COHOMOLOGICAL LENGTH FUNCTIONS

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Abstract. We study certain integer valued length functions on triangulated categories and establish a correspondence between such functions and cohomological functors taking values in the category of finite length modules over some ring.

1. Introduction

Let C be a triangulated category with suspension $\Sigma\colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$. Given a cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Mod}\, k$ into the category of modules over some ring k such that $H(C)$ has finite length for each object C, we consider the function

$$\chi\colon \mathsf{Ob}\,\mathsf{C} \longrightarrow \mathbb{N}, \quad C \mapsto \mathsf{length}_k\, H(C),$$

and ask:

- what are the characteristic properties of such a function $\mathsf{Ob}\,\mathsf{C} \longrightarrow \mathbb{N}$, and
- can we recover H from χ?

Somewhat surprisingly, we can offer fairly complete answers to both questions.

Note that similar questions arise in Boij–Söderberg theory when cohomology tables are studied; recent progress [8] provided some motivation for our work.

Typical examples of cohomological functors are the representable functors of the form $\mathsf{Hom}(-, X)$ for some object X in C. We begin with a result that takes care of this case; it is inspired by recent work of Webb [18]. The proof is elementary and based on a theorem of Bongartz [5].

Theorem 1.1. *Let k be a commutative ring and C a k-linear triangulated category such that each morphism set in C has finite length as a k-module. Suppose also for each pair of objects X, Y that $\mathsf{Hom}(\Sigma^n X, Y) = 0$ for some $n \in \mathbb{Z}$. Then two objects X and Y are isomorphic if and only if the lengths of $\mathsf{Hom}(C, X)$ and $\mathsf{Hom}(C, Y)$ coincide for all C in C.*

Examples of triangulated categories satisfying the assumptions in this theorem arise from bounded derived categories. To be precise, if A is a k-linear exact category such that each morphism set in A has finite length as a k-module, then its bounded derived category $\mathsf{D}^b(\mathsf{A})$ satisfies the above assumptions, since for all objects X, Y in A (viewed as complexes concentrated in degree zero)

$$\mathsf{Hom}(X, \Sigma^n Y) = \mathsf{Ext}^n(X, Y) = 0 \quad \text{for all } n < 0.$$

On the other hand, Auslander and Reiten provided in [3, Section 4.4] simple examples of triangulated categories which do not have the property that objects are determined by the lengths of their morphism spaces.

Now let C be an essentially small triangulated category. Given any additive functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ into the category of abelian groups, we denote by $\mathsf{End}(H)$ the endomorphism ring formed by all natural transformations $H \to H$. Note that

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$\mathsf{End}(H)$ acts on $H(C)$ for all objects C. Moreover, if a ring k acts on $H(C)$ for all objects C, then this action factors through that of $\mathsf{End}(H)$ via a homomorphism $k \to \mathsf{End}(H)$. In particular, when $H(C)$ has finite length over k then it has also finite length over $\mathsf{End}(H)$. This observation motivates the following definition.

Definition 1.2. A cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is called *endofinite*[1] provided that for each object C

(1) $H(C)$ has finite length as a module over the ring $\mathsf{End}(H)$, and
(2) $H(\Sigma^n C) = 0$ for some $n \in \mathbb{Z}$.

If $(H_i)_{i\in I}$ are cohomological functors, then the *direct sum* $\bigoplus_{i\in I} H_i$ is cohomological. A non-zero cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is *indecomposable* if it cannot be written as a direct sum of two non-zero cohomological functors.

An endofinite cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ gives rise to a function

$$\chi_H\colon \mathsf{Ob}\,\mathsf{C} \longrightarrow \mathbb{N}, \quad C \mapsto \mathsf{length}_{\mathsf{End}(H)} H(C)$$

which is cohomological in the following sense; see Lemma 3.4.

