Graphs associated with simplicial complexes

Alexander Grigor'yan*
Department of Mathematics
University of Bielefeld
33501 Bielefeld, Germany

Yuri Muranov†
Department of Mathematics
Grodno State University
Grodno 230023, Republic of Belarus

Shing-Tung Yau
Department of Mathematics
Harvard University
Cambridge MA 02138, USA

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Abstract

To any finite simplicial complex S we associate a digraph G_S in a canonical way and prove that the simplicial homologies of S are isomorphic to the graph homologies of G_S.

Contents

1 Introduction

The notion of a simplicial homology is one of the basic tools of algebraic topology. In a recent paper [6] the authors developed the theory of a graph homology[1], where the notions

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[1]For other approaches to graph cohomology see [1], [2], [7].

of chain and cochain complexes were defined on any finite digraph (=directed graph).

The aim of this paper is to establish a fundamental relation between these two homology theories.

Any graph G can be naturally regarded as an one-dimensional simplicial complex, so that its simplicial homologies of all dimensions $n \geq 2$ are trivial. However, as it was shown in [6] on many examples, the graph homologies of G can be highly non-trivial for any n, as this theory detects automatically higher dimensional substructures of G, for example, a graphical simplex or cube with an appropriate direction of edges.

Generally speaking, a digraph G can be turned into a simplicial complex S in many ways, by spanning on some of its cliques[2] higher dimensional simplexes, that however do not have to match the higher dimensional substructures of G that are predetermined by G.

On the other hand, any simplicial complex S determines naturally a (undirected) graph S_1 that is the 1-skeleton of S. The graph S_1 can be turned into a digraph by choosing arbitrarily directions of the edges. Simple examples show that the simplicial homologies of S and the graph homologies of S_1 can be different regardless of the choice of the digraph structure on S_1 (see Example in Section 3.3).

In this paper, for any finite simplicial complex S we construct in a canonical natural way a finite digraph G_S such that the homology groups $H_*(S)$ and $H_*(G_S)$ over a field $\mathbb{K}$ are isomorphic, where $H_*(S)$ refers to the simplicial homologies of S and $H_*(G_S)$ refers to the graph homologies of G_S.

In fact, the set of vertices of G_S coincides with the set of all simplexes from S, and two simplexes s, t are connected in G_S by a directed edge $s \to t$ if and only if

$$s \supset t \text{ and } \dim s = \dim t + 1. \tag{1.1}$$

The graph G_S can be realized geometrically as follows. Denote by b_s the barycenter of a simplex $s \in S$, and consider the set B_S of the barycenters of all $s \in S$. Define the edges $b_s \to b_t$ between two barycenters by the same rule (1.1), which makes B_S into a digraph (see Fig. 1(b)).

Furthermore, it is not difficult to see that B_S is an 1-skeleton of a natural cubical complex associated with S, that will be denoted by Q_S. More precisely, Q_S can be constructed as follows. For each simplex $s \in S$ consider a full barycentric subdivision s^b of s and for any vertex v of s take the union of all the elements of s^b containing v. This union is topological cube, and the family of all such cubes of all simplexes $s \in S$ forms a cubical complex Q_S that is a cubillage of S (cf. [4, §5]).

The complexes S and Q_S have the same topological realization, which implies that their cell homologies are the same. On the other hand, we prove in Section 5 that the cell homology chain complex of Q_S and the graph homology chain complex of G_S are isomorphic, which implies the isomorphism of $H_*(S)$ and $H_*(G_S)$.

It is worth mentioning that the assignment $S \mapsto G_S$ is a functor from the category of simplicial complexes with inclusion maps to the category of digraphs with inclusion maps. That this functor preserves the homology groups is a very strong property that will undoubtedly have further applications. In this paper we present the proof of the isomorphism of the homology groups while postponing applications to a sequel work.

In Section 2, we give necessary preliminary material about simplicial and cubical complexes and their homology properties following [3] and [9]. In particular, we discuss in

[2]A clique in a graph is a subset of its vertices such that every two vertices in the subset are connected by an (undirected) edge.

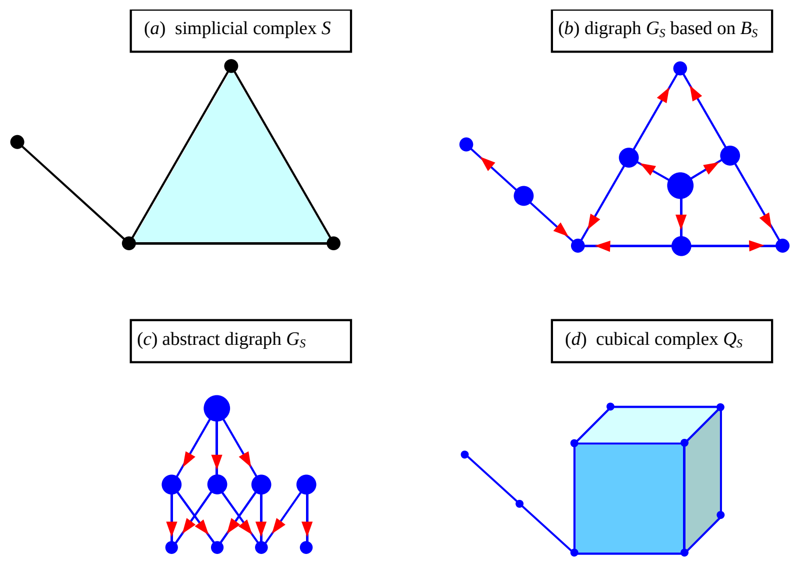

Figure 1: A simplicial complex S, the digraph G_S realized on the barycenters and abstractly, and the cubical complex Q_S

details the procedure of constructing of the cubical complex Q_S mentioned above. In Section 3 we give a brief account of the graph homology theory following [6]. In Section 4, we construct the digraph G_S and describe explicitly the associated chain complex, using specific properties of the graph G_S. Finally, in Section 5 prove the main result – Theorem 5.1. Combining with a result of [5], we obtain an isomorphism of $H^*(G_S)$ and the Hochschild cohomology of a certain algebra that is constructed using the simplexes of S. The authors thank Xueping Huang for pointing out the reference [5].

