ON THE ALGEBRA OF SIEGEL MODULAR FORMS OF GENUS 2

E. B. VINBERG

Abstract. Using the methods of [11], we recover the old result of J. Igusa [3] saying that the algebra of even Siegel modular forms of genus 2 is freely generated by forms of weights $4, 6, 10, 12$. We also determine the structure of the algebra of all Siegel modular forms of genus 2.

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1. Introduction

In 1962, J. Igusa [3] (see also [4], [5], [6]) proved that the algebra of even Siegel modular forms of genus 2 was freely generated by forms of weights $4, 6, 10, 12$. Up to recently, this was essentially the only example, when the structure of the algebra of automorphic forms on a symmetric domain of dimension greater than 2 was completely determined.

In 2010, the author [11] proved that some algebras of automorphic forms on symmetric domains of type IV of dimensions 4,5,6,7 were free, and found the weights of their generators. The proof was based on the interpretation of the corresponding arithmetic quotients of symmetric domains of type IV as the moduli spaces for some classes of quartic surfaces. In dimension 3, the symmetric domain of type IV is nothing else than the Siegel upper half-space $\mathcal{S}_2$ of genus 2. Making use of this fact, we recover the result of Igusa by the method of [11]. This proof has the advantage (or disadvantage?) that it does not involve any "explicit" formulas for automorphic forms like Eisenstein series.

The freeness of the algebra of even Siegel modular forms of genus 2 implies that the action of the Siegel modular group Γ_2 on $\mathcal{S}_2$ is generated by reflections. Since for $g > 2$ the Siegel upper half-space $\mathcal{S}_g$ of genus g does not admit reflections, the algebra of even Siegel modular forms of genus $g > 2$ cannot be free.

We also prove that the algebra of all Siegel modular forms of genus 2 is generated by forms of weights $4, 6, 10, 12, 35$, where the square of the odd generator is expressed as a polynomial in the even generators. (The explicit form of this polynomial can be found in [5].) In a sense, the odd generator is the Jacobian of the even ones (see Section 3). It vanishes exactly on the mirrors of the group Γ_2.

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2. Groups and bundles

The *Siegel upper half-space* $\mathcal{S}_g$ of genus g is (a model of) the Hermitian symmetric space $\mathrm{Sp}_{2g}(\mathbb{R})/\mathrm{U}_g$, the symmetric domain of type III. The *Siegel modular group* of genus g is the group $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$. It is known that it is a maximal discrete subgroup in $\mathrm{Sp}_{2g}(\mathbb{R})$: see, e.g., [1]. (There must be a more adequate reference, but I did not find such.)

A *Siegel modular form* of genus g and weight k is a Γ_g-invariant holomorphic section of the k-th power of a certain $\mathrm{Sp}_{2g}(\mathbb{R})$-equivariant line bundle over $\mathcal{S}_g$. This bundle is uniquely determined by the condition that its $(g+1)$-th power is the canonical bundle.

Another series of symmetric domains is formed by the symmetric domains of type IV, which are the Hermitian symmetric spaces

$$\mathcal{D}_n = \mathrm{SO}^+_{2,n}/(\mathrm{SO}_2 \times \mathrm{SO}_n) = \mathrm{O}^+_{2,n}/(\mathrm{SO}_2 \times \mathrm{O}_n),$$

where $\mathrm{O}^+_{2,n}$ denotes the subgroup of index 2 of $\mathrm{O}_{2,n}$ consisting of the elements whose spinor norm is equal to the determinant, and $\mathrm{SO}^+_{2,n} = \mathrm{O}^+_{2,n} \cap \mathrm{SO}_{2,n}$.

For $n \geq 3$, an *automorphic form* of weight k on $\mathcal{D}_n$ with respect to a lattice $\Gamma \subset \mathrm{O}^+_{2,n}$ is a Γ-invariant holomorphic section of the k-th power of a certain $\mathrm{O}^+_{2,n}$-equivariant line bundle over $\mathcal{D}_n$, whose n-th power is the canonical bundle. If

$$\pi : \mathcal{L}_n \to \mathcal{D}_n$$

is the corresponding $\mathbb{C}^*$-bundle (obtained by deleting the zero section), then automorphic forms of weight k can be viewed as Γ-invariant holomorphic functions on $\mathcal{L}_n$ that are homogeneous of degree $-k$ on each fiber.

In this work, we exploit the isomorphism $\mathcal{S}_2 \simeq \mathcal{D}_3$, which gives rise to some isomorphisms of the algebras of automorphic forms. Below we describe this isomorphism in detail.

Considering the natural representation of $\mathrm{SL}_4(\mathbb{R})$ in $\wedge^2\mathbb{R}^4$, one obtains a homomorphism of $\mathrm{SL}_4(\mathbb{R})$ to $\mathrm{SO}_{3,3}$, whose kernel is $\{\pm E\}$, and the image is the subgroup $\mathrm{SO}^+_{3,3}$ of index 2 of $\mathrm{SO}_{3,3}$ consisting of the elements of spinor norm 1. Restricting this homomorphism to $\mathrm{Sp}_4(\mathbb{R})$ gives rise to an epimorphism

$$\rho : \mathrm{Sp}_4(\mathbb{R}) \to \mathrm{SO}^+_{2,3}.$$

The kernel of ρ is contained in U_2, so ρ induces an isomorphism between the symmetric spaces $\mathcal{S}_2 = \mathrm{Sp}_4/\mathrm{U}_2$ and $\mathcal{D}_3 = \mathrm{SO}^+_{2,3}/(\mathrm{SO}_2 \times \mathrm{SO}_3)$. Note that the equivariant line bundles on these two spaces involved in the definitions of automorphic forms are taken to one another by this isomorphism, since they both are the cubic roots of the canonical bundle.

