A NOTE ON SELF-ADJOINT EXTENSIONS OF THE LAPLACIAN ON WEIGHTED GRAPHS

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ABSTRACT. We study the uniqueness of self-adjoint and Markovian extensions of the Laplacian on weighted graphs. We first show that, for locally finite graphs and a certain family of metrics, completeness of the graph implies uniqueness of these extensions. Moreover, in the case when the graph is not metrically complete and the Cauchy boundary has finite capacity, we characterize the uniqueness of the Markovian extensions.

1. INTRODUCTION

Determining the uniqueness of self-adjoint extensions of a symmetric operator in a certain class is a fundamental topic of functional analysis going back to the work of Friedrichs and von Neumann. If the operator has a unique self-adjoint extension, then it is called *essentially self-adjoint.* A self-adjoint extension whose corresponding form is a Dirichlet form is called *Markovian* and, when such a extension is unique, the operator is said to have a *unique Markovian extension.* It is clear that essential self-adjointness implies the uniqueness of Markovian extensions, but the converse is not necessarily true as can be seen by examples.

In the case of Riemannian manifolds, the Laplacian, whose domain is the space of smooth functions with compact support, has Markovian extensions and generates the Brownian motion. (The Laplacian should satisfy, in addition, the regularity property, but there is always an equivalent operator which has this property [12].) A well-known result going back to the work of Gaffney [15] states that, on a geodesically complete manifold, the Laplacian has a unique Markovian extension. (The essential self-adjointness of the Laplacian in [15] is equivalent to the uniqueness of Markovian extensions of the minimal Laplacian.) Later, it was shown that the Laplacian on a metrically complete Riemannian manifold is essentially self-adjoint [2, 41]. On the other hand, if the manifold is geodesically incomplete, the Laplacian is not essentially self-adjoint in general; however, if the Cauchy boundary, which is the difference between the completion of the manifold and the manifold itself, is "small" in some sense, then it is essentially self-adjoint or has a unique Markov extension depending on how small the Cauchy boundary is [1, 3, 21, 34, 35, 36] (see also the references within). For strongly-local regular Dirichlet forms, the uniqueness of Silverstein extensions was proven by Kawabata and Takeda [29] in the case when the underlying space is metrically complete with respect to the Carnot-Caratheodori distance. This result was extended to general regular Dirichlet forms by Kuwae and Shiozawa [33] using the intrinsic distance defined in [11].

Recently, there has been a tremendous amount of work devoted to the study of self-adjoint extensions of certain operators defined on graphs. More specifically, this problem is studied for adjacency, (magnetic) Laplacian, and Schrödinger-type operators on graphs in [4, 5, 6, 16, 17, 18, 23, 27, 28, 30, 31, 34, 37, 39, 40, 42, 43, 44, 45] among others.

Let us mention, in particular, the series of papers [4, 5, 42] by Colin de Verdière, Torki-Hamza, and Truc. These papers give some relations between metric completeness and essential self-adjointness. However, [23] contains an example of a graph which is metrically complete in one of the distances studied in [4] but for which the corresponding weighted Laplacian does not have a unique Markovian extension and is, therefore, not essentially self-adjoint. One of the reason for this seems to be that the particular metric used in [4] does not take into account the measure on the vertices of the graph. In [39, 40], Milatovic, following [42], shows with a different metric that completeness implies essential self-adjointness under the additional assumption of a uniform bound

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on the vertex degree.

In this paper we investigate these questions for the weighted Laplacian on graphs. Recall that the weighted Laplacian has Markovian extensions and the associated form is one of the most important examples of a non-local Dirichlet form. We use the notion of intrinsic distance with finite jump size introduced in [11] and applied in [20] and show that, if a weighted degree function is bounded on balls defined with respect to any such distance, then the Laplacian is essentially self-adjoint (Theorem 1). As a direct consequence, in the locally finite case, if the graph is metrically complete in any intrinsic path metric with finite jump size, then the Laplacian is essentially self-adjoint (Theorem 2). Compared to the previous results mentioned above we do not assume a uniform bound on the vertex degree and, for Theorem 1, we do not even need local finiteness. These results indicate that intrinsic metrics of finite jump size give the correct notion of distance on graphs when seeking to prove statements analogous to the strongly local case.

In the metrically incomplete case, under some further assumptions, we show that if the Cauchy (or metric) boundary has finite capacity, then the Laplacian has a unique Markovian extension if and only if the Cauchy boundary is polar, that is, has zero capacity, in analogy with [36] (Theorem 3). Moreover, we show that upper Minkowski codimension of the boundary greater than 2 implies zero capacity of the boundary (Theorem 4). We also show by examples that the other implications do not hold. In particular, in the case when the boundary has infinite capacity, the Laplacian may be essentially self-adjoint or might fail to have a unique Markovian extension, see Examples 5.2 and 5.4. In general, if the Laplacian is essentially self-adjoint, then it has a unique Markovian extension, but the opposite implication is not necessarily true, see Example 5.1. In Examples 5.5, 5.6, and 5.7 we discuss the case of upper Minkowski codimension less than or equal to 2 where the boundary may be polar or non-polar.

The paper is organized as follows. In Section 2, we introduce the set up, including background material on Dirichlet forms, Laplacians, intrinsic distances, and Cauchy boundary; and state the main results. In Section 3, we establish the triviality of square integrable eigenfunctions with negative eigenvalue when the weighted degree function is bounded on balls and use this to prove Theorems 1 and 2. Section 4 is devoted to the proofs of Theorems 3 and 4 and Section 5 is devoted to (counter-)examples. In Appendix A, we prove a Hopf-Rinow type property for path metrics on locally finite graphs. This property enables us to prove Theorems 2 and 3. We also present a series of (counter-)examples showing that the property may fail if the graph is not locally finite.

2. The set up and main results

2.1. **Weighted graphs.** We generally follow the setting of [30]. Let X be a countably infinite discrete set. A positive function $\mu : X \to (0, \infty)$ can be viewed as a Radon measure on X with full support so that (X, μ) becomes a measure space.

Let $w : X \times X \to [0, \infty)$ be symmetric, with zero diagonal, and satisfying

$$\sum_{y \in X} w(x, y) < \infty \quad \text{for all } x \in X. \tag{1}$$

The triple (X, w, μ) is called a *weighted graph*. We call $x, y \in X$ *neighbors* if $w(x, y) > 0$ and denote this symmetric relation by $x \sim y$. For $n \geq 1$, we call a sequence of points $(x_0, x_1, \ldots, x_n)$ a *path* connecting x and y if $x_0 = x, x_n = y$, and $x_i \sim x_{i+1}$ for $i = 0, 1, \ldots, n-1$. A weighted graph (X, w, μ) is called *connected* if, for any two distinct points in X, there exists a connecting path. From now on, we only consider connected weighted graphs.

2.2. **Weighted degree and intrinsic metrics.** We call the function $\mathrm{Deg} : X \to [0, \infty)$ given by

$$\mathrm{Deg}(x) := \frac{1}{\mu(x)} \sum_{y \in X} w(x, y)$$

the *weighted degree*. It is, in general, distinct from the usual degree of locally finite graphs which is given by the number of neighbors of a vertex.

A *pseudo metric* is a map $d : X \times X \to [0, \infty)$ that is symmetric, has zero diagonal and satisfies the triangle inequality. A pseudo metric $d = d_\sigma$ is called a *path pseudo metric* if there is a symmetric map $\sigma : X \times X \to [0, \infty)$ such that $\sigma(x, y) > 0$ if and only if $x \sim y$ and

$$d_\sigma(x, y) = \inf\{l_\sigma((x_0, \ldots, x_n)) \mid n \geq 1, (x_0, \ldots, x_n) \text{ is a path connecting } x \text{ and } y\}$$

where the *length* l_σ of a path $(x_0, \ldots, x_n)$ is given by $l_\sigma((x_0, \ldots, x_n)) = \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1})$. We say that d has *jump size* $s \geq 0$ if, for all $x, y \in X$, $w(x, y) = 0$ whenever $d(x, y) > s$.

Following Frank, Lenz and Wingert [11] (see Lemma 4.7 and Theorem 7.3) we make the following definition which has already proven to be useful in several other problems on graphs, see Remark 2.2 below.

Definition. We call a pseudo metric d on (X, w, μ) *intrinsic* if, for all $x \in X$,

$$\frac{1}{\mu(x)} \sum_{y \in X} w(x, y) d(x, y)^2 \leq 1.$$

An intrinsic path pseudo metric d_σ is called *strongly intrinsic* if, for all $x \in X$,

$$\frac{1}{\mu(x)} \sum_{y \in X} w(x, y) \sigma(x, y)^2 \leq 1.$$

The first example below shows that there always exist strongly intrinsic path pseudo metrics with jump size 1 on a connected weighted graph.

Example 2.1. (1) For $x, y \in X$ with $x \sim y$, let $\sigma_0(x, y) = \min\{\mathrm{Deg}^{-\frac{1}{2}}(x), \mathrm{Deg}^{-\frac{1}{2}}(y), 1\}$. Clearly, d_{σ_0} is strongly intrinsic with jump size 1.

