CONTRAHERENT COSHEAVES

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Abstract. Contraherent cosheaves are globalizations of cotorsion (or similar) modules over commutative rings obtained by gluing together over a scheme. The category of contraherent cosheaves over a scheme is a Quillen exact category with exact functors of infinite product; over a quasi-compact semi-separated scheme, it also has enough projectives. We construct the derived co-contra correspondence, meaning an equivalence between appropriate derived categories of quasi-coherent sheaves and contraherent cosheaves, over a quasi-compact semi-separated scheme and, in a different form, over a semi-separated Noetherian scheme with a dualizing complex. The former point of view allows us to obtain a new construction of the extraordinary inverse image functor $f^!$ for a morphism of quasi-compact semi-separated schemes $f\colon X \longrightarrow Y$. The latter approach provides an expanded version of the covariant Serre–Grothendieck duality theory.

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Introduction

Quasi-coherent sheaves resemble comodules. Both form abelian categories with exact functors of infinite direct sum (and in fact, even of filtered inductive limit) and with enough injectives. When one restricts to quasi-coherent sheaves over Noetherian schemes and comodules over (flat) corings over Noetherian rings, both abelian categories are locally Noetherian. Neither has projective objects or exact functors of infinite product, in general.

In fact, quasi-coherent sheaves *are* comodules. Let X be a quasi-compact semi-separated scheme and $\{U_\alpha\}$ be its finite affine open covering. Denote by Y the disconnected union of the schemes U_α; so Y is also an affine scheme and the natural morphism $Y \longrightarrow X$ is affine. Then quasi-coherent sheaves over X can be described as quasi-coherent sheaves $\mathcal{F}$ over Y endowed with an isomorphism $\phi\colon p_1^*(\mathcal{F}) \simeq p_2^*(\mathcal{F})$ between the two inverse images under the natural maps p_1, $p_2\colon Y \times_X Y \rightrightarrows Y$. The isomorphism ϕ has to satisfy a natural associativity constraint.

In other words, this means that the ring of functions $\mathcal{C} = \mathcal{O}(Y \times_X Y)$ has a natural structure of a coring over the ring $A = \mathcal{O}(Y)$. The quasi-coherent sheaves over X are the same thing as (left or right) comodules over this coring. The quasi-coherent sheaves over a (good enough) stack can be also described in such way [6].

There are two kinds of module categories over a coalgebra or coring: in addition to the more familiar comodules, there are also *contramodules.* Introduced originally by Eilenberg and Moore [4] in 1965 (see also the notable paper [1]), they were all but forgotten for four decades, until the author's preprint and then monograph [19] attracted some new attention to them towards the end of 2000's.

Assuming a coring $\mathcal{C}$ over a ring A is a projective left A-module, the category of left $\mathcal{C}$-contramodules is abelian with exact functors of infinite products and enough projectives. Generally, contramodules are "dual-analogous" to comodules in most respects, i. e., they behave as though they formed two opposite categories, which in fact they don't (or otherwise it wouldn't be interesting).

On the other hand, there is an important homological phenomenon of *comodule-contramodule correspondence*, or a *covariant* equivalence between appropriately defined ("exotic") derived categories of left comodules and left contramodules over the same coring. This equivalence is typically obtained by deriving certain adjoint functors which act between the abelian categories of comodules and contramodules and induce a covariant equivalence between their appropriately picked exact or additive subcategories (e. g., the equivalence of *Kleisli categories*, which was empasized in the application to comodules and contramodules in the paper [2]).

Contraherent cosheaves are geometric module objects over a scheme that are similar to (and, sometimes, particular cases of) contramodules in the same way as quasi-coherent sheaves are similar to (or particular cases of) comodules. Thus the simplest way to define contramodules would be to assume one's scheme X to be quasi-compact and semi-separated, pick its finite affine covering $\{U_\alpha\}$, and consider contramodules over the related coring $\mathcal{C}$ over the ring A as constructed above.

This indeed largely agrees with our approach in this paper, but there are several problems to be dealt with. First of all, $\mathcal{C}$ is not a projective A-module, but only a flat one. The most immediate consequence is that one cannot hope for an abelian category of $\mathcal{C}$-contramodules, but at best for an exact category. This is where the *cotorsion modules* [27, 8] (or their generalizations which we call the *contraadjusted* modules) come into play. Secondly, it turns out that the exact category of $\mathcal{C}$-contramodules, however defined, depends on the choice of an affine covering $\{U_\alpha\}$.

In the exposition below, we strive to make our theory as similar (or rather, dual-analogous) to the classical theory of quasi-coherent sheaves as possible, while refraining from the choice of a covering to the extent that it remains practicable. We start with defining cosheaves of modules over a sheaf of rings on a topological space, and proceed to introducing the exact subcategory of contraherent cosheaves in the exact category of cosheaves of $\mathcal{O}_X$-modules on an arbitrary scheme X.

The phenomenon of dependence on the covering manifests itself in this approach in the unexpected predicament of the *contraherence property* of a sheaf of $\mathcal{O}_X$-modules *being not local.* So we have to deal with the *locally contraherent cosheaves*, and the necessity to control the extension of this locality brings the coverings back. Once an open covering in restriction to which our cosheaf becomes contraherent is safely fixed, though, many other cosheaf properties that we consider in this paper become indeed local. And any locally contraherent cosheaf on a quasi-compact semi-separated scheme has a finite left Čech resolution by contraherent cosheaves.

Let us explain the main definition in some more detail. A quasi-coherent sheaf $\mathcal{F}$ on a scheme X can be simply defined as a correspondence assigning to every affine open subscheme $U \subset X$ an $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ and to every pair of embedded affine open subschemes $V \subset U \subset X$ an isomorphism of $\mathcal{O}_X(V)$-modules

$$\mathcal{F}(V) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U).$$

The obvious compatibility condition for three embedded affine open subschemes $W \subset V \subset U \subset X$ needs to be imposed.

Analogously, a contraherent cosheaf $\mathfrak{P}$ on a scheme X is a correspondence assigning to every affine open subscheme $U \subset X$ an $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ and to every pair of embedded affine open subschemes $V \subset U \subset X$ an isomorphism of $\mathcal{O}_X(V)$-modules

$$\mathfrak{P}[V] \simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]).$$

The difference with the quasi-coherent case is that the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ is always flat, but not necessarily projective. So to make one's contraherent cosheaves well-behaved, one has to impose the additional Ext-vanishing condition

$$\operatorname{Ext}^1_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]) = 0$$

for all affine open subschemes $V \subset U \subset X$. Notice that the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ always has projective dimension not exceeding 1, so the condition on Ext^1 is sufficient. This nonprojectivity problem is the reason why one does not have an abelian category of contraherent cosheaves. Imposing the Ext-vanishing requirement allows to obtain, at least, an exact one.

A module P over a commutative ring R is called *contraadjusted* if $\operatorname{Ext}^1_R(R[s^{-1}], P) = 0$ for all $s \in R$. This is equivalent to the vanishing of $\operatorname{Ext}^1_R(S, P)$ for all the R-algebras S of functions on the affine open subschemes of $\operatorname{Spec} R$. More generally, a left module P over a (not necessarily commutative) ring R is said to be *cotorsion* if $\operatorname{Ext}^1_R(F, P) = 0$ (or equivalently, $\operatorname{Ext}^{>0}_R(F, P) = 0$) for any flat left R-module F. It has been proven that cotorsion modules are "numerous enough" (see [7, 9]); one can prove the same for contraadjusted modules in the similar way.

Any quotient module of a contraadjusted module is contraadjusted, so the "contraadjusted dimension" of any module does not exceed 1; while the cotorsion dimension of a module may be infinite if the projective dimensions of flat modules are. This makes the contraadjusted modules useful when working with schemes that are not necessarily Noetherian of finite Krull dimension. Modules of the complementary class to the contraadjusted ones (in the same way as the flats are complementary to the cotorsion modules) we call *very flat.* All very flat modules have projective dimensions not exceeding 1.

One difference between homological theories developed in the settings of exact and abelian categories is that whenever a functor between abelian categories isn't exact, a similar functor between exact categories will tend to have a shrinked domain. Any functor between abelian categories that has an everywhere defined left or right derived functor will tend to be everywhere defined itself, if only because one can always pass to the degree-zero cohomology of the derived category objects. Not so with exact categories, in which complexes may have no cohomology objects in general.

Hence the (sometimes annoying) necessity to deal with multitudes of domains of definitions of various functors in our exposition. On the other hand, a functor with the domain consisting of adjusted objects is typically exact on the exact subcategory where it is defined.

Another difference between the theory of comodules and contramodules over corings as developed in [19] and our present setting is that in *loc. cit.* we considered corings $\mathcal{C}$ over base rings A of finite homological dimension. On the other hand, the ring $A = \mathcal{O}(Y)$ constructed above has infinite homological dimension in most cases, while the coring $\mathcal{C} = \mathcal{O}(Y \times_X Y)$ can be said to have "finite homological dimension relative to A". For this reason, while the comodule-contramodule correspondence theorem [19, Theorem 5.4] was stated in the maximal generality for derived categories of the second kind, our most general co-contra correspondence result in this paper features an equivalence of the conventional derived categories.

One application of this equivalence is a new construction of the extraordinary inverse image functor $f^!$ on the derived categories of quasi-coherent sheaves for any morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$. In fact, we construct a *right derived* functor $\mathbb{R}f^!$, rather than just a triangulated functor $f^!$, as it was usually done before [11, 17].

To a morphism f one assigns the direct and inverse image functors $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ and $f^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ between the abelian categories of quasi-coherent sheaves on X and Y; the functor f^* is left adjoint to the functor f_*. To the same morphism, one also assigns the direct image functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ between the exact categories of colocally projective contraherent cosheaves and the inverse image functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$ between the exact categories of locally injective locally contraherent cosheaves on X and Y. The functor $f^!$ is "partially" right adjoint to the functor $f_!$.

Passing to the derived functors, one obtains the adjoint functors $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ between the (conventional unbounded) derived categories of quasi-coherent cosheaves. One also obtains the adjoint functors $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ and $\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ between the derived categories of contraherent cosheaves on X and Y.

The derived co-contra correspondence (for the conventional derived categories) provides equivalences of triangulated categories $\mathsf{D}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(X\text{–}\mathsf{ctrh})$ and $\mathsf{D}(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}(Y\text{–}\mathsf{ctrh})$ transforming the direct image functor $\mathbb{R}f_*$ into the direct image functor $\mathbb{L}f_!$. So the two inverse image functors $\mathbb{L}f^*$ and $\mathbb{R}f^!$ can be viewed as the adjoints on the two sides to the same triangulated functor of direct image. This finishes our construction of the triangulated functor $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$ right adjoint to $\mathbb{R}f_*$.

As usually in homological algebra, the tensor product and Hom-type operations on the quasi-coherent sheaves and contraherent cosheaves play an important role in our theory. First of all, under appropriate adjustness assumptions one can assign a contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ to a quasi-coherent sheaf $\mathcal{F}$ and a contraherent cosheaf $\mathfrak{P}$ over a scheme X. This operation is the analogue of the tensor product of quasi-coherent sheaves in the contraherent world.

Secondly, to a quasi-coherent sheaf $\mathcal{F}$ and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ over a scheme X one can assign a cosheaf of $\mathcal{O}_X$-modules $\mathcal{F}\otimes_{\mathcal{O}_X}\mathfrak{P}$. Under our duality-analogy, this corresponds to taking the sheaf of $\mathcal{H}om$ from a quasi-coherent sheaf to a sheaf of $\mathcal{O}_X$-modules. When the scheme X is Noetherian, the sheaf $\mathcal{F}$ is coherent, and the cosheaf $\mathfrak{P}$ is contraherent, the cosheaf $\mathcal{F}\otimes_{\mathcal{O}_X}\mathfrak{P}$ is contraherent, too. Under some other assumptions one can apply the (derived or underived) *contraherator* functor $\mathbb{L}\,\mathfrak{C}$ or $\mathfrak{C}$ to the cosheaf $\mathcal{F}\otimes_{\mathcal{O}_X}\mathfrak{P}$ to obtain the (complex of) contraherent cosheaves $\mathcal{F}\otimes_{X\text{-ct}}^{\mathbb{L}}\mathfrak{P}$ or $\mathcal{F}\otimes_{X\text{-ct}}\mathfrak{P}$. These are the analogues of the quasi-coherent internal Hom functor $\mathcal{H}om_{X\text{-qc}}$ on the quasi-coherent sheaves, which can be obtained by applying the coherator functor $\mathcal{Q}$ [26, Appendix B] to the $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ sheaf.

The remaining two operations are harder to come by. Modelled after the comodule-contramodule correspondence functors $\Phi_{\mathcal{C}}$ and $\Psi_{\mathcal{C}}$ from [19], they play a similarly crucial role in our present co-contra correspondence theory. Given a quasi-coherent sheaf sheaf $\mathcal{F}$ and a sheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a semi-separated scheme X, one constructs a quasi-coherent sheaf $\mathcal{F}\odot_X\mathfrak{P}$ on X. Given two quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{P}$ on a semi-separated scheme, under certain adjustness assuptions one can constuct a contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})$.

Derived categories of the second kind, whose roots go back to the work of Husemoller, Moore, and Stasheff on two kinds of differential derived functors [13] and Hinich's paper about DG-coalgebras [12], were introduced in their present form in the author's monograph [19] and memoir [20]. The most important representatives of this class of derived category constructions are known as the *coderived* and the *contraderived* categories; the difference between them consists in the use of the closure with respect to infinite direct sums in one case and with respect to infinite products in the other. Here is a typical example of how they occur.

According to Iyengar and Krause [14], the homotopy category of complexes of projective modules over a Noetherian commutative ring with a dualizing complex is equivalent to the homotopy category of complexes of injective modules. This result to Noetherian schemes with dualizing complexes by Neeman [18] and Murfet [15] in the following form: the derived category $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ of the exact category of flat quasi-coherent sheaves on such a scheme X is equivalent to the homotopy category of injective quasi-coherent sheaves $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$. These results are known as the *covariant Serre–Grothendieck duality* theory.

One would like to reformulate this equivalence so that it connects certain derived categories of modules/sheaves, rather than just subcategories of resolutions. In other words, it would be nice to have some procedure assigning complexes of projective, flat, and/or injective modules/sheaves to arbitrary complexes.

In the case of modules, the homotopy category of projectives is identified with the contraderived category of the abelian category of modules, while the homotopy category of injectives is equivalent to the coderived category of the same abelian category. Hence the Iyengar–Krause result is interpreted as an instance of the "co-contra correspondence"—in this case, an equivalence between the coderived and contraderived categories of the same abelian category.

In the case of quasi-coherent sheaves, however, only a half of the above picture remains true. The homotopy category of injectives $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ is still equivalent to the coderived category of quasi-coherent sheaves $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$. But the attempt to similarly describe the derived category of flats runs into the problem that the infinite products of quasi-coherent sheaves are not exact, so the contraderived category construction does not make sense for them.

This is where the contraherent cosheaves come into play. The covariant Serre–Grothendieck duality for nonaffine schemes is an equivalence of *four* triangulated categories, rather than just two. In addition to the derived category of flat quasi-coherent sheaves $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ and the homotopy category of injective quasi-coherent sheaves $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$, there are also the homotopy category of projective contraherent cosheaves $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ and the derived category of locally injective contraherent cosheaves $\mathsf{D}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$. And just as the homotopy category of injectives $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ is equivalent to the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$, the homotopy category of projectives $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ is identified with the contraderived category $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ of contraherent cosheaves.

To end, let us describe some prospects for future research and applications of contraherent cosheaves. One of such expected applications is related to the $\mathfrak{D}$–$\mathbf{\Omega}$ duality theory. Here $\mathfrak{D}$ stands for the sheaf of rings of differential operators on a smooth scheme and $\mathbf{\Omega}$ denotes the de Rham DG-algebra. The derived $\mathfrak{D}$–$\mathbf{\Omega}$ duality, as developed in [20, Appendix B], happens on two sides. On the "co" side, the functor $\mathcal{H}om_{\mathcal{O}_X}(\mathbf{\Omega},-)$ takes right $\mathfrak{D}$-modules to right DG-modules over $\mathbf{\Omega}$, and there is the adjoint functor $-\otimes_{\mathcal{O}_X}\mathfrak{D}$. These functors induce an equivalence between the derived category of $\mathfrak{D}$-modules and the coderived category of DG-modules over $\mathbf{\Omega}$.

On the "contra" side, over an affine scheme X the functor $\mathbf{\Omega}(X)\otimes_{\mathcal{O}(X)}-$ takes left $\mathfrak{D}$-modules to left DG-modules over $\mathbf{\Omega}$, and the adjoint functor is $\mathrm{Hom}_{\mathcal{O}(X)}(\mathfrak{D}(X),-)$. The induced equivalence is between the derived category of $\mathfrak{D}(X)$-modules and the contraderived category of DG-modules over $\mathbf{\Omega}(X)$. One would like to extend the "contra" side of the story to nonaffine schemes using the contraherent cosheafification, as opposed to the quasi-coherent sheafification on the "co" side. The need for the contraherent cosheaves arises, once again, because the contraderived category construction does not make sense for quasi-coherent sheaves. The contraherent cosheaves are designed for being plugged in to it.

Another direction in which we would like to extend the theory presented below is that of (at least, Noetherian) formal schemes. The idea is to define an exact category of contraherent cosheaves of contramodules over a formal scheme that would serve as the natural "contra"-side counterpart to the abelian category of quasi-coherent torsion sheaves on the "co" side. (For a taste of the contramodule theory over complete Noetherian rings, the reader is referred to [23, Appendix B].)

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1. Contraadjusted and Cotorsion Modules

1.1. Contraadjusted and very flat modules. Let R be a commutative ring. We will say that an R-module P is *contraadjusted* if the R-module $\operatorname{Ext}^1_R(R[r^{-1}],P)$ vanishes for every element $r\in R$. An R-module F is called *very flat* if one has $\operatorname{Ext}^1_R(F,P)=0$ for every contraadjusted R-module P.

By the definition, any injective R-module is contraadjusted and any projective R-module is very flat. Notice that the projective dimension of the R-module $R[r^{-1}]$ never exceeds 1, as it has a natural two-term free resolution $0\longrightarrow\bigoplus_{n=0}^\infty R\longrightarrow\bigoplus_{n=0}^\infty R\longrightarrow R[r^{-1}]\longrightarrow 0$. It follows that any quotient module of a contraadjusted module is contraadjusted, and one has $\operatorname{Ext}^{>0}_R(F,P)=0$ for any very flat R-module F and contraadjusted R-module P.

Computing the Ext^1 in terms of the above resolution, one can more explicitly characterise contraadjusted R-modules as follows. An R-module P is contraadjusted if and only if for any sequence of elements p_0, p_1, p_2, $\ldots\in P$ and $r\in R$ there exists a (not necessarily unique) sequence of elements q_0, q_1, q_2, $\ldots\in P$ such that $q_i=p_i+rq_{i+1}$ for all $i\geqslant 0$.

Furthermore, the projective dimension of any very flat module is equal to 1 or less. Indeed, any R-module M has a two-term right resolution by contraadjusted modules, which can be used to compute $\operatorname{Ext}^*_R(F,M)$ for a very flat R-module F.

It is also clear that the classes of contraadjusted and very flat modules are closed under extensions. Besides, the class of very flat modules is closed under the passage to the kernel of a surjective morphism (i. e., the kernel of a surjective morphism of very flat R-modules is very flat). In addition, we notice that the class of contraadjusted R-modules is closed under infinite products, while the class of very flat R-modules is closed under infinite direct sums.

Theorem 1.1.1. (a) *Any R-module can be embedded into a contraadjusted R-module in such a way that the quotient module is very flat.*

(b) *Any R-module admits a surjective map onto it from a very flat R-module such that the kernel is contraadjusted.*

Proof. Both assertions follow from the results of Eklof and Trlifaj [7, Theorem 10]. It suffices to point out that all the R-modules of the form $R[r^{-1}]$ form a set rather than a proper class. For the reader's convenience and our future use, parts of the argument from [7] are reproduced below. $\square$

The following definition will be used in the sequel. Let A be an abelian category with exact functors of inductive limit, and let $\mathsf{C} \subset \mathsf{A}$ be a class of objects. An object $X \in \mathsf{A}$ is said to be a *transfinitely iterated extension* of objects from C if there exist a well-ordered set Γ and a family of subobjects $X_\gamma \subset X$, $\gamma \in \Gamma$, such that $X_\delta \subset X_\gamma$ whenever $\delta < \gamma$, the union (inductive limit) $\varinjlim_{\gamma\in\Gamma} X_\gamma$ of all X_γ coincides with X, and the quotient objects $X_\gamma / \varinjlim_{\delta<\gamma} X_\delta$ belong to C for all $\gamma \in \Gamma$ (cf. Section 4.1).

By [7, Lemma 1], any transfinitely iterated extension of the R-modules $R[r^{-1}]$, with arbitrary $r \in R$, is a very flat R-module. Proving a converse assertion will be one of our goals. The following lemma is a particular case of [7, Theorem 2].

Lemma 1.1.2. *Any R-module can be embedded into a contraadjusted R-module in such a way that the quotient module is a transfinitely iterated extension of the R-modules $R[r^{-1}]$.*

Proof. The proof is a set-theoretic argument based on the fact that the Cartesian square of any infinite cardinality λ is equicardinal to λ. In our case, let λ be any infinite cardinality no smaller than the cardinality of the ring R. For an R-module L of the cardinality not exceeding λ and an R-module M of the cardinality μ, the set $\mathrm{Ext}^1_R(L, M)$ has the cardinality at most μ^λ, as one can see by computing the Ext^1 in terms of a projective resolution of the first argument. In particular; set $\aleph = 2^\lambda$; then for any R-module M of the cardinality not exceeding $\aleph$ and any $r \in R$ the cardinality of the set $\mathrm{Ext}^1_R(R[r^{-1}], M)$ does not exceed $\aleph$, either.

Let $\daleth$ be the smallest cardinality that is larger than $\aleph$ and let Δ be the smallest ordinal of the cardinality $\daleth$. Notice that the natural map $\varinjlim_{\delta\in\Delta} \mathrm{Ext}^*_R(L, Q_\delta) \longrightarrow \mathrm{Ext}^*_R(L, \varinjlim_{\delta\in\Delta} Q_\delta)$ is an isomorphism for any R-module L of the cardinality not exceeding λ (or even $\aleph$) and any inductive system of R-modules Q_δ indexed by Δ. Indeed, the functor of filtered inductive limit of abelian groups is exact and the natural map $\varinjlim_{\delta\in\Delta} \mathrm{Hom}_R(L, Q_\delta) \longrightarrow \mathrm{Hom}_R(L, \varinjlim_{\delta\in\Delta} Q_\delta)$ is an isomorphism for any R-module L of the cardinality not exceeding $\aleph$. The latter assertion holds because the image of any map of sets $L \longrightarrow \Delta$ is contained in a proper initial segment $\{\delta' \mid \delta' < \delta\} \subset \Delta$ for some $\delta \in \Delta$.

We proceed by induction on Δ constructing for every element $\delta \in \Delta$ an R-module P_δ and a well-ordered set Γ_δ. For any $\delta' < \delta$, we will have an embedding of R-modules $P_{\delta'} \longrightarrow P_\delta$ (such that the three embeddings form a commutative diagram for any three elements $\delta'' < \delta' < \delta$) and an ordered embedding $\Gamma_{\delta'} \subset \Gamma_\delta$ (making $\Gamma_{\delta'}$ an initial segment of Γ_δ). Furthermore, for every element $\gamma \in \Gamma_\delta$, a particular extension class $c(\gamma, \delta) \in \mathrm{Ext}^1_R(R[r(\gamma)^{-1}], P_\delta)$, where $r(\gamma) \in R$, will be defined. For every $\delta' < \delta$ and $\gamma \in \Gamma_{\delta'}$, the class $c(\gamma, \delta)$ will be equal to the image of the class $c(\gamma, \delta')$ with respect to the natural map $\operatorname{Ext}^1_R(R[r(\gamma)^{-1}], P_{\delta'}) \longrightarrow \operatorname{Ext}^1_R(R[r(\gamma)^{-1}], P_\delta)$ induced by the embedding $P_{\delta'} \longrightarrow P_\delta$.

At the starting point $0 \in \Delta$, the module P_0 is our original R-module M and the set Γ_0 is the disjoint union of all the sets $\operatorname{Ext}^1_R(R[r^{-1}], M)$ with $r \in R$, endowed with an arbitrary well-ordering. The elements $r(\gamma)$ and the classes $c(\gamma, 0)$ for $\gamma \in \Gamma_0$ are defined in the obvious way.

Given $\delta = \delta' + 1 \in \Delta$ and assuming that the R-module $P_{\delta'}$ and the set $\Gamma_{\delta'}$ have been constructed already, we produce the module P_δ and the set Γ_δ as follows. It will be clear from the construction below that δ' is always smaller than the well-ordering type of the set $\Gamma_{\delta'}$. So there is a unique element $\gamma_{\delta'} \in \Gamma_{\delta'}$ corresponding to the ordinal δ' (i. e., such that the well-ordering type of the subset all the elements in $\Gamma_{\delta'}$ that are smaller than $\gamma_{\delta'}$ is equivalent to δ').

Define the R-module P_δ as the middle term of the extension corresponding to the class $c(\gamma_{\delta'}, \delta') \in \operatorname{Ext}^1_R(R[r(\gamma_{\delta'})^{-1}], P_{\delta'})$. There is a natural embedding $P_{\delta'} \longrightarrow P_\delta$, as required. Set Γ_δ to be the disjoint union of $\Gamma_{\delta'}$ and the sets $\operatorname{Ext}^1_R(R[r^{-1}], P_\delta)$ with $r \in R$, well-ordered so that $\Gamma_{\delta'}$ is an initial segment, while the well-ordering of the remaining elements is chosen arbitrarily. The elements $r(\gamma)$ for $\gamma \in \Gamma_{\delta'}$ have been defined already on the previous steps and the classes $c(\gamma, \delta)$ for such γ are defined in the unique way consistent with the previous step, while for the remaining $\gamma \in \Gamma_\delta \setminus \Gamma_{\delta'}$ these elements and classes are defined in the obvious way.

When δ is a limit ordinal, set $P_\delta = \varinjlim_{\delta'<\delta} P'_\delta$. Let Γ_δ be the disjoint union of $\bigcup_{\delta'<\delta} \Gamma_{\delta'}$ and the sets $\operatorname{Ext}^1_R(R[r^{-1}], P_\delta)$ with $r \in R$, well-ordered so that $\bigcup_{\delta'<\delta} \Gamma_{\delta'}$ is an initial segment. The elements $r(\gamma)$ and $c(\gamma, \delta)$ for $\gamma \in \Gamma_\delta$ are defined as above.

Arguing by transfinite induction, one easily concludes that the cardinality of the R-module P_δ never exceeds $\aleph$ for $\delta \in \Delta$, and neither does the cardinality of the set Γ_δ. It follows that the well-ordering type of the set $\Gamma = \bigcup_{\delta \in \Delta} \Gamma_\delta$ is equal to Δ. So for every $\gamma \in \Gamma$ there exists $\delta \in \Delta$ such that $\gamma = \gamma_\delta$.

Set $P = \varinjlim_{\delta \in \Delta} P_\delta$. By construction, there is a natural embedding of R-modules $M \longrightarrow P$ and the cokernel is a transfinitely iterated extension of the R-modules $R[r^{-1}]$. As every class $c \in \operatorname{Ext}^1_R(R[r^{-1}], P_\delta)$ corresponds to an element $\gamma \in \Gamma_\delta$, has the corresponding ordinal $\delta' \in \Delta$ such that $\gamma = \gamma_{\delta'}$, and dies in $\operatorname{Ext}^1_R(R[r^{-1}], P_{\delta'+1})$, we conclude that $\operatorname{Ext}^1_R(R[r^{-1}], P) = 0$. ☐

Lemma 1.1.3. *Any R-module admits a surjective map onto it from a transfinitely iterated extension of the R-modules $R[r^{-1}]$ such that the kernel is contraadjusted.*

Proof. The proof follows the second half of the proof of Theorem 10 in [7]. Specifically, given an R-module M, pick a surjective map onto it from a free R-module L. Denote the kernel by K and embed it into a contraadjusted R-module P so that the quotient module Q is a transfinitely iterated extension of the R-modules $R[r^{-1}]$. Then the fibered coproduct F of the R-modules L and P over K is an extension of the R-modules Q and L. It also maps onto M surjectively with the kernel P. ☐

Both assertions of Theorem 1.1.1 are now proven.

Corollary 1.1.4. *An R-module is very flat if and only if it is a direct summand of a transfinitely iterated extension of the R-modules $R[r^{-1}]$.*

Proof. The "if" part has been explained already; let us prove "only if". Given a very flat R-module F, present it as the quotient module of a transfinitely iterated extension E of the R-modules $R[r^{-1}]$ by a contraadjusted R-module P. Since $\operatorname{Ext}^1_R(F,P)=0$, we can conclude that F is a direct summand of E. ☐

In particular, we have proven that any very flat R-module is flat.

Corollary 1.1.5. (a) *Any very flat R-module can be embedded into a contraadjusted very flat R-module in such a way that the quotient module is very flat.*

(b) *Any contraadjusted R-module admits a surjective map onto it from a very flat contraadjusted R-module such that the kernel is contraadjusted.*

Proof. Follows from Theorem 1.1.1 and the fact that the classes of contraadjusted and very flat R-modules are closed under extensions. ☐

1.2. **Affine geometry of contraadjusted and very flat modules.** The results of this section form the module-theoretic background of our main definitions and constructions in Sections 2–3.

Lemma 1.2.1. (a) *The class of very flat R-modules is closed with respect to the tensor products over R.*

(b) *For any very flat R-module F and contraadjusted R-module P, the R-module* $\operatorname{Hom}_R(F,P)$ *is contraadjusted.*

Proof. One approach is to prove both assertions simultaneously using the adjunction isomorphism $\operatorname{Ext}^1_R(F\otimes_R G,\ P)\simeq \operatorname{Ext}^1_R(G,\operatorname{Hom}_R(F,P))$, which clearly holds for any R-module G, any very flat R-module F, and contraadjusted R-module P, and raising the generality step by step. Since $R[r^{-1}]\otimes_R R[s^{-1}]\simeq R[(rs)^{-1}]$, it follows that the R-module $\operatorname{Hom}_R(R[s^{-1}],P)$ is contraadjusted for any contraadjusted R-module P and $s\in R$. Using the same adjunction isomorphism, one then concludes that the R-module $R[s^{-1}]\otimes_R G$ is very flat for any very flat R-module G. From this one can deduce in full generality the assertion (b), and then the assertion (a).

Alternatively, one can use the full strength of Corollary 1.1.4 and check that the tensor product of two transfinitely iterated extensions of flat modules is a transfinitely iterated extension of the pairwise tensor products. Then deduce (b) from (a). ☐

Lemma 1.2.2. *Let $f\colon R\longrightarrow S$ be a homomorphism of commutative rings. Then*

(a) *any contraadjusted S-module is also a contraadjusted R-module in the R-module structure obtained by the restriction of scalars via f;*

(b) *if F is a very flat R-module, then the S-module $S\otimes_R F$ obtained by the extension of scalars via f is also very flat;*

(c) *if F is a very flat R-module and Q is a contraadjusted S-module, then* $\operatorname{Hom}_R(F,Q)$ *is also a contraadjusted S-module;*

(d) *if F is a very flat R-module and G is a very flat S-module, then $F\otimes_R G$ is also a very flat S-module.*

Proof. Part (a): one has $\mathrm{Ext}_R^*(R[r^{-1}],P)\simeq\mathrm{Ext}_S^*(S[f(r)^{-1}],P)$ for any R-module P and $r\in R$. Part (b) follows from part (a), or alternatively, from Corollary 1.1.4. To prove part (c), notice that $\mathrm{Hom}_R(F,Q)\simeq\mathrm{Hom}_S(S\otimes_R F,\,Q)$ and use part (b) together with Lemma 1.2.1(b) (applied to the ring S). Similarly, part (d) follows from part (b) and Lemma 1.2.1(a). □

Lemma 1.2.3. *Let $f\colon R\longrightarrow S$ be a homomorphism of commutative rings such that the localization $S[s^{-1}]$ is a very flat R-module for any element $s\in S$. Then*

(a) *the S-module* $\mathrm{Hom}_R(S,P)$ *obtained by the coextension of scalars via f is contraadjusted for any contraadjusted R-module P;*

(b) *any very flat S-module is also a very flat R-module (in the R-module structure obtained by the restriction of scalars via f);*

(c) *the S-module* $\mathrm{Hom}_R(G,P)$ *is contraadjusted for any very flat S-module G and contraadjusted R-module P.*

Proof. Part (a): one has $\mathrm{Ext}_S^1(S[s^{-1}],\mathrm{Hom}_R(S,P))\simeq\mathrm{Ext}_R^1(S[s^{-1}],P)$ for any R-module P such that $\mathrm{Ext}_R^1(S,P)=0$ and any $s\in S$. Part (b) follows from part (a), or alternatively, from Corollary 1.1.4. Part (c): one has $\mathrm{Ext}_S^1(S[s^{-1}],\mathrm{Hom}_R(G,P))\simeq\mathrm{Ext}_R^1(G\otimes_S S[s^{-1}],\,P)$. By Lemma 1.2.1(a), the S-module $G\otimes_S S[s^{-1}]$ is very flat; by part (b), it is also a very flat R-module; so the desired vanishing follows. □

Lemma 1.2.4. *Let $R\longrightarrow S$ be a homomorphism of commutative rings such that the related morphism of affine schemes* $\operatorname{Spec} S\longrightarrow\operatorname{Spec} R$ *is an open embedding. Then S is a very flat R-module.*

Proof. The open subset $\operatorname{Spec} S\subset\operatorname{Spec} R$, being quasi-compact, can be covered by a finite number of principal affine open subsets $\operatorname{Spec} R[r_\alpha^{-1}]\subset\operatorname{Spec} R$, where $\alpha=1,\dots,N$. The Čech sequence

$$(1)\quad 0\longrightarrow S\longrightarrow\textstyle\bigoplus_\alpha R[r_\alpha^{-1}]\longrightarrow\bigoplus_{\alpha<\beta}R[(r_\alpha r_\beta)^{-1}]\longrightarrow\cdots\longrightarrow R[(r_1\cdots r_N)^{-1}]\longrightarrow 0$$

is an exact sequence of S-modules, since is its localization by every element r_α is exact. It remains to recall that the class of very flat R-modules is closed under the passage to the kernels of surjective morphisms. □

Corollary 1.2.5. *The following assertions hold in the assumptions of Lemma* 1.2.4.

(a) *The S-module $S\otimes_R F$ is very flat for any very flat R-module F.*

(b) *An S-module G is very flat if and only if it is very flat as an R-module.*

(c) *The S-module* $\mathrm{Hom}_R(S,P)$ *is contraadjusted for any contraadjusted R-module P.*

(d) *An S-module Q is contraadjusted if and only if it is contraadjusted as an R-module.*

Proof. Part (a) is a particular case of Lemma 1.2.2(b). Part (b): if G is a very flat S-module, then it is also very flat as an R-module by Lemma 1.2.3(b) and Lemma 1.2.4. Conversely, if G is very flat as an R-module, then $G\simeq S\otimes_R G$ is also a very flat S-module by part (a).

Part (c) follows from Lemma 1.2.3(a) and Lemma 1.2.4. Part (d): if Q is a contraadjusted S-module, then it is also contraadjusted as an R-module by Lemma 1.2.2(a). Conversely, for any S-module Q there are natural isomorphisms of S-modules $Q \simeq \operatorname{Hom}_S(S,Q) \simeq \operatorname{Hom}_S(S\otimes_R S,\ Q) \simeq \operatorname{Hom}_R(S,Q)$; and if Q is contraadjusted as an R-module, then it is also a contraadjusted S-module by part (c). $\square$

Lemma 1.2.6. *Let $R \longrightarrow S_\alpha$, $\alpha = 1, \dots, N$, be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ is a finite open covering. Then*

(a) *an R-module F is very flat if and only if all the S_α-modules $S_\alpha \otimes_R F$ are very flat;*

(b) *for any contraadjusted R-module P, the Čech sequence*

$$(2)\quad 0 \longrightarrow \operatorname{Hom}_R(S_1\otimes_R\cdots\otimes_R S_N,\ P) \longrightarrow \cdots \\ \longrightarrow \textstyle\bigoplus_{\alpha<\beta}\operatorname{Hom}_R(S_\alpha\otimes_R S_\beta,\ P) \longrightarrow \bigoplus_\alpha \operatorname{Hom}_R(S_\alpha,\ P) \longrightarrow P \longrightarrow 0$$

is an exact sequence of R-modules.

Proof. Part (a): by Corollary 1.2.5(a-b), all the R-modules $S_{\alpha_1}\otimes_R\cdots\otimes_R S_{\alpha_k}\otimes_R F$ are very flat whenever the S_α-modules $S_\alpha\otimes_R F$ are very flat. For any R-module F the Čech sequence

$$(3)\quad 0 \longrightarrow F \longrightarrow \textstyle\bigoplus_\alpha S_\alpha\otimes_R F \longrightarrow \bigoplus_{\alpha<\beta} S_\alpha\otimes_R S_\beta\otimes_R F \\ \longrightarrow \cdots \longrightarrow S_1\otimes_R\cdots\otimes_R S_N\otimes_R F \longrightarrow 0$$

is an exact sequence of R-modules (since its localization at any prime ideal of R is). It remains to recall that the class of very flat R-modules is closed with respect to the passage to the kernels of surjections.

Part (b): the exact sequence of R-modules

$$(4)\quad 0 \longrightarrow R \longrightarrow \textstyle\bigoplus_\alpha S_\alpha \longrightarrow \bigoplus_{\alpha<\beta} S_\alpha\otimes_R S_\beta \\ \longrightarrow \cdots \longrightarrow S_1\otimes_R\cdots\otimes_R S_N \longrightarrow 0$$

is composed from short exact sequences of very flat R-modules, so the functor $\operatorname{Hom}_R(-,P)$ into a contraadjusted R-module P preserves its exactness. $\square$

Lemma 1.2.7. *Let $f_\alpha\colon S \longrightarrow T_\alpha$, $\alpha = 1, \dots, N$, be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes $\operatorname{Spec} T_\alpha \longrightarrow \operatorname{Spec} S$ is a finite open covering, and let $R \longrightarrow S$ be a homomorphism of commutative rings. Then*

(a) *if all the R-modules $T_{\alpha_1}\otimes_S\cdots\otimes_S T_{\alpha_k}$ are very flat, then the R-module S is very flat;*

(b) *the R-modules $T_\alpha[t_\alpha^{-1}]$ are very flat for all $t_\alpha\in T_\alpha$, $1\leqslant\alpha\leqslant N$, if and only if the R-module $S[s^{-1}]$ is very flat for all $s\in S$.*

Proof. Part (a) follows from the Čech exact sequence (cf. (4))

$$0 \longrightarrow S \longrightarrow \bigoplus\nolimits_\alpha T_\alpha \longrightarrow \bigoplus\nolimits_{\alpha<\beta} T_\alpha \otimes_S T_\beta \\ \longrightarrow \dotsb \longrightarrow T_1 \otimes_S \dotsb \otimes_S T_N \longrightarrow 0.$$

To prove the "if" assertion in (b), notice that $\operatorname{Spec} T_\alpha[t_\alpha^{-1}]$ as an open subscheme in $\operatorname{Spec} S$ can be covered by a finite number of principal affine open subschemes $\operatorname{Spec} S[s^{-1}]$, and the intersections of these are also principal open affines. The "only if" assertion follows from part (a) applied to the covering of the affine scheme $\operatorname{Spec} S[s^{-1}]$ by the affine open subschemes $\operatorname{Spec} T_\alpha[f_\alpha(s)^{-1}]$, together with the fact that the intersection of every subset of these open affines can be covered by a finite number of principal affine open subschemes of one of the schemes $\operatorname{Spec} T_\alpha$. □

Lemma 1.2.8. *Let $f_\alpha\colon S \longrightarrow T_\alpha$, $\alpha = 1, \dots, N$, be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes $\operatorname{Spec} T_\alpha \longrightarrow \operatorname{Spec} S$ is a finite open covering, and let $R \longrightarrow S$ be a homomorphism of commutative rings. Let F be an S-module. Then*

(a) *if all the R-modules $T_{\alpha_1} \otimes_S \dotsb \otimes_S T_{\alpha_k} \otimes_S F$ are very flat, then the R-module F is very flat;*

(b) *the R-modules $T_\alpha[t_\alpha^{-1}] \otimes_S F$ are very flat for all $t_\alpha \in T_\alpha$, $1 \leqslant \alpha \leqslant N$, if and only if the R-module $F[s^{-1}]$ is very flat for all $s \in S$.*

Proof. Just as in the previous lemma, part (a) follows from the Čech exact sequence (3) constructed for the collection of morphisms of rings $S \longrightarrow T_\alpha$ and the S-module F. The proof of part (b) is similar to that of Lemma 1.2.7(b). □

1.3. **Exact category of contraadjusted modules.** As full subcategories of the abelian category of R-modules closed under extensions, the categories of contraadjusted and very flat R-modules have natural exact category structures. In the exact category of contraadjusted R-modules every morphism has a cokernel, which is, in addition, an admissible epimorphism.

In the exact category of contraadjusted R-modules the functors of infinite product are everywhere defined and exact; they also agree with the infinite products in the abelian category of R-modules. In the exact category of very flat R-modules, the functors of infinite direct sum are everywhere defined and exact, and agree with the infinite direct sums in the abelian category of R-modules.

It is clear from Corollary 1.1.5(b) that there are enough projective objects in the exact category of very flat R-modules; these are precisely the very flat contraadjusted R-modules. Similarly, by Corollary 1.1.5(a) in the exact category of very flat R-modules there are enough injective objects; these are also precisely the very flat contraadjusted modules.

Denote the exact category of contraadjusted R-modules by $R\text{–}\mathsf{mod}^{\mathsf{cta}}$ and the exact category of very flat R-modules by $R\text{–}\mathsf{mod}^{\mathsf{vfl}}$. The tensor product of two very flat R-modules is an exact functor of two arguments $R\text{–}\mathsf{mod}^{\mathsf{vfl}} \times R\text{–}\mathsf{mod}^{\mathsf{vfl}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{vfl}}$. The Hom_R from a very flat R-module into a contraadjusted R-module is an exact functor of two arguments $(R\text{–}\mathsf{mod}^{\mathsf{vfl}})^{\mathsf{op}} \times R\text{–}\mathsf{mod}^{\mathsf{cta}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$ (where C^{op} denotes the opposite category to a category C).

For any homomorphism of commutative rings $f\colon R \longrightarrow S$, the restriction of scalars with respect to f is an exact functor $S\text{–}\mathsf{mod}^{\mathsf{cta}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$. The extension of scalars $F \longmapsto S \otimes_R F$ is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{vfl}} \longrightarrow S\text{–}\mathsf{mod}^{\mathsf{vfl}}$.

For any homomorphism of commutative rings $f\colon R \longrightarrow S$ satisfying the condition of Lemma 1.2.3, the restriction of scalars with respect to f is an exact functor $S\text{–}\mathsf{mod}^{\mathsf{vfl}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{vfl}}$. The coextension of scalars $P \longmapsto \operatorname{Hom}_R(S,P)$ is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{cta}} \longrightarrow S\text{–}\mathsf{mod}^{\mathsf{cta}}$. In particular, these assertions hold for any homomorphism of commutative rings $R \longrightarrow S$ satysfying the assumption of Lemma 1.2.4.

Lemma 1.3.1. *Let $R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ is an open covering. Then*

(a) *a pair of homomorphisms of contraadjusted R-modules $K \longrightarrow L \longrightarrow M$ is a short exact sequence if and only if such are the induced sequences of contraadjusted S_α-modules $\operatorname{Hom}_R(S_\alpha,K) \longrightarrow \operatorname{Hom}_R(S_\alpha,L) \longrightarrow \operatorname{Hom}_R(S_\alpha,M)$ for all α;*

(b) *a homomorphism of contraadjusted R-modules $P \longrightarrow Q$ is an admissible epimorphism in $R\text{–}\mathsf{mod}^{\mathsf{cta}}$ if and only if the induced homomorphisms of contraadjusted S_α-modules $\operatorname{Hom}_R(S_\alpha,P) \longrightarrow \operatorname{Hom}_R(S_\alpha,Q)$ are admissible epimorphisms in $S_\alpha\text{–}\mathsf{mod}^{\mathsf{cta}}$ for all α.*

Proof. Part (a): the "only if" assertion follows from Lemma 1.2.4. For the same reason, if the sequences $0 \longrightarrow \operatorname{Hom}_R(S_\alpha,K) \longrightarrow \operatorname{Hom}_R(S_\alpha,L) \longrightarrow \operatorname{Hom}_R(S_\alpha,M) \longrightarrow 0$ are exact, then so are the sequences obtained by applying the functors $\operatorname{Hom}_R(S_{\alpha_1} \otimes_R \cdots \otimes_R S_{\alpha_k},\, -)$, $k \geqslant 1$, to the sequence $K \longrightarrow L \longrightarrow M$. Now it remains to make use of Lemma 1.2.6(b) in order to deduce exactness of the original sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$.