2 Simplicial and cubical complexes

In this section we state necessary material about simplicial and cubical complexes and describe the construction of a cubical complex associated with a given simplicial complex. The details can be found in [9] and [3].

By an n-dimensional simplex we mean a non-degenerate affine image of a standard simplex

$$\Delta^n = \left\{(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} : x_0 + x_1 + ... + x_n = 1,\ x_i \geq 0 \text{ for all } i = 0, ..., n\right\}$$

in some space $\mathbb{R}^N$. Recall, that a finite simplicial complex S is a finite family of simplexes in $\mathbb{R}^N$ (possibly, of various dimensions) such that the following conditions are satisfied:

1. if S contains a simplex s then S contains all the faces[3] of s;
2. if s_1, s_2 are two simplexes from S then the intersection $s_1 \cap s_2$ is either empty or a simplex from S.

[3]Contrary to a common convention, we do not regard $\emptyset$ as a face.

Let us describe a less known notion of a cubical complex. A standard n-dimensional cube I^n is defined for $n \geq 1$ by:

$$I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1,\ i = 1, \ldots, n\},$$

and for $n = 0$ by $I^0 = \{0\}$. A n-dimensional cube q is a non-degenerate piecewise linear image of I^n in some $\mathbb{R}^N$. A k-dimensional face of I^n is any of the k-cubes

$$\{(x_1, \ldots, x_n) \in I^n : x_{i_1} = \varepsilon_1, ..., x_{i_{n-k}} = \varepsilon_{n-k}\}$$

where $1 \leq i_1 < ... < i_{n-k} \leq n$ and $\varepsilon_j = 0$ or 1, and a k-dimensional face of q is the image under the same mapping $I^n \to \mathbb{R}^N$ of one of the k-dimensional faces of I^n.

A finite cubical complex Q is a finite collections of cubes in some $\mathbb{R}^N$ such that

(i) if Q contains a cube q then Q contains all the faces of q;

(ii) if q_1, q_2 are two cubes from Q then the intersection $q_1 \cap q_2$ is either empty or a cube from Q.

In this paper we will consider only finite simplicial and cubical complexes, so that the adjective "finite" will be omitted. Clearly, both simplicial and cubical complexes have an underlaying structure of a topological space and even a structure of a polyhedron. Denote by $|S|$ the union of all simplexes from a simplicial complex S and similarly by $|Q|$ – the union of all cubes from Q. Both $|S|$ and $|Q|$ will be regarded as topological spaces with the induced topology from the ambient space $\mathbb{R}^N$.

Fix a field $\mathbb{K}$. It is well known that each simplicial complex S gives rise to a chain complex $C_*(S)$ over $\mathbb{K}$ with a boundary operator ∂, and, hence, to the *simplicial homologies* $H_*(C_*(S))$. The construction of cubical homologies is not commonly known and will be outlined below. In in the essence, one obtains a cubical chain complex $C_*(Q)$ over $\mathbb{K}$ with a boundary operator ∂ and the corresponding cubical homologies $H_*(C_*(Q))$. In the both cases one has the fundamental isomorphisms of homology groups

$$H_*(C_*(S)) \cong H_*(|S|) \quad \text{and} \quad H_*(C_*(Q)) \cong H_*(|Q|) \tag{2.1}$$

where $H_*(|S|)$ and $H_*(|Q|)$ are the singular homologies of the topological spaces $|S|$ and $|Q|$, respectively.

For any simplicial complex S, we will construct an associated cubical complex Q_S with the same underlying topological space, that is,

$$|S| = |Q_S|. \tag{2.2}$$

Denote by S^b the barycentric subdivision of S that is defined as follows. For any simplex $s \in S$ let us connect its barycenter by segments to the barycenters of all the faces of s thus dividing s into a collection s^b of smaller simplexes of the same dimension. Then set $S^b = \bigcup_{s \in S} s^b$. It is easy to see that S^b is also a simplicial complex, and $|S| = |S^b|$. Now for any k-simplex $s \in S$ and a vertex v of s define a set $q_{s,v}$ by

$$q_{s,v} = \bigcup_{\{t \in s^b:\ v \in t\}} t,$$

that is, $q_{s,v}$ is the union of all simplexes from s^b that contain the vertex v. It is not difficult to see that $q_{s,v}$ is a k-cube (see [4] and [10] for the details). It is also clear that s is the union of all the cubes $q_{s,v}$ over all vertices v of s (cf. Fig. 2).