More precisely, if $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$, then the scalar product in $\wedge^2\mathbb{R}^4$ is given by

$$(e_1 \wedge e_2, e_3 \wedge e_4) = (e_1 \wedge e_3, e_4 \wedge e_2) = (e_1 \wedge e_4, e_2 \wedge e_3) = 1,$$

all the other products of the basis bivectors $e_i \wedge e_j$ being equal to 0. The group $\mathrm{Sp}_4(\mathbb{R})$ fixes the bivector $e_1 \wedge e_2 + e_3 \wedge e_4$ and leaves invariant its orthogonal complement spanned by

$$f_0 = e_1 \wedge e_2 - e_3 \wedge e_4,\ f_1 = e_1 \wedge e_3,\ f_2 = e_4 \wedge e_2, f_3 = e_1 \wedge e_4, f_4 = e_2 \wedge e_3.$$

In the basis $\{f_0, f_1, f_2, f_3, f_4\}$ the scalar square in the orthogonal complement is given by the quadratic form

$$q = -2x_0^2 + 2x_1x_2 + 2x_3x_4. \tag{1}$$

Set

$$\Gamma = \mathrm{O}^+(q, \mathbb{Z}) \subset \mathrm{O}_{2,3}^+, \tag{2}$$

$$\Gamma_0 = \mathrm{SO}^+(q, \mathbb{Z}) \subset \mathrm{SO}_{2,3}^+. \tag{3}$$

Clearly, the group Γ_2 leaves invariant the lattice generated by f_0, f_1, f_2, f_3, f_4, so $\rho(\Gamma_2) \subset \Gamma_0$. But since Γ_2 is a maximal discrete subgroup in $\mathrm{Sp}_4(\mathbb{R})$, this inclusion cannot be strict, that is

$$\rho(\Gamma_2) = \Gamma_0.$$

This yields an isomorphism between the (graded) algebra of Siegel modular forms of genus 2 and the (graded) algebra of automorphic forms on $\mathcal{D}_3$ with respect to the group Γ_0.

Moreover, we have

$$\Gamma = \Gamma_0 \times \{\pm E\},$$

which means that the algebra of automorphic forms on $\mathcal{D}_3$ with respect to Γ is just the even part of the algebra of automorphic forms with respect to Γ_0, and, hence, is isomorphic to the algebra of even Siegel modular forms of genus 2.

3. Lattices and roots

Following the idea of [11], we are going to find a class of multipolarized $K3$ surfaces parametrized by the points of $\mathcal{D}_3/\Gamma$. To this end, we will introduce and investigate some quadratic lattices, which are intended to play the roles of the lattices of algebraic and transcendental cycles of our $K3$ surfaces.

We use the following notation:

$[A]$: the quadratic lattice, the scalar product in which is given by a (symmetric) matrix A;

$[c]L$: the quadratic lattice obtained from a quadratic lattice L by multiplying all scalar products by c;

$L \oplus M$: the (orthogonal) direct sum of quadratic lattices L and M;

kL: the direct sum of k copies of a quadratic lattice L;

$L^* \subset L \otimes \mathbb{Q}$: the dual lattice of a lattice L.

We denote by $I_{k,l}$ the group $\mathbb{Z}^{k+l}$ equipped with the scalar product

$$(x, y) = x_1y_1 + \cdots + x_ky_k - x_{k+1}y_{k+1} - \cdots - x_{k+l}y_{k+l}.$$

If $k, l > 0$, it is the only (up to isomorphism) odd unimodular quadratic lattice of signature (k, l).

For $k \equiv l \pmod 8$, we define the quadratic lattice

$$J_{k,l} = \{x \in I_{k,l} \oplus \mathbb{Q} : \, 2x_i \in \mathbb{Z}, \, x_i - x_j \in \mathbb{Z} \, (i, j = 1, \ldots, k + l), \sum_i x_i \in 2\mathbb{Z}\}.$$

If $k, l > 0$, it is the only even unimodular quadratic lattice of signature (k, l). In particular, the lattice

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is isomorphic to $J_{1,1}$, and, more generally, the lattice kU is isomorphic to $J_{k,k}$.

Recall that, for a $K3$ surface X, the middle homology lattice $H_2(X,\mathbb{Z})$ is isomorphic to $J_{3,19}$.

The following two even quadratic lattices will be of special interest for us:

$$S_0 = [2] \oplus J_{0,16}, \tag{4}$$

$$T_0 = [-2] \oplus 2U. \tag{5}$$

The scalar square in T_0 is given by the quadratic form q as defined by (1), so the group $\mathrm{O}^+(T_0)$ coincides with the group Γ introduced in the previous section.

It is easy to see that the lattice $[2] \oplus [-2]$ admits only one non-trivial extension (of index 2), which is isomorphic to U. It follows that the lattice $S_0 \oplus T_0$ admits only one non-trivial extension (also of index 2), which is isomorphic to $3U \oplus J_{0,16} \simeq J_{3,19}$. Thereby the lattice $S_0 \oplus T_0$ turns to be embedded into $J_{3,19}$ as a primitive sublattice of index 2. This will allow us to consider the class of $K3$ surfaces having S_0 and T_0 as the lattices of algebraic and transcendental cycles, respectively.

Clearly, any automorphism of the lattice $S_0 \oplus T_0$ (uniquely) extends to an automorphism of $J_{3,19}$. In particular, any automorphism of T_0 extends to an automorphism of $J_{3,19}$ acting trivially on S_0.

