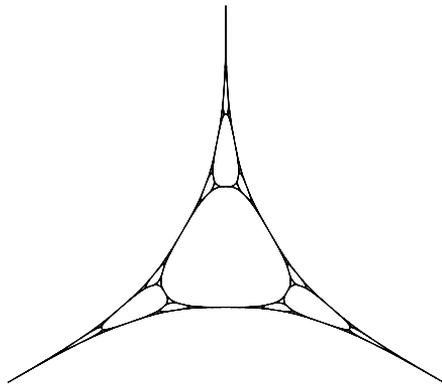
FIGURE 1. Sierpiński gasket

FIGURE 2. Harmonic Sierpiński gasket

to Cheeger [**17**, Section 2] and Shanmugalingam [**81**, Definition 2.5], coincides with the standard Dirichlet form on the Sierpiński gasket. This result is briefly reviewed in Subsection 8.1. On the other hand, the notions of Ricci curvature lower bound for general metric measure spaces due to Lott and Villani [**71, 70**], Sturm [**86, 87**] and Ohta [**76**] are *not* applicable to the case of the measurable Riemannian structure. More precisely, the (harmonic) Sierpiński gasket equipped with the natural geodesic metric and the "Riemannian volume measure" does not satisfy either the curvature dimension condition $\mathsf{CD}(k, N)$ of Lott and Villani [**71, 70**] and Sturm [**86, 87**] or the measure contraction property $\mathsf{MCP}(k, N)$ of Ohta [**76**] and Sturm [**87**] for any $(k, N) \in \mathbb{R} \times [1, \infty]$. We prove this fact in Subsection 8.2 (Theorem 8.23) as an application of the characterization of minimal geodesics (Theorem 4.19) after a review of the precise definitions of $\mathsf{CD}(k, N)$ and $\mathsf{MCP}(k, N)$ and related results.

Finally, we conclude this paper with a short discussion on possibility of (and difficulties in) extending the above-mentioned results to other self-similar fractals.

In the appendix, we provide a brief review of important results for the Brownian motion and the standard Laplacian on the Sierpiński gasket, whose associated heat kernel is known to satisfy the two-sided sub-Gaussian estimate and exhibit various oscillatory behavior. *Those who are not familiar with these results are strongly recommended to read the appendix directly after Section 2.*

NOTATION. In this article, we adopt the following notation and conventions.
(1) $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) The cardinality (the number of all the elements) of a set A is denoted by $\#A$.
(3) We set $\sup \emptyset := 0$ and $\inf \emptyset := \infty$. We write $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$ for $a, b \in [-\infty, \infty]$. We use the same notations also for functions. All functions treated in this paper are assumed to be $[-\infty, \infty]$-valued.
(4) Let $k \in \mathbb{N}$. The Euclidean inner product and norm on $\mathbb{R}^k$ are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively. For a continuous map $\gamma : [a, b] \to \mathbb{R}^k$, where $a, b \in \mathbb{R}$, $a \leq b$, let $\ell_{\mathbb{R}^k}(\gamma)$ be its length with respect to $|\cdot|$. Let $\mathbb{R}^{k \times k}$ be the set of real $k \times k$ matrices, which are also regarded as linear maps from $\mathbb{R}^k$ to itself through the standard basis of $\mathbb{R}^k$, and set $\mathbb{R}_0^{k \times k} := \mathbb{R}^{k \times k} \setminus \{0_{\mathbb{R}^{k \times k}}\}$. For $T \in \mathbb{R}^{k \times k}$, let $\det T$ be its determinant, T^* its transpose, and $\|T\|$ its *Hilbert-Schmidt* norm with respect to $\langle \cdot, \cdot \rangle$. The real orthogonal group of degree k is denoted by $O(k)$.

(5) Let E be a topological space. The Borel σ-field of E is denoted by $\mathcal{B}(E)$. We set $C(E) := \{f \mid f : E \to \mathbb{R}, \ f \text{ is continuous}\}$ and $\|f\|_\infty := \sup_{x \in E} |f(x)|$, $f \in C(E)$. For $A \subset E$, its interior in E is denoted by $\mathrm{int}_E A$ and its boundary in E by $\partial_E A$.

(6) Let (E, ρ) be a metric space. For $r \in (0, \infty)$, $x \in K$ and $A \subset E$, we set $B_r(x, \rho) := \{y \in E \mid \rho(x, y) < r\}$, $\mathrm{diam}_\rho A := \sup_{y, z \in A} \rho(y, z)$ and $\mathrm{dist}_\rho(x, A) := \inf_{y \in A} \rho(x, y)$. For $f : E \to \mathbb{R}$ we set $\mathbf{Lip}_\rho f := \sup_{x, y \in E, \ x \neq y} |f(x) - f(y)| / \rho(x, y)$.

2. Sierpiński gasket and its standard Dirichlet form

In this section, we briefly recall basic facts concerning the Sierpiński gasket and its standard Dirichlet form (resistance form). We mainly follow [**44**, Section 2] for the presentation of this section and refer the reader to [**25, 53, 56, 82**] for further details of each fact.

DEFINITION 2.1 (Sierpiński gasket). Let $V_0 = \{q_1, q_2, q_3\} \subset \mathbb{R}^2$ be the set of the three vertices of an equilateral triangle, set $S := \{1, 2, 3\}$, and for $i \in S$ define $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $f_i(x) := (x + q_i)/2$. The *Sierpiński gasket* (Figure 1) is defined as the *self-similar set associated with* $\{f_i\}_{i \in S}$, i.e. the unique non-empty compact subset K of $\mathbb{R}^2$ that satisfies $K = \bigcup_{i \in S} f_i(K)$. For $i \in S$ we set $F_i := f_i|_K : K \to K$. Define V_m for $m \in \mathbb{N}$ inductively by $V_m := \bigcup_{i \in S} F_i(V_{m-1})$ and set $V_* := \bigcup_{m \in \mathbb{N}} V_m$.

Note that $V_{m-1} \subset V_m$ for any $m \in \mathbb{N}$. K is always regarded as equipped with the relative topology inherited from $\mathbb{R}^2$, so that $F_i : K \to K$ is continuous for each $i \in S$ and V_* is dense in K.

DEFINITION 2.2. (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_m := S^m = \{w_1 \ldots w_m \mid w_i \in S \text{ for } i \in \{1, \ldots, m\}\}$ for $m \in \mathbb{N}$ and $W_* := \bigcup_{m \in \mathbb{N} \cup \{0\}} W_m$. For $w \in W_*$, the unique $m \in \mathbb{N} \cup \{0\}$ with $w \in W_m$ is denoted by $|w|$ and called the *length of w*. Also for $i \in S$ and $n \in \mathbb{N} \cup \{0\}$ we write $i^n := i \ldots i \in W_n$.

(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1 \omega_2 \omega_3 \ldots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, and define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1 \omega_2 \omega_3 \ldots) := \omega_2 \omega_3 \omega_4 \ldots$. Also for $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1 \omega_2 \omega_3 \ldots) := i \omega_1 \omega_2 \omega_3 \ldots$ and set $i^\infty := iii \ldots \in \Sigma$. For $\omega = \omega_1 \omega_2 \omega_3 \ldots \in \Sigma$ and $m \in \mathbb{N} \cup \{0\}$, we write $[\omega]_m := \omega_1 \ldots \omega_m \in W_m$.

(3) For $w = w_1 \ldots w_m \in W_*$, we set $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_m}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$.

Associated with the triple $(K, S, \{F_i\}_{i \in S})$ is a natural projection $\pi : \Sigma \to K$ given by the following proposition, which is used to describe the topological structure of K.

PROPOSITION 2.3. *There exists a unique continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$, and it satisfies $\{\pi(\omega)\} = \bigcap_{m \in \mathbb{N}} K_{[\omega]_m}$ for any $\omega \in \Sigma$. Moreover, $\#\pi^{-1}(x) = 1$ for $x \in K \setminus V_*$, $\pi^{-1}(q_i) = \{i^\infty\}$ for $i \in S$, and for $m \in \mathbb{N}$ and each $x \in V_m \setminus V_{m-1}$ there exist $w \in W_{m-1}$ and $i, j \in S$ with $i \neq j$ such that $\pi^{-1}(x) = \{wij^\infty, wji^\infty\}$.*

Recall the following basic fact ([**53**, Proposition 1.3.5-(2)]) meaning that V_0 should be considered as the *"boundary"* of K, which we will use below without further notice: if $w, v \in W_*$ and $\Sigma_w \cap \Sigma_v = \emptyset$ then $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$.

As studied in [**5, 53, 82**], a standard Dirichlet form (to be precise, a resistance form) $(\mathcal{E}, \mathcal{F})$ is defined on the Sierpiński gasket K, as follows. See [**53**, Chapter

2] and [**56**, Part 1] for general theory of resistance forms. A concise introduction to the theory of resistance forms is found in [**82**, Chapter 1], where the theory is illustrated by treating the particular case of the Sierpiński gasket in detail.

DEFINITION 2.4. Let $m \in \mathbb{N} \cup \{0\}$. We define a non-negative definite symmetric bilinear form $\mathcal{E}_m : \mathbb{R}^{V_m} \times \mathbb{R}^{V_m} \to \mathbb{R}$ on V_m by

$$(2.1) \qquad \mathcal{E}_m(u,v) := \frac{1}{2} \cdot \frac{1}{2} \left(\frac{5}{3}\right)^m \sum_{x,y \in V_m,\, x \overset{m}{\sim} y} (u(x) - u(y))(v(x) - v(y)),$$

where, for $x, y \in V_m$, we write $x \overset{m}{\sim} y$ if and only if $x, y \in F_w(V_0)$ for some $w \in W_m$ and $x \neq y$.

The usual definition of $\mathcal{E}_m$ does not contain the factor $1/2$ so that each edge in the graph $(V_m, \overset{m}{\sim})$ has resistance $(3/5)^m$. Here it has been added for simplicity of the subsequent arguments; see Definition 3.1-(0) below. The factor $3/5$, called the *resistance scaling factor of the Sierpiński gasket*, is specifically chosen for the sake of the validity of the following proposition.

PROPOSITION 2.5. *Let $m, n \in \mathbb{N} \cup \{0\}$, $m \leq n$. Then for each $u \in \mathbb{R}^{V_m}$,*

$$(2.2) \qquad \mathcal{E}_m(u,u) = \min\{\mathcal{E}_n(v,v) \mid v \in \mathbb{R}^{V_n},\, v|_{V_m} = u\}$$

and there exists a unique function $h_{m,n}(u) \in \mathbb{R}^{V_n}$ with $h_{m,n}(u)|_{V_m} = u$ such that $\mathcal{E}_m(u,u) = \mathcal{E}_n(h_{m,n}(u), h_{m,n}(u))$. Moreover, $h_{m,n} : \mathbb{R}^{V_m} \to \mathbb{R}^{V_n}$ is linear.

Let $u : V_* \to \mathbb{R}$. (2.2) implies that $\{\mathcal{E}_m(u|_{V_m}, u|_{V_m})\}_{m \in \mathbb{N} \cup \{0\}}$ is non-decreasing and hence has the limit in $[0, \infty]$. Moreover, if $\lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty$, then it is not difficult to verify that u is uniformly continuous with respect to any metric on K compatible with the original (Euclidean) topology of K, so that u is uniquely extended to a continuous function on K. Based on these observations, we can prove the following theorem; see [**53**, Chapter 2 and Section 3.3] or [**82**, Chapter 1] for details. Let $\mathbf{1} := \mathbf{1}_K$ denote the constant function on K with value 1.

THEOREM 2.6. *Define $\mathcal{F} \subset C(K)$ and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by*

$$(2.3) \qquad \begin{aligned} \mathcal{F} &:= \{u \in C(K) \mid \lim_{m \to \infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\}, \\ \mathcal{E}(u,v) &:= \lim_{m \to \infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}) \in \mathbb{R}, \qquad u, v \in \mathcal{F}. \end{aligned}$$

Then $\mathcal{F}$ is a dense subalgebra of $C(K)$, $\mathcal{E}$ is a non-negative definite symmetric bilinear form on $\mathcal{F}$, and $(\mathcal{E}, \mathcal{F})$ possesses the following properties:

(1) *$\{u \in \mathcal{F} \mid \mathcal{E}(u,u) = 0\} = \{c\mathbf{1} \mid c \in \mathbb{R}\} =: \mathbb{R}\mathbf{1}$, and $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$ is a Hilbert space.*

(2) *$R_{\mathcal{E}}(x,y) := \sup_{u \in \mathcal{F} \setminus \mathbb{R}\mathbf{1}} |u(x) - u(y)|^2 / \mathcal{E}(u,u) < \infty$ for any $x, y \in K$ and $R_{\mathcal{E}} : K \times K \to [0, \infty)$ is a metric on K compatible with the original topology of K.*

(3) *$u^+ \wedge \mathbf{1} \in \mathcal{F}$ and $\mathcal{E}(u^+ \wedge \mathbf{1}, u^+ \wedge \mathbf{1}) \leq \mathcal{E}(u,u)$ for any $u \in \mathcal{F}$.*

(4) *$\mathcal{F} = \{u \in C(K) \mid u \circ F_i \in \mathcal{F} \text{ for any } i \in S\}$, and for any $u, v \in \mathcal{F}$,*

$$(2.4) \qquad \mathcal{E}(u,v) = \frac{5}{3} \sum_{i \in S} \mathcal{E}(u \circ F_i, v \circ F_i).$$

$(\mathcal{E}, \mathcal{F})$ is called the *standard resistance form on the Sierpiński gasket*, which is indeed a resistance form on K with resistance metric $R_{\mathcal{E}}$ by Theorem 2.6-(1),(2),(3) and $\mathcal{F}$ being a dense subalgebra of $C(K)$. Consequently we also have the following theorem by virtue of [**56**, Corollary 6.4, Theorems 9.4 and 10.4], where the strong

locality of $(\mathcal{E}, \mathcal{F})$ follows from (2.4) and $\mathcal{E}(\mathbf{1}, \mathbf{1}) = 0$. See [**25**, Section 1.1] for the notions of regular Dirichlet forms and their strong locality.

THEOREM 2.7. *Let ν be a finite Borel measure on K with full support, i.e. such that $\nu(U) > 0$ for any non-empty open subset U of K. Then $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(K, \nu)$, and its associated Markovian semigroup $\{T_t^\nu\}_{t \in (0, \infty)}$ on $L^2(K, \nu)$ admits a* continuous integral kernel p_ν, *i.e. a continuous function $p_\nu = p_\nu(t, x, y) : (0, \infty) \times K \times K \to \mathbb{R}$ such that for any $f \in L^2(K, \nu)$ and any $t \in (0, \infty)$,*

$$(2.5) \qquad T_t^\nu f = \int_K p_\nu(t, \cdot, y) f(y) d\nu(y) \qquad \nu\text{-a.e.}$$

In the situation of Theorem 2.7, a standard monotone class argument easily shows that such p_ν is unique and satisfies $p_\nu(t, x, y) = p_\nu(t, y, x) \geq 0$ for any $(t, x, y) \in (0, \infty) \times K \times K$. Moreover, p_ν is in fact $(0, \infty)$-valued by [**55**, Theorem A.4]. ν is called the *reference measure of the Dirichlet space* $(K, \nu, \mathcal{E}, \mathcal{F})$, and p_ν is called the *(continuous) heat kernel associated with* $(K, \nu, \mathcal{E}, \mathcal{F})$. See [**56**, Theorem 10.4] for other basic properties of p_ν.

Since we have a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ with compact state space K, by [**25**, (3.2.13) and (3.2.14)] we can define $\mathcal{E}$-energy measures as in the following definition.

DEFINITION 2.8. The $\mathcal{E}$-*energy measure of* $u \in \mathcal{F}$ is defined as the unique Borel measure $\mu_{\langle u \rangle}$ on K such that

$$(2.6) \qquad \int_K f d\mu_{\langle u \rangle} = \mathcal{E}(uf, u) - \frac{1}{2}\mathcal{E}(u^2, f) \qquad \text{for any } f \in \mathcal{F}.$$

We also define $\lambda_{\langle u \rangle}$ to be the unique Borel measure on Σ that satisfies $\lambda_{\langle u \rangle}(\Sigma_w) = (5/3)^{|w|}\mathcal{E}(u \circ F_w, u \circ F_w)$ for any $w \in W_*$, which exists by (2.4) and the Kolmogorov extension theorem. For $u, v \in \mathcal{F}$ we set $\mu_{\langle u,v \rangle} := (\mu_{\langle u+v \rangle} - \mu_{\langle u-v \rangle})/4$ and $\lambda_{\langle u,v \rangle} := (\lambda_{\langle u+v \rangle} - \lambda_{\langle u-v \rangle})/4$, so that they are finite Borel signed measures on K and on Σ respectively and are symmetric and bilinear in $(u, v) \in \mathcal{F} \times \mathcal{F}$.

Let $u \in \mathcal{F}$. According to [**36**, Proposition 3.8] (see also [**14**, Theorem I.7.1.1]), the strong locality of $(\mathcal{E}, \mathcal{F})$ implies that the image measure $\mu_{\langle u \rangle} \circ u^{-1}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. In particular, $\mu_{\langle u \rangle}(\{x\}) = 0$ for any $x \in K$. We also easily see the following proposition by using (2.4) and (2.6). Note that $\pi(A) \in \mathcal{B}(K)$ for any $A \in \mathcal{B}(\Sigma)$ by Proposition 2.3.

PROPOSITION 2.9. $\lambda_{\langle u,v \rangle} = \mu_{\langle u,v \rangle} \circ \pi$ *and* $\lambda_{\langle u,v \rangle} \circ \pi^{-1} = \mu_{\langle u,v \rangle}$ *for any* $u, v \in \mathcal{F}$.

The definition of the measurable Riemannian structure on the Sierpiński gasket involves certain harmonic functions. In the present setting, harmonic functions are formulated as follows.

DEFINITION 2.10. (1) We define $\mathcal{F}_B := \{u \in \mathcal{F} \mid u|_{K \setminus B} = 0\}$ for each $B \subset K$. (2) Let F be a closed subset of K. Then $h \in \mathcal{F}$ is called F-*harmonic* if and only if

$$(2.7) \quad \mathcal{E}(h, h) = \inf_{u \in \mathcal{F}, \, u|_F = h|_F} \mathcal{E}(u, u) \quad \text{or equivalently,} \quad \mathcal{E}(h, u) = 0, \ \forall u \in \mathcal{F}_{K \setminus F}.$$

We set $\mathcal{H}_F := \{h \in \mathcal{F} \mid h \text{ is } F\text{-harmonic}\}$ and $\mathcal{H}_m := \mathcal{H}_{V_m}$ for each $m \in \mathbb{N} \cup \{0\}$.

Note that $\mathcal{H}_F$ is a linear subspace of $\mathcal{F}$ for any closed subset F of K and that $\mathcal{H}_{m-1} \subset \mathcal{H}_m$ for any $m \in \mathbb{N}$. Moreover, we easily have the following proposition by [**56**, Lemma 8.2 and Theorem 8.4].

PROPOSITION 2.11. *Let F be a non-empty closed subset of K.*
(1) *Let $u \in \mathcal{F}$. Then there exists a unique $h_F(u) \in \mathcal{H}_F$ such that $h_F(u)|_F = u|_F$. Moreover, $h_F : \mathcal{F} \to \mathcal{H}_F$ is linear.*
(2) *Let $h \in \mathcal{H}_F$. Then $\min_F h \leq h(x) \leq \max_F h$ for any $x \in K$.*

Proposition 2.5 and (2.4) imply the following useful characterizations of $\mathcal{H}_m$.

PROPOSITION 2.12. *It holds that for any $m \in \mathbb{N} \cup \{0\}$,*

$$(2.8) \qquad \mathcal{H}_m = \{u \in \mathcal{F} \mid \mathcal{E}(u,u) = \mathcal{E}_m(u|_{V_m}, u|_{V_m})\}$$

$$(2.9) \qquad = \{u \in \mathcal{F} \mid u \circ F_w \in \mathcal{H}_0 \text{ for any } w \in W_m\}.$$

For each $h \in \mathcal{H}_0$, by virtue of $h \circ F_w \in \mathcal{H}_0$, $w \in W_*$, $h|_{V_*}$ can be, in principle, explicitly calculated from $h|_{V_0}$ through simple matrix multiplications, as follows.

PROPOSITION 2.13 ([**53**, (3.2.3) and Example 3.2.6]). *Define*

$$(2.10) \quad A_1 := \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad A_2 := \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A_3 := \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix},$$

which we regard as linear maps from $\mathbb{R}^{V_0}$ to itself through the standard basis of $\mathbb{R}^{V_0}$. Then for any $u \in \mathcal{H}_0$ and any $w = w_1 \ldots w_m \in W_*$,

$$(2.11) \qquad u \circ F_w|_{V_0} = A_{w_m} \cdots A_{w_1}(u|_{V_0}).$$

3. Measurable Riemannian structure on the Sierpiński gasket

This section is devoted to a brief introduction to the notion of the measurable Riemannian structure on the Sierpiński gasket and its basic properties. We continue to follow mainly [**44**, Section 2] and refer to [**63, 52, 35**] for further details.

We first define a *"harmonic embedding"* Φ of K into $\mathbb{R}^2$, through which we will regard K as a kind of *"Riemannian submanifold in $\mathbb{R}^2$"* to obtain its measurable Riemannian structure. We also introduce a measure μ which is regarded as the $\mathcal{E}$-energy measure of the "embedding" Φ and will play the role of the *"Riemannian volume measure"*. See [**90**] for an attempt to generalize the framework of harmonic embeddings and their energy measures to other finitely ramified fractals.

Recall that $V_0 = \{q_1, q_2, q_3\}$.

DEFINITION 3.1. (0) We define $h_1, h_2 \in \mathcal{F}$ to be the V_0-harmonic functions satisfying $h_1(q_1) = h_2(q_1) = 0$, $h_1(q_2) = h_1(q_3) = 1$ and $-h_2(q_2) = h_2(q_3) = 1/\sqrt{3}$, so that $\mathcal{E}(h_1, h_1) = \mathcal{E}(h_2, h_2) = 1$ (recall the factor $1/2$ in (2.1)) and $\mathcal{E}(h_1, h_2) = 0$ by (2.8), and $h_1 \circ F_1 = (3/5)h_1$ and $h_2 \circ F_1 = (1/5)h_2$ by (2.11).
(1) We define a continuous map $\Phi : K \to \mathbb{R}^2$ and a compact subset $K_\mathcal{H}$ of $\mathbb{R}^2$ by

$$(3.1) \qquad \Phi(x) := (h_1(x), h_2(x)), \quad x \in K \qquad \text{and} \qquad K_\mathcal{H} := \Phi(K).$$

$K_\mathcal{H}$ is called the *harmonic Sierpiński gasket* (Figure 2). We also set $\hat{q}_i := \Phi(q_i)$ for $i \in S$, so that $\{\hat{q}_1, \hat{q}_2, \hat{q}_3\} = \Phi(V_0)$ is the set of vertices of an equilateral triangle.
(2) We define finite Borel measures μ on K and λ on Σ by

$$(3.2) \qquad \mu := \mu_{\langle h_1 \rangle} + \mu_{\langle h_2 \rangle} \qquad \text{and} \qquad \lambda := \lambda_{\langle h_1 \rangle} + \lambda_{\langle h_2 \rangle},$$

respectively, so that $\lambda = \mu \circ \pi$ and $\lambda \circ \pi^{-1} = \mu$ by Proposition 2.9. μ is called the *Kusuoka measure on the Sierpiński gasket*.

NOTATION. In what follows h_1, h_2 always denote the V_0-harmonic functions given in Definition 3.1-(0). We often regard $\{h_1, h_2\}$ as an orthonormal basis of $(\mathcal{H}_0/\mathbb{R}\mathbf{1}, \mathcal{E})$. Moreover, we set

(3.3) $\|u\|_{\mathcal{E}} := \sqrt{\mathcal{E}(u, u)}, \quad u \in \mathcal{F} \quad$ and $\quad \mathcal{S}_{\mathcal{H}_0} := \{h \in \mathcal{H}_0 \mid \|h\|_{\mathcal{E}} = 1\}$.