(2) For locally finite graphs, let $\sigma_1(x, y) = w(x, y)^{-\frac{1}{2}} \min\{\frac{\mu(x)}{\deg(x)}, \frac{\mu(y)}{\deg(y)}\}^{\frac{1}{2}}$, $x, y \in X$ with $x \sim y$ where deg is the unweighted vertex degree, i.e., the number of neighbors. Clearly, d_{σ_1} is a strongly intrinsic path metric. Moreover, if $\deg \leq K$ for some $K \geq 1$, then d_{σ_1} is equivalent to the metrics used in [4, 5, 39, 40] (in the case of no magnetic field and no potential). This seems to explain why their results always need a global bound on the vertex degree.

(3) Suppose that $\sigma_N \equiv 1$ on neighbors. Then, $d_N = d_{\sigma_N}$ gives the natural graph metric, that is, the distance between x and y is equal to one less than the number of points in the shortest path connecting them. Obviously, d_N is strongly intrinsic if and only if $\mathrm{Deg} \leq 1$. (Clearly, if Deg is bounded by $K > 0$, then $d_N/\sqrt{K}$ is also a strongly intrinsic metric.)

Remark 2.2. Various authors came up with such types of metrics independently. In the context of stochastic completeness for jump processes see the work of Masamune/ Uemura [38], Grigor'yan/Huang/Masamune [20] and also [25, 26]. Independently, Folz [8] came up with similar ideas in the context of heat kernel estimates on locally finite graphs, see also [9, 10]. For further uses of intrinsic metrics, see [24].

For $x_0 \in X$ and $r \geq 0$, we define the distance ball with respect to any pseudo metric d by $B_r(x) := \{x \in X \mid d(x, x_0) \leq r\}$. Moreover, for a subset $U \subseteq X$ and $r \geq 0$, we let $B_r(U) := \bigcup_{x \in U} B_r(x)$.

2.3. **Forms and operators.** Denote by $C(X)$ the set of all functions $X \to \mathbb{R}$ and by $C_c(X)$ the subset of functions which are finitely supported. The Hilbert space $L^2(X, \mu)$ is then defined in the usual way with scalar product

$$\langle u, v \rangle := \sum_X uv\mu := \sum_{x \in X} u(x)v(x)\mu(x)$$

and norm $\|u\| := \langle u, u \rangle^{\frac{1}{2}} = \left(\sum_X u^2 \mu\right)^{\frac{1}{2}}$.

We next introduce a discrete version of the energy measure which can be thought of as a generalized gradient. Let

$$D_{loc} := \{f \in C(X) \mid \sum_{y \in X} w(x, y)(f(x) - f(y))^2 < \infty \text{ for all } x \in X\}.$$

We define, for $f, g \in D_{loc}$ and $x \in X$,

$$(\nabla f \cdot \nabla g)(x) := \sum_{y \in X} w(x, y)(f(x) - f(y))(g(x) - g(y)),$$

$$|\nabla f|^2(x) := (\nabla f \cdot \nabla f)(x) = \sum_{y \in X} w(x, y)(f(x) - f(y))^2.$$

The *generalized form* $\widetilde{Q}$ is a map $C(X) \to [0, \infty]$ given by

$$\widetilde{Q}(f) := \frac{1}{2} \sum_X |\nabla f|^2 = \frac{1}{2} \sum_{x, y \in X} w(x, y)(f(x) - f(y))^2$$

and the *generalized form domain* is given by $\widetilde{D} := \{f \in C(X) \mid \widetilde{Q}(f) < \infty\}$. Clearly, $\widetilde{D} \subseteq D_{\mathrm{loc}}$. By polarization, this gives a sesquilinear form $\widetilde{Q} : \widetilde{D} \times \widetilde{D} \to \mathbb{R}$ as follows

$$\widetilde{Q}(f, g) = \frac{1}{2} \sum_X (\nabla f \cdot \nabla g) = \frac{1}{2} \sum_{x, y \in X} w(x, y)(f(x) - f(y))(g(x) - g(y)).$$

In this context, there are two distinguished restrictions of the generalized form. Let Q be the restriction of $\widetilde{Q}$ to

$$D(Q) := \overline{C_c(X)}^{\|\cdot\|_Q} \quad \text{where} \quad \|\cdot\|_Q := (\widetilde{Q}(\cdot) + \|\cdot\|^2)^{\frac{1}{2}}.$$

The form $(Q, D(Q))$ is then a regular Dirichlet form, see [30]. Furthermore, let Q^{max} be the restriction of $\widetilde{Q}$ to

$$D(Q^{\mathrm{max}}) := \{f \in L^2(X, \mu) \mid \widetilde{Q}(f) < \infty\}.$$

The form $(Q^{\mathrm{max}}, D(Q^{\mathrm{max}}))$ is a Dirichlet form but it is not regular in general. For more discussion of these two forms and the associated self-adjoint operators in our context, see the proof of Theorem 1 below or [23].

The *formal Laplacian* Δ can be introduced on (X, w, μ) as an analogue of the classical Laplace-Beltrami operator on Riemannian manifolds as follows

$$(\Delta f)(x) = \frac{1}{\mu(x)} \sum_{y \in X} w(x, y)(f(x) - f(y)),$$

with domain $F = \{f \in C(X) \mid \sum_{y \in V} w(x, y)|f(y)| < \infty \text{ for all } x \in X\}$. Taking into account (1), Δ is defined pointwise. It is easy to see that F is stable under multiplication by bounded functions on X. It can be shown that the self-adjoint operator L with domain $D(L)$ corresponding to Q is non-negative and is a restriction of Δ, see [30, Theorem 9]. That is,

$$Lu = \Delta u, \qquad u \in D(L).$$

2.4. **Main results.** As discussed in the introduction, it is a classical result that the Laplacian on a weighted manifold is essentially self-adjoint if all geodesic balls are relatively compact which is equivalent to the manifold being metrically complete (see, for example, Theorem 11.5 in [19]). Here we present some counterparts for weighted graphs.

Theorem 1. *Let (X, w, μ) be a weighted graph and let d be an intrinsic pseudo metric with finite jump size. If the weighted degree function* Deg *is bounded on all distance balls, then*

$$D(Q) = D(Q^{\mathrm{max}}),$$
$$D(L) = \{u \in L^2(X, \mu) \cap F \mid \Delta u \in L^2(X, \mu)\}.$$

In particular, if additionally $\Delta C_c(X) \subseteq L^2(X, \mu)$, then $L_c = \Delta|_{C_c(X)}$ is essentially self-adjoint.

Remark 2.3. (a) The result on essential self-adjointness is sharp by Example 5.1 in Section 5.
(b) Let us note that the theorem does not assume that Deg is bounded on X. This would imply that Q is bounded and the statements become trivial.
(c) The condition $\Delta C_c(X) \subseteq L^2(X, \mu)$ holds if and only if $w(x, \cdot)/\mu(\cdot) \in L^2(X, \mu)$ for all $x \in X$, see Proposition 3.3 in [30]. In particular, this always holds in the locally finite case.

Let us note the following immediate consequence.

Corollary 1. *If (X, w, μ) is a weighted graph and d an intrinsic pseudo metric with finite jump size such that all distance balls are finite, then the statements of Theorem 1 hold.*

In the locally finite case, path pseudo metrics are metrics and the analogy to Riemannian manifolds is even more obvious.

Theorem 2. *If (X, w, μ) is a locally finite weighted graph and $d = d_\sigma$ an intrinsic path metric with finite jump size such that (X, d) is metrically complete, then*
$$D(L) = \{u \in L^2(X, \mu) \cap F \mid \Delta u \in L^2(X, \mu)\}$$
and $L_c = \Delta|_{C_c(X)}$ is essentially self-adjoint.

Next, we turn to the metrically incomplete case where we will prove an analogue to results found in [36], see Theorem 3 below. In order to avoid some topological issues when defining the capacity, we now assume that all graphs are locally finite and that we only deal with path metrics.

An extension of $L_c = \Delta|_{C_c(X)}$ is said to be *Markovian*, if the form associated to it is a Dirichlet form. In particular, the operators associated to Q and $Q^{\max}$ are Markovian and, in the locally finite case, having a unique Markovian extension is equivalent to $D(Q) = D(Q^{\max})$ [23, Theorem 4.2].

For a set $U \subseteq X$ define the *capacity* of U by
$$\text{Cap}(U) := \inf\{\|u\|_Q \mid u \in D(Q^{\max}), \ 1_U \leq u\},$$
where 1_U is the characteristic function of U and $\inf \emptyset = \infty$. For a path metric d on X we let $(\overline{X}, \overline{d})$ be the metric completion. We define the *Cauchy boundary* $\partial_C X$ of X to be the difference between $\overline{X}$ and X:
$$\partial_C X := \overline{X} \setminus X.$$
Clearly, X is metrically complete if and only if $\partial_C X$ is empty. For $A \subseteq \overline{X}$ define
$$\text{Cap}(A) := \inf\{\text{Cap}(O \cap X) \mid A \subseteq O \text{ with } O \subseteq \overline{X} \text{ open}\}.$$
Note that, for $U \subseteq X$, the definitions of capacity agree due to the local finiteness and the use of path metrics. We say that $\partial_C X$ is *polar* if $\text{Cap}(\partial_C X) = 0$.

Theorem 3. *Let (X, w, μ) be a locally finite weighted graph and let $d = d_\sigma$ be a strongly intrinsic path metric. If (X, d) is not metrically complete and $\text{Cap}(\partial_C X) < \infty$, then the Cauchy boundary is polar if and only if*
$$D(Q) = D(Q^{\max}),$$
that is, if and only if $L_c = \Delta|_{C_c(X)}$ has a unique Markovian extension.