A primitive vector α of a quadratic lattice L is called a *root* or, more precisely, a *k-root*, if $(\alpha,\alpha) = -k < 0$ and the reflection

$$R_\alpha : x \mapsto x + \frac{2(\alpha,x)}{k}\alpha$$

leaves L invariant, or, equivalently, $k|2(\alpha,x)$ for every $x \in L$. This automatically holds if $k = 1$ or 2. If the lattice L is unimodular, there are no other possibilities, but in general there may be k-roots with $k \neq 1,2$. The reflection R_α defined by a k-root α is called a k-reflection.

In this paper we, however, will deal only with 2-roots and the corresponding reflections, and, moreover, one can prove that the lattices we are going to consider have only 2-roots. For this reason, in order to avoid unnecessary complications, we will use the term "root" for 2-roots only.

A quadratic lattice L is called *hyperbolic*, if its signature is $(1,m)$. In this case, the group $\mathrm{O}^+(L)$ is a cofinite discrete group of motions of the n-dimensional hyperbolic space H^m modelled as a connected component of the hyperboloid $(x,x) = 1$ in the Minkowski space $L \otimes \mathbb{R}$. The linear reflections R_α of the Minkowski space defined by vectors α with negative squares, correspond to reflections in hyperplanes of the space H^m. The group $W(L) \subset \mathrm{O}^+(L)$ generated by all 2-reflections is called the Weyl group of L. Let $P(L)$ be a fundamental polyhedron of $W(L)$ in H^m (which is defined up to the action of $W(L)$). It is bounded by the mirrors of some reflections of $W(L)$. The corresponding roots looking outside $P(L)$ are called the *simple roots* of L. They can be effectively found by means of an algorithm described in [10].

We will use the model of the lattice S_0 described as follows. Consider a real quadratic vector space with a fixed basis $\{e_0, e_1, \dots, e_{16}\}$ and the scalar product

$$(x,y) = 2x_0y_0 - x_1y_1 - \dots - x_{16}y_{16}.$$

The lattice S_0 is embedded into this space as

$$S_0 = \{\sum_{i=0}^{16} x_ie_i : x_0 \in \mathbb{Z},\, 2x_i \in \mathbb{Z},\, x_i - x_j \in \mathbb{Z}\,(i,j = 1,\dots,16),\, \sum_{i=1}^{16} x_i \in 2\mathbb{Z}\}. \tag{6}$$

One can find the simple roots of S_0 by means of the algorithm of [10], taking $e_0/\sqrt{2}$ for the base point. They turn to be the following vectors:

$$\begin{aligned}
&\alpha_i = -e_i + e_{i+1} \quad (i = 1, \dots, 15), \quad \alpha_{16} = -e_{15} - e_{16},\\
&\alpha_{17} = e_0 + 2e_1,\\
&\alpha_{18} = e_0 + e_1 + e_2 + e_3 + e_4,\\
&\alpha_{19} = e_0 + \frac{1}{2}\sum_{i=1}^{16} e_i.
\end{aligned}$$

Their Coxeter diagram (with their numbers indicated) is shown on Fig.1. One can observe that it does not contain Lanner subdiagrams, and every connected parabolic subdiagram is included into a parabolic subdiagram of rank 15. This ensures that the convex polyhedron

$$P = P(S_0) = \{x \in H^{16} : (\alpha_i, x) \geq 0 \quad \text{for} \quad i = 1, \dots, 16\}$$

has finite volume and, hence, is a fundamental polyhedron of the group $W = W(S_0)$: see [10, Proposition 1 and Subsection 2.3].

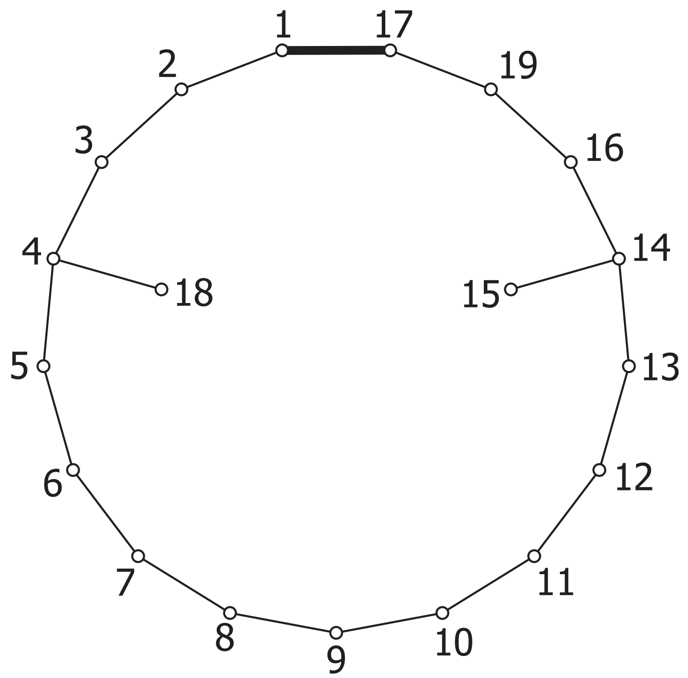

FIGURE 1

4. MULTIPOLARIZED $K3$ SURFACES

For a $K3$ surface X, we denote by $S(X)$ and $T(X)$ the lattices of algebraic and transcendental cycles of X, respectively. Recall that $S(X)$ is a primitive hyperbolic sublattice of the lattice $H_2(X, \mathbb{Z}) \simeq J_{3,19}$, and $T(X)$ is its orthogonal complement. We denote by $A(X)$ the closed convex cone in $S(X) \otimes \mathbb{R}$ generated by the classes of ample divisors on X. If the signature of $S(X)$ is $(1, m)$, then $P(X) = A(X) \cap H^m$ is a fundamental polyhedron of the group $W(X)$ generated by all 2-reflections contained in $\mathrm{O}^+(S(X))$. (See, e.g., [2, Proposition 5.10].)

