STOCHASTIC EVOLUTION OF A CONTINUUM PARTICLE SYSTEM WITH DISPERSAL AND COMPETITION: MICRO- AND MESOSCOPIC DESCRIPTION

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ABSTRACT. A Markov evolution of a system of point particles in $\mathbb{R}^d$ is described at micro-and mesoscopic levels. The particles reproduce themselves at distant points (dispersal) and die, independently and under the influence of each other (competition). The microscopic description is based on an infinite chain of equations for correlation functions, similar to the BBGKY hierarchy used in the Hamiltonian dynamics of continuum particle systems. The mesoscopic description is based on a Vlasov-type kinetic equation for the particle's density obtained from the mentioned chain via a scaling procedure. The main conclusion of the microscopic theory is that the competition can prevent the system from clustering, which makes its description in terms of densities reasonable. A possible homogenization of the solutions to the kinetic equation in the long-time limit is also discussed.

1. INTRODUCTION

Phenomenological physical theories of substances like gases, liquids, etc, are based on macroscopic observations and thus consider these substences as continuous media. However, already in the time of L. Boltzmann and M. Smoluchowski it was understood the importance of the microscopic structure of such and other similar substances. Since then a challenging problem of theoretical and mathematical physics has been to derive the rules of the collective behavior of large systems of interacting particles from their microscopic theory, based on the so called 'first principles'. Achievements in this direction laid the fundamentals of modern statistical physics; see, e.g., [10]. In the Hamiltonian mechanics, the motion of N physical particles in $\mathbb{R}^d$ is described by a system of $2dN$ differential equations, subject to initial conditions. For $N \gg 1$, the abundance of equations, and hence of the initial conditions, makes the point-wise description meaningless since no observation could indicate at which point of the phase space the system actually is. Moreover, the point-wise description would be 'too detailed' for understanding the collective behavior of the system. An alternative can be the statistical approach which provides the possibility to link micro- and macroscopic descriptions to each other. In this approach, one deals with probabilities with which such points lie in given subsets of the phase space. Then instead of 'point-wise evolution' one studies the evolution of probability measures, considered now as the states of the system. In [1], N. N. Bogoliubov suggested another approach, which later became popular, especially in the physical literature. Here, the evolution of probability measures is described indirectly as the evolution of the so called *moment* or *correlation* functions. The latter evolution is obtained from a hierarchy of equations, called now *Bogoliubov*

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or *BBGKY hierarchy* or *chain*, that couples correlation functions of different order with each other, see [10]. The description at this level is *microscopic* since one deals with coordinates of individual particles; cf. the Introduction in [18]. More coarse-grained levels are *meso- and macroscopic* ones. They are attained by choosing appropriate scales for space and time, see [18, 19]. Of course, certain details of the system's behavior are lost due to such coarse-graining. Kinetic equations provide a space-dependent mean-field-like approximate[1] description of the evolution of large particle systems. Such equations are deduced from the BBGKY chain by means of various types of scaling, cf. Section 6 in [10] and also [18, 19].

Along with physics, in modern life sciences one also deals with systems consisting of large number of interacting entities distributed in a continuous habitat and evolving in time. Their collective behavior is observed at a macro-scale, and thus the corresponding mathematical theories which explain this behavior traditionally describe their dynamics by means of phenomenologically (or heuristically) deduced nonlinear equations, mostly differential or integro-differential, involving macroscopic characteristics like density, mobility, etc. However, this kind of macroscopic phenomenology may often be insufficient also in life sciences, as they can collect massive experimental data of high precision characterizing individual behavior of constituting entities. For instance, technical advances in animal tracking and genotyping allow for obtaining rich individual-level data on population dynamics, population genetics, and evolutionary biology. However, mathematical methods are absent that would enable one to derive the population-level consequences of such individual-level observations. From this and many other examples one may see how important is to elaborate a general mathematical theory in which the rules of the collective behavior of a large number of interacting entities can be deduced from the information about their individual behavior and the interactions. This article is aimed at announcing some of our recent achievements in this field based on the concepts of statistical physics mentioned above. The main principles which we follow in our study are:

- The objects are large systems of entities distributed over a continuous habitat (part of $\mathbb{R}^d$, $d \geq 1$) and evolving in continuous time.
- The phase space of the model is the collection Γ of possibly infinite sets of identical points – configurations $\gamma \subset \mathbb{R}^d$.
- In the microscopic theory, the states are probability measures on the phase space Γ, the evolution of which is assumed to be Markovian, described by Kolmogorov-type equations. The main acts of the evolution are particle's birth, death, and spatial motion – jumps or diffusion.
- The evolution of states is described indirectly as the evolution of the corresponding correlation functions obeying equations similar to BBGKY chains.
- The mesoscopic theory is based on kinetic equations, deduced from the equations for correlation functions by means of scaling procedures, in which one scales interactions and densities, whereas time is left unscaled.