Note that the other consequences of Theorem 1 do not necessarily follow if (X, d) is metrically incomplete. Example 5.1 in Section 5 contains a example with polar boundary but where L_c has non-Markovian self-adjoint extensions. Also, in the case when the Cauchy boundary has infinite capacity, the Laplacian may be essentially self-adjoint or may not have a unique Markovian extension, see Examples 5.2 and 5.4.

Next we turn to a criterion which connects polarity of the boundary to co-dimension of the boundary. The *upper Minkowski codimension* $\overline{\text{codim}}_M(\partial_C X)$ of $\partial_C X$ is defined as
$$\overline{\text{codim}}_M(\partial_C X) = \limsup_{r \to 0} \frac{\ln \mu(B_r(\partial_C X))}{\ln r},$$
where $B_r(\partial_C X) = \{x \in \overline{X} \mid \inf_{b \in \partial_C X} \overline{d}(x, b) \leq r\}$. The upper Minkowski codimension (or box counting codimension) is one of the most studied fractal dimensions, especially in practical situations. The relationships between various dimensions in the classical setting can be found in [7].

Theorem 4. *Let (X, w, μ) be a locally finite weighted graph and let $d = d_\sigma$ be an intrinsic path metric. If $\overline{\text{codim}}_M(\partial_C X) > 2$, then $\partial_C X$ is polar.*

In Examples 5.5, 5.6, and 5.7 we show that $\partial_C X$ can be polar or non-polar if $\overline{\operatorname{codim}}_M(\partial_C X) \leq 2$. See the recent paper [21] for some related statements in the case of manifolds.

3. Uniqueness of solutions

Let (X, w, μ) be a weighted graph and let d be an intrinsic pseudo metric with finite jump size s. In this section we will show that under the assumption that Deg is bounded on distance balls there are no L^2 solutions to $(\Delta + \lambda)u = 0$ for $\lambda > 0$. From this fact we can infer Theorems 1 and 2.

3.1. **Leibniz rule and Green's formula.** The following auxiliary lemmas are well known in various other situations, see [11, 22, 23, 27, 28, 30, 31]. However, they do not hold on graphs without further assumptions. Here, we prove them under the assumption that the generalized vertex degree is bounded on balls.

Lemma 3.1. *Let $B \subseteq X$ be such that* Deg *is bounded on B. Then, for all $u, v \in L^2(X, \mu)$,*

$$\sum_{x,y \in B} w(x, y)|u(x)v(x)| < \infty \quad \text{and} \quad \sum_{x,y \in B} w(x, y)|u(x)v(y)| < \infty.$$

Proof. Let $f = \max\{|u|, |v|\}$. Clearly, $f \in L^2(X, \mu)$. We estimate

$$\sum_{x,y \in B} w(x, y)|u(x)||v(x)| \leq \sum_{x \in B} f^2(x) \sum_{y \in B} w(x, y) \leq \sum_{x \in B} f^2(x)\mu(x)\operatorname{Deg}(x)$$
$$\leq \sup_{y \in B} \operatorname{Deg}(y)\|f\|^2 < \infty,$$

since Deg is bounded on B by assumption. On the other hand, by the Cauchy Schwarz inequality and the above, we obtain

$$\sum_{x,y \in B} w(x, y)|u(x)||v(y)| \leq \Big(\sum_{x,y \in B} w(x, y)u^2(x) \Big)^{\frac{1}{2}} \Big(\sum_{x,y \in B} w(x, y)v^2(x) \Big)^{\frac{1}{2}} < \infty.$$

$\square$

The following is an integrated Leibniz rule (see also [11, Theorem 3.7]).

Lemma 3.2. *(Leibniz rule) For all $f, g, h \in D_{loc}$ with $fg \in D_{loc}$*

$$\sum_X (\nabla(fg) \cdot \nabla h) = \sum_X f(\nabla g \cdot \nabla h) + \sum_X g(\nabla f \cdot \nabla h),$$

whenever at least two of the sums converge absolutely. In particular, if Deg *is bounded on $B_s(U)$ for some set $U \subseteq X$, $f \in L^\infty(X)$ with $\operatorname{supp} f \subseteq U$ and $g, h \in L^2(X, \mu)$, the formula applies (where s is the jump size).*

Proof. The first statement follows by the simple algebraic manipulation $fg(x) - fg(y) = f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))$, $x, y \in X$. The 'in particular' follows by basic estimates, Lemma 3.1 and the fact that $w(x, y) = 0$ whenever $d(x, y) > s$. $\square$

The following Green's formula is a variant of [22, Lemma 4.7].

Lemma 3.3. *(Green's formula) Assume that* Deg *is bounded on $B_s(U) \subseteq X$ for some set $U \subseteq X$. Then, for all $u, v \in L^2(X, \mu) \cap F$ with $\operatorname{supp} v \subseteq U$*

$$\sum_X (\Delta u)v\mu = \sum_X u(\Delta v)\mu = \frac{1}{2}\sum_X (\nabla u \cdot \nabla v).$$

Proof. Since $w(x, y) = 0$ whenever $d(x, y) > s$, the statement follows by simple algebraic manipulations, Lemma 3.1 and Fubini's theorem. $\square$

3.2. **A Caccioppoli-type inequality.** The key estimate for the proof of triviality of L^2 solutions to $(\Delta + \lambda)u = 0$ for $\lambda > 0$ is the following Caccioppoli-type inequality. See [11, Theorem 11.1] for a similar result for general Dirichlet forms.

Lemma 3.4. *(Caccioppoli-type inequality) Let $u \in L^2(X, \mu) \cap F$, $U \subseteq X$ and assume that* Deg *is bounded on $B_s(U)$. Then, for all $v \in L^\infty(X)$ with supp $v \subseteq U$,*

$$-\sum_X (\Delta u)uv^2\mu \le \frac{1}{2}\sum_X u^2|\nabla v|^2.$$

Proof. By Lemma 3.2 and Lemma 3.3 we have

$$\sum_X (\Delta u)uv^2\mu = \frac{1}{2}\sum_X (\nabla u \cdot \nabla(uv^2)) = \frac{1}{2}\sum_X v^2|\nabla u|^2 + \frac{1}{2}\sum_X u(\nabla u \cdot \nabla v^2).$$

We focus on the second sum on the right hand side. Since a geometric mean can be estimated by its corresponding arithmetic mean, we have $|ab| \le \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$ for $a, b \in \mathbb{R}$ and $\delta > 0$. We use this estimate with $a = (u(x) - u(y))(v(x) + v(y))$, $b = u(x)(v(x) - v(y))$ and $\delta = 1/2$ for the terms in the second sum on the right hand side above

$$|u(x)(u(x) - u(y))(v^2(x) - v^2(y))| \le \frac{1}{4}(u(x) - u(y))^2(v(x) + v(y))^2 + u^2(x)(v(x) - v(y))^2$$

$$\le \frac{1}{2}(u(x) - u(y))^2(v^2(x) + v^2(y)) + u^2(x)(v(x) - v(y))^2.$$

Multiplying by $w(x, y)$ and summing over $x, y \in X$ yields

$$-\frac{1}{2}\sum_X u(\nabla u \cdot \nabla v^2) \le \frac{1}{2}\sum_X v^2|\nabla u|^2 + \frac{1}{2}\sum_X u^2|\nabla v|^2.$$

The assertion now follows from the equality in the beginning of the proof. $\square$

3.3. **Uniqueness of solutions.** The following is an analogue of [19, Lemma 11.6].

Proposition 3.5. *Assume that the weighted degree function* Deg *is bounded on all distance balls. Then, for all $\lambda > 0$, the equation*

$$(\Delta + \lambda)u = 0,$$

has only the trivial solution in $L^2(X, \mu) \cap F$.

Proof. Let $0 \le r < R$, fix $x_0 \in X$, and consider the cut-off function $\eta = \eta_{R,r} : X \to \mathbb{R}$ given by

$$\eta(x) = \left(\frac{R - d(x, x_0)}{R - r}\right)_+ \wedge 1. \tag{2}$$

Note that $0 \le \eta \le 1$, $\eta|_{B_r} = 1$ and $\eta|_{X \setminus B_R} = 0$. Moreover, η is Lipshitz continuous with Lipshitz constant $\frac{1}{R-r}$ (of course, with respect to the intrinsic pseudo metric d). This immediately implies, as d is an intrinsic pseudo metric, that

$$|\nabla \eta|^2(x) \le \frac{1}{(R-r)^2}\sum_{y \in X} w(x, y)d^2(x, y) \le \frac{\mu(x)}{(R-r)^2}, \qquad x \in X. \tag{3}$$

Fix some $\lambda > 0$ and assume that $u \in L^2(X, \mu) \cap F$ is a solution to the equation $(\Delta + \lambda)u = 0$. Then, we have by Lemma 3.4 and the estimate on $|\nabla \eta|^2$ above, that

$$\lambda \|u1_{B_r}\|^2 \le \lambda \|u\eta\|^2 = -\sum_X (\Delta u)u\eta^2\mu \le \frac{1}{2}\sum_X u^2|\nabla \eta|^2 \le \frac{1}{2(R-r)^2}\|u\|^2.$$

Letting $R \to \infty$, we see that $u \equiv 0$ on B_r. Since r is chosen arbitrarily, $u \equiv 0$ on X. $\square$

Remark 3.6. It would be interesting to prove a similar statement as Proposition 3.5 for L^p.

