

# ON WEIGHTED POINCARÉ INEQUALITIES

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ABSTRACT. The aim of this note is to show that Poincaré inequalities imply corresponding weighted versions in a quite general setting. The proof is short and does not involve covering arguments.

## 1. INTRODUCTION

Let  $(X, \rho)$  be a metric space with a positive  $\sigma$ -finite Borel measure  $dx$ , we will write  $|E| = \int_E dx$  for the measure of a Borel set  $E \subset X$ . We fix some point  $x_0 \in X$  and set  $B_r = \{x \in X : \rho(x, x_0) < r\}$ ,  $\overline{B}_r = \{x \in X : \rho(x, x_0) \leq r\}$ .

We call a function  $\phi : B_1 \rightarrow [0, \infty)$  an *admissible weight*, if  $\phi$  is a radial function, i.e.  $\phi = \Phi(\rho(\cdot, x_0))$  and its profile  $\Phi$  is nonincreasing and right-continuous with left-limits. We assume that  $\phi$  is not identically zero on  $B_1 \setminus \overline{B}_{1/2}$ .

For any admissible weight  $\phi$  there exists a positive, non-zero  $\sigma$ -finite Borel measure  $\nu$  on  $(\frac{1}{2}, 1]$ , such that

$$(1) \quad \phi(x) = \int_{\rho(x, x_0) \vee 1/2}^1 \nu(dt) = \int_{1/2}^1 \chi_{B_t}(x) \nu(dt), \quad x \in B_1 \setminus \overline{B}_{1/2}.$$

(Note that we put  $\int_a^b f(t) \nu(dt) := \int_{(a, b]} f(t) \nu(dt)$ .)

For a function  $u$  we denote by

$$u_E = \frac{1}{|E|} \int_E u(x) dx$$

the mean of  $u$  over the set  $E$ , and by

$$u_E^\phi = \frac{\int_E u(x) \phi(x) dx}{\int_E \phi(x) dx}$$

the mean of  $u$  over the set  $E \subset B_1$  with respect to the weight function  $\phi$ .

Our main result is the following:

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**Theorem 1.** *Let  $1 \leq p < \infty$  and let  $\phi$  be an admissible weight with  $\phi = \Phi(\rho(\cdot, x_0))$ . Let  $F : L^p(X) \times (\frac{1}{2}, 1] \rightarrow [0, \infty]$  be a functional satisfying*

$$(2) \quad F(u + a, r) = F(u, r), \quad a \in \mathbb{R},$$

$$(3) \quad \int_{B_r} |u(x) - u_{B_r}|^p dx \leq F(u, r),$$

for every  $r \in (\frac{1}{2}, 1]$  and every  $u \in L^p(X)$ . Then for  $M = \frac{8^p |B_1| \Phi(0)}{|B_{1/2}| \Phi(1/2)}$

$$(4) \quad \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) dx \leq M \int_{1/2}^1 F(u, t) \nu(dt)$$

for every  $u \in L^p(B_1)$ , where  $\nu$  is as in (1).

By choosing the functional  $F$  appropriately, (4) becomes a Poincaré inequality with weight  $\phi$ , see Section 3. Such inequalities have been studied extensively because of their importance for the regularity theory of partial differential equations, see the exposition in [5].

## 2. PROOF

**Lemma 2.** *Let  $\Omega$  be a finite measure space and  $p \geq 1$ . Assume  $f \in L^p(\Omega)$  with  $\int_\Omega f = 0$ . Then*

$$\|f + a\|_{L^p(\Omega)} \geq \frac{1}{2} \|f\|_{L^p(\Omega)}$$

for every  $a \in \mathbb{R}$ .

*Proof.* We may assume  $a > 0$ . Then

$$\int_{\Omega \cap \{f > 0\}} |f + a|^p \geq \int_{\Omega \cap \{f > 0\}} |f|^p \quad \text{and} \quad \int_{\Omega \cap \{f < -2a\}} |f + a|^p \geq 2^{-p} \int_{\Omega \cap \{f < -2a\}} |f|^p.$$

Furthermore, since  $\int_{\Omega \cap \{f \leq 0\}} |f| = \int_{\Omega \cap \{f > 0\}} |f|$ , we obtain

$$\int_{\Omega \cap \{-2a \leq f \leq 0\}} |f|^p \leq (2a)^{p-1} \int_{\Omega \cap \{-2a \leq f \leq 0\}} |f| \leq (2a)^{p-1} \int_{\Omega \cap \{f > 0\}} |f| \leq 2^{p-1} \int_{\Omega \cap \{f > 0\}} |f + a|^p,$$

where we use  $a^{p-1}b \leq (b+a)^{p-1}(b+a)$  for positive  $a, b$ . Combining these observations we obtain the result.  $\square$