Part (b): it is clear from the very right segment of the exact sequence (2) that surjectivity of the maps $\operatorname{Hom}_R(S_\alpha,P) \longrightarrow \operatorname{Hom}_R(S_\alpha,Q)$ implies surjectivity of the map $P \longrightarrow Q$. It remains to check that the kernel of the latter morphism is a contraadjusted R-module. Denote this kernel by K. Since the morphisms $\operatorname{Hom}_R(S_\alpha,P) \longrightarrow \operatorname{Hom}_R(S_\alpha,Q)$ are admissible epimorphisms, so are all the morphisms obtained by applying the coextension of scalars with respect to the ring homomorphisms $R \longrightarrow S_{\alpha_1} \otimes_R \cdots \otimes_R S_{\alpha_k}$, $k \geqslant 1$, to the morphism $P \longrightarrow Q$.

Now Lemma 1.2.6(b) applied to both sides of the morphism $P \longrightarrow Q$ provides a termwise surjective morphism of finite exact sequences of R-modules. The corresponding exact sequence of kernels has K as its rightmost nontrivial term, while by Lemma 1.2.2(a) all the other terms are contraadjusted R-modules. It follows that the R-module K is also contraadjusted. ☐

1.4. **Cotorsion modules.** Let R be an associative ring. A left R-module P is said to be *cotorsion* [27, 8] if $\operatorname{Ext}^1_R(F,P) = 0$ for any flat left R-module F, or equivalently, $Ext_R^{>0}(F,P) = 0$ for any flat left R-module F. Clearly, the class of cotorsion left

R-modules is closed under extensions and the passage to the cokernels of embeddings, and also under infinite products.

The following theorem, previously known essentially as the "flat cover conjecture", was proven by Eklof–Trlifaj [7] and Bican–Bashir–Enochs [9] (cf. our Theorem 1.1.1).

Theorem 1.4.1. (a) *Any R-module can be embedded into a cotorsion R-module in such a way that the quotient module is flat.*

(b) *Any R-module admits a surjective map onto it from a flat R-module such that the kernel is cotorsion.* ☐

The following results concerning cotorsion (and injective) modules are similar to the results about contraadjusted modules presented in Section 1.2. With a possible exception of the last lemma, all of these are very well known.

Lemma 1.4.2. *Let R be a commutative ring. Then*

(a) *for any flat R-module F and cotorsion R-module P, the R-module* $\mathrm{Hom}_R(F, P)$ *is cotorsion;*

(b) *for any R-module M and any injective R-module J, the R-module* $\mathrm{Hom}_R(M, J)$ *is cotorsion;*

(c) *for any flat R-module M and any injective R-module J, the R-module* $\mathrm{Hom}_R(F, J)$ *is injective.*

Proof. One has $\mathrm{Ext}^1_R(G, \mathrm{Hom}_R(F, P)) \simeq \mathrm{Ext}^1_R(F \otimes_R G,\ P)$ for any R-modules F, G, and P such that $\mathrm{Ext}^1_R(F, P) = 0 = \mathrm{Tor}^R_1(F, G)$. All the three assertions follow from this simple observation. ☐

Our next lemma is a generalization of Lemma 1.4.2 to the noncommutative case.

Lemma 1.4.3. *Let R and S be associative rings. Then*

(a) *for any R-flat R-S-bimodule F and any cotorsion left R-module P, the left S-module* $\mathrm{Hom}_R(F, P)$ *is cotorsion;*

(b) *for any R-S-bimodule M and injective left R-module J, the left S-module* $\mathrm{Hom}_R(M, J)$ *is cotorsion;*

(c) *for any S-flat R-S-bimodule F and any injective left R-module J, the left S-module* $\mathrm{Hom}_R(F, J)$ *is injective.*

Proof. One has $\mathrm{Ext}^1_S(G, \mathrm{Hom}_R(F, P)) \simeq \mathrm{Ext}^1_R(F \otimes_S G,\ P)$ for any R-S-bimodule F, left S-module G, and left R-module P such that $\mathrm{Ext}^1_R(F, P) = 0 = \mathrm{Tor}^S_1(F, G)$. Besides, the tensor product $F \otimes_S G$ is flat over R if F is flat over R and G is flat over S. This proves (a); and (b-c) are even easier. ☐

Lemma 1.4.4. *Let $f\colon R \longrightarrow S$ be a homomorphism of associative rings. Then*

(a) *any cotorsion left S-module is also a cotorsion left R-module in the R-module structure obtained by the restriction of scalars via f;*

(b) *the left S-module* $\mathrm{Hom}_R(S, J)$ *obtained by coextension of scalars via f is injective for any injective left S-module J.*

Proof. Part (a): one has $\mathrm{Ext}^1_R(F, P) \simeq \mathrm{Ext}^1_S(S \otimes_R F,\ P)$ for any flat left R-module F and any left S-module P. Part (b) is left to reader. ☐

Lemma 1.4.5. *Let $f\colon R \longrightarrow S$ be an associative ring homomorphism such that S is a flat left R-module in the induced R-module structure. Then*

(a) *the left S-module* $\operatorname{Hom}_R(S, P)$ *obtained by coextension of scalars via f is cotorsion for any cotorsion left R-module P;*

(b) *any injective right S-module is also an injective right R-module in the R-module structure obtained by the restriction of scalars via f.*

Proof. Part (a): one has $\operatorname{Ext}^1_S(F, \operatorname{Hom}_R(S, P)) \simeq \operatorname{Ext}^1_R(F, P)$ for any left R-module P such that $\operatorname{Ext}^1_R(S, P) = 0$ and any left S-module F. In addition, in the assumptions of Lemma any flat left S-module F is also a flat left R-module. □

Lemma 1.4.6. *Let $R \longrightarrow S_\alpha$ be a collection of commutative ring homomorphisms such that the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is an open covering. Then*

(a) *a contraadjusted R-module P is cotorsion if and only if all the contraadjusted S_α-modules* $\operatorname{Hom}_R(S_\alpha, P)$ *are cotorsion;*

(b) *a contraadjusted R-module J is injective if and only if all the contraadjusted S_α-modules* $\operatorname{Hom}_R(S_\alpha, J)$ *are injective.*

Proof. Part (a): the assertion "only if" follows from Lemma 1.4.5(a). To prove "if", use the Čech exact sequence (2) from Lemma 1.2.6(b). By Lemmas 1.4.4(a) and 1.4.5(a), all the terms of the sequence, except perhaps the rightmost one, are cotorsion R-modules, and since the class of cotorsion R-modules is closed under the cokernels of embeddings, it follows that the rightmost term is cotorsion as well.

Part (b) is proven in the similar way using parts (b) of Lemmas 1.4.4–1.4.5. □

1.5. **Exact category of cotorsion modules.** Let R be an associative ring. As a full subcategory of the abelian category of R-modules closed under extensions, the category of cotorsion left R-modules has a natural exact category structure.

The functors of infinite product are everywhere defined and exact in this exact category, and agree with the infinite products in the abelian category of R-modules. Similarly, the category of flat R-modules has a natural exact category structure with exact functors of infinite direct sum.

It follows from Theorem 1.4.1 that there are enough projective objects in the exact category of cotorsion R-modules; these are precisely the flat cotorsion R-modules. Similarly, there are enough injective objects in the exact category of flat R-modules, and these are also precisely the flat cotorsion R-modules.

Denote the exact category of cotorsion left R-modules by $R\text{–}\mathsf{mod}^{\mathsf{cot}}$ and the exact category of flat left R-modules by $R\text{–}\mathsf{mod}^{\mathsf{fl}}$. The abelian category of left R-modules will be denoted simply by $R\text{–}\mathsf{mod}$, and the additive category of injective R-modules (with the trivial exact category structure) by $R\text{–}\mathsf{mod}^{\mathsf{inj}}$.

For any commutative ring R, the Hom_R from a flat R-module into a cotorsion R-module is an exact functor of two arguments $(R\text{–}\mathsf{mod}^{\mathsf{fl}})^{\mathsf{op}} \times R\text{–}\mathsf{mod}^{\mathsf{cot}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. Analogously, the Hom_R from an arbitrary R-module into an injective R-module is an exact functor $(R\text{–}\mathsf{mod})^{\mathsf{op}} \times R\text{–}\mathsf{mod}^{\mathsf{inj}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. The functors

Hom over a noncommutative ring R mentioned in Lemma 1.4.3 have similar exactness properties.

For any associative ring homomorphism $f\colon R \longrightarrow S$, the restriction of scalars via f is an exact functor $S\text{–}\mathsf{mod}^{\mathsf{cot}} \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. For any associative ring homomorphism $f\colon R \longrightarrow S$ making S a flat left R-module, the coextension of scalars $P \longmapsto \operatorname{Hom}_R(S,P)$ is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{cot}} \longrightarrow S\text{–}\mathsf{mod}^{\mathsf{cot}}$.

Lemma 1.5.1. *Let $R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is an open covering. Then*

(a) *a pair of morphisms of cotorsion R-modules $K \longrightarrow L \longrightarrow M$ is a short exact sequence if and only if such are the sequences of cotorsion S_α-modules* $\operatorname{Hom}(S_\alpha, K) \longrightarrow \operatorname{Hom}(S_\alpha, L) \longrightarrow \operatorname{Hom}(S_\alpha, M)$ *for all α;*

(b) *a morphism of cotorsion R-modules $P \longrightarrow Q$ is an admissible epimorphism if and only if such are the morphisms of cotorsion S_α-modules* $\operatorname{Hom}_R(S_\alpha, P) \longrightarrow \operatorname{Hom}_R(S_\alpha, Q)$ *for all α.*

Proof. Part (a) follows from Lemma 1.3.1(a); part (b) can be proven in the way similar to Lemma 1.3.1(b). □

1.6. **Coherent rings and coadjusted modules.** Recall that an associative ring R is called *left coherent* if all its finitely generated left ideals are finitely presented. Finitely presented left modules over a left coherent ring R form an abelian subcategory in $R\text{–}\mathsf{mod}$ closed under kernels, cokernes, and extensions.

Lemma 1.6.1. *Let R and S be associative rings such that S is left coherent. Let F be a left R-module of finite projective dimension, P be an S-flat R-S-bimodule such that* $\operatorname{Ext}_R^{>0}(F,P) = 0$*, and M be a finitely presented left S-module. Then one has* $\operatorname{Ext}_R^{>0}(F,\ P \otimes_S M) = 0$*, the natural map of abelian groups* $\operatorname{Hom}_R(F,P) \otimes_S M \longrightarrow \operatorname{Hom}_R(F,\ P \otimes_S M)$ *is an isomorphism, and the right S-module* $\operatorname{Hom}_R(F,P)$ *is flat.*

Proof. Let $L_\bullet \longrightarrow M$ be a left resolution of M by finitely generated projective S-modules. Then $P \otimes_S L_\bullet \longrightarrow P \otimes_S M$ is a left resolution of the R-module $P \otimes_S M$ by R-modules annihilated by $\operatorname{Ext}^{>0}(F,-)$. Since the R-module F has finite projective dimension, it follows that $\operatorname{Ext}^{>0}(F,\ P \otimes_S M) = 0$.

Consequently, the functor $M \longmapsto \operatorname{Hom}_R(F,\ P \otimes_S M)$ is exact on the abelian category of finitely presented left S-modules M. Obviously, the functor $M \longmapsto \operatorname{Hom}_R(F,P) \otimes_R M$ is right exact. Since the morphism of functors $\operatorname{Hom}_R(F,P) \otimes_S M \longrightarrow \operatorname{Hom}_R(F,\ P \otimes_S M)$ is an isomorphism for finitely generated projective S-modules M, we can conclude that it is an isomorphism for all finitely presented left S-modules.

Now we have proven that the functor $M \longmapsto \operatorname{Hom}_R(F,P) \otimes_S M$ is exact on the abelian category of finitely presented left S-modules. Since any left S-module is a filtered inductive limit of finitely presented ones and the inductive limits commute with tensor products, it follows that the S-module $\operatorname{Hom}_R(F,P)$ is flat. □

Corollary 1.6.2. *Let R be a commutative ring. Then*

(a) *for any finitely generated R-module M and any contraadjusted R-module P, the R-module $M \otimes_R P$ is contraadjusted;*

(b) *if the ring R is coherent, then for any very flat R-module F and any flat contraadjusted R-module P, the R-module $\operatorname{Hom}_R(F, P)$ is flat and contraadjusted;*

(c) *in the situation of* (b)*, for any finitely presented R-module M the natural morphism of R-modules $\operatorname{Hom}_R(F, P) \otimes_R M \longrightarrow \operatorname{Hom}_F(F, P \otimes_R M)$ is an isomorphism.*

Proof. Part (a) immediately follows from the facts that the class of contraadjusted R-modules is closed under finite direct sums and quotients. Part (b) is provided Lemma 1.6.1 together with Lemma 1.2.1(b), and part (c) is also Lemma 1.6.1. ☐

Corollary 1.6.3. *Let R be a coherent commutative ring such that any flat R-module has a finite projective dimension. Then*

(a) *for any finitely presented R-module M and any flat cotorsion R-module P, the R-module $M \otimes_R P$ is cotorsion;*

(b) *for any flat R-module F and flat cotorsion R-module P, the R-module $\operatorname{Hom}_R(F, P)$ is flat and cotorsion;*

(c) *in the situation of* (a) *and* (b)*, the natural morphism of R-modules $\operatorname{Hom}_R(F, P)$ $\otimes_R M \longrightarrow \operatorname{Hom}_F(F, P \otimes_R M)$ is an isomorphism.*

Proof. Parts (a) and (c) follow from Lemma 1.6.1; part (b) is provided by the same Lemma together with Lemma 1.4.2(a). ☐

Corollary 1.6.4. *Let $f\colon R \longrightarrow S$ be a homomorphism of commutative rings such that the related morphism of affine schemes $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is an open embedding. Assume that the ring R is coherent. Then the S-module $\operatorname{Hom}_R(S, P)$ is flat and contraadjusted for any flat contraadjusted R-module P.*

Proof. The S-module $\operatorname{Hom}_R(S, P)$ is contraadjusted by Corollary 1.2.5(c). The R-module $\operatorname{Hom}_R(S, P)$ is flat by Lemma 1.2.4 and Corollary 1.6.2(b). Since $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is an open embedding, it follows that $\operatorname{Hom}_R(S, P)$ is also flat as an S-module (cf. Corollary 1.2.5(b)). ☐

Let $R\text{–}\mathsf{mod}_{\mathsf{fp}}$ denote the abelian category of finitely presented left modules over a left coherent ring R. For a coherent commutative ring R, the tensor product of a finitely presented R-module with a flat contraadjusted R-module is an exact functor $R\text{–}\mathsf{mod}_{\mathsf{fp}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$ (where the exact category structure on $R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}$ is induced from $R\text{–}\mathsf{mod}$). The Hom_R from a very flat R-module into a flat contraadjusted R-module is an exact functor $(R\text{–}\mathsf{mod}^{\mathsf{vfl}})^{\mathsf{op}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Let R be a coherent commutative ring such that any flat R-module has a finite projective dimension. Then the tensor product of a finitely presented R-module with a flat cotorsion R-module is an exact functor $R\text{–}\mathsf{mod}_{\mathsf{fp}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cot}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cot}}$. The Hom_R from a flat R-module to a flat cotorsion R-module is an exact functor $(R\text{–}\mathsf{mod}^{\mathsf{fl}})^{\mathsf{op}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}$. Here the additive category of flat cotorsion R-modules $R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}$ is endowed with a trivial exact category structure.

Let R be a commutative ring. We will say that an R-module K is *coadjusted* if the functor of tensor product with K over R preserves the class of contraadjusted R-modules. By Corollary 1.6.2(a), any finitely generated R-module is coadjusted.

An R-module K is coadjusted if and only if the R-module $K \otimes_R P$ is contraadjusted for every flat (or very flat) contraadjusted R-module P. Indeed, by Corollary 1.1.5(b), any contraajusted R-module is a quotient module of a very flat contraadjusted R-module; so it remains to recall that any quotient module of a contraadjusted R-module is contraadjusted.

Clearly, any quotient module of a coadjusted R-module is coadjusted. Furthermore, the class of coadjusted R-modules is closed under extensions. One can see this either by applying the above criterion of coadjustness in terms of tensor products with flat contraadjusted R-modules, or straightforwardly from the right exactness property of the functor of tensor product together with the facts that the class of contraadjusted R-modules is closed under quotients and extensions.

Consequently, there is the induced exact category structure on the full subcategory of coadjusted R-modules in the abelian category $R\text{–}\mathsf{mod}$. We denote this exact category by $R\text{–}\mathsf{mod}^{\mathsf{coa}}$. The tensor product of a coadjusted R-module with a flat contraadjusted R-module is an exact functor $R\text{–}\mathsf{mod}^{\mathsf{coa}} \times (R\text{–}\mathsf{mod}^{\mathsf{fl}} \cap R\text{–}\mathsf{mod}^{\mathsf{cta}}) \longrightarrow R\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Over a Noetherian commutative ring R, any injective module J is coadjusted. Indeed, for any R-module P, the tensor product $J \otimes_R P$ is a quotient module of an infinite direct sum of copies of J, which means a quotient module of an injective module, which is contraadjusted. Hence any quotient module of an injective module is coadjusted, too, as is any extension of such modules.

Lemma 1.6.5. (a) *Let $f\colon R \longrightarrow S$ be a homomorphism of commutative rings such that the related morphism of affine schemes* $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ *is an open embedding. Then the S-module $S \otimes_R K$ obtained by the extension of scalars via f is coadjusted for any coadjusted R-module K.*

(b) *Let $f_\alpha\colon R \longrightarrow S_\alpha$ be a collection of homomorphisms of commutative rings for which the corresponding collection of morphisms of affine schemes* $\operatorname{Spec} S_\alpha \longrightarrow \operatorname{Spec} R$ *is a finite open covering. Then an R-module K is coadjusted if and only if all the S_α-modules $S_\alpha \otimes_R K$ are coadjusted.*

Proof. Part (a): any contraadjusted S-module Q is also contraadjusted as an R-module, so the tensor product $(S \otimes_R K) \otimes_S Q \simeq K \otimes_R Q$ is a contraadjusted R-module. By Corollary 1.2.5(d), it is also a contraadjusted S-module.

Part (b): the "only if" assertion is provided by part (a); let us prove "if". Let P be a contraadjusted R-module. Applying the functor $K \otimes_R -$ to the Čech exact sequence (2) from Lemma 1.2.6(b), we obtain a sequence of R-modules that is exact at its rightmost nontrivial term. So it suffices to show that the R-modules $K \otimes_R \operatorname{Hom}_R(S_\alpha, P)$ are contraadjusted.

Now one has $K \otimes_R \mathrm{Hom}_R(S_\alpha, P) \simeq (S_\alpha \otimes_R K) \otimes_{S_\alpha} \mathrm{Hom}_R(S_\alpha, P)$, the S_α-module $\mathrm{Hom}_R(S_\alpha, P)$ is contraadjusted by Corollary 1.2.5(c), and the restriction of scalars from S_α to R preserves contraadjustness by Lemma 1.2.2(a). ☐

2. Contraherent Cosheaves over a Scheme

2.1. **Cosheaves of modules over a sheaf of rings.** Let X be a topological space. A *copresheaf of abelian groups* on X is a covariant functor from the category of open subsets of X (with the identity embeddings as morphisms) to the category of abelian groups.

Given a copresheaf of abelian groups $\mathfrak{P}$ on X, we will denote the abelian group it assigns to an open subset $U \subset X$ by $\mathfrak{P}[U]$ and call it the group of *cosections* of $\mathfrak{P}$ over U. For a pair of embedded open subsets $V \subset U \subset X$, the map $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ that the copresheaf $\mathfrak{P}$ assigns to $V \subset U$ will be called the *corestriction* map.

A copresheaf of abelian groups $\mathfrak{P}$ on X is called a *cosheaf* if for any open subset $U \subset X$ and its open covering $U = \bigcup_\alpha U_\alpha$ the following sequence of abelian groups is exact

$$\bigoplus\nolimits_{\alpha,\beta} \mathfrak{P}[U_\alpha \cap U_\beta] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{P}[U_\alpha] \longrightarrow \mathfrak{P}[U] \longrightarrow 0. \tag{5}$$

Let $\mathcal{O}$ be a sheaf of associative rings on X. A copresheaf of abelian groups $\mathfrak{P}$ on X is said to be a *copresheaf of* (*left*) $\mathcal{O}$*-modules* if for each open subset $U \subset X$ the abelian group $\mathfrak{P}[U]$ is endowed with the structure of a (left) module over the ring $\mathcal{O}(U)$, and for each pair of embedded open subsets $V \subset U \subset X$ the map of corestriction of cosections $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ in the copresheaf $\mathfrak{P}$ is a homomorphism of $\mathcal{O}(U)$-modules. Here the $\mathcal{O}(U)$-module structure on $\mathfrak{P}[V]$ is obtained from the $\mathcal{O}(V)$-module structure by the restriction of scalars via the ring homomorphism $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$.

A copresheaf of $\mathcal{O}$-modules on X is called a *cosheaf of* $\mathcal{O}$*-modules* if its underlying copresheaf of abelian groups is a cosheaf of abelian groups.

Remark 2.1.1. One can define copresheaves with values in any category, and cosheaves with values in any category that has coproducts. In particular, one can speak of cosheaves of sets, etc. Notice, however, that, unlike for (pre)sheaves, the underlying copresheaf of sets of a cosheaf of abelian groups is *not* a cosheaf of sets in general, as the forgetful functor from the abelian groups to sets preserves products, but not coproducts.

Let $\mathbf{B}$ be a base of open subsets of X. We will consider covariant functors from $\mathbf{B}$ (viewed as a full subcategory of the category of open subsets in X) to the category of abelian groups. We say that such a functor $\mathfrak{Q}$ is *endowed with an* $\mathcal{O}$*-module structure* if the abelian group $\mathfrak{Q}[U]$ is endowed with an $\mathcal{O}(U)$-module structure for each $U \in \mathbf{B}$ and the above compatibility condition holds for the corestriction maps $\mathfrak{Q}[V] \longrightarrow \mathfrak{Q}[U]$ assigned by the functor $\mathfrak{Q}$ to all $V, U \in \mathbf{B}$ such that $V \subset U$.

The following result is essentially contained in [5, Section 0.3.2], as is its (more familiar) sheaf version, to which we will turn in due order.

Theorem 2.1.2. *A covariant functor $\mathfrak{Q}$ with an $\mathcal{O}$-module structure on a base* $\mathbf{B}$ *of open subsets of X can be extended to a cosheaf of $\mathcal{O}$-modules $\mathfrak{P}$ on X if and only if the following condition holds. For any open subset $V \in \mathbf{B}$, any its covering $V = \bigcup_\alpha V_\alpha$ by open subsets $V_\alpha \in \mathbf{B}$, and any (or, equivalently, some particular) covering $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ the sequence of abelian groups (or $\mathcal{O}(V)$-modules)*

$$\bigoplus\nolimits_{\alpha,\beta,\gamma} \mathfrak{Q}[W_{\alpha\beta\gamma}] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{Q}[V_\alpha] \longrightarrow \mathfrak{Q}[V] \longrightarrow 0 \tag{6}$$

must be exact. The functor of restriction of cosheaves of $\mathcal{O}$-modules to a base $\mathbf{B}$ *is an equivalence between the category of cosheaves of $\mathcal{O}$-modules on X and the category of covariant functors on* $\mathbf{B}$*, endowed with $\mathcal{O}$-module structures and satisfying* (6).

Proof. The elementary approach taken in the exposition below is to pick an appropriate stage at which one can dualize and pass to (pre)sheaves, where our intuitions work better. First we notice that if the functor $\mathfrak{Q}$ (with its $\mathcal{O}$-module structure) has been extended to a cosheaf of $\mathcal{O}$-modules $\mathfrak{P}$ on X, then for any open subset $U \subset X$ there is an exact sequence of $\mathcal{O}(U)$-modules

$$\bigoplus\nolimits_{W,V',V''} \mathfrak{Q}[W] \longrightarrow \bigoplus\nolimits_V \mathfrak{Q}[V] \longrightarrow \mathfrak{P}[U] \longrightarrow 0, \tag{7}$$

where the summation in the middle term runs over all open subsets $V \in \mathbf{B}$, $V \subset U$, while the summation in the leftmost term is done ove all triples of open subsets W, V', $V'' \in \mathbf{B}$, $W \subset V'$, $V'' \subset U$. Conversely, given a functor $\mathfrak{Q}$ with an $\mathcal{O}$-module structure one can recover the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ as the cokernel of the left arrow.

Clearly, the modules $\mathfrak{P}[U]$ constructed in this way naturally form a copresheaf of $\mathcal{O}$-modules on X. Before proving that it is a cosheaf, one needs to show that for any open covering $U = \bigcup_\alpha V_\alpha$ of an open subset $U \subset X$ by open subsets $V_\alpha \in \mathbf{B}$ and any open coverings $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ the natural map from the cokernel of the morphism

$$\bigoplus\nolimits_{\alpha,\beta,\gamma} \mathfrak{Q}[W_{\alpha\beta\gamma}] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{Q}[V_\alpha] \tag{8}$$

to the (above-defined) $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ is an isomorphism. In particular, it will follow that $\mathfrak{P}[V] \simeq \mathfrak{Q}[V]$ for $V \in \mathbf{B}$.

Notice that it suffices to check both assertions for co(pre)sheaves of abelian groups (though it will not matter in the subsequent argument). Notice also that a copresheaf of $\mathcal{O}$-modules $\mathfrak{P}$ is a cosheaf if and only if the dual presheaf of $\mathcal{O}$-modules $U \longmapsto \operatorname{Hom}_\mathbb{Z}(\mathfrak{P}[U], I)$ is a sheaf on X for every abelian group I (or specifically for $I = \mathbb{Q}/\mathbb{Z}$). Similarly, the condition (6) holds for a covariant functor $\mathfrak{Q}$ on a base $\mathbf{B}$ if and only if the dual condition (9) below holds for the contravariant functor $V \longmapsto \operatorname{Hom}_\mathbb{Z}(\mathfrak{Q}[V], I)$ on $\mathbf{B}$. So it remains to prove the following Proposition 2.1.3. ☐

Now we will consider contravariant functors $\mathcal{G}$ from $\mathbf{B}$ to the category of abelian groups, and say that such a functor is endowed with an $\mathcal{O}$-module structure if the abelian group $\mathcal{G}(U)$ is an $\mathcal{O}(U)$-module for every $U \in \mathbf{B}$ and the restriction morphisms $\mathcal{G}(U) \longrightarrow \mathcal{G}(V)$ are morphisms of $\mathcal{O}(U)$-modules for all V, $U \in \mathbf{B}$ such that $V \subset U$.

Proposition 2.1.3. *A contravariant functor* $\mathcal{G}$ *with an* $\mathcal{O}$*-module structure on a base* $\mathbf{B}$ *of open subsets of* X *can be extended to a sheaf of* $\mathcal{O}$*-modules* $\mathcal{F}$ *on* X *if and only if the following condition holds. For any open subset* $V \in \mathbf{B}$*, any its covering* $V = \bigcup_\alpha V_\alpha$ *by open subsets* $V_\alpha \in \mathbf{B}$*, and any (or, equivalently, some particular) covering* $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ *of the intersections* $V_\alpha \cap V_\beta$ *by open subsets* $W_{\alpha\beta\gamma} \in \mathbf{B}$ *the sequence of abelian groups (or* $\mathcal{O}(V)$*-modules)*

$$0 \longrightarrow \mathcal{G}(V) \longrightarrow \textstyle\prod_\alpha \mathcal{G}(V_\alpha) \longrightarrow \prod_{\alpha,\beta,\gamma} \mathcal{G}(W_{\alpha\beta\gamma}) \tag{9}$$

must be exact. The functor of restriction of sheaves of $\mathcal{O}$*-modules to a base* $\mathbf{B}$ *is an equivalence between the category of sheaves of* $\mathcal{O}$*-modules on* X *and the category of contravariant functors on* $\mathbf{B}$*, endowed with* $\mathcal{O}$*-module structures and satisfying* (9).

Sketch of proof. As above, we notice that if the functor $\mathcal{G}$ (with its $\mathcal{O}$-module structure) has been extended to a sheaf of $\mathcal{O}$-modules $\mathcal{G}$ on X, then for any open subset $U \subset X$ there is an exact sequence of $\mathcal{O}(U)$-modules

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \textstyle\prod_V \mathcal{G}(V) \longrightarrow \prod_{W,V',V''} \mathcal{G}(W),$$

the summation rules being as in (7). Conversely, given a functor $\mathcal{G}$ with an $\mathcal{O}$-module structure one can recover the $\mathcal{O}(U)$-module $\mathcal{F}(U)$ as the kernel of the right arrow.

The rest is a conventional argument with (pre)sheaves and coverings. Recall that a presheaf $\mathcal{F}$ on X is called *separated* if the map $\mathcal{F}(U) \longrightarrow \prod_\alpha \mathcal{F}(U_\alpha)$ is injective for any open covering $U = \bigcup_\alpha U_\alpha$ of an open subset $U \subset X$. Similarly, a contravariant functor $\mathcal{G}$ on a base $\mathbf{B}$ is said to be separated if its sequences (9) are exact at the leftmost nontrivial term.

For any open covering $U = \bigcup_\alpha V_\alpha$ of an open subset $U \subset X$ by open subsets $V_\alpha \in \mathbf{B}$ and any open coverings $V_\alpha \cap V_\beta = \bigcup_\gamma W_{\alpha\beta\gamma}$ of the intersections $V_\alpha \cap V_\beta$ by open subsets $W_{\alpha\beta\gamma} \in \mathbf{B}$ there is a natural map from the (above-defined) $\mathcal{O}(U)$-module $\mathcal{F}(U)$ to the kernel of the morphism

$$\textstyle\prod_\alpha \mathcal{G}(V_\alpha) \longrightarrow \prod_{\alpha,\beta,\gamma} \mathcal{G}(W_{\alpha\beta\gamma}). \tag{10}$$

Let us show that this map is an isomorphism provided that $\mathcal{G}$ satisfies (9). In particular, it will follow that $\mathcal{F}(V) = \mathcal{G}(V)$ for $V \in \mathbf{B}$.

Clearly, when $\mathcal{G}$ is separated, the kernel of (10) does not depend on the choice of the open subsets $W_{\alpha\beta\gamma}$. So we can assume that the collection $\{W_{\alpha\beta\gamma}\}$ for fixed α and β consists of all open subsets $W \in \mathbf{B}$ such that $W \subset V_\alpha \cap V_\beta$.

Furthermore, one can easily see that the map from $\mathcal{F}(U)$ to the kernel of (10) is injective whenever $\mathcal{G}$ is separated. To check surjectivity, suppose that we are given a collection of sections $\phi_\alpha \in \mathcal{G}(V_\alpha)$ representing an element of the kernel.

Fix an open subset $V \in \mathbf{B}$, $V \subset U$, and consider its covering by all the open subsets $W \in \mathbf{B}$ such that $W \subset V \cap V_\alpha$ for some α. Set $\psi_W = \phi_\alpha|_W \in \mathcal{G}(W)$ for every such W; by assumption, if $W \subset V_\alpha \cap V_\beta$, then $\phi_\alpha|_W = \phi_\beta|_W$, so the element ψ_W is well-defined. Applying (9), we conclude that there exists a unique element $\phi_V \in \mathcal{G}(V)$ such that $\phi_V|_W = \psi_W$ for any $W \subset V \cap V_\alpha$. The collection of sections ϕ_V represents an element of $\mathcal{F}(U)$ that is a preimage of our original element of the kernel of (10).

Now let us show that $\mathcal{F}$ is a sheaf. Let $U = \bigcup_\alpha U_\alpha$ be a open covering of an open subset $U \subset X$. First let us see that $\mathcal{F}$ is separated provided that $\mathcal{G}$ is. Let $s \in \mathcal{F}(U)$ be a section whose restriction to all the open subsets U_α vanishes. The element s is represented by a collection of sections $\phi_V \in \mathcal{G}(V)$ defined for all open subsets $V \subset U$, $V \in \mathbf{B}$. The condition $s|_{U_\alpha} = 0$ means that $\phi_W = 0$ whenever $W \subset U_\alpha,\ W \in \mathbf{B}$. To check that $\phi_V = 0$ for all V, we notice that open subsets $W \subset V,\ W \in \mathbf{B}$ for which there exists α such that $W \subset U_\alpha$ form an open covering of V.

Finally, let $s_\alpha \in \mathcal{F}(U_\alpha)$ be a collection of sections such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α and β. Every element s_α is represented by a collection of sections $\phi_V \in \mathcal{G}(V)$ defined for all open subsets $V \subset U_\alpha,\ V \in \mathbf{B}$. Clearly, the element ϕ_V does not depend on the choice of a particular α for which $V \subset U_\alpha$, so our notation is consistent. All the open subsets $V \subset U,\ V \in \mathbf{B}$ for which there exists some α such that $V \subset U_\alpha$ form an open covering of the open subset $U \subset X$. The collection of sections ϕ_V represents an element of the kernel of the morphism (10) for this covering, hence it corresponds to an element of $\mathcal{F}(U)$. ☐

Remark 2.1.4. Let X be a topological space with a topology base $\mathbf{B}$ consisting of quasi-compact open subsets (in the induced topology) for which the intersection of any two open subsets from $\mathbf{B}$ that are contained in a third open subset from $\mathbf{B}$ is quasi-compact as well. E. g., any scheme X with the base of all affine open subschemes has these properties. Then it suffices to check both the conditions (6) and (9) for *finite* coverings V_α and $W_{\alpha\beta\gamma}$ only.

Indeed, let us explain the sheaf case. Obviously, injectivity of the left arrow in (9) for any given covering $V = \bigcup_\alpha V_\alpha$ follows from such injectivity for a subcovering $V = \bigcup_i V_i,\ \{V_i\} \subset \{V_\alpha\}$. Assuming $\mathcal{G}$ is separated, one checks that exactness of the sequence (9) for any given covering follows from the same exactness for a subcovering.

It follows that for a topological space X with a fixed topology base $\mathbf{B}$ satisfying the above condition there is another duality construction relating sheaves to cosheaves in addition to the one we used in the proof of Theorem 2.1.2. Given a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ on X, one restricts it to the base $\mathbf{B}$, obtaining a contravariant functor $\mathcal{G}$ with an $\mathcal{O}$-module structure, defines the dual covariant functor $\mathfrak{Q}$ with an $\mathcal{O}$-module structure on $\mathbf{B}$ by the rule $\mathfrak{Q}[V] = \mathrm{Hom}_\mathbb{Z}(\mathcal{G}(V), I)$, where I is an injective abelian group, and extends the functor $\mathfrak{Q}$ to a cosheaf of $\mathcal{O}$-modules $\mathfrak{P}$ on X.

It is this second duality functor, rather than the one from the proof of Theorem 2.1.2, that will play a role in the sequel.

2.2. **Exact category of contraherent cosheaves.** Let X be a scheme and $\mathcal{O} = \mathcal{O}_X$ be its structure sheaf. A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is called *contraherent* if for any pair of embedded affine open subschemes $V \subset U \subset X$

(i) the morphism of $\mathcal{O}_X(V)$-modules $\mathfrak{P}[V] \longrightarrow \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U])$ induced by the corestriction morphism $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ is an isomorphism; and

(ii) one has $\mathrm{Ext}^{>0}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{P}[U]) = 0$.

It follows from Lemma 1.2.4 that the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ has projective dimension at most 1, so it suffices to require the vanishing of Ext^1 in the condition (ii). We will call (ii) the *contraadjustness condition*, and (i) the *contraherence condition.*

Theorem 2.2.1. *The restriction of cosheaves of $\mathcal{O}_X$-modules to the base of all affine open subschemes of X induces an equivalence between the category of contraherent cosheaves on X and the category of covariant functors $\mathfrak{Q}$ with $\mathcal{O}_X$-module structures on the category of affine open subschemes of X, satisfying the conditions* (i-ii) *for any pair of embedded affine open subschemes $V \subset U \subset X$.*

Proof. According to Theorem 2.1.2, a cosheaf of $\mathcal{O}_X$-modules is determined by its restriction to the base of affine open subsets of X. The contraadjustness and contraherence conditions depend only on this restriction. By Lemma 1.2.4, given any affine scheme U, a module P over $\mathcal{O}(U)$ is contraadjusted if and only if $\operatorname{Ext}^1_{\mathcal{O}(U)}(\mathcal{O}(V), P) = 0$ for all affine open subschemes $V \subset U$. Finally, the key observation is that the contraadjustness and contraherence conditions (i-ii) for a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes of X imply the cosheaf condition (6). This follows from Lemma 1.2.6(b) and Remark 2.1.4. ☐

Remark 2.2.2. Of course, one can similarly define quasi-coherent sheaves $\mathcal{F}$ on X as sheaves of $\mathcal{O}_X$-modules such that for any pair of embedded affine open subschemes $V \subset U \subset X$ the morphism of $\mathcal{O}_X(V)$-modules $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ induced by the restriction morphism $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is an isomorphism. Since the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(V)$ is always flat, no version of the condition (ii) is needed in this case. The analogue of Theorem 2.2.1 is well-known for quasi-coherent sheaves (and can be proven in the same way).

A short sequence of contraherent cosheaves $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ is said to be exact if the sequence of cosection modules $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$ is exact for every affine open subscheme $U \subset X$. Notice that if U_α is an affine open covering of an affine scheme U and $\mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R}$ is a sequence of contraherent cosheaves on U, then the sequence of $\mathcal{O}(U)$-modules $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$ is exact if and only if all the sequences of $\mathcal{O}(U_\alpha)$-modules $0 \longrightarrow \mathfrak{P}[U_\alpha] \longrightarrow \mathfrak{Q}[U_\alpha] \longrightarrow \mathfrak{R}[U_\alpha] \longrightarrow 0$ are. This follows from Lemma 1.3.1(a).

We denote the exact category of contraherent cosheaves on a scheme X by $X\text{–}\mathsf{ctrh}$. By the definition, the functors of cosections over affine open subschemes are exact on this exact category. It is also has exact functors of infinite product, which commute with cosections over affine open subschemes (and in fact, over any quasi-compact quasi-separated open subschemes as well). For a more detailed discussion of this exact category structure, we refer the reader to Section 3.1.

Corollary 2.2.3. *The functor assigning the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ to a contraherent cosheaf $\mathfrak{P}$ on an affine scheme U is an equivalence between the exact category $U\text{–}\mathsf{ctrh}$ of contraherent cosheaves on U and the exact category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{cta}}$ of contraadjusted modules over the commutative ring $\mathcal{O}(U)$.*

Proof. Clear from the above arguments together with Lemmas 1.2.1(b) and 1.2.4. ☐

A contraherent cosheaf $\mathfrak{P}$ on a scheme X is said to be *locally cotorsion* if for any affine open subscheme $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathfrak{P}(U)$ is cotorsion. By Lemma 1.4.6(a), the property of a contraherent cosheaf on an affine scheme to be cotorsion is indeed a local, so our terminology is constent.

A contraherent cosheaf $\mathfrak{J}$ on a scheme X is called *locally injective* if for any affine open subscheme $U \subset X$ the $\mathcal{O}_X(U)$-module $\mathfrak{J}(U)$ is injective. By Lemma 1.4.6(b), local injectivity of a contraherent cosheaf is indeed a local property.

Just as above, one defines the exact categories $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$ of locally cotorsion and locally injective contraherent cosheaves on X. These are full subcategories closed under extensions, infinite products, and cokernels of admissible monomorphisms in $X\text{–}\mathsf{ctrh}$, with the induced exact category structures.

The exact category $U\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ of locally cotorsion contraherent cosheaves on an affine scheme U is equivalent to the exact category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{cot}}$ of cotorsion $\mathcal{O}(U)$-modules. The exact category $U\text{–}\mathsf{ctrh}^{\mathsf{lin}}$ of locally injective contraherent cosheaves on U is equivalent to the additive category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{inj}}$ of injective $\mathcal{O}(U)$-modules endowed with the trivial exact category structure.

Remark 2.2.4. Notice that a morphism of contraherent cosheaves on X is an admissible monomorphism if and only if it acts injectively on the cosection modules over all the affine open subschemes on X. At the same time, the property of a morphism of contraherent cosheaves on X to be an admissible monomorphism is *not* local in X, and *neither* is the property of a cosheaf of $\mathcal{O}_X$-modules to be contraherent (see Section 3.2 below). The property of a morphism of contraherent cosheaves to be an admissible epimorphism is local, though (see Lemma 1.3.1(b)). All of the above applies to locally cotorsion and locally injective contraherent cosheaves as well.

Notice also that a morphism of locally injective or locally cotorsion contraherent cosheaves that is an admissible epimorphism in $X\text{–}\mathsf{ctrh}$ may *not* be an admissible epimorphism in $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$. On the other hand, if a morphism of locally injective or locally cotorsion contraherent cosheaves is an admissible monomorphism in $X\text{–}\mathsf{ctrh}$, then it is also an admissible monomorphism in $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ or $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$, as it is clear from the above.

2.3. **Direct and inverse images of contraherent cosheaves.** Let $\mathcal{O}_X$ be a sheaf of associative rings on a topological space X and $\mathcal{O}_Y$ be such a sheaf on a topological space Y. Furthermore, let $f\colon Y \longrightarrow X$ be a morphism of ringed spaces, i. e., a continuous map $Y \longrightarrow X$ together with a morphism $\mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ of sheaves of rings over X. Then for any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$ the rule $(f_!\mathfrak{Q})[W] = \mathfrak{Q}[f^{-1}(W)]$ for all open subsets $W \subset X$ defines a cosheaf of $\mathcal{O}_X$-modules $f_!\mathfrak{Q}$.

Let $f\colon Y \longrightarrow X$ be an affine morphism of schemes, and let $\mathfrak{Q}$ be a contraherent cosheaf on Y. Then $f_!\mathfrak{Q}$ is a contraherent cosheaf on X. Indeed, for any affine open subscheme $U \subset X$ the $\mathcal{O}_X(U)$-module $(f_!\mathfrak{Q})[U] = \mathfrak{Q}[(f^{-1}(U)]$ is contraadjusted according to Lemma 1.2.2(a) applied to the morphism of commutative rings $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(f^{-1}(U))$. For any pair of embedded affine open subschemes

$V \subset U \subset X$ we have natural isomorphisms of $\mathcal{O}_X(U)$-modules

$$(f_!\mathfrak{Q})[V] = \mathfrak{Q}[f^{-1}(V)] \simeq \operatorname{Hom}_{\mathcal{O}_Y(f^{-1}(U))}(\mathcal{O}_Y(f^{-1}(V)), \mathfrak{Q}[f^{-1}(U)])$$
$$\simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{Q}[f^{-1}(U)]) = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), (f_!\mathfrak{Q})[U]),$$

since $\mathcal{O}_Y(f^{-1}(V)) \simeq \mathcal{O}_Y(f^{-1}(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$.

Recall that a scheme X is called *semi-separated* [26, Appendix B], if it admits an affine open covering with affine pairwise intersections of the open subsets belonging to the covering. Equivalently, a scheme X is semi-separated if and only if the diagonal morphism $X \longrightarrow X \times_{\operatorname{Spec}\mathbb{Z}} X$ is affine, and if and only if the intersection of any two affine open subschemes of X is affine. Any morphism from an affine scheme to a semi-separated scheme is affine, and the fibered product of any two affine schemes over a semi-separated base scheme is an affine scheme.

We will say that a morphism of schemes $f\colon Y \longrightarrow X$ is *coaffine* if for any affine open subscheme $V \subset Y$ there exists an affine open subscheme $U \subset X$ such that $f(V) \subset U$, and for any two such affine open subschemes $f(V) \subset U'$, $U'' \subset X$ there exists a third affine open subscheme $U \subset X$ such that $f(V) \subset U \subset U' \cap U''$. If the scheme X is semi-separated, then the second condition is trivial.

Any morphism into an affine scheme is coaffine. Any embedding of an open subscheme is coaffine. The composition of two coaffine morpisms between semi-separated schemes is a coaffine morphism.

A morphism of schemes $f\colon Y \longrightarrow X$ is called *very flat* if for any two affine open subschemes $V \subset Y$ and $U \subset X$ such that $f(V) \subset U$ the ring $\mathcal{O}_Y(V)$ is a very flat module over the ring $\mathcal{O}_X(U)$. By Lemma 1.2.4, any embedding of an open subscheme is a very flat morphism.

A morphism of affine schemes $\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is very flat if and only if the morphism of commutative rings $R \longrightarrow S$ satisfies the condition of Lemma 1.2.3. According to Lemmas 1.2.6(a) and 1.2.7(b), the property of a morphism to be very flat is local in both the source and the target schemes. By Lemma 1.2.3(b), the composition of very flat morphisms of schemes is a very flat morphism. By Lemma 1.2.2(b), any base change of a very flat morphism is a very flat morphism.

Let $f\colon Y \longrightarrow X$ be a very flat coaffine morphism of schemes, and let $\mathfrak{P}$ be a contraherent cosheaf on X. Define a contraherent cosheaf $f^!\mathfrak{P}$ on Y as follows.

Let $V \subset Y$ be an affine open subscheme. Pick an affine open subscheme $U \subset X$ such that $f(V) \subset U$, and set $(f^!\mathfrak{P})[V] = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(V), \mathfrak{P}[U])$. Due to the contraherence condition on $\mathfrak{P}$, this definition of the $\mathcal{O}_Y(V)$-module $(f^!\mathfrak{P})[V]$ does not depend on the choice of an affine open subscheme $U \subset X$. Since f is a very flat morphism, the $\mathcal{O}_Y(V)$-module $(f^!\mathfrak{P})[V]$ is contraadjusted by Lemma 1.2.3(a). The contraherence condition obviously holds for $f^!\mathfrak{P}$.

Let $f\colon Y \longrightarrow X$ be a flat coaffine morphism of schemes, and $\mathfrak{P}$ be a locally cotorsion contraherent cosheaf on X. Then the same rule as above defines a locally cotorsion contraherent cosheaf $f^!\mathfrak{P}$ on Y. One just uses Lemma 1.4.5(a) in place of Lemma 1.2.3(a). For any coaffine morphism of schemes $f\colon Y \longrightarrow X$ and a locally injective contraherent cosheaf $\mathfrak{J}$ on X the very same rule defines a locally injective contraherent cosheaf $f^!\mathfrak{J}$ on Y.