2. THE SETUP

The object of our study in this paper is a large population of motionless entities (e.g., perennial plants) which reproduce themselves, compete and die at random.

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[1]In the physical literature, it is rather called *random phase approximation.*

The reproduction (dispersal) consists in random independent sending by an entity an offspring to a distant point, which immediately after that becomes a member of the population. Each entity can die independently with a constant rate, as well as under the influence of other population members. The latter event is interpreted as competition-caused. In theoretical biology such models were introduced and studied, e.g., in [2, 3, 16]. A more detailed bibliography can be found in [6].

As was suggested already in [2], the right mathematical context for studying models of this kind is the theory of random point fields in $\mathbb{R}^d$; cf. also page 1311 in [15]. Herein, populations appear as particle configurations constituting the set

$$
(1) \qquad \Gamma := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any bounded } K \subset \mathbb{R}^d\},
$$

where, for a set A, by $|A|$ we denote the number of points in A. For our model, Γ is a phase space. Its metric properties, as well as the main aspects of the analysis on such spaces, were studied in [11]. Along with finite ones the set Γ contains also infinite configurations, which allows for describing 'bulk' properties of a large finite system ignoring boundary and size effects[2]. If the initial configuration γ_0 is fixed, the evolution might be described as a map $t \mapsto \gamma_t \in \Gamma$, which in view of the random character of the events mentioned above ought to be a random process. However, at least so far, for the model considered here this way can be realized only if γ_0 is finite. Hence, to describe mentioned bulk properties one ought to go beyond the point-wise setting, as it has been done in the statistical physics of interacting particle systems. As mentioned above, in the statistical description states of the system are probability measures on Γ. To characterize them one employs *observables* – appropriate functions $F : \Gamma \to \mathbb{R}$. As in the Heisenberg approach in quantum physics, the evolution of states can be described via the evolution of observables. In view of the assumed Markov property, this evolution is obtained from the equation

$$
(2) \qquad \frac{d}{dt} F_t = L F_t, \qquad F_t|_{t=0} = F_0, \qquad t \geq 0,
$$

in which the operator L determines the model. In our case, it is

$$
(3) \qquad (LF)(\gamma) = \sum_{x \in \gamma} \left[m + E^-(x, \gamma \setminus x) \right] \left[F(\gamma \setminus x) - F(\gamma) \right]
$$

$$
+ \int_{\mathbb{R}^d} E^+(y, \gamma) \left[F(\gamma \cup y) - F(\gamma) \right] dy,
$$

where

$$
(4) \qquad E^\pm(x, \gamma) := \sum_{y \in \gamma} a_\pm(x - y).
$$

Here $a_\pm \geq 0$ are suitable functions, see (10) below. As given in (3), L is a typical 'birth-and-death' generator in which the first term corresponds to the death of the particle located at x occurring (a) independently with rate $m \geq 0$, and (b) under the influence of the other particles in γ with rate $E^-(x, \gamma \setminus x) \geq 0$. Here and in the sequel in the corresponding context, we treat each $x \in \mathbb{R}^d$ also as a single-point configuration $\{x\}$. That is, if x belongs to γ (resp. y does not), by $\gamma \setminus x$ (resp. $\gamma \cup y$) we mean the configuration which is obtained from γ by removing x (resp. by adding y). The second term in (3) describes the birth of a particle at $y \in \mathbb{R}^d$ given

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[2] A discussion on how infinite systems provide approximations for large finite systems can be found in [4, 5].

by the whole configuration γ with rate $E^+(y, \gamma) \geq 0$. The model described by (3) is called a *spatial logistic model (SLM)*. A particular case of SLM is the continuous contact model [12, 13] where $a_- \equiv 0$, and hence the competition is absent.