Definition 1.3. A function $\chi\colon \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ is called *cohomological* provided that

(1) $\chi(C \oplus C') = \chi(C) + \chi(C')$ for each pair of objects C and C',
(2) for each object C there is some $n \in \mathbb{Z}$ such that $\chi(\Sigma^n C) = 0$, and
(3) for each exact triangle $A \to B \to C \to$ in C and each labelling

$$\cdots \to X_{-2} \to X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

of the induced sequence

$$\cdots \to \Sigma^{-1}B \to \Sigma^{-1}C \to A \to B \to C \to \Sigma A \to \Sigma B \to \cdots$$

with $\chi(X_0) = 0$ we have

$$\sum_{i=0}^{n} (-1)^{i+n} \chi(X_i) \geq 0 \qquad \text{for all } n \in \mathbb{Z},$$

and equality holds when $\chi(X_n) = 0$.

If $(\chi_i)_{i\in I}$ are cohomological functions and for any C in C the set $\{i \in I \mid \chi_i(C) \neq 0\}$ is finite, then we can define the *locally finite sum* $\sum_{i\in I} \chi_i$. A non-zero cohomological function is *irreducible* if it cannot be written as a sum of two non-zero cohomological functions.

The following theorem is the main result of this note; it builds on work of Crawley-Boevey on finite endolength objects [7].

Theorem 1.4. *Let C be an essentially small triangulated category.*

(1) *Every endofinite cohomological functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ decomposes essentially uniquely into a direct sum of indecomposable endofinite cohomological functors with local endomorphism rings.*
(2) *Every cohomological function $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ can be written uniquely as a locally finite sum of irreducible cohomological functions.*
(3) *The assignment $H \mapsto \chi_H$ induces a bijection between the isomorphism classes of indecomposable endofinite cohomological functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ and the irreducible cohomological functions $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$.*

[1] The term *endofinite* reflects condition (1), while (2) is added for technical reasons.

Examples of endofinite cohomological functors arise from representable functors of the form $\mathsf{Hom}(-,X)$ when C is a Hom-finite k-linear category. Thus Theorem 1.1 can be deduced from Theorem 1.4. The following remark shows that in some appropriate setting each endofinite cohomological functor is representable.

Remark 1.5. Let T be a compactly generated triangulated category and denote by C the full subcategory formed by all compact objects. Then each endofinite cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is isomorphic to $\mathsf{Hom}(-,X)|_{\mathsf{C}}$ for some object X in T, which is unique up to an isomorphism and represents the functor[2]

$$\mathsf{T}^{\mathrm{op}} \longrightarrow \mathsf{Ab}, \quad C \mapsto \mathsf{Hom}(\mathsf{Hom}(-,C)|_{\mathsf{C}}, H).$$

Thus cohomological functions are represented by objects in this setting. For specific examples, see [13].

2. Proof of the first theorem

In this section Theorem 1.1 is proved. We deduce it from the following theorem of Bongartz [5] which generalises a theorem of Auslander [1, 2] and independently of Benson and Parker [4]; its beautiful proof is short and elementary.

Theorem 2.1 (Bongartz)**.** *Let k be a commutative ring and A a k-linear abelian category such that each morphism set in A has finite length as a k-module. Then two objects X and Y are isomorphic if and only if the lengths of $\mathsf{Hom}(C,X)$ and $\mathsf{Hom}(C,Y)$ coincide for all C in A.* □

For the following proof it is convenient to denote for two objects X and Y of a k-linear category by $[X,Y]$ the length of the k-module $\mathsf{Hom}(X,Y)$.