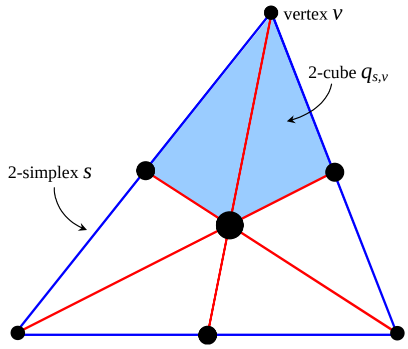

Figure 2: Construction of a cube $q_{s,v}$

The collection of all cubes $\{q_{s,v}\}$ over all $s \in S$ and $v \in s$ is then a cubical complex that will be denoted by Q_S. It is clear from the construction that it satisfies (2.2), which implies by (2.1) that

$$H_*\left(C_*\left(S\right)\right) \cong H_*\left(C_*\left(Q_S\right)\right). \tag{2.3}$$

By construction, the set of vertices of Q_S coincides with the set B_S of the barycenters of all simplexes of S. The one-dimensional skeleton of the cubical complex Q_S can be described as follows. Given two simplexes s, t of S, let us connect their barycenters b_s and b_t by a segment $[b_s, b_t]$ if and only if $s = t \cup \{v\}$ for some vertex $v \notin t$. Then the one-dimensional skeleton of Q_S is given by the union of all such segments $[b_s, b_t]$ (cf. Fig. 1).

Now we briefly describe (to the extend that we need in the proof) construction of homology groups over a field $\mathbb{K}$ of a cubical complex Q that is a particular case of homology groups of cell complexes. An orientation in $\mathbb{R}^n$ is one of the two equivalence classes of the basis in $\mathbb{R}^n$, where two basis $\vec{\mathbf{f}} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ and $\vec{\mathbf{g}} = \{\mathbf{g}_1, \ldots, \mathbf{g}_n\}$ are called equivalent if the matrix A of transformation $A\vec{\mathbf{f}} = \vec{\mathbf{g}}$ has a positive determinant. The orientation that is determined by a basis $\vec{\mathbf{f}}$ will be denoted by $[\vec{\mathbf{f}}]$. An orientation of a cubical complex Q is determined by an arbitrary choice of orientations of all constituent cubes of Q. Let D be an arbitrary n-cube from Q. Let $\varphi : D \to I^n$ be a piecewise linear mapping that exists by definition of a n-cube, and $[\vec{\mathbf{f}}] = [\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}]$ be an orientation of $\mathbb{R}^n$. Then the pair $(\varphi, [\vec{\mathbf{f}}])$ determines an orientation of D.

Let D' be a $(n-1)$-dimensional face of D, and let its orientation be given by a pair $(\psi, [\vec{\mathbf{g}}])$ where $\psi : D' \to I^{n-1}$ is a piecewise linear mapping and $\vec{\mathbf{g}} = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{n-1}\}$ is a basis of $\mathbb{R}^{n-1}$. Let us identify I^{n-1} with a face of I^n by means of the following through map:

$$\Phi : I^{n-1} \overset{\psi^{-1}}{\to} D' \overset{\text{inclusion}}{\to} D \overset{\varphi}{\to} I^n.$$

Considering I^{n-1} as a face of I^n, denote by $\mathbf{g}_0$ the outer normal unit vector to I^{n-1}. The map Φ induces the orientation

$$[\{\mathbf{g}_0, d\Phi\left(\mathbf{g}_1\right), d\Phi\left(\mathbf{g}_2\right), ..., d\Phi\left(\mathbf{g}_{n-1}\right)\}]$$

of $\mathbb{R}^n$. If this orientation is the same as $[\vec{\mathbf{f}}]$ then we set $\mathcal{O}\left(D, D'\right) = 1$,and if it is different, then set $\mathcal{O}\left(D, D'\right) = -1$. We refer to $\mathcal{O}\left(D, D'\right)$ as the *relative orientation* of D' in D.

In order to define the homology groups of a cubical complex Q, fix first an orientation of every cube of the complex, where we do not assume that the orientations of the cubes are agreed in any way. For any $n \geq 0$, let $C_n(Q)$ be the space of n-chains of Q, that is, the $\mathbb{K}$-linear space formed by all formal linear combinations of all n-dimensional cubes of Q. Also set $C_{-1}(Q) = \{0\}$. Define for any $n \geq 1$ the boundary map

$$\partial\colon C_n(Q) \to C_{n-1}(Q)$$

first for any n-cube $D \in Q$ by

$$\partial D = \sum_{D'} \mathcal{O}(D, D') D', \tag{2.4}$$

where the sum is taken over all $(n-1)$-faces D' of D, and then extend ∂ to all elements of $C_n(Q)$ by linearity. For $n = 0$ set by definition $\partial D = 0$.

Then one proves that $\partial^2 = 0$ so that $C_*(Q) = \{C_n(Q)\}$ with the boundary maps ∂ is a chain complex, and the homology groups of the chain complex $C_*(Q)$ are isomorphic to $H_*(|Q|)$ (see [9, §1.5.2]).

3 Homologies of digraphs

In this section we cite a necessary material from [6]. As before, $\mathbb{K}$ is a fixed field.

3.1 Forms and paths on finite sets

Let V be a finite set, whose elements will be called vertices. A *p-form* on V is a $\mathbb{K}$-valued function on V^{p+1}. For example, 0-forms are just functions on V, 1-forms are functions on $V \times V$, etc. The set of all p-forms is a linear space over $\mathbb{K}$ that is denoted by $\Lambda^p(V)$ or simply by Λ^p.

Denote by $e^{i_0\ldots i_p}$ the p-form that takes value 1 at the point $(i_0, i_1, \ldots, i_p)$ and 0 at all other points. Let us refer to $e^{i_0\ldots i_p}$ as an *elementary* p-form. The family $\left\{e^{i_0\ldots i_p}\right\}$ of all elementary p-forms forms a basis in Λ^p and, for any $\omega \in \Lambda^p$, we have an expansion

$$\omega = \sum_{i_0,\ldots,i_p \in V} \omega_{i_0\ldots i_p} e^{i_0\ldots i_p}$$

where $\omega_{i_0\ldots i_p} = \omega(i_0, \ldots, i_p)$.