Let $h_0 \in J_{3,19}$ be a primitive vector with $(h_0, h_0) = d > 0$, and $S_0 \subset J_{3,19}$ be a primitive hyperbolic sublattice containing h_0. Let T_0 be the orthogonal complement of S_0 in $J_{3,19}$. A *multipolarization* of type (h_0, S_0) of a $K3$ surface X is a vector $h \in S(X) \cap A(X)$ and a hyperbolic sublattice $S \subset S(X)$ containing h such that there exists an isomorphism $\varphi : H_2(X, \mathbb{Z}) \to J_{3,19}$ taking h to h_0 and S to S_0.

A non-zero regular differential 2-form ω on X (defined up to a scalar multiplication) by the Poincaré duality can be considered as an element of $H_2(X, \mathbb{C}) = H_2(X, \mathbb{Z}) \otimes \mathbb{C}$. Then

$$\varphi(\omega) \in \tilde{\mathcal{L}}_n =: \{z \in T_0 \otimes \mathbb{C} : (z, z) = 0,\ (z, \bar{z}) > 0\}.$$

We will require in addition that $\varphi(\omega) \in \mathcal{L}_n$, where $\mathcal{L}_n$ is a fixed connected component of the cone $\tilde{\mathcal{L}}_n$ (which has two complex conjugate connected components). Then φ is defined up to a left multiplication by an operator of $\mathrm{O}^+(J_{3,19})$ leaving h_0 and S_0 (or, equivalently, h_0 and T_0) invariant. Denote by $\mathrm{O}^+(T_0, h_0)$ the group formed by the restrictions to T_0 of such operators. It is a subgroup of finite index in $\mathrm{O}^+(T_0)$.

A quadruple (X, h, S, ω), where (X, h, S) is a multipolarized $K3$ surface of type (h_0, S_0) and ω is a non-zero regular differential 2-form on X, will be called a *normed multipolarized* $K3$ *surfaces* of type (h_0, S_0). Via the map $(X, h, S, \omega) \mapsto \varphi(\omega)$ the normed multipolarized $K3$ surfaces of type (h_0, S_0) are parameterized by the points of the variety $\mathcal{L}_n/\mathrm{O}^+(T_0, h_0)$.

The projectivization of $\mathcal{L}_n$ is a model of the n-dimensional symmetric domain $\mathcal{D}_n$ of type IV, where $n = 19 - m$ (or, in other words, $(2, n)$ is the signature of the lattice T_0). Thus, the multipolarized $K3$ surfaces of type (h_0, S_0) are parameterized by the points of the variety $\mathcal{D}_n/\mathrm{O}^+(T_0, h_0)$.

The variety $\mathcal{L}_n/\mathrm{O}^+(T_0, h_0)$ is naturally identified with a Zariski open subset in the spectrum of the algebra $A(\mathcal{D}_n, \mathrm{O}^+(T_0, h_0))$ of automorphic forms on $\mathcal{D}_n$ with respect to the group $\mathrm{O}^+(T_0, h_0)$, the boundary being of dimension ≤ 2. The variety $\mathcal{D}_n/\mathrm{O}^+(T_0, h_0)$ is then identified with a Zariski open subset in the projective spectrum of the algebra $A(\mathcal{D}_n, \mathrm{O}^+(T_0, h_0))$, the boundary being of dimension ≤ 1.

The typical situation for a multipolarized $K3$ surface X is that $S(X) = S$. If this holds, the multipolarization is called *irrational*; otherwise it is called *rational.*

In this paper, we will consider the multipolarized $K3$ surfaces of type (h_0, S_0), where $S_0 \subset J_{3,19}$ is the sublattice introduced in the previous section, and

$$h_0 = 4e_0 + 4e_1 + e_2 + \cdots + e_{13} \in S_0 \tag{7}$$

in the basis used in (6). The non-zero scalar products of h_0 with simple roots are indicated on Fig.2.

As we noted above, in this case any automorphism of T_0 extends to an automorphism of $J_{3,19}$ acting trivially on S_0, so

$$\mathrm{O}^+(T_0, h_0) = \mathrm{O}^+(T_0) = \Gamma,$$

where Γ is the group introduced in Section 2.

Moreover, in the considered case $(h_0, h_0) = 4$. This means that if (X, h, S) is a multipolarized $K3$ surface of type (h_0, S_0), then the linear system $|h|$ defines a

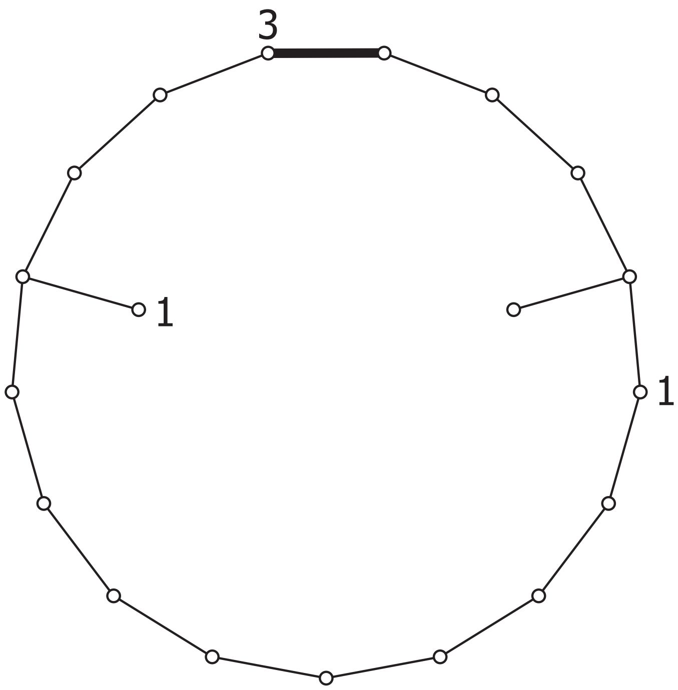

FIGURE 2

morphism

$$\varphi_h : X \to \mathbb{C}P^3.$$

It is known that φ_h is a birational morphism onto its image if and only if the following condition holds:

(*) There are no isotropic vectors $u \in S(X) \cap A(X)$ with $(h, u) = 1$ or 2.