The following proposition, which is in fact an easy consequence of Proposition 2.13, provides an alternative geometric definition of $K_{\mathcal{H}}$, and essentially as its corollary we also see the injectivity of Φ (Theorem 3.3), Proposition 3.4 below and that $\mu_{\langle h \rangle}$ has full support for any $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$.

PROPOSITION 3.2 ([**52**, §3]). *Define*

(3.4) $T_1 := \begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}, T_2 := \begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}, T_3 := \begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix}$

and set $T_w := T_{w_1} \cdots T_{w_m}$ for $w = w_1 \dots w_m \in W_*$ ($T_\emptyset := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$). Also for $i \in S$ define $H_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $H_i(x) := \hat{q}_i + T_i(x - \hat{q}_i)$. Then the following hold:

(1) *$T_2 = R_{\frac{2}{3}\pi} T_1 R_{-\frac{2}{3}\pi}$ and $T_3 = R_{-\frac{2}{3}\pi} T_1 R_{\frac{2}{3}\pi}$, where $R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\theta \in \mathbb{R}$.*
(2) *For any $w \in W_*$, $T_w^* := (T_w)^*$ is equal to the matrix representation of the linear map $F_w^* : \mathcal{H}_0/\mathbb{R}\mathbf{1} \to \mathcal{H}_0/\mathbb{R}\mathbf{1}$, $F_w^* h := h \circ F_w$ by the basis $\{h_1, h_2\}$ of $\mathcal{H}_0/\mathbb{R}\mathbf{1}$.*
(3) *$H_i \circ \Phi = \Phi \circ F_i$ and hence $H_i \circ (\Phi \circ \pi) = (\Phi \circ \pi) \circ \sigma_i$ for any $i \in S$. In particular, $K_{\mathcal{H}} = \bigcup_{i \in S} H_i(K_{\mathcal{H}})$, i.e. $K_{\mathcal{H}}$ is the self-similar set associated with $\{H_i\}_{i \in S}$.*

THEOREM 3.3 ([**52**, Theorem 3.6]). *$\Phi : K \to K_{\mathcal{H}}$ is a homeomorphism.*

PROPOSITION 3.4. *$\mu(K_w) = \lambda(\Sigma_w) = (5/3)^{|w|}\|T_w\|^2$ for any $w \in W_*$.*

Moreover, we have the following theorem due to Kusuoka [**63**] (see [**44**, Theorem 6.8] for an alternative simple proof based on (2.4) and the strong locality of $(\mathcal{E}, \mathcal{F})$). Recall that $\sigma : \Sigma \to \Sigma$ is the shift map defined by $\sigma(\omega_1 \omega_2 \omega_3 \dots) := \omega_2 \omega_3 \omega_4 \dots$.

THEOREM 3.5 ([**63**, §6, Example 1]). *λ is σ-ergodic, that is, $\lambda \circ \sigma^{-1} = \lambda$ and $\lambda(A)\lambda(\Sigma \setminus A) = 0$ for any $A \in \mathcal{B}(\Sigma)$ with $\sigma^{-1}(A) = A$.*

We also remark the following fact due to Hino [**35**].

THEOREM 3.6 ([**35**, Theorem 5.6]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$. Then μ and $\mu_{\langle h \rangle}$ are mutually absolutely continuous.*

Now we can introduce the measurable Riemannian structure on K, which is formulated as a Borel measurable map $Z : K \to \mathbb{R}^{2 \times 2}$, as follows. Recall that $\pi|_{\Sigma \setminus \pi^{-1}(V_* \setminus V_0)}$ is injective by Proposition 2.3.

PROPOSITION 3.7 ([**63**, §1], [**52**, Proposition B.2]). *Define $\Sigma_Z \in \mathcal{B}(\Sigma)$ and $K_Z \in \mathcal{B}(K)$ by*

(3.5) $\Sigma_Z := \left\{ \omega \in \Sigma \; \middle| \; Z_\Sigma(\omega) := \lim_{m \to \infty} \frac{T_{[\omega]_m} T_{[\omega]_m}^*}{\|T_{[\omega]_m}\|^2} \text{ exists in } \mathbb{R}^{2 \times 2} \right\}, \; K_Z := \pi(\Sigma_Z)$.

Then $\lambda(\Sigma \setminus \Sigma_Z) = \mu(K \setminus K_Z) = 0$, $Z_\Sigma(\omega)$ is an orthogonal projection of rank 1 for any $\omega \in \Sigma_Z$, $\pi^{-1}(V_*) \subset \Sigma_Z$ and $Z_\Sigma(\omega) = Z_\Sigma(\tau)$ for $\omega, \tau \in \pi^{-1}(x)$, $x \in V_* \setminus V_0$. Hence setting $Z_x := Z_\Sigma(\omega)$, $\omega \in \pi^{-1}(x)$ for $x \in K_Z$ and $Z_x := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for $x \in K \setminus K_Z$ gives a well-defined Borel measurable map $Z : K \to \mathbb{R}^{2 \times 2}$, $x \mapsto Z_x$.

THEOREM 3.8 ([**52**, §4]). *Set* $C_\Phi^1(K) := \{v \circ \Phi \mid v \in C^1(\mathbb{R}^2)\}$. *Then for each* $u \in C_\Phi^1(K)$, $\nabla u := (\nabla v) \circ \Phi$ *is independent of a particular choice of* $v \in C^1(\mathbb{R}^2)$ *satisfying* $u = v \circ \Phi$. *Moreover,* $C_\Phi^1(K) \subset \mathcal{F}$, $C_\Phi^1(K)/\mathbb{R}\mathbf{1}$ *is dense in* $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$, *and for any* $u, v \in C_\Phi^1(K)$,

$$(3.6) \qquad d\mu_{\langle u,v \rangle} = \langle Z\nabla u, Z\nabla v \rangle d\mu \quad and \quad \mathcal{E}(u,v) = \int_K \langle Z\nabla u, Z\nabla v \rangle d\mu.$$

In view of Theorem 3.8, especially (3.6), we may regard Z as defining a *"one-dimensional tangent space* $\operatorname{Im} Z_x$ *of* K *at* x *with the metric inherited from* $\mathbb{R}^2$*"* for μ-a.e. $x \in K$ in a measurable way, with μ considered as the associated *"Riemannian volume measure"* and $Z\nabla u$ as the *"gradient vector field"* of $u \in C_\Phi^1(K)$. Then the Dirichlet space associated with this *"Riemannian structure"* is $(K, \mu, \mathcal{E}, \mathcal{F})$.

REMARK 3.9. (1) By [**52**, Theorem B.5-(1)], $\Sigma \setminus \Sigma_Z$ *is dense in* Σ and hence $K \setminus K_Z$ *is dense in* K. In other words, there exists a dense set of points x of K where the notion of the tangent space $\operatorname{Im} Z_x$ at x does not make sense.
(2) $Z|_{K_Z} : K_Z \to \mathbb{R}^{2\times 2}$ *is discontinuous.* Indeed, let $n \in \mathbb{N} \cup \{0\}$ and set $x_n := F_{1^n 3}(q_2)$, so that $\lim_{n\to\infty} x_n = q_1$. Then it easily follows from (3.4) and (3.5) that $Z_{q_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Z_{x_n} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, which does not converge to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = Z_{q_1}$ as $n \to \infty$.

As a matter of fact, any $u \in \mathcal{F}$ admits a natural *"gradient vector field"* $\widetilde{\nabla} u$, thereby (3.6) extended to functions in $\mathcal{F}$, as in the following theorem whose essential part is due to Hino [**35**, Theorem 5.4]; see [**44**, Theorem 2.17] for details.

THEOREM 3.10. *Let* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$. *Then for any* $u \in \mathcal{F}$ *the following hold:*
(1) *For* μ-a.e. $x \in K$, *there exists* $\widetilde{\nabla} u(x) \in \operatorname{Im} Z_x$ *such that for any* $\omega \in \pi^{-1}(x)$,

$$(3.7) \qquad \sup_{y \in K_{[\omega]_m}} \left| u(y) - u(x) - \langle \widetilde{\nabla} u(x), \Phi(y) - \Phi(x) \rangle \right| = o(\|T_{[\omega]_m}\|) \quad as \ \ m \to \infty.$$

Such $\widetilde{\nabla} u(x) \in \operatorname{Im} Z_x$ *as in* (3.7) *is unique for each* $x \in K_Z$, *and* $d\mu_{\langle u \rangle} = |\widetilde{\nabla} u|^2 d\mu$.
(2) *For* $\mu_{\langle h \rangle}$-a.e. $x \in K$, *there exists* $\frac{du}{dh}(x) \in \mathbb{R}$ *such that for any* $\omega \in \pi^{-1}(x)$,

$$(3.8) \qquad \sup_{y \in K_{[\omega]_m}} \left| u(y) - u(x) - \frac{du}{dh}(x)(h(y) - h(x)) \right| = o(\|h \circ F_{[\omega]_m}\|_\mathcal{E}) \quad as \ \ m \to \infty.$$

Such $\frac{du}{dh}(x) \in \mathbb{R}$ *as in* (3.8) *is unique for each* $x \in K$, *and* $d\mu_{\langle u \rangle} = \left(\frac{du}{dh}\right)^2 d\mu_{\langle h \rangle}$.

In fact, Theorem 3.10 has been recently improved by Koskela and Zhou [**58**] where the reminder estimates for the derivatives $\widetilde{\nabla} u$ and $\frac{du}{dh}$ are given in terms of the associated geodesic metrics; see Theorem 8.3 below.

REMARK 3.11. (1) As mentioned in [**44**, Remark 2.20], *the "gradient vector field"* $\widetilde{\nabla} u$ *in Theorem 3.10-(1) coincides with the "weak gradient"* $Y(\cdot\,; u)$ *defined by Kusuoka* [**63**, *Lemma* 5.1] (see also [**54**, Definition 4.11]).
(2) The rank of the matrix Z, which is 1 μ-a.e. in the present case, is closely related to the *martingale dimension* of the associated diffusion process. The martingale dimension of a symmetric diffusion process is formally defined as the maximal number of martingale additive functionals which are independent in the sense of stochastic integral representation, and intuitively it corresponds to the *"maximal dimension of the tangent space"* over the state space. For the purpose of analytic characterization of martingale dimension, Kusuoka [**63**, **64**] introduced the notion of *index* for certain strongly local symmetric regular Dirichlet forms on a certain

class of self-similar fractals and identified it as the martingale dimension of the associated diffusion. Hino [**35**, Definitions 2.9, 3.3 and Theorem 3.4] has recently extended these results to general strongly local symmetric regular Dirichlet forms, where the index is defined through certain matrix-valued measurable maps similar to Z as above whose entries are the Radon-Nikodym derivatives of energy measures.

The index of a non-degenerate elliptic symmetric diffusion on a smooth manifold is easily seen to be equal to the dimension of the manifold, whereas it is difficult to determine the exact value of the index for diffusions on fractals. In our case of the standard resistance form $(\mathcal{E}, \mathcal{F})$ on the Sierpiński gasket, it follows from rank $Z = 1$, μ-a.e., that the index is 1, and the same is true also for the k-dimensional Sierpiński gasket with $k \geq 3$, as shown in [**63**, §6, Example 1]. Hino [**34, 36**] has recently proved that the index of a point-recurrent self-similar diffusion (to be precise, the index of the resistance form associated with a regular harmonic structure) on a post-critically finite self-similar set is always 1. This result in particular applies to Brownian motion on affine nested fractals, whose construction is essentailly due to Lindstrøm [**69**]; see [**53**, Section 3.8] and references therein in this regard.

In the case of the canonical Dirichlet form on a generalized Sierpiński carpet, which was constructed in [**6, 8, 65**] and is known to be unique by [**9**], Hino has also proved in [**36**, Theorem 4.15] that the index is less than or equal to the spectral dimension d_{s} of the carpet. Note that this result gives only an upper bound for the index, so that the exact value of the index for generalized Sierpiński carpets is still unknown, except when $d_{\mathrm{s}} < 2$, which implies that the index is 1. (A brief summary of important facts concerning the canonical Dirichlet form on generalized Sierpiński carpets, as well as pictures of some typical generalized Sierpiński carpets, is available in [**46**, Section 5].)

(3) [**35**, Theorem 5.4], from which Theorem 3.10 follows, was stated and proved only for (regular harmonic structures on) post-critically finite self-similar sets. In fact, Hino [**37**, Theorem 3.4] has recently generalized it to general strongly local symmetric regular Dirichlet forms with finite index. See [**37**] for details.

4. Geometry under the measurable Riemannian structure

This section is a brief summary of the results in [**44**, Section 3], which are slight improvements of those in [**54**, Sections 3 and 5] and concern basic geometric properties of K under the measurable Riemannian structure.

We start with the definition of the canonical geodesic metrics associated with the Dirichlet spaces $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$, $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$.

DEFINITION 4.1. Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$. We define the *harmonic geodesic metric* $\rho_{\mathcal{H}}$ *on K* and the *h-geodesic metric* ρ_h *on K* by respectively

$$(4.1) \quad \rho_{\mathcal{H}}(x, y) := \inf\{\ell_{\mathcal{H}}(\gamma) \mid \gamma : [0, 1] \to K, \ \gamma \text{ is continuous}, \ \gamma(0) = x, \gamma(1) = y\},$$

$$(4.2) \quad \rho_h(x, y) := \inf\{\ell_h(\gamma) \mid \gamma : [0, 1] \to K, \ \gamma \text{ is continuous}, \ \gamma(0) = x, \gamma(1) = y\}$$

for $x, y \in K$, where we set $\ell_{\mathcal{H}}(\gamma) := \ell_{\mathbb{R}^2}(\Phi \circ \gamma)$ and $\ell_h(\gamma) := \ell_{\mathbb{R}}(h \circ \gamma)$ for each continuous map $\gamma : [a, b] \to K$, $a, b \in \mathbb{R}$, $a \leq b$.

$\rho_{\mathcal{H}}$ was first introduced by Kigami in [**54**, Section 5], and the author adopted his idea to define ρ_h in [**44**]. As observed in [**44**, Section 3] and reviewed below, ρ_h plays the role of the canonical geodesic metric for the Dirichlet space $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$, as $\rho_{\mathcal{H}}$ does for $(K, \mu, \mathcal{E}, \mathcal{F})$, and $(K, \rho_h, \mu_{\langle h \rangle})$ possesses most of the fundamental geometric

properties in common with $(K, \rho_{\mathcal{H}}, \mu)$. The generalization to $(K, \rho_h, \mu_{\langle h \rangle})$, where in fact *the constants involved are all independent of $h \in \mathcal{S}_{\mathcal{H}_0}$*, played essential roles in the proofs of the main results of [**44**], and it does also in the proofs of the author's recent results in [**47**], which are reviewed in Sections 6 and 7 below.

REMARK 4.2. Note that $\rho_{\mathcal{H}}$ is different from the "harmonic metric" ρ_Φ on K introduced in [**52**, Definition 3.8], which is defined by

$$(4.3) \qquad \rho_\Phi(x, y) := |\Phi(x) - \Phi(y)|, \qquad x, y \in K.$$

ρ_Φ is a metric on K compatible with the original topology of K by Theorem 3.3 and satisfies $\rho_\Phi \leq \rho_{\mathcal{H}}$, but $\rho_{\mathcal{H}}$ is *not* comparable to ρ_Φ. Indeed, as noted in [**54**, p. 800, Remark], $\rho_\Phi\big(F_{1^n}(q_2), F_{1^n}(q_3)\big)/\rho_{\mathcal{H}}\big(F_{1^n}(q_2), F_{1^n}(q_3)\big) = O(3^{-n})$ as $n \to \infty$.

In practice, we need to relate the metrics $\rho_{\mathcal{H}}$ and ρ_h suitably to the cell structure of K to obtain various fundamental inequalities such as volume doubling property of measures and weak Poincaré inequality. In [**55**], Kigami proposed a systematic way of describing the geometry of a self-similar set using the cell-structure and applied it to establish reasonable sufficient conditions for sub-Gaussian bounds of the heat kernel associated with a self-similar Dirichlet form. We follow his framework to describe the relation between the cell structure of K and the metrics $\rho_{\mathcal{H}}$ and ρ_h. Definitions 4.3, 4.4, 4.6, 4.8 and Proposition 4.5 below are adopted from [**55**].

DEFINITION 4.3. (1) Let $w, v \in W_*$, $w = w_1 \ldots w_m$, $v = v_1 \ldots v_n$. We define $wv \in W_*$ by $wv := w_1 \ldots w_m v_1 \ldots v_n$ ($w\emptyset := w$, $\emptyset v := v$). We write $w \leq v$ if and only if $w = v\tau$ for some $\tau \in W_*$. Note that $\Sigma_w \cap \Sigma_v = \emptyset$ if and only if neither $w \leq v$ nor $v \leq w$.
(2) A finite subset Λ of W_* is called a *partition of Σ* if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$.
(3) Let Λ_1, Λ_2 be partitions of Σ. We say that Λ_1 is a *refinement of Λ_2*, and write $\Lambda_1 \leq \Lambda_2$, if and only if for each $w^1 \in \Lambda_1$ there exists $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$.
If $\Lambda_1 \leq \Lambda_2$, then we have a natural surjection $\Lambda_1 \to \Lambda_2$ by which $w^1 \in \Lambda_1$ is mapped to the unique $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$, and in particular, $\#\Lambda_1 \geq \#\Lambda_2$.

DEFINITION 4.4. (1) A family $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ of partitions of Σ is called a *scale on Σ* if and only if $\mathcal{S}$ satisfies the following three properties:

(S1) $\Lambda_1 = W_0 (= \{\emptyset\})$. $\Lambda_{s_1} \leq \Lambda_{s_2}$ for any $s_1, s_2 \in (0, 1]$ with $s_1 \leq s_2$.
(S2) $\min\{|w| \mid w \in \Lambda_s\} \to \infty$ as $s \downarrow 0$.
(Sr) Each $s \in (0, 1)$ admits $\varepsilon \in (0, 1 - s]$ such that $\Lambda_{s'} = \Lambda_s$ for any $s' \in (s, s + \varepsilon)$.
(2) A function $l : W_* \to (0, 1]$ is called a *gauge function on W_** if and only if $l(wi) \leq l(w)$ for any $(w, i) \in W_* \times S$ and $\lim_{m \to \infty} \max\{l(w) \mid w \in W_m\} = 0$.

There is a natural one-to-one correspondence between scales on Σ and gauge functions on W_*, as in the following proposition. See [**55**, Section 1.1] for a proof.

PROPOSITION 4.5. (1) *Let l be a gauge function on W_*. For $s \in (0, 1]$, define*

$$(4.4) \qquad \Lambda_s(l) := \{w \mid w = w_1 \ldots w_m \in W_*, \, l(w_1 \ldots w_{m-1}) > s \geq l(w)\}$$

where $l(w_1 \ldots w_{m-1}) := 2$ when $w = \emptyset$. Then the collection $\mathcal{S}(l) := \{\Lambda_s(l)\}_{s \in (0,1]}$ is a scale on Σ. We call $\mathcal{S}(l)$ the scale induced by the gauge function l.
(2) *Let $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ be a scale on Σ. Then there exists a unique gauge function $l_{\mathcal{S}}$ on W_* such that $\mathcal{S} = \mathcal{S}(l_{\mathcal{S}})$. We call $l_{\mathcal{S}}$ the gauge function of the scale $\mathcal{S}$.*

DEFINITION 4.6. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ. For $s \in (0,1]$ and $x \in K$, we define

$$(4.5) \qquad K_s(x, \mathcal{S}) := \bigcup_{w \in \Lambda_s,\, x \in K_w} K_w, \qquad U_s(x, \mathcal{S}) := \bigcup_{w \in \Lambda_s,\, K_w \cap K_s(x,\mathcal{S}) \neq \emptyset} K_w.$$

$K_s(x, \mathcal{S})$ and $U_s(x, \mathcal{S})$ are clearly non-decreasing in $s \in (0,1]$, and it immediately follows from [**53**, Proposition 1.3.6] that $\{K_s(x, \mathcal{S})\}_{s\in(0,1]}$ and $\{U_s(x, \mathcal{S})\}_{s\in(0,1]}$ are fundamental systems of neighborhoods of x in K.

Proposition 2.3 easily yields the following lemma.

LEMMA 4.7. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ, let $s \in (0,1]$, $x \in K$ and $w \in \Lambda_s$. Then $\#\{v \in \Lambda_s \mid K_v \cap K_s(x,\mathcal{S}) \neq \emptyset\} \leq 6$ and $\#\{v \in \Lambda_s \mid K_w \cap K_v \neq \emptyset\} \leq 4$.

DEFINITION 4.8. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ. A metric ρ on K is called *adapted to* $\mathcal{S}$ if and only if there exist $\beta_1, \beta_2 \in (0,\infty)$ such that

$$(4.6) \qquad B_{\beta_1 s}(x, \rho) \subset U_s(x, \mathcal{S}) \subset B_{\beta_2 s}(x, \rho), \qquad (s,x) \in (0,1] \times K.$$

LEMMA 4.9. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ with gauge function l and let ρ be a metric on K adapted to $\mathcal{S}$. Then ρ is compatible with the original topology of K, and $\operatorname{diam}_\rho K_w \leq \beta_2 l(w)$ for any $w \in W_*$, where $\beta_2 \in (0,\infty)$ is as in (4.6).

PROOF. See [**44**, Lemma 3.7]. □

Next we define scales on Σ to which the metrics $\rho_{\mathcal{H}}$ and ρ_h, $h \in \mathcal{S}_{\mathcal{H}_0}$, are adapted (recall (3.3) for $\mathcal{S}_{\mathcal{H}_0}$).

DEFINITION 4.10. (1) We define $\mathcal{S}^{\mathcal{H}} = \{\Lambda_s^{\mathcal{H}}\}_{s\in(0,1]}$ to be the scale on Σ induced by the gauge function $l_{\mathcal{H}} : W_* \to (0,1]$, $l_{\mathcal{H}}(w) := \|T_w\| \wedge 1 = \sqrt{(3/5)^{|w|}\mu(K_w)} \wedge 1$.
(2) Let $h \in \mathcal{S}_{\mathcal{H}_0}$. We define $\mathcal{S}^h = \{\Lambda_s^h\}_{s\in(0,1]}$ to be the scale on Σ induced by the gauge function $l_h : W_* \to (0,1]$, $l_h(w) := \|h \circ F_w\|_{\mathcal{E}} = \sqrt{(3/5)^{|w|}\mu_{\langle h\rangle}(K_w)}$.

As we will state in Theorem 4.15 below, $\rho_{\mathcal{H}}$ and ρ_h introduced in Definition 4.1, where $h \in \mathcal{S}_{\mathcal{H}_0}$, are indeed metrics on K adapted to $\mathcal{S}^{\mathcal{H}}$ and $\mathcal{S}^h$ respectively and the infimums in (4.1) and (4.2) are achieved by a specific class of paths in K. The key to these results is the next theorem, which requires the following definition.

DEFINITION 4.11. (1) For $x, y \in \mathbb{R}^2$, we set $\overline{xy} := \{x + t(y - x) \mid t \in [0,1]\}$, which is also regarded as the map $[0,1] \ni t \mapsto (1-t)x + ty \in \mathbb{R}^2$.
(2) Let $m \in \mathbb{N}\cup\{0\}$ and $x, y \in V_m$, $x \overset{m}{\sim} y$, where $\overset{m}{\sim}$ is as in Definition 2.4. We define $w(x,y)$ to be the unique $w \in W_m$ such that $x, y \in F_w(V_0)$. Note that $\overline{xy} \subset K_{w(x,y)}$.