If $f\colon Y \longrightarrow X$ is an affine morphism of schemes and $\mathfrak{Q}$ is a locally cotorsion contraherent cosheaf on Y, then $f_!\mathfrak{Q}$ is a locally cotorsion contraherent cosheaf on X. If $f\colon Y \longrightarrow X$ is a flat affine morphism and $\mathfrak{J}$ is a locally injective contraherent cosheaf on Y, then $f_!\mathfrak{J}$ is a locally injective contraherent cosheaf on X.

Let $f\colon Y \longrightarrow X$ be an affine coaffine morphism of schemes. Then for any contraherent cosheaf $\mathfrak{Q}$ on Y and any locally injective contraherent cosheaf $\mathfrak{P}$ on X there is a natural adjunction isomorphism $\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{P}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P})$, where Hom^X and Hom^Y denote the abelian groups of morphisms in the categories of contraherent cosheaves on X and Y.

If, in addition, the morphism f is flat, then such an isomorphism holds for any contraherent cosheaf $\mathfrak{Q}$ on Y and any locally cotorsion contraherent cosheaf $\mathfrak{P}$ on X; in particular, $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories of locally cotorsion contraherent cosheaves $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$ and $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}$. Their restrictions also act as adjoint functors between the exact categories of locally injective contraherent cosheaves $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$ and $Y\text{–}\mathsf{ctrh}^{\mathsf{lin}}$.

If the morphism f is very flat, then the functor $f^!\colon X\text{–}\mathsf{ctrh} \longrightarrow Y\text{–}\mathsf{ctrh}$ is right adjoint to the functor $f_!\colon Y\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{ctrh}$. In all the mentioned cases, both abelian groups $\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{P})$ and $\operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P})$ are identified with the group whose elements are the collections of homomorphisms of $\mathcal{O}_X(U)$-modules $\mathfrak{Q}(V) \longrightarrow \mathfrak{P}(U)$, defined for all affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ and compatible with the corestriction maps.

All the functors constructed in the above section are exact functors between those exact categories where they act. For a construction of the direct image functor $f_!$ (acting between appropriately restricted exact categories of contraherent cosheaves) for a nonaffine morphism of schemes f, see Section 4.4 below.

2.4. $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a contraherent cosheaf. Let X be a scheme over an affine scheme $\operatorname{Spec} R$. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and J be an injective R-module. Then the rule $U \longmapsto \operatorname{Hom}_R(\mathcal{M}(U), J)$ for affine open subschemes $U \subset X$ defines a contraherent cosheaf over X (cf. Remark 2.1.4). We will denote it by $\mathfrak{Cohom}_R(\mathcal{M}, J)$. Since the $\mathcal{O}_X(U)$-module $\operatorname{Hom}_R(\mathcal{M}(U), J)$ is cotorsion by Lemma 1.4.3(b), it is even a locally cotorsion contraherent cosheaf. When $\mathcal{F}$ is a flat quasi-coherent sheaf on X and J is an injective R-module, the contraherent cosheaf $\mathfrak{Cohom}_R(\mathcal{F}, J)$ is locally injective.

Let us call a quasi-coherent sheaf $\mathcal{F}$ over a scheme X *very flat* if the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ is very flat for any affine open subscheme $U \subset X$. According to Lemma 1.2.6(a), very flatness of a quasi-coherent sheaf over a scheme is a local property. By Lemma 1.2.2(b), the inverse image of a very flat quasi-coherent sheaf under any morphism of schemes is very flat. By Lemma 1.2.3(b), the direct image of a very flat quasi-coherent sheaf under a very flat affine morphism of schemes is very flat.

More generally, given a morphism of schemes $f\colon Y \longrightarrow X$, a quasi-coherent sheaf $\mathcal{F}$ on Y is said to be *very flat over* X if for any affine open subschemes $U \subset X$ and

$V\subset Y$ such that $f(V)\subset U$ the module of sections $\mathcal{F}(V)$ is very flat over the ring $\mathcal{O}_X(U)$. According to Lemmas 1.2.6(a) and 1.2.8(b), the property of very flatness of $\mathcal{F}$ over X is local in both X and Y. By Lemma 1.2.3(b), if the scheme Y is very flat over X and a quasi-coherent sheaf $\mathcal{F}$ is very flat on Y, then $\mathcal{F}$ is also very flat over X.

If $X\longrightarrow \operatorname{Spec} R$ is a very flat morphism of schemes and $\mathcal{F}$ is a very flat quasi-coherent sheaf on X, then for any contraadjusted R-module P the rule $U\longmapsto \operatorname{Hom}_R(\mathcal{F}(U),P)$ for affine open subschemes $U\subset X$ defines a contraherent cosheaf on X. The contraadjustness condition on the $\mathcal{O}_X(U)$-modules $\operatorname{Hom}_R(\mathcal{F}(U),P)$ holds by Lemma 1.2.3(c). We will denote the cosheaf so constructed by $\mathfrak{Cohom}_R(\mathcal{F},P)$.

Analogously, if a scheme X if flat over $\operatorname{Spec} R$ and a quasi-coherent sheaf $\mathcal{F}$ on X is flat (or, more generally, the quasi-coherent sheaf $\mathcal{F}$ on X is flat over $\operatorname{Spec} R$, in the obvious sense), then for any cotorsion R-module P the rule $U\longmapsto \operatorname{Hom}_R(\mathcal{F}(U),P)$ for affine open subschemes $U\subset X$ defines a contraherent cosheaf on X. In fact, the $\mathcal{O}_X(U)$-modules $\operatorname{Hom}_R(\mathcal{F}(U),P)$ are cotorsion by Lemma 1.4.3(a), hence the contraherent cosheaf $\mathfrak{Cohom}_R(\mathcal{F},P)$ constructed in this way is locally cotorsion.

Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on a scheme X and $\mathfrak{P}$ be a contraherent cosheaf on X. Then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the rule $U\longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for all affine open subschemes $U\subset X$. For any two embedded affine open subschemes $V\subset U\subset X$ one has

$$\begin{aligned}&\operatorname{Hom}_{\mathcal{O}_X(V)}(\mathcal{F}(V),\mathfrak{P}[V])\\ &\qquad\simeq \operatorname{Hom}_{\mathcal{O}_X(V)}(\mathcal{O}_X(V)\otimes_{\mathcal{O}_X(U)}\mathcal{F}(U),\ \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\mathfrak{P}[U]))\\ &\qquad\qquad\simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])),\end{aligned}$$

so the contraherence condition holds. The contraadjustness condition follows from Lemma 1.2.1(b).

Similarly, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is locally cotorsion contraherent cosheaf on X, then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the same rule $U\longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for all affine open subschemes $U\subset X$. By Lemma 1.4.2(a), $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is a locally cotorsion contraherent cosheaf on X.

Finally, if $\mathcal{M}$ is a quasi-coherent sheaf on X and $\mathfrak{J}$ is a locally injective contraherent cosheaf, then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is defined by the very same rule. One checks the contraherence condition in the same way as above. By Lemma 1.4.2(b), $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is a locally cotorsion contraherent cosheaf on X. If $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective contraherent cosheaf on X, then the contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})$ is locally injective.

For any contraadjusted module P over a commutative ring R, denote by $\widehat{P}$ the corresponding contraherent cosheaf on $\operatorname{Spec} R$. Let $f\colon X\longrightarrow \operatorname{Spec} R$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent sheaf on X. Then whenever $\mathcal{F}$ is a very flat quasi-coherent sheaf and f is a very flat morphism, there is a natural isomorphism of contraherent cosheaves $\mathfrak{Cohom}_R(\mathcal{F},P)\simeq \mathfrak{Cohom}_X(\mathcal{F},f^!\widehat{P})$ on X. Indeed, for any affine open subscheme $U\subset X$ one has

$$\operatorname{Hom}_R(\mathcal{F}(U),P)\simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\operatorname{Hom}_R(\mathcal{O}_X(U),P)\simeq \operatorname{Hom}_R(\mathcal{F}(U),(f^!\widehat{P})[U]).$$

The same isomorphism holds whenever $\mathcal{F}$ is a flat quasi-coherent sheaf, f is a flat morphism, and P is a cotorsion R-module. Finally, for any quasi-coherent sheaf $\mathcal{M}$ on X, any morphism $f\colon X \longrightarrow \operatorname{Spec} R$, and any injective R-module J there is a natural isomorphism of locally cotorsion contraherent cosheaves $\mathfrak{Cohom}_R(\mathcal{M}, J) \simeq \mathfrak{Cohom}_X(\mathcal{M}, f^!\widehat{J})$ on X.

2.5. **Contraherent cosheaves of $\mathfrak{Hom}$ between quasi-coherent sheaves.** A quasi-coherent sheaf $\mathcal{P}$ on a scheme X is said to be *cotorsion* [10] if $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{P}) = 0$ for any flat quasi-coherent sheaf $\mathcal{F}$ on X. Here Ext_X denotes the Ext groups in the abelian category of quasi-coherent sheaves on X. A quasi-coherent sheaf $\mathcal{P}$ on X is called *contraadjusted* if one has $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{P}) = 0$ for any very flat quasi-coherent sheaf $\mathcal{F}$ on X (see Section 2.4 for the definition of the latter).

Clearly the two classes of quasi-coherent sheaves on X so defined are closed under extensions, so they form full exact subcategories in the abelian category of quasi-coherent sheaves. Also, these exact subcategories are closed under the passage to direct summands of objects.

For any affine morphism of schemes $f\colon Y \longrightarrow X$, any flat quasi-coherent sheaf $\mathcal{F}$ on X, and any quasi-coherent sheaf $\mathcal{P}$ on Y there is a natural isomorphism of the extension groups $\operatorname{Ext}^1_Y(f^*\mathcal{F}, \mathcal{P}) \simeq \operatorname{Ext}^1_X(\mathcal{F}, f_*\mathcal{P})$. Hence the classes of contraadjusted and cotorsion quasi-coherent sheaves on schemes are preserved by the direct images with respect to affine morphisms.

Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Suppose that an associative ring R acts on X from the right by quasi-coherent sheaf endomorphisms. Let M be a left R-module. Define a contravariant functor $\mathcal{F} \otimes_R M$ from the category of affine open subschemes $U \subset X$ to the category of abelian groups by the rule $(\mathcal{F} \otimes_R M)(U) = \mathcal{F}(U) \otimes_R M$. The natural $\mathcal{O}_X(U)$-module structures on the groups $(\mathcal{F} \otimes_R M)(U)$ are compatible with the restriction maps $(\mathcal{F} \otimes_R M)(U) \longrightarrow (\mathcal{F} \otimes_R M)(V)$ for embedded affine open subschemes $V \subset U \subset X$, and the quasi-coherence condition

$$(\mathcal{F} \otimes_R M)(V) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} (\mathcal{F} \otimes_R M)(U)$$

holds (see Remark 2.2.2). Therefore, the functor $\mathcal{F} \otimes_R M$ extends uniquely to a quasi-coherent sheaf on X, which we will denote also by $\mathcal{F} \otimes_R M$.

Let $\mathcal{P}$ be a quasi-coherent sheaf on X. Then the abelian group $\operatorname{Hom}_X(\mathcal{F}, \mathcal{P})$ of morphisms in the category of quasi-coherent sheaves on X has a natural left R-module structure. One can easily construct a natural isomorphism of abelian groups $\operatorname{Hom}_X(\mathcal{F} \otimes_R M,\ \mathcal{P}) \simeq \operatorname{Hom}_R(M, \operatorname{Hom}_X(\mathcal{F}, \mathcal{P}))$.

Lemma 2.5.1. *Suppose that* $\operatorname{Ext}^i_X(\mathcal{F}, \mathcal{P}) = 0$ *for* $0 < i \leqslant i_0$ *and either*

(a) *M is a flat left R-module, or*

(b) *the right R-modules $\mathcal{F}(U)$ are flat for all affine open subschemes $U \subset X$.*

Then there is a natural isomorphism of abelian groups $\operatorname{Ext}^i_X(\mathcal{F} \otimes_R M,\ \mathcal{P}) \simeq \operatorname{Ext}^i_R(M, \operatorname{Hom}_X(\mathcal{F}, \mathcal{P}))$ *for all* $0 \leqslant i \leqslant i_0$.

Proof. Replace M by its left projective R-module resolution $L_\bullet$. Then $\operatorname{Ext}^i_X(\mathcal{F} \otimes_R L_j,\ \mathcal{P}) = 0$ for all $0 < i \leqslant i_0$ and all j. Due to the flatness condition (a) or (b), the complex of quasi-coherent sheaves $\mathcal{F}\otimes_R L_\bullet$ is a left resolution of the sheaf $\mathcal{F}\otimes_R M$. Hence the complex of abelian groups $\operatorname{Hom}_X(\mathcal{F}\otimes_R L_\bullet,\,\mathcal{P})$ computes $\operatorname{Ext}^i_X(\mathcal{F}\otimes_R M,\,\mathcal{P})$ for $0\leqslant i\leqslant i_0$. On the other hand, this complex is isomorphic to the complex $\operatorname{Hom}_R(L_\bullet,\operatorname{Hom}_R(\mathcal{F},\mathcal{P}))$, which computes $\operatorname{Ext}^i_R(M,\operatorname{Hom}_X(\mathcal{F},\mathcal{P}))$. ☐

Let $\mathcal{F}$ be a quasi-coherent sheaf with a right action of a ring R on a scheme X, and let $f\colon Y\longrightarrow X$ be a morphism of schemes. Then $f^*\mathcal{F}$ is a quasi-coherent sheaf on Y with a right action of R, and for any left R-module M there is a natural isomorphism of quasi-coherent sheaves $f^*(\mathcal{F}\otimes_R M)\simeq f^*\mathcal{F}\otimes_R M$. Analogously, if $\mathcal{G}$ is a quasi-coherent sheaf on Y with a right action of R and f is a quasi-compact quasi-separated morphism, then $f_*\mathcal{G}$ is a quasi-coherent sheaf on X with a right action of R, and for any left R-module M there is a natural morphism of quasi-coherent sheaves $f_*\mathcal{G}\otimes_R M\longrightarrow f_*(\mathcal{G}\otimes_R M)$ on X. If the morphism f is affine, then this map is an isomorphism of quasi-coherent sheaves on X.

Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on a semi-separated scheme X, and let $\mathcal{P}$ be a contraadjusted quasi-coherent sheaf on X. Define a contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})$ by the rule $U\longmapsto\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})$ for any affine open subscheme $U\subset X$, where $j\colon U\longrightarrow X$ denotes the identity open embedding. Given two embedded affine open subschemes $V\subset U\subset X$ with the identity embeddings $j\colon U\longrightarrow X$ and $k\colon V\longrightarrow X$, the adjunction provides a natural map of quasi-coherent sheaves $j_*j^*\mathcal{F}\longrightarrow k_*k^*\mathcal{F}$. There is also a natural action of the ring $\mathcal{O}_X(U)$ on the quasi-coherent sheaf $j_*j^*\mathcal{F}$. Thus our rule defines a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes in X.

Let us check that the contraadjustness and contraherence conditions are satisfied. For a very flat $\mathcal{O}_X(U)$-module G, we have

$$\operatorname{Ext}^1_{\mathcal{O}_X(U)}(G,\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P}))\\ \simeq\operatorname{Ext}^1_X((j_*j^*\mathcal{F})\otimes_{\mathcal{O}_X(U)}G,\,\mathcal{P})\simeq\operatorname{Ext}^1_X(j_*(j^*\mathcal{F}\otimes_{\mathcal{O}_X(U)}G),\,\mathcal{P})=0,$$

since $j_*(j^*\mathcal{F}\otimes_{\mathcal{O}_X(U)}G)$ is a very flat quasi-coherent sheaf on X. For a pair of embedded affine open subschemes $V\subset U\subset X$, we have

$$\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P}))\simeq\operatorname{Hom}_X((j_*j^*\mathcal{F})\otimes_{\mathcal{O}_X(U)}\mathcal{O}_X(V),\,\mathcal{P})\\ \simeq\operatorname{Hom}_X(j_*(j^*\mathcal{F}\otimes_{\mathcal{O}_X(U)}\mathcal{O}_X(V)),\,\mathcal{P})\simeq\operatorname{Hom}_X(k_*k^*\mathcal{F},\mathcal{P}).$$

Lemma 2.5.2. *Let $Y\subset X$ be a quasi-compact open subscheme in a semi-separated scheme such that the identity open embedding $j\colon Y\longrightarrow X$ is an affine morphism. Then*

(a) *for any very flat quasi-coherent sheaf $\mathcal{F}$ and contraaadjusted quasi-coherent sheaf $\mathcal{P}$ on X, there is a natural isomorphism of $\mathcal{O}(Y)$-modules $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[Y]\simeq\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})$;*

(b) *for any flat quasi-coherent sheaf $\mathcal{F}$ and cotorsion quasi-coherent sheaf $\mathcal{P}$ on X, there is a natural isomorphism of $\mathcal{O}(Y)$-modules $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[Y]\simeq\operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P})$;*

(c) *for any quasi-coherent sheaf $\mathcal{M}$ and injective quasi-coherent sheaf $\mathcal{J}$ on X, there is a natural isomorphism of $\mathcal{O}(Y)$-modules $\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})[Y]\simeq\operatorname{Hom}_X(j_*j^*\mathcal{M},\mathcal{J})$.*

Proof. Part (a): let $Y = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering. Denote by $k_{\alpha_1,\dots,\alpha_i}$ the open embeddings $U_{\alpha_1} \cap \dots \cap U_{\alpha_i} \longrightarrow X$. Then there is a finite exact sequence

$$0 \longrightarrow j_*j^*\mathcal{F} \longrightarrow \bigoplus\nolimits_\alpha k_{\alpha*}k_\alpha^*\mathcal{F} \longrightarrow \bigoplus\nolimits_{\alpha<\beta} k_{\alpha,\beta*}k_{\alpha,\beta}^*\mathcal{F} \\ \longrightarrow \dotsb \longrightarrow k_{1,\dots,N*}k_{1,\dots,N}^*\mathcal{F} \longrightarrow 0$$

of very flat quasi-coherent sheaves on X. The functor $\operatorname{Hom}_X(-,\mathcal{P})$ transforms it into an exact sequence of $\mathcal{O}(Y)$-modules ending in

$$\bigoplus\nolimits_{\alpha<\beta} \mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha\cap U_\beta] \longrightarrow \bigoplus\nolimits_\alpha \mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha] \\ \longrightarrow \operatorname{Hom}_X(j_*j^*\mathcal{F},\mathcal{P}) \longrightarrow 0;$$

and it remains to compare it with the construction (8) of the $\mathcal{O}(Y)$-module $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[Y]$ in terms of the modules $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha]$ and $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})[U_\alpha\cap U_\beta]$. The proofs of parts (b) and (c) are similar. ☐

Similarly one defines a locally cotorsion contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{P})$ for a flat quasi-coherent sheaf $\mathcal{F}$ and a cotorsion quasi-coherent sheaf $\mathcal{P}$ on X. Finally, a locally cotorsion contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})$ for any quasi-coherent sheaf $\mathcal{M}$ and an injective quasi-coherent sheaf $\mathcal{J}$ on X is defined by the very same rule. When $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathcal{J}$ is an injective quasi-coherent sheaf on X, the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{F},\mathcal{J})$ is locally injective.

For any affine morphism $f\colon Y \longrightarrow X$ and any quasi-coherent sheaves $\mathcal{M}$ on X and $\mathcal{N}$ on Y there is a natural isomorphism

$$f_*(f^*\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{N}) \simeq \mathcal{M}\otimes_{\mathcal{O}_X} f_*\mathcal{N} \tag{11}$$

of quasi-coherent sheaves on X ("the projection formula"). In particular, for any quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{K}$ on X there is a natural isomorphism

$$f_*f^*(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{K}) \simeq \mathcal{M}\otimes_{\mathcal{O}_X} f_*f^*\mathcal{K} \tag{12}$$

of quasi-coherent sheaves on X. For any embedding $j\colon U \longrightarrow X$ of an affine open subscheme into a semi-separated scheme X and any quasi-coherent sheaves $\mathcal{K}$ and $\mathcal{M}$ on X there is a natural isomorphism

$$j_*j^*(\mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{M}) \simeq j_*j^*\mathcal{K}\otimes_{\mathcal{O}_X(U)}\mathcal{M}(U) \tag{13}$$

of quasi-coherent sheaves on X.

Recall that the *quasi-coherent internal Hom* sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{P})$ for quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{P}$ on a scheme X is defined as the quasi-coherent sheaf for which there is a natural isomorphism of abelian groups $\operatorname{Hom}_X(\mathcal{K},\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{P})) \simeq \operatorname{Hom}_X(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{K},\mathcal{P})$ for any quasi-coherent sheaf $\mathcal{K}$ on X. The sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{P})$ can be constructed by applying the coherator functor [26, Sections B.12–B.14] to the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{P})$.

Lemma 2.5.3. *Let X be a scheme. Then*

(a) *for any very flat quasi-coherent sheaf $\mathcal{F}$ and contraadjusted quasi-coherent sheaf $\mathcal{P}$ on X, the quasi-coherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})$ on X is contraadjusted;*

(b) *for any flat quasi-coherent sheaf* $\mathcal{F}$ *and cotorsion quasi-coherent sheaf* $\mathcal{P}$ *on* X*, the quasi-coherent sheaf* $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})$ *on* X *is cotorsion;*

(c) *for any quasi-coherent sheaf* $\mathcal{M}$ *and any injective quasi-coherent sheaf* $\mathcal{J}$ *on* X*, the quasi-coherent sheaf* $\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{J})$ *on* X *is cotorsion;*

(d) *for any flat quasi-coherent sheaf* $\mathcal{F}$ *and any injective quasi-coherent sheaf* $\mathcal{J}$ *on* X*, the quasi-coherent sheaf* $\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{J})$ *is injective.*

Proof. We will prove part (a); the proofs of the other parts are similar. Let $\mathcal{G}$ be a very flat quasi-coherent sheaf on X. We will show that the functor $\mathrm{Hom}_X(-,\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P}))$ transforms any short exact sequence of quasi-coherent sheaves $0\longrightarrow\mathcal{K}\longrightarrow\mathcal{L}\longrightarrow\mathcal{G}\longrightarrow 0$ into a short exact sequence of abelian groups. Indeed, the sequence of quasi-coherent sheaves $0\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{K}\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}\longrightarrow\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}\longrightarrow 0$ is exact, because $\mathcal{F}$ is flat (or because $\mathcal{G}$ is flat). Since $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ is very flat by Lemma 1.2.1(a) and $\mathcal{P}$ is contraadjusted, the functor $\mathrm{Hom}_X(-,\mathcal{P})$ transforms the latter sequence of sheaves into a short exact sequence of abelian groups. ☐

It follows from the isomorphism (12) that for any very flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a semi-separated scheme X and any contraadjusted quasi-coherent sheaf $\mathcal{P}$ on X there is a natural isomorphism of contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\ \mathcal{P})\simeq\mathfrak{Hom}_X(\mathcal{G},\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{P})). \tag{14}$$

Similarly, for any flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a cotorsion quasi-coherent sheaf $\mathcal{P}$ on X there is a natural isomorphism (14) of locally cotorsion contraherent cosheaves. Finally, for any flat quasi-coherent sheaf $\mathcal{F}$, quasi-coherent sheaf $\mathcal{M}$, and injective quasi-coherent sheaf $\mathcal{J}$ on X there are natural isomorphisms of locally cotorsion contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{J})\simeq\mathfrak{Hom}_X(\mathcal{M},\mathcal{H}om_{X\text{-qc}}(\mathcal{F},\mathcal{J}))\simeq\mathfrak{Hom}_X(\mathcal{F},\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{J})). \tag{15}$$

It follows from the isomorphism (13) that for any very flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a contraadjusted quasi-coherent sheaf $\mathcal{P}$ on a semi-separated scheme X there is a natural isomorphism of contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{G}\otimes_{\mathcal{O}_X}\mathcal{F},\ \mathcal{P})\simeq\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{Hom}_X(\mathcal{G},\mathcal{P})). \tag{16}$$

Similarly, for any flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a cotorsion quasi-coherent sheaf $\mathcal{P}$ on X there is a natural isomorphism (14) of locally cotorsion contraherent cosheaves. Finally, for any flat quasi-coherent sheaf $\mathcal{F}$, quasi-coherent sheaf $\mathcal{K}$, and injective quasi-coherent sheaf $\mathcal{J}$ on X there are natural isomorphisms of locally cotorsion contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{F},\ \mathcal{J})\simeq\mathfrak{Cohom}_X(\mathcal{K},\mathfrak{Hom}_X(\mathcal{F},\mathcal{J}))\simeq\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{Hom}_X(\mathcal{K},\mathcal{J})). \tag{17}$$

Remark 2.5.4. One can slightly generalize the constructions and results of this section by weakening the definitions of contraadjusted and cotorsion quasi-coherent sheaves. Namely, a quasi-coherent sheaf $\mathcal{P}$ on X may be called weakly cotorsion if the functor $\mathrm{Hom}_X(-,\mathcal{P})$ transforms short exact sequences of flat quasi-coherent sheaves on X into short exact sequences of abelian groups. The weakly contraadjusted quasi-coherent sheaves are defined similarly (with the flat quasi-coherent sheaves replaced by very flat ones). Appropriate versions of Lemmas 2.5.1 and 2.5.3 can be proven in this setting, and the contraherent cosheaves $\mathfrak{Hom}$ can be defined.

On a quasi-compact semi-separated scheme X (or more generally, on a scheme where there are enough flat or very flat quasi-coherent sheaves), there is no difference between the weak and ordinary cotorsion/contraadjusted quasi-coherent sheaves (see Section 4.1 below; cf. [19, Sections 5.1.4 and 5.3]). One reason why we chose to use the stronger versions of these conditions here rather that the weaker ones is that it is not immediately clear whether the classes of weakly cotorsion/contraadjusted quasi-coherent sheaves are closed under extensions, or how the exact categories of such sheaves should be defined.

2.6. **Contratensor product of sheaves and cosheaves.** Let X be a semi-separated scheme and $\mathbf{B}$ be an (initially fixed) base of open subsets of X consisting of affine open subschemes. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathfrak{P}$ be a cosheaf of $\mathcal{O}_X$-modules.

The *contratensor product* $\mathcal{M} \odot_X \mathfrak{P}$ (computed on the base $\mathbf{B}$) is a quasi-coherent sheaf on X defined as the (nonfiltered) inductive limit of the following diagram of quasi-coherent sheaves on X indexed by affine open subschemes $U \in \mathbf{B}$ (cf. [5, Section 0.3.2] and Section 2.1 above).

To any affine open subscheme $U \in \mathbf{B}$ with the identity open embedding $j\colon U \longrightarrow X$ we assign the quasi-coherent sheaf $j_*(j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U])$ on X. For any pair of embedded affine open subschemes $V \subset U$, V, $U \in \mathbf{B}$ with the embedding maps $j\colon U \longrightarrow X$ and $h\colon V \longrightarrow U$ there is the morphism of quasi-coherent sheaves

$$j_*h_*(h^*j^*\mathcal{M} \otimes_{\mathcal{O}_X(V)} \mathfrak{P}[V]) \longrightarrow j_*(j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U])$$

defined in terms of the $\mathcal{O}_X(U)$-module morphism $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ and of the natural isomorphism $h_*(h^*\mathcal{K} \otimes_{\mathcal{O}(V)} N) \simeq \mathcal{K} \otimes_{\mathcal{O}(U)} N$, which holds for any quasi-coherent sheaf $\mathcal{K}$ on U and any $\mathcal{O}(V)$-module N.

Let $\mathcal{M}$ and $\mathcal{J}$ be quasi-coherent sheaves on X for which the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{M}, \mathcal{J})$ is defined (i. e., one of the sufficient conditions given in Section 2.5 for the construction of $\mathfrak{Hom}$ to make sense is satisfied). Then for any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ there is a natural isomorphism of abelian groups

$$\operatorname{Hom}_X(\mathcal{M} \odot_X \mathfrak{P},\ \mathcal{J}) \simeq \operatorname{Hom}^X(\mathfrak{P}, \mathfrak{Hom}_X(\mathcal{M}, \mathcal{J})), \tag{18}$$

where Hom^X now denotes the group of homomorphisms of cosheaves of $\mathcal{O}_X$-modules. In other words, the functor $\mathcal{M} \odot_X -$ is left adjoint to the functor $\mathfrak{Hom}_X(\mathcal{M}, -)$ "wherever the latter is defined".

Indeed, both groups of homomorphisms consist of all the compatible collections of morphisms of quasi-coherent sheaves

$$j_*(j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U]) \simeq j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{P}[U] \longrightarrow \mathcal{J}$$

on X, or equivalently, all the compatible collections of morphisms of $\mathcal{O}_X(U)$-modules

$$\mathfrak{P}[U] \longrightarrow \operatorname{Hom}_X(j_*j^*\mathcal{M}, \mathcal{J})$$

defined for all the identity embeddings $j\colon U \longrightarrow X$ of affine open subschemes $U \in \mathbf{B}$. The compatibility is with respect to the identity embeddings of affine open subschemes $h\colon V \longrightarrow U$, $V, U \in \mathbf{B}$, into one another.

In particular, the adjunction isomorphism (18) holds for any quasi-coherent sheaf $\mathcal{M}$, cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$, and injective quasi-coherent sheaf $\mathcal{J}$. Since there are enough injective quasi-coherent sheaves, it follows that the quasi-coherent sheaf of contratensor product $\mathcal{M} \odot_X \mathfrak{P}$ does not depend on the base of open affines $\mathbf{B}$ that was used to construct it.

The isomorphism $j_*j^*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K}) \simeq \mathcal{M} \otimes_{\mathcal{O}_X} j_*j^*\mathcal{K}$ for an embedding of affine open subscheme $j\colon U \longrightarrow X$ and quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{K}$ on X (see (12)) allows to construct a natural isomorphism of quasi-coherent sheaves

$$\mathcal{M} \otimes_{\mathcal{O}_X} (\mathcal{K} \odot_X \mathfrak{P}) \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{K}) \odot_X \mathfrak{P} \tag{19}$$

for any quasi-coherent sheaves $\mathcal{M}$ and $\mathcal{K}$ and any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a semi-separated scheme X.

3. Locally Contraherent Cosheaves

3.1. **Exact category of locally contraherent cosheaves.** Let X be a scheme and $W \subset X$ be its open subscheme. For any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on X, the restriction $\mathfrak{P}|_W$ of $\mathfrak{P}$ onto W is a cosheaf of $\mathcal{O}_W$-modules defined by the rule $\mathfrak{P}|_W(U) = \mathfrak{P}(U)$ for any open subset $U \subset W$. For a contraherent cosheaf $\mathfrak{P}$ on X one has $\mathfrak{P}|_W \simeq j^!\mathfrak{P}$, where $j\colon W \longrightarrow X$ is the identity open embedding.

A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a scheme X is called *locally contraherent* if every point $x \in X$ has an open neighborhood $x \in W \subset X$ such that the cosheaf of $\mathcal{O}_W$-modules $\mathfrak{P}|_W$ is contraherent. Given an open covering $\mathbf{W} = \{W\}$ a scheme X, a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is called $\mathbf{W}$-locally contraherent if for any open subscheme $W \subset X$ belonging to $\mathbf{W}$ the cosheaf of $\mathcal{O}_W$-modules $\mathfrak{P}|_W$ is contraherent on W. Obviously, a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is locally contraherent if and only if there exists an open covering $\mathbf{W}$ of the scheme X such that $\mathfrak{P}$ is $\mathbf{W}$-locally contraherent.

Let us call an open subscheme of a scheme X *subordinate* to an open covering $\mathbf{W}$ if it is contained in one of the open subsets of X belonging to $\mathbf{W}$. Notice that, by the definition of a contraherent cosheaf, the property of a cosheaf of $\mathcal{O}_X$-modules to be $\mathbf{W}$-locally contraherent only depends on the collection of all affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$.

Theorem 3.1.1. *Let $\mathbf{W}$ be an open covering of a scheme X. Then the restriction of cosheaves of $\mathcal{O}_X$-modules to the base of open subsets of X consisting of all the affine open subschemes subordinate to $\mathbf{W}$ induces an equivalence between the category of $\mathbf{W}$-locally contraherent cosheaves on X and the category of covariant functors with $\mathcal{O}_X$-module structures on the category of affine open subschemes of X subordinate to $\mathbf{W}$, satisfying the contraadjustness and contraherence conditions* (i-ii) *of Section* 2.2 *for all affine open subschemes $V \subset U \subset X$ subordinate to $\mathbf{W}$.*

Proof. The same as in Theorem 2.2.1, except that the base of affine open subschemes of X subordinate to $\mathbf{W}$ is considered throughout. $\square$

Let X be a scheme and $\mathbf{W}$ be its open covering. By Theorem 2.1.2, the category of cosheaves of $\mathcal{O}_X$-modules is a full subcategory of the category of covariant functors with $\mathcal{O}_X$-module structures on the category of affine open subschemes of X subordinate to $\mathbf{W}$. The category of such functors with $\mathcal{O}_X$-module structures is clearly abelian, has exact functors of infinite direct sum and infinite product, and the functors of cosections over a particular affine open subscheme subordinate to $\mathbf{W}$ are exact on it and preserve infinite direct sums and products.

The full subcategory of cosheaves of $\mathcal{O}_X$-modules in this abelian category is closed under extensions, cokernels, and infinite direct sums. For the quasi-compactness reasons explained in Remark 2.1.4, it is also closed under infinite products.

Therefore, the category of cosheaves of $\mathcal{O}_X$-modules acquires the induced exact category structure with exact functors of infinite direct sum and product, and exact functors of cosections on affine open subschemes subordinate to $\mathbf{W}$. Of course, this exact category structure depends on the choice of a covering $\mathbf{W}$. Along the way we have proven that infinite products exist in the additive category of cosheaves of $\mathcal{O}_X$-modules on a scheme X, and the functors of cosections over quasi-compact quasi-separated open subschemes of X preserve them.

The full subcategory of $\mathbf{W}$-locally contraherent cosheaves is closed under extensions and infinite products in the category of cosheaves of $\mathcal{O}_X$-modules with the above exact category structure. Thus the category of $\mathbf{W}$-locally contrantraherent cosheaves has the induced exact category structure with exact functors of infinite product, and exact functors of cosections over affine open subschemes subordinate to $\mathbf{W}$. We denote this exact category of $\mathbf{W}$-locally contraherent cosheaves on a scheme X by $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

More explicitly, a short sequence of $\mathbf{W}$-locally contraherent cosheaves $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ is exact in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ if the sequence of cosection modules $0 \longrightarrow \mathfrak{P}[U] \longrightarrow \mathfrak{Q}[U] \longrightarrow \mathfrak{R}[U] \longrightarrow 0$ is exact for every affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Passing to the inductive limit with respect to refinements of the coverings $\mathbf{W}$, we obtain the exact category structure on the category of locally contraherent cosheaves $X\text{–}\mathsf{lcth}$ on the scheme X.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is said to be *locally cotorsion* if for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is cotorsion. By Lemma 1.4.6(a), this definition can be equivalently rephrased by saying that a locally contraherent cosheaf $\mathfrak{P}$ on X is locally cotorsion if and only if for any affine open subscheme $U \subset X$ such that the cosheaf $\mathfrak{P}|_U$ is contraherent on the scheme U the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ is cotorsion.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X is called *locally injective* if for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ the $\mathcal{O}_X(U)$-module $\mathfrak{J}[U]$ is injective. By Lemma 1.4.6(b), a locally contraherent cosheaf $\mathfrak{J}$ on X is locally injective if and only if for any affine open subscheme $U \subset X$ such that the cosheaf $\mathfrak{J}|_U$ is contraherent on the scheme U the $\mathcal{O}(U)$-module $\mathfrak{J}[U]$ is injective.

One defines the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ of locally cotorsion and locally injective $\mathbf{W}$-locally contraherent cosheaves on X in the same way as above. These are full subcategories closed under extensions, infinite products, and cokernels of admissible monomorphisms in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, with the induced exact category structures. Passing to the inductive limit with respect to refinements, we obtain the exact categories $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}^{\mathsf{lin}}$ of locally cotorsion and locally injective locally contraherent cosheaves on X.

3.2. **Contraherent and locally contraherent cosheaves.** By Lemma 1.3.1(a), a short sequence of $\mathbf{W}$-locally contraherent cosheaves on X is exact in $X\text{–}\mathsf{lcth}$ (i. e., after some refinement of the covering) if and only if it is exact in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. By Lemma 1.3.1(b), a morphism of $\mathbf{W}$-locally contraherent cosheaves is an admissible epimorphism in $X\text{–}\mathsf{lcth}$ if and only if it is an admissible epimorphism in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

Analogously, by Lemma 1.5.1(a), a short sequence of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X is exact in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ if and only if it is exact in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. By Lemma 1.5.1(b), a morphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X is an admissible epimorphism in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ if and only if it is an admissible epimorphism in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. The similar assertions hold for locally injective locally contraherent cosheaves, and they are provable in the same way.

On the other hand, a morphism in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$, or $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ is an admissible monomorphism if and only if it acts injectively on the modules of cosections over all the affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. The following counterexample shows that this condition *does* change when the covering $\mathbf{W}$ is refined.

In other words, the full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}$ is closed under the passage to the kernels of admissible epimorphisms, but not to the cokernels of admissible monomorpisms in $X\text{–}\mathsf{lcth}$. Once we show that, it will also follow that there *do* exist locally contraherent cosheaves that are not contraherent. The locally cotorsion and locally injective contraherent cosheaves have all the same problems.

Example 3.2.1. Let R be a commutative ring and f, $g \in R$ be two elements generating the unit ideal. Let M be an R-module containing no f-divisible or g-divisible elements, i. e., $\operatorname{Hom}_R(R[f^{-1}], M) = 0 = \operatorname{Hom}_R(R[g^{-1}], M)$.

Let $M \longrightarrow P$ be an embedding of M into a contraadjusted R-module P, and let Q be the cokernel of this embedding. Then Q is also a contraadjusted R-module. One can take R to be a Dedekind domain, so that it has homological dimension 1; then whenever P is a cotorsion or injective R-module, Q has the same property.

Consider the morphism of contraherent cosheaves $\widehat{P} \longrightarrow \widehat{Q}$ on $\operatorname{Spec} R$ related to the surjective morphism of contraadjusted (cotorsion, or injective) R-modules $P \longrightarrow Q$. In restriction to the covering of $\operatorname{Spec} R$ by the two principal affine open subsets $\operatorname{Spec} R[f^{-1}]$ and $\operatorname{Spec} R[g^{-1}]$, we obtain two morphisms of contraherent cosheaves related to the two morphisms of contraadjusted modules $\operatorname{Hom}_R(R[f^{-1}], P) \longrightarrow \operatorname{Hom}_R(R[f^{-1}], Q)$ and $\operatorname{Hom}_R(R[g^{-1}], P) \longrightarrow \operatorname{Hom}_R(R[g^{-1}], Q)$ over the rings $\operatorname{Spec} R[f^{-1}]$ and $\operatorname{Spec} R[g^{-1}]$.

Due to the condition imposed on M, the latter two morphisms of contraadjusted modules are injective. On the other hand, the morphism of contraadjusted R-modules $P \longrightarrow Q$ is not. It follows that the cokernel $\mathfrak{R}$ of the morphism of contraherent cosheaves $\widehat{P} \longrightarrow \widehat{Q}$ taken in the category of all cosheaves of $\mathcal{O}_{\operatorname{Spec} R}$-modules (or equivalently, in the category of copresheaves of $\mathcal{O}_{\operatorname{Spec} R}$-modules) is contraherent in restriction to $\operatorname{Spec} R[f^{-1}]$ and $\operatorname{Spec} R[g^{-1}]$, but not over $\operatorname{Spec} R$. In fact, one has $\mathfrak{R}[\operatorname{Spec} R] = 0$ (since the morphism $P \longrightarrow Q$ is surjective).

Let us point out that for any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a scheme X such that the $\mathcal{O}_X(U)$-modules $\mathfrak{P}[U]$ are contraadjusted for all affine open subschemes $U \subset X$ subordinate to a particular open covering $\mathbf{W}$, the $\mathcal{O}_X(U)$-modules $\mathfrak{P}[U]$ are contraadjusted for *all* affine open subschemes $U \subset X$. This is so simply because the class of contraadjusted modules is closed under finite direct sums, restrictions of scalars, and cokernels. So the contraadjustness condition (ii) of Section 2.2 is, in fact, local; it is the contraherence condition (i) that isn't.

In the rest of the section we will explain how to distinguish the contraherent cosheaves among all the locally contraherent ones. Let X be a semi-separated scheme, $\mathbf{W}$ be its open covering, and $\{U_\alpha\}$ be an affine open covering subordinate to $\mathbf{W}$ (i. e., consisting of affine open subschemes subordinate to $\mathbf{W}$).

Let $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Consider the homological Čech complex of abelian groups (or $\mathcal{O}(X)$-modules) $C_\bullet(\{U_\alpha\}, \mathfrak{P})$ of the form

$$(20)\quad \cdots \longrightarrow \textstyle\bigoplus_{\alpha<\beta<\gamma} \mathfrak{P}[U_\alpha \cap U_\beta \cap U_\gamma] \longrightarrow \bigoplus_{\alpha<\beta} \mathfrak{P}[U_\alpha \cap U_\beta] \longrightarrow \bigoplus_\alpha \mathfrak{P}[U_\alpha].$$

Here (as in the sequel) our notation presumes the indices α to be linearly ordered. Let $\Delta(X, \mathfrak{P}) = \mathfrak{P}[X]$ denote the functor of global cosections of (locally contraherent) cosheaves on X; then, by the definition, we have $\Delta(X, \mathfrak{P}) \simeq H_0C_\bullet(\{U_\alpha\}, \mathfrak{P})$.

Lemma 3.2.2. *Let U be an affine scheme with an open covering $\mathbf{W}$ and a finite affine open covering $\{U_\alpha\}$ subordinate to $\mathbf{W}$. Then a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on U is contraherent if and only if $H_{>0}C_\bullet(\{U_\alpha\}, \mathfrak{P}) = 0$.*

Proof. The "only if" part is provided by Lemma 1.2.6(b). Let us prove "if". If the Čech complex $C_\bullet(\{U_\alpha\}, \mathfrak{P})$ has no higher homology, then it is a finite left resolution of the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ by contraadjusted $\mathcal{O}(U)$-modules. As we have explained above, the $\mathcal{O}(U)$-module $\mathfrak{P}[U]$ is contraadjusted, too.

For any affine open subscheme $V \subset U$, consider the Čech complex $C_\bullet(\{V \cap U_\alpha\}, \mathfrak{P}|_V)$ related to the restrictions of our cosheaf $\mathfrak{P}$ and our covering U_α to the open subscheme V. The complex $C_\bullet(\{V \cap U_\alpha\}, \mathfrak{P}|_V)$ can be obtained from the complex $C_\bullet(\{U_\alpha\}, \mathfrak{P})$ by applying the functor $\operatorname{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V), -)$. We have

$$H_0C_\bullet(\{U_\alpha\}, \mathfrak{P}) \simeq \mathfrak{P}[U] \quad \text{and} \quad H_0C_\bullet(\{V \cap U_\alpha\}, \mathfrak{P}|_V) \simeq \mathfrak{P}[V].$$

Since the functor $\operatorname{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V), -)$ preserves exactness of short sequences of contraadjusted $\mathcal{O}(U)$-modules, we conclude that $\mathfrak{P}[V] \simeq \operatorname{Hom}_{\mathcal{O}(U)}(\mathcal{O}(V), \mathfrak{P}[U])$. Both the contraadjustness and contraherence conditions have been now verified. □

Corollary 3.2.3. *If a* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{Q}$ *on an affine scheme* U *is an extension of two contraherent cosheaves* $\mathfrak{P}$ *and* $\mathfrak{R}$*, then* $\mathfrak{Q}$ *is also a contraherent cosheaf on* U*.*

Proof. Pick a finite affine open covering $\{U_\alpha\}$ of the affine scheme U subordinate to the covering $\mathbf{W}$. Then the complex of abelian groups $C_\bullet(\{U_\alpha\},\mathfrak{Q})$ is an extension of the complexes of abelian groups $C_\bullet(\{U_\alpha\},\mathfrak{P})$ and $C_\bullet(\{U_\alpha\},\mathfrak{R})$. Hence whenever the latter two complexes have no higher homology, neither does the former one. □

Corollary 3.2.4. *For any scheme* X *and any its open covering* $\mathbf{W}$*, the full exact subcategory of* $\mathbf{W}$*-locally contraherent cosheaves on* X *is closed under extensions in the exact category of locally contraherent cosheaves on* X*. In particular, the full exact subcategory of contraherent cosheaves on* X *is closed under extensions in the exact category of locally contraherent (or* $\mathbf{W}$*-locally contraherent) cosheaves on* X*.*

Proof. Follows easily from Corollary 3.2.3. □

3.3. **Direct and inverse images of locally contraherent cosheaves.** Let $\mathbf{W}$ be an open covering of a scheme X and $\mathbf{T}$ be an open covering of a scheme Y. A morphism of schemes $f\colon Y\longrightarrow X$ is called $(\mathbf{W},\mathbf{T})$-*affine* if for any affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$ the open subscheme $f^{-1}(U)\subset Y$ is affine and subordinate to $\mathbf{T}$. Any $(\mathbf{W},\mathbf{T})$-affine morphism is affine.