The sums in (3) and (4) may contain infinite number of summands and hence may diverge. This means that the direct study of the corresponding equation (2) for our model is rather problematic. Following the way suggested by N. N. Bogoliubov, the evolution of states will be studied via the evolution of the corresponding correlation functions $k_0 \mapsto k_t$. By a correlation function k we mean an infinite collection of symmetric positive functions $k^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$, $n \in \mathbb{N}_0$, which determine the state in a certain way, see [6, 7, 8, 9] for more detail. Such k can also be viewed as infinite-dimensional (Fock-type) vectors. Note that $k^{(0)} = 1$ and $k^{(1)}$ is the particle density. For the Poisson measure with intensity $\varkappa > 0$, one has $k^{(n)}(x_1, \ldots, x_n) = \varkappa^n$, $n \in \mathbb{N}_0$. We say that a correlation function k, and hence the corresponding state, are *sub-Poissonian* if

$$(5) \qquad k^{(n)}(x_1, \ldots, x_n) \leq C^n,$$

which holds for some $C > 0$, all $n \in \mathbb{N}_0$, and (Lebesgue) almost all $(x_1, \ldots, x_n)$. Then a sub-Poissonian state is, in a way, similar to the Poissonian state in which the particles are independently scattered over $\mathbb{R}^d$. At the same time, the increase of $k^{(n)}$ with n as $n!$ (see (9) below), may correspond to the formation of *clusters*. The evolution $k_0 \mapsto k_t$ is obtained from the equation

$$(6) \qquad \frac{d}{dt} k_t = L^\Delta k_t, \qquad k_t|_{t=0} = k_0,$$

which, in fact, is a chain of equations for particular $k_t^{(n)}$, analogous to the Bogoliubov hierarchy for physical particles. The generator L^Δ is constructed from L as in (3) by a certain procedure, see [7, 9]. In particular, the first equations in the chain (6) have the following form: $dk_t^{(0)}/dt = 0$ and

$$(7) \quad \frac{d}{dt} k_t^{(1)}(x) = -m k_t^{(1)}(x) - \int_{\mathbb{R}^d} a_-(x-y) k_t^{(2)}(x,y) dy + \int_{\mathbb{R}^d} a_+(x-y) k_t^{(1)}(y) dy.$$

The RHS of the equation with $dk_t^{(2)}/dt$ contains $k_t^{(n)}$ with $n = 1, 2, 3$, etc. Theoretical biologists try to solve chains like (6) by decoupling them; cf. [14]. In the simplest version, one sets

$$k^{(2)}(x, y) \simeq k^{(1)}(x) k^{(1)}(y),$$

which is equivalent to neglecting spatial pair correlations. This turns (7) into the following nonlinear equation

$$(8)$$
$$\frac{d}{dt} k_t^{(1)}(x) = -m k_t^{(1)}(x) - k_t^{(1)}(x) \int_{\mathbb{R}^d} a_-(x-y) k_t^{(1)}(y) dy + \int_{\mathbb{R}^d} a_+(x-y) k_t^{(1)}(y) dy,$$

which, in fact, is a kinetic equation for our model. Its eventual solution provides a mean-field-like approximation of the evolution of the density. Note that the question of whether the evolution of states preserves sub-Poissonicity, and hence clusters do not occur, can be answered only by studying the whole chain in (6). For the contact model, which one obtains from (3) by setting $a_- \equiv 0$, it is known [7] that

$$(9) \qquad \text{const} \cdot n! \, c_t^n \leq k_t^{(n)}(x_1, \ldots, x_n) \leq \text{const} \cdot n! \, C_t^n,$$

where the left-hand inequality holds if all x_i belong to a ball of small enough radius. Hence, in spite of the fact that $C_t \to 0$ as $t \to +\infty$ if the mortality dominates the dispersal, k_t are definitely not sub-Poissonian if $a_- \equiv 0$.

In this article we address the following questions:

(1) Can the competition in (3) provide that the problem (6) has a solution that obeys (5)?
(2) Can one deduce (8) rigorously; e.g., by a kind of scaling procedure, which would allow for error control?
(3) What can be said of solutions to (8) in that case?

The answers are given in the form of theorems stated below. The proof of Theorem 1 can be found in [6]; the remaining theorems will be proven in our work under preparation.