Following Freyd [9, §3] and Verdier [17, II.3], we consider the *abelianisation* $\mathsf{Ab}(\mathsf{C})$ of C which is the abelian category of additive functors $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ that admit a presentation

$$0 \longrightarrow F \longrightarrow \mathsf{Hom}(-,A) \longrightarrow \mathsf{Hom}(-,B).$$

Completing the representing morphism of such a presentation to an exact triangle $A \to B \to C \to$ in C yields in $\mathsf{Ab}(\mathsf{C})$ two resolutions

$$0 \to F \to \mathsf{Hom}(-,A) \to \mathsf{Hom}(-,B) \to \mathsf{Hom}(-,C) \to \mathsf{Hom}(-,\Sigma A) \to \cdots$$
$$\cdots \to \mathsf{Hom}(-,\Sigma^{-1}A) \to \mathsf{Hom}(-,\Sigma^{-1}B) \to \mathsf{Hom}(-,\Sigma^{-1}C) \to F \to 0.$$

Proof of Theorem 1.1. Let X,Y be objects in C satisfying $[C,X] = [C,Y]$ for all C in C. Fix F in $\mathsf{Ab}(\mathsf{C})$ with presentation as above and choose $n \in \mathbb{Z}$ such that $\mathsf{Hom}(\Sigma^n(A \oplus B \oplus C), X) = 0$. Suppose first that $n \geq 0$. Applying the functor $\mathsf{Hom}(-,\mathsf{Hom}(-,X))$ to the first resolution of F and using Yoneda's lemma gives

$$\begin{aligned}[F,\mathsf{Hom}(-,X)] &= \sum_{i=0}^{n} \big((-1)^i[\Sigma^i A,X] - (-1)^i[\Sigma^i B,X] + (-1)^i[\Sigma^i C,X]\big) \\ &= [F,\mathsf{Hom}(-,Y)].\end{aligned}$$

If $n < 0$, the second resolution yields a similar identity. It follows from Bongartz' theorem that $\mathsf{Hom}(-X)$ and $\mathsf{Hom}(-,Y)$ are isomorphic in $\mathsf{Ab}(\mathsf{C})$, and therefore X and Y are isomorphic in C, by Yoneda's lemma. □

[2]The functor is cohomological since H is an injective object in the category of additive functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$; see the proof of Theorem 1.4. Thus Brown's representabilty theorem applies.

Remark 2.2. (1) The dual statement of the theorem holds true, of course.

(2) Refining the proof given above and following [5], one can show that for given X and Y in C there is a finite set S of objects in C (depending on X and Y) such that X and Y are isomorphic if and only if the lengths of $\mathsf{Hom}(C, X)$ and $\mathsf{Hom}(C, Y)$ coincide for all C in S.

(3) There is a completely different proof based on the notion of endofiniteness [13]. In fact, the arguments used in the proof of [13, Theorem 3.4] yield the following:

Given any Hom-finite k-linear triangulated category C, two indecomposable objects X and Y in C are isomorphic if and only if both objects annihilate the same morphisms in C, that is, $\mathsf{Hom}(f, X) = 0$ iff $\mathsf{Hom}(f, Y) = 0$ for all f in C.

Then one observes that for each object X in C, the values of $[C, X]$ for all C in C determine the morphisms f in C such that $\mathsf{Hom}(f, X) = 0$, provided one assumes for all C in C that $\mathsf{Hom}(\Sigma^n C, X) = 0$ for some $n \in \mathbb{Z}$.

3. Proof of the second theorem

In this section Theorem 1.4 is proved. We deduce it from work of Crawley-Boevey on endofinite objects in locally finitely presented abelian categories [6, 7].

As before, we use the abelianisation $\mathsf{Ab}(\mathsf{C})$ of C which is the abelian category of additive functors $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ that admit a presentation

$$0 \longrightarrow F \longrightarrow \mathsf{Hom}(-, A) \longrightarrow \mathsf{Hom}(-, B).$$

In addition, we work in the category $\mathsf{Mod}\,\mathsf{C}$ of additive functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$. This is a locally finitely presented abelian category and the abelianisation $\mathsf{Ab}(\mathsf{C})$ identifies with the full subcategory of finitely presented objects of $\mathsf{Mod}\,\mathsf{C}$; see [7] for details.