Let $f\colon Y\longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine morphism of schemes and $\mathfrak{Q}$ be a $\mathbf{T}$-locally contraherent cosheaf on Y. Then the cosheaf of $\mathcal{O}_X$-modules $f_!\mathfrak{Q}$ on X is $\mathbf{W}$-locally contraherent. The proof of this assertion is similar to that of its global version in Section 2.3. We have constructed an exact functor of direct image $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}}\longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ between the exact categories of $\mathbf{T}$-locally contraherent cosheaves on Y and $\mathbf{W}$-locally contraherent cosheaves on X.

A morphism of schemes $f\colon Y\longrightarrow X$ is called $(\mathbf{W},\mathbf{T})$-*coaffine* if for any affine open subscheme $V\subset Y$ subordinate to $\mathbf{T}$ there exists an affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$ such that $f(V)\subset U$, and for any two such affine open subschemes $f(V)\subset U'$, $U''\subset X$ subordinate to $\mathbf{W}$ there exists a third affine open subscheme $U\subset X$ such that $f(V)\subset U\subset U'\cap U''$. If the scheme X is semi-separated, then the second condition is trivial.

Notice that for any fixed open covering $\mathbf{W}$ of a semi-separated scheme X and any morphism of schemes $f\colon Y\longrightarrow X$ the covering $\mathbf{T}$ of the scheme Y consisting of all the full preimages $f^{-1}(U)$ of affine open subschemes $U\subset X$ has the property that the morphism $f\colon Y\longrightarrow X$ is $(\mathbf{W},\mathbf{T})$-coaffine. If the morphism f is affine, it is also $(\mathbf{W},\mathbf{T})$-affine with respect to the covering $\mathbf{T}$ constructed in this way.

Let $f\colon Y\longrightarrow X$ be a very flat $(\mathbf{W},\mathbf{T})$-coaffine morphism, and let $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Define a $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{P}$ on Y in the following way. Let $V\subset Y$ be an affine open subscheme subordinate to $\mathbf{T}$. Pick an affine open subscheme $U\subset X$, subordinate to $\mathbf{T}$ and such that $f(V)\subset U$, and set $(f^!\mathfrak{P})[V]=\operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_Y(V),\mathfrak{P}[U])$. One checks that the $\mathcal{O}_Y(V)$-module $(f^!\mathfrak{P})[V]$ is well-defined, and the contraadjustness and the ($\mathbf{T}$-local) contraherence conditions hold for $f^!\mathfrak{P}$ in the same way as in Section 2.3. We have constructed an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}$.

Let $f\colon Y \longrightarrow X$ be a flat $(\mathbf{W},\mathbf{T})$-coaffine morphism of schemes, and let $\mathfrak{P}$ be a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. Then the same rule as above defines a locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{P}$ on Y. So we obtain an exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$. Finally, for any $(\mathbf{W},\mathbf{T})$-coaffine morphism of schemes $f\colon Y \longrightarrow X$ and any locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X the same rule defines a locally injective $\mathbf{T}$-locally contraherent cosheaf $f^!\mathfrak{J}$ on Y. We obtain an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$.

Passing to the inductive limits of exact categories with respects to the refinements of coverings and taking into account the above remark about $(\mathbf{W},\mathbf{T})$-coaffine morphisms, we obtain an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth} \longrightarrow Y\text{–}\mathsf{lcth}$ for any very flat morphism of schemes $f\colon Y \longrightarrow X$. For an open embedding $j\colon Y \longrightarrow X$, the direct image $j^!$ coincides with the restriction functor $\mathfrak{P} \longmapsto \mathfrak{P}|_Y$ on the locally contraherent cosheaves $\mathfrak{P}$ on X. For a flat morphism f, we have an exact functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}$, and for an arbitrary morphism of schemes $f\colon Y \longrightarrow Y$ there is an exact functor of inverse image of locally injective locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$.

If $f\colon Y \longrightarrow X$ is a $(\mathbf{W},\mathbf{T})$-affine morphism and $\mathfrak{Q}$ is a locally cotorsion $\mathbf{T}$-locally contraherent cosheaf on Y, then $f_!\mathfrak{Q}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. So the direct image functor $f_!$ restricts to an exact functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. If f is a flat $(\mathbf{W},\mathbf{T})$-affine morphism and $\mathfrak{J}$ is a locally injective $\mathbf{T}$-locally contraherent cosheaf on Y, then $f_!\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X. Hence in this case the direct image also restricts to an exact functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine $(\mathbf{W},\mathbf{T})$-coaffine morphism. Then for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y and locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X there is an adjunction isomorphism

$$\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{J}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{J}), \tag{21}$$

where Hom^X and Hom^Y denote the abelian groups of morphisms in the categories of locally contraherent cosheaves on X and Y.

If the morphism f is, in addition, flat then the isomorphism

$$\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{P}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P}) \tag{22}$$

holds for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ and locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X. In particular, $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. When the morphism f is very flat, the functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}$ is right adjoint to the functor $f_!\colon Y\text{–}\mathsf{lcth}_{\mathbf{T}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

In all the mentioned cases, both abelian groups $\operatorname{Hom}^X(f_!\mathfrak{Q},\mathfrak{P})$ and $\operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{P})$ are identified with the group consisting of all the compatible collections of homomorphisms of $\mathcal{O}_X(U)$-modules $\mathfrak{Q}[V] \longrightarrow \mathfrak{P}[U]$ defined for all affine open subschemes $U \subset X$ and $V \subset Y$ subordinate to, respectively, $\mathbf{W}$ and $\mathbf{T}$, and such that $f(V) \subset U$.

Let $f\colon Y \longrightarrow X$ be a morphism of schemes and $j\colon U \longrightarrow X$ be an open embedding. Set $V = U \times_X Y$, and denote by $j'\colon V \longrightarrow Y$ and $f'\colon V \longrightarrow U$ the natural morphisms. Then for any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$, there is a natural isomorphism of cosheaves of $\mathcal{O}_U$-modules $(f_!\mathfrak{Q})|_U \simeq f'_!(\mathfrak{Q}|_V)$.

In particular, suppose f is a $(\mathbf{W},\mathbf{T})$-affine morphism. Define the open coverings $\mathbf{W}|_U$ and $\mathbf{T}|_V$ as the collections of all intersections of the open subsets $W \in \mathbf{W}$ and $T \in \mathbf{T}$ with U and V, respectively. Then f' is a $(\mathbf{W}|_U,\mathbf{T}|_V)$-affine morphism. For any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y there is a natural isomorphism of $\mathbf{W}|_U$-locally contraherent cosheaves $j^! f_!\mathfrak{Q} \simeq f'_! j'^! \mathfrak{Q}$ on U.

More generally, let $f\colon Y \longrightarrow X$ and $g\colon x \longrightarrow X$ be morphisms of schemes. Set $y = x \times_X Y$; let $f'\colon y \longrightarrow x$ and $g'\colon y \longrightarrow Y$ be the natural morphisms. Let $\mathbf{W}$, $\mathbf{T}$, and $\mathbf{w}$ be open coverings of, respectively, X, Y, and x such that the morphism f is $(\mathbf{W},\mathbf{T})$-affine, while the morphism g is $(\mathbf{W},\mathbf{w})$-coaffine. Assume that the scheme Y is semi-separated.

Define two coverings $\mathbf{t}'$ and $\mathbf{t}''$ of the scheme y by the rules that $\mathbf{t}'$ consists of all the full preimages of affine open subschemes in x subordinate to $\mathbf{w}$, while $\mathbf{t}''$ is the collection of all the full preimages of affine open subschemes in Y subordinate to $\mathbf{T}$. One can easily see that the covering $\mathbf{t}'$ is subordinate to $\mathbf{t}''$. Let $\mathbf{t}$ be any covering of y such that $\mathbf{t}'$ is subordinate to $\mathbf{t}$ and $\mathbf{t}$ is subordinate to $\mathbf{t}''$. Then the former condition guarantees that the morphism f' is $(\mathbf{w},\mathbf{t})$-affine, while under the latter condition the morphism g' is $(\mathbf{T},\mathbf{t})$-coaffine.

Assume that the morphism g is very flat. Then for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{P}$ on Y there is a natural isomorphism $g^! f_!\mathfrak{P} \simeq f'_! g'^! \mathfrak{P}$ of $\mathbf{w}$-locally contraherent cosheaves on x.

Alternatively, assume that the morphism g is flat. Then for any locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{P}$ on Y there is a natural isomorphism $g^! f_!\mathfrak{P} \simeq f'_! g'^! \mathfrak{P}$ of locally cotorsion $\mathbf{w}$-locally contraherent cosheaves on x.

As a third alternative, assume that the morphism f is flat (while g may be arbitrary). Then for any locally injective $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{J}$ on Y there is a natural isomorphism $g^! f_!\mathfrak{J} \simeq f'_! g'^! \mathfrak{J}$ of locally injective $\mathbf{w}$-locally contraherent cosheaves on x.

All these isomorphisms of locally contraherent cosheaves are constructed using the natural isomorphism of r-modules $\mathrm{Hom}_R(r,P) \simeq \mathrm{Hom}_S(S\otimes_R r,\ P)$ for any commutative ring homomorphisms $R \longrightarrow S$ and $R \longrightarrow r$, and any S-module P.

In other words, the direct images of $\mathbf{T}$-locally contraherent cosheaves under $(\mathbf{W},\mathbf{T})$-affine morphisms commute with the inverse images in those base change situations when all the functors involved are defined.

The following particular case will be important for us. Let $\mathbf{W}$ be an open covering of a scheme X and $j\colon Y \longrightarrow X$ be an affine open embedding subordinate to $\mathbf{W}$ (i. e., Y is contained in one of the open subsets of X belonging to $\mathbf{W}$). Then one can endow the scheme Y with the open covering $\mathbf{T} = \{Y\}$ consisting of the only open subset $Y \subset Y$. This makes the embedding j both $(\mathbf{W},\mathbf{T})$-affine and $(\mathbf{W},\mathbf{T})$-coaffine. Also, the morphism j, being an open embedding, is very flat.

Therefore, the inverse and direct images $j^!$ and $j_!$ form a pair of adjoint exact functors between the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ of $\mathbf{W}$-locally contraherent cosheaves on X and the exact category $Y\text{–}\mathsf{ctrh}$ of contraherent cosheaves on Y. Moreover, the image of the functor $j_!$ is contained in the full exact subcategory of (globally) contraherent cosheaves $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$. Both functors preserve the subcategories of locally cotorsion and locally injective cosheaves.

Now let $\mathbf{W}$ be an open covering of a quasi-compact semi-separated scheme X and let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine covering of X subordinate to $\mathbf{W}$. Denote by $j_{\alpha_1,\dots,\alpha_k}$ the open embeddings $U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \longrightarrow X$. Then for any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X the cosheaf Čech sequence (cf. (45))

$$
(23)\quad 0 \longrightarrow j_{1,\dots,N!}j_{1,\dots,N}^!\mathfrak{P} \longrightarrow \cdots \\
\longrightarrow \textstyle\bigoplus_{\alpha<\beta} j_{\alpha,\beta!}j_{\alpha,\beta}^!\mathfrak{P} \longrightarrow \bigoplus_\alpha j_{\alpha!}j_\alpha^!\mathfrak{P} \longrightarrow \mathfrak{P} \longrightarrow 0
$$

is exact in the exact category of $\mathbf{W}$-locally contraherent cosheaves on X. Indeed, the corresponding sequence of cosections over every affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$ is exact by Lemma 1.2.6(b). We have constructed a finite left resolution of a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ by contraherent cosheaves. When $\mathfrak{P}$ is a locally cotorsion or locally injective $\mathbf{W}$-locally contraherent cosheaf, the sequence (23) is exact in the category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ or $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$, respectively.

3.4. **Contrahereable cosheaves and the contraherator.** A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on a scheme X is said to be *derived contrahereable* if

(i°) for any affine open subscheme $U \subset X$ and its finite affine open covering $U = \bigcup_{\alpha=1}^N U_\alpha$ the homological Čech sequence (cf. (20))

$$
(24)\quad 0 \longrightarrow \mathfrak{P}[U_1 \cap \dots \cap U_N] \longrightarrow \cdots \\
\longrightarrow \textstyle\bigoplus_{\alpha<\beta} \mathfrak{P}[U_\alpha \cap U_\beta] \longrightarrow \bigoplus_\alpha \mathfrak{P}[U_\alpha] \longrightarrow \mathfrak{P}[U] \longrightarrow 0
$$

is exact; and

(ii) for any affine open subscheme $U \subset X$, the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is contraadjusted.

We will call (i°) the *exactness condition* and (ii) the *contraadjustness condition.*

Notice that the present contraadjustness condition (ii) is an equivalent restatement of the contraadjustness condition (ii) of Section 2.2, while the exactness condition (i°) is weaker than the contraherence condition (i) of Section 2.2 (provided that the condition (ii) is assumed). By Remark 2.1.4, the exactness condition (i°) can be also thought of as a strengthening of the cosheaf condition (6) on a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes of X; any such functor satisfying (i°) can be extended to a cosheaf of $\mathcal{O}_X$-modules.

Let $\mathbf{W}$ be an open covering of a scheme X. A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on X is called $\mathbf{W}$*-locally derived contrahereable* if its restrictions $\mathfrak{P}|_W$ to all the open subschemes $W \in \mathbf{W}$ are derived contrahereable on W. In other words, this means that the conditions (i°) and (ii) must hold for all the affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. A cosheaf of $\mathcal{O}_X$-modules is called locally derived contrahereable if it is $\mathbf{W}$-locally derived contrahereable for some open covering $\mathbf{W}$. Any contraherent cosheaf is derived contrahereable, and any $\mathbf{W}$-locally contraherent cosheaf is $\mathbf{W}$-locally derived contrahereable. Conversely, according to Lemma 3.2.2, if a $\mathbf{W}$-locally contraherent cosheaf is derived contrahereable, then it is contraherent.

A $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{P}$ on X is called *locally cotorsion* (respectively, *locally injective*) if the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is cotorsion (resp., injective) for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. A locally derived contrahereable cosheaf $\mathfrak{P}$ is locally cotorsion (resp., locally injective) if and only if the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ is cotorsion (resp., injective) for every affine open subscheme $U \subset X$ such that the cosheaf $\mathfrak{P}|_U$ is derived contrahereable.

In the $\mathbf{W}$-related exact category structure on the category of cosheaves of $\mathcal{O}_X$-modules discussed in Section 3.1, $\mathbf{W}$-locally derived contrahereable cosheaves form an exact subcategory closed under extensions, infinite products, and cokernels of admissible monomorphisms. Hence there is the induced exact category structure on the category of $\mathbf{W}$-locally derived contrahereable cosheaves on X.

Let U be an affine scheme and $\mathfrak{Q}$ be a derived contrahereable cosheaf on U. The *contraherator* $\mathfrak{C}\mathfrak{Q}$ of the cosheaf $\mathfrak{Q}$ is defined in this simplest case as the contraherent cosheaf on U corresponding to the contraadjusted $\mathcal{O}(U)$-module $\mathfrak{Q}[U]$, that is $\mathfrak{C}\mathfrak{Q} = \widehat{\mathfrak{Q}[U]}$. There is a natural morphism of derived contrahereable cosheaves $\mathfrak{Q} \longrightarrow \mathfrak{C}\mathfrak{Q}$ on U. For any affine open subscheme $V \subset U$ there is a natural morphism of contraherent cosheaves $\mathfrak{C}(\mathfrak{Q}|_V) \longrightarrow (\mathfrak{C}\mathfrak{Q})|_V$ on V. Our next goal is to extend this construction to an appropriate class of cosheaves of $\mathcal{O}_X$-modules on quasi-compact semi-separated schemes X (cf. [26, Appendix B]).

Let X be such a scheme with an open covering $\mathbf{W}$ and a finite affine open covering $X = \bigcup_{\alpha=1}^N U_\alpha$ subordinate to $\mathbf{W}$. The *contraherator complex* $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P})$ of a $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{P}$ on X is a finite Čech complex of contraherent cosheaves on X of the form

$$(25)\quad 0 \longrightarrow j_{1,\dots,N!}\,\mathfrak{C}(\mathfrak{P}|_{U_1\cap\dots\cap U_N}) \longrightarrow \cdots \\ \longrightarrow \textstyle\bigoplus_{\alpha<\beta} j_{\alpha,\beta!}\,\mathfrak{C}(\mathfrak{P}|_{U_\alpha\cap U_\beta}) \longrightarrow \bigoplus_\alpha j_{\alpha!}\,\mathfrak{C}(\mathfrak{P}|_{U_\alpha}),$$

where $j_{\alpha_1,\dots,\alpha_k}$ is the open embedding $U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \longrightarrow X$ and the notation $\mathfrak{C}(\mathfrak{Q})$ was explained above. The differentials in this complex are constructed using the adjunction of the direct and inverse images of contraherent cosheaves and the above morphisms $\mathfrak{C}(\mathfrak{Q}|_V) \longrightarrow (\mathfrak{C}\mathfrak{Q})|_V$.

Lemma 3.4.1. *Let $\mathfrak{P}$ be a $\mathbf{W}$-locally derived contrahereable cosheaf on a quasi-compact semi-separated scheme X. Then the object of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{ctrh})$ of the exact category of contraherent cosheaves on X represented by the complex $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P})$ does not depend on the choice of a finite affine open covering $\{U_\alpha\}$ of X subordinate to the covering $\mathbf{W}$.*

Proof. Let us adjoin another affine open subscheme $V \subset X$, subordinate to $\mathbf{W}$, to the covering $\{U_\alpha\}$. Then the complex $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P})$ embeds into the complex

$\mathfrak{C}_\bullet(\{V, U_\alpha\}, \mathfrak{P})$ by a termwise split morphism of complexes with the cokernel isomorphic to the complex $k_! \, \mathfrak{C}_\bullet(\{V \cap U_\alpha\}, \mathfrak{P}|_V) \longrightarrow k_! \, \mathfrak{C}(\mathfrak{P}|_V)$.

The complex of contraherent cosheaves $\mathfrak{C}_\bullet(\{V \cap U_\alpha\}, \mathfrak{P}|_V) \longrightarrow \mathfrak{C}(\mathfrak{P}|_V)$ on V corresponds to the acyclic complex of contraadjusted $\mathcal{O}(V)$-modules (24) for the covering of an affine open subscheme $V \subset X$ by the affine open subschemes $V \cap U_\alpha$. Hence the cokernel of the admissible monomorphism of complexes $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P}) \longrightarrow \mathfrak{C}_\bullet(\{V, U_\alpha\}, \mathfrak{P})$ is an acyclic complex of contraherent cosheaves on X.

Now, given two affine open coverings $X = \bigcup_\alpha U_\alpha = \bigcup_\beta V_\beta$ subordinate to $\mathbf{W}$, one compares both complexes $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P})$ and $\mathfrak{C}_\bullet(\{V_\beta\}, \mathfrak{P})$ with the complex $\mathfrak{C}_\bullet(\{U_\alpha, V_\beta\}, \mathfrak{P})$ corresponding to the union of the two coverings $\{U_\alpha, V_\beta\}$. □

A $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{P}$ on a quasi-compact semi-separated scheme X is called $\mathbf{W}$-*locally contrahereable* if the complex $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P})$ for some particular (or equivalently, for any) finite affine open covering $X = \bigcup_\alpha U_\alpha$ is quasi-isomorphic in $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ to a $\mathbf{W}$-locally contraherent cosheaf on X (viewed as a complex concentrated in homological degree 0). The $\mathbf{W}$-locally contraherent cosheaf that appears here is called the ($\mathbf{W}$-*local*) *contraherator* of $\mathfrak{P}$ and denoted by $\mathfrak{C}\,\mathfrak{P}$. A derived contrahereable cosheaf $\mathfrak{P}$ on X is called *contrahereable* if it is locally contrahereable with respect to the covering $\{X\}$.

Any derived contrahereable cosheaf $\mathfrak{Q}$ on an affine scheme U is contrahereable, because the contraherator complex $\mathfrak{C}_\bullet(\{U\}, \mathfrak{Q})$ is concentrated in homological degree 0. The contraherator cosheaf $\mathfrak{C}\,\mathfrak{Q}$ constructed in this way coincides with the one defined above specifically in the affine scheme case; so our notation and terminology is consistent. Any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on a quasi-compact semi-separated scheme X is $\mathbf{W}$-locally contrahereable; the corresponding contraherator complex $\mathfrak{C}_\bullet(\{U_\alpha\}, \mathfrak{P})$ is the contraherent Čech resolution (23) of a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P} = \mathfrak{C}\,\mathfrak{P}$.

The contraherator complex construction $\mathfrak{C}_\bullet(\{U_\alpha\}, -)$ is an exact functor from the exact category of $\mathbf{W}$-locally contrahereable cosheaves to the exact category of finite complexes of contraherent cosheaves on X, or to the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{ctrh})$. The full subcategory of $\mathbf{W}$-locally contrahereable cosheaves in the exact category of $\mathbf{W}$-locally derived contrahereable cosheaves is closed under extensions and infinite products. Hence it acquires the induced exact category structure. The contraherator $\mathfrak{C}$ is an exact functor from the exact category of $\mathbf{W}$-locally contrahereable cosheaves to that of $\mathbf{W}$-locally contraherent ones.

All the above applies to locally cotorsion and locally injective $\mathbf{W}$-locally derived contrahereable cosheaves as well. These form full exact subcategories closed under extensions, infinite products, and cokernels of admissible monomorphisms in the exact category of $\mathbf{W}$-locally derived contrahereable cosheaves. The contraherator complex construction $\mathfrak{C}_\bullet(\{U_\alpha\}, -)$ takes locally cotorsion (resp., locally injective) $\mathbf{W}$-locally derived contrahereable cosheaves to finite complexes of locally cotorsion (resp., locally injective) contraherent cosheaves on X.

A locally cotorsion (resp., locally injective) $\mathbf{W}$-locally derived contrarehreable cosheaf is called $\mathbf{W}$-locally contrahereable if it is $\mathbf{W}$-locally contrahereable as a

$\mathbf{W}$-locally derived contrahereable cosheaf. The contraherator functor $\mathfrak{C}$ takes locally cotorsion (resp., locally injective) $\mathbf{W}$-locally contrahereable cosheaves to locally cotorsion (resp., locally injective) $\mathbf{W}$-locally contraherent cosheaves.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine morphism of schemes. Then the direct image functor $f_!$ takes $\mathbf{T}$-locally derived contrahereable cosheaves on Y to $\mathbf{W}$-locally derived contrahereable cosheaves on X. For a $(\mathbf{W},\mathbf{T})$-affine morphism f of quasi-compact semi-separated schemes, the functor $f_!$ also commutes with the contraherator complex construction, as there is a natural isomorphism of complexes of contraherent cosheaves

$$f_!\,\mathfrak{C}_\bullet(\{f^{-1}(U_\alpha)\},\mathfrak{P}) \simeq \mathfrak{C}_\bullet(\{U_\alpha\}, f_!\mathfrak{P})$$

for any finite affine open covering U_α of X subordinate to $\mathbf{W}$. It follows that the functor $f_!$ takes $\mathbf{T}$-locally contrahereable cosheaves to $\mathbf{W}$-locally contrahereable cosheaves and commutes with the functor $\mathfrak{C}$.

For any $\mathbf{W}$-locally contrahereable cosheaf $\mathfrak{P}$ and $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Q}$ on a quasi-compact semi-separated scheme X, there is a natural isomorphism of the groups of morphisms

$$\operatorname{Hom}^X(\mathfrak{P},\mathfrak{Q}) \simeq \operatorname{Hom}^X(\mathfrak{C}\mathfrak{P},\mathfrak{Q}). \tag{26}$$

In other words, the functor $\mathfrak{C}$ is left adjoint to the identity embedding functor of the category of $\mathbf{W}$-locally contrahereable cosheaves into the category of $\mathbf{W}$-locally contraherent ones. Indeed, applying the contraherator complex construction $\mathfrak{C}_\bullet(\{U_\alpha\},-)$ to a morphism $\mathfrak{P}\longrightarrow\mathfrak{Q}$ and passing to the zero homology, we obtain the corresponding morphism $\mathfrak{C}\mathfrak{P}\longrightarrow\mathfrak{Q}$. Conversely, any cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ is the cokernel of the rightmost arrow of the complex

$$\begin{aligned} 0 \longrightarrow j_{1,\dots,N!}(\mathfrak{P}|_{U_1\cap\dots\cap U_N}) \longrightarrow \cdots \qquad\qquad \\ \longrightarrow \textstyle\bigoplus_{\alpha<\beta} j_{\alpha,\beta!}(\mathfrak{P}|_{U_\alpha\cap U_\beta}) \longrightarrow \textstyle\bigoplus_\alpha j_{\alpha!}(\mathfrak{P}|_{U_\alpha}) \end{aligned} \tag{27}$$

in the additive category of cosheaves of $\mathcal{O}_X$-modules. A cosheaf $\mathfrak{P}$ satisfying the exactness condition (i°) for affine open subschemes $U\subset X$ subordinate to $\mathbf{W}$ is also quasi-isomorphic to the whole complex (27) in the $\mathbf{W}$-related exact category structure on the category of cosheaves of $\mathcal{O}_X$-modules. Passing to the zero homology of the natural morphism between the complexes (27) and (25), we produce the desired adjunction morphism $\mathfrak{P}\longrightarrow\mathfrak{C}\mathfrak{P}$.

For any $\mathbf{W}$-locally contrahereable cosheaf $\mathfrak{P}$ and any quasi-coherent sheaf $\mathcal{M}$ on X the natural morphism of cosheaves of $\mathcal{O}_X$-modules $\mathfrak{P}\longrightarrow\mathfrak{C}\mathfrak{P}$ induces an isomorphism of the contratensor products

$$\mathcal{M}\odot_X\mathfrak{P} \simeq \mathcal{M}\odot_X\mathfrak{C}(\mathfrak{P}). \tag{28}$$

Indeed, for any injective quasi-coherent sheaf $\mathcal{J}$ on X one has

$$\begin{aligned} \operatorname{Hom}_X(\mathcal{M}\odot_X\mathfrak{P},\ \mathcal{J}) &\simeq \operatorname{Hom}^X(\mathfrak{P},\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})) \\ &\simeq \operatorname{Hom}^X(\mathfrak{C}\mathfrak{P},\mathfrak{Hom}_X(\mathcal{M},\mathcal{J})) \simeq \operatorname{Hom}_X(\mathcal{M}\odot_X\mathfrak{C}\mathfrak{P},\ \mathcal{J}) \end{aligned}$$

in view of the isomorphism (18).

3.5. **$\mathfrak{Cohom}$ into a locally derived contrahereable cosheaf.** We start with discussing the $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a locally contraherent cosheaf.

Let $\mathbf{W}$ be an open covering of a scheme X. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X, and let $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. The $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ on the scheme X is defined by the rule $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. The contraadjustness and $\mathbf{W}$-local contraherence conditions can be verified in the same way as it was done in Section 2.4.

Similarly, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the same rule $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$.

Finally, if $\mathcal{M}$ is a quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is defined by the same rule as above. If $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X, then the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})$ is locally injective.

For any two very flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on a scheme X and any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X there is a natural isomorphism of $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G},\,\mathfrak{P}) \simeq \mathfrak{Cohom}_X(\mathcal{G},\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})). \tag{29}$$

Similarly, for any two flat quasi-coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X there is a natural isomorphism (29) of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves. Finally, for any flat quasi-coherent sheaf $\mathcal{F}$, quasi-coherent sheaf $\mathcal{M}$, and locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X there are natural isomorphisms of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\begin{aligned} &\mathfrak{Cohom}_X(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F},\,\mathfrak{J}) \\ &\qquad\simeq \mathfrak{Cohom}_X(\mathcal{M},\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})) \simeq \mathfrak{Cohom}_X(\mathcal{F},\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})). \end{aligned}\tag{30}$$

More generally, let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X, and let $\mathfrak{P}$ be a $\mathbf{W}$-locally derived contrahereable cosheaf on X. The $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ on the scheme X is defined by the rule $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. For any pair of embedded affine open subschemes $V \subset U$ the restriction and corestriction morphisms $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ and $\mathfrak{P}[V] \longrightarrow \mathfrak{P}[U]$ induce a morphism of $\mathcal{O}_X(U)$-modules

$$\operatorname{Hom}_{\mathcal{O}_X(V)}(\mathcal{F}(V),\mathfrak{P}[V]) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U]),$$

so our rule defines a covariant functor with $\mathcal{O}_X$-module structure on the category of affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$. The contraherence condition clearly holds; and to check the exactness condition (i°) of Section 3.4 for this covariant functor, it suffices to apply the functor $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),-)$ to the exact sequence of contraadjusted $\mathcal{O}_X(U)$-modules (24) for the cosheaf $\mathfrak{P}$.

Similarly, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally derived contrahereable cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P})$ is defined by the same rule $U\longmapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathfrak{P}[U])$ for any affine open subscheme $U\subset X$ subordinate to $\mathbf{W}$. Finally, if $\mathcal{M}$ is a quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf on X, then the locally cotorsion $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ is defined by the very same rule. If $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf on X, then the $\mathbf{W}$-locally derived cotrahereable cosheaf $\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J})$ is locally injective.

The associativity isomorphisms (29–30) hold for the $\mathfrak{Cohom}$ into $\mathbf{W}$-locally derived contrahereable cosheaves under the assumptions similar to the ones made above in the locally contraherent case.

3.6. **Contraherent tensor product.** Let $\mathfrak{F}$ be a cosheaf of $\mathcal{O}_X$-modules on a scheme X and $\mathcal{M}$ be a quasi-coherent cosheaf on X. Define a covariant functor with an $\mathcal{O}_X$-module structure on the category of affine open subschemes of X by the rule $U\longmapsto \mathcal{M}(U)\otimes_{\mathcal{O}_X(U)}\mathfrak{F}[U]$. To a pair of embedded affine open subschemes $V\subset U\subset X$ this functor assigns the $\mathcal{O}_X(U)$-module homomorphism

$$\mathcal{M}(V)\otimes_{\mathcal{O}_X(V)}\mathcal{F}[V]\simeq \mathcal{M}(U)\otimes_{\mathcal{O}_X(U)}\mathfrak{F}[V]\longrightarrow \mathcal{M}(U)\otimes_{\mathcal{O}_X(U)}\mathfrak{F}[U].$$

Obviously, this functor satisfies the condition (6) of Theorem 2.1.2 (since the restriction of the cosheaf $\mathfrak{F}$ to affine open subschemes of X does). Hence the functor $U\longmapsto \mathcal{M}(U)\otimes_{\mathcal{O}_X(U)}\mathfrak{F}[U]$ extends uniquely to a cosheaf of $\mathcal{O}_X$-modules on X, which we will denote by $\mathcal{M}\otimes_X\mathfrak{F}$.

For any quasi-coherent sheaf $\mathcal{M}$, cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$, and locally injective $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{J}$ on a scheme X there is a natural isomorphism of abelian groups

$$\mathrm{Hom}^X(\mathcal{M}\otimes_X\mathfrak{F},\ \mathfrak{J})\simeq \mathrm{Hom}^X(\mathfrak{F},\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})). \tag{31}$$

The analogous adjunction isomorphism holds in the other cases mentioned in Section 3.5 when the functor $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a locally derived contrahereable cosheaf is defined. In other words, the functors $\mathfrak{Cohom}_X(\mathcal{M},-)$ and $\mathcal{M}\otimes_X-$ between subcategories of the category of cosheaves of $\mathcal{O}_X$-modules are adjoint "wherever the former functor is defined".

For a locally free sheaf of finite rank $\mathcal{E}$ and a $\mathbf{W}$-locally contraherent (resp., $\mathbf{W}$-locally derived contrahereable) cosheaf $\mathfrak{P}$ on a scheme X, the cosheaf of $\mathcal{O}_X$-modules $\mathcal{E}\otimes_X\mathfrak{P}$ is $\mathbf{W}$-locally contraherent (resp., $\mathbf{W}$-locally derived contrahereable). There is a natural isomorphism of $\mathbf{W}$-locally contraherent (resp., $\mathbf{W}$-locally derived contrahereable) cosheaves $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)\otimes_X\mathfrak{P}\simeq\mathfrak{Cohom}_X(\mathcal{E},\mathfrak{P})$ on X.

The isomorphism $j_*j^*(\mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{M})\simeq j_*j^*\mathcal{K}\otimes_{\mathcal{O}_X(U)}\mathcal{M}(U)$ (13) for quasi-coherent sheaves $\mathcal{K}$ and $\mathcal{M}$ and the embedding of an affine open subscheme $j\colon U\longrightarrow X$ allows to construct a natural isomorphism of quasi-coherent sheaves

$$(\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{M}) \odot_X \mathfrak{F} \simeq \mathcal{K} \odot_X (\mathcal{M} \otimes_X \mathfrak{F}) \tag{32}$$

for any quasi-coherent sheaves $\mathcal{K}$ and $\mathcal{M}$ and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ on a semi-separated scheme X.

Let $\mathbf{W}$ be an open covering of a scheme X. We will call a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ $\mathbf{W}$-*flat* if the $\mathcal{O}_X(U)$-module $\mathfrak{F}[U]$ is flat for every affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ is said to be *flat* if it is $\{X\}$-flat. Clearly, the direct image of a $\mathbf{T}$-flat cosheaf of $\mathcal{O}_Y$-modules with respect to a flat $(\mathbf{W}, \mathbf{T})$-affine morphism of schemes $f\colon Y \longrightarrow X$ is $\mathbf{W}$-flat.

One can easily see that whenever a cosheaf $\mathfrak{F}$ is $\mathbf{W}$-flat and satisfies the "exactness condition" (i°) of Section 3.4 for finite affine open coverings of affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$, the cosheaf $\mathcal{M} \otimes_X \mathfrak{F}$ also satisfies the condition (i°) for such open affines $U \subset X$. Similarly, whenever a quasi-coherent sheaf $\mathcal{F}$ is flat and a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ satisfies the condition (i°), so does the cosheaf $\mathcal{F} \otimes_X \mathfrak{P}$.

The full subcategory of $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves is closed under extensions and kernels of admissible epimorphisms in the exact category of $\mathbf{W}$-locally contraherent cosheaves $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ on X. Hence it acquires the induced exact category structure, which we will denote by $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{fl}}$. Similarly, there is the exact category structure on the category of $\mathbf{W}$-flat $\mathbf{W}$-locally derived contrahereable cosheaves on X induced from the exact category of $\mathbf{W}$-locally derived contrahereable cosheaves.

A quasi-coherent sheaf $\mathcal{K}$ on a scheme X is called *coadjusted* if the $\mathcal{O}_X(U)$-module $\mathcal{K}(U)$ is coadjusted (see Section 1.6) for every affine open subscheme $U \subset X$. By Lemma 1.6.5, the coadjustness of a quasi-coherent sheaf is a local property. By the definition, if a cosheaf of $\mathcal{O}_X$-modules $\mathfrak{P}$ on X satisfies the contraherence condition (ii) of Section 2.2 or 3.4 and a quasi-coherent sheaf $\mathcal{K}$ on X is coadjusted, then the cosheaf of $\mathcal{O}_X$-modules $\mathcal{K} \otimes_X \mathfrak{P}$ also satisfies the condition (ii).

The full subcategory of coadjusted quasi-coherent sheaves is closed under extensions and the passage to quotient objects in the abelian category of quasi-coherent sheaves $X\text{–}\mathsf{qcoh}$ on X. Hence it acquires the induced exact category structure, which we will denote by $X\text{–}\mathsf{qcoh}^{\mathsf{coa}}$.

Let $\mathcal{K}$ be a coadjusted quasi-coherent sheaf and $\mathfrak{F}$ be a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent (or more generally, $\mathbf{W}$-locally derived contrahereable) cosheaf on a scheme X. Then the tensor product $\mathcal{K} \otimes_X \mathfrak{F}$ satisfies both conditions (i°) and (ii) for affine open subschemes $U \subset X$ subordinate to $\mathbf{W}$, i. e., it is $\mathbf{W}$-locally derived contrahereable. (Of course, the cosheaf $\mathcal{K} \otimes_X \mathfrak{F}$ is *not* in general locally contraherent, even if the cosheaf $\mathfrak{F}$ was $\mathbf{W}$-locally contraherent.)

Assuming the scheme X is quasi-compact and semi-separated, the contraherator complex construction now allows to assign to this cosheaf of $\mathcal{O}_X$-modules a complex of contraherent cosheaves $\mathfrak{C}_\bullet(\{U_\alpha\},\ \mathcal{K} \otimes_X \mathfrak{F})$ on X. To a short exact sequence of coadjusted quasi-coherent sheaves or $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent (or $\mathbf{W}$-locally derived contrahereable) cosheaves on X, the functor $\mathfrak{C}_\bullet(\{U_\alpha\},\ - \otimes_X -)$ assigns a short exact sequence of complexes of contraherent cosheaves.

By Lemma 3.4.1, the corresponding object of the bounded derived category $\mathsf{D^b}(X\text{–}\mathsf{ctrh})$ does not depend on the choice of a finite affine open covering $\{U_\alpha\}$. We will denote it by $\mathcal{K} \otimes_{X\text{-ct}}^{\mathbb{L}} \mathfrak{F}$ and call the *derived contraherent tensor product* of a coadjusted quasi-coherent sheaf $\mathcal{K}$ and a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on a quasi-compact semi-separated scheme X.

When the derived category object $\mathcal{K} \otimes_{X\text{-ct}}^{\mathbb{L}} \mathfrak{F}$, viewed as an object of the derived category $\mathsf{D^b}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ via the embedding of exact categories $X\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$, turns out to be isomorphic to an object of the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, we say that the (underived) *contraherent tensor product* of $\mathcal{K}$ and $\mathfrak{F}$ is defined, and denote the corresponding object by $\mathcal{K} \otimes_{X\text{-ct}} \mathfrak{F} \in X\text{–}\mathsf{lcth}_{\mathbf{W}}$. In other words, for a coadjusted quasi-coherent sheaf $\mathcal{K}$ and a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf $\mathcal{F}$ on a quasi-compact semi-separated scheme X one sets $\mathcal{K} \otimes_{X\text{-ct}} \mathfrak{F} = \mathfrak{C}(\mathcal{K} \otimes_X \mathfrak{F})$ whenever the right hand side is defined (where $\mathfrak{C}$ denotes the $\mathbf{W}$-local contraherator).

Now assume that the scheme X is Noetherian. Then, by Corollary 1.6.4, a contraherent cosheaf $\mathfrak{F}$ on an affine open subscheme $U \subset X$ is flat if and only if the contraadjusted $\mathcal{O}(U)$-module $\mathfrak{F}[U]$ is flat. Besides, the full subcategory of $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is closed under infinite products. In addition, any coherent sheaf on X is coadjusted, as is any injective quasi-coherent sheaf and any quasi-coherent quotient sheaf of an injective one.

For any injective quasi-coherent sheaf $\mathcal{J}$ and any $\mathbf{W}$-flat $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{F}$ on X, the tensor product $\mathcal{J} \otimes_X \mathfrak{F}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf on X. For any flat quasi-coherent sheaf $\mathcal{F}$ and any locally injective $\mathbf{W}$-locally derived contrahereable cosheaf $\mathfrak{J}$ on X, the tensor product $\mathcal{F} \otimes_X \mathfrak{J}$ is a locally injective $\mathbf{W}$-locally derived contrahereable cosheaf. These assertions hold since the tensor product of a flat module and an injective module over a Noetherian ring is injective.

Let $\mathcal{M}$ be a coherent sheaf on X. Let $\mathfrak{F}$ be a $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaf on X. Then the cosheaf of $\mathcal{O}_X$-modules $\mathcal{M} \otimes_X \mathfrak{F}$ is $\mathbf{W}$-locally contraherent. Indeed, by Corollary 1.6.2(a), the $\mathcal{O}_X(U)$-module $\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}(U)$ is contraadjusted for any affine open subscheme $U \subset X$. For a pair of embedded affine open subschemes $V \subset U \subset X$ subordinate to the covering $\mathbf{W}$, one has

$$\begin{aligned}\mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathfrak{F}[V] &\simeq \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[V] \\ &\simeq \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V), \mathfrak{F}[U]) \\ &\simeq \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\, \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathfrak{F}[U])\end{aligned}$$

according to Corollary 1.6.2(c). Consequently, the contraherent tensor product $\mathcal{M} \otimes_{X\text{-ct}} \mathfrak{F}$ is defined and isomorphic to the tensor product $\mathcal{M} \otimes_X \mathfrak{F}$.

Finally, assume that the scheme X is Noetherian of finite Krull dimension. Then, according to [24, Corollaire II.3.3.2], for any affine open subscheme $U \subset X$, any flat $\mathcal{O}(U)$-module has finite projective dimension, so Corollary 1.6.3 is applicable. It follows that for any coherent sheaf $\mathcal{M}$ and $\mathbf{W}$-flat locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X, the tensor product $\mathcal{M} \otimes_X \mathfrak{F}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X.

3.7. **Compatibility of direct and inverse images with the tensor operations.** Let $\mathbf{W}$ be an open covering of a scheme X and $\mathbf{T}$ be an open covering of a scheme Y. Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-coaffine morphism.

Let $\mathcal{F}$ be a flat quasi-coherent sheaf and $\mathfrak{J}$ be a locally injective $\mathbf{W}$-locally contraherent cosheaf on the scheme X. Then there is a natural isomorphism of locally injective $\mathbf{T}$-locally contraherent cosheaves

$$f^! \,\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{J}) \simeq \mathfrak{Cohom}_Y(f^*\mathcal{F}, f^!\mathfrak{J}) \tag{33}$$

on the scheme Y.

Assume additionally that f is a flat morphism. Let $\mathcal{M}$ be a quasi-coherent sheaf and $\mathfrak{J}$ be a locally injective $\mathbf{W}$-locally contraherent cosheaf on X. Then there is a natural isomorphism of locally cotorsion $\mathbf{T}$-locally contraherent cosheaves

$$f^! \,\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J}) \simeq \mathfrak{Cohom}_Y(f^*\mathcal{M}, f^!\mathfrak{J}) \tag{34}$$

on Y. Analogously, if $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathfrak{P}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X, then there is a natural isomorphism of locally cotorsion $\mathbf{T}$-locally contraherent cosheaves

$$f^! \,\mathfrak{Cohom}_X(\mathcal{F},\mathfrak{P}) \simeq \mathfrak{Cohom}_Y(f^*\mathcal{F}, f^!\mathfrak{P}) \tag{35}$$

on the scheme Y.

Assume that, moreover, f is a very flat morphism. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf and $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Then there is the natural isomorphism (35) of $\mathbf{T}$-locally contraherent cosheaves on Y.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W},\mathbf{T})$-affine $(\mathbf{W},\mathbf{T})$-coaffine morphism. Let $\mathcal{N}$ be a quasi-coherent cosheaf on Y and $\mathfrak{J}$ be a locally injective $\mathbf{W}$-locally contraherent cosheaf on X. Then there is a natural isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(f_*\mathcal{N},\mathfrak{J}) \simeq f_! \,\mathfrak{Cohom}_Y(\mathcal{N}, f^!\mathfrak{J}) \tag{36}$$

on the scheme X. This is one version of the projection formula for the $\mathfrak{Cohom}$ from a quasi-coherent sheaf to a contraherent cosheaf.

Assume additionally that f is a flat morphism. Let $\mathcal{G}$ be a flat quasi-coherent sheaf on Y and $\mathfrak{P}$ be a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X. Then there is a natural isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(f_*\mathcal{G},\mathfrak{P}) \simeq f_! \,\mathfrak{Cohom}_Y(\mathcal{G}, f^!\mathfrak{P}) \tag{37}$$

on the scheme X.

Assume that, moreover, f is a very flat morphism. Let $\mathcal{G}$ be a very flat quasi-coherent sheaf on Y and $\mathfrak{P}$ be a $\mathbf{W}$-locally contraherent cosheaf on X. Then there is the natural isomorphism (37) of $\mathbf{W}$-locally contraherent cosheaves on X.

Let $f\colon Y \longrightarrow X$ be a $(\mathbf{W}, \mathbf{T})$-affine morphism. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X and $\mathfrak{Q}$ be a $\mathbf{T}$-locally contraherent cosheaf on Y. Then there is a natural isomorphism of $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(\mathcal{F}, f_!\mathfrak{Q}) \simeq f_!\,\mathfrak{Cohom}_Y(f^*\mathcal{F}, \mathfrak{Q}) \tag{38}$$

on the scheme X. The similar isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves on X holds for any flat quasi-coherent sheaf $\mathcal{F}$ on X and locally cotorsion $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{Q}$ on Y. This is another version of the projection formula for $\mathfrak{Cohom}$.

Assume additionally that f is a flat morphism. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathfrak{J}$ be a locally injective $\mathbf{T}$-locally contraherent cosheaf on Y. Then there is a natural isomorphism of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves

$$\mathfrak{Cohom}_X(\mathcal{M}, f_!\mathfrak{J}) \simeq f_!\,\mathfrak{Cohom}_Y(f^*\mathcal{M}, \mathfrak{J}) \tag{39}$$

on the scheme X.

Let $f\colon Y \longrightarrow X$ be a quasi-compact morphism of semi-separated schemes. Let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X and $\mathcal{Q}$ be a contraadjusted quasi-coherent sheaf on Y. Then there is a natural isomorphism of contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{F}, f_*\mathcal{Q}) \simeq f_!\,\mathfrak{Hom}_Y(f^*\mathcal{F}, \mathcal{Q}) \tag{40}$$

on X. Here the right hand side is, by construction, a contraherent cosheaf if the morphism f is affine, and a cosheaf of $\mathcal{O}_X$-modules otherwise (see Section 2.3). Both sides are, in fact, contraherent in the general case, because the isomorphism holds and the left hand side is. This is a version of the projection formula for $\mathfrak{Hom}$.