3. THE RESULTS

We assume that the competition and dispersal rates are additive, cf. (4), and the corresponding *kernels* $a_\pm$ are symmetric, essentially bounded, and integrable. That is,

$$(10) \qquad a_\pm \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \qquad a_\pm(x) = a_\pm(-x) \geq 0,$$

and thus we set

$$(11) \qquad \langle a_\pm \rangle = \int_{\mathbb{R}^d} a_\pm(x)dx, \qquad \|a_\pm\| = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} a_\pm(x).$$

Let Γ_0 be the subset of Γ as in (1) consisting of finite configurations only. For $\gamma \in \Gamma_0$, by $|\gamma|$ we denote the number of points in γ. One observes that, for $\gamma \in \Gamma_0$, the sums in (4) and in the expressions below are well-defined. Set

$$E^\pm(\gamma) = \sum_{x \in \gamma} E^\pm(x, \gamma \setminus x) = \sum_{x \in \gamma} \sum_{y \in \gamma \setminus x} a_\pm(x - y), \quad \gamma \in \Gamma_0.$$

Then the operator in (6) acts as follows

$$(12)\, (L^\Delta k)(\gamma) \quad = \quad -(m|\gamma| + E^-(\gamma))k(\gamma) + \sum_{x \in \gamma} E^+(x, \gamma \setminus x)k(\gamma \setminus x)$$

$$- \int_{\mathbb{R}^d} E^-(x, \gamma)k(\gamma \cup x)dx + \int_{\mathbb{R}^d} \sum_{x \in \gamma} a_+(x - y)k(\gamma \setminus x \cup y)dy,$$

where $\gamma \in \Gamma_0$. For a one-point $\gamma = \{x\}$, one readily derives from (12) the right-hand side of (7).

So far, (12) is just a formal expression. Then our next aim is to introduce a Banach space, in which L^Δ can be defined as a linear operator, and hence (6) can be studied. For $k : \Gamma_0 \to \mathbb{R}$, let us assume that each $k^{(n)} : (\mathbb{R}^d) \to \mathbb{R}$ is essentially bounded, and then, for $\alpha \in \mathbb{R}$, set

$$(13) \qquad \|k\|_\alpha = \sup_{n \in \mathbb{N}_0} e^{\alpha n} q_n(k),$$

where $q_0(k) = |k^{(0)}|$, and

$$q_n(k) := \operatorname*{ess\,sup}_{(x_1,\ldots,x_n) \in (\mathbb{R}^d)^n} |k^{(n)}(x_1,\ldots,x_n)|, \qquad n \in \mathbb{N}.$$

Afterwards, we set

$$(14) \qquad \mathcal{K}_\alpha = \{k : \Gamma_0 \to \mathbb{R} : \|k\|_\alpha < \infty\}.$$

That is, $\mathcal{K}_\alpha$ is an infinite dimensional real vector space equipped with norm (13), complete with respect to this norm. For $\alpha'' < \alpha'$, we have $\|k\|_{\alpha''} \leq \|k\|_{\alpha'}$; and hence,

$$(15) \qquad \mathcal{K}_{\alpha'} \subset \mathcal{K}_{\alpha''}, \qquad \text{for } \alpha'' < \alpha'.$$

That is, the bigger is α, the 'smaller' is the corresponding space (14). For $k \in \mathcal{K}_\alpha$, by (13) we readily have $q_n(k) \leq \|k\|_\alpha \exp(-\alpha n)$, which yields that k is sub-Poissonian; cf. (5). Thus, for each $\alpha \in \mathbb{R}$, the space (14) contains only sub-Poissonian k, and each such a k is contained in some $\mathcal{K}_\alpha$. Therefore, the collection of spaces (14) with all possible $\alpha \in \mathbb{R}$, cf. (15), provides an appropriate framework for describing the evolution of sub-Poissonian correlation functions.

Now let us turn to defining the operator (12). First of all, we note that, for $k \in \mathcal{K}_\alpha$, $L^\Delta k$ need not be in the same $\mathcal{K}_\alpha$, which means that L^Δ, as a map from $\mathcal{K}_\alpha$ into itself, cannot be defined on the whole $\mathcal{K}_\alpha$. On the other hand, it is possible to obtain the following estimate, see Section 4 in [6],

$$\|L^\Delta k\|_\alpha \leq \psi(\alpha' - \alpha)\|k\|_{\alpha'}, \qquad \alpha' > \alpha,$$

where $\psi : (0, +\infty) \to (0, +\infty)$ is a certain function such that $\psi(\tau) \to +\infty$ as $\tau \to 0^+$. That is L^Δ acts as a bounded operator to $\mathcal{K}_\alpha$ from any smaller $\mathcal{K}_{\alpha'}$. Then the domain of L^Δ in $\mathcal{K}_\alpha$ is defined to be the set of all those elements of this space which belong to smaller spaces $\mathcal{K}_{\alpha'}$. That is

$$\mathcal{D}_\alpha = \bigcup_{\alpha' : \alpha' > \alpha} \mathcal{K}_{\alpha'}.$$