Under this condition, $Y = \varphi_h(X)$ is a quartic surface with at most simple singularities. The morphism φ_h retracts into singular points the smooth rational curves on X whose classes are orthogonal to h, and it is an isomorphism beyond them.

As follows from the next proposition, property (*) holds for all multipolarized $K3$ surfaces considered in this paper.

Proposition 1. *There are no isotropic vectors $u \in J_{3,19}$ with $(h_0, u) = 1$ or 2 such that the sublattice generated by S_0 and u is hyperbolic.*

Proof. First, we will show that there are no such vectors u in the very lattice S_0. According to [11, Lemma 1], it suffices to test the isotropic vectors from $S_0 \cap A(S_0)$. Using the algorithm described in [11, Subsection 1.6], we see that the cone $A(S_0)$ has exactly three isotropic edges, corresponding to a parabolic subdiagram of type $\tilde{D}_{14} + \tilde{A}_1$ and two parabolic subdiagrams of type $\tilde{E}_8 + \tilde{E}_7$, and if u_1, u_2, u_3 are the primitive vectors of these edges, then $(h_0, u_1) = (h_0, u_2) = 3$, $(h_0, u_3) = 5$.

Suppose now that there is an isotropic vector $u \in J_{3,19} \setminus S_0$ satisfying the conditions of the proposition. We have $u = u' + u''$ with $u' \in S_0^*$, $u'' \in T_0^*$. By our assumption the sublattice generated by S_0 and u is hyperbolic. This means that $(u'', u'') < 0$, whence $(u', u') > 0$. Moreover, since $2S_0^* \subset S_0$, we have

$(2u', 2u') \in 2\mathbb{Z}$. Taking into account that the Gram determinant of h_0 and u' cannot be positive, we obtain that $(h_0, u') = 2$ and $(u', u') = 1$ or $\frac{1}{2}$.

In the first case the Gram determinant equals 0, which means that $u' = \frac{1}{2}h_0$. But it is easy to see that $\frac{1}{2}h_0 \notin S_0^*$, a contradiction.

In the second case we have $(2u', 2u') = 2$, so $2u' = h_0 + \alpha$, where

$$(h_0, \alpha) = 0, \quad (\alpha, \alpha) = -2,$$

i.e. α is a root orthogonal to h_0. Every such root is obtained from a simple root orthogonal to h_0, by applying some element of $W(S_0)$ fixing h_0. For any simple root α orthogonal to h_0, there is another simple root β orthogonal to h_0 such that $(\alpha, \beta) = 1$ (see Fig.2), whence $h_0 + \alpha \notin 2S_0^*$. But then this is also true for any root α orthogonal to h_0, a contradiction. □

5. Quartics

We will use the following notation from [11]:

Q: the SL_4-module of quartic forms in 4 variables;

T: the maximal torus of SL_4 consisting of diagonal matrices;

$Y(F) \subset \mathbb{C}P^3\,(F \in Q, F \neq 0)$: the surface defined in homogeneous coordinates by the equation $F = 0$;

$Q^\circ \subset Q$: the Zariski open subset consisting of the forms $F \in Q$ such that the surface $Y(F)$ is irreducible and has at most simple singularities;

$X(F)\,(F \in Q^\circ)$: the $K3$ surface obtained by the desingularization of $Y(F)$;

$\omega(F)$: the regular differential 2-form on $X(F)$ canonically defined as in [11, Subsection 2.2] (recall that $\omega(tF) = t^{-1}\omega(F)$ for $t \in \mathbb{C}^*$).

Let now (X, h, S, ω) be a normed multipolarized $K3$ surface of type (h_0, S_0), where $S_0 \subset J_{3,19}$ and $h_0 \in S_0$ are defined by (6) and (7). Proposition 1 assures that the linear system $|h|$ defines a birational morphism

$$\varphi_h : X \to Y \subset \mathbb{C}P^3,$$

where $Y = Y(F)$ is an irreducible quartic surface with at most simple singularities. The form F is normalized by the condition $\omega(F) = \omega$.

Proposition 2. *Let $Y = Y(F)\,(F \in Q^\circ)$, and let $X = X(F)$ be the corresponding $K3$ surface. Assume that X admits an irrational multipolarization (h, S) of type (h_0, S_0), where $h \in H_2(X, \mathbb{Z})$ is the class of (the pullback of) a plane section of $Y(F)$. Then the form F is SL_4-equivalent to a form*

$$F_0 = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3), \tag{8}$$

where f_2 is a quadratic form not containing x_1^2 and x_3^2, and $A, B, C \neq 0$. The form F_0 is uniquely defined up to the action of T.

Proof. By our assumption $S(X) = S$. Considering the scalar products of h with the simple roots $\alpha_1, \dots, \alpha_{19}$ of S (see Fig.2), we see that φ_h retracts to points all the smooth rational curves on X but α_1, α_{13}, α_{18}, and maps α_{13} and α_{18} to some lines on Y, say, l_1 and l_2. (We denote smooth rational curves on X as the corresponding simple roots.) The quartic Y has two singular points, say, o and p, of types A_{11} and A_5, both lying on l_1, and o lying also on l_2.