3.4. **Proofs of Theorems 1 and 2.** The proof of Theorem 1 follows by standard techniques used in [30] and [23].

Proof of Theorem 1. The equality $D(Q) = D(Q^{\max})$ follows from Proposition 3.5 and [23, Corollary 4.3]. Define $D_{\max} = \{u \in L^2(X, \mu) \cap F \mid \Delta u \in L^2(X, \mu)\}$. By Theorem 9 in [30], L is a restriction of Δ which implies that $D(L) \subseteq D_{\max}$. Letting $f \in D_{\max}$ we see that $g := (\Delta + \lambda)f \in L^2(X, \mu)$ for all $\lambda > 0$, so that $u := (L + \lambda)^{-1}g \in D(L)$. As u solves the equation $(\Delta + \lambda)u = g$ (see Lemma 2.8 in [30]), we conclude that $f = u$ by the uniqueness of solutions, Proposition 3.5 (as f solves the equation by definition). Thus, $f \in D(L)$ and, therefore, $D(L) = D_{\max}$.

Let $L_{Q^{\max}}$ be the self-adjoint operator associated to $Q^{\max}$. Let us show that $D(L_{Q^{\max}}) \subseteq F$ and that $L_{Q^{\max}}$ is a restriction of Δ. For $u \in D(L_{Q^{\max}})$ and $x \in X$ we apply the triangle inequality $|u(x)| \leq |u(x) - u(y)| + |u(y)|$ and Cauchy Schwarz inequality to estimate

$$\sum_{y \in X} w(x, y)|u(y)| \leq \mu(x)^{\frac{1}{2}}\mathrm{Deg}(x)^{\frac{1}{2}}(2Q^{\max}(u))^{\frac{1}{2}} + \mu(x)\mathrm{Deg}(x)|u(x)| < \infty.$$

Hence, $D(L_{Q^{\max}}) \subseteq D(Q^{\max}) \subseteq F$. Then, for all $u \in D(L_{Q^{\max}})$, $v \in C_c(X)$,

$$\sum_X (\Delta u)v\mu = Q^{\max}(u, v) = \langle L_{Q^{\max}}u, v \rangle.$$

Hence, $L_{Q^{\max}}$ is a restriction of Δ. Since $D(L) = D_{\max}$ as we just proved, $D(L_{Q^{\max}}) \subseteq D(L)$ and thus $L_{Q^{\max}}$ is a restriction of L which implies that $L_{Q^{\max}} = L$.

If $\Delta C_c(X) \subseteq L^2(X, \mu)$, then essential self-adjointness is a rather immediate consequence of $L = L_c^*$ (see Proposition 3.5). Since L_c can be easily seen to be a restriction of L and $D(L) = D_{\max}$, the statement follows (cf. [23, Theorem 6.2]). □

Proof of Theorem 2. As we assume local finiteness, it is clear that $\Delta C_c(X) \subseteq L^2(X, \mu)$. Furthermore, by Theorem A.1, it follows that the metric completeness of (X, d) implies that distance balls are finite. Hence, Deg is bounded on distance balls which implies the statement by Theorem 1. □

4. Cauchy boundary and equilibrium potentials

Let (X, w, μ) be a locally finite weighted graph and let d be a path metric. Recall that $(\overline{X}, \overline{d})$ denotes the metric completion of (X, d) and $\partial_C X = \overline{X} \setminus X$ is the Cauchy boundary. In this section we prove Theorems 3 and 4.

4.1. **Existence of equilibrium potentials.** The following is well known and follows directly from [13, Lemma 2.1.1.].

Lemma 4.1. *If $\mathrm{Cap}(O) < \infty$ for an open set $O \subset \overline{X}$, then there is a unique element $e \in D(Q^{\max})$ such that $0 \leq e \leq 1$, $e|_{O \cap X} \equiv 1$, and $\mathrm{Cap}(O) = \|e\|_Q$.*

Proof. From [13, Lemma 2.1.1.] it follows that for any $U \subseteq X$ there is such an $e \in D(Q^{\max})$ (as we consider X equipped with the discrete topology). Note that [13, Lemma 2.1.1.] assumes regularity of the form but this is not needed for the proof. Now, for an open set $O \subseteq \overline{X}$ one easily checks that $\mathrm{Cap}(O) = \mathrm{Cap}(O \cap X)$ (as any function $u \in D(Q^{\max})$ with $1_{O' \cap X} \leq u$ for $O \subseteq O' \subseteq \overline{X}$ open satisfies $1_{O \cap X} \leq u$) which finishes the proof. □

We call such an e the *equilibrium potential* associated to O.

4.2. **The boundary alternative.** The following lemma shows that if the minimal and the maximal forms agree, then the capacity of any subset of the boundary is either zero or infinite.

Lemma 4.2. *Let $A \subseteq \partial_C X$. If $D(Q) = D(Q^{\max})$, then either $\mathrm{Cap}(A) = \infty$ or $\mathrm{Cap}(A) = 0$.*

Proof. Assume that $D(Q) = D(Q^{\max})$ and A has finite capacity. Then, there exists an open set $O \subseteq \overline{X}$ such that $A \subseteq O$ and $\mathrm{Cap}(O) < \infty$. Let e be the equilibrium potential associated to O. Since $D(Q) = D(Q^{\max})$ there exists a sequence of functions e_n in $C_c(X)$ converging to e as $n \to \infty$ in the $\|\cdot\|_Q$ norm. Clearly, $(e - e_n)_+ \wedge 1$ belongs to $D(Q^{\max})$ and equals 1 on a neighborhood O_n of A. Therefore,

$$\mathrm{Cap}(A) \leq \liminf_{n \to \infty} \mathrm{Cap}(O_n) \leq \lim_{n \to \infty} \|(e - e_n)_+ \wedge 1\|_Q \leq \lim_{n \to \infty} \|e - e_n\|_Q = 0.$$

□

4.3. **Approximation by equilibrium potentials.** Next, we show that every bounded function in $D(Q^{\max})$ can be approximated via equilibrium potentials if the boundary has capacity zero.

Lemma 4.3. *Assume that $\partial_C X$ is polar and let e_n be the equilibrium potentials associated to open sets $O_n \subseteq \overline{X}$ with $\partial_C X \subseteq O_n$ and $\mathrm{Cap}(O_n) \to 0$ as $n \to \infty$. Then $\|u - (1 - e_n)u\|_Q \to 0$ as $n \to \infty$ for all $u \in D(Q^{\max}) \cap L^\infty(X)$.*

Proof. Note that $u - (1 - e_n)u = e_n u$ and $\|e_n u\| \leq \|u\|_\infty \|e_n\| \to 0$ as $n \to \infty$. Moreover, we have

$$\widetilde{Q}(e_n u) = \frac{1}{2} \sum_X |\nabla(e_n u)|^2 \leq \sum_X e_n^2 |\nabla u|^2 + \sum_X u^2 |\nabla e_n|^2$$

$$\leq \sum_X e_n^2 |\nabla u|^2 + 2 \|u\|_\infty^2 \widetilde{Q}(e_n) \to 0 \quad \text{as } n \to \infty$$

by noting that $e_n(x) \to 0$ for all x and applying the Lebesgue dominated convergence theorem. □

4.4. **Restriction to complete subgraphs.** In the next lemma we show that bounded functions in $D(Q^{\max})$ that are zero close to the boundary can be approximated by finitely supported functions. Here we need to assume that the metric is strongly intrinsic.

Lemma 4.4. *Assume that $d = d_\sigma$ is a strongly intrinsic path metric. Let $O \subseteq \overline{X}$ be open with $\partial_C X \subseteq O$, $\mathrm{Cap}(O) < \infty$ and let e be the equilibrium potential associated to O. Then, $C_c(X)$ is form dense in $(1 - e)(D(Q^{\max}) \cap L^\infty(X)) = \{(1 - e)u \mid u \in D(Q^{\max}) \cap L^\infty(X)\}$.*

Proof. Let $Y = X \setminus O$, μ_Y be the restriction of μ to Y and σ_Y and w_Y be the restrictions of σ and w to $Y \times Y$. We first assume that (Y, w_Y, μ_Y) is connected. From this it follows that $d_Y = d_{\sigma_Y}$ is a strongly intrinsic path metric on (Y, w_Y, μ_Y) with $d_Y \geq d$.

Claim: (Y, d_Y) is metrically complete.
Proof of the claim: If (x_n) is a Cauchy sequence in Y, then, by $d_Y \geq d$, it is a Cauchy sequence in X and has a limit point in $\overline{X}$. However, as $Y = X \setminus O$, the limit point is not in $\partial_C X$. As (X, d_σ) is a discrete metric space, see Lemma A.3 (a), (x_n) must be eventually constant which proves the claim.

For $R > 0$ and fixed $x_0 \in Y$, let $\eta_R : Y \to [0, 1]$ given by

$$\eta_R(x) = \left(\frac{2R - d_Y(x, x_0)}{R} \right)_+ \wedge 1.$$

By completeness, B_{2R} in (Y, d_Y) is finite, see Theorem A.1 in Appendix A, and it follows that $\eta_R(1 - e)D(Q^{\max}) \subseteq C_c(X)$.