Now we can make precise what do we mean by solving (6). Namely, for $0 < T \leq +\infty$, a map $[0, T) \ni t \mapsto k_t \in \mathcal{K}_\alpha$ is said to be a *classical solution*[3] to the problem (6) in space $\mathcal{K}_\alpha$ on time interval $[0, T)$ if: (a) it is continuously differentiable on $(0, T)$; (b) $k_t \in \mathcal{D}_\alpha$ for all $t \in (0, T)$; (c) it obeys both the equation and the initial condition in (6).

Let us take some $\alpha_* \in \mathbb{R}$, and then any $\alpha^* > \alpha_*$, and then set

$$(16) \qquad T_* = \frac{\alpha^* - \alpha_*}{\langle a_+ \rangle + \langle a_- \rangle e^{-\alpha_*}},$$

where $\langle a_\pm \rangle$ are the same as in (11). Our first result is given in the next statement.

Theorem 1. *Suppose that there exists $\theta > 0$ such that, for almost all $x \in \mathbb{R}^d$,*

$$(17) \qquad a_+(x) \leq \theta a_-(x).$$

Then, for each $\alpha^* \in \mathbb{R}$ such that

$$(18) \qquad \theta e^{\alpha^*} < 1,$$

and any $\alpha_* < \alpha^*$, the problem (6) with $k_0 \in \mathcal{K}_{\alpha^*}$ has a unique classical solution in $\mathcal{K}_{\alpha_*}$ on time interval $[0, T_*)$, with T_* given in (16).

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[3]In infinite dimensional Banach spaces, there can be solutions in weaker senses.

Let us make some comments on this result. First of all we note that the solution to (6) is sub-Poissonian, as hence clusters do not occur, if the initial k_0 shares this property. Second, the solution lies in a 'bigger' space than k_0 does. The choice of this initial space is restricted by (18). For a fixed α^*, T_* has a unique maximum as a function of α_*, that means that there exists an optimal choice of α_*, for which the time interval $[0, T_*)$ has maximal length. In (17) we, in fact, assume that the dispersal is dominated by the competition. The essence of this condition can be formulated as follows: each particle can 'kill' its offspring and can be 'killed' by it.

Now let us turn to the mesoscopic theory, which we base on a Vlasov-type scaling; cf. Section 6 in [5]. According to the general scheme developed in [8], this amounts to passing to the scale at which the microscopic density gets of order ε^{-1} and the interaction of order ε, where $\varepsilon \in (0, 1]$ is a scaling parameter. This means that the initial correlation function $k_0(\gamma)$ gets of order $\varepsilon^{-|\gamma|}$; and hence, one assumes that

$$(19) \qquad \varepsilon^{|\gamma|} k_0(\gamma) \to r_0(\gamma), \qquad \text{as } \varepsilon \to 0^+, \qquad \gamma \in \Gamma_0,$$

where r_0 is a correlation function. Additionally, one assumes that also k_t for $t > 0$, have the same divergence, and hence should be 'renormalized' as in (19). The interaction in L^Δ is represented by the competition kernel a_- which thus should be replaced by εa_- (weak interaction limit). Let L_ε^Δ be the operator which we obtain by such a replacement. Then we set, cf. (19),

$$(20) \qquad k_{t,\text{ren}}^{(\varepsilon)}(\gamma) = \varepsilon^{|\gamma|} k_t(\gamma), \qquad (L_{\varepsilon,\text{ren}}^\Delta k)(\gamma) = \varepsilon^{|\gamma|}(L_\varepsilon^\Delta k)(\gamma).$$

Thereafter, the problem (6) turns into the following

$$(21) \qquad \frac{d}{dt} k_{t,\text{ren}}^{(\varepsilon)} = L_{\varepsilon,\text{ren}}^\Delta k_{t,\text{ren}}^{(\varepsilon)}, \qquad k_{t,\text{ren}}^{(\varepsilon)}|_{t=0} = k_{0,\text{ren}}^{(\varepsilon)}.$$