Definition 3.1. A function $\chi\colon \mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \to \mathbb{N}$ is called *additive*[3] provided that

(1) $\chi(F) = \chi(F') + \chi(F'')$ if $0 \to F' \to F \to F'' \to 0$ is an exact sequence, and
(2) for each object F there is some $n \in \mathbb{Z}$ such that $\chi(\Sigma^n F) = 0$.

We show that additive and cohomological functions are closely related.

Lemma 3.2. *Restricting a function $\chi\colon \mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \to \mathbb{N}$ to $\mathsf{Ob}\,\mathsf{C}$ by setting $\chi(C) = \chi(\mathsf{Hom}(-, C))$ gives a natural bijection between*

- *the additive functions $\mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \to \mathbb{N}$, and*
- *the cohomological functions $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$.*

Proof. Let $\chi\colon \mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \to \mathbb{N}$ be an additive function. An exact triangle $A \to B \to C \to$ in C yields in $\mathsf{Ab}(\mathsf{C})$ an exact sequence

$$\cdots \to \mathsf{Hom}(-, A) \to \mathsf{Hom}(-, B) \to \mathsf{Hom}(-, C) \to \mathsf{Hom}(-, \Sigma A) \to \cdots.$$

Using this sequence it is easily checked that the restriction of χ to C is a cohomological function.

Conversely, given a cohomological function $\chi\colon \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$, we extend it to a function $\mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \to \mathbb{N}$ which again we denote by χ. Fix F in $\mathsf{Ab}(\mathsf{C})$ with presentation as above induced by an exact triangle $A \to B \to C \to$ in C, and choose $n \in \mathbb{Z}$ such that $\chi(\Sigma^n(A \oplus B \oplus C)) = 0$. Then define

$$\chi(F) = \begin{cases} \sum_{i=0}^{n} \big((-1)^i \chi(\Sigma^i A) - (-1)^i \chi(\Sigma^i B) + (-1)^i \chi(\Sigma^i C)\big), & \text{if } n \geq 0; \\ \sum_{i=-1}^{n} \big((-1)^{i+1} \chi(\Sigma^i A) - (-1)^{i+1} \chi(\Sigma^i B) + (-1)^{i+1} \chi(\Sigma^i C)\big), & \text{if } n < 0. \end{cases}$$

[3]The term *additive* reflects condition (1), while (2) is added for technical reasons.

This gives a non-negative integer and does not depend on n since χ is a cohomological function. Also, $\chi(F)$ does not depend on the choice of the exact triangle which presents F by a variant of Schanuel's lemma; see Lemma A.1. Note that $\chi(\mathsf{Hom}(-,C)) = \chi(C)$ for each C in C. Standard arguments involving resolutions show that χ is additive.

Clearly, restricting from $\mathsf{Ab}(\mathsf{C})$ to C and extending from C to $\mathsf{Ab}(\mathsf{C})$ are mutually inverse operations. □

There is a parallel between functions and functors. The analogue of Lemma 3.2 for functors is due to Freyd [9].

Lemma 3.3 (Freyd)**.** *Restricting a functor* $F\colon \mathsf{Ab}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$ *to* C *by setting* $F(C) = F(\mathsf{Hom}(-,C))$ *gives a natural bijection between*

- *the exact functors* $\mathsf{Ab}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$*, and*
- *the cohomological functors* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$*.*

Proof. The inverse map sends a cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ to the exact functor $\mathsf{Hom}(-,H)\colon \mathsf{Ab}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$. □

An endofinite cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ induces two functions:

$$\chi_H\colon \mathsf{Ob}\,\mathsf{C} \longrightarrow \mathbb{N}, \quad C \mapsto \mathsf{length}_{\mathsf{End}(H)}\, H(C),$$
$$\hat{\chi}_H\colon \mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \longrightarrow \mathbb{N}, \quad F \mapsto \mathsf{length}_{\mathsf{End}(H)}\, \mathsf{Hom}(F,H).$$

Lemma 3.4. *The function* χ_H *is cohomological and* $\hat{\chi}_H$ *is additive.*