THEOREM 4.12 ([**89**], [**54**, Theorem 5.4]). Set $I := [-1/\sqrt{3}, 1/\sqrt{3}]$. Then $\Phi(\overline{q_2 q_3}) = \{(\varphi(t), t) \mid t \in I\}$ for some $\varphi : I \to \mathbb{R}$, and the following hold:
(1) φ is C^1 but not C^2, φ' is strictly increasing and $\varphi'(\pm 1/\sqrt{3}) = \pm 1/\sqrt{3}$.
(2) $\overline{q_2 q_3} \subset K_Z$ and $(\varphi'(t), 1) \in \operatorname{Im} Z_{\Phi^{-1}(\varphi(t),t)}$ for any $t \in I$.
(3) $K_{\mathcal{H}} \subset \{(s,t) \in \mathbb{R} \times I \mid s \leq \varphi(t)\}$, or equivalently, $h_1 \leq \varphi \circ h_2$.

DEFINITION 4.13. (1) Let $m \in \mathbb{N} \cup \{0\}$. A sequence $\Gamma = \{x_k\}_{k=0}^N \subset V_m$, where $N \in \mathbb{N}$, is called an *m-walk* if and only if $x_{k-1} \overset{m}{\sim} x_k$ for $k \in \{1, \ldots, N\}$ and $w(x_{k-1}, x_k) \neq w(x_k, x_{k+1})$ for $k \in \{1, \ldots, N-1\}$. For such Γ we define continuous maps $\overline{\Gamma} : [0, N] \to K$ and $\widehat{\Gamma} : [0, \ell_{\mathcal{H}}(\overline{\Gamma})] \to K$ by

$$(4.7) \qquad \overline{\Gamma}(t) := (k-t)x_{k-1} + (t-k+1)x_k, \qquad t \in [k-1, k], \; k \in \{1, \ldots, N\},$$

and $\widehat{\Gamma} := \overline{\Gamma} \circ \varphi_\Gamma^{-1}$, where φ_Γ is the homeomorphism $\varphi_\Gamma : [0, N] \to [0, \ell_\mathcal{H}(\overline{\Gamma})]$, $\varphi_\Gamma(t) := \ell_\mathcal{H}(\overline{\Gamma}|_{[0,t]})$; note that $\ell_\mathcal{H}(\overline{\Gamma}) < \infty$ by Theorem 4.12 and Proposition 3.2.
(2) Let $\gamma : [a, b] \to K$ be continuous, $a, b \in \mathbb{R}$, $a < b$. For $m \in \mathbb{N} \cup \{0\}$, γ is called a *harmonic m-geodesic* if and only if $\gamma(t) = \widehat{\Gamma}\left(\ell_\mathcal{H}(\overline{\Gamma}) \frac{t-a}{b-a}\right)$, $t \in [a, b]$ for some m-walk Γ. γ is called a *harmonic geodesic* if and only if there exist $n \in \mathbb{N} \cup \{0\}$ and $\{a_m\}_{m \geq n}, \{b_m\}_{m \geq n} \subset [a, b]$ with $\lim_{m \to \infty} a_m = a$ and $\lim_{m \to \infty} b_m = b$ such that $a_{m+1} \leq a_m < b_m \leq b_{m+1}$ and $\gamma|_{[a_m, b_m]}$ is a harmonic m-geodesic for each $m \geq n$.

Then Theorem 4.12 together with Propositions 3.2 and 3.7 immediately yields the following proposition.

PROPOSITION 4.14. *If $\gamma : [0, 1] \to K$ is a harmonic geodesic, then $\Phi \circ \gamma|_{(0,1)}$ is C^1 and for any $t \in (0, 1)$, $\gamma(t) \in K_Z$, $(\Phi \circ \gamma)'(t) \in \operatorname{Im} Z_{\gamma(t)}$ and $|(\Phi \circ \gamma)'(t)| = \ell_\mathcal{H}(\gamma)$.*

THEOREM 4.15 ([**54**, Theorems 5.1 and 5.11], [**44**, Propositions 3.15 and 3.16]). *Let $h \in \mathcal{S}_{\mathcal{H}_0}$ and let $\mathbf{h}$ denote any one of $\mathcal{H}$ and h.*
(1) $\rho_\mathbf{h}$ is a metric on K satisfying

$$(4.8) \qquad B_{\sqrt{2}s/50}(x, \rho_\mathbf{h}) \subset U_s(x, \mathcal{S}^\mathbf{h}) \subset B_{10s}(x, \rho_\mathbf{h}), \qquad (s, x) \in (0, 1] \times K.$$

(2) For each $x, y \in K$ with $x \neq y$, there exists a harmonic geodesic $\gamma_{xy}^\mathbf{h} : [0, 1] \to K$ such that $\gamma_{xy}^\mathbf{h}(0) = x$, $\gamma_{xy}^\mathbf{h}(1) = y$ and $\rho_\mathbf{h}(x, y) = \ell_\mathbf{h}(\gamma_{xy}^\mathbf{h})$. Moreover, if $m \in \mathbb{N} \cup \{0\}$ and $x, y \in V_m$ then we can take a harmonic m-geodesic as $\gamma_{xy}^\mathbf{h}$.

REMARK 4.16. Let $\gamma : [0, 1] \to K$ be a harmonic geodesic and let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$. Then we easily see from Theorem 4.12 that the set $\{t \in (0, 1) \mid (h \circ \gamma)'(t) = 0\}$ is discrete (see [**44**, (3.15)]), so that $\varphi_h : [0, 1] \to [0, \ell_h(\gamma)]$, $\varphi_h(t) := \ell_h(\gamma|_{[0,t]})$, is strictly increasing. Therefore γ admits a parametrization by ℓ_h-length given by $\gamma \circ \varphi_h^{-1}$: $\ell_h(\gamma \circ \varphi_h^{-1}|_{[0,t]}) = t$ for any $t \in [0, \ell_h(\gamma)]$.

The proof of Theorem 4.15 is based on the following lemma, which in turn is an easy consequence of Theorem 4.12.

LEMMA 4.17 ([**54**, Lemma 5.6], [**44**, Lemma 3.18]). *Let $h \in \mathcal{S}_{\mathcal{H}_0}$ and let $\mathbf{h}$ denote any one of $\mathcal{H}$ and h. Let $w \in W_*$ and $x, y \in F_w(V_0)$, $x \neq y$. Then*

$$(4.9) \quad \ell_\mathbf{h}(\overline{xy}) = \inf\{\ell_\mathbf{h}(\gamma) \mid \gamma : [0, 1] \to K_w, \gamma \text{ is continuous}, \gamma(0) = x, \gamma(1) = y\},$$

$$(4.10) \qquad\qquad \frac{\sqrt{2}}{10} l_\mathbf{h}(w) \leq \ell_\mathbf{h}(\overline{xy}) \leq \frac{4\sqrt{6}}{3} l_\mathbf{h}(w).$$

The proof of (4.9) for $\ell_\mathcal{H}$ utilizes the following theorem from convex geometry as well, which is quoted in [**54**, Theorem 5.2] without reference. Recall that a subset Γ of $\mathbb{R}^2$ is called a *Jordan curve* if and only if it is homeomorphic to $\{x \in \mathbb{R}^2 \mid |x| = 1\}$. For such Γ, its length $\ell_{\mathbb{R}^2}(\Gamma)$ with respect to the Euclidean norm $|\cdot|$ is naturally defined, and according to the Jordan curve theorem (see e.g. [**73**, Section 4]), $\mathbb{R}^2 \setminus \Gamma$ consists of two connected components whose boundaries in $\mathbb{R}^2$ are both equal to Γ.

THEOREM 4.18. *Let C be a compact convex subset of $\mathbb{R}^2$ with $\operatorname{int}_{\mathbb{R}^2} C \neq \emptyset$.*
(1) $\partial_{\mathbb{R}^2} C$ is a Jordan curve and $\ell_{\mathbb{R}^2}(\partial_{\mathbb{R}^2} C) < \infty$.
(2) Let D be a compact subset of $\mathbb{R}^2$ such that $\partial_{\mathbb{R}^2} D$ is a Jordan curve and $C \subset D$. If $C \neq D$, or equivalently if $\partial_{\mathbb{R}^2} C \neq \partial_{\mathbb{R}^2} D$, then $\ell_{\mathbb{R}^2}(\partial_{\mathbb{R}^2} C) < \ell_{\mathbb{R}^2}(\partial_{\mathbb{R}^2} D)$.

PROOF. (1) An elementary argument shows that $\partial_{\mathbb{R}^2} C$ is a Jordan curve (see e.g. [**13**, Theorems 6.7 and 11.9]), and it follows from (2) that $\ell_{\mathbb{R}^2}(\partial_{\mathbb{R}^2} C) < \infty$.

(2) Note that for $q \in \text{int}_{\mathbb{R}^2} D$, $\partial_{\mathbb{R}^2} D$ regarded as a closed curve in $\mathbb{R}^2 \setminus \{q\}$ is not homotopic in $\mathbb{R}^2 \setminus \{q\}$ to a constant map, by virtue of the Schönflies theorem [**73**, Section 10, Theorem 4] saying that every injective continuous map from a Jordan curve to $\mathbb{R}^2$ is the restriction of a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. On the basis of this fact, the assertion can be verified by using [**13**, Theorem 7.9] to approximate $\partial_{\mathbb{R}^2} C$ by convex polygons whose vertices belong to $\partial_{\mathbb{R}^2} C$ and by applying the arguments in [**13**, Proof of Theorem 7.11 and Exercise 7.4]. $\square$

In fact, we can also prove the following characterization of shortest paths with respect to $\ell_{\mathcal{H}}$ by using Theorems 4.12 and 4.18, as follows.

THEOREM 4.19. *Let* $\gamma : [0,1] \to K$ *be continuous and satisfy* $\rho_{\mathcal{H}}(\gamma(0), \gamma(1)) = \ell_{\mathcal{H}}(\gamma) > 0$. *Then there exist a unique harmonic geodesic* $\widehat{\gamma} : [0,1] \to K$ *and a unique non-decreasing continuous surjection* $\varphi_\gamma : [0,1] \to [0,1]$ *such that* $\gamma = \widehat{\gamma} \circ \varphi_\gamma$. *Moreover, if* $m \in \mathbb{N} \cup \{0\}$ *and* $\gamma(0), \gamma(1) \in V_m$, *then* $\widehat{\gamma}$ *is a harmonic m-geodesic.*

Theorem 4.19 has important applications to the *invalidity* of various geometric conditions on $(K, \rho_{\mathcal{H}}, \mu)$ related to Ricci curvature lower bound; see Subsection 8.2 below.

We need the following lemma for the proof of Theorem 4.19.

LEMMA 4.20. *Let* $w \in W_*$ *and* $x, y \in F_w(V_0)$, $x \neq y$. *Let* $a, b \in \mathbb{R}$, $a < b$ *and let* $\gamma : [a,b] \to K_w$ *be a continuous injective map with* $\gamma(a) = x$ *and* $\gamma(b) = y$. *If* $\ell_{\mathcal{H}}(\gamma) \leq \ell_{\mathcal{H}}(\overline{xy})$, *then* $\gamma([a,b]) = \overline{xy}$.

PROOF. Let C, D be the compact subsets of $\mathbb{R}^2$ with $\text{int}_{\mathbb{R}^2} C \neq \emptyset \neq \text{int}_{\mathbb{R}^2} D$ whose boundaries in $\mathbb{R}^2$ are the Jordan curves $\overline{\Phi(x)\Phi(y)} \cup \Phi(\overline{xy})$ and $\overline{\Phi(x)\Phi(y)} \cup \Phi \circ \gamma([a,b])$, respectively. Then C is convex and $C \subset D$ by virtue of the rotational symmetry of $K_{\mathcal{H}}$, Proposition 3.2-(3), Theorem 4.12 and the Jordan curve theorem. Therefore if $\ell_{\mathcal{H}}(\gamma) \leq \ell_{\mathcal{H}}(\overline{xy})$, then $\ell_{\mathbb{R}^2}(\partial_{\mathbb{R}^2} D) \leq \ell_{\mathbb{R}^2}(\partial_{\mathbb{R}^2} C)$, hence $\partial_{\mathbb{R}^2} D = \partial_{\mathbb{R}^2} C$ by Theorem 4.18-(2), and thus $\gamma([a,b]) = \overline{xy}$. $\square$

PROOF OF THEOREM 4.19. For uniqueness, let $\widehat{\gamma}$ and φ_γ be as in the assertion, so that $\varphi_\gamma(0) = 0$, $\varphi_\gamma(1) = 1$ and $\ell_{\mathcal{H}}(\widehat{\gamma}) = \ell_{\mathcal{H}}(\gamma)$. Let $t \in [0,1]$ and choose $s \in [0,1]$ so that $t = \ell_{\mathcal{H}}(\gamma|_{[0,s]})/\ell_{\mathcal{H}}(\gamma)$. Then $\ell_{\mathcal{H}}(\gamma|_{[0,t]}) = \ell_{\mathcal{H}}(\widehat{\gamma}|_{[0,\varphi_\gamma(t)]}) = \ell_{\mathcal{H}}(\gamma)\varphi_\gamma(t)$ by Proposition 4.14, hence $t = \varphi_\gamma(s)$ and $\widehat{\gamma}(t) = \gamma(s)$, proving the uniqueness assertion.

For existence, define a non-decreasing continuous surjection $\varphi_\gamma : [0,1] \to [0,1]$ by $\varphi_\gamma(t) := \ell_{\mathcal{H}}(\gamma|_{[0,t]})/\ell_{\mathcal{H}}(\gamma)$. Since $\gamma(s) = \gamma(t)$ for any $s, t \in [a,b]$ with $\varphi_\gamma(s) = \varphi_\gamma(t)$, there exists a unique map $\widehat{\gamma} : [0,1] \to K$ such that $\gamma = \widehat{\gamma} \circ \varphi_\gamma$, and then $\widehat{\gamma}$ is continuous and $\ell_{\mathcal{H}}(\widehat{\gamma}|_{[0,t]}) = t\ell_{\mathcal{H}}(\gamma)$ for any $t \in [0,1]$. In particular, $\ell_{\mathcal{H}}(\widehat{\gamma}|_{[s,t]}) > 0$ for any $s, t \in [0,1]$ with $s < t$, which together with $\ell_{\mathcal{H}}(\widehat{\gamma}) = \ell_{\mathcal{H}}(\gamma) = \rho_{\mathcal{H}}(\gamma(0), \gamma(1))$, $\widehat{\gamma}(0) = \gamma(0)$ and $\widehat{\gamma}(1) = \gamma(1)$ shows that $\widehat{\gamma}$ is injective.

To see that $\widehat{\gamma}$ is a harmonic geodesic, set $n := \inf\{m \in \mathbb{N} \cup \{0\} \mid \#\widehat{\gamma}^{-1}(V_m) \geq 2\}$ ($n < \infty$ by $\widehat{\gamma}(0) \neq \widehat{\gamma}(1)$), and for $m \geq n$ set $a_m := \min\widehat{\gamma}^{-1}(V_m)$ and $b_m := \max\widehat{\gamma}^{-1}(V_m)$, so that $a_{m+1} \leq a_m < b_m \leq b_{m+1}$ for any $m \geq n$. The injectivity of $\widehat{\gamma}$ easily yields $\lim_{m\to\infty} a_m = 0$, $\lim_{m\to\infty} b_m = 1$ and $\#\widehat{\gamma}^{-1}(V_m) < \infty$.

Let $m \geq n$, $N := \#\widehat{\gamma}^{-1}(V_m) - 1$ and let $\{t_k\}_{k=0}^N$ be the strictly increasing enumeration of the elements of $\widehat{\gamma}^{-1}(V_m)$, so that $t_0 = a_m$ and $t_N = b_m$. Also set $x_k := \widehat{\gamma}(t_k)$ for $k \in \{0, \ldots, N\}$ and $\Gamma := \{x_k\}_{k=0}^N \subset V_m$. We claim that Γ is an m-walk and that $\widehat{\gamma}(t) = \widehat{\Gamma}\big(\ell_{\mathcal{H}}(\overline{\Gamma})\frac{t-a_m}{b_m-a_m}\big)$ for any $t \in [a_m, b_m]$. Indeed, let $k \in \{1, \ldots, N\}$. It easily follows from $\widehat{\gamma}((t_{k-1}, t_k)) \cap V_m = \emptyset$ that $x_{k-1} \overset{m}{\sim} x_k$ and that

$\widehat{\gamma}([t_{k-1}, t_k]) \subset K_{w(x_{k-1}, x_k)}$, and then we also have $\ell_{\mathcal{H}}(\widehat{\gamma}|_{[t_{k-1}, t_k]}) = \rho_{\mathcal{H}}(x_{k-1}, x_k)$ by $\ell_{\mathcal{H}}(\widehat{\gamma}) = \rho_{\mathcal{H}}(\widehat{\gamma}(0), \widehat{\gamma}(1))$. If $k < N$ and $w(x_{k-1}, x_k) = w(x_k, x_{k+1})$, then Lemma 4.20 implies $\ell_{\mathcal{H}}(\overline{x_{k-1} x_{k+1}}) < \ell_{\mathcal{H}}(\widehat{\gamma}|_{[t_{k-1}, t_{k+1}]})$, contradicting $\ell_{\mathcal{H}}(\widehat{\gamma}) = \rho_{\mathcal{H}}(\widehat{\gamma}(0), \widehat{\gamma}(1))$. Thus Γ is an m-walk. Moreover, since $\ell_{\mathcal{H}}(\widehat{\gamma}|_{[t_{k-1}, t_k]}) = \rho_{\mathcal{H}}(x_{k-1}, x_k) \le \ell_{\mathcal{H}}(\overline{x_{k-1} x_k})$, Lemma 4.20 yields $\widehat{\gamma}([t_{k-1}, t_k]) = \overline{x_{k-1} x_k}$, which together with $\ell_{\mathcal{H}}(\widehat{\gamma}|_{[0, t]}) = t \ell_{\mathcal{H}}(\widehat{\gamma})$, $t \in [0, 1]$, easily implies the above claim. Thus $\widehat{\gamma}|_{[a_m, b_m]}$ is a harmonic m-geodesic for any $m \ge n$ and hence $\widehat{\gamma}$ is a harmonic geodesic. If $m \in \mathbb{N} \cup \{0\}$ and $\gamma(0), \gamma(1) \in V_m$, then $m \ge n$, $a_m = 0$ and $b_m = 1$, so that $\widehat{\gamma}$ is a harmonic m-geodesic. $\square$

REMARK 4.21. *If $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, then the assertions of Theorem 4.19 and Lemma 4.20 are not valid for ℓ_h and ρ_h.*

PROOF. Noting that $\min_{V_0} h < \max_{V_0} h$ by Proposition 2.11-(2), let $i, j \in S$ be such that $h(q_i) = \min_{V_0} h$ and $h(q_j) = \max_{V_0} h$ and let $\{k\} = S \setminus \{i, j\}$. We first assume $h(q_i) + h(q_j) \ne 2h(q_k)$. By considering $-h$ and q_j instead of h and q_i if necessary, we may assume that $h(q_i) + h(q_j) < 2h(q_k)$. Let U be the connected component of $h^{-1}((-\infty, h(q_k)))$ with $q_i \in U$. Then Theorem 4.12 and the rotational symmetry of $K_{\mathcal{H}}$ easily imply that h is strictly decreasing on $\overline{q_k q_i}$ and on $\overline{q_k F_{k^n}(q_j)}$ for some $n \in \mathbb{N}$. Therefore $K_{k^n} \setminus \{q_k\} \subset U$ by the strong maximum principle [**53**, Theorem 3.2.14] for $h \circ F_{k^n}$, and hence $\rho_h(q_i, x) = h(x) - h(q_i)$ for any $x \in K_{k^n}$ by [**44**, (4.13)] (see (5.8) below). A similar argument for $-h \circ F_{k^n}$ together with [**44**, (4.13)] also implies that $\rho_h(q_k, x) = h(q_k) - h(x)$ for any $x \in K_{k^m}$ for some $m \ge n$. Thus by Theorem 4.15 and Remark 4.16, for any $x \in K_{k^m} \setminus \{q_k\}$ there exists a continuous injective map $\gamma_x : [0, 1] \to K$ with $\gamma_x(0) = q_i$, $\gamma_x(1/2) = x$ and $\gamma_x(1) = q_k$ such that $\ell_h(\gamma_x) = h(q_k) - h(q_i) = \ell_h(\overline{q_i q_k}) = \rho_h(q_i, q_k)$. Now if $x \in K_{k^m} \setminus \bigcup_{w \in W_*} F_w(\overline{q_1 q_2} \cup \overline{q_2 q_3} \cup \overline{q_3 q_1})$, then the conclusions of Theorem 4.19 and Lemma 4.20 are not valid for γ_x since $\widehat{\gamma}((0, 1)) \subset \bigcup_{w \in W_*} F_w(\overline{q_1 q_2} \cup \overline{q_2 q_3} \cup \overline{q_3 q_1})$ and hence $x \notin \widehat{\gamma}((0, 1))$ for any harmonic geodesic $\widehat{\gamma} : [0, 1] \to K$.

If $h(q_i) + h(q_j) = 2h(q_k)$, then since $h \in \mathcal{H}_{\{q_i, q_j\}}$ by the axial symmetry of K and $(\mathcal{E}, \mathcal{F})$, it follows similarly to [**44**, Proposition 4.9] (see Proposition 5.8 below) that $\rho_h(q_i, x) = h(x) - h(q_i)$ and $\rho_h(q_j, x) = h(q_j) - h(x)$ for any $x \in K$. The rest of the proof goes in exactly the same way as in the previous paragraph. $\square$

At the last of this section, we state the volume doubling property and the weak Poincaré inequality of $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$ under the metrics $\rho_{\mathcal{H}}$ and ρ_h, respectively. The following lemma is essential for the proofs of those properties.

LEMMA 4.22 ([**44**, Lemma 3.9], cf. [**54**, Section 3]). *Let $h \in \mathcal{S}_{\mathcal{H}_0}$.*
(1) *For any $(w, i) \in W_* \times S$,*

$$(4.11) \qquad \frac{1}{15} \mu(K_w) \le \mu(K_{wi}) \le \frac{3}{5} \mu(K_w), \qquad \frac{1}{5} \|T_w\| \le \|T_{wi}\| \le \frac{3}{5} \|T_w\|,$$

$$(4.12) \qquad \frac{1}{15} \mu_{\langle h \rangle}(K_w) \le \mu_{\langle h \rangle}(K_{wi}) \le \frac{3}{5} \mu_{\langle h \rangle}(K_w), \qquad \frac{1}{5} l_h(w) \le l_h(wi) \le \frac{3}{5} l_h(w).$$

(2) *If $w, v \in W_*$ satisfies $|w| = |v|$ and $K_w \cap K_v \ne \emptyset$, then*

$$(4.13) \qquad \mu_{\langle h \rangle}(K_w) \le 9 \mu_{\langle h \rangle}(K_v), \quad l_h(w) \le 3 l_h(v) \quad and \quad l_{\mathcal{H}}(w) \le 3 l_{\mathcal{H}}(v).$$

Then we can verify the following proposition on the basis of Lemma 4.22 in exactly the same ways as [**55**, Proofs of Theorems 1.3.5 and 1.4.3].

PROPOSITION 4.23 ([**44**, Proposition 3.10], cf. [**54**, Theorem 6.2]). (1) *There exists* $c_{\mathrm{G}} \in (0, \infty)$ *such that for any* $g, h \in \mathcal{S}_{\mathcal{H}_0}$,

$$(4.14) \qquad \mu_{\langle g \rangle}(K_w) \leq c_{\mathrm{G}} \mu_{\langle g \rangle}(K_v)$$

whenever either $w, v \in \Lambda_s^{\mathcal{H}}$ *or* $w, v \in \Lambda_s^h$ *for some* $s \in (0, 1]$ *and* $K_w \cap K_v \neq \emptyset$.
(2) *Let* $\kappa := \log_5 15$ *and* $\hat{\kappa} := \log_{5/3} 15$. *Then there exists* $c_{\mathrm{v}} \in (0, \infty)$ *such that for any* $g, h \in \mathcal{S}_{\mathcal{H}_0}$, *any* $x \in K$ *and any* $s, t \in (0, 1]$ *with* $s \leq t$,

$$(4.15) \quad \nu(U_t(x, \mathcal{S})) \leq c_{\mathrm{v}} \left(\frac{t}{s}\right)^{\kappa} \nu(U_s(x, \mathcal{S})), \quad \mu_{\langle g \rangle}(U_t(x, \mathcal{S})) \leq c_{\mathrm{v}} \left(\frac{t}{s}\right)^{\hat{\kappa}} \mu_{\langle g \rangle}(U_s(x, \mathcal{S})),$$

where $(\nu, \mathcal{S})$ *denotes any one of* $(\mu, \mathcal{S}^{\mathcal{H}})$ *and* $(\mu_{\langle h \rangle}, \mathcal{S}^h)$.