Indeed, let $j\colon U \longrightarrow X$ be an embedding of an affine open subscheme; set $V = U \times_X Y$. Let $j'\colon V \longrightarrow Y$ and $f'\colon V \longrightarrow U$ be the natural morphisms. Then one has

$$\begin{aligned}\mathfrak{Hom}_X(\mathcal{F}, f_*\mathcal{Q})[U] &\simeq \mathrm{Hom}_X(j_*j^*\mathcal{F}, f_*\mathcal{Q}) \simeq \mathrm{Hom}_Y(f^*j_*j^*\mathcal{F}, \mathcal{Q}) \\ &\simeq \mathrm{Hom}_Y(j'_*f'^*j^*\mathcal{F}, \mathcal{Q}) \simeq \mathrm{Hom}_Y(j'_*j'^*f^*\mathcal{F}, \mathcal{Q}) \simeq \mathfrak{Hom}_Y(f^*\mathcal{F}, \mathcal{Q})[V].\end{aligned}$$

Here we are using the fact that the direct images of quasi-coherent sheaves with respect to affine morphisms of schemes commute with the inverse images in the base change situations. Notice that, when the morphism f is not affine, neither is the scheme V; however, the scheme V is quasi-compact and the open embedding morphism $j'\colon V \longrightarrow Y$ is affine, so Lemma 2.5.2 applies.

The similar isomorphism of locally cotorsion contraherent cosheaves on X holds for any flat quasi-coherent sheaf $\mathcal{F}$ on X and any cotorsion quasi-coherent sheaf $\mathcal{Q}$ on Y. Now assume additionally that f is a flat morphism. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathcal{J}$ be an injective quasi-coherent sheaf on Y. Then there is a natural isomorphism of locally cotorsion contraherent cosheaves

$$\mathfrak{Hom}_X(\mathcal{M}, f_*\mathcal{J}) \simeq f_!\,\mathfrak{Hom}_Y(f^*\mathcal{M}, \mathcal{J}) \tag{41}$$

on the scheme Y. The proof is similar to the above.

Let $f\colon Y \longrightarrow X$ be an affine morphism of semi-separated schemes. Let $\mathcal{M}$ be a quasi-coherent sheaf on X and $\mathfrak{Q}$ be a cosheaf of $\mathcal{O}_Y$-modules. Then there is a natural isomorphism of quasi-coherent sheaves

$$\mathcal{M} \odot_X f_!\mathfrak{Q} \simeq f_*(f^*\mathcal{M} \odot_Y \mathfrak{Q}) \tag{42}$$

on the scheme X. This is a version of the projection formula for the contratensor product of quasi-coherent sheaves and cosheaves of $\mathcal{O}_X$-modules.

Indeed, in the notation above, for any affine open subscheme $U \subset X$ we have

$$\begin{aligned} & j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} (f_!\mathfrak{Q})[U] \simeq j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathfrak{Q}[V] \\ & \simeq (j_*j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V)) \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \simeq (j_*(j^*\mathcal{M} \otimes_{\mathcal{O}_X(U)} \mathcal{O}_Y(V))) \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \\ & \simeq j_*f'_*f'^*j^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \simeq f_*j'_*j'^*f^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V] \\ & \simeq f_*(j'_*j'^*f^*\mathcal{M} \otimes_{\mathcal{O}_Y(V)} \mathfrak{Q}[V]). \end{aligned}$$

As we have shown in Section 2.6, the contratensor product $f^*\mathcal{M}\odot_Y\mathfrak{Q}$ can be computed on the base of all the affine open subschemes of Y whose images are contained in some affine open subschemes of X. In view of the cofinality, we can further restrict ourselves to (the diagram indexed by) the affine open subschemes $V \subset Y$ of the form $V = U \times_X Y$, where U are affine open subschemes in X, as above. It remains to use the fact that the direct image of quasi-coherent sheaves with respect to an affine morphism is an exact functor.

Let $f\colon Y \longrightarrow X$ be an affine morphism of schemes. Then for any quasi-coherent sheaf $\mathcal{M}$ on X and any cosheaf of $\mathcal{O}_Y$-modules $\mathfrak{Q}$ there is a natural isomorphism of cosheaves of $\mathcal{O}_X$-modules

$$f_!(f^*\mathcal{M} \otimes_Y \mathfrak{Q}) \simeq \mathcal{M} \otimes_X f_!\mathfrak{Q}. \tag{43}$$

This is a version of the projection formula for the tensor product of quasi-coherent sheaves and cosheaves of $\mathcal{O}_X$-modules.

4. Quasi-compact Semi-separated Schemes

4.1. **Contraadjusted and cotorsion quasi-coherent sheaves.** Recall that the definition of a very flat quasi-coherent sheaf was given in Section 2.4, and the definition of a contraadjusted quasi-coherent sheaf in Section 2.5 (cf. Remark 2.5.4).

In particular, a quasi-coherent sheaf $\mathcal{P}$ over an affine scheme U is very flat (respectively, contraadjusted) if and only if the $\mathcal{O}(U)$-module $\mathcal{P}(U)$ is very flat (respectively, contraadjusted). The class of very flat quasi-coherent sheaves is preserved by inverse images with respect to arbitrary morphisms of schemes and direct images with respect to very flat affine morphisms (which includes affine open embeddings). The class of contraadjusted quasi-coherent sheaves is preserved by direct images with respect to affine morphisms of schemes.

The class of very flat quasi-coherent sheaves on any scheme X is closed under the passage to the kernel of a surjective morphism. Both full subcategories of very flat and contraadjusted quasi-coherent sheaves are closed under extensions in the abelian category of quasi-coherent sheaves. Hence they acquire the induced exact category structures, which we denote by $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ and $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$, respectively.

Let us introduce one bit of categorical terminology. Given an exact category E and a class of objects $\mathsf{C} \subset \mathsf{E}$, we say that an object $X \in \mathsf{E}$ is a *finitely iterated extension* of objects from C if there exists a nonnegative integer N and a sequence of admissible monomorphisms $0 = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{N-1} \longrightarrow X_N = X$ in E such that the cokernels of all the morphisms $X_{i-1} \longrightarrow X_i$ belong to C (cf. Section 1.1).

Let X be a quasi-compact semi-separated scheme.

Lemma 4.1.1. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$, where $\mathcal{F}$ is a very flat quasi-coherent sheaf and $\mathcal{P}$ is a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes in X.*

Proof. The proof is based on the construction from [22, Section A.1] and Theorem 1.1.1(b). We argue by a kind of induction in the number of affine open subschemes covering X. Assume that for some open subscheme $h\colon V \longrightarrow X$ there is a short exact sequence $0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow 0$ of quasi-coherent sheaves on X such that the restriction $h^*\mathcal{K}$ of the sheaf $\mathcal{K}$ to the open subscheme V is very flat, while the sheaf $\mathcal{Q}$ is a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes in X. Let $j\colon U \longrightarrow X$ be an affine open subscheme; we will construct a short exact sequence $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$ having the same properties with respect to the open subscheme $U \cup V \subset X$.

Pick an short exact sequence $0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{G} \longrightarrow j^*\mathcal{K} \longrightarrow 0$ of quasi-coherent sheaves on the affine scheme U such that the sheaf $\mathcal{G}$ is very flat and the sheaf $\mathcal{R}$ is contraadjusted. Consider its direct image $0 \longrightarrow j_*\mathcal{R} \longrightarrow j_*\mathcal{G} \longrightarrow j_*j^*\mathcal{K} \longrightarrow 0$ with respect to the affine open embedding j, and take its pull-back with respect to the adjunction morphism $\mathcal{K} \longrightarrow j_*j^*\mathcal{K}$. Let $\mathcal{F}$ denote the middle term of the resulting short exact sequence of quasi-coherent sheaves on X.

By Lemma 1.2.6(a), it suffices to show that the restrictions of $\mathcal{F}$ to U and V are very flat in order to conclude that the restriction to $U \cup V$ is. We have $j^*\mathcal{F} \simeq \mathcal{G}$, which is very flat by the construction. On the other hand, the sheaf $j^*\mathcal{K}$ is very flat over $V \cap U$, hence so is the sheaf $\mathcal{R}$, as the kernel of a surjective map $\mathcal{G} \longrightarrow j^*\mathcal{K}$. The embedding $U \cap V \longrightarrow V$ is a very flat affine morphism, so the sheaf $j_*\mathcal{R}$ is very flat over V. Now it is clear from the short exact sequence $0 \longrightarrow j_*\mathcal{R} \longrightarrow \mathcal{F} \longrightarrow \mathcal{K} \longrightarrow 0$ that the sheaf $\mathcal{F}$ is very flat over V.

Finally, the kernel $\mathcal{P}$ of the composition of surjective morphisms $\mathcal{F} \longrightarrow \mathcal{K} \longrightarrow \mathcal{M}$ is an extension of the sheaves $\mathcal{Q}$ and $j_*\mathcal{R}$, the latter of which is the direct image of a contraadjustested quasi-coherent sheaf from an affine open subscheme of X, and the former is a finitely iterated extension of such. ☐

Corollary 4.1.2. (a) *A quasi-coherent sheaf $\mathcal{P}$ on X is contraadjusted if and only if the functor* $\operatorname{Hom}_X(-,\mathcal{P})$ *takes short exact sequences of very flat quasi-coherent sheaves on X to short exact sequences of abelian groups.*

(b) *A quasi-coherent sheaf* $\mathcal{P}$ *on* X *is contraadjusted if and only if* $\operatorname{Ext}_X^{>0}(\mathcal{F},\mathcal{P}) = 0$ *for any very flat quasi-coherent sheaf* $\mathcal{F}$ *on* X*.*

(c) *The class of contraadjusted quasi-coherent sheaves on* X *is closed with respect to the passage to the cokernels of injective morphisms.*

Proof. While the condition in part (a) is *a priori* weaker and the condition in part (b) is *a priori* stronger than our definition of a contraherent cosheaf $\mathcal{P}$ by the condition $\operatorname{Ext}_X^1(\mathcal{F},\mathcal{P}) = 0$ for any very flat $\mathcal{F}$, all the three conditions are easily seen to be equivalent provided that every quasi-coherent sheaf on X is the quotient sheaf of a very flat one. That much we know from Lemma 4.1.1. The condition in (b) clearly has the property (c). □

Lemma 4.1.3. *Any quasi-coherent sheaf* $\mathcal{M}$ *on* X *can be included in a short exact sequence* $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow 0$*, where* $\mathcal{F}$ *is a very flat quasi-coherent sheaf and* $\mathcal{P}$ *is a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes in* X*.*

Proof. Any quasi-coherent sheaf on a quasi-compact quasi-separated scheme can be embedded into a finite direct sum of direct images of injective quasi-coherent sheaves from affine open subschemes constituting a finite covering. So an embedding $\mathcal{M} \longrightarrow \mathcal{J}$ of a sheaf $\mathcal{M}$ into a sheaf $\mathcal{J}$ with the desired (an even stronger) properties exists, and it remains to make sure that the quotient sheaf has the desired properties.

One does this using Lemma 4.1.1 and (the dual version of) the procedure used in the second half of the proof of Theorem 10 in [7] (see the proof of Lemma 1.1.3). Present the quotient sheaf $\mathcal{J}/\mathcal{M}$ as the quotient sheaf of a very flat sheaf $\mathcal{F}$ by a subsheaf $\mathcal{Q}$ representable as a finitely iterated extension of the desired kind. Set $\mathcal{P}$ to be the fibered product of $\mathcal{J}$ and $\mathcal{F}$ over $\mathcal{J}/\mathcal{M}$; then $\mathcal{P}$ is an extension of $\mathcal{J}$ and $\mathcal{Q}$, and there is a natural injective morphism $\mathcal{M} \longrightarrow \mathcal{P}$ with the cokernel $\mathcal{F}$. □

Corollary 4.1.4. (a) *Any quasi-coherent sheaf on* X *admits a surjective map onto it from a very flat quasi-coherent sheaf such that the kernel is contraadjusted.*

(b) *Any quasi-coherent sheaf on* X *can be embedded into a contraadjusted quasi-coherent sheaf in such a way that the cokernel is very flat.*

(c) *A quasi-coherent sheaf on* X *is contraadjusted if and only if it is a direct summand of a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes of* X*.*

Proof. Parts (a) and (b) follow from Lemmas 4.1.1 and 4.1.3, respectively. The proof of part (c) uses (the dual version of) the argument from the proof of Corollary 1.1.4. Given a contraadjusted quasi-coherent sheaf $\mathcal{P}$, use Lemma 4.1.3 to embed it into a finitely iterated extension $\mathcal{Q}$ of the desired kind in such a way that the cokernel $\mathcal{F}$ is a very flat quasi-coherent sheaf. Since $\operatorname{Ext}_X^1(\mathcal{F},\mathcal{P}) = 0$ by the definition, we can conclude that $\mathcal{P}$ is a direct summand of $\mathcal{Q}$. □

Lemma 4.1.5. *A quasi-coherent sheaf on* X *is very flat and contraadjusted if and only if it is a direct summand of a finite direct sum of the direct images of very flat contraadjusted quasi-coherent sheaves from affine open subschemes of* X*.*

Proof. The "if" assertion is clear. To prove "only if", notice that the very flat contraadjusted quasi-coherent sheaves are the injective objects of the exact category of very flat quasi-coherent sheaves (cf. Section 1.3). So it remains to show that there are enough injectives of the kind described in the formulation of Lemma in the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$.

Indeed, let $\mathcal{F}$ be a very flat quasi-coherent sheaf on X and $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering. Denote by j_α the identity open embeddings $U_\alpha \longrightarrow X$. For each α, pick an injective morphism $j_\alpha^*\mathcal{F} \longrightarrow \mathcal{G}_\alpha$ from a very flat quasi-coherent sheaf $j_\alpha^*\mathcal{F}$ to a very flat contraadjusted quasi-coherent sheaf $\mathcal{G}_\alpha$ on U_α such that the cokernel $\mathcal{G}_\alpha/j_\alpha^*\mathcal{F}_\alpha$ is a very flat. Then $\bigoplus_\alpha j_{\alpha*}\mathcal{G}_\alpha$ is a very flat contraadjusted quasi-coherent sheaf on X and the cokernel of the natural morphism $\mathcal{F} \longrightarrow \bigoplus_\alpha j_{\alpha*}\mathcal{G}_\alpha$ is very flat (since its restriction to each U_α is). □

The following corollary provides equivalent definitions of contraadjusted and very flat quasi-coherent sheaves on a quasi-compact semi-separated scheme resembling the corresponding definitions for modules over a ring in Section 1.1.

Corollary 4.1.6. (a) *A quasi-coherent sheaf $\mathcal{P}$ on X is contraadjusted if and only if $\operatorname{Ext}_X^{>0}(j_*j^*\mathcal{O}_X, \mathcal{P}) = 0$ for any affine open embedding of schemes $j\colon Y \longrightarrow X$.*

(b) *A quasi-coherent sheaf $\mathcal{F}$ on X is very flat if and only if $\operatorname{Ext}_X^1(\mathcal{F}, \mathcal{P}) = 0$ for any contraadjusted quasi-coherent sheaf $\mathcal{P}$ on X.*

Proof. Part (a): the "only if" assertion follows from Corollary 4.1.2(b). To prove "if", notice that any very flat sheaf $\mathcal{F}$ on X has a finite right Čech resolution by finite direct sums of sheaves of the form $j_*j^*\mathcal{F}$, where $j\colon U \longrightarrow X$ are embeddings of affine open subschemes. Hence the condition $\operatorname{Ext}_X^{>0}(j_*j^*\mathcal{F}, \mathcal{P}) = 0$ for all such j implies $\operatorname{Ext}_X^{>0}(\mathcal{F}, \mathcal{P}) = 0$.

Furthermore, a very flat quasi-coherent sheaf $j^*\mathcal{F}$ on U is a direct summand of a transfinitely iterated extension of the direct images of the structure sheaves of principal affine open subschemes $V \subset U$ (by Corollary 1.1.4). Since the direct images with respect to affine morphisms preserve transfinitely iterated extensions, it remains to use the quasi-coherent sheaf version of the result that Ext^1-orthogonality is preserved by transfinitely iterated extensions in the first argument [7, Lemma 1].

Part (b): "only if" holds by the definition of contraadjusted sheaves, and "if" can be deduced from Corollary 4.1.4(a) by an argument similar to the proof of Corollary 1.1.4 (and dual to that of Corollary 4.1.4(c)). □

Now we proceed to formulate the analogues of the above assertions for cotorsion quasi-coherent sheaves. The definition of these was given in Section 2.5. The class of cotorsion quasi-coherent sheaves is closed under extensions in the abelian category of quasi-coherent sheaves on a scheme and under the direct images with respect to affine morphisms of schemes. We denote the induced exact category structure on the category of of cotorsion quasi-coherent sheaves on a scheme X by $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$.

As above, in the sequel X denotes a quasi-compact semi-separated scheme.

Lemma 4.1.7. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$, where $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathcal{P}$ is a finitely iterated extension of the direct images of cotorsion quasi-coherent sheaves from affine open subschemes in X.*

Proof. Similar to that of Lemma 4.1.1, except that Theorem 1.4.1(b) is used in place of Theorem 1.1.1(b). □

Corollary 4.1.8. (a) *A quasi-coherent sheaf $\mathcal{P}$ on X is cotorsion if and only if the functor* $\mathrm{Hom}_X(-,\mathcal{P})$ *takes short exact sequences of flat quasi-coherent sheaves on X to short exact sequences of abelian groups.*

(b) *A quasi-coherent sheaf $\mathcal{P}$ on X is cotorsion if and only if* $\mathrm{Ext}_X^{>0}(\mathcal{F},\mathcal{P}) = 0$ *for any flat quasi-coherent sheaf $\mathcal{F}$ on X.*

(c) *The class of cotorsion quasi-coherent sheaves on X is closed with respect to the passage to the cokernels of injective morphisms.*

Proof. Similar to that of Corollary 4.1.2. □

Lemma 4.1.9. *Any quasi-coherent sheaf $\mathcal{M}$ on X can be included in a short exact sequence $0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \longrightarrow 0$, where $\mathcal{F}$ is a flat quasi-coherent sheaf and $\mathcal{P}$ is a finitely iterated extension of the direct images of cotorsion quasi-coherent sheaves from affine open subschemes in X.*

Proof. Similar to that of Lemma 4.1.3. □

Corollary 4.1.10. (a) *Any quasi-coherent sheaf on X admits a surjective map onto it from a flat quasi-coherent sheaf such that the kernel is cotorsion.*

(b) *Any quasi-coherent sheaf on X can be embedded into a cotorsion quasi-coherent sheaf in such a way that the cokernel is flat.*

(c) *A quasi-coherent sheaf on X is cotorsion if and only if it is a direct summand of a finitely iterated extension of the direct images of cotorsion quasi-coherent sheaves from affine open subschemes of X.*

Proof. Similar to that of Corollary 4.1.4. □

Lemma 4.1.11. *A quasi-coherent sheaf on X is flat and cotorsion if and only if it is a direct summand of a finite direct sum of the direct images of flat cotorsion quasi-coherent sheaves from affine open subschemes of X.*

Proof. Similar to that of Lemma 4.1.5. □

The following result shows that contraadjusted (and in particular, cotorsion) quasi-coherent sheaves are adjusted to direct images with respect to nonaffine morphisms of quasi-compact semi-separated schemes (cf. Corollary 4.4.2 below).

Corollary 4.1.12. *Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then*

(a) *the functor* $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ *takes the full exact subcategory* $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset Y\text{–}\mathsf{qcoh}$ *into the full exact subcategory* $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset X\text{–}\mathsf{qcoh}$*, and induces an exact functor between these exact categories;*

(b) *the functor* $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ *takes the full exact subcategory* $Y\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset Y\text{–}\mathsf{qcoh}$ *into the full exact subcategory* $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset X\text{–}\mathsf{qcoh}$, *and induces an exact functor between these exact categories.*

Proof. For any affine morphism $g\colon V \longrightarrow Y$ into a quasi-compact semi-separated scheme Y, the inverse image functor g^* takes quasi-coherent sheaves that can be represented as finitely iterated extensions of the direct images of quasi-coherent sheaves from affine open subschemes in Y to quasi-coherent sheaves of the similar type on V. This follows easily from the fact that direct images of quasi-coherent sheaves with respect to affine morphisms of schemes commute with inverse images in the base change situations. In particular, it follows from Corollary 4.1.4(c) that the functor g^* takes contraadjusted quasi-coherent sheaves on Y to quasi-coherent sheaves that are direct summands of finitely iterated extensions of the direct images of quasi-coherent sheaves from affine open subschemes $W \subset V$.

The quasi-coherent sheaves on V that can be represented as such iterated extensions form a full exact subcategory in the abelian category of quasi-coherent sheaves. The functor of global sections $\Gamma(V,-)$ is exact on this exact category. Indeed, there is a natural isomorphism of the Ext groups $\mathrm{Ext}_V^*(\mathcal{F}, h_*\mathcal{G}) \simeq \mathrm{Ext}_W^*(h^*\mathcal{F}, \mathcal{G})$ for any quasi-coherent sheaves $\mathcal{F}$ on V and $\mathcal{G}$ on W, and a flat affine morphism $h\colon W \longrightarrow V$. Applying this isomorphism in the case when h is the embedding of an affine open subscheme and $\mathcal{F} = \mathcal{O}_V$, one concludes that $\mathrm{Ext}_V^{>0}(\mathcal{O}_V, \mathcal{G}) = 0$ for all quasi-coherent sheaves $\mathcal{G}$ from the exact category in question.

Specializing to the case of the open subschemes $V = U \times_X Y \subset Y$, where U are affine open subschemes in X, we deduce the assertion that the functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{qcoh}$ is exact. It remains to recall that the direct images of contraadjusted quasi-coherent sheaves with respect to affine morphisms of schemes are contraadjusted in order to show that f_* takes $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$. Since the direct images of cotorsion quasi-coherent sheaves with respect to affine morphisms of schemes are cotorsion, it similarly follows that f_* takes $Y\text{–}\mathsf{qcoh}^{\mathsf{cot}}$ to $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$. ☐

4.2. **Colocally projective contraherent cosheaves.** Let X be a scheme and $\mathbf{W}$ be its open covering. A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is called *colocally projective* if for any short exact sequence $0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{K} \longrightarrow 0$ of locally injective $\mathbf{W}$-locally contraherent cosheaves on X the short sequence of abelian groups $0 \longrightarrow \mathrm{Hom}^X(\mathfrak{P}, \mathfrak{I}) \longrightarrow \mathrm{Hom}^X(\mathfrak{P}, \mathfrak{J}) \longrightarrow \mathrm{Hom}^X(\mathfrak{P}, \mathfrak{K}) \longrightarrow 0$ is exact.

Obviously, the class of colocally projective $\mathbf{W}$-locally contraherent cosheaves on X is closed under direct summands. It follows from the adjunction isomorphism (21) of Section 3.3 that the functor of direct image of $\mathbf{T}$-locally contraherent cosheaves $f_!$ with respect to any $(\mathbf{W},\mathbf{T})$-affine $(\mathbf{W},\mathbf{T})$-coaffine morphism of schemes $f\colon Y \longrightarrow X$ takes colocally projective $\mathbf{T}$-locally contraherent cosheaves on Y to colocally projective $\mathbf{W}$-locally contraherent cosheaves on X. It is also clear that *any* contraherent cosheaf on an affine scheme U with the covering $\{U\}$ is colocally projective.

Generally speaking, according to the above definition the colocal projectivity property of a locally contraherent cosheaf $\mathfrak{P}$ on a scheme X may depend not only on the cosheaf $\mathfrak{P}$ itself, but also on the covering $\mathbf{W}$. No such dependence occurs on quasi-compact semi-separated schemes. Indeed, we will see below in this section that on such a scheme any colocally projective $\mathbf{W}$-locally contraherent cosheaf is (globally) contraherent. Moreover, the class of colocally projective $\mathbf{W}$-locally contraherent cosheaves coincides with the class of colocally projective contraherent cosheaves and does not depend on the covering $\mathbf{W}$.

Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering.

Lemma 4.2.1. *Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then*

(a) *any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{P} \longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$;*

(b) *any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{P} \longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of locally cotorsion contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$.*

Proof. The argument is a dual version of the proofs of Lemmas 4.1.1 and 4.1.7. Let us prove part (a); the proof of part (b) is completely analogous.

Arguing by induction in $1 \leqslant \beta \leqslant N$, we consider the open subscheme $V = \bigcup_{\alpha<\beta} U_\alpha$ with the induced covering $\mathbf{W}|_V = \{V \cap W \mid W \in \mathbf{W}\}$ and the identity embedding $h\colon V \longrightarrow X$. Assume that we have constructed an exact triple $0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{Q} \longrightarrow 0$ of $\mathbf{W}$-locally contraherent cosheaves on X such that the restriction $h^!\mathfrak{K}$ of the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{K}$ to the open subscheme $V \subset X$ is locally injective, while the cosheaf $\mathfrak{Q}$ on X is a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$, $\alpha < \beta$. When $\beta = 1$, it suffices to take $\mathfrak{K} = \mathfrak{F}$ and $\mathfrak{Q} = 0$ for the induction base. Set $U = U_\beta$ and denote by $j\colon U \longrightarrow X$ the identity open embedding.

Let $0 \longrightarrow j^!\mathfrak{K} \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{R} \longrightarrow 0$ be an exact triple of contraherent cosheaves on the affine scheme U such that the contraherent cosheaf $\mathfrak{I}$ is (locally) injective. Consider its direct image $0 \longrightarrow j_!j^!\mathfrak{K} \longrightarrow j_!\mathfrak{I} \longrightarrow j_!\mathfrak{R} \longrightarrow 0$ with respect to the affine open embedding j, and take its push-forward with respect to the adjunction morphism $j_!j^!\mathfrak{K} \longrightarrow \mathfrak{K}$. Let us show that in the resulting exact triple $0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{J} \longrightarrow j_!\mathfrak{R} \longrightarrow 0$ the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ is locally injective in the restriction to $U \cup V$. By Lemma 1.4.6(b), it suffices to show that the restrictions of $\mathfrak{J}$ to U and V are locally injective.

Indeed, in the restriction to U we have $j^!j_!j^!\mathfrak{K} \simeq j^!\mathfrak{K}$, hence $j^!\mathfrak{J} \simeq j^!j_!\mathfrak{I} \simeq \mathfrak{I}$ is a (locally) injective contraherent cosheaf. On the other hand, if $j'\colon U \cap V \longrightarrow V$ and $h'\colon U \cap V \longrightarrow U$ denote the embeddings of $U \cap V$, then $h^!j_!\mathfrak{R} \simeq j'_!h'^!\mathfrak{R}$ (as explained in the end of Section 3.3). Notice that the contraherent cosheaf $h'^!j^!\mathfrak{K} \simeq j'^!h^!\mathfrak{K}$ is locally injective, hence the contraherent cosheaf $h'^!\mathfrak{R}$ is locally injective as the cokernel of the admissible monomorphism of locally injective contraherent cosheaves $h'^!j^!\mathfrak{K} \longrightarrow h'^!\mathfrak{I}$. Since the local injectivity of $\mathbf{T}$-locally contraherent cosheaves is preserved by the direct images with respect to flat $(\mathbf{W}, \mathbf{T})$-affine morphisms, the contraherent cosheaf $j'_! h'^! \mathfrak{R}$ is locally injective, too. Now in the exact triple $0 \longrightarrow h^! \mathfrak{K} \longrightarrow h^! \mathfrak{J} \longrightarrow h^! j_! \mathfrak{R} \longrightarrow 0$ of $\mathbf{W}|_V$-locally contraherent cosheaves on V the middle term is locally injective, because so are the other two terms.

Finally, the composition of admissible monomorphisms of $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{F} \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{J}$ on X is an admissible monomorphism with the cokernel isomorphic to an extension of the contraherent cosheaves $j_! \mathfrak{R}$ and $\mathfrak{Q}$, hence also a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$, $\alpha \leqslant \beta$. The induction step is finished, and the whole lemma is proven. ☐

We denote by $\operatorname{Ext}^{X,*}(-,-)$ the Ext groups in the exact category of $\mathbf{W}$-locally contraherent cosheaves on X. Notice that these do not depend on the covering $\mathbf{W}$ and coincide with the Ext groups in the whole category of locally contraherent cosheaves $X\text{–}\mathsf{lcth}$. Indeed, the full exact subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is closed under extensions and the passage to kernels of admissible epimorphisms in $X\text{–}\mathsf{lcth}$ (see Section 3.2), and for any object $\mathfrak{P} \in X\text{–}\mathsf{lcth}$ there exists an admissible epimorphism onto $\mathfrak{P}$ from an object of $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (see the resolution (23) in Section 3.3).

For the same reasons (up to duality), the Ext groups computed in the exact subcategories of locally cotorsion and locally injective $\mathbf{W}$-locally contraherent cosheaves $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ agree with those in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (and also in $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}^{\mathsf{lin}}$). Indeed, the full exact subcategories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ are closed under extensions and the passage to cokernels of admissible monomorphisms in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ (see Section 3.1), and we have just constructed in Lemma 4.2.1 an admissible monomorphism from any $\mathbf{W}$-locally contraherent cosheaf to a locally injective one. We refer to Section A.2 for further details.

Corollary 4.2.2. (a) *A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X is colocally projective if and only if $\operatorname{Ext}^{X,1}(\mathfrak{P},\mathfrak{J}) = 0$ and if and only if $\operatorname{Ext}^{X,>0}(\mathfrak{P},\mathfrak{J}) = 0$ for all locally injective $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{J}$ on X.*

(b) *The class of colocally projective $\mathbf{W}$-locally contraherent cosheaves on X is closed under extensions and the passage to kernels of admissible epimorphisms in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.*

Proof. Part (a) follows from the existence of an admissible monomorphism from any $\mathbf{W}$-locally contraherent cosheaf on X into a locally injective $\mathbf{W}$-locally contraherent cosheaf (a weak form of Lemma 4.2.1(a)). Part (b) follows from part (a). ☐

Lemma 4.2.3. *Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then*

(a) *any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally contraherent cosheaf on X and $\mathfrak{P}$ is a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$;*

(b) *any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X can be included in an exact triple $0 \longrightarrow \mathfrak{J} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{F} \longrightarrow 0$, where $\mathfrak{J}$ is a locally injective $\mathbf{W}$-locally* *contraherent cosheaf on* X *and* $\mathfrak{P}$ *is a finitely iterated extension of the direct images of locally cotorsion contraherent cosheaves from the affine open subschemes* $U_\alpha \subset X$.

Proof. There is an admissible epimorphism $\bigoplus_\alpha j_{\alpha !} j_\alpha^! \mathfrak{F} \longrightarrow \mathfrak{F}$ (see (23) for the notation and explanation) onto any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ from a finite direct sum of the direct images of contraherent cosheaves from the affine open subschemes U_α. When $\mathfrak{F}$ is a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf, this is an admissible epimorphism in the category of locally cotorsion $\mathbf{W}$-locally contraherent cosheaves, and $j_\alpha^! \mathfrak{F}$ are locally cotorsion contraherent cosheaves on U_α.

Given that, the desired exact triples in Lemma can be obtained from those of Lemma 4.2.1 by the construction from the second half of the proof of Theorem 10 in [7] (see the proof of Lemma 1.1.3; cf. the proofs of Lemmas 4.1.3 and 4.1.9). ☐

Corollary 4.2.4. (a) *For any* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{F}$ *on* X *there exists an admissible monomorphism from* $\mathfrak{F}$ *into a locally injective* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{J}$ *on* X *such that the cokernel* $\mathfrak{P}$ *is a colocally projective* $\mathbf{W}$*-locally contraherent cosheaf.*

(b) *For any* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{F}$ *on* X *there exists an admissible epimorphism onto* $\mathfrak{F}$ *from a colocally projective* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{P}$ *on* X *such that the kernel* $\mathfrak{J}$ *is a locally injective* $\mathbf{W}$*-locally contraherent cosheaf.*

(c) *Let* $X = \bigcup_\alpha U_\alpha$ *is a finite affine open covering subordinate to* $\mathbf{W}$*. Then a* $\mathbf{W}$*-locally contraherent cosheaf on* X *is colocally projective if and only if it is a contraherent cosheaf and a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from the affine open subschemes* $U_\alpha \subset X$.

Proof. The "if" assertion in part (c) follows from Corollary 4.2.2(b) together with our preliminary remars in the beginning of this section. This having been shown, part (a) follows from Lemma 4.2.1(a) and part (b) from Lemma 4.2.3(a).

The "only if" assertion in (c) follows from Corollary 4.2.2(a) and Lemma 4.2.3(a) by the argument from the proof of Corollary 1.1.4 (cf. Corollaries 4.1.4(c) and 4.1.10(c)). Notice that the functors of direct image with respect to the open embeddings $U_\alpha \longrightarrow X$ take contraherent cosheaves to contraherent cosheaves, and the full subcategory of contraherent cosheaves $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}$ is closed under extensions. ☐

By a colocally projective locally cotorsion $\mathbf{W}$-locally contraherent cosheaf we will mean a $\mathbf{W}$-locally contraherent cosheaf that is simultaneously colocally projective and locally cotorsion.

Corollary 4.2.5. (a) *For any locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{F}$ *on* X *there exists an admissible monomorphism from* $\mathfrak{F}$ *into a locally injective* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{J}$ *on* X *such that the cokernel* $\mathfrak{P}$ *is a colocally projective locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf.*

(b) *For any locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{F}$ *on* X *there exists an admissible epimorphism onto* $\mathfrak{F}$ *from a colocally projective locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf* $\mathfrak{P}$ *on* X *such that the kernel* $\mathfrak{J}$ *is a locally injective* $\mathbf{W}$*-locally contraherent cosheaf.*

(c) *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X is colocally projective if and only if it is (a contraherent cosheaf and) a direct summand of a finitely iterated extension of the direct images of locally cotorsion contraherent cosheaves from the affine open subschemes $U_\alpha \subset X$.*

Proof. Same as Corollary 4.2.4, except that parts (b) of Lemmas 4.2.1 and 4.2.3 need to be used. Parts (a-b) can be also easily deduced from Corollary 4.2.4(a-b). □

Corollary 4.2.6. *The full subcategory of colocally projective $\mathbf{W}$-locally contraherent cosheaves in the exact category of all locally contraherent cosheaves on X does not depend on the choice of the open covering $\mathbf{W}$.*

Proof. Given two open coverings $\mathbf{W}'$ and $\mathbf{W}''$ of the scheme X, pick a finite affine open covering $X = \bigcup_{\alpha=1}^N U_\alpha$ subordinate to both $\mathbf{W}'$ and $\mathbf{W}''$, and apply Corollary 4.2.4(c). □

As a full subcategory closed under extensions and kernels of admissible epimorphisms in X–ctrh, the category of colocally projective contraherent cosheaves on X acquires the induced exact category structure. We denote this exact category by X–$\mathsf{ctrh}_{\mathsf{clp}}$. The (similarly constructed) exact category of colocally projective locally cotorsion contraherent cosheaves on X is denoted by X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$.

The full subcategory of contraherent sheaves that are simultaneously colocally projective and locally injective will be denoted by X–$\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$. Clearly, any extension of two objects from this subcategory is trivial in X–ctrh, so the category of colocally projective locally injective contraherent cosheaves is naturally viewed as an additive category endowed with the trivial exact category structure.

It follows from Corollary 4.2.4(a-b) that the additive category X–$\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ is simultaneously the category of projective objects in X–$\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ and the category of injective objects in X–$\mathsf{ctrh}_{\mathsf{clp}}$, and that it has enough of both such projectives and injectives.

Corollary 4.2.7. *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering. Then a contraherent cosheaf on X is colocally projective and locally injective if and only if it is isomorphic to a direct summand of a finite direct sum of the direct images of (locally) injective contraherent cosheaves from the open embeddings $U_\alpha \longrightarrow X$.*

Proof. For any locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{J}$ on X, the map $\bigoplus_\alpha j_{\alpha!} j_\alpha^! \mathfrak{J} \longrightarrow \mathfrak{J}$ is an admissible epimorphism in the category of locally injective $\mathbf{W}$-locally contraherent cosheaves. Now if $\mathfrak{J}$ is also colocally projective, then the extension splits. □

Corollary 4.2.8. *The three full subcategories of colocally projective cosheaves X–$\mathsf{ctrh}_{\mathsf{clp}}$, X–$\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$, and X–$\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ are closed with respect to infinite products in the category X–ctrh.*

Proof. The assertions are easily deduced from the descriptions of the full subcategories of colocally projective cosheaves given in Corollaries 4.2.4(c), 4.2.5(c), and 4.2.7

together with the fact that the functor of direct image of contraherent cosheaves with respect to an affine morphism of schemes preserves infinite products. ☐

Corollary 4.2.9. *Let $f\colon Y \longrightarrow X$ be an affine morphism of quasi-compact semi-separated schemes. Then*

(a) *the functor of inverse image of locally injective locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$ takes the full subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$ into $Y\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$;*

(b) *assuming that the morphism f is also flat, the functor of inverse image of locally cotorsion locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lct}}$ takes the full subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$ into $Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$;*

(c) *assuming that the morphism f is also very flat, the functor of inverse image of locally contraherent cosheaves $f^!\colon X\text{–}\mathsf{lcth} \longrightarrow Y\text{–}\mathsf{lcth}$ takes the full subcategory $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ into $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.*

Proof. Parts (a-c) follow from Corollaries 4.2.4(c), 4.2.5(c), and 4.2.7, respectively, together with the base change results from the second half of Section 3.3. ☐

4.3. **Projective contraherent cosheaves.** Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its affine open covering.

Lemma 4.3.1. (a) *The exact category of $\mathbf{W}$-locally contraherent cosheaves on X has enough projective objects.*

(b) *Let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then a $\mathbf{W}$-locally contraherent cosheaf on X is projective if and only if it is a direct summand of a direct sum over α of the direct images of contraherent cosheaves on U_α corresponding to very flat contraadjusted $\mathcal{O}(U_\alpha)$-modules.*

Proof. The assertion "if" in part (b) follows from the adjunction of the direct and inverse image functors for the embeddings $U_\alpha \longrightarrow X$ together with the fact that the very flat contraadjusted modules are the projective objects of the exact categories of contraadjusted modules over $\mathcal{O}(U_\alpha)$ (see Section 1.3).

It remains to show that there exists an admissible epimorphism onto any $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on X from a direct sum of the direct images of contraherent cosheaves on U_α corresponding to very flat contraadjusted modules. Here is suffices to pick admissible epimorphisms from such contraherent cosheaves $\mathfrak{P}_\alpha$ on U_α onto the restrictions $j_\alpha^!\mathfrak{F}$ of $\mathfrak{F}$ to U_α and consider the corresponding morphism $\bigoplus_\alpha j_{\alpha!}\mathfrak{P}_\alpha \longrightarrow \mathfrak{F}$ of $\mathbf{W}$-locally contraherent cosheaves on X (see the proof of Lemma 4.2.3). To check that this is an admissible epimorphism, one can, e. g., notice that it is so in the restriction to each U_α and recall that being an admissible epimorphism of $\mathbf{W}$-locally contraherent cosheaves is a local property (see Section 3.2). ☐

Corollary 4.3.2. (a) *There are enough projective objects in the exact category $X\text{–}\mathsf{lcth}$ of locally contraherent cosheaves on X, and all these projective objects belong to the full subcategory of contraherent cosheaves $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}$.*

(b) *The full subcategories of projective objects in the three exact categories $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}$ coincide.* ☐

Lemma 4.3.3. (a) *The exact category of locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaves on* X *has enough projective objects.*

(b) *Let* $X = \bigcup_\alpha U_\alpha$ *be a finite affine open covering subordinate to* $\mathbf{W}$*. Then a locally cotorsion* $\mathbf{W}$*-locally contraherent cosheaf on* X *is projective if and only if it is a direct summand of a direct sum over* α *of the direct images of locally cotorsion contraherent cosheaves on* U_α *corresponding to flat cotorsion* $\mathcal{O}(U_\alpha)$*-modules.*

Proof. Similar to the proof of Lemma 4.3.1. $\square$

Corollary 4.3.4. (a) *There are enough projective objects in the exact category* $X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ *of locally cotorsion locally contraherent cosheaves on* X*, and all these projective objects belong to the full subcategory of locally cotorsion contraherent cosheaves* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}$*.*

(b) *The full subcategories of projective objects in the three exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ *coincide.* $\square$

We denote the additive category of projective (objects in the category of) contraherent cosheaves on X by $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$, and the additive category of projective (objects in the category of) locally cotorsion contraherent cosheaves X by $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$.

Let us issue a *warning* that both the terminology and notation are misleading here: a projective locally cotorsion contraherent cosheaf on X does *not* have to be a projective contraherent cosheaf. Indeed, a flat cotorsion module over a commutative ring would not be in general very flat. Of course, both additive categories $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ are contained in $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ (hence the second one also in $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$).

A more general version of parts (a-b) of the following Corollary will be proven in the next Section 4.4.

Corollary 4.3.5. *Let* $f\colon Y \longrightarrow X$ *be an affine morphism of quasi-compact semi-separated schemes. Then*

(a) *if the morphism* f *is very flat, then the direct image functor* $f_!$ *takes projective contraherent cosheaves to projective contraherent cosheaves;*

(b) *if the morphism* f *is flat, then the direct image functor* $f_!$ *takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves;*

(c) *if the scheme* X *is Noetherian and the morphism* f *is an open embedding, then the inverse image functor* $f^!$ *takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves.*

Proof. Part (a) holds, since in its assumptions the functor $f_!\colon Y\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{ctrh}$ is "parially left adjoint" to the exact functor $f^!\colon X\text{–}\mathsf{lcth} \longrightarrow Y\text{–}\mathsf{lcth}$. The proof of part (b) is similar. The proof of part (c) is analogous to that of Corollary 4.2.9(b), except that one also has to use Corollary 1.6.4. $\square$

Corollary 4.3.6. *Over a semi-separated Noetherian scheme* X*, the full subcategory* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ *of projective locally cotorsion contraherent cosheaves is closed under infinite products in* $X\text{–}\mathsf{ctrh}$*.*

Proof. In addition to what have been said in the proof of Corollary 4.2.8, it is also important here that infinite products of flat modules over a coherent ring are flat. $\square$

Corollary 4.3.7. *Let X be a semi-separated Noetherian scheme. Then*
(a) *any cosheaf from $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ is flat;*
(b) *any cosheaf from $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$ is flat.*

Proof. Follows from Lemmas 4.3.1(b) and 4.3.3(b) together with the remarks about flat contraherent cosheaves over affine Noetherian schemes and the direct images of flat cosheaves of $\mathcal{O}$-modules in Section 3.6. $\square$

4.4. **Homology of locally contraherent cosheaves.** The functor $\Delta(X,-)$ of global cosections of locally contraherent cosheaves on a scheme X, which assigns to a cosheaf $\mathfrak{F}$ the abelian group (or even the $\mathcal{O}(X)$-module) $\mathfrak{F}[X]$, is right exact as a functor on the exact category of locally contraherent cosheaves $X\text{–}\mathsf{lcth}$ on X. In other workds, if $0 \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{R} \longrightarrow 0$ is a short exact sequence of locally contraherent cosheaves on X, then the sequence of abelian groups

$$\Delta(X,\mathfrak{P}) \longrightarrow \Delta(X,\mathfrak{Q}) \longrightarrow \Delta(X,\mathfrak{R}) \longrightarrow 0$$

is exact. Indeed, the procedure recovering the groups of cosections of cosheaves $\mathcal{F}$ on X from their groups of cosections over affine open subschemes $U \subset X$ subordinate to a particular covering $\mathbf{W}$ and the corestriction maps between such groups uses the operations of the infinite direct sum and the cokernel of a morphism (or in other words, the nonfiltered inductive limit) only (see (5), (7), or (20)).

Notice that for any $(\mathbf{W},\mathbf{T})$-affine morphims of schemes $f\colon Y \longrightarrow X$ and a $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{F}$ on X there is a natural isomorphism of $\mathcal{O}(X)$-modules $\mathfrak{F}[Y] \simeq (f_!\mathfrak{F})[X]$. This follows from the formulas cited above, together with the facts that the full preimages of open subsets forming an affine open covering of X subordinate to $\mathbf{W}$ form an affine open covering of Y subordinate to $\mathbf{T}$. Besides, the full preimages of open subsets forming affine open coverings of pairwise intersections of the open subsets of a particular affine open covering of X form affine open coverings of the intersections of the corresponding preimages in Y.

Now let X be a quasi-compact semi-separated scheme. Then the left derived functor of the functor of global sections of locally contraherent cosheaves on X can be defined in the conventional way using left projective resolutions in the exact category $X\text{–}\mathsf{lcth}$ (see Lemma 4.3.1 and Corollary 4.3.2). Notice that the derived functors of $\Delta(X,-)$ (and in fact, any left derived functors) computed in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ for a particular open covering $\mathbf{W}$ and in the whole category $X\text{–}\mathsf{lcth}$ agree. We denote this derived functor by $\mathbb{L}_*\Delta(X,-)$. The groups $\mathbb{L}_i\Delta(X,\mathfrak{F})$ are called the *homology groups* of a locally contraherent cosheaf $\mathfrak{F}$ on the scheme X.

Let us point out that the functor $\Delta(U,-)$ of global cosections of contraherent cosheaves on an affine scheme U is exact, so the groups $\mathbb{L}_{>0}\Delta(U,\mathfrak{F})$ vanish when U is affine and $\mathfrak{F}$ is contraherent.