Let $\widetilde{\nabla}$ be the generalized gradient for (Y, w_Y, μ_Y). Let $v \in (1 - e)(D(Q^{\max}) \cap L^\infty(X))$ and set $g_R = v - \eta_R v = (1 - \eta_R)v$. Now,

$$\widetilde{Q}(g_R) = \frac{1}{2} \sum_Y |\widetilde{\nabla} g_R|^2 + \sum_{x \in Y} \sum_{y \in O \cap X} w(x, y)g_R(x)^2.$$

For the first term we get, using that d_Y is intrinsic and therefore $|\widetilde{\nabla}\eta_R|^2 \leq \mu_Y/R^2$,

$$\sum_Y |\widetilde{\nabla} g_R|^2 \leq \sum_Y v^2 |\widetilde{\nabla}\eta_R|^2 + \sum_Y (1 - \eta_R)^2 |\widetilde{\nabla} v|^2 \leq \frac{1}{R^2} \|v\|^2 + \sum_Y (1 - \eta_R)^2 |\widetilde{\nabla} v|^2 \to 0,$$

as $R \to \infty$. For the second term let $u \in (D(Q^{\max}) \cap L^\infty(X))$ such that $v = (1 - e)u$. Then,

$$\sum_{x \in Y} \sum_{y \in O \cap X} w(x, y)g_R(x)^2 \leq \|u\|_\infty^2 \sum_{x \in Y} (1 - \eta_R(x))^2 \sum_{y \in O \cap X} w(x, y)(1 - e(x))^2 \to 0$$

by the Lebesgue dominated convergence theorem. This follows, since $\eta_R \to 1$ pointwise as $R \to \infty$ and $\sum_Y \sum_{O \cap X} w(x, y)(1 - e(x))^2 \leq \widetilde{Q}(e) \leq \mathrm{Cap}(O)^2 < \infty$ (as $e(y) = 1$ for $y \in O \cap X$). Moreover, as g_R converges pointwise to zero it also converges to zero in L^2.

In the case where Y is not connected there are at most countably many connected components Y_i, $i \geq 0$. For $v \in (1 - e)(D(Q^{\max}) \cap L^\infty(X))$ let $v_i = v|_{Y_i}$. Since $\widetilde{Q}(v) = \sum_{i \geq 0} \widetilde{Q}(v_i)$, the statement follows by a diagonal sequence argument. □

4.5. **Proof of Theorem 3.**

Proof of Theorem 3. Since we assume local finiteness, L_c having a unique Markovian extension is equivalent to $D(Q) = D(Q^{\mathrm{max}})$ by Theorem 4.2 in [23].

If $D(Q) = D(Q^{\mathrm{max}})$, then the assumption $\mathrm{Cap}(\partial_C X) < \infty$ implies $\mathrm{Cap}(\partial_C X) = 0$ by Lemma 4.2.

If, on the other hand, $\partial_C X$ has zero capacity, then $C_c(X)$ is form dense in $D(Q^{\mathrm{max}}) \cap L^\infty(X)$ by Lemmas 4.3 and 4.4. But $D(Q^{\mathrm{max}}) \cap L^\infty(X)$ is form dense in $D(Q^{\mathrm{max}})$ since if $u \in D(Q^{\mathrm{max}})$, then $u_l = (u \vee -l) \wedge l$ converges to u in $\|\cdot\|_Q$ as $l \to \infty$. This implies $D(Q) = D(Q^{\mathrm{max}})$. $\qquad\square$

4.6. **Proof of Theorem 4.**

Proof of Theorem 4. As $\overline{\mathrm{codim}}_M(\partial_C X) > 2$, there exists an $\epsilon > 0$ and a sequence $r_n \to 0$ as $n \to \infty$ such that

$$\mu(B_{r_n}(\partial_C X)) < r_n^{2+\epsilon}.$$

For $R > 0$ and $x \in X$, let

$$\eta_R(x) = \left(\frac{2R - \overline{d}(x, \partial_C X)}{R} \right)_+ \wedge 1.$$

In particular, $0 \le \eta_R \le 1$, $\eta_R|_{B_R(\partial_C X)} \equiv 1$ and $\eta_R|_{X \setminus B_{2R}(\partial_C X)} \equiv 0$. It follows that

$$\|\eta_R\|^2 \le \mu(B_{2R}(\partial_C X))$$

and

$$\widetilde{Q}(\eta_R) \le \frac{1}{2} \sum_{x,y \in B_{2R}(\partial_C X)} w(x,y) \left(\frac{\overline{d}(x, \partial_C X)}{R} - \frac{\overline{d}(y, \partial_C X)}{R} \right)^2$$

$$\le \frac{1}{2R^2} \sum_{x,y \in B_{2R}(\partial_C X)} w(x,y) d(x,y)^2 \le \frac{\mu(B_{2R}(\partial_C X))}{2R^2}$$

since d is intrinsic.

Letting $2R = r_n$, it follows that

$$\mathrm{Cap}(\partial_C X) \le \left\| \eta_{r_n/2} \right\|_Q \le \left(\mu(B_{r_n}(\partial_C X)) + \frac{2}{r_n^2} \mu(B_{r_n}(\partial_C X)) \right)^{\frac{1}{2}}$$

$$\le \left(r_n^{2+\epsilon} + 2r_n^\epsilon \right)^{\frac{1}{2}} \to 0 \quad \text{as } n \to \infty.$$

$\qquad\square$

5. (COUNTER-)EXAMPLES

In this section we present the examples mentioned in Section 2.4. In particular, these examples show that Markov uniqueness does not imply essential self-adjointness, that no conclusion can be drawn concerning uniqueness in the infinite capacity case and that the boundary can be polar or non-polar for any upper Minkowski codimension less than or equal to 2.

As we often use a graph with $X = \mathbb{N}_0$ and $x \sim y$ if and only if $|x - y| = 1$ we make several preliminary observations concerning graphs of this type with a given path metric d. First, in this case,

$$\partial_C X \ne \emptyset \quad \text{if and only if} \quad l(X) := \sum_{x=0}^{\infty} d(x, x+1) < \infty,$$

see Theorem A.1 in Appendix A. Second, if $\partial_C X \ne \emptyset$, then

$$\mathrm{Cap}(\partial_C X) < \infty \quad \text{if and only if} \quad \mu(X) < \infty.$$

This can be seen as follows: if $\mu(X) = \infty$, then every neighborhood of the boundary must have infinite measure so that $\mathrm{Cap}(\partial_C X) = \infty$. If $\mu(X) < \infty$, then $1 \in D(Q^{\mathrm{max}})$ which implies that $\mathrm{Cap}(\partial_C X) \le \|1\|_Q = \mu(X) < \infty$. These two observations will be used repeatedly below.

Example 5.1 (Polar Cauchy boundary (and consequently $D(Q) = D(Q^{\max})$) but no essential self-adjointness)**.** Let $X = \mathbb{Z}$ with $w(x, y) = 1$ if $|x-y| = 1$ and 0 otherwise. The strongly intrinsic path metric $d = d_{\sigma_0}$ introduced in Example 2.1 satisfies $d(x, x+1) = \min\{\sqrt{\mu(x)/2}, \sqrt{\mu(x+1)/2}, 1\}$ for an arbitrary measure μ. Therefore, if the measure is chosen so that it satisfies $\sum_{x=-\infty}^{\infty} x^2 \sqrt{\mu(x)} < \infty$, then (X, d) is metrically incomplete, the Cauchy boundary consists of two points and $h : x \mapsto x^2$ is in $\ell^2(X, \mu)$ which we will use later.

Define e_n by

$$e_n(x) := (|x|/n - 1)_+ \wedge 1.$$

One checks that $e_n \in D(Q^{\max})$ with

$$\widetilde{Q}(e_n) = \sum_{x=-\infty}^{\infty} (e_n(x) - e_n(x+1))^2 = 2n\frac{1}{n^2} \to 0$$

and that $e_n \to 0$ in $L^2(X, \mu)$ as $n \to \infty$ by the Lebesgue dominated convergence theorem. Thus, the Cauchy boundary of X is polar and $L_c = \Delta|_{C_c(X)}$ has a unique Markovian extension.

On the other hand, the formal Laplacian Δ acts as $\Delta f(x) = \frac{1}{\mu(x)}(f(x) - f(x-1) + f(x) - f(x+1))$. Clearly, $h(x) = x$ is harmonic, square integrable by the choice of μ, and $h \notin D(Q^{\max})$. This shows that $h \in D(L_c^*) \setminus D(Q^{\max})$, that is, L_c is not essentially self-adjoint.

Let p_L and p_R be the two points forming $\partial_C X$ and let χ_L and χ_R be two cut-off functions in $D(Q^{\max})$ which equal 1 on a neighborhood of p_L and p_R, respectively. By restricting Δ to $\{a\chi_L + f : a \in \mathbb{R}, \ f \in D(L)\}$, $\{a\chi_R + f : a \in \mathbb{R}, \ f \in D(L)\}$, and $\{a\chi_L + f + b\chi_R : a, b \in \mathbb{R}, \ f \in D(L)\}$, we obtain a family of non-Markovian self-adjoint extensions of L_c.