REMARK 4.24. *The powers* κ *and* $\hat{\kappa}$ *in* (4.15) *are best possible. See* [**44**, Remark 3.11] *for details.*

Now we conclude the volume doubling property of $(K, \rho_{\mathcal{H}}, \mu)$ and $(K, \rho_h, \mu_{\langle h \rangle})$ as an immediate consequence of Proposition 4.23-(2) and Theorem 4.15-(1).

THEOREM 4.25 ([**54**, Theorem 6.2], [**44**, Theorem 3.19]). *As in Proposition 4.23-(2) let* $\kappa := \log_5 15$ *and* $\hat{\kappa} := \log_{5/3} 15$. *Then there exists* $c_{\mathrm{V}} \in (0, \infty)$ *such that for any* $g, h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, *any* $x \in K$ *and any* $r, s \in (0, \infty)$ *with* $r \leq s$,

$$(4.16) \quad \nu(B_s(x, \rho)) \leq c_{\mathrm{V}} \left(\frac{s}{r}\right)^{\kappa} \nu(B_r(x, \rho)), \; \mu_{\langle g \rangle}(B_s(x, \rho)) \leq c_{\mathrm{V}} \left(\frac{s}{r}\right)^{\hat{\kappa}} \mu_{\langle g \rangle}(B_r(x, \rho)),$$

where (ν, ρ) *denotes any one of* $(\mu, \rho_{\mathcal{H}})$ *and* $(\mu_{\langle h \rangle}, \rho_h)$.

Finally we state the weak Poincaré inequality of $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$.

PROPOSITION 4.26 ([**44**, Proposition 3.20]). *Let* $c_{\mathrm{P}} := 3^4 10^6 c_{\mathrm{G}}^4$ *with* c_{G} *as in Proposition 4.23-(1). Let* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ *and let* (ν, ρ) *denote any one of* $(\mu, \rho_{\mathcal{H}})$ *and* $(\mu_{\langle h \rangle}, \rho_h)$. *Then for any* $(r, x) \in (0, \infty) \times K$, *with* $\overline{u}_{r,x}^{\nu,\rho} := \nu(B_r(x, \rho))^{-1} \int_{B_r(x,\rho)} u \, d\nu$,

$$(4.17) \qquad \int_{B_r(x,\rho)} \left| u - \overline{u}_{r,x}^{\nu,\rho} \right|^2 d\nu \leq c_{\mathrm{P}} r^2 \mu_{\langle u \rangle} \left(B_{250\sqrt{2}r}(x, \rho)\right), \qquad u \in \mathcal{F}.$$

Proposition 4.26 is easily proved by using Lemma 4.7, Theorem 4.15-(1), Lemma 4.22-(1), Proposition 4.23-(1) and the following fact implied by the definition of the resistance metric $R_{\mathcal{E}}$: for any $w \in W_*$ and any $x, y \in K_w$,

$$(4.18) \quad |u(x) - u(y)|^2 \leq R_{\mathcal{E}}\left(F_w^{-1}(x), F_w^{-1}(y)\right) \mathcal{E}(u \circ F_w, u \circ F_w) \leq 3\left(\frac{3}{5}\right)^{|w|} \mu_{\langle u \rangle}(K_w);$$

note that we easily have $\mathrm{diam}_{R_{\mathcal{E}}} K \leq 3$ by using [**53**, Lemma 3.3.5].

NOTATION. In what follows we will use the constants $\kappa = \log_5 15$, $\hat{\kappa} = \log_{5/3} 15$, c_{G} and c_{V} appearing in Proposition 4.23 and Theorem 4.25 without further notice.

5. Short time asymptotics of the heat kernels

In this section, we review known results on short time asymptotic behavior of the heat kernels p_{μ} and $p_{\mu_{\langle h \rangle}}$, $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, mainly following [**44**, Sections 4–6]. The results concern three different aspects of the asymptotics: off-diagonal Gaussian behavior, one-dimensional behavior at vertices and non-integer-dimensional μ-a.e. on-diagonal behavior, which are reviewed separately in each of the following three subsections.

5.1. Intrinsic metrics and off-diagonal Gaussian behavior. Let us start this subsection with the following standard definition.

DEFINITION 5.1. Let ν be a finite Borel measure on K with full support. Define

$$(5.1) \qquad \rho_\nu(x, y) = \sup\{u(x) - u(y) \mid u \in \mathcal{F}, \ \mu_{\langle u \rangle} \leq \nu\}, \qquad x, y \in K.$$

Clearly, $\rho_\nu(x, y) = \rho_\nu(y, x) \in [0, \infty)$, $\rho_\nu(x, x) = 0$ and $\rho_\nu(x, y) \leq \rho_\nu(x, z) + \rho_\nu(z, y)$ for any $x, y, z \in K$; in fact, $\rho_\nu(x, y)^2 \leq \nu(K)R_\mathcal{E}(x, y)$. ρ_ν is called the *intrinsic metric of the Dirichlet space* $(K, \nu, \mathcal{E}, \mathcal{F})$ or simply the *ν-intrinsic metric on K*.

As suggested by the results of [**84, 85, 78, 40**], off-diagonal Gaussian behavior of the Markovian semigroup of a strong local Dirichlet space is described best by the associated intrinsic metric. On the other hand, it is highly non-trivial to give a reasonable geometric characterization of the intrinsic metric for concrete examples. For the canonical Dirichlet space associated with a smooth Riemannian manifold, it is not difficult to see that the intrinsic metric is equal to its Riemannian distance; see [**75**] and references therein for related results on Riemannian manifolds. The same is in fact true also for $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$, $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, as follows.

THEOREM 5.2 ([**44**, Theorem 4.2]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) denote any one of $(\mu, \rho_\mathcal{H})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Then $\rho = \rho_\nu$. Moreover, $\rho(x, \cdot) \in \mathcal{F}$ and $\mu_{\langle \rho(x, \cdot) \rangle} = \nu$ for any $x \in K$.*

Then in view of Theorem 4.25 and Proposition 4.26, the general results of Sturm [**84, 85**] and Ramírez [**78**] together with $\rho = \rho_\nu$ imply the following Gaussian bounds and Varadhan's asymptotic relation.

COROLLARY 5.3 ([**54**, Theorem 6.3], [**44**, Corollary 4.3]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) denote any one of $(\mu, \rho_\mathcal{H})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Let $n \in \mathbb{N}$. Then there exist $c_\mathrm{L}, c_\mathrm{U} \in (0, \infty)$ determined solely by $\kappa, c_\mathrm{G}, c_\mathrm{V}$ and $c_\mathrm{U}(n) \in (0, \infty)$ determined solely by $n, \kappa, c_\mathrm{G}, c_\mathrm{V}$ such that for any $(t, x, y) \in (0, \infty) \times K \times K$,*

$$(5.2) \qquad c_\mathrm{L} \frac{\exp\left(-\frac{\rho(x,y)^2}{c_\mathrm{L} t}\right)}{\nu\left(B_{\sqrt{t}}(x, \rho)\right)} \leq p_\nu(t, x, y) \leq c_\mathrm{U} \frac{\left(1 + \frac{\rho(x,y)^2}{t}\right)^{\kappa/2} \exp\left(-\frac{\rho(x,y)^2}{4t}\right)}{\sqrt{\nu\left(B_{\sqrt{t}}(x, \rho)\right) \nu\left(B_{\sqrt{t}}(y, \rho)\right)}},$$

$$(5.3) \qquad \left| \partial_t^n p_\nu(t, x, y) \right| \leq c_\mathrm{U}(n) \frac{\left(1 + \frac{\rho(x,y)^2}{t}\right)^{\kappa/2+n} \exp\left(-\frac{\rho(x,y)^2}{4t}\right)}{t^n \sqrt{\nu\left(B_{\sqrt{t}}(x, \rho)\right) \nu\left(B_{\sqrt{t}}(y, \rho)\right)}}.$$

COROLLARY 5.4 ([**44**, Corollary 4.4]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) denote any one of $(\mu, \rho_\mathcal{H})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Then*

$$(5.4) \qquad \lim_{t \downarrow 0} 4t \log p_\nu(t, x, y) = -\rho(x, y)^2, \qquad x, y \in K.$$

Moreover, according to a recent result [**58**, Theorem 7.1] of Koskela and Zhou, we have the following asymptotic behavior of the "logarithmic derivatives of the heat kernels" by virtue of $\rho(x, \cdot) \in \mathcal{F}$ and $\mu_{\langle \rho(x, \cdot) \rangle} = \nu$, $x \in K$; see [**72, 83**] for the corresponding pointwise results for the heat kernels on Riemannian manifolds. Note that $\mathrm{dist}_\rho(\cdot, A) \in \mathcal{F}$ for any $A \subset K$ with $A \neq \emptyset$ by Proposition 5.6 below.

COROLLARY 5.5. *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) denote any one of $(\mu, \rho_\mathcal{H})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Then for any $A \in \mathcal{B}(K)$ with $\nu(A) > 0$, $\mu_{\langle 4t \log T_t^\nu \mathbf{1}_A \rangle}$ converges weakly*

to $\mu_{\langle \mathrm{dist}_\rho(\cdot, A)^2 \rangle}$ as $t \downarrow 0$, that is, for any $f \in C(K)$,

$$(5.5) \qquad \lim_{t \downarrow 0} \int_K f \, d\mu_{\langle 4t \log T_t^\nu \mathbf{1}_A \rangle} = \int_K f \mu_{\langle \mathrm{dist}_\rho(\cdot, A)^2 \rangle}.$$

The proof of $\rho \leq \rho_\nu$ and that of $\rho(x, \cdot) \in \mathcal{F}$ and $\mu_{\langle \rho(x, \cdot) \rangle} = \mu$ for $x \in K$ are based on Theorem 3.10 and Theorem 4.15-(2), whereas the converse inequality $\rho_\nu \leq \rho$ is an immediate consequence of the following proposition.

PROPOSITION 5.6 ([**44**, Proposition 4.10]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) be any one of $(\mu, \rho_\mathcal{H})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Then $\{u \in \mathcal{F} \mid \mu_{\langle u \rangle} \leq \nu\} = \{u \in C(K) \mid \mathbf{Lip}_\rho u \leq 1\}$.*

Proposition 5.6 is proved by using Theorem 4.15-(2) and Proposition 5.8 below to reduce the proof to the case of the heat kernels on one-dimensional intervals. We need the following lemma for the statement of Proposition 5.8.

LEMMA 5.7 (cf. [**56**, Theorem 10.4]). *Let ν be a finite Borel measure on K with full support, let U be a non-empty open subset of K and set $\nu|_U := \nu|_{\mathcal{B}(U)}$ and $\mathcal{E}^U := \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U}$. Then $(\mathcal{E}^U, \mathcal{F}_U)$ is a strong local regular Dirichlet form on $L^2(U, \nu|_U)$ whose associated Markovian semigroup $\{T_t^{\nu, U}\}_{t \in (0, \infty)}$ admits a unique continuous integral kernel $p_\nu^U = p_\nu^U(t, x, y) : (0, \infty) \times U \times U \to [0, \infty)$, and p_ν^U is extended to a continuous function on $(0, \infty) \times K \times K$ by setting $p_\nu^U := 0$ on $(0, \infty) \times (K \times K \setminus U \times U)$. p_ν^U is called the* heat kernel *associated with $(U, \nu|_U, \mathcal{E}^U, \mathcal{F}_U)$.*

PROPOSITION 5.8 ([**44**, Proposition 4.9]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, $i \in S$, $b \in (h(q_i), \infty)$ and set $a := h(q_i)$. Suppose that the connected component U of $h^{-1}((-\infty, b))$ with $q_i \in U$ satisfies $U \cap V_0 = \{q_i\}$. Let $p_{[a,b]} = p_{[a,b]}(t, x, y) : (0, \infty) \times [a, b] \times [a, b] \to [0, \infty)$ be the heat kernel for d^2/dx^2 on $[a, b]$ with Neumann (reflecting) boundary condition at a and Dirichlet (absorbing) boundary condition at b. Then*

$$(5.6) \qquad \mu_{\langle h \rangle} \circ (h|_U)^{-1} = \mathcal{E}(h, h_1^i) \mathbf{1}_{[a,b]} dx \qquad (dx \text{ is the Lebesgue measure on } \mathbb{R}),$$

$$(5.7) \qquad p_{\mu_{\langle h \rangle}}^U(t, q_i, x) = \mathcal{E}(h, h_1^i)^{-1} p_{[a,b]}(t, a, h(x)), \qquad (t, x) \in (0, \infty) \times \overline{U},$$

$$(5.8) \qquad \rho_h(q_i, x) = \rho_{\mu_{\langle h \rangle}}(q_i, x) = h(x) - a, \qquad x \in \overline{U}.$$

5.2. One-dimensional asymptotics at vertices. As observed from the picture of the harmonic Sierpiński gasket $K_\mathcal{H}$ (Figure 2), for $x \in V_*$, sufficiently small neighborhoods of $\Phi(x)$ in $K_\mathcal{H}$ are geometrically very close to the *"tangent line of $K_\mathcal{H}$ at $\Phi(x)$"*. As reflections of this geometric intuition, the Kusuoka measure μ and the associated heat kernel p_μ exhibit sharp one-dimensional behavior, as follows.

THEOREM 5.9 ([**44**, Theorem 5.3]). *The limit $\lim_{r \downarrow 0} \mu(B_r(x, \rho_\mathcal{H}))/r =: 2\xi_x \in (0, \infty)$ exists for any $x \in V_*$.*

THEOREM 5.10 ([**44**, Theorem 5.8]). *Let $x \in V_*$. Choose $\zeta_x = (\zeta_x^1, \zeta_x^2) \in \mathrm{Im}\, Z_x$ so that $|\zeta_x| = 1$ and set $h_x := \zeta_x^1(h_1 - h_1(x)\mathbf{1}) + \zeta_x^2(h_2 - h_2(x)\mathbf{1})$. Then there exist $t_x, r_x, c_x \in (0, \infty)$ such that for any $\delta \in (0, 1]$ and any $(t, y) \in (0, t_x] \times B_{r_x}(x, \rho_\mathcal{H})$,*

$$(5.9) \qquad \left| p_\mu(t, x, y) - \frac{\exp\left(-\frac{h_x(y)^2}{4t}\right)}{\xi_x \sqrt{4\pi t}} \right| \leq c_{x,\delta} \left(t^{\frac{\hat{\kappa}-1}{2}} + \delta^\kappa |h_x(y)|^{\frac{2(\hat{\kappa}-1)}{\kappa+1}} \right) \frac{\exp\left(-\frac{h_x(y)^2}{4(1+\delta)t}\right)}{\xi_x \sqrt{4\pi t}},$$

where $c_{x,\delta} := c_x / \delta^{\frac{15}{4}\kappa + \frac{\hat{\kappa}}{2} + 2}$. In particular, $\lim_{t \downarrow 0} \sqrt{4\pi t} p_\mu(t, x, x) = 1/\xi_x$.

THEOREM 5.11 ([**44**, Theorem 5.16]). *Let $x \in V_*$ and $\alpha \in (-1, \infty)$. Then*

$$(5.10) \qquad \lim_{t \downarrow 0} \frac{1}{t^{\alpha/2}} \int_K \rho_{\mathcal{H}}(x,y)^\alpha p_\mu(t,x,y) d\mu(y) = \int_{\mathbb{R}} |y|^\alpha \frac{e^{-y^2/4}}{\sqrt{4\pi}} dy.$$

The key to the proofs of Theorems 5.9, 5.10 and 5.11 is again reduction to the "direction of the tangent line $\operatorname{Im} Z_x = \mathbb{R}\zeta_x$ of $K_{\mathcal{H}}$ at $\Phi(x)$", that is, to the case of $\mu_{\langle h_x \rangle}$, $p_{\mu_{\langle h_x \rangle}}$ and ρ_{h_x}, based on a suitable modification [**44**, Proposition 5.4] of Proposition 5.8 for h_x on a sufficiently small neighborhood of x.

5.3. On-diagonal asymptotics at almost every point. As we saw in Subsections 5.1 and 5.2, the heat kernel p_μ of $(K, \mu, \mathcal{E}, \mathcal{F})$ satisfies the Gaussian bounds and Varadhan's asymptotic relation of exactly the same forms as those for the heat kernels on Riemannian manifolds, and $p_\mu(t,x,x)$ is asymptotically equivalent to a constant multiple of $1/\sqrt{4\pi t}$ as $t \downarrow 0$ for each $x \in V_*$. On the other hand, we cannot expect such a smooth behavior of $p_\mu(t,x,x)$ for generic $x \in K$. Indeed, we have the following result. Recall that $\lambda = \mu \circ \pi$. Note also that $2 \log_{25/3} 5 = 1.5181 \ldots < 2$.

THEOREM 5.12 ([**44**, Theorem 6.1 and Proposition 6.6]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) denote any one of $(\mu, \rho_{\mathcal{H}})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Define*

$$(5.11) \qquad \eta := \inf_{m \in \mathbb{N}} \frac{1}{2m} \sum_{w \in W_m} \lambda(\Sigma_w) \log \|T_w\| \qquad and \qquad d^{\mathrm{loc}} := 2 + \eta^{-1} \log \frac{5}{3}.$$

Then $\eta = \lim_{m \to \infty} \frac{1}{2m} \sum_{w \in W_m} \lambda(\Sigma_w) \log \|T_w\| \in \left[\log \frac{\sqrt{3}}{5}, \log \frac{3}{5} \right)$, $d^{\mathrm{loc}} \in (1, 2 \log_{25/3} 5]$ and

$$(5.12) \qquad \lim_{r \downarrow 0} \frac{\log \nu(B_r(x,\rho))}{\log r} = \lim_{t \downarrow 0} \frac{2 \log p_\nu(t,x,x)}{-\log t} = d^{\mathrm{loc}} \qquad \nu\text{-}a.e.\ x \in K.$$

The key step for the proof of Theorem 5.12 is the following proposition, which can be verified by using Lemma 4.7, Proposition 4.23-(1), (4.8), (4.16) and (4.11).

PROPOSITION 5.13 ([**44**, Proposition 6.4]). *Let $x \in K$ and $\omega \in \pi^{-1}(x)$. Then* $\lim_{r \downarrow 0} \log \mu(B_r(x, \rho_{\mathcal{H}}))/\log r$ *exists if and only if* $\lim_{m \to \infty} \frac{1}{m} \log \|T_{[\omega]_m}\|$ *exists, and if either of these two limits exists then*

$$(5.13) \qquad \frac{\lim_{r \downarrow 0} \log \mu(B_r(x, \rho_{\mathcal{H}}))}{\log r} = 2 + \frac{\log \frac{5}{3}}{\lim_{m \to \infty} \frac{1}{m} \log \|T_{[\omega]_m}\|} \in [1, 2\log_{25/3} 5].$$

It is immediate from (5.11) that $\eta < \log \frac{3}{5}$. Since $\eta = \lim_{m \to \infty} \frac{1}{m} \log \|T_{[\omega]_m}\|$ for λ-a.e. $\omega \in \Sigma$ by the σ-ergodicity of λ (Theorem 3.5) and Kingman's subadditive ergodic theorem [**21**, Theorem 10.7.1], Theorem 5.12 is now easily proved by using Proposition 5.13 (as well as its counterpart for $(\mu_{\langle h \rangle}, \rho_h)$) and (5.2).

6. Ahlfors regularity and singularity of Hausdorff measure

In this and the next sections, we review the author's recent unpublished results to be treated in a forthcoming paper [**47**] which mainly concerns Weyl's Laplacian eigenvalue asymptotics for the Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$. A crucial fact for Weyl's asymptotics is the *Ahlfors regularity of $(K, \rho_{\mathcal{H}})$ and of (K, ρ_h) uniform in $h \in \mathcal{S}_{\mathcal{H}_0}$*, which we explain in some detail in this section as a preparation for the next section. We also see that the Hausdorff measures on $(K, \rho_{\mathcal{H}})$ and (K, ρ_h) (of the appropriate dimension) are singular with respect to the Kusuoka measure μ.

Let us first recall the following standard notations and definitions. See e.g. [**22**, Chapters 2 and 3] and references therein for details of Hausdorff measure, Hausdorff dimension and box-counting dimension; note that the definitions there apply to any metric space although they are stated only for the Euclidean spaces.

NOTATION. Let (E, ρ) be a metric space and let $A \subset E$ be non-empty.
(1) For $\alpha \in (0, \infty)$, the α-dimensional Hausdorff measure on E with respect to ρ is denoted by $\mathcal{H}_\rho^\alpha$ and the Hausdorff dimension of A with respect to ρ by $\dim_H(A, \rho)$.
(2) The lower and upper box-counting dimensions of A with respect to ρ are denoted by $\underline{\dim}_B(A, \rho)$ and $\overline{\dim}_B(A, \rho)$, respectively. If they are equal, their common value, called the box-counting dimension of A with respect to ρ, is denoted by $\dim_B(A, \rho)$.

Note that $0 \leq \dim_H(A, \rho) \leq \underline{\dim}_B(A, \rho) \leq \overline{\dim}_B(A, \rho) \leq \infty$ by [**22**, (3.17)].

The following theorem was obtained in [**44**]. Recall (5.11) for the constant d^{loc}.

THEOREM 6.1 ([**44**, Theorem 7.2 and Proposition 7.6]). *Set* $d := \dim_H(K, \rho_\mathcal{H})$. *Then* $d^{\mathrm{loc}} \leq d \leq 2\log_{25/3} 5$ *and* $\mathcal{H}_{\rho_\mathcal{H}}^d(K) \in (0, \infty)$. *Moreover, for any* $h \in \mathcal{S}_{\mathcal{H}_0}$,

$$(6.1) \qquad d = \dim_B(K, \rho_\mathcal{H}) = \dim_B(K, \rho_h),$$

$$(6.2) \qquad 3^{-10} s^{-d} \leq \#\Lambda_s^h \leq \#\Lambda_s^\mathcal{H} \leq 3^{19} s^{-d}, \qquad s \in (0, 1].$$

The proof of (6.2) heavily relies on the rotational symmetry of K and $(\mathcal{E}, \mathcal{F})$, whereas (6.1) follows from (6.2) by virtue of Lemma 4.7, (4.8) and [**43**, Proposition 2.24]. In fact, we can further prove the following theorem which asserts that $(K, \rho_\mathcal{H})$ and (K, ρ_h), $h \in \mathcal{S}_{\mathcal{H}_0}$, are Ahlfors regular with Hausdorff dimension d.

THEOREM 6.2. *There exist* $c_{6.1}, c_{6.2} \in (0, \infty)$ *such that for any* $h \in \mathcal{S}_{\mathcal{H}_0}$ *and any* $(r, x) \in (0, 1] \times K$,

$$(6.3) \qquad c_{6.1} r^d \leq \mathcal{H}_{\rho_\mathcal{H}}^d(B_r(x, \rho_\mathcal{H})) \leq c_{6.2} r^d, \qquad c_{6.1} r^d \leq \mathcal{H}_{\rho_h}^d(B_r(x, \rho_h)) \leq c_{6.2} r^d.$$

In particular, $d = \dim_H(K, \rho_h)$ *for any* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$.

The following propositions are the key steps for the proof of Theorem 6.2.