By Corollary 4.3.5(a), for any very flat $(\mathbf{W},\mathbf{T})$-affine morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$ the functor $f_!$ takes projective contraherent cosheaves on Y to projective contraherent cosheaves on X. It also makes a commutative diagram with the restrictions of the functors $\Delta(X,-)$ and $\Delta(Y,-)$ to the categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $Y\text{–}\mathsf{lcth}_{\mathbf{T}}$. Hence one has $\mathbb{L}_*\Delta(Y,\mathfrak{G}) \simeq \mathbb{L}_*\Delta(X, f_!\mathfrak{G})$ for any $\mathbf{T}$-locally contraherent cosheaf $\mathfrak{G}$ on Y.

In particular, the latter assertion applies to the embeddings of affine open subschemes $j\colon U \longrightarrow X$, so $\mathbb{L}_{>0}\Delta(X, j_!\mathfrak{G}) = 0$ for all contraherent cosheaves $\mathfrak{G}$ on U. Since the derived functor $\mathbb{L}_*\Delta$ takes short exact sequences of locally contraherent cosheaves to long exact sequences of abelian groups, it follows from Corollary 4.2.4(c) that one has $\mathbb{L}_{>0}\Delta(X,\mathfrak{P}) = 0$ for any colocally projective contraherent cosheaf $\mathfrak{P}$ on X.

Therefore, the derived functor $\mathbb{L}_*\Delta$ can be computed using colocally projective left resolutions. Now we also see that the derived functors $\mathbb{L}_*\Delta$ defined in the theories of arbitrary (i. e., locally contraadjusted) contraherent cosheaves and of locally cotorsion contraherent cosheaves agree.

Let $\mathfrak{F}$ be a $\mathbf{W}$-locally contraherent cosheaf on X, and let $X = \bigcup_\alpha U_\alpha$ be a finite affine open covering of X subordinate to $\mathbf{W}$. Then the contraherent Čech resolution (23) for $\mathfrak{F}$ is a colocally projective left resolution of a locally contraherent cosheaf $\mathfrak{F}$, and one can use it to compute the derived functor $\mathbb{L}_*(X,\mathfrak{F})$. In other words, the homology of a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{F}$ on a quasi-compact semi-separated scheme X are computed by the homological Čech complex $C_*(\{U_\alpha\},\mathfrak{F})$ (see (20)) related to any finite affine open covering $X = \bigcup_\alpha U_\alpha$ subordinate to $\mathbf{W}$.

Corollary 4.4.1. *A locally contraherent cosheaf $\mathfrak{F}$ on an affine scheme U is contraherent if and only if its higher homology $\mathbb{L}_{>0}\Delta(X,\mathfrak{F})$ vanish.*

Proof. See Lemma 3.2.2. □

The following result is to be compared with Corollary 4.1.12 (for another comparison, see Corollary 4.2.9).

Corollary 4.4.2. *Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then*

(a) *the functor of direct image of cosheaves of $\mathcal{O}_X$-modules takes colocally projective contraherent cosheaves to colocally projective contraherent cosheaves, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ between these exact categories;*

(b) *the functor of direct image of cosheaves of $\mathcal{O}_X$-modules takes colocally projective locally cotorsion contraherent cosheaves to colocally projective locally cotorsion contraherent cosheaves, and induces an exact functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$ between these exact categories;*

(c) *if the morphism f is flat, then the functor of direct image of cosheaves of $\mathcal{O}_X$-modules takes colocally projective locally injective contraherent cosheaves to colocally projective locally injective contraherent cosheaves.*

Proof. Part (a): by Corollary 4.2.9(a), the inverse image of a colocally projective contraherent cosheaf on Y with respect to an affine open embedding $j\colon V \longrightarrow Y$ is colocally projective. As we have seen above, the global cosections of colocally projective contraherent cosheaves is an exact functor. It follows that the functor $f_!$ takes short exact sequences in $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ to short exact sequences in the exact category of cosheaves of $\mathcal{O}_X$-modules (with the exact category structure related to the covering $\{X\}$; see Section 3.1).

Since $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ is a full exact subcategory closed under extensions in $X\text{–}\mathsf{ctrh}$, and the latter exact category is such a subcategory in the exact category of cosheaves of $\mathcal{O}_X$-modules, in view of Corollary 4.2.4(c) it remains to recall that the direct images of colocally projective contraherent cosheaves with respect to affine morphisms of schemes are colocally projective (see the remarks in the beginning of Section 4.2).

The proof of part (b) is similar; and to prove part (c) one only needs to recall that the direct images of locally injective contraherent cosheaves with respect to flat affine morphisms of schemes are locally injective. □

Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Just as it was done in Sections 2.3 and 3.3, one shows that the adjunction isomorphism (21) holds for any colocally projective contraherent cosheaf $\mathfrak{Q}$ on Y and any locally injective locally contraherent cosheaf $\mathfrak{J}$ on X. In particular, when the morphism f is affine and flat, the restrictions of the functors $f_!$ and $f^!$ form an adjoint pair of functors between the additive categories $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$.

If the morphism f is flat, the adjunction isomorphism

$$\operatorname{Hom}^X(f_!\mathfrak{Q}, \mathfrak{F}) \simeq \operatorname{Hom}^Y(\mathfrak{Q}, f^!\mathfrak{F}) \tag{44}$$

holds for any colocally projective contraherent cosheaf $\mathfrak{Q}$ on Y and any locally cotorsion locally contraherent cosheaf $\mathfrak{F}$ on X. When the morphism f is also affine, the restrictions of $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$.

If the morphism f is very flat, the same adjunction isomorphism (44) holds for any colocally projective contraherent cosheaf $\mathfrak{Q}$ on Y and any locally contraherent cosheaf $\mathfrak{F}$ on X. When the morphism f is also affine, the restrictions of $f_!$ and $f^!$ form an adjoint pair of functors between the exact categories $Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.

Corollary 4.4.3. *Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then*

(a) *if the morphism f is very flat, then the direct image functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ takes projective contraherent cosheaves to projective contraherent cosheaves;*

(b) *if the morphism f is flat, then the direct image functor $f_!\colon Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}}$ takes projective locally cotorsion contraherent cosheaves to projective locally cotorsion contraherent cosheaves.*

Proof. Follows from the above partial adjunctions (44) between the exact functors $f_!$ and $f^!$. □

4.5. **The "naïve" co-contra correspondence.** Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering. We refer to Section A.1 for the definitions of the derived categories mentioned below.

Theorem 4.5.1. (a) *For any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{ctr}$, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}$ *is an equivalence of triangulated categories.*

The reason most unbounded derived categories aren't mentioned in part (b) is because one needs a uniform restriction on the extension of locality of locally contraherent cosheaves in order to work with simultaneously with infinite collections of these. In particular, infinite products exist in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$, but not necessarily in $X\text{–}\mathsf{lcth}$, so the contraderived category of the latter exact category is not well-defined.

Theorem 4.5.2. *For any symbol* $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-$, *or* abs *there is a natural equivalence of triangulated categories* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$. *These equivalences of derived categories form commutative diagrams with the natural functors* $\mathsf{D}^{\mathsf{b}} \longrightarrow \mathsf{D}^{\pm} \longrightarrow \mathsf{D}$, $\mathsf{D}^{\mathsf{b}} \longrightarrow \mathsf{D}^{\mathsf{abs}\pm} \longrightarrow \mathsf{D}^{\mathsf{abs}}$, $\mathsf{D}^{\mathsf{abs}\pm} \longrightarrow \mathsf{D}^{\pm}$, $\mathsf{D}^{\mathsf{abs}} \longrightarrow \mathsf{D}$ *between different versions of derived categories of the same exact category.*

Notice that Theorem 4.5.2 does not say anything about the coderived and contraderived categories D^{co} and $\mathsf{D}^{\mathsf{ctr}}$ of quasi-coherent sheaves and contraherent cosheaves (neither does Theorem 4.5.1 mention the coderived categories). The reason is that infinite products are not exact in the abelian category of quasi-coherent sheaves and infinite direct sums may not exist in the exact category of contraherent cosheaves. So only the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ are well-defined. Comparing these two requires a different approach; the entire Section 5 will be devoted to that.

Recall the definition of the left homological dimension $\mathrm{ld}_{\mathsf{F}/\mathsf{E}} E$ of an object E of an exact category E with respect to a full exact subcategory $\mathsf{F} \subset \mathsf{E}$, given (under a specific set of assumptions on F and E) in Section A.4. The *right homological dimension with respect to an exact subcategory* F (or the right F-homological dimension) $\mathrm{rd}_{\mathsf{F}/\mathsf{E}} E$ is defined in the dual way (and under the dual set of assumptions).

Lemma 4.5.3. (a) *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering, then the right homological dimension of any quasi-coherent sheaf on* X *with respect to the exact subcategory of contraadjusted quasi-coherent sheaves* $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset X\text{–}\mathsf{qcoh}$ *(is well-defined and) does not exceed* N.

(b) *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering subordinate to* $\mathbf{W}$, *then the left homological dimension of any* $\mathbf{W}$*-locally contraherent cosheaf on* X *with respect to the exact subcategory of colocally projective contraherent cosheaves* $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ *does not exceed* $N-1$. *Consequently, the same bound holds for the left homological dimension of any object of* $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ *with respect to the exact subcategory* $X\text{–}\mathsf{ctrh}$.

Proof. Part (a): first of all, the assumptions on the pair of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset X\text{–}\mathsf{qcoh}$ making the right homological dimension well-defined hold by Corollaries 4.1.2(c) and 4.1.4(b).

Furthermore, the right homological dimension of any module over a commutative ring R with respect to the exact category of contraadjusted R-modules $R\text{–}\mathsf{mod}^{\mathsf{cta}} \subset R\text{–}\mathsf{mod}$ does not exceed 1. It follows easily that the homological dimension of any quasi-coherent sheaf of the form $j_*\mathcal{G}$, where $j\colon U \longrightarrow X$ is an affine open subscheme, with respect to the exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \subset X\text{–}\mathsf{qcoh}$ does not exceed 1, either. Now any quasi-coherent sheaf $\mathcal{F}$ on X has a Čech resolution

$$(45)\quad 0 \longrightarrow \mathcal{F} \longrightarrow \textstyle\bigoplus_\alpha j_{\alpha*}j_\alpha^*\mathcal{F} \longrightarrow \bigoplus_{\alpha<\beta} j_{\alpha,\beta*}j_{\alpha,\beta}^*\mathcal{F} \longrightarrow \cdots \longrightarrow j_{1,\dots,N*}j_{1,\dots,N}^*\mathcal{F} \longrightarrow 0$$

of length $N-1$ by finite direct sums of quasi-coherent sheaves of the above form. It remains to use the dual version of Corollary A.4.5(a).

Part (b): the assumptions on the exact categories $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ making the left homological dimension well-defined hold by Corollaries 4.2.2(b) and 4.2.4(b). The pair of exact categories $X\text{–}\mathsf{ctrh} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ satisfies the same assumptions for the reasons explained in Section 3.2. It remains to recall the resolution (23). ☐

Proof of Theorem 4.5.1. Part (a) follows from Proposition A.4.6 together with Lemma 4.5.3(b). Part (b) in the case $\star = \mathsf{b}$ is obtained from part (a) by passing to the inductive limit over refinements of coverings, while in the case $\star = -$ it is provided by Proposition A.2.1(a). ☐

Proof of Theorem 4.5.2. By Proposition A.4.6 and its dual version, together with Lemma 4.5.3, the functors $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$ induced by the corresponding embedings of exact categories are all equivalences of triangulated categories. Hence it suffices to construct a natural equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ in order to prove all assertions of Theorem.

According to Sections 2.5 and 2.6, there are natural functors

$$\mathfrak{Hom}_X(\mathcal{O}_X,-)\colon X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{ctrh}$$

and

$$\mathcal{O}_X \odot_X -\colon X\text{–}\mathsf{lcth} \longrightarrow X\text{–}\mathsf{qcoh}$$

related by the adjunction isomorphism (18), which holds for those objects for which the former functor is defined. So it remains to prove the following lemma. ☐

Lemma 4.5.4. *On a quasi-compact semi-separated scheme X, the functor $\mathfrak{Hom}_X(\mathcal{O}_X,-)$ takes $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, the functor $\mathcal{O}_X \odot_X -$ takes $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$, and the restrictions of these functors to these subcategories are mutually inverse equivalences of exact categories.*

Proof. Obviously, on an affine scheme U the functor $\mathfrak{Hom}_U(\mathcal{O}_U,-)$ takes a contraadjusted quasi-coherent sheaf $\mathcal{Q}$ with the contraadjusted $\mathcal{O}(U)$-module of global sections $\mathcal{Q}(U)$ to the contraherent cosheaf $\mathfrak{Q}$ with the contraadjusted $\mathcal{O}(U)$-module of global cosections $\mathfrak{Q}[U] = \mathcal{Q}(U)$. Furthermore, if $j\colon U \longrightarrow X$ is the embedding of an affine open subscheme, then by the formula (40) of Section 3.7 there is a natural isomorphism $\mathfrak{Hom}_X(\mathcal{O}_X, j_*\mathcal{Q}) \simeq j_!\mathfrak{Q}$ of contraherent cosheaves on X.

Analogously, the functor $\mathcal{O}_U \odot_U -$ takes a contraherent cosheaf $\mathfrak{Q}$ with the contraadjusted $\mathcal{O}(U)$-module of global cosections $\mathfrak{Q}[U]$ to the contraadjusted quasi-coherent sheaf $\mathcal{Q}$ with the $\mathcal{O}(U)$-module of global sections $\mathcal{Q}(U)$ on U. If an embedding of affine open subscheme $j\colon U \longrightarrow X$ is given, then by the formula (42) there is a natural isomorphism $\mathcal{O}_X \odot_X j_!\mathfrak{Q} \simeq j_*\mathcal{Q}$ of quasi-coherent sheaves on X.

By Corollary 4.1.4(c), any sheaf from $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ is a direct summand of a finitely iterated extension of the direct images of contraadjusted quasi-coherent sheaves from affine open subschemes of X. It is clear from the definition of the functor $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ that it preserves exactness of short sequences of contraadjusted quasi-coherent cosheaves; hence it preserves, in particular, such iterated extensions.

By Corollary 4.2.4(c), any cosheaf from $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from affine open subschemes of X. Let us show that the functor $\mathcal{O}_X \odot_X -$ preserves exactness of short sequences of colocally projective contraherent cosheaves on X, and therefore, in particular, preserves such extensions. Indeed, the adjunction isomorphism

$$\mathrm{Hom}_X(\mathcal{O}_X \odot_X \mathfrak{P},\ \mathcal{F}) \simeq \mathrm{Hom}^X(\mathfrak{P}, \mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{F}))$$

holds for any contraherent cosheaf $\mathfrak{P}$ and contraadjusted quasi-coherent sheaf $\mathcal{F}$. Besides, the contraherent cosheaf $\mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{J})$ is locally injective for any injective quasi-coherent sheaf $\mathcal{J}$ on X. By Corollary 4.2.2(a), it follows that the functor $\mathfrak{P} \longmapsto \mathrm{Hom}_X(\mathcal{O}_X \odot_X \mathfrak{P},\ \mathcal{J})$ preserves exactness of short sequences of colocally projective contraherent cosheaves on X, and consequently so does the functor $\mathfrak{P} \longmapsto \mathcal{O}_X \odot_X \mathfrak{P}$.

Now one can easily deduce that the adjunction morphisms

$$\mathfrak{P} \longrightarrow \mathfrak{Hom}_X(\mathcal{O}_X,\ \mathcal{O}_X \odot_X \mathfrak{P}) \quad \text{and} \quad \mathcal{O}_X \odot_X \mathfrak{Hom}_X(\mathcal{O}_X, \mathcal{F}) \longrightarrow \mathcal{F}$$

are isomorphisms for any colocally projective contraherent cosheaf $\mathfrak{P}$ and contraadjusted quasi-coherent sheaf $\mathcal{F}$, as a morphism of finitely filtered objects inducing an isomorphism of the associated graded objects is also itself an isomorphism. The proof of Lemma, and hence also of Theorem 4.5.2, is finished. □

Given an exact category E, let $\mathsf{Hot}^\star(\mathsf{E})$ denote the homotopy category $\mathsf{Hot}(\mathsf{E})$ if $\star = \mathsf{abs}$, co, ctr, or $\varnothing$; the category $\mathsf{Hot}^+(\mathsf{E})$, if $\star = \mathsf{abs}+$ or $+$; the category $\mathsf{Hot}^-(\mathsf{E})$, if $\star = \mathsf{abs}-$ or $-$; and the category $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E})$ if $\star = \mathsf{b}$. Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering.

The following lemma is a variation of Lemma 4.5.3(b).

Lemma 4.5.5. *Let $X = \bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering subordinate to $\mathbf{W}$. Then* (a) *the left homological dimension of any locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory of colocally projective locally cotorsion contraherent cosheaves $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lct}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ does not exceed $N-1$. Consequently, the same bound holds for the left homological dimension of any object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ with respect to the exact subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}$;*

(b) *the left homological dimension of any locally injective $\mathbf{W}$-locally contraherent cosheaf on X with respect to the exact subcategory of colocally projective locally* *injective contraherent cosheaves* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}} \subset X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ *does not exceed* $N-1$*. Consequently, the same bound holds for the left homological dimension of any object of* $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ *with respect to the exact subcategory* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}$*.* ☐

The following two corollaries are similar to Theorem 4.5.1.

Corollary 4.5.6. (a) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lct}}$ *is an equivalence of triangulated categories.* ☐

Corollary 4.5.7. (a) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$, *the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}} \longrightarrow X\text{–}\mathsf{lcth}^{\mathsf{lin}}$ *is an equivalence of triangulated categories.* ☐

The following two corollaries provide, essentially, several restricted versions of Theorem 4.5.2.

Corollary 4.5.8. (a) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, *or* abs, *there is a natural equivalence of triangulated categories* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cot}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}})$.

(b) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, *or* abs, *there is a natural equivalence of triangulated categories* $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$.

Proof. By Proposition A.4.6 together with Lemma 4.5.5, the functors $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ and $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ induced by the corresponding embeddings of exact categories are equivalences of triangulated categories. Hence it suffices to show that the equivalence of exact categories from Lemma 4.5.4 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{clp}}$ and $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}}$. The former of these assertions follows from Corollaries 4.1.10(c) and 4.2.5(c), while the latter one is obtained from Corollary 4.2.7 together with the fact that any injective quasi-coherent sheaf on X is a direct summand of a finite direct sum of the direct images of injective quasi-coherent sheaves from open embeddings $U_\alpha \longrightarrow X$ forming a covering. ☐

Lemma 4.5.9. *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering, then the right homological dimension of any very flat quasi-coherent sheaf on* X *with respect to the exact subcategory of contraadjusted very flat quasi-coherent sheaves* $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \subset X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ *does not exceed* N*.*

Proof. The right homological dimension is well-defined due to Corollary 4.1.4(b), so it remains to apply Lemma 4.5.3(a) and the dual version of Corollary A.4.3. ☐

Corollary 4.5.10. (a) *For any symbol* $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, *or* abs, *there is a natural equivalence of triangulated categories* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$.

(b) *There is a natural equivalence of triangulated categories* $\mathsf{D}^+(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}^+(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}})$.

Proof. Part (a): assuming $\star \neq \mathsf{co}$, by Lemma 4.5.9 together with the dual version of Proposition A.4.6 the triangulated functor $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ is an equivalence of categories. In particular, we have proven that $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) = \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$, so the previous assertion holds for $\star = \mathsf{co}$ as well. Hence it suffices to show that the equivalence of exact categories from Lemma 4.5.4 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ with $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$. This follows from Lemmas 4.1.5 and 4.3.1(b).

To prove part (b), notice that the triangulated functor $\mathsf{Hot}^+(X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^+(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is an equivalence of categories by Corollary 4.1.10(b) and the dual version of Proposition A.2.1(a). So it remains to show that the equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ with $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}$. This follows from Lemmas 4.1.11 and 4.3.3(b). ☐

4.6. **Homotopy locally injective complexes.** Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering. The goal of this section is to construct a full subcategory in the homotopy category $\mathsf{Hot}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ that would be equivalent to the unbounded derived category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$. The significance of this construction is best illustrated using the duality-analogy between the contraherent cosheaves and the quasi-coherent sheaves.

As usually, the notation $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ refers to the unbounded derived category of the exact category of the exact category of flat quasi-coherent sheaves on X. The full triangulated subcategory $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}} \subset \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ of *homotopy flat complexes* of flat quasi-coherent sheaves on X is defined as the minimal triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ containing the objects of $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and closed under infinite direct sums (cf. Section A.3). The following result is essentially due to Spaltenstein [25].

Theorem 4.6.1. (a) *The composition of natural triangulated functors* $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ *is an equivalence of triangulated categories.*

(b) *A complex of flat quasi-coherent sheaves* $\mathcal{F}^\bullet$ *on* X *belongs to* $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}$ *if and only if its tensor product* $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}^\bullet$ *with any acyclic complex of quasi-coherent sheaves* $\mathcal{M}^\bullet$ *on* X *is also an acyclic complex of quasi-coherent sheaves.*

Proof. Part (a) is a particular case of Proposition A.3.3. To prove part (b), let us first show that the tensor product of any complex of quasi-coherent sheaves and a complex of sheaves acyclic with respect to the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ is acyclic. Indeed, any complex in an abelian category is a locally stabilizing inductive limit of finite complexes; so it suffices to notice that the tensor product of any quasi-coherent sheaf with a complex acyclic with respect to $X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ is acyclic. Hence the class of all complexes of flat quasi-coherent sheaves satisfying the condition in part (b) can be viewed as a strictly full triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$.

Now the "only if" assertion easily follows from the facts that the tensor products of quasi-coherent sheaves preserve infinite direct sums and the tensor product with a flat quasi-coherent sheaf is an exact functor. In view of (the proof of) part (a), it suffices to show that any complex $\mathcal{F}^\bullet$ over X–$\mathsf{qcoh}^{\mathsf{fl}}$ satisfying the tensor product condition of part (b) and acyclic with respect to X–qcoh is also acyclic with respect to X–$\mathsf{qcoh}^{\mathsf{fl}}$ in order to prove "if".

Notice that the tensor product of a bounded above complex of flat quasi-coherent sheaves and an acyclic complex of quasi-coherent sheaves is an acyclic complex. Since any quasi-coherent sheaf $\mathcal{M}$ over X has a flat left resolution, it follows that the complex of quasi-coherent sheaves $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{F}^\bullet$ is acyclic. One easily concludes that the complex $\mathcal{F}^\bullet$ is acyclic with respect to X–$\mathsf{qcoh}^{\mathsf{fl}}$. □

The full triangulated subcategory $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$ of *homotopy locally injective complexes* of localy injective $\mathbf{W}$-locally contraherent cosheaves on X is defined as the minimal full triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ containing the objects of X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ and closed under infinite products. Given a complex of quasi-coherent sheaves $\mathcal{M}^\bullet$ and a complex of $\mathbf{W}$-locally contraherent cosheaves $\mathfrak{P}^\bullet$ on X such that the $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{M}^i,\mathfrak{P}^j)$ is defined for all i, $j\in\mathbb{Z}$ (see Sections 2.4 and 3.5), we define the complex $\mathfrak{Cohom}_X(\mathcal{M}^\bullet,\mathfrak{P}^\bullet)$ as the total complex of the bicomplex $\mathfrak{Cohom}_X(\mathcal{M}^i,\mathfrak{P}^j)$ constructed by taking infinite products of $\mathbf{W}$-locally contraherent cosheaves along the diagonals.

Theorem 4.6.2. (a) *The composition of natural triangulated functors* $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}} \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *is an equivalence of triangulated categories.*

(b) *A complex of locally injective* $\mathbf{W}$*-locally contraherent cosheaves* $\mathfrak{J}^\bullet$ *on* X *belongs to* $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ *if and only if the complex* $\mathfrak{Cohom}_X(\mathcal{M},\mathfrak{J})$ *into it from any acyclic complex of quasi-coherent sheaves* $\mathcal{M}^\bullet$ *is an acyclic complex in the exact category* X–$\mathsf{lcth}_{\mathbf{W}}$ *(or, at one's choice, in the exact category* X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$*).*

Proof. Part (a): the argument goes along the lines of the proof of Theorem 4.6.1(a), but Proposition A.3.3 is not directly applicable, the category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ being not abelian; so there are some complications. First of all, we will need another definition. The full triangulated subcategory $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}\subset\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$ of homotopy locally injective complexes of colocally projective locally injective contraherent cosheaves on X is defined as the minimal full triangulated subcategory containing the objects of X–$\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ and closed under infinite products.

It was shown in Section 4.5 that the natural functor $\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})\longrightarrow\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of triangulated categories. Analogously one shows (using, e. g., Corollary A.4.3) that the natural functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})\longrightarrow\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ is an equivalence of categories, as are the similar functors $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})\longrightarrow\mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ and $\mathsf{Hot}^+(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})\longrightarrow\mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$. Therefore, the equivalence $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})\simeq\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ identifies the subcategories generated by bounded or bounded below complexes. Thus the natural functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}\longrightarrow\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$ is also an equivalence of triangulated categories, and it remains to show that the functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}\longrightarrow\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$ is an equivalence of categories.

We will show that any complex over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ admits a quasi-isomorphism with respect to the exact category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ into a complex belonging to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}$. In particular, by the dual version of [20, Lemma 1.6] applied to the homotopy category $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$ with the full triangulated subcategory $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$ and the thick subcategory of complexes acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ it will follow that the category $\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$ is equivalent to the localization of $\mathsf{D}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$ by the thick subcategory of complexes acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$. By the dual version of Corollary A.3.2, the latter subcategory is semiorthogonal to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}$ in $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$. In view of the same construction of a quasi-isomorphism with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, these two subcategories form a semiorthogonal decomposition of $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})$, which implies the desired assertion.

Lemma 4.6.3. *There exists a positive integer d such that for any complex $\mathfrak{P}^0 \longrightarrow \cdots \longrightarrow \mathfrak{P}^{d+1}$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ there exists a complex $\mathfrak{Q}^0 \longrightarrow \cdots \longrightarrow \mathfrak{Q}^{d+1}$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ together with a morphism of complexes $\mathfrak{P}^\bullet \longrightarrow \mathfrak{Q}^\bullet$ such that $\mathfrak{Q}^0 = 0$, while the morphisms of contraherent cosheaves $\mathfrak{P}^d \longrightarrow \mathfrak{Q}^d$ and $\mathfrak{P}^{d+1} \longrightarrow \mathfrak{Q}^{d+1}$ are isomorphisms.*

Proof. In view of Lemma 4.5.4, it suffices to prove the assertion of Lemma for a complex $\mathcal{P}^0 \longrightarrow \cdots \longrightarrow \mathcal{P}^{d+1}$ over the category $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$. Set $\mathcal{Q}^0 = 0$. Consider the quasi-coherent sheaf $\mathcal{R}^1 = \operatorname{coker}(\mathcal{P}^0 \to \mathcal{P}^1)$ and embed it into a contraadjusted quasi-coherent sheaf $\mathcal{Q}^1$. Denote by $\mathcal{R}^2$ the fibered coproduct of the quasi-coherent sheaves $\mathcal{Q}^1$ and $\mathcal{P}^2$ over $\mathcal{R}^1$, embed it into a contraadjusted quasi-coherent sheaf $\mathcal{Q}^2$, and proceed by applying the dual version of the construction of Lemma A.2.2(a) up to producing the quasi-coherent sheaves $\mathcal{Q}^{d-2}$ and $\mathcal{R}^{d-1}$.

The sequence $0 \longrightarrow \mathcal{R}^1 \longrightarrow \mathcal{Q}^1 \oplus \mathcal{P}^2 \longrightarrow \mathcal{Q}^2 \oplus \mathcal{P}^3 \longrightarrow \cdots \longrightarrow \mathcal{Q}^{d-2} \oplus \mathcal{P}^{d-1} \longrightarrow \mathcal{R}^{d-1} \longrightarrow 0$ is a right resolution of the quasi-coherent sheaf $\mathcal{R}^1$, all of whose terms, except perhaps the rightmost one, are contraadjusted quasi-coherent sheaves. By Lemma 4.5.3(a) and the dual version of Corollary A.4.2, for d large enough the quasi-coherent sheaf $\mathcal{R}^{d-1}$ will be contraadjusted. It remains to set $\mathcal{Q}^{d-1} = \mathcal{R}^{d-1}$, $\mathcal{Q}^d = \mathcal{P}^d$, and $\mathcal{Q}^{d+1} = \mathcal{P}^{d+1}$. □

Now let $\mathfrak{P}^\bullet$ be a complex over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$. For each fragment of $d+2$ consequtive terms $\mathfrak{P}^{i-d-1} \longrightarrow \cdots \longrightarrow \mathfrak{P}^i$ in $\mathfrak{P}^\bullet$ we construct the corresponding complex ${}^{(i)}\mathfrak{Q}^{i-d-1} \longrightarrow \cdots \longrightarrow {}^{(i)}\mathfrak{Q}^i$ as in Lemma 4.6.3. Pick an admissible monomorphism ${}^{(i)}\mathfrak{Q}^{i-d} \longrightarrow {}^{(i)}\mathfrak{J}^{i-d}$ from a colocally projective contraherent cosheaf ${}^{(i)}\mathfrak{Q}^{i-d}$ into a colocally projective locally injective contraherent cosheaf ${}^{(i)}\mathfrak{J}^{i-d}$ on X (see Corollaries 4.2.4(b) and 4.2.2(a)).

Proceeding with the dual version of the construction of Lemma A.2.2(a) (see also the above proof of Lemma 4.6.3), we obtain a termwise admissible monomorphism with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ from the complex ${}^{(i)}\mathfrak{Q}^{i-d} \longrightarrow \cdots \longrightarrow {}^{(i)}\mathfrak{Q}^i$ into a complex ${}^{(i)}\mathfrak{J}^{i-d} \longrightarrow \cdots \longrightarrow {}^{(i)}\mathfrak{J}^i$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ such that the cone of this morphism is quasi-isomorphic to an object of $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ placed in the cohomological degree i.

Set ${}^{(i)}\mathfrak{J}^j=0$ for j outside of the segment $[i-d,\,i]$. We obtain a finite complex ${}^{(i)}\mathfrak{J}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ endowed with a morphism of complexes $\mathfrak{P}^\bullet\longrightarrow{}^{(i)}\mathfrak{J}^\bullet$ with the following property. For any affine open subscheme $U\subset X$, the induced morphism of cohomology modules $H^i(\mathfrak{P}^\bullet[U])\longrightarrow H^i(\mathfrak{J}^\bullet[U])$ is injective.

Denote by ${}^0\mathfrak{J}^\bullet$ the direct product of all the complexes ${}^{(i)}\mathfrak{J}^\bullet$. The morphism of complexes $\mathfrak{P}^\bullet\longrightarrow{}^0\mathfrak{J}^\bullet$ over the exact category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ is a termwise admissible monomorphism. Consider the corresponding complex of cokernels and apply the same procedure to it, constructing a termwise acyclic complex of complexes $0\longrightarrow\mathfrak{P}^\bullet\longrightarrow{}^0\mathfrak{J}^\bullet\longrightarrow{}^1\mathfrak{J}^\bullet\longrightarrow\cdots$ over the exact category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ in which all the complexes ${}^i\mathfrak{J}^\bullet$ are infinite products of finite complexes over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.

By Lemma A.2.4 applied to the projective system of quotient complexes of silly filtration with respect to the left index of the bicomplex ${}^\bullet\mathfrak{J}^\bullet$, the total complex of ${}^\bullet\mathfrak{J}^\bullet$ constructed by taking infinite products along the diagonals belongs to $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}})^{\mathsf{lin}}$. It remains to show that the cone $\mathfrak{F}^\bullet$ of the morphism from $\mathfrak{P}^\bullet$ to the total complex of ${}^\bullet\mathfrak{J}^\bullet$ is acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.

Indeed, by the dual version of Lemma A.3.4, the complex of cosections $\mathfrak{F}^\bullet[U]$ is an acyclic complex of $\mathcal{O}_X(U)$-modules for any affine open subscheme $U\subset X$. Since the $\mathcal{O}_X(U)$-modules $\mathfrak{F}^i[U]$ are contraadjusted and quotient modules of contraadjusted modules are contraadjusted, the complex $\mathfrak{F}^\bullet[U]$ is also acyclic with respect to the exact category $\mathcal{O}(U)\text{–}\mathsf{mod}^{\mathsf{cta}}$.

Since the functor $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),-)$ preserves exactness of short sequences of contraadjusted $\mathcal{O}_X(U)$-modules for any pair of embedded affine open subschemes $V\subset U\subset X$, one easily concludes that the rules $U\longmapsto\operatorname{coker}(\mathfrak{F}^{i-1}[U]\to\mathfrak{F}^i[U])$ define contraherent cosheaves on X. Hence the complex of contrahent cosheaves $\mathfrak{F}^\bullet$ is acyclic over $X\text{–}\mathsf{ctrh}$. Since the contraherent cosheaves $\mathfrak{F}^i$ belong to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, this complex is also acyclic over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ by Lemma 4.5.3(b) and Corollary A.4.2.

Part (a) is proven; let us prove part (b). The argument is similar to the proof of Theorem 4.6.1(b). First we show that $\mathfrak{Cohom}_X$ from any complex of quasi-coherent sheaves into a complex of locally contraherent cosheaves acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ is acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$. Indeed, any complex of quasi-coherent sheaves is a locally stabilizing inductive limit of a sequence of finite complexes. So it remains to recall that $\mathfrak{Cohom}_X$ from a quasi-coherent sheaf into a complex acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ is a complex acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$, and use Lemma A.2.4 again.

Hence the class of all complexes over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ satisfying the $\mathfrak{Cohom}$ condition in part (b) can be viewed as a strictly full triangulated subcategory in $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$. Now the "only if" assertion follows from the preservation of infinite products in the second argument by the functor $\mathfrak{Cohom}_X$ and its exactness as a functor on the category $X\text{–}\mathsf{qcoh}$ for any fixed locally injective locally contraherent cosheaf in the second argument. In view of (the proof of) part (a), in order to prove "if" it suffices to show that any complex $\mathfrak{J}^\bullet$ over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ satisfying the $\mathfrak{Cohom}$ condition in (b) and acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is also acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$.

Notice that the $\mathfrak{Cohom}_X$ from a bounded above complex of very flat quasi-coherent sheaves into an acyclic complex of $\mathbf{W}$-locally contraherent cosheaves is an acyclic complex of $\mathbf{W}$-locally contraherent cosheaves. Since any quasi-coherent sheaf $\mathcal{M}$ on X has a very flat left resolution (see Lemma 4.1.1), it follows that the complex $\mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J}^\bullet)$ is acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.

Now let $U \subset X$ be an affine open subscheme subordinate to $\mathbf{W}$, let N be an $\mathcal{O}(U)$-module, viewed also as a quasi-coherent sheaf on U, and $\mathcal{M}$ be any quasi-coherent extension (e. g., the direct image) of N to X. Then acyclicity of the complex $\mathfrak{Cohom}_X(\mathcal{M}, \mathfrak{J}^\bullet)$ with respect to the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ implies, in particular, exactness of the complex of $\mathcal{O}_X(U)$-modules $\operatorname{Hom}_{\mathcal{O}_X(U)}(N, \mathfrak{J}^\bullet[U])$. Since this holds for all $\mathcal{O}_X(U)$-modules N, it follows that all the $\mathcal{O}_X(U)$-modules of cocycles in the acyclic complex of $\mathcal{O}_X(U)$-modules $\mathfrak{J}^\bullet[U]$ are injective. □

The following lemma will be needed in Section 4.7.

Lemma 4.6.4. *Let $\mathfrak{P}^\bullet$ be a complex over the exact category $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ and $\mathfrak{J}^\bullet$ be a complex over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ belonging to $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$. Then the natural morphism of graded abelian groups $H^* \operatorname{Hom}^X(\mathfrak{P}^\bullet, \mathfrak{J}^\bullet) \simeq \operatorname{Hom}_{\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}})}(\mathfrak{P}^\bullet, \mathfrak{J}^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})}(\mathfrak{P}^\bullet, \mathfrak{J}^\bullet)$ is an isomorphism (in other words, the complex $\operatorname{Hom}^X(\mathfrak{P}^\bullet, \mathfrak{J}^\bullet)$ computes the groups of morphisms in the derived category $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathsf{w}})$).*

Proof. Since any complex over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ admits a quasi-isomorphism into it from a complex over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ and any complex over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ acyclic over $X\text{–}\mathsf{lcth}_{\mathsf{w}}$ is also acyclic over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$, it suffices to show that the complex of abelian groups $\operatorname{Hom}^X(\mathfrak{P}^\bullet, \mathfrak{J}^\bullet)$ is acyclic for any complex $\mathfrak{P}^\bullet$ acyclic with respect to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ and any complex $\mathfrak{J}^\bullet \in \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$. For a complex $\mathfrak{J}^\bullet$ obtained from objects of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ by iterating the operations of cone and infinite direct sum the latter assertion is obvious (see Corollary 4.2.2(a)), so it remains to consider the case of a complex $\mathfrak{J}^\bullet$ acyclic with respect to $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$. In this case we will show that the complex $\operatorname{Hom}^X(\mathfrak{P}^\bullet, \mathfrak{J}^\bullet)$ is acyclic for any complex $\mathfrak{P}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$.

Let i be an integer. Applying Lemma 4.6.3 to the fragment $\mathfrak{P}^{i-d-1} \longrightarrow \cdots \longrightarrow \mathfrak{P}^i$ of the complex $\mathfrak{P}^\bullet$, we obtain a morphism of complexes from $\mathfrak{P}^\bullet$ to a finite complex $\mathfrak{Q}^\bullet$ over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ such that the morphisms $\mathfrak{P}^{i-1} \longrightarrow \mathfrak{Q}^{i-1}$ and $\mathfrak{P}^i \longrightarrow \mathfrak{Q}^i$ are isomorphisms. The cocone of this morphism splits naturally into a direct sum of two complexes concentrated in cohomological degrees $\leqslant i$ and $\geqslant i$, respectively. We are interested in the former complex. Its subcomplex of silly truncation $\mathfrak{R}(j,i)^\bullet$ is a finite complex over $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ concentrated in the cohomological degrees between j and i and endowed with a morphism of complexes $\mathfrak{R}(j,i)^\bullet \longrightarrow \mathfrak{P}^\bullet$, which is a termwise isomorphism in the degrees between j and $i - d$.

The complex $\mathfrak{P}^\bullet$ is a termwise stabilizing inductive limit of the sequence of complexes $\mathfrak{R}(j,i)^\bullet$ as the degree j decreases, while the degree i increases (fast enough). It remains to recall that the functor Hom^X from a colocally projective contraherent cosheaf takes acyclic complexes over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ to acyclic complexes of abelian groups, and, e. g., use Lemma A.2.4 once again. □

4.7. **Derived functors of direct and inverse image.** For the rest of Section 4, unless otherwise mentioned, the upper index $\star$ in the notation for derived categories denotes one of the symbols b, $+$, $-$, $\varnothing$, $\mathsf{abs}{+}$, $\mathsf{abs}{-}$, co, ctr, or abs.

Let $f\colon Y \longrightarrow X$ be a morphism of quasi-compact semi-separated schemes. Then for any symbol $\star \neq \mathsf{ctr}$ the right derived functor of direct image

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \tag{46}$$

is constructed in the following way. By Lemma 4.5.3(a) together with the dual version of Proposition A.4.6, the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{cta}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories (as is the similar functor for sheaves over X). By Corollary 4.1.12(a), the restriction of the functor of direct image $f_*\colon Y\text{–}\mathsf{qcoh} \longrightarrow X\text{–}\mathsf{qcoh}$ provides an exact functor $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$. Now the derived functor $\mathbb{R}f_*$ is defined by by restricting the functor of direct image $f_*\colon \mathsf{Hot}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{Hot}(X\text{–}\mathsf{qcoh})$ to the full subcategory of complexes of contraadjusted quasi-coherent sheaves on Y (with the appropriate boundedness conditions).

For any symbol $\star \neq \mathsf{co}$, the left derived functor of direct image

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \tag{47}$$

is constructed in the following way. By Lemma 4.5.3(b) (for the covering $\{Y\}$ of the scheme Y) together with Proposition A.4.6, the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh})$ is an equivalence of triangulated categories (as is the similar functor for cosheaves over X). By Corollary 4.4.2(a), there is an exact functor of direct image $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{clp}}$. The derived functor $\mathbb{L}f_!$ is defined as the induced functor $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}_{\mathsf{clp}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}})$.

Similarly one defines the left derived functor of direct image

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}). \tag{48}$$

Theorem 4.7.1. *For any symbol* $\star \neq \mathsf{co}$, ctr, *the equivalences of categories* $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(Y\text{–}\mathsf{ctrh})$ *and* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^\star(X\text{–}\mathsf{ctrh})$ *from Theorem* 4.5.2 *transform the right derived functor* $\mathbb{R}f_*$ (46) *into the left derived functor* $\mathbb{L}f_!$ (47).

Proof. It suffices to show that the equivalences of exact categories $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq Y\text{–}\mathsf{ctrh}^{\mathsf{clp}}$ and $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}^{\mathsf{clp}}$ from Lemma 4.5.4 transform the functor f_* into the functor $f_!$. The isomorphism (40) of Section 3.7 proves as much. $\square$

Let $f\colon Y \longrightarrow X$ be a morphism of schemes into a semi-separated scheme X. Let $\mathbf{W}$ and $\mathbf{T}$ be open coverings of the schemes X and Y, respectively, for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine. According to Section 3.3, there is an exact functor of inverse image $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$; for a flat morphism f, there is also an exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lct}}$, and for a very flat morphism f, an exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}$.

For a quasi-compact semi-separated scheme X, it follows from Corollary 4.1.10(a) and Proposition A.2.1(a) that the natural functor $\mathsf{D}^-(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^-(X\text{–}\mathsf{qcoh})$ is an equivalence of categories. Similarly, it follows from Corollary 4.2.4(a) and the dual version of Proposition A.2.1(a) that the natural functor $\mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow$

$\mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of triangulated categories. This allows to define, for any morphism $f\colon Y\longrightarrow X$ into a quasi-compact semi-separated scheme X and coverings $\mathbf{W}$, $\mathbf{T}$ as above, the derived functors of inverse image

$$\mathbb{L}f^*\colon \mathsf{D}^-(X\text{–}\mathsf{qcoh})\longrightarrow \mathsf{D}^-(Y\text{–}\mathsf{qcoh}) \tag{49}$$

and

$$\mathbb{R}f^!\colon \mathsf{D}^+(X\text{–}\mathsf{lcth}_{\mathbf{W}})\longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{lcth}_{\mathbf{T}})$$

by applying the functors f^* and $f^!$ to (appropriately bounded) complexes of flat sheaves and locally injective cosheaves. When both schemes are quasi-compact and semi-separated, one can take into account the equivalences of categories from Theorem 4.5.1(a) in order to produce the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^+(X\text{–}\mathsf{ctrh})\longrightarrow \mathsf{D}^+(Y\text{–}\mathsf{ctrh}), \tag{50}$$

which clearly does not depend on the choice of the coverings $\mathbf{W}$ and $\mathbf{T}$.

According to Section 4.6, the natural functors $\mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}\longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}\longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ are equivalences of categories for any quasi-compact semi-separated scheme X with an open covering $\mathbf{W}$. This allows to define, for any morphism $f\colon Y\longrightarrow X$ into a quasi-compact semi-separated scheme X and coverings $\mathbf{W}$, $\mathbf{T}$ as above, the derived functors of inverse image

$$\mathbb{L}f^*\colon \mathsf{D}(X\text{–}\mathsf{qcoh})\longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh}) \tag{51}$$

and

$$\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})\longrightarrow \mathsf{D}(Y\text{–}\mathsf{lcth}_{\mathbf{T}})$$

by applying the functors $f^*\colon \mathsf{Hot}(X\text{–}\mathsf{qcoh})\longrightarrow \mathsf{Hot}(Y\text{–}\mathsf{qcoh})$ and $f^!\colon \mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{Hot}(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}})$ to homotopy flat complexes of flat quasi-coherent sheaves and homotopy locally injective complexes of locally injective $\mathbf{W}$-locally contraherent cosheaves, respectively. Of course, this construction is well-known for quasi-coherent sheaves [25, 17]; we discuss here the sheaf and cosheaf situations together in order to emphasize the duality-analogy between them.

When both schemes are quasi-compact and semi-separated, one can use the equivalences of categories from Theorem 4.5.1(a) in order to obtain the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh})\longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh}) \tag{52}$$

which does not depend on the choice of the coverings $\mathbf{W}$ and $\mathbf{T}$. Notice also that the restriction of the functor f^* takes $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}$ into $\mathsf{Hot}(Y\text{–}\mathsf{qcoh}^{\mathsf{fl}})^{\mathsf{fl}}$ and the restriction of the functor $f^!$ takes $\mathsf{Hot}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})^{\mathsf{lin}}$ into $\mathsf{Hot}(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}})^{\mathsf{lin}}$.

It is easy to see that for any morphism of quasi-compact semi-separated schemes $f\colon Y\longrightarrow X$ the functor $\mathbb{L}f^*$ (49) is left adjoint to the functor $\mathbb{R}f_*\colon \mathsf{D}^-(Y\text{–}\mathsf{qcoh})\longrightarrow \mathsf{D}^-(X\text{–}\mathsf{qcoh})$ (46) and the functor $\mathbb{R}f^!$ (50) is right adjoint to the functor $\mathbb{L}f_!\colon \mathsf{D}^+(Y\text{–}\mathsf{ctrh})\longrightarrow \mathsf{D}^+(X\text{–}\mathsf{ctrh})$ (47). Essentially, one uses the partial adjunctions on the level of exact categories together with the fact that the derived functor constructions are indeed those of the "left" and "right" derived functors, as stated (cf. [19, Lemma 8.3]).