Define *the exterior derivative* $d : \Lambda^p \to \Lambda^{p+1}$ by

$$(d\omega)_{i_0\ldots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0\ldots\widehat{i_q}\ldots i_{p+1}}, \tag{3.1}$$

where $\omega \in \Lambda^p$ and the hat $\widehat{i_q}$ means omission of the index i_q. For example, for a function $f \in \Lambda^0$ we have

$$(df)_{ij} = f_j - f_i,$$

and for 1-form $\omega \in \Lambda^1$

$$(d\omega)_{ijk} = \omega_{jk} - \omega_{ik} + \omega_{ij}.$$

It follows from (3.1) that

$$de^{i_0\ldots i_p} = \sum_{k \in V} \sum_{q=0}^{p+1} (-1)^q e^{i_0 i_1\ldots i_{q-1} k i_q \ldots i_p}.$$

For example, we have

$$\begin{aligned} de^i &= \sum_{k\in V}\left(e^{ki}-e^{ik}\right), \\ de^{ij} &= \sum_{k\in V}\left(e^{kij}-e^{ikj}+e^{ijk}\right). \end{aligned}$$

An easy calculation shows that, for any $p\geq 0$ and all $\omega\in\Lambda^p$,

$$d^2\omega = 0.$$

An *elementary* p*-path* on a finite set V is any (ordered) sequence $i_0,...,i_p$ of $p+1$ vertices of V that will be denoted by $i_0...i_p$ or by $e_{i_0...i_p}$. Denote by $\Lambda_p=\Lambda_p(V)$ the linear space of all formal linear combination of all elementary p-paths $e_{i_0...i_p}$ with coefficients from $\mathbb{K}$. The elements of Λ_p are called p*-paths.* By definition, each p-path $v\in\Lambda_p$ has the form

$$v=\sum_{i_0,...,i_p\in V} v^{i_0i_1...i_p}e_{i_0i_1...i_p},$$

where $v^{i_0i_1...i_p}$ are the coefficients of v. For example, 0-paths are linear combinations of the vertices e_i:

$$v=\sum_{i\in V}v^ie_i,$$

and 1-paths are linear combinations of pairs of vertices e_{ij} :

$$v=\sum_{i,j\in V}v^{ij}e_{ij}.$$

For any p-form $\omega\in\Lambda^p$ and p-path $v\in\Lambda_p$ there is a natural pairing

$$(\omega,v):=\sum_{i_0,...,i_p\in V}\omega_{i_0...i_p}v^{i_0...i_p},$$

which implies, in particular, that the spaces Λ^p and Λ_p are dual.

The operator $d:\Lambda^p\to\Lambda^{p+1}$ has then the dual *boundary* operator $\partial:\Lambda_{p+1}\to\Lambda_p$ that is given by

$$\partial e_{i_0...i_{p+1}}=\sum_{q=0}^{p+1}(-1)^q\, e_{i_0...\widehat{i_q}...i_{p+1}}. \tag{3.2}$$

For example,

$$\begin{aligned} \partial e_{ij} &= e_j-e_i \\ \partial e_{ijk} &= e_{jk}-e_{ik}+e_{ij}. \end{aligned}$$

It follows from (3.2) that, for any $v\in\Lambda_{p+1}$,

$$(\partial v)^{i_0...i_p}=\sum_{k\in V}\sum_{q=0}^{p+1}(-1)^q\, v^{i_0...i_{q-1}ki_q...i_p}$$

This formula holds for all $p\geq 0$. We need also the operator $\partial:\Lambda_0\to\Lambda_{-1}$ where we set $\Lambda_{-1}=\{0\}$ and $\partial v=0$ for all $v\in\Lambda_0$.

If v is an 1-path, then ∂v is given by

$$(\partial v)^i = \sum_{k \in V} \left(v^{ki} - v^{ik} \right).$$

If v is a 2-path then

$$(\partial v)^{ij} = \sum_{k \in V} \left(v^{kij} - v^{ikj} + v^{ijk} \right).$$

By duality, we have $(d\omega, v) = (\omega, \partial v)$ for any $\omega \in \Lambda^{p-1}$ and any $v \in \Lambda_p$. It follows that, for any p-path v,

$$\partial^2 v = 0.$$

An elementary p-path $e_{i_0 \ldots i_p}$ (the same is $i_0 \ldots i_p$) is called *regular* if $i_k \neq i_{k+1}$ for all k. We would like to define the boundary operator ∂ on the subspace of Λ_p spanned by regular elementary paths. Just restriction of ∂ does not work as ∂ is not invariant on this subspace.

Let I_p be the subspace of Λ_p that is spanned by all irregular $e_{i_0 \ldots i_p}$. Consider the quotient space

$$\mathcal{R}_p = \mathcal{R}_p(V) = \Lambda_p / I_p.$$

The elements of $\mathcal{R}_p$ are the equivalence classes $v \bmod I_p$ where $v \in \Lambda_p$, and they are called *regularized* p-paths. One verifies that the boundary operator ∂ is well-defined for regularized paths. Clearly, $\mathcal{R}_p$ is linearly isomorphic to the space of regular p-paths:

$$\operatorname{span}\left\{ e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is regular} \right\}.$$

For simplicity of notation, we will identify $\mathcal{R}_p$ with this space, by setting all irregular p-paths to be equal to 0.

3.2 Forms and paths on digraphs

A digraph is a pair $G = (V, E)$ where V is an arbitrary set and E is a subset of $V \times V \setminus \mathrm{diag}$. In this paper the set V will be always assumed non-empty and finite. The elements of V are called *vertices* and the elements of E are called (*directed*) *edges*.

The edge starting at a vertex a and ending at b will be denoted by ab. The fact that there exists an edge starting at a and ending at b will be denoted by $a \to b$.