Proof. The functor $\mathsf{Hom}(-,H)\colon \mathsf{Ab}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Ab}$ is exact since H is cohomological. It follows that $\hat{\chi}_H$ is additive. The restriction of $\hat{\chi}_H$ to C equals χ_H. Thus χ_H is cohomological by Lemma 3.2. □

Proof of Theorem 1.4. An endofinite cohomological functor $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is a finite endolength injective object in $\mathsf{Mod}\,\mathsf{C}$, as defined in [7]. The term 'finite endolength' refers to the fact that $\mathsf{Hom}(F,H)$ has finite length as $\mathsf{End}(H)$-module for each F in $\mathsf{Ab}(\mathsf{C})$; see Lemma 3.4. The injectivity follows from the proof of [12, Theorem 1.2], using that $\mathsf{Ext}^1(-,H)$ vanishes on $\mathsf{Ab}(\mathsf{C})$ for any cohomological functor H.

In [7, Theorem 3.5.2] it is shown that each finite endolength object decomposes into a direct sum of indecomposable objects with local endomorphism rings; see [12, Theorem 1.2] for an alternative proof. This yields part (1).

In [6], additive functions on locally finitely presented Grothendieck categories are studied. In particular, there it is shown that every additive function on $\mathsf{Ab}(\mathsf{C})$ can be written uniquely as a locally finite sum of irreducible additive functions. This proves part (2), in view of Lemma 3.2. For an alternative proof, see Proposition B.1.

Finally, part (3) follows from the main theorem in [6] which establishes the bijection between isomorphism classes of indecomposable finite endolength injective objects in $\mathsf{Mod}\,\mathsf{C}$ and irreducible additive functions $\mathsf{Ob}\,\mathsf{Ab}(\mathsf{C}) \to \mathbb{N}$. For an alternative proof, see Proposition B.2, using the bijection between cohomological and exact functors from Lemma 3.3. □

Remark 3.5. Theorem 1.4 yields a remarkable duality between cohomological functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ and cohomological functors $\mathsf{C} \to \mathsf{Ab}$. This follows from the fact that the definition of a cohomological function $\mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$ is self-dual; it is an analogue of the *elementary duality* between left and right modules over a ring studied by Herzog [11].

The duality links indecomposable endofinite cohomological functors $H\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ and $H'\colon \mathsf{C} \to \mathsf{Ab}$ when $\chi_H = \chi_{H'}$. In that case

$$(\mathsf{End}(H)/\operatorname{rad}\mathsf{End}(H))^{\mathrm{op}} \cong \mathsf{End}(H')/\operatorname{rad}\mathsf{End}(H')$$

where $\operatorname{rad} A$ denotes the Jacobson radical of a ring A. This follows from Remark B.3.

The duality specialises to Serre duality when C is a Hom-finite k-linear category with k a field. More precisely, if $F\colon \mathsf{C} \to \mathsf{C}$ is a Serre functor [16] and $D = \mathsf{Hom}(-,k)$, then

$$\mathsf{Hom}(-,FX) \cong D\,\mathsf{Hom}(X,-)$$

for each object X, and therefore $\chi_{\mathsf{Hom}(-,FX)} = \chi_{\mathsf{Hom}(X,-)}$.

The assignment $H \mapsto \chi_H$ between endofinite cohomological functors and cohomological functions satisfies some weighted additivity. For instance, $\chi_{H\oplus H'} = \chi_H + \chi_{H'}$ provided that H and H' have no common indecomposable summand, but $\chi_{H\oplus H} = \chi_H$. We have the following concise formula.