*Proof of Theorem 1.* First we observe that it is enough to prove that

$$(5) \quad \int_{B_1} |u(x) - u_{B_1}^{\tilde{\phi}}|^p \tilde{\phi}(x) dx \leq \frac{2^{2p} |B_1|}{|B_{1/2}|} \int_{1/2}^1 F(u, t) \nu(dt),$$

where  $\tilde{\phi}(x) = \phi(x) \wedge \Phi(\frac{1}{2})$ . Indeed, we have

$$\frac{\Phi(\frac{1}{2})}{\Phi(0)} \phi(x) \leq \phi(x) \wedge \Phi(\frac{1}{2}) \leq \phi(x).$$

Hence if (5) holds, then

$$\begin{aligned} \int_{B_1} |u(x) - u_{B_1}^{\tilde{\phi}}|^p \tilde{\phi}(x) dx &\geq \frac{\Phi(\frac{1}{2})}{\Phi(0)} \int_{B_1} |u(x) - u_{B_1}^{\tilde{\phi}}|^p \phi(x) dx \\ &\geq \frac{\Phi(\frac{1}{2})}{\Phi(0)} 2^{-p} \int_{B_1} |u(x) - u_{B_1}^{\phi}|^p \phi(x) dx, \end{aligned}$$

where in the last line we have used Lemma 2.

Now we prove (5). To simplify the notation, we assume that  $\phi(x) = \Phi(\frac{1}{2})$  for  $x \in B_{1/2}$ , so that  $\tilde{\phi} = \phi$ .

Because of (2), by subtracting a constant from  $u$ , we may and do assume that  $u_{B_1}^{\phi} = 0$ , which means that

$$(6) \quad 0 = \int_{B_1} u(x) \phi(x) dx = \int_{1/2}^1 \int_{B_t} u(x) dx \nu(dt) = \int_{1/2}^1 u_{B_t} |B_t| \nu(dt).$$

We start from the integral on the right hand side of (4) and use (3)

$$\begin{aligned} R &:= \int_{1/2}^1 F(u, t) \nu(dt) \geq \int_{1/2}^1 \int_{B_t} |u(x) - u_{B_t}|^p dx \nu(dt) \\ &= \frac{1}{2} \int_{1/2}^1 \int_{B_t} |u(x) - u_{B_t}|^p dx \nu(dt) + \frac{1}{2} \int_{B_1} \int_{1/2}^1 |u(x) - u_{B_t}|^p \chi_{B_t}(x) \nu(dt) dx \\ &=: I_1 + I_2 \end{aligned}$$

(In fact  $I_1 = I_2$ , but we treat them differently.) We now deal with the inner integral in  $I_2$ . For  $x \in B_{1/2}$  we have

$$\int_{1/2}^1 |u(x) - u_{B_t}|^p \chi_{B_t}(x) \nu(dt) \geq \frac{1}{|B_1|} \int_{1/2}^1 |u(x) - u_{B_t}|^p |B_t| \nu(dt).$$

Since  $\int_{1/2}^1 u_{B_t} |B_t| \nu(dt) = 0$ , by Lemma 2 we obtain

$$\int_{1/2}^1 |u(x) - u_{B_t}|^p |B_t| \nu(dt) \geq 2^{-p} \int_{1/2}^1 |u_{B_t}|^p |B_t| \nu(dt).$$

Therefore

$$I_2 \geq \frac{2^{-p}}{2|B_1|} \int_{B_{1/2}} \int_{1/2}^1 |u_{B_t}|^p |B_t| \nu(dt) dx = \frac{2^{-p}|B_{1/2}|}{2|B_1|} \int_{1/2}^1 |u_{B_t}|^p |B_t| \nu(dt).$$