The non-connected parabolic subdiagrams of the Coxeter diagram yield linear dependences between simple roots (see [11, Subsection 1.6]). In our case, we obtain

$$\alpha_1 + \alpha_{17} = \alpha_{18} + \alpha_3 + 2(\alpha_4 + \cdots + \alpha_{14}) + \alpha_{15} + \alpha_{16}, \tag{9}$$

$$\begin{aligned} &2\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 + \alpha_9 + 3\alpha_{18} = \\ &\alpha_{11} + 2\alpha_{12} + 3\alpha_{13} + 4\alpha_{14} + 3\alpha_{16} + 2\alpha_{19} + \alpha_{17} + 2\alpha_{15}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_{18} = \\ &\alpha_9 + 2\alpha_{10} + 3\alpha_{11} + 4\alpha_{12} + 5\alpha_{13} + 6\alpha_{14} + 4\alpha_{16} + 2\alpha_{10} + 3\alpha_{15}. \end{aligned} \tag{11}$$

The vector h can be expressed as a linear combination of simple roots as follows:

$$h = 3\alpha_2 + 6\alpha_3 + 9\alpha_4 + 8\alpha_5 + 7\alpha_6 + 6\alpha_7 + 5\alpha_8 + 4\alpha_9 + 3\alpha_{10} + 2\alpha_{11} + \alpha_{12} + 4\alpha_{18}. \tag{12}$$

Other expressions can be found using (9)-(11).

Denote by $\bar{S}$ the subgroup of S generated by the simple roots orthogonal to h. It follows from (12) and (9)-(11) that

$$h \equiv 4\alpha_{18} \equiv 3\alpha_{13} + \alpha_{18} \pmod{\bar{S}}.$$

This means that there are a plane $P_1 \subset \mathbb{C}P^3$ such that

$$P_1 \cap Y = 4l_2 \tag{13}$$

and a plane $P_2 \subset \mathbb{C}P^3$ such that

$$P_2 \cap Y = 3l_1 + l_2. \tag{14}$$

The configuration $(o, p, l_1, l_2, P_1, P_2)$ is canonically associated with Y. Applying a suitable projective transformation, we may (and will) assume that

$$\begin{aligned} &o = (1:0:0:0), \quad p = (0:0:0:1), \\ &l_1 : x_1 = x_2 = 0, \quad l_2 : x_2 = x_3 = 0, \\ &P_1 : x_3 = 0, \quad P_2 : x_2 = 0. \end{aligned} \tag{15}$$

(See Fig.3).

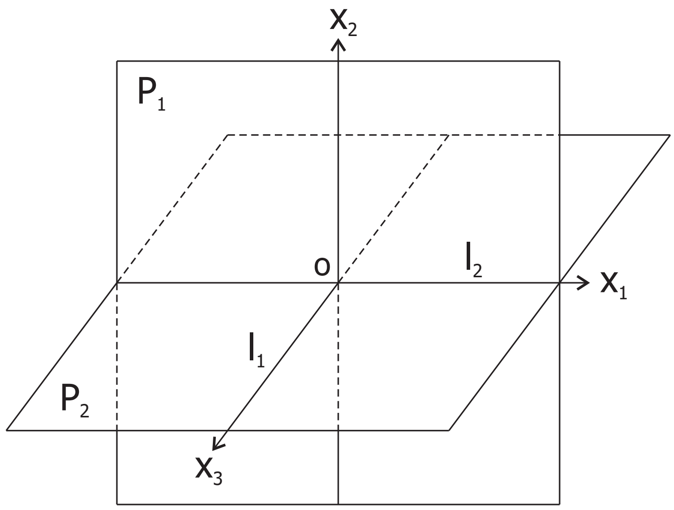

FIGURE 3

Under these assumptions, the conditions (13) and (14) mean that

$$F = Bx_1^3x_3 + Cx_2^4 + x_2x_3g_2(x_0, x_1, x_2, x_3), \tag{16}$$

where g_2 is a quadratic form not containing x_3^2. (The latter follows from the assumption that $p = (0 : 0 : 0 : 1)$ is a singular point of Y.) Clearly, $B, C \neq 0$, since otherwise F would be divisible by x_2 or x_3.

There is an automorphism of the lattice $J_{3,19}$ acting trivially on S_0 (in particular, fixing h_0) and as -1 on T_0. According to [8], this means that there is an involution δ of X leaving invariant all smooth rational curves on X and multiplying ω by -1. Moreover, δ is realized as a projective involution of Y (which we will denote with the same letter), leaving invariant all the elements of the configuration (o, p, l_1, l_2, P_1, P_2).

Again applying a suitable projective transformation, we may assume that δ is defined by a diagonal linear operator D with ± 1 on the diagonal. Since δ multiplies ω by -1, we have two possibilities: either $DF = F$ and $\det D = -1$, or $DF = -F$ and $\det D = 1$. However, the second possibility is not realized because of the non-zero term Cx_2^4 of F. Hence, $\det D = -1$, and, replacing D with $-D$ if needed, we may assume that D just multiplies one coordinate by -1. Considering the condition $DF = F$, we see from (16) that the only possibility is that

$$D = \mathrm{diag}(-1, 1, 1, 1), \tag{17}$$

which means that x_0 occurs with even exponents in all non-zero terms of F, i.e.,

$$F = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4 + x_2x_3f_2(x_1, x_2, x_3),$$

where f_2 is a quadratic form not containing x_3^2. Here $A \neq 0$, since otherwise $o = (1 : 0 : 0 : 0)$ is not a simple singularity of Y (the second differential of the local equation vanishes at o).