Proposition 3.6. *Let $H = \bigoplus_{i\in I} H_i$ be the decomposition of an endofinite cohomological functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ into indecomposables. If $J \subseteq I$ is a subset such that $(H_i)_{i\in J}$ contains each isomorphism class from $(H_i)_{i\in I}$ exactly once then $\chi_H = \sum_{i\in J} \chi_{H_i}$.*

Proof. Adapt the proofs of Propositions 4.5 and 4.6 in [13]. Alternatively, use Remark B.4. □

Remark 3.7. Let C be a k-linear category such that each morphism set in C has finite length as a k-module. Then we have two maps $\mathsf{Ob}\,\mathsf{C} \times \mathsf{Ob}\,\mathsf{C} \to \mathbb{N}$, taking (X,Y) either to $\mathsf{length}_k \mathsf{Hom}(X,Y)$ or to $\mathsf{length}_{\mathsf{End}(Y)} \mathsf{Hom}(X,Y)$. While the first map preserves sums in both arguments, the second one does not in the second argument, but it satisfies the above 'weighted additivity'.

Example 3.8. (1) Let k be a field and Γ a quiver with underlying diagram of Dynkin type. Denote by $\mathsf{rep}(\Gamma,k)$ the category of finite dimensional k-linear representations of Γ. The cohomological functors $\mathsf{D}^b(\mathsf{rep}(\Gamma,k))^{\mathrm{op}} \to \mathsf{Ab}$ that are indecomposable and endofinite are precisely (up to isomorphism) the representable functors $\mathsf{Hom}(-,\Sigma^n X)$ with X an indecomposable object in $\mathsf{rep}(\Gamma,k)$ (viewed as complex concentrated in degree zero) and $n \in \mathbb{Z}$. More generally, if C is any locally finite triangulated category [14], then each indecomposable endofinite cohomological functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ is representable.

(2) Let $\mathsf{coh}\,\mathbb{P}^1_k$ be the category of coherent sheaves on the projective line over a field k. The indecomposable endofinite cohomological functors $\mathsf{D}^b(\mathsf{coh}\,\mathbb{P}^1_k)^{\mathrm{op}} \to \mathsf{Ab}$ are precisely the representable functors $\mathsf{Hom}(-,\Sigma^n X)$ with X an indecomposable object in $\mathsf{coh}\,\mathbb{P}^1_k$ or $X = k(t)$ the function field and $n \in \mathbb{Z}$.

Appendix A. Schanuel's lemma for triangulated categories

Let C be a triangulated category. An exact triangle $A \to B \to C \to$ induces a *presentation* of a functor $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Ab}$ provided there exists an exact sequence

$$\mathsf{Hom}(-,B) \longrightarrow \mathsf{Hom}(-,C) \longrightarrow F \longrightarrow 0.$$

Two exact triangles are called *homotopy equivalent*[4] if they induce presentations of the same functor.

Lemma A.1. *Let $A \to B \to C \to$ and $A' \to B' \to C' \to$ be two homotopy equivalent exact triangles. Then $A \oplus B' \oplus C \cong A' \oplus B \oplus C'$.*

[4]This notion is consistent with the homotopy relation introduced in [15, Section 1.3].

Proof. The triangles induce exact sequences

$$0 \to \Sigma^{-1}F \to \mathsf{Hom}(-, A) \to \mathsf{Hom}(-, B) \to \mathsf{Hom}(-, C) \to F \to 0$$

and

$$0 \to \Sigma^{-1}F \to \mathsf{Hom}(-, A') \to \mathsf{Hom}(-, B') \to \mathsf{Hom}(-, C') \to F \to 0$$

which represent the same class in $\mathsf{Ext}^3(F, \Sigma^{-1}F)$, since the presentations induce a morphism between both triangles. Now apply the variant of Schanuel's lemma which is given below. □

Lemma A.2. *Let*

$$0 \to M \to P_r \to \cdots \to P_1 \to P_0 \to N \to 0$$

and

$$0 \to M \to Q_r \to \cdots \to Q_1 \to Q_0 \to N \to 0$$

be exact sequences in some abelian category which represent the same class in $\mathsf{Ext}^{r+1}(N, M)$*. If all* P_i *and* Q_i *are projective, then*

$$\bigoplus_{i\geq 0}(P_{2i} \oplus Q_{2i+1}) \cong \bigoplus_{i\geq 0}(P_{2i+1} \oplus Q_{2i}).$$