Let $i_0 \ldots i_p$ be a regular elementary p-path on V. It is called *allowed* if $i_{k-1} \to i_k$ for any $k = 1, \ldots, p$, and *non-allowed* otherwise. We say that an elementary p-form $e^{i_0 \ldots i_p}$ is allowed if $i_0 \ldots i_p$ is allowed, and non-allowed if $i_0 \ldots i_p$ is non-allowed.

We would like to reduce the space $\mathcal{R}_p$ of regular p-paths on V to adapt it to the digraph structure G. Denote by $\mathcal{A}_p = \mathcal{A}_p(G)$ the subspace of $\mathcal{R}_p$ spanned by the allowed elementary p-paths, that is,

$$\mathcal{A}_p = \operatorname{span}\left\{ e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is allowed} \right\}.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Note that $\mathcal{A}_0$ consists of linear combination of vertices, and $\mathcal{A}_1$ consists of linear combinations of the edges.

In general, the spaces $\mathcal{A}_p$ are not invariant for operator ∂. For example, if ab and bc are edges then $e_{abc} \in \mathcal{A}_2$ while

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab}$$

is non-allowed if ac is not an edge.

Consider the following subspace of $\mathcal{A}_p$

$$\Omega_p = \Omega_p(G) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}. \tag{3.3}$$

Then the family $\{\Omega_p\}$ is ∂-invariant. Indeed, if $v \in \Omega_p$ then $\partial v \in \mathcal{A}_{p-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{p-2}$ whence $\partial v \in \Omega_{p-1}$. The elements of Ω_p are called *∂-invariant* p-paths.

We obtain a chain complex

$$0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots \tag{3.4}$$

and the notion of homology groups of the digraph G:

$$H_p(G) := \ker \partial|_{\Omega_p} \big/ \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

Let $G' = (V', E')$ be a subgraph of G, that is, V' is a subset of V and

$$E' = \left\{ab \in E : a, b \in V'\right\}. \tag{3.5}$$

It is frequently useful to know that any ∂-invariant path v in G' is also ∂-invariant in G. Indeed, any allowed path in G' is allowed in G by (3.5). Denoting by ∂' the boundary operator in G', let us verify that $\partial' v = \partial v$. Indeed, it follows from (3.2) that, for an elementary p-path $e_{i_0 \dots i_p}$ in G', both $\partial' e_{i_0 \dots i_p}$ and $\partial e_{i_0 \dots i_p}$ are determined by the $(p-1)$-paths $e_{i_0 \dots \widehat{i_q} \dots i_{p+1}}$ that are the same in G' and G. Hence, $\partial' v = \partial v$ follows, which by (3.3) implies that v is ∂-invariant in G.

Now we would like to reduce the space $\mathcal{R}^p$ of regular p-forms on V according to the digraph structure. Denote by $\mathcal{A}^p = \mathcal{A}^p(G)$ the subspace of $\mathcal{R}^p$, spanned by the allowed elementary p-forms:

$$\mathcal{A}^p = \operatorname{span}\left\{e^{i_0 \dots i_p} : i_0 \dots i_p \text{ is allowed}\right\},$$

and by $\mathcal{N}^p = \mathcal{N}^p(G)$ the subspace of $\mathcal{R}^p$, spanned by the non-allowed elementary p-forms:

$$\mathcal{N}^p = \operatorname{span}\left\{e^{i_0 \dots i_p} : i_0 \dots i_p \text{ is non-allowed}\right\}.$$

Consider the following subspace of $\mathcal{R}^p$:

$$\mathcal{J}^p = \mathcal{J}^p(G) = \mathcal{N}^p + d\mathcal{N}^{p-1}, \tag{3.6}$$

and observe that $d\mathcal{J}^p \subset \mathcal{J}^{p+1}$. Hence, the operator d is well-defined on the quotient spaces

$$\Omega^p = \Omega^p(G) = \mathcal{R}^p / \mathcal{J}^p.$$

The elements of Ω^p are called d-invariant p-forms. In other words, the elements of Ω^p are the equivalence classes of regular p-forms under the following equivalence relations:

$$\omega_1 \simeq \omega_2 \Leftrightarrow \omega_1 - \omega_2 \in \mathcal{J}^p. \tag{3.7}$$

Using (3.6), we can rewrite the definition of $\simeq$ more explicitly as follows:

$$\omega_1 \simeq \omega_2 \Leftrightarrow \omega_1 - \omega_2 = \varphi + d\psi \text{ for some } \varphi \in \mathcal{N}^p, \psi \in \mathcal{N}^{p-1}. \tag{3.8}$$

Since $\mathcal{R}^p = \mathcal{A}^p \oplus \mathcal{N}^p$, every equivalence class contains a representative from $\mathcal{A}^p$, so that Ω^p is the space of equivalence classes of allowed p-forms.

We obtain a cochain complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \ldots \tag{3.9}$$

which allows us to define the cohomologies of the digraph G by

$$H^p(G) := \ker d|_{\Omega^p} / \operatorname{Im} d|_{\Omega^{p-1}} .$$

It is possible to show that the spaces $\Omega^p(G)$ and $\Omega_p(G)$ are dual (in particular, their dimensions are the same), and so are the operators d and ∂. Therefore, the cochain complex (3.9) and the chain complex (3.4) are dual, and so are the $\mathbb{K}$-linear spaces $H^p(G)$ and $H_p(G)$. We will refer to $H^p(G)$ and $H_p(G)$ as the *graph* (co)homologies, in order to distinguish from other theories of (co)homologies.