Adding to x_1 a suitable multiple of x_2, one can kill the term x_1^2 in f_2, which gives the desired form (8). After that, one can only act by elements of T. □

6. Invariants

The forms (8) with arbitrary A, B, C constitute a 7-dimensional T-invariant subspace of Q. Denote it by R. It consists of the forms

$$F = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4 + Dx_1x_2^2x_3 + Ex_2^3x_3 + Gx_1x_2x_3^2 + Hx_2^2x_3^2. \tag{18}$$

It is easy to find the algebra $\mathbb{C}[R]^T$ of T-invariant polynomial functions on R. Fortunately, it turns to be free.

Proposition 3. *The algebra $\mathbb{C}[R]^T$ is freely generated by the invariants*

$$I_4 = A^2BD,\ I_6 = A^3B^2E,\ I_{10} = A^5B^3CG,\ I_{12} = A^6B^4CH \tag{19}$$

of degrees $4, 6, 10, 12$, respectively.

Proof. Denote by α, β, γ the weights of the monomials $x_0^2x_2x_3$, $x_1^3x_3$, x_2^4 with respect to T. They are linearly independent, and the weights of the other four monomials of R are $-2\alpha - \beta$, $-3\alpha - 2\beta$, $-5\alpha - 3\beta - \gamma$, $-6\alpha - 4\beta - \gamma$, whence the result follows. □

The next and more difficult step is to understand the relation between the algebra $\mathbb{C}[R]^T$ and the algebra $\mathbb{C}[Q]^{\mathrm{SL}_4}$ of SL_4-invariant polynomial functions on Q.

The restriction of invariants defines a homomorphism

$$\mu : \mathbb{C}[Q]^{\mathrm{SL}_4} \to \mathbb{C}[R]^T$$

and, thereby, a morphism

$$\mu^* : R/\!\!/T \to Q/\!\!/\mathrm{SL}_4$$

of the categorical quotients.

Proposition 4. *The morphism μ^* is finite.*

Proof. This is a particular case of [11, Proposition 11]. (Our space R plays the role of the space denoted by R_0 in that proposition.) □

The image of μ^* is $\overline{(\mathrm{SL}_4)R}/\!\!/\mathrm{SL}_4$. It follows from Propositions 2 and 4 that $R/\!\!/T$ is its normalization. Since

$$\dim R/\!\!/T = 4 = \dim \mathcal{L}_3,$$

the set of forms F for which $X(F)$ admits a multipolarization of type (h_0, S_0), is dense (and open) in $\overline{(\mathrm{SL}_4)R}$.

7. Singularities

It is known (see, e.g., [11, Proposition 9]) that the forms of Q° are stable in the sense of Mumford [7]. It follows that the image of Q° under the quotient morphism $\pi : Q \to Q/\!\!/\mathrm{SL}_4$ is open and the restriction of π to Q° is a geometric quotient as a morphism onto its image: see, e.g., [9, Theorem 4.2]. For this reason, we denote this image as Q°/SL_4.

Following the lines of [11], we are to prove now that the intersection $R^\circ =: R \cap Q^\circ$ is big enough in a sense.

Proposition 5. *The complement of R°/T in $R/\!\!/T$ is of codimension ≥ 2.*

Proof. Let $R_1 \subset R$ be the 5-dimensional subspace defined by the equations $D = E = 0$. Clearly, its image $R_1/\!\!/T \subset R/\!\!/T$ is the 2-dimensional subspace defined by the equations $I_4 = I_6 = 0$.

It follows from [11, Proposition 13] (which is a slightly more general result) that any form

$$F = Ax_0^2x_2x_3 + Bx_1^3x_3 + Cx_2^4 + Gx_1x_2x_3^2 + Hx_2^2x_3^2 \in R_1$$

with $A, B, C \neq 0$ and $G \neq 0$ or $H \neq 0$ belongs to Q°. Note that all the forms $F \in R_1$ that do not satisfy this condition goes to 0 under the quotient morphism $R \to R/\!\!/T$. Thus, $R^\circ/T \supset R_1/\!\!/T \setminus \{0\}$. This implies that the complement of R°/T in $R/\!\!/T$ cannot contain divisors, since any such divisor would intersect the subspace $R_1/\!\!/T$ along a line. □

8. Main theorem

Theorem 1. *The algebra of even Siegel modular forms of genus* 2 *is freely generated by forms of weights* $4, 6, 10, 12$.

Proof. As was explained in Section 2, the algebra of even Siegel modular forms of genus 2 is isomorphic (as a graded algebra) to the algebra $A(\mathcal{D}_3, \Gamma)$ of automorphic forms on the symmetric domain $\mathcal{D}_3$ of type IV with respect to the group Γ defined by (2). As in [11, Subsection 4.4], it follows from Propositions 2, 4, 5 that this algebra is isomorphic to $\mathbb{C}[R]^T$. The structure of the latter algebra is described by Proposition 3, whence the result follows. □

9. Jacobian

Let f_4, f_6, f_{10}, f_{12} be free generators of degrees $4, 6, 10, 12$ of the algebra $A(\mathcal{D}_3, \Gamma)$. We will consider them as holomorphic functions on

$$\mathcal{L}_3 \subset \mathbb{C}^{2,3} =: T_0 \otimes \mathbb{C}.$$

Let f_2 be the scalar square on $\mathbb{C}^{2,3}$, and z_1, z_2, z_3, z_4, z_5 be the coordinates on $\mathbb{C}^{2,3}$ with respect to some basis.