Example 5.2 (Cauchy boundary with infinite capacity and essential self-adjointness)**.** Let $X = \mathbb{N}_0$ with $\mu(X) = \infty$ and w symmetric such that $w(x, y) > 0$ if and only if $|x - y| = 1$. By [30, Theorem 6] the operator $\Delta|_{C_c(X)}$ is essentially selfadjoint. (This can be also seen directly as there are no non-trivial solutions to $(\Delta + \lambda)u = 0$ in $L^2(X, \mu)$ for $\lambda > 0$. This follows as any positive solution to this equation must be increasing by a minimum principle, see also equation (4) below.)
If $d = d_{\sigma_0}$ and w satisfies

$$l(X) = \lim_{x \to \infty} d(0, x) \leq \sum_{x=0}^{\infty} \left(\frac{1}{\mathrm{Deg}(x)}\right)^{\frac{1}{2}} \leq \sum_{x=0}^{\infty} \left(\frac{\mu(x)}{w(x, x+1)}\right)^{\frac{1}{2}} < \infty,$$

then it follows that (X, d) is not metrically complete and the boundary consists of a single point. Since $\mu(X) = \infty$, $\mathrm{Cap}(\partial_C X) = \infty$ as noted above.

Example 5.3 (Cauchy boundary with finite positive capacity and consequently $D(Q) \neq D(Q^{\max})$)**.** Let $X = \mathbb{N}_0$ with $\mu(X) < \infty$ and let w be symmetric with $w(x, y) > 0$ if and only if $|x - y| = 1$ and satisfying

$$l(X) = \lim_{x \to \infty} d(0, x) \leq \sum_{x=0}^{\infty} \left(\frac{\mu(x)}{w(x, x+1)}\right)^{\frac{1}{2}} < \infty \quad \text{and} \quad \sum_{x=0}^{\infty} \frac{1}{w(x, x+1)} < \infty$$

where $d = d_{\sigma_0}$. In particular, the Cauchy boundary $\partial_C X$ of X consists of one point and has finite capacity.

Recall that $D(Q) \neq D(Q^{\max})$ is equivalent to $(\Delta + \lambda)u = 0$ having a non-trivial solution in $D(Q^{\max})$ for any $\lambda > 0$ [23, Corollary 3.3]. By [32, Lemma 4.2], the equation $(\Delta + \lambda)u = 0$ on X translates to

$$u(x + 1) - u(x) = \frac{\lambda}{w(x, x+1)} \sum_{y=0}^{x} u(y)\mu(y) \tag{4}$$

from which it follows, see [32, Lemma 5.4], that u is bounded if and only if

$$\sum_{x=0}^{\infty} \frac{\sum_{y=0}^{x} \mu(y)}{w(x, x+1)} < \infty.$$

As $\mu(X) < \infty$, this is equivalent to

$$\sum_{x=0}^{\infty} \frac{1}{w(x, x+1)} < \infty.$$

Furthermore, as $\mu(X) < \infty$, $u \in L^{\infty}(X)$ implies that $u \in L^2(X, \mu)$. It is also not difficult to see that $\widetilde{Q}(u) < \infty$ in this case as, by (4), we get that

$$w(x, x+1)(u(x+1) - u(x))^2 \le \frac{1}{w(x, x+1)} \left(\lambda \mu(X) \|u\|_{\infty} \right)^2.$$

Therefore, $D(Q) \neq D(Q^{\max})$. Finally, $\mathrm{Cap}(\partial_C X) > 0$ follows by combining $\mathrm{Cap}(\partial_C X) < \infty$, $D(Q) \neq D(Q^{\max})$, and Theorem 3.

Example 5.4 (Cauchy boundary with infinite capacity and $D(Q) \neq D(Q^{\max})$ (and consequently no essential self-adjointness)). We consider $X = \mathbb{Z}$ with $X = X_- \cup X_+$ where $X_- = -\mathbb{N}_0$ is taken as in Example 5.2 and $X_+ = \mathbb{N}_0$ is taken as in Example 5.3. In particular, the Cauchy boundary $\partial_C X$ of X consists of two points, p_L and p_R, and has infinite capacity as $\mathrm{Cap}(p_L) = \infty$ by $\mu(X_-) = \infty$. On the other hand, by Example 5.3 we have $0 < \mathrm{Cap}(p_R) < \infty$ which gives $D(Q) \neq D(Q^{\max})$ by Lemma 4.2.

Example 5.5 (Polar Cauchy boundary with upper Minkowski codimension 2). Let $X = \mathbb{N}_0$ with $w(x, y) = 1/8$ if $|x-y| = 1$ and 0 otherwise and $\mu(x) = 4^{-x}$. Therefore, for $x > 0$, $\mathrm{Deg}(x) = 4^{x-1}$ so that, with $d = d_{\sigma_0}$, we get $d(x, x+1) = 2^{-x}$. Furthermore, by using the technique of Example 5.1, we can show that the Cauchy boundary consists of a single point, p_R, and that $\mathrm{Cap}(p_R) = 0$. Let $r(x) := \overline{d}(x, p_R) = \sum_{y=x}^{\infty} 2^{-y} = 2^{-(x-1)}$ so that $\mu(B_{r(x)}(p_R)) = \sum_{y=x}^{\infty} 4^{-y} = 4^{-(x-1)}/3 = r(x)^2/3$. Therefore,

$$\frac{\ln \mu(B_{r(x)}(p_R))}{\ln r(x)} = \frac{2 \ln r(x) - \ln 3}{\ln r(x)} \to 2 \quad \text{as } x \to \infty.$$

Example 5.6 (Non-polar Cauchy boundary with upper Minkowski codimension 2). Let $X = \mathbb{N}_0$, with w symmetric, satisfying $w(x, x+1) = (x+1)^2$ and $w(x, y) = 0$ if $|x - y| \ge 2$ with

$$d(x, x+1) = \frac{1}{2^{x+2}} \qquad \text{and} \qquad \mu(x) = \frac{(x+1)^2}{4^x}.$$

It is easy to check that this metric is intrinsic. As $l(X) < \infty$, $\mu(X) < \infty$, and $\sum_{x=0}^{\infty} \frac{1}{w(x,x+1)} < \infty$ it follows that

$$0 < \mathrm{Cap}(\partial_C X) < \infty$$

by the reasoning of Example 5.3. Therefore, $\overline{\mathrm{codim}}_M(\partial_C X) \le 2$ by Theorem 4. By definition,

$$r(x) := \overline{d}(x, \partial_C X) = \frac{1}{2^{x+1}} \quad \text{and} \quad \mu(B_{r(x)}(\partial_C X)) = \sum_{y=x}^{\infty} \frac{(y+1)^2}{4^y}.$$

Now, for every $\beta > 1/4$, there exists an M such that $(x+1)^2 \le (4\beta)^x$ for all $x \ge M$. Hence, for all $x \ge M$ and $1/4 < \beta < 1$, we have $\mu(B_{r(x)}(\partial_C X)) \le \sum_{y=x}^{\infty} \beta^y = \beta^x (1 - \beta)^{-1}$. Therefore, for all $x \ge M$,

$$\frac{\ln \mu(B_{r(x)}(\partial_C X))}{\ln r(x)} \ge \frac{\ln \left(\beta^x (1 - \beta)^{-1} \right)}{\ln 2^{-(x+1)}}$$

which implies that $\overline{\mathrm{codim}}_M(\partial_C X) \ge -\ln \beta / \ln 2$. As $\beta > 1/4$ was chosen arbitrarily, it follows that $\overline{\mathrm{codim}}_M(\partial_C X) \ge 2$ yielding that $\overline{\mathrm{codim}}_M(\partial_C X) = 2$.

Example 5.7 (Upper Minkowski codimension between 0 and 2 with polar and non-polar Cauchy boundary). Let $X = \mathbb{N}_0$ with $w(x, y) > 0$ if and only if $|x - y| = 1$. Let

$$d(x, x+1) = \frac{1}{2^{\alpha(x+1)}} \qquad \text{and} \qquad \mu(x) = \frac{1}{2^{(2\alpha-1)x}}.$$

Then, $l(X) < \infty$ for $\alpha > 0$ and $\mu(X) < \infty$ for $\alpha > 1/2$. Thus, $\mathrm{Cap}(\partial_C X) < \infty$ for $\alpha > 1/2$. Now,

$$r(x) := \overline{d}(x, \partial_C X) = \sum_{y=x}^{\infty} \frac{1}{2^{\alpha(y+1)}} = \left(\frac{1}{2^{\alpha} - 1} \right) \frac{1}{2^{\alpha x}}$$

and

$$\mu\left(B_{r(x)}(\partial_C X)\right) = \sum_{y=x}^{\infty} \frac{1}{2^{(2\alpha-1)y}} = \left(\frac{1}{2^{2\alpha-1}-1}\right)\frac{1}{2^{(2\alpha-1)(x-1)}}$$

so that

$$\overline{\operatorname{codim}}_M(\partial_C X) = 2 - \frac{1}{\alpha}.$$

We now specify two choices of weights w:

Case 1 - polar Cauchy boundary: Let $w(x, x+1) = 1$ for all $x \in \mathbb{N}_0$. Clearly, d is intrinsic for all $\alpha > 0$. Furthermore, if $\operatorname{Cap}(\partial_C X) < \infty$, then $\operatorname{Cap}(\partial_C X) = 0$ as in Example 5.1. Hence, there exist examples of graphs with polar Cauchy boundary such that $0 < \overline{\operatorname{codim}}_M(\partial_C X) < 2$.