3.3 Some examples

Let $G = (V, E)$ be a finite digraph as before. The space Ω_0 has always the basis $\{e_a\}_{a\in V}$ and Ω_1 has the basis $\{e_{ab}\}_{ab\in E}$. Let us give examples of ∂-invariant paths in Ω_n with $n \geq 2$.

Example 3.1 Let us call by a *triangle* a sequence $\{a, b, c\}$ of three distinct vertices a, b, c of G such that ab, bc, ac are edges:

$$\begin{array}{ccc} {}^a\bullet & \rightarrow & \bullet^c \\ & \searrow \ \nearrow & \\ & \underset{b}{\bullet} & \end{array} . \tag{3.10}$$

The triangle determines a 2-path $e_{abc} \in \Omega_2$ as $e_{abc} \in \mathcal{A}_2$ and

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1.$$

More generally, a graphical n-simplex is a sequence $\{a_k\}_{k=0}^n$ of $n+1$ distinct vertices from V such that $a_i \to a_j$ for all $i < j$. Then $e_{a_0\ldots a_n}$ and $\partial e_{a_0\ldots a_n}$ are allowed so that the n-path $e_{a_0\ldots a_n}$ is ∂-invariant. One can say that this n-path determines the simplex.

Example 3.2 Let us called by a *square* a sequence $\{a, b, b', c\}$ of four distinct vertices $a, b, b', c \in V$ such that $ab, bc, ab', b'c$ are edges:

$$\begin{array}{ccc} {}^{b'}\bullet & \longrightarrow & \bullet^c \\ \uparrow & & \uparrow \\ {}^a\bullet & \longrightarrow & \bullet^b \end{array}$$

The square determines a 2-path

$$v = e_{abc} - e_{ab'c} \in \Omega_2$$

as $v \in \mathcal{A}_2$ and

$$\begin{aligned} \partial v &= (e_{bc} - e_{ac} + e_{ab}) - (e_{b'c} - e_{ac} + e_{ab'}) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1. \end{aligned}$$

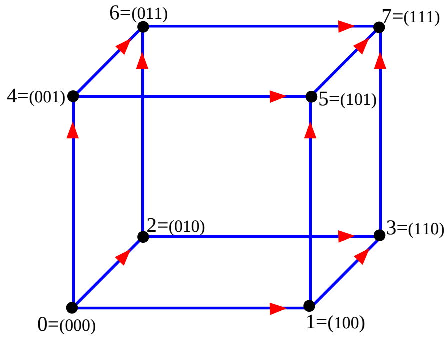

Figure 3: A graphical 3-cube. The binary representations of the vertices are shown in brackets.

Example 3.3 More generally, a graphical n-cube is a set C of 2^n vertices of V that any vertex $\alpha \in C$ can be identified with a sequences $(\alpha_1...\alpha_n)$ of binary digits so that $\alpha \to \beta$ if and only if the sequence $(\beta_1...\beta_n)$ is obtained from $(\alpha_1...\alpha_n)$ by replacing a digit 0 by 1 at exactly one position. The digraph $\bullet \to \bullet$ is an 1-cube, a square is a 2-cube, and a 3-cube is shown on Fig. 3.

With any graphical n-cube one can associate a ∂-invariant n-path as it was shown in [6, Example 6.7] (cf. Section 4.2 below). For example, for 3-cube as on Fig. 3 this is

$$v = e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}.$$

It is easy to see that

$$\begin{aligned} \partial v &= (e_{457} - e_{467}) - (-e_{013} - e_{023}) + (e_{015} - e_{045}) \\ &\quad - (e_{237} - e_{267}) + (e_{137} - e_{157}) - (e_{026} - e_{046}) . \end{aligned}$$

In other words, ∂v is an alternating sum of six 2-paths each of them corresponding to a geometric face of the cube. This observation will be put in a general context in Section 4.2, and it is a key to the proof of our main Theorem 5.1.

Example 3.4 It is clear that the ∂-invariant 2-paths associated to different triangles, are linearly independent. Let us give an example showing that the ∂-invariant 2-paths associated to different squares can form a linear dependence. Consider the digraph on Fig. 4.

It has three squares $\{0, 1, 2, 4\}$, $\{0, 1, 3, 4\}$, $\{0, 2, 3, 4\}$ that give rise to the following three ∂-invariant 2-paths

$$e_{014} - e_{024}, \; e_{014} - e_{034}, \; e_{024} - e_{034},$$

that are obviously linearly dependent. It is possible to show that in this case $\dim \Omega_2 = 2$ (cf. [6, Proposition 5.2]).

Example 3.5 Consider a (undirected) graph G on Fig. 5 with 6 vertices and 12 edges.

As an one-dimensional simplicial complex, G has simplicial homologies $H_* (C_* (G))$. On the other hand, let us introduce arbitrarily a set D of directions on the edges of G, so

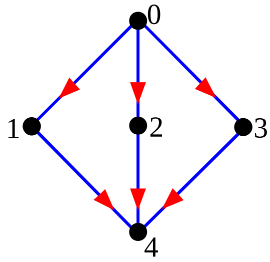

Figure 4: A digraph with linearly dependent squares

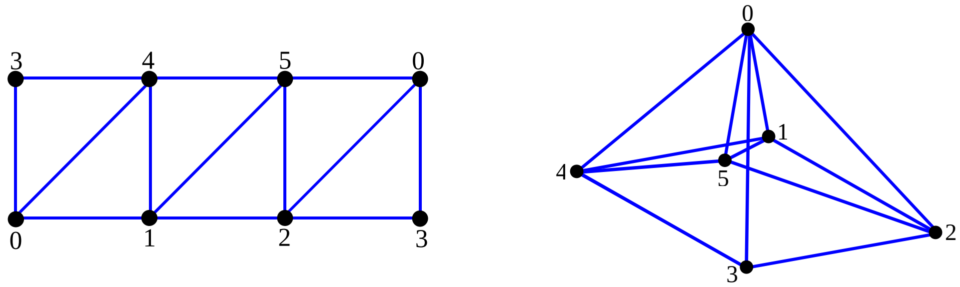

Figure 5: Graph G in two representations: embedded on the Möbius band (left) and in $\mathbb{R}^3$ (right). On the left picture the vertices with the same number are merged.