PROPOSITION 6.3. *Let* $h \in \mathcal{S}_{\mathcal{H}_0}$ *and let* Λ *be a partition of* Σ. *Then*

$$(6.4) \qquad 3^{-29} \leq \sum_{w \in \Lambda} \|h \circ F_w\|_\mathcal{E}^d \leq 3^{29}.$$

PROPOSITION 6.4. *There exist* $c_{6.3}, c_{6.4} \in (0, \infty)$ *such that for any* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ *and any* $w \in W_*$,

$$(6.5) \qquad c_{6.3} \|T_w\|^d \leq \mathcal{H}_{\rho_\mathcal{H}}^d(K_w) \leq c_{6.4} \|T_w\|^d,$$

$$(6.6) \qquad c_{6.3} \|h \circ F_w\|_\mathcal{E}^d \leq \mathcal{H}_{\rho_h}^d(K_w) \leq c_{6.4} \|h \circ F_w\|_\mathcal{E}^d.$$

Theorem 6.2 is now an easy consequence of Proposition 6.4 together with (4.8), Lemma 4.7 and Lemma 4.22-(1). Proposition 6.3 follows by applying (6.2) to $\Lambda_{s/\|h \circ F_w\|_\mathcal{E}}^{h \circ F_w/\|h \circ F_w\|_\mathcal{E}}$ with $s := \min_{v \in \Lambda} \|h \circ F_v\|_\mathcal{E}$ for $w \in \Lambda$ and using $\#\Lambda_{s/\|h \circ F_w\|_\mathcal{E}}^{h \circ F_w/\|h \circ F_w\|_\mathcal{E}} = \#\{v \in \Lambda_s^h \mid v \leq w\}$ to sum up the resulting inequalities, *which is possible since the constants in* (6.2) *is independent of* h. Then we can also verify Proposition 6.4 on the basis of Proposition 6.3, Lemma 4.9 and the following lemma, by considering $h \circ F_w/\|h \circ F_w\|_\mathcal{E}$ instead of h to localize the argument to K_w (or alternatively, by using Lemma 7.6 below).

LEMMA 6.5 (cf. [**44**, Lemma 7.8]). *Let* $\mathcal{S} = \{\Lambda_s\}_{s \in (0,1]}$ *be a scale on* Σ *with gauge function* l *and let* ρ *be a metric on* K *adapted to* $\mathcal{S}$ *with* $\beta_1, \beta_2 \in (0, \infty)$ *as in* (4.6). *Let* $\alpha, \delta, \varepsilon \in (0, \infty)$ *and let* $\mathcal{H}_{\rho, \delta}^\alpha$ *be the* α-*dimensional pre-Hausdorff measure on* K *with respect to* ρ *as defined in* [**22**, (2.1)] *and* [**53**, Definition 1.5.1]. *If* $\delta \in (0, \beta_1)$ *and* $\mathcal{H}_{\rho, \delta}^\alpha(K) < \varepsilon$, *then there exists a partition* Λ *of* Σ *such that* $\sum_{w \in \Lambda} l(w)^\alpha < 4\beta_1^{-\alpha}\varepsilon$ *and* $\max_{w \in \Lambda} l(w) \leq \beta_1^{-1}\delta$.

For $h, h^\perp \in \mathcal{S}_{\mathcal{H}_0}$ with $\mathcal{E}(h, h^\perp) = 0$, a monotone class argument using Proposition 6.4 easily shows that $2^{-d/2}c_{6.3}c_{6.4}^{-1}\mathcal{H}_{\rho_{\mathcal{H}}}^d \leq \mathcal{H}_{\rho_h}^d + \mathcal{H}_{\rho_{h^\perp}}^d \leq 2^{1-d/2}c_{6.3}^{-1}c_{6.4}\mathcal{H}_{\rho_{\mathcal{H}}}^d$. Then the following conjecture seems natural in view of Proposition 3.4, Theorem 3.6 and Proposition 6.4 although the author has no particular idea how to prove it.

CONJECTURE 6.6. *For any* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, $\mathcal{H}_{\rho_{\mathcal{H}}}^d$ *and* $\mathcal{H}_{\rho_h}^d$ *are mutually absolutely continuous.*

Recall that $d^{\mathrm{loc}} \leq d$ by Theorem 6.1. In fact, here we have the strict inequality, which also implies the singularity of $\mathcal{H}_{\rho_{\mathcal{H}}}^d$ and $\mathcal{H}_{\rho_h}^d$, $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, with respect to μ.

THEOREM 6.7. $d^{\mathrm{loc}} < d$.

COROLLARY 6.8. $\mathcal{H}_{\rho_{\mathcal{H}}}^d$ *is singular to* μ, *and so is* $\mathcal{H}_{\rho_h}^d$ *for any* $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$.

PROOF. Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and let (ν, ρ) denote any one of $(\mu, \rho_{\mathcal{H}})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Define $K_{d^{\mathrm{loc}}}^{\nu, \rho} \in \mathcal{B}(K)$ by

$$(6.7) \qquad K_{d^{\mathrm{loc}}}^{\nu, \rho} := \left\{ x \in K \ \middle| \ \lim_{r \downarrow 0} \frac{\log \nu(B_r(x, \rho))}{\log r} = d^{\mathrm{loc}} \right\}.$$

Then $\mu\big(K \setminus K_{d^{\mathrm{loc}}}^{\nu, \rho}\big) = 0$ by Theorems 5.12 and 3.6, whereas $\dim_{\mathrm{H}}\big(K_{d^{\mathrm{loc}}}^{\nu, \rho}, \rho\big) = d^{\mathrm{loc}}$ by [**22**, Proposition 4.9], where (4.16) is used to verify [**22**, Covering lemma 4.8] for (K, ρ), and hence $\mathcal{H}_\rho^d\big(K_{d^{\mathrm{loc}}}^{\nu, \rho}\big) = 0$ by $d^{\mathrm{loc}} < d$. Thus $\mathcal{H}_\rho^d$ is singular to μ. □

In the rest of this section, we briefly explain the idea of the proof of Theorem 6.7. A similar idea was also used in [**33**, Theorem 4.1 and Proof of Theorem 2.1] to establish singularity of energy measures on self-similar sets. For $m \in \mathbb{N}$, we set

$$(6.8) \qquad \eta_m := \frac{1}{2(2 - d)} \sum_{w \in W_m} \left(\frac{5}{3}\right)^m \|T_w\|^2 \log \frac{\|T_w\|^2}{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w)},$$

so that $\lim_{m \to \infty} \frac{1}{m}\eta_m = \eta$ by Theorem 5.12 and Proposition 6.4. Then for $m, n \in \mathbb{N}$,

$$\eta_{(m+1)n} = \frac{(\frac{5}{3})^{(m+1)n}}{2(2 - d)} \sum_{w \in W_{mn}} \mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w) \sum_{v \in W_n} \frac{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_{wv})}{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w)} \frac{\|T_{wv}\|^2}{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_{wv})} \log \frac{\|T_{wv}\|^2}{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_{wv})}$$

$$(6.9) \qquad \geq \frac{(\frac{5}{3})^{(m+1)n}}{2(2 - d)} \sum_{w \in W_{mn}} \mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w) \frac{(\frac{3}{5})^n \|T_w\|^2}{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w)} \log \frac{(\frac{3}{5})^n \|T_w\|^2}{\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w)} = \eta_{mn} + \frac{n \log \frac{3}{5}}{2 - d},$$

where the inequality is due to the convexity of the function $(0, \infty) \ni t \mapsto t \log t$ and $\sum_{v \in W_n} \|T_{wv}\|^2 = (3/5)^n \|T_w\|^2$. (6.9) in particular yields

$$(6.10) \qquad \frac{\log \frac{3}{5}}{2 - d^{\mathrm{loc}}} = \eta = \lim_{m \to \infty} \frac{\eta_m}{m} \geq \lim_{m \to \infty} \frac{1}{m}\left(\eta_1 + (m - 1)\frac{\log \frac{3}{5}}{2 - d}\right) = \frac{\log \frac{3}{5}}{2 - d}$$

and hence $d^{\mathrm{loc}} \leq d$, which was proved in [**44**, Proof of Theorem 7.2] by a different method.

Note that $c_{6.4}^{-1} \|T_w\|^{2-d} \leq \|T_w\|^2/\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_w) \leq c_{6.3}^{-1}\|T_w\|^{2-d}$ for any $w \in W_*$ by Proposition 6.4 and that the image $Z(K_Z)$ of the map $Z|_{K_Z} : K_Z \to \mathbb{R}^{2\times2}$ is equal to the set of *all* orthogonal projections on $\mathbb{R}^2$ of rank 1 by Theorem 4.12 and the rotational symmetry of $K_{\mathcal{H}}$. By using these facts and the definition of Z in Proposition 3.7, we can verify that if we fix a sufficiently large $n \in \mathbb{N}$, then $\left(\|T_{wv}\|^2/\mathcal{H}_{\rho_{\mathcal{H}}}^d(K_{wv})\right)_{v\in W_n} \in \mathbb{R}^{W_n}$ appearing in the first line of (6.9) is some uniform distance away from constant vectors for m large enough and for sufficiently many $w \in W_{mn}$. Then the second line of (6.9) has to be smaller than the first by a uniform constant for m large enough, which together with the same limiting procedure as in (6.10) yields $d^{\mathrm{loc}} < d$. See [47] for the complete proof and further details.

7. Weyl's Laplacian eigenvalue asymptotics

As already mentioned in the last section, our main concern in this section is Weyl's Laplacian eigenvalue asymptotics for the Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$, which is the main result of the author's forthcoming paper [47].

Let us start with the following basic definition. See Lemma 5.7 above for the definitions of $\nu|_U$, $(\mathcal{E}^U, \mathcal{F}_U)$, $\{T_t^{\nu,U}\}_{t\in(0,\infty)}$ and p_ν^U.

DEFINITION 7.1. Let ν be a finite Borel measure on K with full support and let U be a non-empty open subset of K. Noting that the non-positive self-adjoint operator $\Delta_{\nu,U}$ of $(U, \nu|_U, \mathcal{E}^U, \mathcal{F}_U)$ (the generator of $\{T_t^{\nu,U}\}_{t\in(0,\infty)}$) has discrete spectrum and that $\operatorname{tr} T_t^{\nu,U} < \infty$ for $t \in (0,\infty)$ by [19, Theorem 2.1.4], let $\{\lambda_n^{\nu,U}\}_{n\in\mathbb{N}}$ be the non-decreasing enumeration of all the eigenvalues of $-\Delta_{\nu,U}$, where each eigenvalue is repeated according to its multiplicity. The *eigenvalue counting function* $\mathcal{N}_{\nu,U}$ and the *partition function* $\mathcal{Z}_{\nu,U}$ *of the Dirichlet space* $(U, \nu|_U, \mathcal{E}^U, \mathcal{F}_U)$ are defined respectively by, for $\lambda \in \mathbb{R}$ and $t \in (0,\infty)$,

$$(7.1) \qquad\qquad \mathcal{N}_{\nu,U}(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^{\nu,U} \leq \lambda\},$$

$$(7.2) \quad \mathcal{Z}_{\nu,U}(t) := \operatorname{tr} T_t^{\nu,U} = \sum_{n\in\mathbb{N}} e^{-\lambda_n^{\nu,U} t} = \int_{\mathbb{R}} e^{-\lambda t} d\mathcal{N}_{\nu,U}(\lambda) = \int_K p_\nu^U(t,x,x) d\nu(x).$$

In the situation of Definition 7.1, $\mathcal{N}_{\nu,U}(\lambda) < \infty$ for $\lambda \in \mathbb{R}$ by $\lim_{n\to\infty} \lambda_n^{\nu,U} = \infty$, $\mathcal{N}_{\nu,U}(\lambda) = 0$ for $\lambda \in (-\infty, 0)$ by $\lambda_1^{\nu,U} \geq 0$, and $\mathcal{Z}_{\nu,U}$ is $(0,\infty)$-valued and continuous.

The main result of [47] is stated as follows.

THEOREM 7.2. *There exist* $c_{\mathcal{N}}, c_{\mathcal{Z}} \in (0,\infty)$ *such that for any non-empty open subset* U *of* K *with* $\mathcal{H}_{\rho_{\mathcal{H}}}^d(\partial_K U) = 0$,

$$(7.3) \qquad \lim_{\lambda\to\infty} \frac{\mathcal{N}_{\mu,U}(\lambda)}{\lambda^{d/2}} = c_{\mathcal{N}} \mathcal{H}_{\rho_{\mathcal{H}}}^d(U) \qquad and \qquad \lim_{t\downarrow 0} t^{d/2} \mathcal{Z}_{\mu,U}(t) = c_{\mathcal{Z}} \mathcal{H}_{\rho_{\mathcal{H}}}^d(U).$$

Recall that for the eigenvalue counting function $\mathcal{N}_U$ and the partition function $\mathcal{Z}_U$ associated with the Dirichlet Laplacian on a non-empty open subset U of $\mathbb{R}^k$ with $\operatorname{vol}_k(U) < \infty$, where vol_k denotes the Lebesgue measure on $\mathbb{R}^k$, it holds that

$$(7.4) \lim_{\lambda\to\infty} \frac{\mathcal{N}_U(\lambda)}{\lambda^{k/2}} = (2\pi)^{-k} v_k \operatorname{vol}_k(U) \quad and \quad \lim_{t\downarrow 0} t^{k/2} \mathcal{Z}_U(t) = (4\pi)^{-k/2} \operatorname{vol}_k(U)$$

with $v_k := \operatorname{vol}_k(\{x \in \mathbb{R}^k \mid |x| < 1\})$. See e.g. [15, 16, 66, 67, 68, 74] and references therein for known results concerning Weyl's asymptotics on Euclidean domains and Riemannian manifolds. According to Theorem 7.2, the Dirichlet Laplacian $\Delta_{\mu,U}$ on a non-empty open subset U of K satisfies Weyl's eigenvalue asymptotics similar to

(7.4) unless $\partial_K U$ is "too rough", but the limit is given by a constant multiple of the Hausdorff measure $\mathcal{H}^d_{\rho_{\mathcal{H}}}$ which is, contrary to (7.4), *singular to the "Riemannian volume measure"* μ by Corollary 6.8.

The rest of this section is devoted to a sketch of the proof of Theorem 7.2. The main idea of the proof is to follow the method due to Kigami and Lapidus [**57**] of obtaining a renewal equation for the eigenvalue counting function (or the partition function) from the self-similarity of $(\mathcal{E}, \mathcal{F})$ to apply a suitable renewal theorem. The problem in doing so for the present setting of $(K, \mu, \mathcal{E}, \mathcal{F})$ is that the reference measure μ is *not* self-similar, but it can be resolved by incorporating the information on the pair $\{h_1 \circ F_w, h_2 \circ F_w\}$ of V_0-harmonic functions appearing in $\mu \circ F_w = (5/3)^{|w|}\big(\mu_{\langle h_1 \circ F_w \rangle} + \mu_{\langle h_2 \circ F_w \rangle}\big)$ as the second variable, in the following way.

DEFINITION 7.3. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}_0$. We define $\Phi_M := M\Phi : K \to \mathbb{R}^2$,

$$(7.5) \qquad \mu_M := \mu_{\langle ah_1 + bh_2 \rangle} + \mu_{\langle ch_1 + dh_2 \rangle},$$

(7.6) $\rho_M(x,y) := \inf\{\ell_M(\gamma) \mid \gamma : [0,1] \to K, \ \gamma \text{ is continuous}, \ \gamma(0) = x, \ \gamma(1) = y\}$

for $x, y \in K$, where $\ell_M(\gamma) := \ell_{\mathbb{R}^2}(\Phi_M \circ \gamma)$ for each continuous map $\gamma : [a_\gamma, b_\gamma] \to K$, $a_\gamma, b_\gamma \in \mathbb{R}$, $a_\gamma \leq b_\gamma$. Note that $\Phi_M(x) = ((ah_1 + bh_2)(x), (ch_1 + dh_2)(x))$ for $x \in K$.

Namely, we identify $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}_0$ with the pair $\{ah_1 + bh_2, ch_1 + dh_2\}$ of V_0-harmonic functions and define the counterpart of $\Phi, \mu, \rho_{\mathcal{H}}$ for M by replacing $\{h_1, h_2\}$ with $\{ah_1 + bh_2, ch_1 + dh_2\}$. Then for $M \in \mathbb{R}^{2\times 2}_0$ with $\det M \neq 0$, Theorems 3.8 and 3.10-(1) are valid with $\Phi_M, \mu_M, Z_M := \|MZ\|^{-2}MZM^*$ in place of Φ, μ, Z (Theorem 3.10-(1) remains valid for $\det M = 0$ if we set $Z_M := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for $MZ = 0$); note that $Z_M(x)$ is the orthogonal projection onto $M(\operatorname{Im} Z_x)$ for each $x \in K$.

We also define the corresponding scale on Σ as follows. Note that $\|M\|^2 = \|ah_1 + bh_2\|^2_{\mathcal{E}} + \|ch_1 + dh_2\|^2_{\mathcal{E}}$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}_0$.

DEFINITION 7.4. (1) We set $\mathbb{S}^{2\times 2} := \{M \in \mathbb{R}^{2\times 2} \mid \|M\| = 1\}$.
(2) Let $M \in \mathbb{S}^{2\times 2}$. We define $\mathcal{S}^M = \{\Lambda^M_s\}_{s \in (0,1]}$ to be the scale on Σ induced by the gauge function $l_M : W_* \to (0,1]$, $l_M(w) := \|MT_w\| = \sqrt{(3/5)^{|w|}\mu_M(K_w)}$.

Then for $M \in \mathbb{S}^{2\times 2}$, by exactly the same proofs we still have Theorem 4.15 and Lemma 4.17 with M in place of $\mathbf{h}$, and Lemma 4.22 and Proposition 4.23 with $\mu_M, l_M, \mathcal{S}^M = \{\Lambda^M_s\}_{s \in (0,1]}$ in place of $\mu_{\langle h \rangle}, l_h, \mathcal{S}^h = \{\Lambda^h_s\}_{s \in (0,1]}$ and $c_{\mathrm{G}}, c_{\mathrm{v}}$ the same. Consequently Theorem 4.25, Proposition 4.26, Theorem 5.2, Corollaries 5.3, 5.4, 5.5 and Proposition 5.6 hold for $(\nu, \rho) = (\mu_M, \rho_M)$, $M \in \mathbb{R}^{2\times 2}_0$, with $c_{\mathrm{V}}, c_{\mathrm{P}}, c_{\mathrm{L}}, c_{\mathrm{U}}, c_{\mathrm{U}}(n)$ unchanged. Theorem 4.19 and Lemma 4.20 with ρ_M, ℓ_M in place of $\rho_{\mathcal{H}}, \ell_{\mathcal{H}}$ are also valid for any $M \in \mathbb{R}^{2\times 2}_0$ *with* $\det M \neq 0$ by exactly the same proofs.

REMARK 7.5. Here *we do* **not** *exclude the case of* $\det M = 0$ where μ_M, ρ_M, ℓ_M are equal to $\mu_{\langle h \rangle}, \rho_h, \ell_h$ for some $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ with $\|h\|_{\mathcal{E}} = \|M\|$ and $l_M, \mathcal{S}^M$ are equal to $l_h, \mathcal{S}^h$ when $M \in \mathbb{S}^{2\times 2}$. This consideration is absolutely necessary for the proof of Theorem 7.2, whose reason will be described at the very end of this section.

Noting that $\#\partial_K K_w < \infty$ for $w \in W_*$ by $\partial_K K_w \subset F_w(V_0)$, we can easily verify the following lemma from Proposition 3.2 and the definitions of ρ_M and $\mathcal{H}^d_{\rho_M}$.

LEMMA 7.6. $\mathcal{H}^d_{\rho_M} \circ F_w = \mathcal{H}^d_{\rho_{MT_w}}$ *for any* $M \in \mathbb{R}^{2\times 2}_0$ *and any* $w \in W_*$.

The following proposition is a crucial step for the proof of Theorem 7.2.

PROPOSITION 7.7. (1) *There exist* $c_{7.1}, c_{7.2} \in (0, \infty)$ *such that*

$$(7.7) \qquad\qquad c_{7.1} \|M\|^d \le \mathcal{H}^d_{\rho_M}(K) \le c_{7.2} \|M\|^d, \qquad M \in \mathbb{R}^{2\times 2}_0.$$

(2) *Define* $\mathcal{T} : \mathbb{R}^{2\times 2}_0 \to (0, \infty)$ *by* $\mathcal{T}(M) := \mathcal{H}^d_{\rho_M}(K)^{1/d}$. *Then* $\mathcal{T}$ *is continuous, and for any* $M \in \mathbb{R}^{2\times 2}_0$, *any* $U \in O(2)$ *and any* $a \in \mathbb{R} \setminus \{0\}$,

$$(7.8) \quad \mathcal{T}(M)^d = \sum_{i \in S} \mathcal{T}(MT_i)^d, \quad \mathcal{T}(UM) = \mathcal{T}(M) \quad and \quad \mathcal{T}(aM) = |a|\mathcal{T}(M).$$

REMARK 7.8. (1) Let $M \in \mathbb{R}^{2\times 2}_0$, $\det M \ne 0$ and $c_M := \inf_{T \in \mathbb{R}^{2\times 2}_0} \|MT\|/\|T\|$, so that $c_M > 0$. Then a monotone class argument using (6.5), Lemma 7.6 and (7.7) easily shows that $c_{6.4}^{-1} c_{7.1} c_M^d \mathcal{H}^d_{\rho_{\mathcal{H}}} \le \mathcal{H}^d_{\rho_M} \le c_{6.3}^{-1} c_{7.2} \|M\|^d \mathcal{H}^d_{\rho_{\mathcal{H}}}$. In particular, $\mathcal{H}^d_{\rho_M}$ is singular to μ by Corollary 6.8.
(2) In fact, for each $r, s \in (0, \infty)$ with $r \le s$, $\mathcal{T}$ is $\frac{d}{d+1}$-Hölder continuous on $\{M \in \mathbb{R}^{2\times 2}_0 \mid r \le \|M\| \le s\}$ with respect to the norm metric.
(3) The properties stated in Proposition 7.7-(2) characterizes $\mathcal{T}$ uniquely up to constant multiples: if $\mathcal{T}_0 : \mathbb{R}^{2\times 2}_0 \to [0, \infty)$ *is continuous and satisfies* (7.8) *for any* $M \in \mathbb{R}^{2\times 2}_0$, *any* $U \in O(2)$ *and any* $a \in \mathbb{R} \setminus \{0\}$, *then* $\mathcal{T}_0 = c_0 \mathcal{T}$ *for some* $c_0 \in [0, \infty)$.

Proposition 7.7-(1) follows from Proposition 6.3, Lemma 6.5 and (4.8) with M in place of $\mathbf{h}$, and (7.8) is immediate from Lemma 7.6 and the definition of $\mathcal{T}$. On the other hand, *the proof of the continuity of* $\mathcal{T}$ *at* $M \in \mathbb{R}^{2\times 2}_0$ *satisfying* $\det M = 0$ *is quite involved and relies heavily on Theorem 4.15-(2)*.