Proof. We use induction on r. The case $r = 0$ is clear and we suppose that $r > 0$. The pullback of $\eta\colon P_0 \to N$ and $\theta\colon Q_0 \to N$ induces an exact sequence

$$0 \longrightarrow K \longrightarrow P_0 \oplus Q_0 \longrightarrow N \longrightarrow 0$$

with $Q_0 \oplus \mathsf{Ker}\,\eta \cong K \cong P_0 \oplus \mathsf{Ker}\,\theta$, by Schanuel's lemma. Adding complexes of the form $Q_0 \xrightarrow{\mathrm{id}} Q_0$ and $P_0 \xrightarrow{\mathrm{id}} P_0$ yields two exact sequences

$$0 \to M \to P_r \to \cdots \to P_2 \to P_1 \oplus Q_0 \to K \to 0$$

and

$$0 \to M \to Q_r \to \cdots \to Q_2 \to Q_1 \oplus P_0 \to K \to 0$$

which represent the same class in $\mathsf{Ext}^r(K, M)$. Now the assertion follows from the induction hypothesis. □

Appendix B. Additive functions

Let A be an abelian category. A function $\chi\colon \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ is called *additive* if $\chi(X) = \chi(X') + \chi(X'')$ for each exact sequence $0 \to X' \to X \to X'' \to 0$.

We give a quick proof of the following result from [6] using the localisation theory for abelian categories [10].

Proposition B.1 (Crawley-Boevey)**.** *Every additive function* $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ *can be written uniquely as a locally finite sum of irreducible additive functions.*

Proof. Fix an additive function $\chi\colon \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$. The objects X satisfying $\chi(X) = 0$ form a Serre subcategory of A which we denote by S_χ. The quotient category $\mathsf{A}/\mathsf{S}_\chi$ is an abelian length category since the length of each object X is bounded by $\chi(X)$. Let $\mathsf{Sp}\,\chi$ (the *spectrum* of χ) denote a representative set of simple objects in $\mathsf{A}/\mathsf{S}_\chi$. For each S in $\mathsf{Sp}\,\chi$ let S_S denote the Serre subcategory of A formed by all objects X such that a composition series of X in $\mathsf{A}/\mathsf{S}_\chi$ has no factor isomorphic to S. Define $\chi_S\colon \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ by sending X to the length of X in A/S_S. From the construction it follows that

$$\chi = \sum_{S\in\mathsf{Sp}\,\chi} \chi(S)\chi_S. \tag{B.1}$$

We claim that each χ_S is irreducible and that the above expression is unique. To see this, write $\chi = \chi' + \chi''$ as a sum of two additive functions. This implies $\mathsf{S}_\chi \subseteq \mathsf{S}_{\chi'}$, and if $\chi' \neq 0$, then for some $S \in \mathsf{Sp}\,\chi$ the object S is non-zero in $\mathsf{A}/\mathsf{S}_{\chi'}$. In that case χ_S arises as a summand of χ' with multiplicity $\chi'(S)$. □

Now suppose that A is essentially small. An exact functor $F\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ is called *endofinite* if $F(X)$ has finite length as $\mathsf{End}(F)$-module for each object X. An endofinite exact functor F induces an additive function

$$\chi_F\colon \mathsf{Ob}\,\mathsf{A} \longrightarrow \mathbb{N}, \quad X \mapsto \mathsf{length}_{\mathsf{End}(F)}\, F(X).$$

Proposition B.2. *The assignment $F \mapsto \chi_F$ induces a bijection between the isomorphism classes of indecomposable endofinite exact functors $\mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ and the irreducible additive functions $\mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$.*