that (G, D) is a digraph and, hence, has the graph homologies $H_*(G, D)$. We claim that for, any choice of D,

$$H_1(C_*(G)) \neq H_1(G, D).$$

Indeed, it is easy to see that G as an 1-dimensional simplicial complex is homotopy equivalent to a wedge of seven circles $\mathbb{S}^1$, whence by the homotopy invariance of simplicial homologies we obtain

$$\dim H_1(C_*(G)) = 7.$$

It remains to verify that, for any choice D of the edge directions,

$$\dim H_1(G, D) \leq 6.$$

As above let $\{\Omega_n\}$ be the chain complex of the digraph (G, D). In particular, $\dim \Omega_0 = 6$ that is the number of vertices, and $\dim \Omega_1 = 12$ that is the number of edges. We have the following universal identity

$$\dim H_1(\Omega) - \dim H_0(\Omega) = \dim \Omega_1 - \dim \Omega_0 - \dim \partial\Omega_2$$

that follows from [6, Lemma 3.4]. On the other hand, $\dim H_0(\Omega) = 1$ as the graph G is connected (cf. [6, Proposition 4.2]). Therefore, we obtain

$$\dim H_1(\Omega) = 7 - \dim \partial\Omega_2.$$

It remains to show that the space $\partial\Omega_2$ is non-trivial. For that it suffices to verify that there is a triangle $\{a, b, c\}$ in (G, D) in the sense of Example 3.1 since then $e_{abc} \in \Omega_2$

and $\partial e_{abc} \neq 0$. Indeed, let us try to define directions D on the edges so that (G, D) contains no triangles. Then any undirected triangle in G must become a cycle $\begin{array}{ccc}\bullet & \longleftarrow & \bullet \\ & \searrow \ \ \nearrow & \\ & \bullet & \end{array}$ or $\begin{array}{ccc}\bullet & \longrightarrow & \bullet \\ & \nwarrow \ \ \swarrow & \\ & \bullet & \end{array}$ rather than a triangle (3.10).

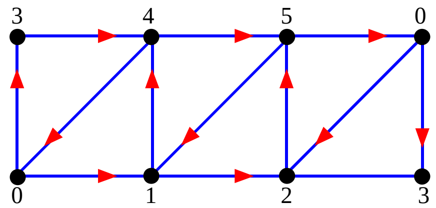

Figure 6: An attempt to introduce on G the direction of edges. Any direction of the edge 23 will create a triangle.

Given a direction of the edge 03, this requirement determines uniquely the directions of all other edges (cf. Fig. 6), up to the edge 23. However, with any direction on 23 the sequence $\{0, 2, 3\}$ will become a triangle, which finishes the proof.

4 Digraphs associated with simplicial complexes

4.1 Cubical graphs

Let M be a finite set with m elements. Let us introduce in the power set 2^M of M the structure of a digraph as follows: for arbitrary two sets $s_1, s_2 \in 2^M$ define the edge between them by the rule

$$s_1 \to s_2 \Leftrightarrow s_2 \text{ is obtained from } s_1 \text{ by removing of exactly one element.} \tag{4.1}$$

Denote this digraph by G_M. Let us fix an enumeration of the elements of M by integers $0, 1, ..., m-1$, in fact, identify M with the set $\{0, 1, ..., m-1\}$. For any set $s \in 2^M$ define its *anti-indicator* $N(s)$ by

$$N(s) = \sum_{i \in M \setminus s} 2^i.$$

For example, $N(\emptyset) = 2^m - 1$ and $N(M) = 0$. Clearly, if $s_1 \to s_2$ then

$$N(s_2) = N(s_1) + 2^i \tag{4.2}$$

where i is the unique element in $s_1 \setminus s_2$.

Let S be a family of subsets of M, that is, $S \subset 2^M$. Denote by $G_{S,M}$ the digraph with the vertex set S, whose edges are all the edges from G_M with the endpoints in S. If no confusion arrises, we write shortly G_S instead of $G_{S,M}$.

Definition 4.1 The digraph G_S is called *cubical* if the family $S \subset 2^M$ possesses the following property: if s, t are two elements of S then any subset u of M such that $s \subset u \subset t$, is also an element of S.

For example, the full digraph G_M is a cubical graph. The reason for the term "cubical" is that G_M is, in fact, a graphical m-cube. Indeed, with each element $s \in 2^M$ consider $N(s)$ as a binary number, which provides an one-to-one correspondence between 2^M and the sequences of m binary digits. Moreover, $s_1 \to s_2$ means by (4.2) that $N(s_2)$ is obtained from $N(s_1)$ by replacing one binary digit 0 by 1. Hence, G_M is a graphical m-cube (cf. Fig. 7). In fact, G_M is nothing other than the inverted Hasse diagram of the partially ordered set 2^M.