Case 2 - non-polar Cauchy boundary: Let $w(x, x+1) = 2^x$ for all $x \in \mathbb{N}_0$. It is easy to see that d is intrinsic for all $\alpha \geq \frac{1}{2}$. Furthermore, since $\sum_{x=0}^{\infty} \frac{1}{w(x,x+1)} < \infty$ one can show that $\operatorname{Cap}(\partial_C X) > 0$ as in Example 5.4. Consequently, there exists a family of graphs with non-polar Cauchy boundary such that $0 < \overline{\operatorname{codim}}_M(\partial_C X) < 2$.

APPENDIX A. A HOPF-RINOW TYPE THEOREM

Let (X, w, μ) be a weighted graph. Recall that a graph is called *locally finite* if for all $x \in X$ the number of $y \in X$ with $w(x, y) > 0$ is finite.

A metric space (X, d) is said to be *metrically complete* if every Cauchy sequence converges to a point in X. A path (x_n) (finite or infinite) is called a *geodesic* with respect to the path metric $d = d_\sigma$ if $d(x_0, x_n) = l_\sigma((x_0, \ldots, x_n))$ for all $n \geq 0$. A weighted graph (X, w, μ) with a path metric $d = d_\sigma$ is said to be *geodesically complete* if all infinite geodesics have infinite lengths, i.e., $l_\sigma((x_n)) = \lim_k l_\sigma((x_0, \ldots, x_k)) = \infty$ for all infinite geodesics (x_n).

We prove the following Hopf-Rinow type theorem.

Theorem A.1. *Let (X, w, μ) be a locally finite weighted graph and d be a path pseudo metric. Then, (X, d) is a discrete metric space. Moreover, the following are equivalent:*

(i) *(X, d) is metrically complete.*
(ii) *(X, d) is geodesically complete.*
(iii) *Every distance ball is finite.*
(iv) *Every bounded and closed set is compact.*

In particular, if (X, d) is complete, then for all $x, y \in X$ there is a path $(x_0, \ldots, x_n)$ connecting x and y such that $d_\sigma(x, y) = l_\sigma((x_0, \ldots, x_n))$.

Remark A.2. (a) Anytime a path pseudo metric d induces the discrete topology on X the following implications hold: (iii)$\Leftrightarrow$(iv)$\Rightarrow$(i)$\Rightarrow$(ii). This will be the case if and only if $\inf_{y \sim x} \sigma(x, y) > 0$ for all $x \in X$. In fact, (iv)$\Rightarrow$(i) holds for general metric spaces. The stronger assumption of local finiteness is needed for the implications (ii)$\Rightarrow$(i), (i)$\Rightarrow$(iii) (or (iv)) and (ii)$\Rightarrow$(iii) (or (iv)). See Example A.5 below.

(b) A similar statement as (i)$\Rightarrow$(iii) can also be found in [39].

We prove the theorem in several steps through the following lemmas.

Lemma A.3. *Let (X, w, μ) be a locally finite weighted graph and d be a path pseudo metric. Then, the following hold:*

(a) *(X, d) is a discrete metric space. In particular, it is locally compact.*
(b) *A set is compact in (X, d) if and only if it is finite.*

Proof. The local finiteness implies that for all $x \in X$ there is an $r > 0$ such that $d(x, y) > r$ for all $y \in X$ with $y \sim x$. First, by the definition of d, we have that for all $x, z \in X$ there is $y \sim x$ with $d(x, y) \leq d(x, z)$. Thus, $d(x, y) = 0$ implies $x = y$, therefore, d is a metric. Second, it yields that $B_r(x) = \{x\}$ and $\{x\}$ is an open set which shows (a). From this we conclude that for any infinite set U the cover $\{\{x\} \mid x \in U\}$ has no finite subcover. The other direction of (b) is clear. □

The authors are grateful to Florentin Münch for a crucial idea in the proof of the following lemma.

Lemma A.4. *Let (X, w, μ) be a locally finite weighted graph and d be a path metric. Assume that B_r is infinite for some $r \geq 0$. Then, there exists an infinite geodesic of bounded length.*

Proof. Let $o \in X$ be the center of the infinite ball B of radius r and let d_N be the natural graph distance. Let P_n, $n \geq 0$, be the set of finite paths $(x_0, \ldots, x_k)$ such that $x_0 = o$, $x_i \neq x_j$ for $i \neq j$, $d_N(x_k, o) = n$ and $d_N(x_j, o) \leq n$ for $j = 0, \ldots, k$.

Claim: $\Gamma_n = \{\gamma \in P_n \mid \gamma \text{ geodesic with respect to } d, l(\gamma) \leq r\} \neq \emptyset$ for all $n \geq 0$.

Proof of the claim: The set P_n is finite by local finiteness of the graph and, thus, contains a minimal element $\gamma = (x_0, \ldots, x_K)$ with respect to the length l, i.e., for all $\gamma' \in P_n$ we have $l(\gamma') \geq l(\gamma)$. Then, γ is a geodesic: for every path $(x_0', \ldots, x_M')$ with $x_0' = o$ and $x_M' = x_K$, we let $m \in \{n, \ldots, M\}$ be such that $(x_0', \ldots, x_m') \in P_n$. By the minimality of γ we infer

$$l((x_0', \ldots, x_M')) \geq l((x_0', \ldots, x_m')) \geq l(\gamma).$$

It follows that γ is a geodesic. Clearly, $l(\gamma) \leq r$, as otherwise $B \subseteq \{y \in X \mid d_N(y, o) \leq n - 1\}$ which would imply the finiteness of B by the local finiteness of the path space. Thus, $\gamma \in \Gamma_n$ which proves the claim.

We inductively construct an infinite geodesic (x_k) with bounded length: We set $x_0 = o$. Since $\Gamma_n \neq \emptyset$, there is a geodesic in Γ_n for every $n \geq 0$ such that x_0 is a subgeodesic. Suppose we have constructed a geodesic $(x_1, \ldots, x_k)$ such that for all $n \geq k$ there is a geodesic in Γ_n that has $(x_1, \ldots, x_k)$ as a subgeodesic. By local finiteness x_k has finitely many neighbors. Thus, there must be a neighbor x_{k+1} of x_k such that for infinitely many n the path $(x_0, \ldots, x_k, x_{k+1})$ is a subpath of a geodesic in Γ_n. However, a subpath of geodesic is a geodesic. Thus, there is an infinite geodesic $\gamma = (x_n)_{n \geq 0}$ with $l(\gamma) = \lim_{n \to \infty} l((x_0, \ldots, x_n)) \leq r$ as $(x_0, \ldots, x_n) \in \Gamma_n$ for all $n \geq 0$. $\square$

Proof of Theorem A.1. The fact that (X, d) is a discrete metric space follows from Lemma A.3. We now turn to the proof of the equivalences. We start with (i)$\Rightarrow$(ii). If there is a bounded geodesic, then it is a Cauchy sequence. Since a geodesic is a path it is not eventually constant, thus it does not converge by discreteness. Hence, (X, d) is not metrically complete. To prove (ii)$\Rightarrow$(iii) suppose that there is a distance ball that is infinite. By Lemma A.4 there is a bounded infinite geodesic and (X, d) is not geodesically complete. From Lemma A.3 (b) we deduce (iii)$\Leftrightarrow$(iv). Finally, we consider the direction (iv)$\Rightarrow$(i). If every bounded and closed set is compact, then every closed distance ball is compact. Then, by Lemma A.3 (b) every distance ball is finite and it follows that (X, d) is metrically complete. $\square$

We finish this appendix by giving several (counter-)examples to show that some of the statements above fail to be true in the case of non-locally finite graphs. We present the examples with respect to the path metric with $\sigma = \sigma_0$ (see Example 2.1). Another example of this type can be found in [11, Example 14.1].

Example A.5. Let $\mu \equiv 1$, $\sigma = \sigma_0$, and $d = d_\sigma$.

(1) *A metrically and geodesically complete graph with non compact distance balls.*
This example can be thought of as a star graph where the rays are two subsequent edges. Let $X = \mathbb{N}_0$ and let w be symmetric with $w(0, 2n) = 1/2^n$ and $w(2n - 1, 2n) = 1 - 1/2^n$ for $n \in \mathbb{N}$ and $w \equiv 0$ otherwise. We have $d(0, 2n) = 1$ for $n \in \mathbb{N}$. Then, (X, d) is metrically (and geodesically) complete but $B_1(0)$ is not compact.

(2) *A non locally compact graph.*
This example can be thought of as a star graph where the rays are copies of $\mathbb{N}$ whose lengths become shorter. Let $X = \mathbb{N}_0^2$ and let w be symmetric with $w((0, 0), (m, 0)) = 1/2^m$ for $m \in \mathbb{N}$ and $w((m, n - 1), (m, n)) = 2^{2(m+n)}/5$ for $m, n \in \mathbb{N}$ and $w \equiv 0$ otherwise. Then, $\mathrm{Deg}((0, 0)) = 1$, $\mathrm{Deg}((m, 0)) = 1/2^m + 2^{2(m+1)}/5$ and $\mathrm{Deg}((m, n)) = 2^{2(m+n)}$ for $m, n \in \mathbb{N}$. Hence, $d((m, n - 1), (m, n)) = 2^{-(m+n)}$ for $m, n \in \mathbb{N}$ and $1/2^{m+1} \leq d((0, 0), (m, n)) \leq 3/2^{m+1}$. Let $B_\varepsilon(0, 0)$ be a ball about $(0, 0)$ and let $U_\varepsilon(0, 0)$ denote its interior. Then $\{U_{\varepsilon/2}(0, 0)\} \cup \{U_{1/2^{(m+n+1)}}(m, n) \mid m \geq 1, n \geq 0\}$ is an open cover with no finite subcover.