Similarly, one concludes from the construction in the proof of Theorem 4.6.2 that the functor $\mathbb{L}f^*$ (51) is left adjoint to the functor $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ (46). And in order to show that the functor $\mathbb{R}f^!$ (52) is right adjoint to the functor $\mathbb{L}f_!\colon \mathsf{D}(Y\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{ctrh})$ (47), one can use Lemma 4.6.4. So we have obtained a new proof of the following classical result [11, 17].

Corollary 4.7.2. *For any morphism of quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$, the derived direct image functor $\mathbb{R}f_*\colon \mathsf{D}(Y\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ has a right adjoint functor $f^!\colon \mathsf{D}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{qcoh})$.*

Proof. We have (more or less) explicitly constructed the functor $f^!$ as the right derived functor $\mathbb{R}f^!\colon \mathsf{D}(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}(Y\text{–}\mathsf{ctrh})$ (52) of the exact functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}$ between exact subcategories of the exact categories of locally contraherent cosheaves on X and Y. The above construction of the functor $f^!$ for bounded below complexes (50) is particularly explicit. In either case, the construction is based on the identification of the functor $\mathbb{R}f_*$ of derived direct image of quasi-coherent sheaves with the functor $\mathbb{L}f_!$ of derived direct image of contraherent cosheaves, which is provided by Theorems 4.5.2 and 4.7.1. ☐

4.8. **Finite flat and locally injective dimension.** A morphism of schemes $f\colon Y \longrightarrow X$ is said to have *flat dimension not exceeding D* if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ the $\mathcal{O}_X(U)$-module $\mathcal{O}_Y(V)$ has flat dimension not exceeding D. The morphism f has *very flat dimension not exceeding D* if the similar bound holds for the very flat dimension of the $\mathcal{O}_X(U)$-modules $\mathcal{O}_Y(V)$. Here the *very flat dimension* of a module over commutative ring is defined as the minimal length of its very flat left resolution (cf. Section A.4); it cannot differ by more than 1 from the module's projective dimension.

For any morphism of finite flat dimension $f\colon Y \longrightarrow X$ into a quasi-compact semi-separated scheme X and any symbol $\star \neq \mathsf{ctr}$, the left derived functor

$$\mathbb{L}f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}) \tag{53}$$

is constructed in the following way. Let us call a quasi-coherent sheaf $\mathcal{F}$ on X *adjusted to f* if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$ one has $\operatorname{Tor}_{>0}^{\mathcal{O}_X(U)}(\mathcal{O}_Y(V), \mathcal{F}(U)) = 0$. Quasi-coherent sheaves $\mathcal{F}$ on X adjusted to f form a full subcategory $X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}} \subset X\text{–}\mathsf{qcoh}$ closed under extensions, kernels of surjective morphisms and infinite direct sums, and such that any quasi-coherent sheaf on X has a finite left resolution by sheaves from $X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}}$. By Proposition A.4.6, it follows that the natural functor $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh})$ is an equivalence of triangulated categories.

The right exact functor $f^*\colon X\text{–}\mathsf{qcoh} \longrightarrow Y\text{–}\mathsf{qcoh}$ restricts to an exact functor $f^*\colon X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}} \longrightarrow Y\text{–}\mathsf{qcoh}$. In view of the above equivalence of categories, the induced functor on the derived categories $f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{f\text{–}\mathsf{adj}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh})$ provides the desired derived functor $\mathbb{L}f^*$. For any morphism of finite flat dimension $f\colon Y \longrightarrow X$ between quasi-compact semi-separated schemes Y and X, the functor $\mathbb{L}f^*$ is left adjoint to the functor $\mathbb{R}f_*$ (46) from Section 4.7 (cf. [22, Section 1.9]).

For any morphism of finite very flat dimension $f\colon Y \longrightarrow X$ into a quasi-compact semi-separated scheme X, any open coverings $\mathbf{W}$ and $\mathbf{T}$ of the schemes X and Y for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine, and any symbol $\star \ne \mathsf{co}$, the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth}_{\mathbf{T}}) \tag{54}$$

is constructed in the following way. Let us call a $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X *adjusted to* f if for any affine open subschemes $U \subset X$ and $V \subset Y$ such that U is subordinate to $\mathbf{W}$ and $f(V) \subset U$ one has $\operatorname{Ext}^{>0}_{\mathcal{O}_X(U)}(\mathcal{O}_X(V),\mathfrak{P}[U]) = 0$. One can easily see that the adjustness condition does not change when restricted to open subschemes V subordinate to $\mathbf{T}$, nor it is changed by a refinement of the covering $\mathbf{W}$.

Locally contraherent cosheaves on X adjusted to f form a full subcategory $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}} \subset X\text{–}\mathsf{lcth}$ closed under extensions and cokernels of admissible monomorphisms; the category $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}} = X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}} \cap X\text{–}\mathsf{lcth}_{\mathbf{w}}$ is also closed under infinite products in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and such that any $\mathbf{W}$-locally contraherent cosheaf on X has a finite right resolution by objects of $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}$. By the dual version of Proposition A.4.6, the natural functor $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}})$ is an equivalence of triangulated categories. The construction of the exact functor $f^!\colon X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{T}}$ from Section 3.3 extends without any changes to the case of cosheaves from $X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}$, defining an exact functor

$$f^!\colon X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}} \longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}.$$

Instead of Lemma 1.2.3(a) used in Section 2.3, one uses the following lemma in order to check that the contraadjustness condition is preserved.

Lemma 4.8.1. *Let $f\colon R \longrightarrow S$ be a homomorphism of commutative rings and P be an R-module such that $\operatorname{Ext}^1_R(S[s^{-1}],P) = 0$ for all elements $s \in s$. Then the S-module $\operatorname{Hom}_R(S,P)$ is contraadjusted.*

Proof. See proof of Lemma 1.2.3(a). ☐

In view of the above equivalence of triangulated categories, the induced functor $f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{f\text{–}\mathsf{adj}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{lcth})$ provides the desired functor $\mathbb{R}f^!$ (54). When both schemes X and Y are quasi-compact and semi-separated, one can use the equivalences of categories from Theorem 4.5.1(a) in order to obtain the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}), \tag{55}$$

which is right adjoint to the functor $\mathbb{L}f_!$ (47) from Section 4.7.

For a morphism of finite flat dimension betweeen quasi-compact semi-separated schemes $f\colon Y \longrightarrow X$ one can similarly construct the right derived functor

$$\mathbb{R}f^!\colon \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{lct}}), \tag{56}$$

which is right adjoint to the functor $\mathbb{L}f_!$ (48).

A quasi-coherent sheaf $\mathcal{F}$ on a scheme X is said to have *flat dimension not exceeding* d if the flat dimension of the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ does not exceed d for any affine open subscheme $U \subset X$. Assuming that any quasi-coherent sheaf on X is the quotient sheaf of a flat quasi-coherent sheaf (e. g., when X is quasi-compact and semi-separated), a quasi-coherent sheaf has flat dimension $\leqslant d$ if and only if it admits a flat left resolution of length $\leqslant d$. Clearly, the property of a quasi-coherent sheaf to have flat dimension not exceeding d is local. Quasi-coherent sheaves of finite flat dimension form a full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}} \subset X\text{–}\mathsf{qcoh}$ closed under extensions and kernels of surjective morphisms; the full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d} \subset X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ of quasi-coherent sheaves of flat dimension not exceeding d is closed under the same operations, and also under infinite direct sums.

Let us say that a quasi-coherent sheaf $\mathcal{F}$ on a scheme X has *very flat dimension not exceeding* d if the very flat dimension of the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)$ does not exceed d for any affine open subscheme $U \subset X$. Here the *very flat dimension* of a module over a commutative ring R is defined its left homological dimension (in the sense of Section A.4) with respect to the full exact subcategory of very flat modules $R\text{–}\mathsf{mod}^{\mathsf{vfl}} \subset R\text{–}\mathsf{mod}$; in other words, it is the minimal length of its left very flat resolution. Over a quasi-compact semi-separated scheme X, a quasi-coherent sheaf has very flat dimension $\leqslant d$ if and only if it admits a very flat left resolution of length $\leqslant d$. Since the property of a quasi-coherent sheaf to be very flat is local, so is its property to have flat dimension not exceeding d. Quasi-coherent sheaves of very flat dimension $\leqslant d$ form a full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d} \subset X\text{–}\mathsf{qcoh}$ closed under extensions, kernels of surjective morphisms, and infinite direct sums. We denote the inductive limit of the exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ as $d \to \infty$ by $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}$.

A $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on a scheme X is said to have *locally injective dimension not exceeding* d if the injective dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{P}[U]$ does not exceed d for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. Assuming that any $\mathbf{W}$-locally contraherent cosheaf on X has an admissible monomorphisms into a locally injective $\mathbf{W}$-locally contraherent cosheaf (e. g., when X is quasi-compact and semi-separated), a $\mathbf{W}$-locally contraherent cosheaf has locally injective dimension $\leqslant d$ if and only if it admits a locally injective right resolution of length $\leqslant d$ in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. The property of a locally contraherent cosheaf to have locally injective dimension not exceeding d is local and refinements of the covering $\mathbf{W}$ do not change it. $\mathbf{W}$-locally contraherent cosheaves of finite locally injective dimension form a full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ closed under extensions and cokernels of admissible monomorphisms; the full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ of quasi-coherent sheaves of locally injective dimension not exceeding d is closed under the same operations, and also under infinite products. We set $X\text{–}\mathsf{ctrh}^{\mathsf{flid}} = X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{flid}}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d} = X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{flid}\text{–}d}$.

For the rest of the section, let X be a quasi-compact semi-separated scheme.

Corollary 4.8.2. (a) *For any symbol* $\star \neq \mathsf{ctr}$ *and any (finite) integer* $d \geqslant 0$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star \neq \mathsf{ctr}$ *and any (finite) integer* $d \geqslant 0$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ *is an equivalence of triangulated categories.*

(c) *For any symbol* $\star \neq \mathsf{co}$ *and any (finite) integer* $d \geqslant 0$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ *is an equivalence of triangulated categories.*

Proof. Parts (a-b) follow from Proposition A.4.6, while part (c) follows from the dual version of the same. □

Corollary 4.8.3. (a) *For any symbol* $\star = \mathsf{b}$ *or* $-$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(b) *For any symbol* $\star = \mathsf{b}$ *or* $-$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}}$ *is an equivalence of triangulated categories.*

(c) *For any symbol* $\star = \mathsf{b}$ *or* $+$*, the triangulated functor* $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}}$ *is an equivalence of triangulated categories.*

Proof. The assertions concerning the case $\star = \mathsf{b}$ follow from the respective assertions of Corollary 4.8.2 by passage to the inductive limit as $d \to \infty$. The assertions concerning the case $\star = -$ in parts (a-b) follow from Proposition A.2.1(a), while the assertion about $\star = +$ in part (c) follows from the dual version of it. □

Lemma 4.8.4. *If* $X = \bigcup_{\alpha=1}^N U_\alpha$ *is a finite affine open covering subordinate to* $\mathbf{W}$*, then the left homological dimension of any object of the exact category* $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ *with respect to the full exact subcategory* $X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$ *does not exceed* $N-1$*.*

Proof. In view of Corollary A.4.3, it suffices to show that any object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ admits an admissible epimorphism with respect to the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ from an object of $X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$. We will do more and show that the exact sequence (23) is a left resolution of an object $\mathfrak{P} \in X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ by objects of $X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$. Indeed, the functor of inverse image with respect to a very flat $(\mathbf{W},\mathbf{T})$-coaffine morphism $f\colon Y \longrightarrow X$ takes $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ into $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{flid}\text{–}d}$, while the functor of direct image with respect to a flat $(\mathbf{W},\mathbf{T})$-affine morphism f takes $Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{flid}\text{–}d}$ into $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$. The sequence (23) is exact over $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$, since it is exact over $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ is closed under admissible monomorphisms in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. □

As a matter of notational convenience, set the triangulated category $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ to be the inductive limit of (the equivalences of categories of) $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d})$ as $d \to \infty$ for any symbol $\star \neq \mathsf{ctr}$. For any morphism $f\colon Y \longrightarrow X$ into a quasi-compact semi-separated scheme X one can construct the left derived functor

$$\mathbb{L}f^*\colon \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \tag{57}$$

as the functor on the derived categories induced by the exact functor $f^*\colon X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow Y\text{–}\mathsf{qcoh}^{\mathsf{fl}}$.

Analogously, set $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ to be the inductive limit of (the equivalences of categories) $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d})$ as $d\to\infty$ for any symbol $\star\ne\mathsf{co}$. For any morphism $f\colon Y\longrightarrow X$ into a quasi-compact semi-separated scheme X, any open coverings $\mathbf{W}$ and $\mathbf{T}$ of the schemes Y and X for which the morphism f is $(\mathbf{W},\mathbf{T})$-coaffine, the right derived functor

$$\mathbb{R}f^!\colon\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})\longrightarrow\mathsf{D}^\star(Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{flid}})$$

is constructed as the functor on the derived categories induced by the exact functor $f^!\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}\longrightarrow Y\text{–}\mathsf{lcth}_{\mathbf{T}}^{\mathsf{lin}}$.

As usually, we set $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}})=\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{flid}})$. Now Lemma 4.8.4 together with Proposition A.4.6 provide a natural equivalence of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}})\simeq\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$. For a morphism $f\colon Y\longrightarrow X$ of quasi-compact semi-separated schemes, such equivalences allow to define the derived functor

$$\mathbb{R}f^!\colon\mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}})\longrightarrow\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}}),\tag{58}$$

which clearly does not depend on the choice of the coverings $\mathbf{W}$ and $\mathbf{T}$.

Lemma 4.8.5. (a) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering, then the right homological dimension of any quasi-coherent sheaf of flat dimension $\leqslant d$ on X with respect to the full exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}\subset X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ does not exceed N.*

(b) *If $X=\bigcup_{\alpha=1}^N U_\alpha$ is a finite affine open covering subordinate to $\mathbf{W}$, then the left homological dimension of any $\mathbf{W}$-locally contraherent cosheaf of locally injective dimension $\leqslant d$ on X with respect to the full exact subcategory $X\text{–}\mathsf{ctrh}^{\mathsf{clp}}\cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}\subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ does not exceed $N-1$.*

Proof. Part (a): in view of Lemma 4.5.3(a) and the dual version of Corollary A.4.3, it suffices to show that there exists an injective morphism from any given quasi-coherent sheaf belonging to $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ into a quasi-coherent sheaf belonging to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ with the cokernel belonging to $X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$. This follows from Corollary 4.1.4(b) or 4.1.10(b). The proof of part (b) is similar up to duality, and based on Lemma 4.5.3(b) and Corollary 4.2.4(b) (alternatively, the argument from the proof of Lemma 4.8.4 is sufficient in this case). ☐

Lemma 4.8.6. *Let $X=\bigcup_{\alpha=1}^N U_\alpha$ be a finite affine open covering. Then*

(a) *a quasi-coherent sheaf on X belongs to $X\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of quasi-coherent sheaves from $U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{cta}}\cap U_\alpha\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$;*

(b) *a contraherent cosheaf on X belongs to $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}\cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}$ if and only if it is a direct summand of a finitely iterated extension of the direct images of contraherent cosheaves from $U_\alpha\text{–}\mathsf{ctrh}^{\mathsf{flid}-d}$.*

Proof. One has to repeat the arguments in Sections 4.1 and 4.2 working with, respectively, quasi-coherent sheaves of flat dimension $\leqslant d$ only or locally contraherent cosheaves of locally injective dimension $\leqslant d$ only throughout. ☐

Lemma 4.8.7. *Let $R \longrightarrow S$ be a morphism of associative rings such that S is a left R-module of flat dimension $\leqslant D$. Then*

(a) *any left S-module F of flat dimension $\leqslant d$ over S has flat dimension $\leqslant d+D$ over R;*

(b) *any right S-module P of injective dimension $\leqslant d$ over S has injective dimension $\leqslant d+D$ over R.*

Proof. Part (a) follows from the spectral sequence $\mathrm{Tor}^S_p(\mathrm{Tor}^R_q(G,S),F) \Longrightarrow \mathrm{Tor}^R_{p+q}(G,F)$, which holds for any right R-module G. Part (b) follows from the spectral sequence $\mathrm{Ext}^p_{S^{\mathsf{op}}}(\mathrm{Tor}^R_q(M,S),P) \Longrightarrow \mathrm{Ext}^{p+q}_{R^{\mathsf{op}}}(M,P)$, which holds for any right R-module M (where S^{op} and R^{op} denote the rings opposite to S and R). $\square$

Let $f\colon Y \longrightarrow X$ be a morphism of flat dimension $\leqslant D$ between quasi-compact semi-separated schemes X and Y.

Corollary 4.8.8. (a) *The exact functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}}$ takes objects of $Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}$ to objects of $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}(d+D)}$.*

(b) *The exact functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{clp}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{clp}}$ takes objects of $Y\text{–}\mathsf{ctrh}^{\mathsf{clp}} \cap Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}$ to objects of $X\text{–}\mathsf{ctrh}^{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}(d+D)}$.*

Proof. Part (a) follows from Lemma 4.8.6(a) together with the fact that the direct image with respect to an affine morphism of flat dimension $\leqslant D$ takes quasi-coherent sheaves of flat dimension $\leqslant d$ to quasi-coherent sheaves of flat dimension $\leqslant d+D$. The latter is provided by Lemma 4.8.7(a).

Part (b) similarly follows from Lemma 4.8.6(b) together with the fact that the direct image with respect to a $(\mathbf{W},\mathbf{T})$-affine morphism of flat dimension $\leqslant D$ takes $\mathbf{T}$-locally contraherent cosheaves of locally injective dimension $\leqslant d$ to $\mathbf{W}$-locally contraherent cosheaves of locally injective dimension $\leqslant d+D$. The latter is provided by Lemma 4.8.7(b). $\square$

According to Lemma 4.8.5(a) and the dual version of Proposition A.4.6, for any symbol $\star \neq \mathsf{ctr}$ the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d})$ is an equivalence of triangulated categories (as is the similar functor for sheaves over X). So one can construct the right derived functor

$$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}(d+D)})$$

as the functor on the derived categories induced by the exact functor $f_*\colon Y\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}d} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}\text{–}(d+D)}$ from Corollary 4.8.8(a). Passing to the inductive limits as $d \to \infty$, we obtain the right derived functor

(59) $$\mathbb{R}f_*\colon \mathsf{D}^\star(Y\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}),$$

which is right adjoint to the functor $\mathbb{L}f^*$ (57).

Analogously, according to Lemma 4.8.5(b) and Proposition A.4.6, for any symbol $\star \neq \mathsf{co}$ the natural functor $\mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{clp}} \cap Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d})$ is an equivalence of triangulated categories (as is the similar functor for cosheaves over X). Thus one can construct the left derived functor

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}(d+D)})$$

as the functor on the derived categories induced by the exact functor $f_!\colon Y\text{–}\mathsf{ctrh}^{\mathsf{clp}} \cap Y\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}d} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}\text{–}(d+D)}$ from Corollary 4.8.8(b). Passing to the inductive limits as $d \to \infty$, we obtain the left derived functor

$$\mathbb{L}f_!\colon \mathsf{D}^\star(Y\text{–}\mathsf{ctrh}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}), \tag{60}$$

which is left adjoint to the functor $\mathbb{R}f^!$ (58).

4.9. **Background equivalences of triangulated categories.** The results of this section complement those of Sections 4.5 and 4.8. They will be needed in Section 5.

Let X be a quasi-compact semi-separated scheme and $\mathbf{W}$ be its open covering.

Lemma 4.9.1. (a) *If $X = \bigcup_{\alpha=1}^N U_\alpha$ is a finite affine covering, then the homological dimension of the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ does not exceed $N + d$.*

(b) *If $X = \bigcup_{\alpha=1}^N U_\alpha$ is a finite affine covering subordinate to $\mathbf{W}$, then the homological dimension of the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}$ does not exceed $N - 1 + d$.*

Proof. Part (a): in fact, one proves the stronger assertion that $\mathrm{Ext}_X^{\geq N+d}(\mathcal{F}, \mathcal{M}) = 0$ for any quasi-coherent sheaf $\mathcal{M}$ and any quasi-coherent sheaf of very flat dimension $\leqslant d$ over X (also, the Ext groups in the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ agree with those in the abelian category $X\text{–}\mathsf{qcoh}$). Since any object of $X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}$ has a finite left resolution of length $\leqslant d$ by objects $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$, it suffices to consider the case of $\mathcal{F} \in X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$.

The latter can be dealt with using the Čech resolution (45) of the sheaf $\mathcal{M}$ and the adjunction of exact functors j^* and j_* for the embedding of an affine open subscheme $j\colon U \longrightarrow X$, inducing the similar adjunction on the level of Ext groups (cf. the proof of Lemma 5.1.1(b) below). Alternatively, the desired assertion can be deduced from Lemma 4.5.3(a). The proof of part (b) is similar and can be based either on the Čech resolution (23), or on Lemma 4.5.3(b). ☐

Corollary 4.9.2. (a) *The natural triangulated functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d})$ and $\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}\text{–}d})$ are equivalences of triangulated categories. In particular, such functors between the derived categories of the exact category $X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}$ are equivalences.*

(b) *The natural triangulated functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d})$ and $\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}\text{–}d})$ are equivalences of triangulated categories. In particular, such functors between the derived categories of the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ are equivalences.*

Proof. Follows from the respective parts of Lemma 4.9.1 together with the result of [19, Remark 2.1]. ☐

Denote the full subcategory of objects of injective dimension $\leqslant d$ in the abelian category $X\text{–}\mathsf{qcoh}$ by $X\text{–}\mathsf{qcoh}^{\mathsf{fid}\text{–}d}$, the full subcategory of objects of projective dimension $\leqslant d$ in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ by $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}\text{–}d}$, and the full subcategory of objects of projective dimension $\leqslant d$ in the exact category $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ by $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}$. We set $X\text{–}\mathsf{ctrh}_{\mathsf{fpd}-d} = X\text{–}\mathsf{lcth}_{\{X\},\,\mathsf{fpd}-d}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{fpd}-d} = X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\{X\},\,\mathsf{fpd}-d}$. Clearly, the projective dimension of an object of $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ or $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ does not change when the open covering $\mathbf{W}$ is replaced by its refinement.

One can easily see that the full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d} \subset X\text{–}\mathsf{qcoh}$ is closed under extensions and cokernels of admissible monomorphisms; when the scheme X is Noetherian, it is also closed under infinite direct sums. The full subcategory $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}}$ is closed under extensions and kernels of admissible epimorphisms; when the scheme X is Noetherian, it is also closed under infinite products (see Corollary 4.3.6). The full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is closed under extensions and kernels of admissible epimorphisms.

Corollary 4.9.3. (a) *For any scheme X, the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$ *are equivalences of categories.*

(b) *For any Noetherian scheme X, the natural triangulated functors* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-d})$ *are equivalences of categories.*

(c) *For any quasi-compact semi-separated scheme X, the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$ *are equivalences of categories.*

(d) *For any semi-separated Noetherian scheme X, the natural triangulated functors* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}-d})$ *are equivalences of categories.*

(e) *For any quasi-compact semi-separated scheme X, the natural triangulated functors* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$, *and* $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}-d})$ *are equivalences of categories.*

Proof. Parts (c-e) follow from Proposition A.4.6, while parts (a-b) follows from the dual version of it. □

5. Co-Contra Correspondence over a Noetherian Scheme with a Dualizing Complex

5.1. **More background equivalences of triangulated categories.** Let X be a semi-separated Noetherian scheme and $\mathbf{W}$ be its open covering.

The *cotorsion dimension* of a quasi-coherent sheaf on X is defined as its right homological dimension with respect to the full exact subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \subset X\text{–}\mathsf{qcoh}$, i. e., the minimal length of a right resolution by cotorsion quasi-coherent sheaves. Similarly, the *locally cotorsion dimension* of a $\mathbf{W}$-locally contraherent cosheaf on X is its right homological dimension with respect to the full exact subcategory

X–$\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \subset X$–$\mathsf{lcth}_{\mathbf{W}}$, i. e., the minimal length of a right resolution by locally cotorsion $\mathbf{W}$-locally contraherent cosheaves. Clearly, the locally cotorsion dimension of a $\mathbf{W}$-locally contraherent cosheaf is not changed by refinements of the covering $\mathbf{W}$.

Lemma 5.1.1. *Let $X = \bigcup_{\alpha=1}^{N} U_\alpha$ be a finite affine open covering, and let D denote the Krull dimension of X. Then*

(a) *the very flat dimension of any flat quasi-coherent sheaf on X does not exceed D;*

(b) *the homological dimension of the exact category of flat quasi-coherent sheaves on X does not exceed $N-1+D$;*

(c) *the cotorsion dimension of any quasi-coherent sheaf on X does not exceed $N-1+D$;*

(d) *the right homological dimension of any flat quasi-coherent sheaf on X with respect to the exact subcategory of flat cotorsion quasi-coherent sheaves X–$\mathsf{qcoh}^{\mathsf{cot}} \cap X$–$\mathsf{qcoh}^{\mathsf{fl}} \subset X$–$\mathsf{qcoh}^{\mathsf{fl}}$ does not exceed $N-1+D$;*

(e) *the locally cotorsion dimension of any locally contraherent cosheaf on X does not exceed D.*

Proof. Part (a): by [24, Corollaire II.3.2.7], the projective dimension of any flat module over a Noetherian commutative ring of Krull dimension D does not exceed D. Part (b): one proves that $\operatorname{Ext}_X^{\geq N-1+D}(\mathcal{F}, \mathcal{M}) = 0$ for any flat quasi-coherent sheaf $\mathcal{F}$ and any quasi-coherent $\mathcal{M}$ on X. It suffices to use the Čech resolution (45) of the sheaf $\mathcal{M}$ and the natural isomorphisms $\operatorname{Ext}_X^*(\mathcal{F}, j_*\mathcal{G}) \simeq \operatorname{Ext}_U^*(j^*\mathcal{F}, \mathcal{G})$ for the embeddings of affine open subschemes $j\colon U \longrightarrow X$ and any quasi-coherent sheaves $\mathcal{F}$, $\mathcal{G}$ in order to reduce the question to the case of an affine scheme U. Then it remains to apply the same result from [24]. Taking into account Corollaries 4.1.8(c) and 4.1.10(b) (guaranteeing that the cotorsion dimension is well-defined), this Ext vanishing also implies part (c). The right homological dimension in part (d) is well-defined due to Corollary 4.1.10(b), so (d) follows from (c) in view of the dual version of Corollary A.4.3 (cf. Lemma 4.5.9). Since the locally cotorsion dimension is well-defined due to, e. g., Corollary 4.2.4(a) and the results of Section 3.2, part (e) follows from part (c) applied to the case of an affine scheme. ☐

Corollary 5.1.2. *Let X be a semi-separated Noetherian scheme of finite Krull dimension and $d \geqslant 0$ be any (finite) integer. Then the natural triangulated functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d})$ and $\mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d})$ are equivalences of triangulated categories. In particular, such functors between the derived categories of the exact category X–$\mathsf{qcoh}^{\mathsf{fl}}$ are equivalences of categories.*

Proof. Follows from Lemma 5.1.1(b) together with the result of [19, Remark 2.1]. ☐

Corollary 5.1.3. *Let X be a semi-separated Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, ctr, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ induced by the embedding of exact categories X–$\mathsf{qcoh}^{\mathsf{vfl}} \longrightarrow X$–$\mathsf{qcoh}^{\mathsf{fl}}$ is an equivalence of triangulated categories.*

Proof. Follows from Lemma 5.1.1(a) together with Proposition A.4.6. ☐

Corollary 5.1.4. *Let X be a semi-separated Noetherian scheme of finite Krull dimension. Then*

(a) *for any symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{ctr}$, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ induced by the embedding of exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is an equivalence of triangulated categories;*

(b) *for any symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-$, or abs, the triangulated functor $\mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{lct}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth})$ induced by the embedding of exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}$ is an equivalence of triangulated categories.*

Proof. Both assertions follow from Lemma 5.1.1(e) together with the dual version of Proposition A.4.6. □

The following corollary is another restricted version of Theorem 4.5.2; it is to be compared with Corollary 4.5.10.

Corollary 5.1.5. *Let X be a semi-separated Noetherian scheme of finite Krull dimension. Then for any symbol $\star = \mathsf{b}, +, -, \varnothing, \mathsf{abs}+, \mathsf{abs}-, \mathsf{co}$, or abs there is a natural equivalence of triangulated categories $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})$.*

Proof. Assuming $\star \neq \mathsf{co}$, by Lemma 5.1.1(d) together with the dual version of Proposition A.4.6 the triangulated functor $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ is an equivalence of categories. In view of Corollary 5.1.2, the same assertion holds for $\star = \mathsf{co}$. Hence it remains to recall that the equivalence of categories from Lemma 4.5.4 identifies $X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ with $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ (see the proof of Corollary 4.5.10(b)). □

A cosheaf of $\mathcal{O}_X$-modules $\mathfrak{F}$ on a scheme X is said to have **W**-*flat dimension not exceeding* d if the flat dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{F}[U]$ does not exceed d for any affine open subscheme $U \subset X$ subordinate to $\mathbf{W}$. The flat dimension of a cosheaf of $\mathcal{O}_X$-modules is defined as its $\{X\}$-flat dimension. The flat dimension of a contraherent cosheaf $\mathfrak{F}$ on an affine Noetherian scheme U is equal to the flat dimension of the $\mathcal{O}_X(U)$-module $\mathfrak{F}[U]$ (cf. Section 3.6).

Over a semi-separated Noetherian scheme X, a $\mathbf{W}$-locally contraherent cosheaf has $\mathbf{W}$-flat dimension $\leqslant d$ if and only if it admits a left resolution of length $\leqslant d$ by $\mathbf{W}$-flat $\mathbf{W}$-locally contraherent cosheaves. A locally cotorsion $\mathbf{W}$-locally contraherent cosheaf has $\mathbf{W}$-flat dimension $\leqslant d$ if and only if it admits a left resolution of length $\leqslant d$ by $\mathbf{W}$-flat locally cotorsion $\mathbf{W}$-locally contraherent cosheaves (see Corollary 4.3.7). $\mathbf{W}$-locally contraherent cosheaves of $\mathbf{W}$-flat dimension not exceeding d form a full subcategory $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}$ closed under extensions, kernels of admissible epimorphisms, and infinite products.

We denote the intersection $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}} \cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d}$ by $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d}$, and also set $X\text{–}\mathsf{ctrh}^{\mathsf{ffd}\text{–}d} = X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{ffd}\text{–}d}$ and $X\text{–}\mathsf{ctrh}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d} = X\text{–}\mathsf{lcth}_{\{X\}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d}$.

Lemma 5.1.6. *Let U be a Noetherian affine scheme of Krull dimension D. Then any flat contraherent cosheaf on U has projective dimension not exceeding D in the exact category $U\text{–}\mathsf{ctrh}$.*

Proof. Essentially, we have to prove that for a Noetherian ring $R = \mathcal{O}(U)$ of Krull dimension D any flat contraadjusted R-module has a resolution of length at most D by very flat contraadjusted R-modules in the exact category $R\text{–}\mathsf{mod}^{\mathsf{cta}}$. In view of [24, Corollaire II.3.2.7] or Lemma 5.1.1(a) (applied in the case of an affine scheme U) and Corollary A.4.3, it suffices to show that there exists a surjective morphism onto any flat contraadjusted R-module from a very flat contraadjusted R-module with a flat contraadjusted kernel. This follows from Corollary 1.1.5(b). □

Lemma 5.1.7. *Let X be a semi-separated Noetherian scheme and $X = \bigcup_{\alpha=1}^N U_\alpha$ be its finite affine open covering subordinate to $\mathbf{W}$. Then*

(a) *a locally cotorsion $\mathbf{W}$-locally contraherent cosheaf on X has finite projective dimension in $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ if and only if it has finite $\mathbf{W}$-flat dimension; more precisely, the inclusions of full subcategories $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}\text{–}(d+N-1)}$ hold in the category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$;*

(b) *assuming the scheme X has finite Krull dimension D, a $\mathbf{W}$-locally contraherent cosheaf on X has finite projective dimension in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$ if and only if it has finite $\mathbf{W}$-flat dimension; more precisely, the inclusions of full subcategories $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}\text{–}(d+N-1+D)}$ hold in the category $X\text{–}\mathsf{lcth}_{\mathbf{W}}$.*

Proof. The inclusions $X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}\text{–}d}$, $X\text{–}\mathsf{lcth}_{\mathbf{W},\,\mathsf{fpd}\text{–}d} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d}$ hold due to Corollary 4.3.7. To prove the inclusion $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d} \subset X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W},\,\mathsf{fpd}\text{–}(d+N-1)}$, it suffices to notice that the Čech sequence (23) provides a resolution of length $N-1$ by projective locally cotorsion contraherent cosheaves for any flat locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X in the exact category $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$.

Finally, let us show that any flat $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{P}$ on X has projective dimension at most $N-1+D$ in $X\text{–}\mathsf{lcth}_{\mathbf{W}}$. The sequence (23) is a resolution of length $N-1$ of $\mathfrak{P}$ by finite direct sums of the direct images of flat contraherent cosheaves from the affine open subschemes U_α. By Lemma 5.1.6, each of these has projective dimension at most D, and it remains to apply Corollary A.4.5(a). □

Corollary 5.1.8. (a) *For any semi-separated Noetherian scheme X and any (finite) integer $d \geqslant 0$, the natural triangulated functors $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d})$, and $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct},\,\mathsf{ffd}\text{–}d})$ are equivalences of triangulated categories.*

(b) *For any semi-separated Noetherian scheme X of finite Krull dimension and any (finite) integer $d \geqslant 0$, the natural triangulated functors $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d})$, $\mathsf{Hot}^{\pm}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{abs}\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d}) \longrightarrow \mathsf{D}^{\pm}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d})$, and $\mathsf{Hot}^{\mathsf{b}}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d})$ are equivalences of triangulated categories.*

Proof. It is clear from Lemma 5.1.7 that the homological dimensions of the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d} \cap X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}$ and $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{ffd}\text{–}d}$ are finite, so it remains to apply [19, Remark 2.1] (to obtain the equivalences between various derived categories of these exact categories) and Proposition A.4.6 (to identify the absolute derived categories with the homotopy categories of projective objects). Alternatively, one can use [20, Theorem 3.6 and Remark 3.6]. ☐

The following theorem is the main result of this section.

Theorem 5.1.9. (a) *For any Noetherian scheme X, the natural functor* $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ *is an equivalence of triangulated categories.*

(b) *For any semi-separated Noetherian scheme X, the natural functor* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}^{\mathsf{lct}}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lct}}_{\mathbf{W}})$ *is an equivalence of triangulated categories.*

(c) *For any semi-separated Noetherian scheme X of finite Krull dimension, the natural functor* $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ *is an equivalence of triangulated categories.*

Proof. Part (a) is a standard result (see, e. g., [22, Lemma 1.7(b)]) which is a particular case of [20, Theorem 3.7 and Remark 3.7] and can be also obtained from the dual version of Proposition A.2.1(c). The key observation is that there are enough injectives in $X\text{–}\mathsf{qcoh}$ and the full subcategory $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$ they form is closed under infinite direct sums. Similarly, part (b) can be obtained either from Proposition A.2.1(c), or from the dual version of [20, Theorem 3.7 and Remark 3.7] (see also [20, Section 3.8]). In any case the argument is based on Lemma 4.3.3(a) and Corollary 4.3.6.

Finally, to prove part (c) one has to use the more advanced features of the results of [20, Sections 3.7–3.8] involving the full generality of the conditions ($*$)–($**$). Specifically, if $X = \bigcup_\alpha U_\alpha$ is a finite affine open covering, then it follows from Lemma 4.3.1(b) that an infinite product of projective contraherent cosheaves on X is a direct summand of a direct sum over α of the direct images of contraherent cosheaves on U_α correspoding to infinite products of very flat contraadjusted $\mathcal{O}(U_\alpha)$-modules. Infinite products of such modules may not be very flat, but they are certainly flat and contraadjusted. By Lemma 5.1.6, one can conclude that the projective dimensions of infinite products of projective objects in $X\text{–}\mathsf{ctrh}$ do not exceed the Krull dimension D of the scheme X. So the contraherent cosheaf analogue of the condition ($**$) holds for $X\text{–}\mathsf{ctrh}$, and one can apply the dual version of [20, Remark 3.7]. ☐

5.2. **Co-contra correspondence over a regular scheme.** Let X be a regular Noetherian scheme of finite Krull dimension.

Theorem 5.2.1. (a) *The triangulated functor* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}$ *is an equivalence of triangulated categories.*

(b) *The triangulated functor* $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *induced by the embedding of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{ctrh}$ *is an equivalence of triangulated categories.*

(c) *There is a natural equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *provided by the derived functors* $\mathbb{R}\mathfrak{Hom}_X(\mathcal{O}_X, -)$ *and* $\mathcal{O}_X \odot^{\mathbb{L}}_X -$.

Proof. Part (a) actually holds for any symbol $\star \neq \mathsf{ctr}$ in the upper indices of the derived category signs, and is a particular case of Corollary 4.8.2(a). Indeed, one has

$X\text{–}\mathsf{qcoh} = X\text{–}\mathsf{qcoh}^{\mathsf{ffd}-d}$ provided that d is greater or equal to the Krull dimension of X. Similarly, part (b) actually holds for any symbol $\star \neq \mathsf{co}$ in the upper indices, and is a particular case of Corollary 4.8.2(c). Indeed, one has $X\text{–}\mathsf{lcth}_{\mathbf{W}} = X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}-d}$ provided that d is greater or equal to the Krull dimension of X.

To prove part (c), notice that all the triangulated functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh})$ and $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ are equivalences of categories by Corollary 4.9.2 (since one also has $X\text{–}\mathsf{qcoh} = X\text{–}\mathsf{qcoh}^{\mathsf{fvfd}-d}$ provided that d is greater or equal to the Krull dimension of X). So it remains to apply Theorem 4.5.2. □

5.3. **Co-contra correspondence over a Gorenstein scheme.** Let X be a Gorenstein Noetherian scheme of finite Krull dimension. We will use the following formulation of the Gorenstein condition: for any affine open subscheme $U \subset X$, the classes of $\mathcal{O}_X(U)$-modules of finite flat dimension, of finite projective dimension, and of finite injective dimension coincide.

Notice that neither of these dimensions can exceed the Krull dimension D of the scheme X. Accordingly, the class of $\mathcal{O}_X(U)$-modules defined by the above finite homological dimension conditions is closed under both infinite direct sums and infinite products. It is also closed under extensions and the passages to the cokernels of embeddings and the kernels of surjections.

Moreover, since the injectivity of a quasi-coherent sheaf on a Notherian scheme is a local property, the full subcategories of quasi-coherent sheaves of finite flat dimension and of finite injective dimension coincide in $X\text{–}\mathsf{qcoh}$. Similarly, the full subcategories of locally contraherent cosheaves of finite flat dimension and of finite locally injective dimension coincide in $X\text{–}\mathsf{lcth}$. Neither of these dimensions can exceed D.

Theorem 5.3.1. (a) *The triangulated functors* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ *induced by the embeddings of exact categories* $X\text{–}\mathsf{qcoh}^{\mathsf{fl}} \longrightarrow X\text{–}\mathsf{qcoh}^{\mathsf{ffd}} \longrightarrow X\text{–}\mathsf{qcoh}$ *are equivalences of triangulated categories.*

(b) *The triangulated functors* $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{flid}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *induced by the embeddings of exact categories* $X\text{–}\mathsf{ctrh}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{ctrh}^{\mathsf{flid}} \longrightarrow X\text{–}\mathsf{ctrh}$ *are equivalences of triangulated categories.*

(c) *There is a natural equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{flid}})$ *provided by the derived functors* $\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X, -)$ *and* $\mathcal{O}_X \odot_X^{\mathbb{L}} -$.

Proof. Parts (a-b): by Corollary 4.8.2(a,c), the functors $\mathsf{D}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ are equivalences of categories for any symbol $\star \neq \mathsf{ctr}$ and the functors $\mathsf{D}^{\star}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\star}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{flid}})$ are equivalences of categories for any symbol $\star \neq \mathsf{co}$.

To prove that the functor $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ is an equivalence of categories, notice that one has $X\text{–}\mathsf{qcoh}^{\mathsf{inj}} \subset X\text{–}\mathsf{qcoh}^{\mathsf{fid}-D} = X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ and the functor $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-D})$ is an equivalence of categories by Corollary 4.9.3(a-b), while the composition $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fid}-D}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$ is an equivalence of categories by Theorem 5.1.9(a).

Similarly, to prove that the functor $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equialence of categories, notice that one has $X\text{–}\mathsf{ctrh}_{\mathsf{prj}} \subset X\text{–}\mathsf{lcth}^{\mathsf{ffd}\text{–}D}_{\mathbf{W}} = X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}}$ and the functor $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{lcth}^{\mathsf{ffd}\text{–}D}_{\mathbf{W}})$ is an equivalence of categories by Corolary 5.1.8(b), while the composition $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{ffd}\text{–}D}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ is an equivalence of categories by Theorem 5.1.9(c).

To prove part (c), notice that the functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ are equivalences of categories by Corollary 5.1.2, while the functors $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{w}}) \longrightarrow \mathsf{D}(X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}})$ are equivalences of categories by Corollary 4.9.2(b).

Furthermore, consider the intersections $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}}$. As was explained in Section 4.8, the functor $\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{ffd}})$ is an equivalence of triangulated categories for any $\star \neq \mathsf{ctr}$, while the functor $\mathsf{D}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}^{\mathsf{flid}}_{\mathbf{W}})$ is an equivalence of triangulated categories for any $\star \neq \mathsf{co}$.

Finally, it is clear from Lemma 4.8.6 that the equivalence of exact categories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \simeq X\text{–}\mathsf{ctrh}_{\mathsf{clp}}$ of Lemma 4.5.4 identifies their full exact subcategories $X\text{–}\mathsf{qcoh}^{\mathsf{cta}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{ffd}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{clp}} \cap X\text{–}\mathsf{ctrh}^{\mathsf{flid}}$. So the induced equivalence of the derived categories $\mathsf{D}^{\mathsf{abs}}$ or D provides the desired equivalence of triangulated categories in part (c). ☐

5.4. **Co-contra correspondence over a scheme with a dualizing complex.** Let X be a semi-separated Noetherian scheme with a dualizing complex $\mathcal{D}^\bullet_X$, which will be viewed as a finite complex of injective quasi-coherent sheaves on X. The following result complements the covariant Serre–Grothendieck duality theory as developed in the papers and the thesis [14, 18, 15, 22].

Theorem 5.4.1. *There are natural equivalences between the four triangulated categories* $\mathsf{D}^{\mathsf{abs}=\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh})$, $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$, *and* $\mathsf{D}^{\mathsf{abs}=\mathsf{ctr}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$. *(Here the notation* $\mathsf{abs} = \mathsf{co}$ *and* $\mathsf{abs} = \mathsf{ctr}$ *presumes the assertions that the corresponding derived categories of the second kind coincide for the exact category in question.) Among these, the equivalences* $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{ctrh})$ *and* $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{ctrh}^{\mathsf{lin}})$ *do not require a dualizing complex and do not depend on it; all the remaining equivalences do and do.*

Proof. For any quasi-compact semi-separated scheme X with an open covering $\mathbf{W}$, one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}}) = \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ by Corollary 4.9.2(b). For any Noetherian scheme X of finite Krull dimension, one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) = \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ by Corollary 5.1.2.

For any semi-separated Noetherian scheme X, one has $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}})$ by Theorem 5.1.9(a) and $\mathsf{Hot}(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$ by Corollary 4.5.8(b). Hence the desired equivalence $\mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$, which is provided by the derived functors

$$\mathbb{R}\mathfrak{Hom}_X(\mathcal{O}_X, -)\colon \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}^{\mathsf{lin}}_{\mathbf{W}})$$

and

$$\mathcal{O}_X \odot_X^{\mathbb{L}} -\colon \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{–}\mathsf{qcoh}).$$

For any semi-separated Noetherian scheme X of finite Krull dimension, one has $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}})$ by Corollary 5.1.5, $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}})$ by Theorem 5.1.9(b), and $\mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lct}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ by Corollary 5.1.4. Alternatively, one can refer to the equivalence $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}})$ holding by Corollary 5.1.3, $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{vfl}}) \simeq \mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}})$ by Corollary 4.5.10(a), and $\mathsf{Hot}(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$ by Theorem 5.1.9(c). Either way, one gets the same desired equivalence $\mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$, which is provided by the derived functors

$$\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X, -)\colon \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}})$$

and

$$\mathcal{O}_X \odot_X^{\mathbb{L}} -\colon \mathsf{D}^{\mathsf{ctr}}(X\text{–}\mathsf{lcth}_{\mathbf{W}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}).$$

Now we are going to construct a commutative diagram of equivalences of triangulated categories

$$\begin{array}{ccc}
\mathsf{D}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{fl}}) & \overset{\mathcal{D}_X^\bullet\otimes_{\mathcal{O}_X}-\ \to}{\underset{\leftarrow\ \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet,-)}{=\!=\!=\!=\!=}} & \mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{inj}}) \\
\mathcal{O}_X\odot_X-\uparrow\ \Big\|\ \downarrow\mathbb{R}\,\mathfrak{Hom}_X(\mathcal{O}_X,-) & \overset{\mathfrak{Hom}_X(\mathcal{D}_X^\bullet,-)\ \longleftarrow}{\underset{\mathcal{D}_X^\bullet\odot_X^{\mathbb{L}}-\ \longrightarrow}{\diagup\!\!\diagup}} & \mathcal{O}_X\odot_X^{\mathbb{L}}-\uparrow\ \Big\|\ \downarrow\mathfrak{Hom}_X(\mathcal{O}_X,-) \\
\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}) & \overset{\mathcal{D}_X^\bullet\otimes_{X\text{-ct}}-\ \to}{\underset{\leftarrow\ \mathfrak{Cohom}_X(\mathcal{D}_X^\bullet,-)}{=\!=\!=\!=\!=}} & \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})
\end{array}$$

for any symbol $\star = \mathsf{b}$, $\mathsf{abs}+$, $\mathsf{abs}-$, or abs.