The expression for $L_{\varepsilon,\text{ren}}^\Delta$ can be calculated from (20) explicitly, which yields it in the form

$$(22) \qquad L_{\varepsilon,\text{ren}}^\Delta = V + \varepsilon B,$$

where, cf. (12),

$$(23) \qquad \begin{aligned} (Vk)(\gamma) &= -mk(\gamma) - \int_{\mathbb{R}^d} E^-(x, \gamma) k(\gamma \cup x) dx \\ &+ \int_{\mathbb{R}^d} \sum_{x \in \gamma} a_+(x - y) k(\gamma \setminus x \cup y) dy, \end{aligned}$$

and

$$(Bk)(\gamma) = -E^-(\gamma) k(\gamma) + \sum_{x \in \gamma} E^+(x, \gamma \setminus x) k(\gamma \setminus x).$$

Note that $L_{\varepsilon,\text{ren}}^\Delta|_{\varepsilon=1} = L^\Delta$. For each $\varepsilon \in (0, 1]$, (21) has a unique classical solution in the same space and on the same time interval as (6) does. This can be proven exactly as Theorem 1. For r_0 as in (19) and V as in (23), let us consider

$$(24) \qquad \frac{d}{dt} r_t = Vr_t, \qquad r_t|_{t=0} = r_0.$$

For this problem, one can probe an analog of Theorem 1, even without assuming (17) because the latter condition relates to the operator B, cf. (22).

Theorem 2. *Let α^*, α_*, and T_* be as in Theorem 1, and let $k_{t,\mathrm{ren}}^{(\varepsilon)}$ (resp. r_t) be solutions to (20) (resp. (24)) in $\mathcal{K}_{\alpha_*}$ on time interval $[0, T_*)$, with the initial conditions $k_{t,\mathrm{ren}}^{(\varepsilon)}|_{t=0} = r_t|_{t=0} = r_0 \in \mathcal{K}_{\alpha^*}$. Then, for every positive $T < T_*$, it follows that*

$$\sup_{t \in [0,T]} \|k_{t,\mathrm{ren}}^{(\varepsilon)} - r_t\|_{\alpha_*} \to 0, \qquad \text{as } \varepsilon \to 0^+.$$

The equation in (24) is, in fact, a chain of equations for $r^{(n)}$, $n \in \mathbb{N}_0$. It is called a *Vlasov hierarchy.* Theorem 2 states that, in the scaling limit $\varepsilon \to 0^+$, the solutions to (21) converge in $\mathcal{K}_{\alpha_*}$ to the solution to the Vlasov hierarchy, uniformy on closed subintervals of $[0, T_*)$. The reason to pass from $k_t = k_{t,\mathrm{ren}}^{(\varepsilon)}|_{\varepsilon=1}$ to $r_t = k_{t,\mathrm{ren}}^{(\varepsilon)}|_{\varepsilon=0}$ can be seen from the following arguments. Suppose that r_0 in (24) is such that $r_0^{(0)} = 1$ and

$$r_0^{(n)}(x_1, \ldots, x_n) = \varrho_0(x_1) \cdots \varrho_0(x_n), \qquad n \in \mathbb{N},$$

for some positive $\varrho_0 \in L^\infty(\mathbb{R}^d)$. In other words, we assume that r_0 is the correlation function of a heterogeneous Poisson measure, with density function ϱ_0. Let us now seek the solution to (24) also as the product of certain unknown ϱ_t. The peculiarity of V as in (23) is that it is possible to do, and that ϱ_t can be found from the following equation

$$(25) \quad \frac{d}{dt}\varrho_t(x) = -m\varrho_t(x) - \varrho_t(x)\int_{\mathbb{R}^d} a_-(x-y)\varrho_t(y)dy + \int_{\mathbb{R}^d} a_+(x-y)\varrho_t(y)dy,$$

with the initial condition $\varrho_t|_{t=0} = \varrho_0$, which is exactly the kinetic equation (8). In particular, the passage from $\varepsilon = 1$ to $\varepsilon = 0$ has led us from (7) to (8). Since we know that the Vlasov hierarchy (24) has a unique solution, then this solution has the form $r_t^{(n)}(x_1, \ldots, x_n) = \varrho_t(x_1) \cdots \varrho_t(x_n)$, $n \in \mathbb{N}$, whenever (25) has a unique solution in $L^\infty(\mathbb{R}^d)$ on time interval $[0, T_*)$, which obeys the condition

$$\|\varrho_t\|_{L^\infty(\mathbb{R}^d)} \le e^{-\alpha_*}.$$

The latter is to guarantee that r_t lies in $\mathcal{K}_{\alpha_*}$; cf. (13) and (14).