By virtue of Proposition 7.7, now we can associate to the function $(t, M) \mapsto \mathcal{Z}_{\mu_M, K\setminus V_0}(t)$ a certain renewal equation to which *Kesten's renewal theorem* [**51**, *Theorem* 2] *for functionals of Markov chains* is applicable, as follows. Define

$$(7.9) \quad \mathbb{S}_{\mathcal{T}} := \{M \in \mathbb{R}^{2\times 2}_0 \mid \mathcal{T}(M) = 1\} \quad and \quad M^{\mathcal{T}} := \mathcal{T}(M)^{-1} M \in \mathbb{S}_{\mathcal{T}}, \ M \in \mathbb{R}^{2\times 2}_0,$$

so that $\mathbb{S}_{\mathcal{T}}$ is compact by Proposition 7.7, and further define $\mathcal{N} : \mathbb{S}_{\mathcal{T}} \times \mathbb{R} \to \mathbb{N} \cup \{0\}$ and $\mathcal{Z} : \mathbb{S}_{\mathcal{T}} \times (0, \infty) \to (0, \infty)$ by

$$(7.10) \qquad \mathcal{N}(M, \lambda) := \mathcal{N}_{\mu_M, K\setminus V_0}(\lambda) \quad and \quad \mathcal{Z}(M, t) := \mathcal{Z}_{\mu_M, K\setminus V_0}(t).$$

Let $M \in \mathbb{S}_{\mathcal{T}}$. Then $\mu_M \le c_{7.1}^{-d/2} \mu$ by (7.7) and hence $\lambda_1^{\mu_M, K\setminus V_0} \ge c_{7.1}^{2/d} \lambda_1^{\mu, K\setminus V_0} > 0$. Since $\mathcal{N}_{\mu_M, K_i\setminus F_i(V_0)}(\lambda) = \mathcal{N}_{\mu_{MT_i}, K\setminus V_0}(\lambda) = \mathcal{N}\big((MT_i)^{\mathcal{T}}, \mathcal{T}(MT_i)^2 \lambda\big)$ for $\lambda \in \mathbb{R}$ and $i \in S$ by (2.4) and $\mu_M \circ F_i = (5/3)\mu_{MT_i}$, [**53**, Corollary 4.1.8] yields

$$0 \le \mathcal{N}(M, \lambda) - \sum_{i \in S} \mathcal{N}\big((MT_i)^{\mathcal{T}}, \mathcal{T}(MT_i)^2 \lambda\big) = \mathcal{N}_{\mu_M, K\setminus V_0}(\lambda) - \mathcal{N}_{\mu_M, K\setminus V_1}(\lambda) \le 3$$

for $\lambda \in \mathbb{R}$, and hence for any $t \in (0, \infty)$,

$$(7.11) \qquad\qquad 0 \le \mathcal{Z}(M, t) - \sum_{i \in S} \mathcal{Z}\big((MT_i)^{\mathcal{T}}, \mathcal{T}(MT_i)^{-2} t\big) \le 3.$$

Now we define $f, g : \mathbb{S}_{\mathcal{T}} \times \mathbb{R} \to \mathbb{R}$ by $f(M, s) := e^{-ds} \mathcal{Z}(M, e^{-2s})$ and

$$(7.12) \qquad g(M, s) := e^{-ds}\left(\mathcal{Z}(M, e^{-2s}) - \sum_{i \in S} \mathcal{Z}\big((MT_i)^{\mathcal{T}}, \mathcal{T}(MT_i)^{-2} e^{-2s}\big) \right),$$

so that for any $(M, s) \in \mathbb{S}_{\mathcal{T}} \times \mathbb{R}$, $0 \le g(M, s) \le 3e^{-ds}$ and

$$(7.13) \qquad f(M, s) = g(M, s) + \sum_{i \in S} \mathcal{T}(MT_i)^d f\big((MT_i)^{\mathcal{T}}, s + \log \mathcal{T}(MT_i)\big).$$

Since $\sum_{i \in S} \mathfrak{T}(MT_i)^d = \mathfrak{T}(M)^d = 1$ by (7.8), we observe that the sum in (7.13) involves a conservative Markovian kernel (see Definition 8.2 below) $\mathcal{P}$ on $(\mathbb{S}_{\mathfrak{T}}, \mathcal{B}(\mathbb{S}_{\mathfrak{T}}))$ given by

$$(7.14) \qquad \mathcal{P}(M, \cdot) := \sum_{i \in S} \mathfrak{T}(MT_i)^d \delta_{(MT_i)^{\mathfrak{T}}}, \qquad M \in \mathbb{S}_{\mathfrak{T}},$$

where $\delta_M(A) := \mathbf{1}_A(M)$, $M \in \mathbb{S}_{\mathfrak{T}}$, $A \subset \mathbb{S}_{\mathfrak{T}}$. Then by considering a Markov chain $X = (\Omega, \mathcal{M}, \{X_n\}_{n \in \mathbb{N} \cup \{0\}}, \{\mathbf{P}_M\}_{M \in \mathbb{S}_{\mathfrak{T}}})$ on $\mathbb{S}_{\mathfrak{T}}$ such that $\mathbf{P}_M \circ X_1^{-1} = \mathcal{P}(M, \cdot)$ for $M \in \mathbb{S}_{\mathfrak{T}}$ and another sequence $\{u_n\}_{n \in \mathbb{N}}$ of real random variables on $(\Omega, \mathcal{M})$ such that $u_n = -\log \mathfrak{T}(X_n T_i)$ on $\{X_{n+1} = (X_n T_i)^{\mathfrak{T}}\}$, from (7.13) we obtain

$$(7.15) \qquad f(M, s) = \mathbf{E}_M \left[\sum_{k=0}^{n-1} g(X_k, s - \mathcal{V}_k) \right] + \mathbf{E}_M[f(X_n, s - \mathcal{V}_n)]$$

for any $(M, s) \in \mathbb{S}_{\mathfrak{T}} \times \mathbb{R}$ and any $n \in \mathbb{N}$, where $\mathcal{V}_n := \sum_{k=0}^{n-1} u_k$. It is not difficult to see that $0 \le f(X_n, s - \mathcal{V}_n) \le c_{7.3} e^{ds} (3/5)^{dn}$ for n large enough (depending only on s) for some $c_{7.3} \in (0, \infty)$, and hence letting $n \to \infty$ in (7.15) results in

$$(7.16) \qquad f(M, s) = \mathbf{E}_M \left[\sum_{n=0}^{\infty} g(X_n, s - \mathcal{V}_n) \right], \qquad (M, s) \in \mathbb{S}_{\mathfrak{T}} \times \mathbb{R}.$$

Kesten's renewal theorem [51, Theorem 2] asserts the convergence as $s \to \infty$ of a function of the form (7.16) to a finite limit *which is independent of M*. Therefore once the assumptions of [51, Theorem 2] are verified, we can conclude (7.3) for $U = K_w \setminus F_w(V_0)$, $w \in W_*$, where the strict positivity of the limit is guaranteed by an a priori bound [44, Theorem 7.2] for $\mathcal{Z}_{\mu_{\langle h \rangle}, K}$, $h \in \mathcal{S}_{\mathcal{H}_0}$, and then the extension to a general U is straightforward. The assumptions of [51, Theorem 2] consist of conditions [51, I.1–I.4] on the random variables $\{(X_n, \mathcal{V}_n)\}_{n \in \mathbb{N} \cup \{0\}}$, the direct Riemann integrability [51, Definition 1] of the function $g : \mathbb{S}_{\mathfrak{T}} \times \mathbb{R} \to \mathbb{R}$ and the continuity of g. The direct Riemann integrability of g follows from the estimates $\lambda_1^{\mu_M, K \setminus V_0} \ge c_{7.1}^{2/d} \lambda_1^{\mu, K \setminus V_0} > 0$ and $0 \le g(M, s) \le 3e^{-ds}$ for $(M, s) \in \mathbb{S}_{\mathfrak{T}} \times \mathbb{R}$, and for the sake of the continuity we need to consider the partition functions $\mathcal{Z}_{\mu_M, K \setminus V_0}$ and not the eigenvalue counting functions $\mathcal{N}_{\mu_M, K \setminus V_0}$.

To verify part of the conditions [51, I.1 and I.2], we need the existence and the uniqueness of an invariant probability measure for $\mathcal{P}$. The existence follows by the classical theorem of Krylov and Bogolioubov [59] (see [29, Theorem 1.10]) since $\mathbb{S}_{\mathfrak{T}}$ is compact and $\mathcal{P}$ is a Feller Markovian kernel, i.e. $\mathcal{P}u \in C(\mathbb{S}_{\mathfrak{T}})$ for any $u \in C(\mathbb{S}_{\mathfrak{T}})$, by the continuity of $\mathfrak{T}$. The uniqueness is implied by a recent powerful result [94, Theorem 6.4] of Worm and Hille (see also [93, Theorem 7.4.6]). (To be precise, the uniqueness is assured only after taking the quotient $\check{\mathbb{S}}_{\mathfrak{T}} := O(2) \backslash \mathbb{S}_{\mathfrak{T}}$ of $\mathbb{S}_{\mathfrak{T}}$ by the canonical left action $O(2) \times \mathbb{S}_{\mathfrak{T}} \ni (U, M) \mapsto UM \in \mathbb{S}_{\mathfrak{T}}$ of $O(2)$ on $\mathbb{S}_{\mathfrak{T}}$. Accordingly $\mathbb{S}_{\mathfrak{T}}$ in the above argument has to be replaced by $\check{\mathbb{S}}_{\mathfrak{T}}$.) Note that *the state space $\mathbb{S}_{\mathfrak{T}}$ of the Markov chain X would not be compact if $\mathbb{S}_{\mathfrak{T}}^0 := \{M \in \mathbb{S}_{\mathfrak{T}} \mid \det M = 0\}$ were removed*. Moreover, since $\mathbb{S}_{\mathfrak{T}}^0$ is closed in $\mathbb{S}_{\mathfrak{T}}$ and $\mathcal{P}$-invariant (i.e. $\mathcal{P}(M, \mathbb{S}_{\mathfrak{T}}^0) = 1$ for any $M \in \mathbb{S}_{\mathfrak{T}}^0$), *the invariant measure for $\mathcal{P}$ has to be supported on $\mathbb{S}_{\mathfrak{T}}^0$* (to be precise, $\mathcal{P}$ needs to be regarded as a Markovian kernel on $(\check{\mathbb{S}}_{\mathfrak{T}}, \mathcal{B}(\check{\mathbb{S}}_{\mathfrak{T}}))$). For these reasons we cannot exclude $\mathbb{S}_{\mathfrak{T}}^0$ from the state space of the Markov chain X.

8. Connections to general theories on metric measure spaces

In this section, we briefly mention some connections to general theories of analysis and geometry on metric measure spaces. In Subsection 8.1, we state some recent results of Koskela and Zhou [**58**, Section 4] on connections to the theory of differential calculus on metric measure spaces initiated by Cheeger [**17**] and developed further by many people, e.g. Shanmugalingam [**81**] and Keith [**48, 49, 50**]. Subsection 8.2 concerns the theory of Ricci curvature lower bound for metric measure spaces established by Lott and Villani [**71, 70**] and Sturm [**86, 87**] and partially also by Ohta [**76**]. In fact, very recently there have been attempts to unify methods and ideas developed in those fields to establish differential calculus on an even wider range of metric measure spaces, e.g. Ambrosio, Gigli and Savaré [**2, 3, 4**], Gigli [**27**] and Koskela and Zhou [**58**]. There are also a huge number of other related results and it is beyond the author's ability to review even just the central achievements of these fields. For further details, we refer the reader to the above-mentioned works, monographs [**31, 1, 91, 92**] and references therein.

We need the following definitions for the discussions below.

DEFINITION 8.1. Let (E, ρ) be a metric space and let $\gamma : [a, b] \to E$, $a, b \in \mathbb{R}$, $a \leq b$, be continuous.
(1) The ρ-length $\ell_\rho(\gamma)$ of γ is defined by
(8.1)
$$\ell_\rho(\gamma) := \sup\{\textstyle\sum_{k=1}^n \rho(\gamma(t_{k-1}), \gamma(t_k)) \mid n \in \mathbb{N}, \{t_k\}_{k=0}^n \subset [a, b] \text{ is non-decreasing}\}.$$
(2) Suppose $\ell_\rho(\gamma) < \infty$ and define $\varphi_\gamma : [a, b] \to [0, \ell_\rho(\gamma)]$ by $\varphi_\gamma(t) := \ell_\rho(\gamma|_{[a,t]})$. Since $\gamma(s) = \gamma(t)$ for any $s, t \in [a, b]$ with $\varphi_\gamma(s) = \varphi_\gamma(t)$, there exists a unique map $\gamma_\rho : [0, \ell_\rho(\gamma)] \to E$ such that $\gamma = \gamma_\rho \circ \varphi_\gamma$, and then γ_ρ is continuous and $\ell_\rho(\gamma_\rho|_{[0,t]}) = t$ for any $t \in [0, \ell_\rho(\gamma)]$. γ_ρ is called the ρ-length parametrization of γ.
(3) Suppose $a < b$. Then γ is called a minimal ρ-geodesic if and only if $\ell_\rho(\gamma|_{[a,t]}) = \frac{t-a}{b-a}\rho(\gamma(a), \gamma(b))$ for any $t \in [a, b]$, or equivalently $\rho(\gamma(s), \gamma(t)) = \frac{|t-s|}{b-a}\rho(\gamma(a), \gamma(b))$ for any $s, t \in [a, b]$. Let $\mathcal{G}(E, \rho)$ denote the set of minimal ρ-geodesics with domain $[0, 1]$, which is equipped with the metric $\rho_{\mathcal{G}(E,\rho)}(\gamma_1, \gamma_2) := \sup_{t \in [0,1]} \rho(\gamma_1(t), \gamma_2(t))$.

Also recall the following standard definition.

DEFINITION 8.2. Let $(E, \mathcal{B})$ be a measurable space. Then $\mathcal{P} : E \times \mathcal{B} \to [0, 1]$ is called a Markovian kernel on $(E, \mathcal{B})$ if and only if $\mathcal{P}(x, \cdot) : \mathcal{B} \to [0, 1]$ is a measure on $(E, \mathcal{B})$ for any $x \in E$ and $\mathcal{P}(\cdot, A) : E \to [0, 1]$ is $\mathcal{B}$-measurable for any $A \in \mathcal{B}$. Such $\mathcal{P}$ is called conservative if and only if $\mathcal{P}(x, E) = 1$ for any $x \in E$.

8.1. Identification of Dirichlet form as Cheeger energy. In [**17**] Cheeger established a theory of differential calculus on a general metric measure space which admits the volume doubling property and the weak $(1, p)$-Poincaré inequality (in terms of upper gradients) for some $p \in [1, \infty)$. Since $(K, \rho_{\mathcal{H}}, \mu)$ and $(K, \rho_h, \mu_{\langle h \rangle})$, $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, satisfy the volume doubling property (Theorem 4.25) and the weak Poincaré inequality in terms of (the densities of) the $\mathcal{E}$-energy measures (Proposition 4.26), it is natural to expect that Cheeger's results in [**17**] are applicable to them. Koskela and Zhou [**58**, Section 4] have recently proved that this is indeed the case, that Cheeger's Rademacher theorem [**17**, Theorem 4.38] takes an explicit form using Φ and h as the coordinate functions, and that the associated Cheeger 2-energies coincide with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. To be precise, they have proved the following results. Recall Theorem 3.10 for the derivatives $\widetilde{\nabla}u$ and $\frac{du}{dh}$ of $u \in \mathcal{F}$.

THEOREM 8.3 ([**58**, Theorems 4.2 and 4.3]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and $u \in \mathcal{F}$. Then for μ-a.e. $x \in K$, or equivalently for $\mu_{\langle h \rangle}$-a.e. $x \in K$ (recall Theorem 3.6),*

$$(8.2) \qquad u(y) - u(x) = \langle \widetilde{\nabla} u(x), \Phi(y) - \Phi(x) \rangle + o(\rho_{\mathcal{H}}(x, y)) \quad \text{as } y \to x,$$

$$(8.3) \qquad u(y) - u(x) = \frac{du}{dh}(x)(h(y) - h(x)) + o(\rho_h(x, y)) \qquad \text{as } y \to x.$$

DEFINITION 8.4. *Let (E, ρ) be a metric space. For $f : E \to \mathbb{R}$, we define*

$$(8.4) \qquad \mathrm{Lip}_\rho f(x) := \lim_{r \downarrow 0} \sup_{y \in B_r(x, \rho) \setminus \{x\}} \frac{|f(y) - f(x)|}{\rho(x, y)}, \quad x \in E.$$

Note that $\mathrm{Lip}_\rho f : E \to [0, \infty]$ is Borel measurable if $f \in C(E)$.

THEOREM 8.5 ([**58**, Theorems 4.1 and 4.3]). *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and $u \in \mathcal{F}$. Then*

$$(8.5) \qquad |\widetilde{\nabla} u| = \mathrm{Lip}_{\rho_{\mathcal{H}}} u \quad \mu\text{-a.e.} \quad \text{and} \quad \left| \frac{du}{dh} \right| = \mathrm{Lip}_{\rho_h} u \quad \mu_{\langle h \rangle}\text{-a.e.}$$

As a consequence, we obtain the identification of $(\mathcal{E}, \mathcal{F})$ as the Cheeger 2-energy and $(1, 2)$-Sobolev space, for which let us recall Cheeger's definitions in [**17**].

DEFINITION 8.6 ([**32**]). *Let (E, ρ) be a metric space and let $f : E \to \mathbb{R}$. A Borel measurable function $g : X \to [0, \infty]$ is called an upper ρ-gradient for f if and only if for any continuous map $\gamma : [0, 1] \to E$ with $\ell_\rho(\gamma) < \infty$,*

$$(8.6) \qquad |f(\gamma(1)) - f(\gamma(0))| \leq \int_0^{\ell_\rho(\gamma)} g(\gamma_\rho(s)) ds.$$

DEFINITION 8.7 ([**17**, Section 2]). *Let (E, ρ) be a metric space and let $\mathfrak{m}$ be a Borel measure on E such that $\mathfrak{m}(B_r(x, \rho)) \in (0, \infty)$ for any $(r, x) \in (0, \infty) \times E$. For $p \in [1, \infty)$, we define the Cheeger p-energy $\mathrm{Ch}_p^{\rho, \mathfrak{m}} : L^p(E, \mathfrak{m}) \to [0, \infty]$ by*
(8.7)

$$\mathrm{Ch}_p^{\rho, \mathfrak{m}}(f) := \inf \left\{ \liminf_{n \to \infty} \int_E g_n^p d\mathfrak{m} \; \middle| \; \begin{array}{l} g_n \text{ is an upper } \rho\text{-gradient for an } \mathfrak{m}\text{-version} \\ \text{of } f_n \text{ for each } n \in \mathbb{N} \text{ for some } \{f_n\}_{n \in \mathbb{N}} \subset \\ L^p(E, \mathfrak{m}) \text{ with } \lim_{n \to \infty} \int_E |f_n - f|^p d\mathfrak{m} = 0 \end{array} \right\}$$

and the Cheeger $(1, p)$-Sobolev space $H^{1,p}(E, \rho, \mathfrak{m})$ over $(E, \rho, \mathfrak{m})$ by

$$(8.8) \qquad H^{1,p}(E, \rho, \mathfrak{m}) := \{ f \in L^p(E, \mathfrak{m}) \mid \mathrm{Ch}_p^{\rho, \mathfrak{m}}(f) < \infty \}.$$

REMARK 8.8. (1) $\mathrm{Ch}_p^{\rho, \mathfrak{m}}$ was originally termed the "upper gradient p-energy" in [**17**]. Here we have followed the notation and terminology adopted in [**2, 3, 27**]. (2) Shanmugalingam [**81**, Definition 2.5] proposed another way of defining a $(1, p)$-Sobolev space under the same framework, independently of Cheeger's work [**17**, Section 2]. Her Sobolev space is denoted as $N^{1,p}(E, \rho, \mathfrak{m})$ and called the *Newtonian space of index $p \in [1, \infty)$*, and she proved in [**81**, Theorem 4.10] that for $p \in (1, \infty)$, $H^{1,p}(E, \rho, \mathfrak{m}) = N^{1,p}(E, \rho, \mathfrak{m})$ and they are equipped with exactly the same norm.

COROLLARY 8.9. *Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$. Then $\mathcal{E}(u, u) = \mathrm{Ch}_2^{\rho_{\mathcal{H}}, \mu}(u) = \mathrm{Ch}_2^{\rho_h, \mu_{\langle h \rangle}}(u)$ for any $u \in \mathcal{F}$ and $\mathcal{F} = H^{1,2}(K, \rho_{\mathcal{H}}, \mu) = H^{1,2}(K, \rho_h, \mu_{\langle h \rangle})$.*

PROOF. On the basis of Theorem 4.25 and Proposition 4.26, the first assertion follows from (8.5), [**48**, Theorem 2], [**17**, Theorem 6.1] and [**58**, Theorem 2.2-(i)], whereas [**58**, Theorem 2.2-(i)] and [**81**, Theorem 4.10] yield the latter. □

REMARK 8.10. Let $M \in \mathbb{R}_0^{2 \times 2}$ and let Φ_M, μ_M, ρ_M be as in Definition 7.3. Then *Theorems 8.3, 8.5 and Corollary 8.9 with Φ_M, μ_M, ρ_M in place of $\Phi, \mu, \rho_\mathcal{H}$ are still valid* with exactly the same proofs as those in [**58**, Section 4], where $\widetilde{\nabla} u$ is given by the version of Theorem 3.10-(1) for Φ_M, μ_M with Z and $\|T_{[\omega]_m}\|$ replaced by $Z_M := \|MZ\|^{-2} MZM^*$ ($Z_M := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ when $MZ = 0$) and $\|MT_{[\omega]_m}\|$, respectively.

8.2. Invalidity of Ricci curvature lower bound. In the last decade, Lott and Villani [**71**, **70**] and Sturm [**86**, **87**] formulated Ricci curvature lower bound for general metric measure spaces and showed that for a complete Riemannian manifold equipped with Riemannian distance and volume, their formulations are equivalent to the usual lower bound of its Ricci curvature tensor. Around the same period, Ohta [**76**] and Sturm [**87**] proposed another formulation of Ricci curvature lower bound which is in principle weaker but easier to handle. Their main idea was to make use of notions from optimal transport theory, and they also derived various analytic and geometric consequences of their formulations.

The purpose of this subsection is to show that $(K, \rho_\mathcal{H}, \mu)$ does *not* satisfy any of those conditions for Ricci curvature lower bound. We need to introduce several notions from optimal transport theory to state the Ricci bound conditions precisely.

Throughout this subsection, we fix a complete separable metric space (E, ρ) and a Borel measure $\mathfrak{m}$ on E such that $\mathfrak{m}(E) > 0$ and $E = \bigcup_{U \subset E \text{ open in } E, \, \mathfrak{m}(U) < \infty} U$.

DEFINITION 8.11 (Wasserstein space). Let $\mathcal{P}(E)$ denote the set of all Borel probability measures on E. Let $p \in [1, \infty)$ and define

$$(8.9) \qquad \mathcal{P}_p(E, \rho) := \big\{ \nu \in \mathcal{P}(E) \mid \int_E \rho(x, y)^p d\nu(y) < \infty \text{ for some (any) } x \in E \big\},$$

$$(8.10) \quad W_p^\rho(\nu_1, \nu_2)^p := \inf \big\{ \int_{E \times E} \rho^p d\varpi \mid \varpi \in \mathcal{P}(E \times E), \, \varpi \circ \mathrm{pr}_j^{-1} = \nu_j, \, j = 1, 2 \big\}$$

for $\nu_1, \nu_2 \in \mathcal{P}(E)$, where $\mathrm{pr}_j : E \times E \to E$ is given by $\mathrm{pr}_j(x_1, x_2) := x_j$. Then W_p^ρ is indeed a metric on $\mathcal{P}_p(E, \rho)$ by [**91**, Theorem 7.3-(i)] and called the *p-Wasserstein distance over* (E, ρ). $(\mathcal{P}_p(E, \rho), W_p^\rho)$ is called the *p-Wasserstein space over* (E, ρ), which is a complete separable metric space by [**92**, Theorem 6.18].

DEFINITION 8.12 (Relative entropy). For $\nu \in \mathcal{P}(E)$, we write $\nu \ll \mathfrak{m}$ if and only if ν is absolutely continuous with respect to $\mathfrak{m}$. We define the *relative entropy* $\mathrm{Ent}_\mathfrak{m} : \mathcal{P}_2(E, \rho) \to [-\infty, \infty]$ *with respect to* $\mathfrak{m}$ by

$$(8.11) \qquad \mathrm{Ent}_\mathfrak{m}(\nu) := \begin{cases} \int_E \frac{d\nu}{d\mathfrak{m}} \log \frac{d\nu}{d\mathfrak{m}} d\mathfrak{m} & \text{if } \nu \ll \mathfrak{m} \text{ and } \int_E \big(\frac{d\nu}{d\mathfrak{m}} \log \frac{d\nu}{d\mathfrak{m}} \big)^+ d\mathfrak{m} < \infty, \\ \infty & \text{otherwise} \end{cases}$$

for $\nu \in \mathcal{P}_2(E, \rho)$, and we set $\mathcal{P}_2^*(E, \rho, \mathfrak{m}) := \{v \in \mathcal{P}_2(E, \rho) \mid \mathrm{Ent}_\mathfrak{m}(\nu) < \infty\}$.