Using inequality  $|a|^p + |b|^p \geq 2^{1-p}|a+b|^p$  we obtain

$$\begin{aligned} I_1 + I_2 &\geq \frac{1}{2} \int_{1/2}^1 \int_{B_t} \left( |u(x) - u_{B_t}|^p + \frac{2^{-p}|B_{1/2}|}{|B_1|} |u_{B_t}|^p \right) dx \nu(dt) \\ &\geq \frac{2^{-p}|B_{1/2}|}{2|B_1|} 2^{1-p} \int_{1/2}^1 \int_{B_t} |u(x)|^p dx \nu(dt) \\ &= \frac{|B_{1/2}|}{|B_1|} 2^{-2p} \int_{B_1} |u(x)|^p \phi(x) dx \end{aligned}$$

and the proof is finished.  $\square$

## 3. APPLICATIONS

Let us discuss some corollaries. Corollary 3 is well-known [5], although our class of admissible weights is more general. Proposition 4 allows to deduce a weighted Poincaré inequality for fractional Sobolev norms from an unweighted version. Corollaries 5 and 6 give a more concrete result for fractional Sobolev norms. The first allows for more general kernels and exponents  $p$ . Corollary 6 improves [2, Theorem 5.1] because the result is robust for  $s \rightarrow 1-$  and allows for general weights and exponents  $p$ .

**Corollary 3.** *Let  $p \geq 1$  and  $\phi$  be an admissible weight. Consider  $X = \mathbb{R}^d$  equipped with the Lebesgue measure and the Euclidean metric. There exists a positive constant  $C$  depending on  $p, d$  and  $\phi$  such that*

$$(7) \quad \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) dx \leq C \int_{B_1} |\nabla u(x)|^p \phi(x) dx,$$

for every  $u \in W^{1,p}(B_1)$ .

**Proposition 4.** *Let  $p \geq 1$  and let  $\phi$  be an admissible weight of the form  $\phi = \Phi(\rho(\cdot, x_0))$ . Assume that for some kernel  $k : B_1 \times B_1 \rightarrow [0, \infty)$  and some positive constant  $C$  the following inequality holds*

$$(8) \quad \int_{B_r} |u(x) - u_{B_r}|^p dx \leq C \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) dy dx,$$

whenever  $r \in (\frac{1}{2}, 1]$  and  $u \in L^p(X)$ . Then with  $M = \frac{8^p |B_1| \Phi(0)}{|B_{1/2}| \Phi(1/2)}$

$$(9) \quad \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) dx \leq CM \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) (\phi(y) \wedge \phi(x)) dy dx$$

for  $u \in L^p(X)$ .

**Corollary 5.** *Let  $\phi$  be an admissible weight of the form  $\phi = \Phi(\rho(\cdot, x_0))$  and  $p \geq 1$ . Let  $k : B_1 \times B_1 \rightarrow [0, \infty)$  be a kernel satisfying  $k \geq c$  for some constant  $c > 0$ . There is a positive constant  $M$  depending on  $d, p$  and  $\Phi$  such that for  $u \in L^p(X)$*

$$(10) \quad \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) dx \leq \frac{M}{c} \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) (\phi(y) \wedge \phi(x)) dy dx$$

for  $u \in L^p(X)$ .

**Corollary 6.** *Let  $p \geq 1$ ,  $r \geq 1$  and  $0 < s_0 \leq s < 1$ . Consider  $X = \mathbb{R}^d$  equipped with the Lebesgue measure and the Euclidean metric. Let  $\phi$  be an admissible weight of the form  $\phi = \Phi(|\cdot|)$ . Then there exists a positive constant  $C$  depending on  $p, d, s_0$  and  $\Phi$  such that*

$$(11) \quad \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) dx \leq C(1-s) r^{p(1-s)} \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \chi_{\{|x-y| \leq \frac{1}{r}\}} (\phi(y) \wedge \phi(x)) dy dx$$

for all  $u \in L^p(B_1)$ .

*Proof of Corollary 3.* It is well-known that the following Poincaré inequality holds

$$(12) \quad \int_{B_r} |u(x) - u_{B_r}|^p dx \leq c r^p \int_{B_r} |\nabla u(x)|^p dx$$

for every  $u \in W^{1,p}(B_r)$  and  $r > 0$  where  $c > 0$  depends on  $p$  and  $d$ . Set

$$F(u, r) = c r^p \int_{B_r} |\nabla u(x)|^p dx,$$

for  $u \in W^{1,p}(B_1)$  and  $F(u, r) = \infty$  otherwise. Then for  $u \in W^{1,p}(B_1)$

$$\begin{aligned} \int_{1/2}^1 F(u, t) \nu(dt) &= c \int_{1/2}^1 t^p \int_{B_1} |\nabla u(x)|^p \chi_{B_t}(x) dx \nu(dt) \\ &\leq c \int_{B_1} |\nabla u(x)|^p \int_{1/2}^1 \chi_{B_t}(x) \nu(dt) dx = c \int_{B_1} |\nabla u(x)|^p \phi(x) dx. \end{aligned}$$