Although the functions f_4, f_6, f_{10}, f_{12} are defined only on (an open subset of) the hypersurface $f_2 = 0$, one can consistently define the Jacobian

$$J = \frac{D(f_2, f_4, f_6, f_{10}, f_{12})}{D(z_1, z_2, z_3, z_4, z_5)}$$

as a holomorphic function on $\mathcal{L}_3$. Indeed, the differentials of f_4, f_6, f_{10}, f_{12} at some point of $\mathcal{L}_3$ are linear forms on the tangent space of the hypersurface $f_2 = 0$. In order to define the Jacobian, these forms should be extended to linear forms on the whole space $\mathbb{C}^{2,3}$. These extensions are not unique, but they are defined up to addition of some multiples of df_2, which does not affect the Jacobian.

Up to a scalar multiple, the Jacobian J does not depend on the choice of free generators of the algebra $A(\mathcal{D}_3, \Gamma)$. It is homogeneous of degree

$$2 - 4 - 6 - 10 - 12 - 5 = -35$$

and skew-invariant under the group Γ in the following sense:

$$\gamma J = (\det \gamma) J \quad \text{for} \quad \gamma \in \Gamma.$$

Thus, J is an automorphic form of weight 35 for the subgroup Γ_0 of Γ, and J^2 is an automorphic form of weight 70 for the very group Γ.

Proposition 6. *The null divisor of J is the sum of all the mirrors of (the reflections of) the group Γ taken with multiplicity* 1.

Proof. Since the quotient map $\pi : \mathcal{L}_3 \to \mathcal{L}_3/\Gamma$ locally at a point $z \in \mathcal{L}_3$ looks as the quotient by the stabilizer Γ_z of z in Γ, the Jacobian J vanishes at z if and only if $\Gamma_z \neq \{e\}$. Since $\mathcal{L}_3/\Gamma$ is a complex manifold (an open subset in $\mathbb{C}^4$), each stabilizer is generated by reflections. It follows that J vanishes exactly on the mirrors of Γ. If z belongs to just one mirror, the group Γ_z is generated by one reflection. It follows that the multiplicity of each mirror in the null divisor of J is 1. □

Theorem 2. *The algebra* $A(\mathcal{D}_3, \Gamma_0)$ *is generated by the forms* I_4, I_6, I_{10}, I_{12}*, and* J*, with the defining relation*

$$J^2 = P(I_4, I_6, I_{10}, I_{12}),$$

where P *is some polynomial.*

Proof. Any automorphic form f for Γ_0 decomposes as $f = f_+ + f_-$, where f_+ (resp. f_-) is a Γ-invariant (resp. Γ-skew-invariant) homogeneous holomorphic function of the same degree. The function f_+ is an automorphic form for Γ, while f_- vanishes on all the mirrors of Γ and, hence, is divisible by J, the quotient being an automorphic form for Γ. Thus, as a vector space, the algebra $A(\mathcal{D}_3, \Gamma_0)$ decomposes as

$$A(\mathcal{D}_3, \Gamma_0) = A(\mathcal{D}_3, \Gamma) \oplus A(\mathcal{D}_3, \Gamma)J,$$

whence the result follows. □

Corollary. *The algebra of all Siegel modular forms of genus* 2 *is generated by forms* $\Phi_4, \Phi_6, \Phi_{10}, \Phi_{12}, \Phi_{35}$ *of weights* $4, 6, 10, 12, 35$*, with the defining relation*

$$\Phi_{35}^2 = P(\Phi_4, \Phi_6, \Phi_{10}, \Phi_{12}).$$

The polynomial P is written down in [5, p.849].

The roots of the lattice T_0 are the vectors $e \in T_0$ with $(e, e) = -2$. One can prove that they decompose into two Γ-orbits depending on whether $e \in 2T_0^*$ or not. This means that there are two conjugacy classes of reflections in Γ, say, C_1 and C_2. The quotient map $\pi : \mathcal{L}_3 \to \mathcal{L}_3/\Gamma$ takes the mirrors of these two classes of reflections to two irreducible components of the discriminant of π given by the equation $P = 0$ in $\mathcal{L}_3/\Gamma \subset \mathbb{C}^4$. It follows that the polynomial P decomposes into a product of two irreducible polynomials, and, correspondingly, the Jacobian J decomposes into a product of two holomorphic functions, say, J_1 and J_2, so that J_1 (resp. J_2) vanishes (with multiplicity 1) exactly on the mirrors of the reflections of C_1 (resp. C_2). These functions are automorphic forms for some index 2 subgroups of Γ.

Let $e \in T_0$ be a root of the class C_1, i.e. $e \in 2T_0^*$. Then $T_0 = \mathbb{Z}e \oplus 2U$. As in [11, Section 3], one can prove that the morphism

$$\nu : \mathcal{L}_2/\mathrm{O}^+(2U) \to \mathcal{L}_3/\Gamma,$$

induced by the embedding $2U \subset T_0$, is a closed embedding. This means that the restriction of automorphic forms defines a surjective homomorphism

$$\rho : A(\mathcal{D}_3, \Gamma) \to A(\mathcal{D}_2, \mathrm{O}^+(2U)).$$

Taking into account that $\mathcal{D}_2$ is the direct product of two copies of the upper half-plane and $\mathrm{O}^+(2U)$ is an index 2 extension of the direct product of two copies of the Klein modular group, it is easy to prove that the algebra $A(\mathcal{D}_2, \mathrm{O}^+(2U))$ is freely generated by forms of weights $4, 6, 12$. It follows that the kernel of ρ is the principal ideal generated by I_{10}. This means that $J_1^2 = I_{10}$, which agrees with the explicit form of the polynomial P given in [5, p.849].

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Prof. E. B. Vinberg, Department of Mechanics and Mathematics, Moscow State University, Moscow 119992, GSP–2, Russia