(3) *A non Hausdorff space.*
This example can be thought as two vertices which are connected by infinitely many paths that become shorter. Let $X = \mathbb{N}_0 \cup \{\infty\}$ and let w be symmetric with $w(0, 2n) = w(\infty, 2n) = 1/2^n$

and $w(2n-1, 2n) = 2^{2n}$ and $w(n, m) = 0$ all other $m, n \in \mathbb{N}_0$. Then, $\sigma(0, 2n) = \sigma(\infty, 2n) \leq 1/2^n$. Hence, $d(0, \infty) = 0$.

(4) *An infinite ball and non discreteness.*

This example is a modification of (1). Let $X = \mathbb{N}_0$ and let w be symmetric with $w(0, 2n) = 1/2^n$ and $w(2n-1, 2n) = 2^n$ and $w(n, m) = 0$ all other $m, n \in \mathbb{N}_0$. Then, every d-ball about 0 is compact but it contains infinitely many vertices. Moreover, the vertices $x_n = 2n$ converge to $x = 0$ with respect to d. (This is, in particular, a counterexample to Lemma A.3).

(5) *A geodesically complete graph which is not metrically complete.*

This example is an extension of (3) and can be thought as a "line graph" where between each two points on the line there are infinitely many "line segments" that become shorter. Let $X = \mathbb{N}_0^2$ and let w be symmetric with $w((m, 0), (m, 2n)) = 1/2^n = w((m+1, 0), (m, 2n))$ and $w((m, 2n), (m, 2n-1)) = 2^{2(m+1)} - 3/2^n$ for $m \in \mathbb{N}_0, n \in \mathbb{N}$ and $w \equiv 0$ otherwise. It follows that $\mathrm{Deg}((m, 2n)) = 2^{2(m+1)} - 1/2^n$ implying that $d((m, 0), (m+1, 0)) = 1/2^m$. Thus $(x_m) = ((m, 0))$ is a Cauchy sequence which does not converge. On the other hand, the space is geodesically complete as there are no infinite geodesics.

References

[1] J. Cheeger, On the Hodge theory of Riemannian pseudomanifolds, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146.

[2] P. R. Chernoff, Essential self-adjointness of powers of generators of hyperbolic equations, J. Funct. Anal. **12** (1973), 401–414.

[3] Y. Colin de Verdière, Pseudo-laplaciens. I, Ann. Inst. Fourier, **32** (1982), no. 3, 275–286.

[4] Y. Colin de Verdière, N. Torki-Hamza, F. Truc, Essential self-adjointness for combinatorial Schrödinger operators II-Metrically non complete graphs, Math. Phys. Anal. Geom. **14** (2011), no. 1, 21–38.

[5] Y. Colin de Verdière, N. Torki-Hamza, F. Truc, Essential self-adjointness for combinatorial Schrödinger operators III-Magnetic fields, Ann. Fac. Sci. Toulouse Math. (6), **20**, no. 3 (2011), 599–611.

[6] J. Dodziuk, Elliptic operators on infinite graphs, Analysis, geometry and topology of elliptic operators, World Sci. Publ., Hackensack, NJ, 2006, pp. 353–368.

[7] K. Falconer, Fractal geometry, 2nd ed., John Wiley & Sons Inc., Hoboken, NJ, 2003. Mathematics foundations and applications.

[8] M. Folz, Gaussian upper bounds for heat kernels of continuous time simple random walks, Elec. J. Prob. **16** (2011), 1693–1722.

[9] M. Folz, Volume growth and stochastic completeness of graphs preprint, 2012, arXiv:1201.5908.

[10] M. Folz, Volume growth and spectrum for general graph Laplacians, preprint, 2012 arXiv:1204.4770.

[11] R. L. Frank, D. Lenz, D. Wingert, Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory, to appear in: J. Funct. Anal.

[12] M. Fukushima, Regular representations of Dirichlet spaces, Trans. Amer. Math. Soc. **155** (1971), 455–473.

[13] M. Fukushima, Y. Oshima, M. Takeda, Dirichlet forms and symmetric Markov processes, de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011.

[14] M. P. Gaffney, The harmonic operator for exterior differential forms, Proc. Nat. Acad. Sci. U. S. A. **37** (1951) 48–50.

[15] M. P. Gaffney, A special Stokes's theorem for complete Riemannian manifolds, Ann. of Math. (2) **60** (1954), 140–145.

[16] S. Golénia, Unboundedness of adjacency matrices of locally finite graphs, Lett. Math. Phys. **93** (2010), no. 2, 127–140.

[17] S. Golénia, Hardy inequality and Weyl asymptotic for discrete Laplacians, preprint, 2011, arXiv:1106.0658.

[18] S. Golénia, C. Schumacher, The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs, J. Math. Phys. **52** (2011), no. 6, 063512, 17 pp.

[19] A. Grigor'yan, Heat kernel and analysis on manifolds, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society, Providence, RI, 2009.

[20] A. Grigor'yan, X. Huang, J. Masamune, On stochastic completeness of jump processes, Math. Z. **271** (2012), no. 3-4, 1211–1239.

[21] A. Grigor'yan, J. Masamune, Parabolicity and stochastic completeness of manifolds in terms of the Green formula, preprint, 2012, 29 pages.

[22] S. Haeseler, M. Keller, Generalized solutions and spectrum for Dirichlet forms on graphs, Random Walks, Boundaries and Spectra, Progress in Probability, vol. 64, 2011 Birkhäuser, 181–201.

[23] S. Haeseler, M. Keller, D. Lenz, R. Wojciechowski, Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions, J. Spectr. Theory **2** (2012), no. 4, 397–432.

[24] S. Haeseler, M. Keller, R. K. Wojciechowski, Volume growth and bounds for the essential spectrum for Dirichlet forms, preprint, 2012, arXiv:1205.4985.

[25] X. Huang, On stochastic completeness of weighted graphs, Ph.D. thesis, 2011.

[26] X. Huang, On uniqueness class for a heat equation on graphs, preprint, 2011.

[27] P. E. T. Jorgensen, Essential self-adjointness of the graph-Laplacian, J. Math. Phys. **49** (2008), no. 7, 073510, 33 pp.

[28] P. E. T. Jorgensen, E. P. J. Pearse, Spectral reciprocity and matrix representations of unbounded operators, J. Funct. Anal. **261** (2011), no. 3, 749–776.

[29] T. Kawabata, M. Takeda, On uniqueness problem for local Dirichlet forms, Osaka J. Math. **33** (1996), no. 4, 881–893.

[30] M. Keller, D. Lenz, Dirichlet forms and stochastic completeness of graphs and subgraphs, J. Reine Angew. Math. **666** (2012), 189–223.

[31] M. Keller, D. Lenz, Unbounded Laplacians on graphs: basic spectral properties and the heat equation, Math. Model. Nat. Phenom. **5** (2010), no. 4, 198–224.

[32] M. Keller, D. Lenz, R. K. Wojciechowski, Volume growth, spectrum and stochastic completeness of infinite graphs, preprint, 2011, arXiv:1105.0395.

[33] K. Kuwae, Y. Shiozawa, A remark on the uniqueness of Silverstein extensions of symmetric Dirichlet forms, preprint, 2012.

[34] K. Kuwae, T. Shioya, Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry, Comm. Anal. Geom. **11** (2003), 599–673.

[35] P. Li, G. Tian, On the heat kernel of the Bergmann metric on algebraic varieties, J. Amer. Math. Soc. **8** (1995), no. 4, 857–877.

[36] J. Masamune, Essential self-adjointness of Laplacians on Riemannian manifolds with fractal boundary, Comm. Partial Differential Equations **24** (1999), no. 3-4, 749–757.

[37] J. Masamune, A Liouville property and its application to the Laplacian of an infinite graph, Spectral analysis in geometry and number theory, Contemp. Math., vol. 484, Amer. Math. Soc., Providence, RI, 2009, pp. 103–115.

[38] J. Masamune, T. Uemura, Conservation property of symmetric jump processes, Ann. Inst. Henri. Poincare Probab. Statist. 47 (2011), no. 3, 650–662.

[39] O. Milatovic, Essential self-adjointness of discrete magnetic Schrödinger operators on locally finite graphs, Integr. Equ. Oper. Theory **71** (2011), 13–27.

[40] O. Milatovic, A Sears-type self-adjointness result for discrete magnetic Schrödinger operators, to appear in: J. Math. Anal. Appl.

[41] R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, J. Funct. Anal. **52** (1983), no. 1, 48–79.

[42] N. Torki-Hamza, Laplaciens de graphes infinis I-Graphes métriquement complets, Confluentes Math. **2** (2010), no. 3, 333–350.

[43] A. Weber, Analysis of the physical Laplacian and the heat flow on a locally finite graph, J. Math. Anal. Appl. **370** (2010), no. 1, 146–158.

[44] R. K. Wojciechowski, Stochastic completeness of graphs, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)– City University of New York.

[45] R. K. Wojciechowski, Heat kernel and essential spectrum of infinite graphs, Indiana Univ. Math. J. **58** (2009), no. 3, 1419–1441.

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