By Theorem 1 the assertion follows with  $C = 2^{3p+d} \frac{\Phi(0)}{\Phi(1/2)} c$ .  $\square$

*Proof of Proposition 4.* Let

$$F(u, r) = C \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) dy dx.$$

Then

$$\begin{aligned} \int_{1/2}^1 F(u, t) \nu(dt) &= C \int_{1/2}^1 \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \chi_{B_t}(y) \chi_{B_t}(x) dy dx \nu(dt) \\ &= C \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \int_{1/2}^1 \chi_{B_t}(y) \chi_{B_t}(x) \nu(dt) dy dx \\ &= C \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) (\phi(y) \wedge \phi(x)) dy dx. \end{aligned}$$

The assertion now follows from Theorem 1.  $\square$

**Lemma 7.** *Let  $r \geq 1$ ,  $p \geq 1$  and  $0 < s < 1$ . Then*

$$(13) \quad \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} dy dx \leq (3r)^{p(1-s)} \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} \chi_{\{|x-y| \leq \frac{1}{r}\}} dy dx$$

for all  $u \in L^p(B_1)$ .

*Proof.* Let  $n$  be a natural number such that  $n \geq 2r > n - 1$ . We introduce

$$A_k = A_k(x, y) = \frac{k}{n}y + \frac{n-k}{n}x, \quad k = 0, 1, \dots, n.$$

Then

$$\begin{aligned} I &= \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} dy dx = \int_{B_1} \int_{B_1} \frac{|\sum_{k=1}^n (u(A_{k-1}) - u(A_k))|^p}{|x - y|^{d+ps}} dy dx \\ &\leq n^{p-1} \sum_{k=1}^n \int_{B_1} \int_{B_1} \frac{|u(A_{k-1}) - u(A_k)|^p}{|x - y|^{d+ps}} dy dx. \end{aligned}$$

Note that  $|A_{k-1} - A_k| = \frac{1}{n}|x - y|$ . If we substitute  $\tilde{x} = A_{k-1}$ ,  $\tilde{y} = A_k$ , then  $d\tilde{y} d\tilde{x} = n^{-d} dy dx$  (which follows by an elementary calculation, see also [3, page 570]). Moreover,  $\tilde{x}, \tilde{y} \in B_1$

with  $|\tilde{x} - \tilde{y}| \leq \frac{2}{n} \leq \frac{1}{r}$ . Hence

$$I \leq n^{p-ps} \int_{B_1} \int_{B_1} \frac{|u(\tilde{x}) - u(\tilde{y})|^p}{|\tilde{x} - \tilde{y}|^{d+ps}} \chi_{\{|\tilde{x}-\tilde{y}| \leq \frac{1}{r}\}} d\tilde{y} d\tilde{x}.$$

Since  $n < 2r + 1 \leq 3r$ , the assertion follows.  $\square$

*Proof of Corollary 5.* First we use a well-known argument to obtain a nonweighted Poincaré inequality. By the convexity of the function  $x \mapsto |x|^p$  it holds  $|a + b|^p \geq |a|^p + bp|a|^{p-1} \operatorname{sgn}(a)$ . Thus

$$\begin{aligned} \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) dy dx &\geq c \int_{B_r} \int_{B_r} |(u(x) - u_{B_r}) + (u_{B_r} - u(y))|^p dy dx \\ &\geq c |B_r| \int_{B_r} |u(x) - u_{B_r}|^p dx \\ &\geq c |B_{1/2}| \int_{B_r} |u(x) - u_{B_r}|^p dx, \end{aligned}$$

whenever  $u \in L^p(B_r)$  and  $\frac{1}{2} < r \leq 1$ .

The assertion follows now from Proposition 4.  $\square$

*Proof of Corollary 6.* From [4] and [1, page 80] we know that there exists a constant  $C = C(p, d, s_0)$ , such that for  $s_0 \leq s < 1$

$$(14) \quad \int_{B_r} |u(x) - u_{B_r}|^p dx \leq C(1-s)r^{ps} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} dy dx,$$

for all  $u \in L^p(B_1)$ . The assertion follows now from (14), Proposition 4 and Lemma 7.  $\square$

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