DEFINITION 8.13 (Curvature-dimension condition, [**86**, Definition 4.5-(i)]). Let $k \in \mathbb{R}$. We say that $(E, \rho, \mathfrak{m})$ satisfies the *curvature-dimension condition* $\mathsf{CD}(k, \infty)$ or $(E, \rho, \mathfrak{m})$ *has curvature* $\geq k$ if and only if for any $\nu_0, \nu_1 \in \mathcal{P}_2^*(E, \rho, \mathfrak{m})$ there exists a minimal W_2^ρ-geodesic $\alpha : [0, 1] \to \mathcal{P}_2(E, \rho)$ such that $\alpha(0) = \nu_0$, $\alpha(1) = \nu_1$ and

$$(8.12) \quad \mathrm{Ent}_\mathfrak{m}(\alpha(t)) \leq (1 - t) \mathrm{Ent}_\mathfrak{m}(\alpha(0)) + t \, \mathrm{Ent}_\mathfrak{m}(\alpha(1)) - \frac{k}{2} t(1 - t) W_2^\rho(\alpha(0), \alpha(1))^2$$

for any $t \in [0, 1]$. Note that $\alpha(t) \in \mathcal{P}_2^*(E, \rho, \mathfrak{m})$ for any $t \in [0, 1]$ for such α.

The curvature-dimension condition $\mathsf{CD}(k, \infty)$ is a generalization of the notion of Ricci curvature lower bound adapted to the setting of a general metric measure space. Indeed, the following equivalence holds for complete Riemannian manifolds.

THEOREM 8.14 ([**80**, Theorem 1.1], [**86**, Theorem 4.9]). *Let (M, g) be a complete Riemannian manifold with Riemannian geodesic distance ρ_g and Riemannian volume measure $\mathfrak{m}_g$. Let $k \in \mathbb{R}$. Then $(M, \rho_g, \mathfrak{m}_g)$ satisfies* $\mathsf{CD}(k, \infty)$ *if and only if* $\mathrm{Ric}_g \geq kg$, *i.e.* $\mathrm{Ric}_g(v, v) \geq kg(v, v)$ *for any $x \in M$ and any $v \in T_x M$, where $T_x M$ denotes the tangent space of M at x and Ric_g the Ricci curvature tensor of (M, g).*

DEFINITION 8.15. For $A, B \subset E$ and $t \in [0, 1]$, we define

$$(8.13) \qquad [A, B]_t^\rho := \left\{ z \in E \;\middle|\; \begin{array}{l} \rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1-t)\rho(x, y) \\ \text{for some } (x, y) \in A \times B \end{array} \right\}.$$

Note that $[A, B]_t^\rho \in \bigcap_{\nu \in \mathcal{P}(E)} \overline{\mathcal{B}(E)}^\nu$ if $A, B \in \mathcal{B}(E)$ by [**20**, III.13, III.17 and III.33], where $\overline{\mathcal{B}(E)}^\nu$ denotes the completion of $\mathcal{B}(E)$ by ν. We also set $[x, y]_t^\rho := [\{x\}, \{y\}]_t^\rho$ for $x, y \in E$ and $t \in [0, 1]$, so that $[x, y]_t^\rho$ is a closed subset of E.

Let $A, B \in \mathcal{B}(E)$ be such that $\mathfrak{m}(A), \mathfrak{m}(B) \in (0, \infty)$ and set $\nu_0 := \mathfrak{m}(A)^{-1} \mathbf{1}_A \mathfrak{m}$ and $\nu_1 := \mathfrak{m}(B)^{-1} \mathbf{1}_B \mathfrak{m}$. By [**86**, Lemma 2.11-(ii)], if $\nu_0, \nu_1 \in \mathcal{P}_2(E, \rho)$ and $\alpha : [0, 1] \to \mathcal{P}_2(E, \rho)$ is a minimal W_2^ρ-geodesic with $\alpha(0) = \nu_0$ and $\alpha(1) = \nu_1$, then $\alpha(t)([A, B]_t^\rho) = 1$ for any $t \in [0, 1]$. Thus an application of Jensen's inequality to the left-hand side of (8.12), together with the lower semicontinuity of W_2^ρ under weak convergence ([**92**, Remark 6.12]), yields the following generalized Brunn-Minkowski inequality, which was new even for complete Riemannian manifolds.

PROPOSITION 8.16 (Brunn-Minkowski inequality, cf. [**87**, Proposition 2.1]). *Let $k \in \mathbb{R}$ and suppose that $(E, \rho, \mathfrak{m})$ satisfies* $\mathsf{CD}(k, \infty)$. *Then for any $A, B \in \mathcal{B}(E)$ with $\mathfrak{m}(A), \mathfrak{m}(B) \in (0, \infty)$ and any $t \in [0, 1]$,*
(8.14)

$$\log \mathfrak{m}([A, B]_t^\rho) \geq (1 - t) \log \mathfrak{m}(A) + t \log \mathfrak{m}(B) + \frac{k}{2} t(1 - t) W_2^\rho \left(\frac{\mathbf{1}_A}{\mathfrak{m}(A)} \mathfrak{m}, \frac{\mathbf{1}_B}{\mathfrak{m}(B)} \mathfrak{m} \right)^2.$$

REMARK 8.17. (1) Some time after [**86**], Sturm introduced in [**87**, Definition 1.3] a variant of the curvature-dimension condition for $(E, \rho, \mathfrak{m})$ denoted as $\mathsf{CD}(k, N)$, where $k \in \mathbb{R}$ and $N \in [1, \infty)$. Roughly speaking, $(E, \rho, \mathfrak{m})$ satisfying $\mathsf{CD}(k, N)$ has Ricci curvature bounded from below by k and dimension at most N. Indeed, according to [**87**, Theorem 1.7], for $(M, g), \rho_g, \mathfrak{m}_g$ as in Theorem 8.14, $(M, \rho_g, \mathfrak{m}_g)$ satisfies $\mathsf{CD}(k, N)$ if and only if $\mathrm{Ric}_g \geq kg$ and $\dim M \leq N$. As naturally expected from its meaning, if $(E, \rho, \mathfrak{m})$ satisfies $\mathsf{CD}(k, N)$ then it also satisfies $\mathsf{CD}(k', N')$ for any $k' \in (-\infty, k]$ and any $N' \in [N, \infty)$ by [**87**, Proposition 1.6-(i)]. Moreover by [**87**, Proposition 1.6-(ii)], *if $k \in \mathbb{R}$, $(E, \rho, \mathfrak{m})$ satisfies $\mathsf{CD}(k, N)$ for some $N \in [1, \infty)$ and $\mathfrak{m}(E) < \infty$, then $(E, \rho, \mathfrak{m})$ also satisfies $\mathsf{CD}(k, \infty)$.*
(2) Independently of Sturm's work [**86, 87**], Lott and Villani [**70**, Definition 4.7] (see also [**71**, Definitions 5.12 and 5.13]) also defined the notion of $(E, \rho, \mathfrak{m})$ having *N-Ricci curvature bounded below by k* for $k \in \mathbb{R}$ and $N \in [1, \infty]$, which is essentially the same as the curvature-dimension condition $\mathsf{CD}(k, N)$. As noted in [**70**, Remark 4.11], this notion gets weaker as K decreases and N increases. (In the main texts of [**71, 70**], the compactness of the metric space is assumed for the sake of simplicity, but many results there can be extended to the non-compact case; see [**71**, Appendix E] and [**92**, Chapters 29 and 30].) As remarked in [**70**, discussions before and after Lemma 4.14], *for $k \in \mathbb{R}$, $N \in [1, \infty]$ and $(E, \rho, \mathfrak{m})$ with E compact and $\mathfrak{m}(E) < \infty$, the condition of N-Ricci curvature bounded below by k implies $\mathsf{CD}(k, N)$.*
(3) An important feature of the notions of $\mathsf{CD}(k, N)$ and of N-Ricci curvature bounded below by k is that *they are preserved under measured Gromov-Hausdorff*

limits; see [**86**, Subsection 4.5], [**87**, Section 3], [**71**, Subsections 4.2, 5.3 and E.5] and [**92**, Chapter 29] for details. They also give rise to a lot of fundamental analytic and geometric consequences; see e.g. [**87**, Section 2], [**71**, Section 6], [**70**, Section 5] and [**92**, Chapter 30].

There is yet another formulation of Ricci curvature lower bound for $(E, \rho, \mathfrak{m})$ due to Ohta [**76**] and Sturm [**87**], which is given as follows.

DEFINITION 8.18 (Measure contraction property, [**87**, Section 5], cf. [**76**]). Let $k \in \mathbb{R}$ and $N \in [1, \infty)$. $(E, \rho, \mathfrak{m})$ is said to satisfy the *measure contraction property* $\mathsf{MCP}(k, N)$ if and only if for each $t \in (0, 1)$, for $\mathfrak{m}$-a.e. $x \in E$ there exists a Markovian kernel $\mathcal{P}_{t,x}$ on $(E, \mathcal{B}(E))$ such that $\mathcal{P}_{t,x}(y, [x, y]_t^\rho) = 1$ for $\mathfrak{m}$-a.e. $y \in E$ and

$$(8.15) \qquad \int_E \varsigma_{k,N}^{(t)}(\rho(x,y)) \mathcal{P}_{t,x}(y, \cdot) d\mathfrak{m}(y) \leq \mathfrak{m},$$

where $\varsigma_{k,N}^{(t)} : [0, \infty) \to (0, \infty]$ is defined by

$$(8.16) \qquad \varsigma_{k,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } k\theta^2 \geq (N-1)\pi^2, \\ t\left(\dfrac{\sin\left(t\theta\sqrt{k/(N-1)}\right)}{\sin\left(\theta\sqrt{k/(N-1)}\right)}\right)^{N-1} & \text{if } 0 < k\theta^2 < (N-1)\pi^2, \\ t^N & \text{if } k\theta^2 = 0, \\ t\left(\dfrac{\sinh\left(t\theta\sqrt{-k/(N-1)}\right)}{\sinh\left(\theta\sqrt{-k/(N-1)}\right)}\right)^{N-1} & \text{if } k\theta^2 < 0 \end{cases}$$

for $N > 1$ and $\varsigma_{k,1}^{(t)}(\theta) := \infty \mathbf{1}_{(0,\infty)}(k\theta^2) + t\mathbf{1}_{(-\infty,0]}(k\theta^2)$.

REMARK 8.19. (1) Let $t \in (0, 1)$, $k, k' \in \mathbb{R}$, $k \geq k'$ and $N, N' \in [1, \infty)$, $N \leq N'$. Then it is easy to see that $\varsigma_{k,N}^{(t)} \geq \varsigma_{k',N'}^{(t)}$, so that $\mathsf{MCP}(k, N)$ *implies* $\mathsf{MCP}(k', N')$. (2) *Sturm's original version* [**87**, *Definition* 5.1] *of* $\mathsf{MCP}(k, N)$ *is slightly stronger than as in Definition* 8.18, requiring additionally the Borel measurability of $\mathcal{P}_{t,x}(y, \cdot)$ in $(x, y) \in E \times E$ and $\int_E \varsigma_{k,N}^{(1-t)}(\rho(x,y)) \mathcal{P}_{t,x}(y, \cdot) d\mathfrak{m}(x) \leq \mathfrak{m}$ for $\mathfrak{m}$-a.e. $y \in E$. (3) $\mathsf{MCP}(k, N)$ *as in Definition* 8.18 *is implied by Ohta's definition* [**76**, *Definition* 2.1] *of* $\mathsf{MCP}(k, N)$ under his standing assumption in [**76**] that $\mathfrak{m}(B_r(x, \rho)) \in (0, \infty)$ for any $(r, x) \in (0, \infty) \times E$ and (E, ρ) is a *length space*, i.e. for any $x, y \in E$, $\rho(x, y) = \inf\{\ell_\rho(\gamma) \mid \gamma[0, 1] \to E, \gamma \text{ is continuous}, \gamma(0) = x, \gamma(1) = y\}$. Indeed, this observation is easily verified by using [**21**, Theorems 10.2.1 and 10.2.2] (and $\mathfrak{m}\left(E \setminus B_{\pi\sqrt{(N-1)/k}}(x, \rho)\right) = 0$ for any $x \in E$ when $k(N-1) > 0$, which follows from [**76**, Theorem 4.3 and Lemma 4.4-(i)]).

In principle, $\mathsf{MCP}(k, N)$ is weaker than $\mathsf{CD}(k, N)$ but still means Ricci curvature lower bound by k and dimension upper bound by N. Indeed, by [**76**, Corollary 2.7] or [**87**, Remark 5.3], if $(E, \rho, \mathfrak{m})$ satisfies $\mathsf{MCP}(k, N)$ then $\dim_{\mathrm{H}}(\mathrm{supp}_E[m], \rho) \leq N$, where $\mathrm{supp}_E[m] := \{x \in E \mid \mathfrak{m}(U) > 0 \text{ for any open subset } U \text{ of } E \text{ with } x \in U\}$. Furthermore $\mathsf{CD}(k, N)$ implies $\mathsf{MCP}(k, N)$ under certain geometric conditions on $(E, \rho, \mathfrak{m})$, and for a complete Riemannian manifold (M, g), $\mathsf{MCP}(k, N)$ corresponds to $\mathrm{Ric}_g \geq kg$ and $\dim M \leq N$ in a weaker sense than $\mathsf{CD}(k, N)$ does, as follows.

THEOREM 8.20 ([**87**, Lemma 4.1 and Theorem 5.4]). *Let* $k \in \mathbb{R}$, $N \in [1, \infty)$ *and suppose that* $(E, \rho, \mathfrak{m})$ *satisfies* $\mathsf{CD}(k, N)$.

(1) If (E, ρ) is non-branching, i.e. $x_1 = x_2$ for any $z, x_0, x_1, x_2 \in E$ with $2\rho(z, x_i) = \rho(x_0, x_1) = \rho(x_0, x_2)$, $i \in \{0, 1, 2\}$, then there exists a Borel measurable map $\gamma :$ $E \times E \to \mathcal{G}(E, \rho)$ such that for $\mathfrak{m} \times \mathfrak{m}$-a.e. $(x, y) \in E \times E$, $\gamma(x, y)$ is the unique minimal ρ-geodesic with $\gamma(x, y)(0) = x$ and $\gamma(x, y)(1) = y$.
(2) If a map $\gamma : E \times E \to \mathcal{G}(E, \rho)$ as in (1) exists, then $(E, \rho, \mathfrak{m})$ satisfies $\mathsf{MCP}(k, N)$.

THEOREM 8.21 ([**76**, Theorem 3.2 and Corollary 3.3], [**87**, Corollary 5.5]). Let (M, g) be a complete Riemannian manifold with Riemannian geodesic distance ρ_g and Riemannian volume measure $\mathfrak{m}_g$. (Note that (M, ρ_g) is non-branching.) Let $k \in \mathbb{R}$. Then $(M, \rho_g, \mathfrak{m}_g)$ satisfies $\mathsf{MCP}(k, \dim M)$ if and only if $\mathrm{Ric}_g \geq kg$.

REMARK 8.22. (1) In principle, $\mathsf{MCP}(k, N)$ should hold for a wider range of metric measure spaces than $\mathsf{CD}(k, N)$. Indeed, for $(M, g), \rho_g, \mathfrak{m}_g$ as in Theorem 8.21, $\mathsf{MCP}(k, N)$ with $\dim M < N$ does *not* imply $\mathsf{CD}(k, N)$ nor $\mathrm{Ric}_g \geq kg$ in general, as noted in [**87**, Remark 5.6]. Furthermore for the $(2n + 1)$-dimensional Heisenberg group $H_n(\mathbb{R}) := \mathbb{R}^{2n+1}$ equipped with the Carnot-Carathéodory metric and the Lebesgue measure, where $n \in \mathbb{N}$, $\mathsf{MCP}(k, N)$ holds if and only $(k, N) \in (-\infty, 0] \times [2n+3, \infty)$ by [**42**, Theorem 2.3 and Remark 3.3], whereas $\mathsf{CD}(k, N)$ does *not* hold for any $(k, N) \in \mathbb{R} \times [1, \infty]$ by [**42**, Theorem 3.2 and Remark 3.3].

Another advantage of $\mathsf{MCP}(k, N)$ is that it is easier to handle than $\mathsf{CD}(k, N)$. For example, $\mathsf{MCP}(k, N)$ is preserved by the operations of taking products and cones of metric measure spaces by [**77**, Proposition 3.3 and Theorem 4.2], but similar properties for $\mathsf{CD}(k, N)$ are not known except for products of non-branching compact $\mathsf{CD}(k, \infty)$-spaces due to [**86**, Proposition 4.16].
(2) Similarly to $\mathsf{CD}(k, N)$, $\mathsf{MCP}(k, N)$ *is preserved by measured Gromov-Hausdorff limits*; see [**76**, Section 6] and [**87**, Theorem 5.9 and Corollary 5.10]. $\mathsf{MCP}(k, N)$ also admits almost the same analytic and geometric consequences as $\mathsf{CD}(k, N)$; see [**76**, Sections 2, 4 and 5], [**87**, Sections 5 and 6] and [**79**, Section 3] for details.

Now we turn to the case of the Sierpiński gasket K equipped with the harmonic geodesic metric $\rho_{\mathcal{H}}$ and the Kusuoka measure μ. We have the following result.

THEOREM 8.23. Let $k \in \mathbb{R}$ and $N \in [1, \infty)$. Then $(K, \rho_{\mathcal{H}}, \mu)$ does **not** satisfy any one of $\mathsf{CD}(k, \infty), \mathsf{CD}(k, N), \mathsf{MCP}(k, N)$.

The rest of this section is devoted to the proof of Theorem 8.23. We start with the following easy lemma. Recall that we set $\overline{xy} := \{(1 - t)x + ty \mid t \in [0, 1]\}$ for $x, y \in \mathbb{R}^2$ (Definition 4.11-(1)).

LEMMA 8.24. Set $\triangle_* := \bigcup_{w \in W_*} F_w(\overline{q_1 q_2} \cup \overline{q_2 q_3} \cup \overline{q_3 q_1})$. Then $\mu(\triangle_*) = 0$.

PROOF. Let $h \in \mathcal{H}_0$, $w \in W_*$, $i \in S$ and $S \setminus \{i\} = \{j, k\}$. By [**41**, Lemma 4.2],

$$(8.17) \qquad \mu_{\langle h \rangle}(K_{vj} \cup K_{vk}) = \mu_{\langle h \rangle}(K_v) - \mu_{\langle h \rangle}(K_{vi}) \leq \frac{14}{15} \mu_{\langle h \rangle}(K_v), \qquad v \in W_*$$

(see [**38**, Proposition 3.8] and [**36**, Proposition 5.27] for similar estimates in more general settings), and then an inductive use of (8.17) easily shows that

$$\mu_{\langle h \rangle}(F_w(\overline{q_j q_k})) = \lim_{m \to \infty} \mu_{\langle h \rangle}\Big(\bigcup_{v \in \{j, k\}^m} K_{wv} \Big) \leq \lim_{m \to \infty} \Big(\frac{14}{15} \Big)^m \mu_{\langle h \rangle}(K_w) = 0.$$

Thus $\mu_{\langle h \rangle}(\triangle_*) = 0$ and hence $\mu(\triangle_*) = \mu_{\langle h_1 \rangle}(\triangle_*) + \mu_{\langle h_2 \rangle}(\triangle_*) = 0$. $\qquad \square$

The key to the proof of Theorem 8.23 is the following proposition, which is an easy consequence of Theorem 4.19.

PROPOSITION 8.25. *Let* $A, B \subset K$ *and* $t \in (0, 1)$. *Then* $[A, B]_t^{\rho_{\mathcal{H}}} \subset \triangle_* \cup (A \cap B)$.

PROOF. Let $z \in [A, B]_t^{\rho_{\mathcal{H}}}$. Take $(x, y) \in A \times B$ such that $\rho_{\mathcal{H}}(x, z) = t\rho_{\mathcal{H}}(x, y)$ and $\rho_{\mathcal{H}}(z, y) = (1 - t)\rho_{\mathcal{H}}(x, y)$. If $x = y$, then $\rho_{\mathcal{H}}(x, z) = 0$ and $z = x = y \in A \cap B$. Assume $x \neq y$, so that $z \notin \{x, y\}$. By Theorem 4.15-(2), there exist harmonic geodesics $\gamma_{xz} : [0, t] \to K$ and $\gamma_{zy} : [t, 1] \to K$ such that $\gamma_{xz}(0) = x$, $\gamma_{xz}(t) = z = \gamma_{zy}(t)$, $\gamma_{zy}(1) = y$, $\ell_{\mathcal{H}}(\gamma_{xz}) = \rho_{\mathcal{H}}(x, z) = t\rho_{\mathcal{H}}(x, y)$ and $\ell_{\mathcal{H}}(\gamma_{zy}) = \rho_{\mathcal{H}}(z, y) = (1 - t)\rho_{\mathcal{H}}(x, y)$. Define $\gamma : [0, 1] \to K$ by $\gamma|_{[0,t]} := \gamma_{xz}$ and $\gamma|_{[t,1]} := \gamma_{zy}$. Then γ is continuous, $\gamma(0) = x$, $\gamma(t) = z$, $\gamma(1) = y$ and $\ell_{\mathcal{H}}(\gamma) = \ell_{\mathcal{H}}(\gamma_{xz}) + \ell_{\mathcal{H}}(\gamma_{zy}) = \rho_{\mathcal{H}}(x, y)$. Now Theorem 4.19 implies that $\gamma = \widehat{\gamma} \circ \varphi_\gamma$ for some harmonic geodesic $\widehat{\gamma} : [0, 1] \to K$ and a non-decreasing continuous surjection $\varphi_\gamma : [0, 1] \to [0, 1]$, and then $z = \widehat{\gamma}(\varphi_\gamma(t)) \in \triangle_*$ by $z \notin \{x, y\} = \{\widehat{\gamma}(0), \widehat{\gamma}(1)\}$ and the definition of $\widehat{\gamma}$ being a harmonic geodesic (Definition 4.13-(2)). Thus $[A, B]_t^{\rho_{\mathcal{H}}} \subset \triangle_* \cup (A \cap B)$. $\qquad\square$

PROOF OF THEOREM 8.23. Let $A, B \in \mathcal{B}(K)$ be such that $\mu(A)\mu(B) > 0$ and $\mu(A \cap B) = 0$. Let $t \in (0, 1)$. Then $\mu([A, B]_t^{\rho_{\mathcal{H}}}) = 0$ by Proposition 8.25 and Lemma 8.24, so that (8.14) does not hold and hence neither does $\mathsf{CD}(k, \infty)$ by Proposition 8.16. [**87**, Proposition 1.6-(ii)] (see Remark 8.17-(1) above) further implies that $(K, \rho_{\mathcal{H}}, \mu)$ does not satisfy $\mathsf{CD}(k, N)$, either.