The exterior vertical functors are constructed by applying the additive functors $\mathcal{O}_X \odot_X -$ and $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ to the given complexes termwise. The interior (derived) vertical functors have been defined in Corollaries 4.5.8(b) and 5.1.5. All the functors invoking the dualizing complex $\mathcal{D}_X^\bullet$ are constructed by applying the respective exact functors of two arguments to $\mathcal{D}_X^\bullet$ and the given unbounded complex termwise and totalizing the bicomplexes so obtained.

First of all, one notices that the functors in the interior upper triangle are right adjoint to the ones in the exterior. This follows from the adjunction (18) together with the adjunction of the tensor product of quasi-coherent sheaves and the quasi-coherent internal Hom.

The upper horizontal functors $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -$ and $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, -)$ are mutually inverse for the reasons explained in [15, Theorem 8.4 and Proposition 8.9] and [22, Theorem 2.5]. The argument in [22] is based on the observations that the morphism of finite complexes of flat quasi-coherent sheaves

$$\mathcal{F} \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet,\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$$

is a quasi-isomorphism for any sheaf $\mathcal{F} \in X\text{–}\mathsf{qcoh}^{\mathsf{fl}}$ and the morphism of finite complexes of injective quasi-coherent sheaves

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow \mathcal{J}$$

is a quasi-isomorphism for any sheaf $\mathcal{J} \in X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$.

Let us additionally point out that, according to Lemma 2.5.3(c) and [15, Lemma 8.7], the complex $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{J}^\bullet)$ is a complex of flat cotorsion quasi-coherent sheaves for any complex $\mathcal{J}^\bullet$ over $X\text{–}\mathsf{qcoh}^{\mathsf{inj}}$. So the functor $\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, -)$ actually lands in $\mathsf{Hot}^\star(X\text{–}\mathsf{qcoh}^{\mathsf{cot}} \cap X\text{–}\mathsf{qcoh}^{\mathsf{fl}})$ (as does the functor $\mathcal{O}_X \odot_X -$ on the diagram, according to the proof of Corollary 5.1.5). The interior upper triangle is commutative due to the natural isomorphism (15). The exterior upper triangle is commutative due to the natural isomorphism (19).

In order to discuss the equivalence of categories in the lower horizontal line, we will need the following lemma. It is based on the definitions of the $\mathfrak{Cohom}$ functor in Section 3.5 and the contraherent tensor product functor $\otimes_{X\text{-ct}}$ in Section 3.6.

Lemma 5.4.2. *Let $\mathcal{J}$ be an injective quasi-coherent sheaf on a semi-separated Noetherian scheme X with an open covering $\mathbf{W}$. Then there are two well-defined exact functors*

$$\mathfrak{Cohom}_X(\mathcal{J}, -)\colon X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$$

and

$$\mathcal{J} \otimes_{X\text{-ct}} -\colon X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$$

between the exact categories $X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ and $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ (with the trivial exact category structure on the latter). The functor $\mathcal{J} \otimes_{X\text{-ct}} -$ is left adjoint to the functor $\mathfrak{Cohom}_X(\mathcal{J}, -)$. Besides, the functor $\mathcal{J} \otimes_{X\text{-ct}} -$ takes values in $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}} \subset X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$. For any quasi-coherent sheaf $\mathcal{M}$ and a projective locally cotorsion contraherent cosheaf $\mathfrak{P}$ on X there is a natural isomorphism

$$\mathcal{M} \odot_X (\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{P}) \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{J}) \odot_X \mathfrak{P} \tag{61}$$

of quasi-coherent sheaves on X.

Proof. Let us show that the locally cotorsion $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{Cohom}_X(\mathcal{J}, \mathfrak{K})$ is projective for any locally injective $\mathbf{W}$-locally contraherent cosheaf $\mathfrak{K}$ on X. Indeed, $\mathcal{J}$ is a direct summand of a finite direct sum of the direct images of injective quasi-coherent sheaves $\mathcal{I}$ from the embeddings of affine open subschemes $j\colon U \longrightarrow X$ subordinate to $\mathbf{W}$. So it suffices to consider the case of $\mathcal{J} = j_*\mathcal{I}$.

According to (36), there is a natural isomorphism of locally cotorsion ($\mathbf{W}$-locally) contraherent cosheaves $\mathfrak{Cohom}_X(j_*\mathcal{I}, \mathfrak{K}) \simeq j_!\, \mathfrak{Cohom}_U(\mathcal{I}, j^!\mathfrak{K})$ on X. The $\mathcal{O}(U)$-modules $\mathcal{I}(U)$ and $\mathfrak{K}[U]$ are injective, so $\mathrm{Hom}_{\mathcal{O}(U)}(\mathcal{I}(U), \mathfrak{K}[U])$ is a flat cotorsion $\mathcal{O}(U)$-module. In other words, the locally cotorsion contraherent cosheaf $\mathfrak{Cohom}_U(\mathcal{I}, j^!\mathfrak{K})$ is projective on U, and therefore its direct image with respect to j is projective on X (see Lemma 4.3.3(b), or Corollaries 4.3.5(b) and/or 4.4.3(b)).

Now let $\mathfrak{P}$ be a contraherent cosheaf from $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ or $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$. In both cases $\mathfrak{P}$ is a flat contraherent cosheaf (Lemma 4.3.7), so the tensor product $\mathcal{J} \otimes_X \mathfrak{P}$ is a locally injective derived contrahereable cosheaf on X.

Moreover, by Lemmas 4.3.1(b) and 4.3.3(b) $\mathfrak{P}$ is a direct summand of a finite direct sum of the direct images of flat contraherent cosheaves from affine open subschemes of X. It was explained in Section 3.4 that derived contrahereable cosheaves on affine schemes are contrahereable and the direct images of cosheaves with respect to affine morphisms preserve contrahereability. So it follows from the isomorphism (43) that $\mathcal{J} \otimes_X \mathfrak{P}$ is a locally injective contrahereable cosheaf. Its contraherator $\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{P} = \mathfrak{C}(\mathcal{J} \otimes_X \mathfrak{P})$ is consequently a locally injective contraherent cosheaf on X.

Furthermore, according to Section 3.4 the (global) contraherator construction commutes with the direct images with respect to affine morphisms. Hence the contraherent cosheaf $\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{P}$ is a direct summand of a finite direct sum of the direct images of (locally) injective contraherent cosheaves from affine open subschemes of X, i. e., $\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{P}$ is a colocally projective locally injective contraherent cosheaf.

We have constructed the desired exact functors. A combination of the adjunction isomorphisms (31) and (26) makes them adjoint to each other. Finally, for any $\mathcal{M} \in X\text{–}\mathsf{qcoh}$ and $\mathfrak{P} \in X\text{–}\mathsf{ctrh}_{\mathsf{prj}}$ or $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ one has

$$\mathcal{M} \odot_X (\mathcal{J} \otimes_{X\text{-ct}} \mathfrak{P}) = \mathcal{M} \odot_X \mathfrak{C}(\mathcal{J} \otimes_X \mathfrak{P}) \simeq \mathcal{M} \odot_X (\mathcal{J} \otimes_X \mathfrak{P}) \simeq (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{J}) \odot_X \mathfrak{P}$$

according to the isomorphisms (28) and (32). □

Now we can return to the proof of Theorem 5.4.1. The functors in the interior lower triangle are left adjoint to the ones in the exterior, as it follows from the adjunction (18) and Lemma 5.4.2. Let us show that the lower horizontal functors are mutually inverse.

According to the proof of Corollary 4.5.8(b), the functor $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}) \longrightarrow \mathsf{D}^\star(X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}})$ induced by the embedding $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}} \longrightarrow X\text{–}\mathsf{lcth}_{\mathbf{W}}^{\mathsf{lin}}$ is an equivalence of triangulated categories. Therefore, it suffices to show that the for any cosheaf $\mathfrak{J} \in X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ the morphism of complexes of contraherent cosheaves

$$\mathcal{D}_X^\bullet \otimes_{X\text{-ct}} \mathfrak{Cohom}_X(\mathcal{D}_X^\bullet, \mathfrak{J}) \longrightarrow \mathfrak{J}$$

is a homotopy equivalence (or just a quasi-isomorphism in $X\text{–}\mathsf{ctrh}$), and for any cosheaf $\mathfrak{P} \in X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ the morphism of complexes of contraherent cosheaves

$$\mathfrak{P} \longrightarrow \mathfrak{Cohom}_X(\mathcal{D}_X^\bullet,\ \mathcal{D}_X^\bullet \otimes_{X\text{-ct}} \mathfrak{P})$$

is a homotopy equivalence (or just a quasi-isomorphism in $X\text{–}\mathsf{ctrh}$).

According to Corollaries 4.2.7 and 4.3.3, any object of $X\text{–}\mathsf{ctrh}_{\mathsf{clp}}^{\mathsf{lin}}$ or $X\text{–}\mathsf{ctrh}_{\mathsf{prj}}^{\mathsf{lct}}$ is a direct summand of a finite direct sum of direct images of objects in the similar categories on affine open subschemes of X. According to (39) and (43) together with the results of Section 3.4, both functors $\mathfrak{Cohom}_X(\mathcal{D}_X^\bullet, -)$ and $\mathcal{D}_X^\bullet \otimes_{X\text{-ct}} -$ commute with such direct images. So the question reduces to the case of an affine scheme U, for which the distinction between quasi-coherent sheaves and contraherent cosheaves mostly loses its significance, as both are identified with (appropriate classes of) $\mathcal{O}(U)$-modules. For this reason, the desired quasi-isomorphisms follow from the similar quasi-isomorphisms for quasi-coherent sheaves obtained in [22, proof of Theorem 2.5] (as quoted above).

According again to the proof of Corollary 4.5.8(b), the functor $\mathfrak{Hom}_X(\mathcal{O}_X, -)$ on the diagram actually lands in $\mathsf{Hot}^\star(X\text{–}\mathsf{ctrh}^{\mathsf{lin}}_{\mathsf{clp}})$ (as does the functor $\mathcal{D}^\bullet_X \otimes_{X\text{-ct}} -$, according to Lemma 5.4.2). The exterior upper triangle is commutative due to the natural isomorphism (17). The interior upper triangle is commutative due to the natural isomorphism (61).

The assertion that the two diagonal functors on the diagram are mutually inverse follows from the above. It can be also proven directly in the manner of the above proof of the assertion that the two lower horizontal functors are mutually inverse. One needs to use the natural isomorphisms (41) and (42) for commutation with the direct images. ☐

Appendix A. Derived categories of exact categories and resolutions

In this appendix we recall and review some general results about the derived categories of the first and second kind of abstract exact categories and their full subcategories, in presence of finite or infinite resolutions. There is nothing essentially new here. Two or three most difficult arguments are omitted and references to the author's previous works containing detailed proofs of similar results in different (but more concrete) settings are given instead. The results of this appendix are used in the main body of the paper starting from Section 4.5.

A.1. **Derived categories of the second kind.** Let E be an exact category. The homotopy categories of (finite, bounded above, bounded below, and unbounded) complexes over E will be denoted by $\mathsf{Hot}^{\mathsf{b}}(\mathsf{E})$, $\mathsf{Hot}^-(\mathsf{E})$, $\mathsf{Hot}^+(\mathsf{E})$, and $\mathsf{Hot}(\mathsf{E})$, respectively. For the definitions of the conventional derived categories (of the first kind) $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$, $\mathsf{D}^-(\mathsf{E})$, $\mathsf{D}^+(\mathsf{E})$, and $\mathsf{D}(\mathsf{E})$ we refer to [16] and [21, Section A.7]. Here are the definitions of the derived categories of the second kind [19, 20, 22].

An (unbounded) complex over E is said to be *absolutely acyclic* if it belongs to the minimal thick subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the all the total complexes of short exact sequences of complexes over E. Here a short exact sequence $0 \longrightarrow {}'K^\bullet \longrightarrow K^\bullet \longrightarrow {}''K^\bullet \longrightarrow 0$ of complexes over E is viewed as a bicomplex with three rows and totalized as such. The *absolute derived category* $\mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ of the exact category E is defined as the quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by the thick subcategory of absolutely acyclic complexes.

Similarly, a bounded above (respectively, below) complex over E is called absolutely acyclic if it belongs to the minimal thick subcategory of $\mathsf{Hot}^-(\mathsf{E})$ (resp., $\mathsf{Hot}^+(\mathsf{E})$) containing all the total complexes of short exact sequences of bounded above (resp., below) complexes over E. We will see below that a bounded above (resp., below) complex over E is absolutely acyclic if and only if it is absolutely acyclic as an unbounded complex, so there is no ambiguity in our terminology. The bounded above (resp., below) absolute derived category of E is defined as the quotient category of $\mathsf{Hot}^-(\mathsf{E})$ (resp., $\mathsf{Hot}^+(\mathsf{E})$) by the thick subcategory of absolutely acyclic complexes and denoted by $\mathsf{D}^{\mathsf{abs}-}(\mathsf{E})$ (resp., $\mathsf{D}^{\mathsf{abs}+}(\mathsf{E})$).

We do not define the "absolute derived category of finite complexes over E", as it would not be any different from the conventional bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{E})$. Indeed, any (bounded or unbounded) absolutely acyclic complex is acyclic; and any finite acyclic complex over an exact category is absolutely acyclic, since it is composed of short exact sequences. Moreover, any acyclic complex over an exact category of finite homological dimension is absolutely acyclic [19, Remark 2.1].

Lemma A.1.1. *For any exact category* E*, the functors* $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}-}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}+}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ *induced by the natural embeddings of the categories of bounded complexes into those of unbounded ones are all fully faithful.*

Proof. We will show that any morphism in $\mathsf{Hot}(\mathsf{E})$ (in a certain direction) between a complex bounded from a particular side (or from both sides) and a complex absolutely acyclic with respect to the class of complexes unbounded from a particular side (or both sides) factorizes through a complex absolutely acyclic with respect to the class of correspondingly bounded complexes. For this purpose, it suffices to demonstrate that any absolutely acyclic complex can be presented as a termwise stabiling filtered inductive (or projective) limit of complexes absolutely acyclic with respect to the class of complexes bounded in a particular way.

Indeed, any short exact sequence of complexes over E is the inductive limit of short exact sequences of their subcomplexes of silly filtration, which are bounded below. One easily concludes that any absolutely acyclic complex is a termwise stabilizing inductive limit of complexes absolutely acyclic with respect to the class of complexes bounded below, and any absolutely acyclic complex bounded above is a termwise stabilizing inductive limit of finite acyclic complexes. This proves that the functors $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}-}(\mathsf{E})$ and $\mathsf{D}^{\mathsf{abs}+}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ are fully faithful.

On the other hand, any absolutely acyclic complex bounded below, being, by implication, an acyclic complex bounded below, is the inductive limit of its subcomplexes of canonical filtration, which are finite acyclic complexes. This shows that the functor $\mathsf{D}^{\mathsf{b}}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}+}(\mathsf{E})$ is fully faithful, too. Finally, to prove that the functor $\mathsf{D}^{\mathsf{abs}-}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ is fully faithful, one presents any absolutely acyclic complex as a termwise stabilizing projective limit of complexes absolutely acyclic with respect to the class of complexes bounded above. □

Assume that infinite direct sums exist and are exact functors in the exact category E. Then a complex over E is called *coacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the total complexes of short exact sequences of complexes over E and closed under infinite direct sums. The *coderived category* $\mathsf{D}^{\mathsf{co}}(\mathsf{E})$ of the exact category E is defined as the quotient category of the homotopy category $\mathsf{Hot}(\mathsf{E})$ by the thick subcategory of coacyclic complexes.

Similarly, if the functors of infinite product are everywhere defined and exact in the exact category E, one calls a complex over E *contraacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(\mathsf{E})$ containing the total complexes of short exact sequences of complexes over E and closed under infinite products. The *contraderived category* $\mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ of the exact category E is the quotient category of $\mathsf{Hot}(\mathsf{E})$ by the thick subcategory of contraacyclic complexes [19, Sections 2.1 and 4.1].

A.2. **Infinite left resolutions.** Let E be an exact category and $\mathsf{F} \subset \mathsf{E}$ be a full subcategory closed under extensions and the passage to the kernels of admissible epimorphisms. Suppose further that every object of E is the image of an admissible epimorphism from an object belonging to F. We endow F with the induced structure of an exact category.

Proposition A.2.1. (a) *The triangulated functor* $\mathsf{D}^-(\mathsf{F}) \longrightarrow \mathsf{D}^-(\mathsf{E})$ *induced by the exact embedding functor* $\mathsf{F} \longrightarrow \mathsf{E}$ *is an equivalence of triangulated categories.*

(b) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{E})$ *induced by the exact embedding functor* $\mathsf{F} \longrightarrow \mathsf{E}$ *is fully faithful.*

(c) *If the infinite products are everywhere defined and exact in the exact category* E*, and preserve the full subcategory* $\mathsf{F} \subset \mathsf{E}$*, then the triangulated functor* $\mathsf{D}^{\mathsf{ctr}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(\mathsf{E})$ *induced by the embedding* $\mathsf{F} \longrightarrow \mathsf{E}$ *is an equivalence of categories.*

Proof. The proof of part (a) is based on part (b) of the following lemma.

Lemma A.2.2. (a) *For any finite complex* $E^{-d} \longrightarrow \cdots \longrightarrow E^0$ *over* E *there exists a finite complex* $F^{-d} \longrightarrow \cdots \longrightarrow F^0$ *over* F *together with a morphism of complexes* $F^\bullet \longrightarrow E^\bullet$ *over* E *such that the morphisms* $F^i \longrightarrow E^i$ *are admissible epimorphisms in* E *and the cocone (or equivalently, the termwise kernel) of the morphism* $F^\bullet \longrightarrow E^\bullet$ *is quasi-isomorphic to an object of* E *placed in the cohomological degree* $-d$*.*

(b) *For any bounded above complex* $\cdots \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^0$ *over* E *there exists a bounded above complex* $\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0$ *over* F *together with a quasi-isomorphism of complexes* $F^\bullet \longrightarrow E^\bullet$ *over* E *such that the morphisms* $F^i \longrightarrow E^i$ *are admissible epimorphisms in* E*.*

Proof. Pick an admissible epimorphism $F^0 \longrightarrow E^0$ with $F^0 \in \mathsf{F}$ and consider the fibered product G^{-1} of the objects E^{-1} and F^0 over E^0 in E. Then there exists a unique morphism $E^{-2} \longrightarrow G^{-1}$ having a zero composition with the morphism $G^{-1} \longrightarrow F^0$ and forming a commutative diagram with the morphisms $E^{-2} \longrightarrow E^{-1}$ and $G^{-1} \longrightarrow E^{-1}$. Continuing the construction, pick an admissible epimorphism $F^{-1} \longrightarrow G^{-1}$ with $F^{-1} \in \mathsf{F}$, consider the fibered product G^{-2} of E^{-2} and F^{-1} over G^{-1}, etc. In the case (a), proceed in this way until the object F^{-d} is constructed; in the case (b), proceed indefinitely. The desired assertions follow from the observation that natural morphism between the complexes $G^{-d} \longrightarrow F^{-d+1} \longrightarrow \cdots \longrightarrow F^0$ and $E^{-d} \longrightarrow E^{-d+1} \longrightarrow \cdots \longrightarrow E^0$ is a quasi-isomorphism for any $d \geqslant 1$. ☐

In view of [20, Lemma 1.6], in order to finish the proof of part (a) of Proposition it remains to show that any bounded above complex over F that is acyclic over E is also acyclic over F. This follows immediately from the condition that F is closed with respect to the passage to the kernels of admissible epimorphisms in E.

The proof of part (b) is similar to that of [22, Proposition 1.5], and part (c) is proven along the lines of [22, Remark 1.5] and [23, Theorem 4.2.1]. Not to reiterate here the rather involved arguments from [22, 23], let us just briefly explain how to construct a morphism with contraacyclic cone onto a given complex over E from an appropriately chosen complex over F in the situation (c).

Lemma A.2.3. (a) *For any complex $E^\bullet$ over E, there exists a complex $P^\bullet$ over F together with a morphism of complexes $P^\bullet \longrightarrow E^\bullet$ such that the morphism $P^i \longrightarrow E^i$ is an admissible epimorphism for each $i \in \mathbb{Z}$.*

(b) *For any complex $E^\bullet$ over E, there exists a bicomplex $P_\bullet^\bullet$ over F together with a morphism of bicomplexes $P_\bullet^\bullet \longrightarrow E^\bullet$ over E such that the complexes $P_j^\bullet$ vanish for all $j < 0$, while for each $i \in \mathbb{Z}$ the complex $\cdots \longrightarrow P_2^i \longrightarrow P_1^i \longrightarrow P_0^i \longrightarrow E^i \longrightarrow 0$ is acyclic with respect to the exact category E.*

Proof. To prove part (a), pick admissible epimorphisms $F^i \longrightarrow E^i$ onto all the objects $E^i \in \mathsf{E}$ from some objects $F^i \in \mathsf{F}$. Then the contractible complex $P^\bullet$ with the terms $P^i = F^i \oplus F^{i-1}$ (that is the complex freely generated by the graded object F^* over F) comes together with a natural morphism of complexes $P^\bullet \longrightarrow E^\bullet$ with the desired property. Part (b) is easily deduced from (a) by passing to the termwise kernel of the morphism of complexes $P_0^\bullet = P^\bullet \longrightarrow E^\bullet$ and iterating the construction. ☐

Lemma A.2.4. *Let A be an additive category with countable direct products. Let $\cdots \longrightarrow P^{\bullet\bullet}(2) \longrightarrow P^{\bullet\bullet}(1) \longrightarrow P^{\bullet\bullet}(0)$ be a projective system of bicomplexes over A. Suppose that for every pair of integers i, $j \in \mathbb{Z}$ the projective system $\cdots P^{ij}(2) \longrightarrow P^{ij}(1) \longrightarrow P^{ij}(0)$ stabilizes, and let $P^{ij}(\infty)$ denote the corresponding limit. Then the total complex of the bicomplex $P^{\bullet\bullet}(\infty)$ constructed by taking infinite products along the diagonals is homotopy equivalent to a complex obtained from the total complexes of the bicomplexes $P^{\bullet\bullet}(n)$ (constructed in the same way) by iterated application of the operations of shift, cone, and countable product.*

Proof. Denote by $T(n)$ and $T(\infty)$ the total complexes of, respectively, the bicomplexes $P^{\bullet\bullet}(n)$ and $P^{\bullet\bullet}(\infty)$. Then the short sequence of telescope construction

$$0 \longrightarrow T(\infty) \longrightarrow \prod_n T(n) \longrightarrow \prod_n T(n) \longrightarrow 0$$

is a termwise split short exact sequence of complexes over A. Indeed, at every term of the complexes the sequences decomposes into a countable product of sequences corresponding to the projective systems $P^{ij}(*)$ with fixed indices i, j and their limits $P^{ij}(\infty)$. It remains to notice that the telescope sequence of a stabilizing projective system is split exact, and a product of split exact sequences is split exact. ☐

Returning to part (c) of Proposition, given a complex $E^\bullet$ over E, one applies Lemma A.2.3(b) to obtain a bicomplex $P_\bullet^\bullet$ over F mapping onto $E^\bullet$. Let us show that the cone of the morphism onto $E^\bullet$ from the total complex $T^\bullet$ constructed by taking infinite products along the diagonals of the bicomplex $P_\bullet^\bullet$ is a contraacyclic complex over E. For this purpose, augment the bicomplex $P_\bullet^\bullet$ with the complex $E^\bullet$ and represent the resulting bicomplex as the termwise stabilizing projective limit of its quotient bicomplexes of canonical filtration with respect to the lower indices. The latter bicomplexes being finite exact sequences of complexes over E, the assertion follows from Lemma A.2.4. □

A.3. **Homotopy adjusted complexes.** The following simple construction (cf. [25]) will be useful for us when working with the conventional unbounded derived categories. Let E be an exact category.

If the functors of infinite direct sums exist and are exact in E, we denote by $\mathsf{D}(\mathsf{E})^{\mathsf{lh}} \subset \mathsf{D}(\mathsf{E})$ the minimal full triangulated subcategory in $\mathsf{D}(\mathsf{E})$ containing the objects of E and closed under infinite direct sums. Similarly, if the functors of infinite product exist and are exact in E, we denote by $\mathsf{D}(\mathsf{E})^{\mathsf{rh}}$ the minimal full triangulated subcategory of $\mathsf{D}(\mathsf{E})$ containing the objects of E and closed under infinite products. It is not difficult to show (see the proof of Proposition A.3.3) that, in the assumptions of the respective definitions, $\mathsf{D}^-(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})^{\mathsf{lh}}$ and $\mathsf{D}^+(\mathsf{E}) \subset \mathsf{D}(\mathsf{E})^{\mathsf{rh}}$.

From now on the assumptions of Section A.2 concerning the pair of exact categories $\mathsf{F} \subset \mathsf{E}$ are enforced.

Lemma A.3.1. *Let $B^\bullet$ be a bounded above complex over F and $C^\bullet$ be an acyclic complex over E. Then for any morphism of complexes $B^\bullet \longrightarrow C^\bullet$ over E there exists a quasi-isomorphism of bounded above complexes $L^\bullet \longrightarrow B^\bullet$ over F such that the composition of morphisms $L^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet$ factorizes through a bounded above acyclic complex $K^\bullet$ over F.*

Proof. Clearly, one can assume the complex $C^\bullet$ to be bounded above. Let $K^\bullet \longrightarrow C^\bullet$ be a termwise admissible epimorphism and a quasi-isomorphism of complexes over E such that $K^\bullet$ is a bounded above complex over F (see Lemma A.2.2). As a bounded above complex over F acyclic over E, the complex $K^\bullet$ is also acyclic over F.

Let $M^\bullet$ denote the termwise fibered product of the complexes $B^\bullet$ and $K^\bullet$ over the complex $C^\bullet$; then $M^\bullet$ is a bounded above complex over E and $M^\bullet \longrightarrow B^\bullet$ is a termwise admissible epimorphism and a quasi-isomorphism over E. Let $L^\bullet \longrightarrow M^\bullet$ be a termwise admissible epimorphism and a quasi-isomorphism of complexes over E such that $L^\bullet$ is a bounded above complex over F.

Then the composition $L^\bullet \longrightarrow M^\bullet \longrightarrow B^\bullet$, being (a termwise admissible epimorphism and) a quasi-isomorphism over E between bounded above complexes over F, is consequently also (a termwise admissible epimorphism and) a quasi-isomorphism of bounded complexes over F. Now the composition $L^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet$ factorizes naturally through the acyclic complex $K^\bullet$ over F. (For a slightly different version of this argument, see [22, proof of Lemma 2.9].) □

Corollary A.3.2. *Let $B^\bullet$ be a bounded above complex over F and $C^\bullet$ be a complex over F that is acyclic as a complex over E. Then the group $\mathrm{Hom}_{\mathsf{D}(\mathsf{F})}(B^\bullet, C^\bullet)$ of morphisms in the derived category $\mathsf{D}(\mathsf{F})$ of the exact category F vanishes.*

Proof. Indeed, any morphism from $B^\bullet$ to $C^\bullet$ in $\mathsf{D}(\mathsf{F})$ can be represented as a fraction $B^\bullet \longrightarrow {}'C^\bullet \longleftarrow C^\bullet$, where $B^\bullet \longrightarrow {}'C^\bullet$ is a morphism of complexes over F and ${}'C^\bullet \longrightarrow C^\bullet$ is a quasi-isomorphism of such complexes. Then the complex ${}'C^\bullet$ is also acyclic over E, and it remains to apply Lemma A.3.1 to the morphism $B^\bullet \longrightarrow C^\bullet$. □

Proposition A.3.3. *Suppose that the exact category* E *is actually abelian, and that infinite direct sums are everywhere defined and exact in the category* E *and preserve the full exact subcategory* $\mathsf{F}\subset\mathsf{E}$. *Then the composition of natural triangulated functors* $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}\longrightarrow\mathsf{D}(\mathsf{F})\longrightarrow\mathsf{D}(\mathsf{E})$ *is an equivalence of triangulated categories.*

Proof. We will show that any complex over E is the target of a quasi-isomorphism with the source belonging to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$. By [20, Lemma 1.6], it will follow, in particular, that $\mathsf{D}(\mathsf{E})$ is isomorphic to the localization of $\mathsf{D}(\mathsf{F})$ by the thick subcategory of complexes over F acyclic over E. By Corollary A.3.2, the latter subcategory is semiorthogonal to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$, so the same constuction of a quasi-isomorphism with respect to E will also imply that these two subcategories form a semiorthogonal decomposition of $\mathsf{D}(\mathsf{F})$. This would clearly suffice to prove the desired assertion.

Let $C^\bullet$ be a complex over E. Consider all of its subcomplexes of canonical truncation, pick a termwise surjective quasi-isomorphism onto each of them from a bounded above complex over F, and replace the latter with its finite subcomplex of silly filtration with, say, only two nonzero terms. Take the direct sum $B_0^\bullet$ of all the obtained complexes over F and consider the natural morphism of complexes $B_0^\bullet\longrightarrow C^\bullet$. This is a termwise surjective morphism of complexes which also acts surjectively on all the objects of coboundaries, cocycles, and cohomology. Next we apply the same construction to the kernel of this morphism of complexes, etc.

We have constructed an exact complex of complexes $\cdots\longrightarrow B_2^\bullet\longrightarrow B_1^\bullet\longrightarrow B_0^\bullet\longrightarrow C^\bullet\longrightarrow 0$ which remains exact after replacing all the complexes $B_i^\bullet$ and $C^\bullet$ with their cohomology objects (taken in the abelian category E). All the complexes $B_i^\bullet$ belong to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$ by the construction. It remains to show that the totalization of the bicomplex $B_\bullet^\bullet$ obtained by taking infinite direct sums along the diagonals also belongs to $\mathsf{D}(\mathsf{F})^{\mathsf{lh}}$ and maps quasi-isomorphically onto $C^\bullet$.

The totalization of the bicomplex $B_\bullet^\bullet$ is a direct limit of the totalizations of its subbicomplexes of silly filtration in the lower indices $B_n^\bullet\longrightarrow\cdots\longrightarrow B_0^\bullet$. By the dual version of Lemma A.2.4, the former assertion follows. To prove the latter one, it suffices to apply the following result due to Eilenberg and Moore [3] to the bicomplex obtained by augmenting $B_\bullet^\bullet$ with $C^\bullet$. ☐

Lemma A.3.4. *Let* A *be an abelian category with exact functors of countable direct sum, and let* $D_\bullet^\bullet$ *be a bicomplex over* A *such that the complexes* $D_j^\bullet$ *vanish for all* $j<0$, *while the complexes* $D_\bullet^i$ *are acyclic for all* $i\in\mathbb{Z}$, *as are the complexes* $H^i(D_\bullet^\bullet)$. *Then the total complex of the bicomplex* $D_\bullet^\bullet$ *obtained by taking infinite direct sums along the diagonals is acyclic.*

Proof. Denote by $S^\bullet(\infty)$ the totalization of the bicomplex $D_\bullet^\bullet$ and by $S^\bullet(n)$ the totalizations of its subbicomplexes of silly filtration $D_n^\bullet\longrightarrow\cdots\longrightarrow D_0^\bullet$. Consider the telescope short exact sequence

$$0\longrightarrow\textstyle\bigoplus_n S^\bullet(n)\longrightarrow\bigoplus S^\bullet(n)\longrightarrow S^\bullet\longrightarrow 0$$

and pass to the long exact sequence of cohomology associated with this short exact sequence of complexes. The morphisms $\bigoplus_n H^i(S^\bullet(n))\longrightarrow\bigoplus_n H^i(S^\bullet(n))$ in this long exact sequence are the differentials in the two-term complexes computing the derived functor of inductive limit $\varinjlim_n^* H^i(S^\bullet(n))$. It is clear from the conditions on the bicomplex $D^\bullet_\bullet$ that the morphisms of cohomology $H^i(S^\bullet(n-1)) \longrightarrow H^i(S^\bullet(n))$ induced by the embeddings of complexes $S^\bullet(n-1) \longrightarrow S^\bullet(n)$ vanish. Hence the morphisms $\bigoplus_n H^i(S^\bullet(n)) \longrightarrow \bigoplus_n H^i(S^\bullet(n))$ are isomorphisms and $H^*(S^\bullet) = 0$. □

Remark A.3.5. In particular, it follows from Proposition A.3.3 that $\mathsf{D}(\mathsf{E}) = \mathsf{D}(\mathsf{E})^{\mathsf{lh}}$ for any abelian category E with exact functors of infinite direct sum. On the other hand, the following example is instructive. Let $\mathsf{E} = R\text{–}\mathsf{mod}$ be the abelian category of left modules over an associative ring R and $\mathsf{F} = R\text{–}\mathsf{mod}^{\mathsf{prj}}$ be the full additive subcategory of projective R-modules (with the induced trivial exact category structure). Then we have $\mathsf{D}(\mathsf{F}) = \mathsf{Hot}(\mathsf{F}) \neq \mathsf{D}(\mathsf{E})$, while $\mathsf{D}(\mathsf{F})^{\mathsf{lh}} \subsetneq \mathsf{D}(\mathsf{F})$ is the full subcategory of homotopy projective complexes in $\mathsf{Hot}(R\text{–}\mathsf{mod}^{\mathsf{prj}})$ [25]. Hence one can see (by considering $\mathsf{E} = \mathsf{F} = R\text{–}\mathsf{mod}^{\mathsf{prj}}$) that the assertion of Proposition A.3.3 is not generally true when the exact category E is not abelian.

Corollary A.3.6. *In the assumptions of Proposition* A.3.3*, the fully faithful functor* $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F})^{\mathsf{lh}} \longrightarrow \mathsf{D}(\mathsf{F})$ *is left adjoint to the triangulated functor* $\mathsf{D}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{E})$ *induced by the embedding of exact categories* $\mathsf{F} \longrightarrow \mathsf{E}$.

Proof. Clear from the proof of Proposition A.3.3. □

Keeping the assumptions of Proposition A.3.3, assume additionally that the exact category F has finite homological dimension. Then the natural functor $\mathsf{D}^{\mathsf{co}}(\mathsf{F}) \longrightarrow \mathsf{D}(\mathsf{F})$ is an equivalence of triangulated categories [19, Remark 2.1]. Consider the composition of triangulated functors $\mathsf{D}(\mathsf{E}) \simeq \mathsf{D}(\mathsf{F})^{\mathsf{lh}} \longrightarrow \mathsf{D}(\mathsf{F}) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{F}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$. The following result is a generalization of [22, Lemma 2.9].

Corollary A.3.7. *The functor* $\mathsf{D}(\mathsf{E}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathsf{E})$ *so constructed is left adjoint to the Verdier localization functor* $\mathsf{D}^{\mathsf{co}}(\mathsf{E}) \longrightarrow \mathsf{D}(\mathsf{E})$.

Proof. One has to show that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathsf{E})}(B^\bullet, C^\bullet) = 0$ for any complex $B^\bullet \in \mathsf{D}(\mathsf{F})^{\mathsf{lh}}$ and any acyclic complex $C^\bullet$ over E. This vanishing easily follows from Lemma A.3.1. □

A.4. **Finite left resolutions.** We keep the assumptions of Section A.2. Assume additionally that the additive category E is, in the terminology of [16, 21], "semi-saturated" (i. e., it contains the kernels of its split epimorphisms, or equivalently, the cokernels of its split monomorphisms). Then the additive category F has the same property. The following results elaborate upon the ideas of [22, Remark 2.1].

We will say that an object of the derived category $\mathsf{D}^-(\mathsf{E})$ has *left* F*-homological dimension not exceeding* m if its isomorphism class can be represented by a bounded above complex $F^\bullet$ over F such that $F^i = 0$ for $i < -m$. By the definition, the full subcategory of objects of finite left F-homological dimension in $\mathsf{D}^-(\mathsf{E})$ is the image of the fully faithful triangulated functor $\mathsf{D}^{\mathsf{b}}(\mathsf{F}) \longrightarrow \mathsf{D}^-(\mathsf{F}) \simeq \mathsf{D}^-(\mathsf{E})$.

Lemma A.4.1. *If a bounded above complex* $G^\bullet$ *over the exact subcategory* F*, viewed as an object of the derived category* $\mathsf{D}^-(\mathsf{E})$*, has left* F*-homological dimension not* *exceeding m, then the differential $G^{-m-1} \longrightarrow G^{-m}$ has a cokernel $'G^{-m}$ in the additive category F, and the complex $\dotsb \longrightarrow G^{-m-1} \longrightarrow G^{-m} \longrightarrow {}'G^{-m} \longrightarrow 0$ over F is acyclic. Consequently, the complex $G^\bullet$ over F is quasi-isomorphic to the finite complex $0 \longrightarrow {}'G^{-m} \longrightarrow G^{-m+1} \longrightarrow G^{-m+2} \longrightarrow \dotsb \longrightarrow 0$.*

Proof. In view of the equivalence of categories $\mathsf{D}^-(\mathsf{F}) \simeq \mathsf{D}^-(\mathsf{E})$, the assertion really depends on the exact subcategory F only. By the definition of the derived category, two complexes representing isomorphic objects in it are connected by a pair of quasi-isomorphisms. Thus it suffices to consider two cases when there is a quasi-isomorphism acting either in the direction $G^\bullet \longrightarrow F^\bullet$, or $F^\bullet \longrightarrow G^\bullet$ (where $F^\bullet$ is a bounded above complex over F such that $F^i = 0$ for $i < -m$).

In the former case, acyclicity of the cone of the morphism $G^\bullet \longrightarrow F^\bullet$ implies the existence of cokernels of its differentials and the acyclicity of canonical truncations, which provides the desired conclusion.

In the latter case, from acyclicity of the cone of the morphism $F^\bullet \longrightarrow G^\bullet$ one can similarly see that the morphism $G^{-m-2} \longrightarrow G^{-m-1} \oplus F^{-m}$ with the vanishing component $G^{-m-2} \longrightarrow F^{-m}$ has a cokernel, and it follows that the morphism $G^{-m-2} \longrightarrow G^{-m-1}$ also does. Denoting the cokernel of the latter morphism by $'G^{-m-1}$, one easily concludes that the complex $\dotsb \longrightarrow G^{-m-2} \longrightarrow G^{-m-1} \longrightarrow {}'G^{-m-1} \longrightarrow 0$ is acyclic, and it remains to show that the morphism $'G^{-m-1} \longrightarrow G^{-m}$ is an admissible monomorphism in the exact category F. Indeed, the morphism $'G^{-m-1} \oplus F^{-m} \longrightarrow G^{-m} \oplus F^{-m+1}$ is; and hence so is its composition with the embedding of a direct summand $'G^{-m-1} \longrightarrow {}'G^{-m-1} \oplus F^{-m}$. □

Corollary A.4.2. *Let $G^\bullet$ be a finite complex over the exact category E such that $G^i = 0$ for $i < -m$ and $G^i \in \mathsf{F}$ for $i > -m$. Assume that the object represented by $G^\bullet$ in $\mathsf{D}^-(\mathsf{E})$ has left F-homological dimension not exceeding m. Then the object G^{-m} also belongs to F.*

Proof. Replace the object G^{-m} with its left resolution by objects from F and apply Lemma A.4.1. □

We say that an object $E \in \mathsf{E}$ has left F-homological dimension not exceeding m if the corresponding object of the derived category $\mathsf{D}^-(\mathsf{E})$ does. In other words, E must have a left resolution by objects of F of the length not exceeding m. Let us denote the left F-homological dimension of an object E by $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E$.

Corollary A.4.3. *Let $\mathsf{E}' \subset \mathsf{E}$ be a (strictly) full semi-saturated additive subcategory with an induced exact category structure. Set $\mathsf{F}' = \mathsf{E}' \cap \mathsf{F}$, and assume that every object of E' is the image of an admissible epimorphism in the exact category E' acting from an object belonging to F'. Then for any object $E \in \mathsf{E}'$ one has $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E = \operatorname{ld}_{\mathsf{F}'/\mathsf{E}'} E$.*

Proof. Follows from Corollary A.4.2. □

Lemma A.4.4. *Let $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ be an exact triple in E. Then*

(a) *if $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E' \leqslant m$ and $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E'' \leqslant m$, then $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E \leqslant m$;*

(b) *if $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E \leqslant m$ and $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E'' \leqslant m+1$, then $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E' \leqslant m$;*

(c) *if $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E \leqslant m$ and $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E' \leqslant m-1$, then $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E'' \leqslant m$.*

Proof. Let us prove part (a); the proofs of parts (b) and (c) are similar. The morphism $E''[-1] \longrightarrow E'$ in $\mathsf{D}^-(\mathsf{E}) \simeq \mathsf{D}^-(\mathsf{F})$ can be represented by a morphism of complexes $''F^\bullet \longrightarrow {}'F^\bullet$ in $\mathsf{Hot}^-(\mathsf{F})$. By Lemma A.4.1, both complexes can be replaced by their canonical truncations at the degree $-m$. Obviously, there is the induced morphism between the complexes truncated in this way, so we can simply assume that $''F^i = 0 = {}'F^i$ for $i < -m$. Moreover, the complex $''F^\bullet$ could be truncated even one step further, i. e., the morphism $''F^{-m} \longrightarrow {}''F^{-m+1}$ is an admissible monomorphism in the exact category F. From this one easily concludes that for the cone $G^\bullet$ of the morphism of complexes $''F^\bullet \longrightarrow {}'F^\bullet$ one has $G^i = 0$ for $i < -m-1$ and the morphism $G^{-m-1} \longrightarrow G^{-m}$ is an admissible monomorphism in F. $\square$

Corollary A.4.5. (a) *Let $0 \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_0 \longrightarrow E \longrightarrow 0$ be an exact sequence in E. Then the left F-homological dimension $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E$ does not exceed the supremum of the expressions $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E_i + i$ over $0 \leqslant i \leqslant n$ (where we set $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} 0 = -1$).*

(b) *Let $0 \longrightarrow E \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0$ be an exact sequence in E. Then the left F-homological dimension $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E$ does not exceed the supremum of the expressions $\operatorname{ld}_{\mathsf{F}/\mathsf{E}} E^i - i$ over $0 \leqslant i \leqslant n$.*

Proof. Part (a) follows by induction from Lemma A.4.4(c), and part (b) similarly follows from Lemma A.4.4(b). $\square$

Proposition A.4.6. *Suppose that the left F-homological dimension of all objects $E \in \mathsf{E}$ does not exceed a fixed constant d. Then the triangulated functor $\mathsf{D}^\star(\mathsf{F}) \longrightarrow \mathsf{D}^\star(\mathsf{E})$ induced by the exact embedding functor $\mathsf{F} \longrightarrow \mathsf{E}$ is an equivalence of triangulated categories for any symbol $\star = \mathsf{b}$, $+$, $-$, $\varnothing$, $\mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs.*

When $\star = \mathsf{co}$ (respectively, $\star = \mathsf{ctr}$), it is presumed here that the functors of infinite direct sum (resp., infinite product) are everywhere defined and exact in the category E and preserve the full subcategory $\mathsf{F} \subset \mathsf{E}$.

Proof. The cases $\star = -$ or ctr were considered in Proposition A.2.1(a,c) (and hold in its weaker assumptions). They can be also treated together with the other cases, as it is explained below.

In the cases $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$ one can argue as follows. Using the construction of Lemma A.2.3 and taking into account Corollary A.4.2, one produces for any $\star$-bounded complex $E^\bullet$ over E its finite left resolution $P_\bullet^\bullet$ of length d (in the lower indices) by $\star$-bounded (in the upper indices) complexes over F. The total complex of $P_\bullet^\bullet$ maps by a quasi-isomophism (in fact, a morphism with an absolutely acyclic cone) over the exact category E onto the complex $E^\bullet$.

By [20, Lemma 1.6], it remains to show that any complex $C^\bullet$ over F that is acyclic as a complex over E is also acyclic as a complex over F. For this purpose, we apply the same construction of the resolution $P_\bullet^\bullet$ to the complex $C^\bullet$. The complex $P_0^\bullet$ is acyclic over F and maps by a termwise admissible epimorphism onto the complex $C^\bullet$; it follows that the induced morphisms of the objects of cocycles are admissible epimorphisms, too. The passage to the cocycle objects of acyclic complexes also commutes with the passage to the kernels of termwise admissible epimorphisms.

We conclude that the cocycle objects of the acyclic complexes $P_i^\bullet$ form resolutions of the cocycle objects of the complex $C^\bullet$. Since the left F-homological dimension of objects of E does not exceed d and for $i < d$ the cocycle objects of $P_i^\bullet$ belong to F, so do the cocycle objects of the complex $P_d^\bullet$. Now the total complex of $P_\bullet^\bullet$ is acyclic over F and maps onto $C^\bullet$ with a cone acyclic over F.

The rather involved argument in the cases $\star = \mathsf{abs}+$, $\mathsf{abs}-$, co, ctr, or abs is similar to that in [22, Theorem 1.4] and goes back to the proof of [19, Theorem 7.2.2]. We do not reiterate the details here. □

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