Our next problem is to study the solvability of (25). For some technical reasons, it is more convenient to solve it in the space $C_{\mathrm{b}}(\mathbb{R}^d)$ of all bounded continuous functions $\phi : \mathbb{R}^d \to \mathbb{R}$. We equip this space with the norm

$$\|\phi\| = \sup_{x \in \mathbb{R}^d} |\phi(x)|,$$

which turns it into a Banach space, that can be isomorphically embedded into $L^\infty(\mathbb{R}^d)$. We say that $\phi \in C_{\mathrm{b}}(\mathbb{R}^d)$ is *positive* if $\phi(x) \ge 0$ for all $x \in \mathbb{R}^d$. Our first result in this domain is the following statement.

Theorem 3. *For each positive $\varrho_0 \in C_{\mathrm{b}}(\mathbb{R}^d)$, the problem (25) with the initial condition $\varrho_t|_{t=0} = \varrho_0$ has a unique classical positive solution $\varrho_t \in C_{\mathrm{b}}(\mathbb{R}^d)$ on time interval $[0, +\infty)$.*

Now let $\varrho_0 \in C_{\mathrm{b}}(\mathbb{R}^d)$ be constant as a function of x. Then the solution as in Theorem 3 will also be independent of x and can be obtained explicitly. Set $\varrho_t(x) \equiv u_t$. Then u_t has to solve the following

$$(26) \qquad \frac{d}{dt}u_t = (\langle a_+ \rangle - m)u_t - \langle a_- \rangle u_t^2,$$

see (11), which is a Bernoulli equation. For $m > \langle a_+ \rangle$, its solution decays to zero exponentially as $t \to +\infty$. For $m = \langle a_+ \rangle$, the solution is $u_t = u_0/(1 + \langle a_- \rangle u_0 t)$, and hence also decays to zero as $t \to +\infty$. For $m < \langle a_+ \rangle$, we set

$$q = \frac{\langle a_+ \rangle - m}{\langle a_- \rangle}.$$

In this case the solution to (26) has the form

$$(27) \qquad u_t = \frac{u_0 q}{u_0 + (q - u_0)\exp(-q\langle a_- \rangle t)},$$

which, in particular, means that $u_t \to q$ as $t \to +\infty$.

Theorem 4. *Suppose that $q > 0$ and, for almost all $x \in \mathbb{R}^d$, the following holds*

$$(28) \qquad \frac{a_+(x)}{\langle a_+ \rangle} \geq \left(1 - \frac{m}{\langle a_+ \rangle}\right) \frac{a_-(x)}{\langle a_- \rangle}.$$

Let also the initial condition $\varrho_0 \in C_{\mathrm{b}}(\mathbb{R}^d)$ to the problem (25) obeys

$$0 < \delta \leq \varrho_0(x) < q,$$

which holds for some δ and all $x \in \mathbb{R}^d$. Then, for each $x \in \mathbb{R}^d$ and $t > 0$, the solution as in Theorem 3 obeys the bounds $u_t \leq \varrho_t(x) < q$, where u_t is given in (27) with $u_0 = \delta$. Hence $\varrho_t(x) \to q$ point-wise as $t \to +\infty$.

The result just obtained can be interpreted as an asymptotic homogenization of the density. Let us add some comments to (28). If $a_+(x) = \theta a_-(x)$, for some $\theta > 0$ and almost all x, cf. (17), then (28) holds for all $m \in [0, \langle a_+ \rangle)$. Note that the case of $a_+(x) = a_-(x)$ was studied in [17]. In population biology, competition often has the range shorter than that of dispersal. In such a case, the mentioned homogenization occurs at nonzero mortality m. For instance, let $a_+ = \alpha \mathbb{I}_R$ and $a_- = \beta \mathbb{I}_r$ for some positive α, β, r, and R, and also $R \geq r$. Here $\mathbb{I}_r(x) = 1$ if $|x| \leq r$ and $\mathbb{I}_r(x) = 0$ otherwise. Then (28) holds if

$$1 - \frac{m}{\langle a_+ \rangle} \leq \left(\frac{r}{R}\right)^d.$$

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