For $\mathsf{MCP}(k, N)$, let $t \in (0, 1)$, $x \in K$ and suppose that there exists a Markovian kernel $\mathcal{P}_{t,x}$ on $(K, \mathcal{B}(K))$ satisfying (8.15) and $\mathcal{P}_{t,x}(y, [x, y]_t^{\rho_{\mathcal{H}}}) = 1$, μ-a.e. $y \in K$. Then since $[x, y]_t^{\rho_{\mathcal{H}}} \subset \triangle_*$ for any $K \setminus \{x\}$ by Proposition 8.25, $\mathcal{P}_{t,x}(y, \triangle_*) = 1$ for μ-a.e. $y \in K$ and hence (8.15) yields $\mu(\triangle_*) \geq \int_K \varsigma_{k,N}^{(t)}(\rho_{\mathcal{H}}(x, y))d\mu(y) > 0$, contradicting Lemma 8.24. Therefore $(K, \rho_{\mathcal{H}}, \mu)$ does not satisfy $\mathsf{MCP}(k, N)$. $\qquad\square$

REMARK 8.26. Unfortunately, the above proof of Theorem 8.23 does not work for $(K, \rho_h, \mu_{\langle h \rangle})$, $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$, since the assertion of Theorem 4.19 is not valid for ℓ_h and ρ_h by Remark 4.21. Theorem 8.23 is quite likely to be true also for $(K, \rho_h, \mu_{\langle h \rangle})$, but the author has no idea at this moment how to manage this case.

On the other hand, for any $M \in \mathbb{R}_0^{2 \times 2}$ *with* $\det M \neq 0$ and with μ_M, ρ_M, ℓ_M as in Definition 7.3, Theorem 4.19 with ρ_M, ℓ_M in place of $\rho_{\mathcal{H}}, \ell_{\mathcal{H}}$ is valid and hence *Theorem* 8.23 *and Proposition* 8.25 *hold also for* (K, ρ_M, μ_M).

9. Possible generalizations to other self-similar fractals

We conclude this paper with some remarks on possible generalizations to other self-similar fractals. In this paper, we have restricted our attention to the particular case of the 2-*dimensional standard* Sierpiński gasket. In fact, it is almost the *only* self-similar fractal that possesses all the required properties for the framework and the results of this paper. Extension to other self-similar fractals would be *only partially* possible and involve essential difficulties which would vary depending on each fractal. Below we illustrate the actual (complicated) situation by mentioning each of the concrete examples in Figure 3 separately.

9.1. Sierpiński gaskets. Let $k \in \mathbb{N}$, $k \geq 3$. The k-*dimensional standard Sierpiński gasket* is the direct k-dimensional analogue of the Sierpiński gasket. The results in [**63, 52, 35**] except those in [**52**, Lemma 4.2 and Appendix] are stated and proved including this case and therefore Section 3 is immediately extended to this case (with obvious changes). As shown in [**54**, Section 3] for the Kusuoka measure μ, Lemma 4.22 and Proposition 4.23 can be verified similarly with suitable changes of the constants. On the other hand, the proof of Lemma 4.17 for the

Sierpiński gasket heavily relied on the 2-dimensionality of the space, as is observed from its dependence on Theorem 4.18, and extension to the k-dimensional case is not straightforward at all. Since the proof of Theorem 4.15 was based on Lemma 4.17, it is still unclear how we can verify Theorem 4.15 for the k-dimensional case.

Recall that Lemma 4.17 has two assertions (4.9) and (4.10). It seems possible to prove (4.9) by taking the projection of the harmonic k-dimensional Sierpiński gasket onto a suitable 2-dimensional subspace and then applying Theorem 4.18. To the contrary, *the lower inequality of* (4.10) *is **not** valid for the k-dimensional case.* Indeed, if h is a V_0-harmonic function taking 1 at $x \in V_0$, -1 at $y \in V_0$ and 0 on $V_0 \setminus \{x, y\}$, then $h = 0$ on the hyperplane containing $V_0 \setminus \{x, y\}$, from which we can easily show that the lower inequality of (4.10) does not hold. This degeneracy causes a lot of troubles in the proofs of various geometric inequalities and therefore extension to the k-dimensional case should require significant effort, although most of the results in this paper are quite likely to hold also for the k-dimensional case.

Another possible extension is the case of the 2-*dimensional level-l Sierpiński gasket* with $l \geq 3$ (see Figure 3 for a picture of the level-3 case; the Sierpiński gasket is regarded as the level-2 case). For simplicity we consider here the level-3 case only. Then the measurable Riemannian structure can be introduced in exactly the same manner, and by virtue of the 2-dimensionality and Theorem 4.18 we can prove Lemma 4.17, thereby Theorem 4.15, and also Theorems 4.19 and 8.23. Interestingly, however, it is also possible to show that *Lemma 4.22-(2) does **not** hold*, on the basis of the denseness of vertices from which the space spreads away in *three* directions. Consequently Proposition 4.23 is not valid by [**55**, Theorems 1.3.5 and 1.4.3] and hence neither is the volume doubling property (Theorem 4.25). Therefore by [**55**, Theorem 3.2.3], *even the on-diagonal upper bound $p_\mu(t, x, x) \leq c_U / \mu\big(B_{\sqrt{t}}(x, \rho_{\mathcal{H}})\big)$ of the heat kernel p_μ is false in this case.*

It is an interesting challenging problem to establish a sharp two-sided estimate for the heat kernel p_μ in this case, but the actual behavior of p_μ is expected to be very wild and this problem should be difficult. On the other hand, it is still likely that we can extend Theorem 5.2 and the results in Sections 6 and 7 to this case, but the actual proofs will be much more involved. In the case of the (level-2) Sierpiński gasket, Lemma 4.22-(2) is used especially in the proof of the continuity of the function $\mathfrak{T} : \mathbb{R}_0^{2 \times 2} \to (0, \infty)$ defined in Proposition 7.7-(2), and the extension to the level-3 case, where Lemma 4.22-(2) is invalid, will require a non-trivial improvement in the proof.

9.2. Other nested fractals and Sierpiński carpets. For most other nested fractals such as the N-polygasket with $N \geq 5$, $N/4 \notin \mathbb{N}$ (see [**45**, Subsection 5.2] for its precise definition), the snowflake and the Vicsek set shown in Figure 3, the situation is much worse than in the case of Sierpiński gaskets. The problem is that *the harmonic structure is degenerate*, i.e. non-constant V_0-harmonic functions can be constant on some K_w, or in other words, the family $\{A_i\}_{i \in S} \subset \mathbb{R}^{(\#V_0) \times (\#V_0)}$ of matrices defined by (2.11) contains non-invertible ones. In such cases it is highly non-trivial whether the Kusuoka measure $\mu := \sum_{i=1}^{\#V_0 - 1} \mu_{\langle h_i \rangle}$, where $\{h_i\}_{i=1}^{\#V_0 - 1} \subset \mathcal{H}_0$ is arbitrarily chosen to be orthonormal in $(\mathcal{H}_0/\mathbb{R}\mathbf{1}, \mathcal{E})$, is *energy-dominant* in the sense that $\mu_{\langle u \rangle}$ is absolutely continuous with respect to μ for any $u \in \mathcal{F}$. The method of introducing a measurable Riemannian structure on the basis of μ makes sense only if μ is energy-dominant, which may or may not be the case depending on each self-similar fractal.
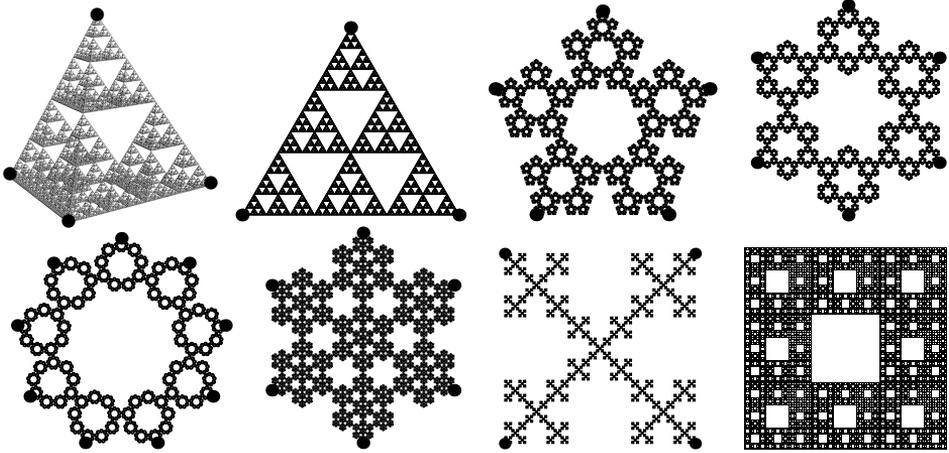
FIGURE 3. Typical self-similar fractals. From the *upper left*, three-dimensional standard (level-2) Sierpiński gasket, two-dimensional level-3 Sierpiński gasket, pentagasket (5-polygasket), hexagasket (6-polygasket), nonagasket (9-polygasket), snowflake, the Vicsek set and Sierpiński carpet. In each fractal except the last one, the set V_0 of its boundary points is marked by solid circles.

For example, in the case of the Vicsek set, *any* V_0-harmonic function is constant on each connected component of the complement of the two diagonals, so that the Kusuoka measure μ is supported only on the union of the diagonals, which is much smaller than the whole Vicsek set. For the N-polygasket with $N \neq 6, 9$ and the snowflake, it is still *not* known whether the Kusuoka measure μ is energy-dominant.

On the other hand, in the case of the hexagasket (6-polygasket) and the nonagasket (9-polygasket), by virtue of their dihedral symmetry we can prove that *the Kusuoka measure μ is energy-dominant*, which was essentially stated and proved in [**88**, Section 7]. Therefore μ gives rise to a measurable Riemannian structure. We can extend Lemma 4.17 and thereby Theorem 4.15 to this case with a suitable notion of harmonic geodesics, whereas the degeneracy of the harmonic structure easily implies that Lemma 4.22-(2) and the lower inequalities in Lemma 4.22-(1) are not true. Consequently *Proposition* 4.23 *and Theorem* 4.25 *do* **not** *hold* and hence **neither** *does the on-diagonal upper bound* $p_\mu(t, x, x) \leq c_\mathrm{U}/\mu\big(B_{\sqrt{t}}(x, \rho_\mathcal{H})\big)$ *of the heat kernel* p_μ by [**55**, Theorem 3.2.3]. (To be precise, [**55**, Theorem 3.2.3] is not directly applicable to this case due to the lack of the lower inequalities in Lemma 4.22-(1), but we can still verify [**55**, Lemmas 3.5.4 and 3.5.5] by using certain specific properties of the hexagasket and the nonagasket.) Similarly to the case of the 2-dimensional level-3 Sierpiński gasket, it is likely that we can extend Theorem 5.2 and the results in Sections 6 and 7. In fact, since the linear map $F_w^* : \mathcal{H}_0/\mathbb{R}\mathbf{1} \to \mathcal{H}_0/\mathbb{R}\mathbf{1}$, $F_w^* h := h \circ F_w$, has *rank one with the same image* (same up to symmetry of the space) for *"most"* $w \in W_*$, just the usual renewal theorem [**23**, Section XI.1, Renewal theorem (Alternative form)] should suffice for the proof of Theorem 7.2 and therefore Proposition 7.7-(2) should not be required.

Finally, for the Sierpiński carpet, and its generalizations called generalized Sierpiński carpets, nothing is known about non-degeneracy of V_0-harmonic functions and possibility of introducing a measurable Riemannian structure by using the energy measures of V_0-harmonic functions. Note that the set V_0 of boundary points is an *infinite* set in this case; for example, $V_0 = ([0,1] \times \{0,1\}) \cup (\{0,1\} \times [0,1])$ for the Sierpiński carpet. On one hand, this property gives us plenty of choices of V_0-harmonic functions and it is very likely that some choice of V_0-harmonic functions should work for the purpose of introducing a measurable Riemannian structure. On the other hand, the infiniteness of V_0 makes any kind of explicit calculations for the canonical Dirichlet form $(\mathcal{E}, \mathcal{F})$ impossible, so that non-degeneracy of V_0-harmonic functions is very difficult to verify despite plentifulness of V_0-harmonic functions.

In any of the above cases, we could use instead of V_0-harmonic functions a general family $\boldsymbol{g} = \{g_n\}_{n=1}^{N} \subset \mathcal{F}$ of functions such that $\mu_{\langle \boldsymbol{g} \rangle} := \sum_{n=1}^{N} \mu_{\langle g_n \rangle}$ is energy dominant, as is done in [**35, 37**], but then it would become much more difficult to establish fundamental geometric properties like those in Section 4.

Appendix A. Case of the standard Laplacian on the Sierpiński gasket

We follow the notation introduced in Section 2 throughout this appendix. Here we briefly review some important results for the so-called *standard Laplacian* Δ_0 *on* K, which is the non-positive self-adjoint operator of the Dirichlet space $(K, \mu_0, \mathcal{E}, \mathcal{F})$ with μ_0 the *self-similar measure on* K *with weight* $(1/3, 1/3, 1/3)$. Namely, μ_0 is the unique Borel measure on K such that $\mu_0(K_w) = 3^{-|w|}$ for any $w \in W_*$. μ_0 is in fact a constant multiple of the d_{f}-dimensional Hausdorff measure on K with respect to the Euclidean metric, where $d_{\mathrm{f}} := \log_2 3$ is the Hausdorff dimension of K with respect to the Euclidean metric; for details see e.g. [**53**, Section 1.5].

The *Brownian motion on the Sierpiński gasket*, which is the diffusion process corresponding to $(K, \mu_0, \mathcal{E}, \mathcal{F})$, was first constructed by Goldstein [**28**] and Kusuoka [**62**] and then intensively studied in a seminal work [**11**] by Barlow and Perkins. The most important result in [**11**] is the following *sub-Gaussian bound* for the transition density of the Brownian motion on K with respect to μ_0, which is nothing but the heat kernel p_{μ_0} associated with $(K, \mu_0, \mathcal{E}, \mathcal{F})$ in our notation. Let

(A.1) $\quad \rho_0(x,y) := \inf\{\ell_{\mathbb{R}^2}(\gamma) \mid \gamma : [0,1] \to K, \ \gamma \text{ is continuous}, \ \gamma(0) = x, \ \gamma(1) = y\}$

for $x, y \in K$, so that ρ_0 is a metric on K comparable to the Euclidean metric.

THEOREM A.1 ([**11**, Theorem 1.5]). *Let* $d_{\mathrm{w}} := \log_2 5$ *and set* $d_{\mathrm{s}} := d_{\mathrm{f}}/d_{\mathrm{w}}$. *Then there exist* $c_{\mathrm{A.1}}, c_{\mathrm{A.2}} \in (0, \infty)$ *such that for any* $(t, x, y) \in (0,1] \times K \times K$,
(A.2)
$$\frac{c_{\mathrm{A.1}}}{t^{d_{\mathrm{s}}/2}} \exp\left(-\left(\frac{\rho_0(x,y)^{d_{\mathrm{w}}}}{c_{\mathrm{A.1}}t}\right)^{\frac{1}{d_{\mathrm{w}}-1}}\right) \leq p_{\mu_0}(t,x,y) \leq \frac{c_{\mathrm{A.2}}}{t^{d_{\mathrm{s}}/2}} \exp\left(-\left(\frac{\rho_0(x,y)^{d_{\mathrm{w}}}}{c_{\mathrm{A.2}}t}\right)^{\frac{1}{d_{\mathrm{w}}-1}}\right).$$

Note that $d_{\mathrm{w}} > 2$, which is why an estimates of the form (A.2) is called *sub-Gaussian*. Roughly speaking, (A.2) says that heat on K diffuses up to the distance comparable to $t^{1/d_{\mathrm{w}}}$ at time t on average, which is not the case (at least for small t) for the heat kernels on Riemannian manifolds. The following theorem and proposition are part of the reasons for such a non-classical behavior of p_{μ_0}.

THEOREM A.2 ([**63**, §6, Example 1]). $\mu_{\langle u \rangle}$ *is singular to* μ_0 *for any* $u \in \mathcal{F}$.

PROPOSITION A.3. $\rho_0(x, \cdot) \notin \mathcal{F}$ *for any* $x \in K$.

Theorems A.1 and A.2 have been extended to a wide range of self-similar fractals including *(affine) nested fractals*, a class of post-critically finite self-similar fractals with certain good symmetry, and *generalized Sierpiński carpets*, a natural generalization of the Sierpiński carpet. See e.g. [**60, 24, 7, 8, 9, 55, 56**] for extensions of Theorem A.1 and [**12, 33, 39**] for those of Theorem A.2. (See Figure 3 above for pictures of several typical nested fractals and the Sierpiński carpet.) On the other hand, the author is not sure whether a proof of Proposition A.3 is given in any reference and to what extent it can be generalized, although it is essentially known to experts at least in simple cases. For the reader's convenience, we give here a proof of Proposition A.3 in the present setting of the Sierpiński gasket.

PROOF OF PROPOSITION A.3. We first prove the assertion for $x \in V_0$. Recall that $V_0 = \{q_1, q_2, q_3\}$. Without loss of generality we may assume that $q_1 = (0,0)$, $q_2 = (1, -1/\sqrt{3})$ and $q_3 = (1, 1/\sqrt{3})$, and by the rotational symmetry of K and $(\mathcal{E}, \mathcal{F})$ it suffices to show for $x = q_1$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(y,z) := (2/\sqrt{3})y$. Then an induction in m easily shows that $\rho_0(q_1, y) = f(y)$ for any $m \in \mathbb{N} \cup \{0\}$ and any $y \in V_m$ and hence $\rho_0(q_1, \cdot) = f|_K$ by the denseness of V_* in K. Now suppose $f|_K \in \mathcal{F}$. Then $f|_K \circ F_i = (1/2)f|_K + (\mathbf{1}_{\{2,3\}}(i)/\sqrt{3})\mathbf{1} \in \mathcal{F}$ for any $i \in S$ and hence $\mathcal{E}(f|_K, f|_K) = (5/3)\sum_{i \in S} \mathcal{E}(f|_K/2, f|_K/2) = (5/4)\mathcal{E}(f|_K, f|_K)$ by (2.4), so that $\mathcal{E}(f|_K, f|_K) = 0$ which contradicts $f|_K \notin \mathbb{R}\mathbf{1}$. Thus $\rho_0(q_1, \cdot) = f|_K \notin \mathcal{F}$.

Next for general $x \in K$, choose $i \in S$ so that $x \in K_i$ and let $j, k \in S \setminus \{i\}$ be such that $j \neq k$ and $\rho_0(x, F_i(q_j)) \leq \rho_0(x, F_i(q_k))$. Then we easily see that $\rho_0(x, F_j(\cdot)) = (1/2)\rho_0(q_i, \cdot) + \rho_0(x, F_i(q_j))\mathbf{1} \notin \mathcal{F}$ and hence $\rho_0(x, \cdot) \notin \mathcal{F}$. $\qquad\square$

Also as opposed to the case of the heat kernels on Riemannian manifolds, p_{μ_0} is known to exhibit various oscillatory asymptotic behavior, as follows.

THEOREM A.4 ([**61**, Theorem 1.2-a)]). *There exists a continuous* $\log(5/2)$-*periodic* **non-constant** *function* $G : \mathbb{R} \to (0, \infty)$ *such that*

$$(A.3) \quad \lim_{n \to \infty} \left((2/5)^n s\right)^{\frac{1}{d_w - 1}} \log p_{\mu_0}\left((2/5)^n s, x, y\right) = -\rho_0(x,y)^{\frac{d_w}{d_w - 1}} G\left(\log \frac{s}{\rho_0(x,y)}\right)$$

for any $(s, x, y) \in (0, \infty) \times K \times K$ *with* $x \neq y$. *In particular, for any* $x, y \in K$ *with* $x \neq y$, *the limit* $\lim_{t \downarrow 0} t^{\frac{1}{d_w - 1}} \log p_{\mu_0}(t, x, y)$ *does* **not** *exist.*

THEOREM A.5 ([**45**, Corollary 6.2]). *The limit* $\lim_{t \downarrow 0} t^{d_s/2} p_{\mu_0}(t, x, x)$ *does* **not** *exist for any* $x \in K$.

A similar oscillation is observed also in the Laplacian eigenvalue asymptotics.

THEOREM A.6 ([**57, 10**], cf. [**26**], [**53**, Theorems 4.1.5, 4.3.4, 4.4.10 and B.4.3]). *Let* $\{\lambda_n^0\}_{n \in \mathbb{N}}$ *be the non-decreasing enumeration of all the eigenvalues of* $-\Delta_0$, *where each eigenvalue is repeated according to its multiplicity, and define the* eigenvalue counting function $\mathcal{N}_0$ *of* $(K, \mu_0, \mathcal{E}, \mathcal{F})$ *by* $\mathcal{N}_0(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^0 \leq \lambda\}$, $\lambda \in \mathbb{R}$. *Then there exists a right-continuous* $\log 5$-*periodic* **discontinuous** *function* $G_0 : \mathbb{R} \to \mathbb{R}$ *with* $0 < \inf_{s \in \mathbb{R}} G_0(s) < \sup_{s \in \mathbb{R}} G_0(s) < \infty$, *such that*

$$(A.4) \qquad\qquad \mathcal{N}_0(\lambda) = \lambda^{d_s/2} G_0(\log \lambda) + O(1) \qquad as \ \lambda \to \infty.$$

Roughly speaking, the asymptotic log-periodicity stated in Theorems A.4 and A.6 is more or less implied by the self-similarity of μ_0 and $(\mathcal{E}, \mathcal{F})$, whereas it is highly non-trivial to prove that there does exist oscillation in the asymptotics as in Theorems A.4, A.5 and A.6. Theorem A.4 was proved by utilizing a very detailed

description of the behavior of the Brownian motion on K provided in [**11**], and no essential extension to other fractals is known for this result because such detailed information of the Brownian motion is not available for most fractals.

The existence of G_0 in Theorem A.6 except its discontinuity was proved by Kigami and Lapidus in [**57**, Theorem 2.4 and Corollary 2.5] in the general framework of a self-similar regular Dirichlet form on a post-critically finite self-similar set equipped with a self-similar measure. In [**57**] they established the method of obtaining a certain renewal equation for the eigenvalue counting function $\mathcal{N}_0(\lambda)$ to apply the renewal theorem [**53**, Theorems B.4.2 and B.4.3] (see also [**23**, Section XI.1]) for Borel probability measures on $(0, \infty)$. Then the same method has been used by many authors in the context of analysis on fractals; see e.g. [**61, 30, 43, 47**]. (A variant of the method of [**57**] is described in some detail in Section 7 above.)

The discontinuity of G_0 in Theorem A.6 was proved by Barlow and Kigami [**10**] for affine nested fractals by showing the existence of *localized eigenfunctions* of the Laplacian. For example, in our present setting of the Sierpiński gasket, the dihedral symmetry of $(K, \mu_0, \mathcal{E}, \mathcal{F})$ implies that there exists an eigenfunction φ of $-\Delta_0$ with eigenvalue $\lambda \in (0, \infty)$ such that $\varphi|_{V_0} = 0$, and then for any $w \in W_*$,

$$(A.5) \qquad \varphi_w := \begin{cases} \varphi \circ F_w^{-1} & \text{on } K_w, \\ 0 & \text{on } K \setminus K_w \end{cases}$$

is also an eigenfunction of $-\Delta_0$ with eigenvalue $5^{|w|}\lambda$. This fact immediately implies the discontinuity of G_0, and Theorem A.5 for "generic" $x \in K$ is also proved on the basis of this fact and the eigenfunction expansion [**19**, (2.1.4)] of the heat kernel p_{μ_0}, by noticing that φ_w in (A.5) has amplitude comparable to $3^{|w|/2}$ after it is normalized in $L^2(K, \mu_0)$. Some more work is required to show the non-existence of the limit $\lim_{t \downarrow 0} t^{d_s/2} p_{\mu_0}(t, x, x)$ for *any* $x \in K$ and it has been done in [**45**] only for a limited class of concrete nested fractals, whereas the non-existence of this limit for "generic" points can be still verified for general affine nested fractals, as stated in [**45**, Theorem 4.5]. In fact, very recently, the author has proved in [**46**] that $t^{d_s/2} p_{\mu_0}(t, x, x)$ is *neither* regularly varying at 0 *nor* asymptotically log-periodic as $t \downarrow 0$ for "generic" points x, for a wide range of self-similar fractals including most (but not all) nested fractals and all generalized Sierpiński carpets.

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