Proof. We construct the inverse map. Let $\chi\colon \mathsf{Ob}\,\mathsf{A} \to \mathbb{N}$ be an irreducible additive function. Following the proof of Proposition B.1, we consider the Serre subcategory S_χ of A consisting of the objects X satisfying $\chi(X) = 0$. The quotient category $\mathsf{B} = \mathsf{A}/\mathsf{S}_\chi$ is an abelian length category, and $\chi(X)$ equals the length of X in B for each object X, since χ is irreducible. Now consider the abelian category $\mathsf{Lex}(\mathsf{B}^{\mathrm{op}}, \mathsf{Ab})$ of left exact functors $\mathsf{B}^{\mathrm{op}} \to \mathsf{Ab}$; see [10] for details. The Yoneda functor

$$\mathsf{B} \longrightarrow \mathsf{Lex}(\mathsf{B}^{\mathrm{op}}, \mathsf{Ab}), \quad X \mapsto H_X = \mathsf{Hom}(-, X)$$

identifies B with the full subcategory of finite length objects. Thus there is a unique simple object in $\mathsf{Lex}(\mathsf{B}^{\mathrm{op}}, \mathsf{Ab})$ and we denote by F its injective envelope. It follows that F is indecomposable, and the injectivity implies that F is exact. For each X in B we have

$$\mathsf{length}_{\mathsf{End}(F)}\, F(X) = \mathsf{length}_{\mathsf{End}(F)}\, \mathsf{Hom}(H_X, F) = \mathsf{length}_{\mathsf{B}}\, X = \chi(X)$$

since each finitely generated $\mathsf{End}(F)$-submodule of $\mathsf{Hom}(H_X, F)$ is of the form $\mathsf{Hom}(H_X/H_{X'}, F)$ for some subobject $X' \subseteq X$. Let $F'\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be the composite of F with the quotient functor $\mathsf{A} \to \mathsf{B}$ and observe that $\mathsf{End}(F') \cong \mathsf{End}(F)$. Then F' has the desired properties: it is indecomposable endofinite exact and $\chi_{F'} = \chi$.

It remains to show for an indecomposable endofinite exact functor $F\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ that the function χ_F is irreducible. Set $\mathsf{B} = \mathsf{A}/\mathsf{S}_{\chi_F}$ and view F as an exact functor $\mathsf{B}^{\mathrm{op}} \to \mathsf{Ab}$. Note that $\mathsf{Hom}(H_S, F) = F(S) \neq 0$ for each simple object S in B. The indecomposability of F implies that all simple objects in B are isomorphic, and the equation (B.1) then implies that χ is irreducible since for each simple object S

$$\chi_F(S) = \mathsf{length}_{\mathsf{End}(F)}\, F(S) = \mathsf{length}_{\mathsf{End}(F)}\, \mathsf{Hom}(H_S, F) = \mathsf{length}_{\mathsf{B}}\, S = 1. \qquad \square$$

Remark B.3. Let $F\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be an indecomposable endofinite exact functor and S the corresponding simple object in $\mathsf{A}/\mathsf{S}_{\chi_F}$. Then the endomorphism ring $\mathsf{End}(F)$ is local and

$$\mathsf{End}(F)/\,\mathsf{rad}\,\mathsf{End}(F) \cong \mathsf{End}(S)$$

since F identifies with an injective envelope of S. Here, $\mathsf{rad}\,A$ denotes the Jacobson radical of a ring A.

Remark B.4. Let $F\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be an endofinite exact functor and $\mathsf{B} = \mathsf{A}/\mathsf{S}_{\chi_F}$. For each object X in A we have

$$\chi_F(X) = \mathsf{length}_{\mathsf{End}(F)}\, F(X) = \mathsf{length}_{\mathsf{End}(F)}\, \mathsf{Hom}(H_X, F) = \mathsf{length}_{\mathsf{B}}\, X,$$

and this can be used to compute $\sum_i \chi_{F_i}$ for any decomposition $F = \bigoplus_i F_i$ into exact functors.

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