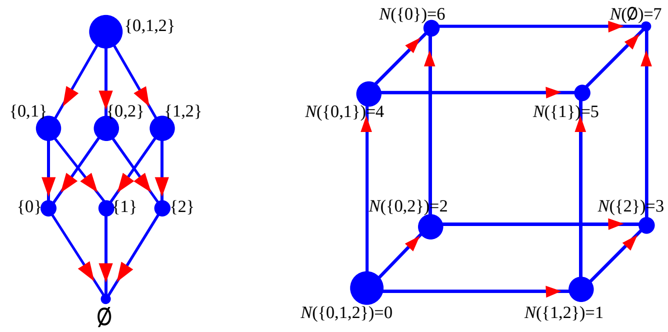

Figure 7: The cubical graph G_M for $M = \{0, 1, 2\}$ drawn in two ways. On the right picture each vertex s is assigned the number $N(s)$.

Example 4.2 With any simplicial complex S we associate a cubical digraph as follows. Denote by M the set of all vertices of S (with a fixed enumeration as above). Then any k-simplex in S can be regarded as a $(k+1)$-subset of M, and S can be regarded as a subset of 2^M. By the above construction, we obtain a digraph G_S. It satisfies the definition of a cubical graph because by definition of a simplicial complex, if a subset s of M is a simplex from S then any non-empty subset s' of s is also a simplex of S.

Equivalently, one can describe the graph G_S of a simplicial complex S as follows. The set of vertices of G_S coincides with the set of all simplexes from of S. The edges in G_S are defined by (4.1) or, equivalently, by

$$s_1 \to s_2 \Leftrightarrow s_1 \supset s_2 \text{ and } \dim s_1 = \dim s_2 + 1, \tag{4.3}$$

where $s_1 . s_2$ are simplexes from S (see Fig. 1 in Introduction).

In this section we prove certain properties of general cubical digraphs that will be applied in the proof of Theorem 5.1 to special cubical digraphs that arise from simplicial complexes.

4.2 ∂-invariant paths associated with cubes

Fix a set $M = \{0, 1, ..., m-1\}$ as above, and consider the digraph G_M. Let $\{\alpha_k\}_{k=0}^n$ be an allowed path in G_M, that is, $\alpha_{k-1} \to \alpha_k$ for all $k = 1, ..., n$. Define a non-negative integer $\sigma(\alpha)$ as follows. Since $\alpha_{k-1} \to \alpha_k$, there is a unique value $i_k \in \{0, 1, ..., m-1\}$ such that

$$\alpha_{k-1} \setminus \alpha_k = \{i_k\},$$

or, equivalently,

$$N(\alpha_k) = N(\alpha_{k-1}) + 2^{i_k}. \tag{4.4}$$

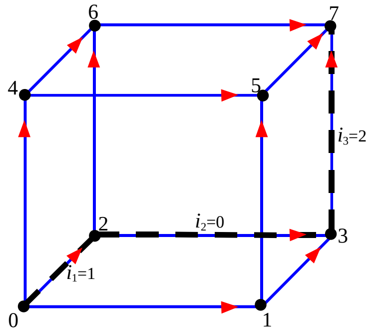

Figure 8: For the path $\alpha = 0237$, the sequence $\{i_1, i_2, i_3\}$ is $\{1, 0, 2\}$, and it has one inversion. Hence, $\sigma(\alpha) = 1$.

Then define $\sigma(\alpha)$ as the number of inversions in the sequence $\{i_1, ..., i_n\}$ (cf. Fig. 8).

Lemma 4.3 *Let $\alpha = \{\alpha_k\}_{k=0}^n$ be an allowed path in G_M.*

(a) *Denote by α' the truncated sequence $\{\alpha_k\}_{k=1}^n$ so that α' is an allowed path. Then the difference $\sigma(\alpha) - \sigma(\alpha')$ depends only on $\alpha_0, \alpha_1, \alpha_n$.*

(b) *Denote by α' the truncated sequence $\{\alpha_k\}_{k=0}^{n-1}$ so that α' is an allowed path. Then the difference $\sigma(\alpha) - \sigma(\alpha')$ depends only on $\alpha_0, \alpha_{n-1}, \alpha_n$*

Proof. Indeed, let i_k be as in (4.4). Then $\sigma(\alpha)$ is the number of inversions in the sequence $\{i_1, i_2, ..., i_n\}$ while $\sigma(\alpha')$ is the number of inversions in the sequence $\{i_2, i_3, ..., i_n\}$. Therefore, the difference $\sigma(\alpha) - \sigma(\alpha')$ is the number of inversions of i_1 in $\{i_1, i_2, ..., i_n\}$, that is, the number of the values $i_2, ..., i_n$ that are smaller than i_1. Since by (4.4)

$$N(\alpha_n) - N(\alpha_1) = 2^{i_2} + 2^{i_3} + ... + 2^{i_n},$$

and all i_k are different, the values of $i_2, ..., i_n$ (but not the order) are uniquely determined by $N(\alpha_n) - N(\alpha_1)$. Since i_1 is determined by $N(\alpha_1) - N(\alpha_0)$, the number of the values $i_2, ..., i_n$ that are smaller than i_1 is determined by $N(\alpha_n) - N(\alpha_1)$ and $N(\alpha_1) - N(\alpha_0)$, which finishes the proof of (a). Part (b) is proved similarly. ∎

For any two subsets s, t of M, such that $t \subset s$, denote by $D_{s,t}$ the family of all subsets $u \subset M$ such that $t \subset u \subset s$. We consider $D_{s,t}$ as a digraph with the edges as in (4.1). Clearly, $D_{s,t}$ is a subgraph of G_M and $D_{s,t}$ is isomorphic to the full digraph $G_{s \setminus t}$ so that $D_{s,t}$ is a graphical n-cube, where $n = |s| - |t|$. Note that if $S \subset 2^M$ satisfies the property of Definition 4.1 and s, t are two elements of S such that $t \subset s$ then $D_{s,t}$ is a subgraph of S.

For any n-cube $D_{s,t} \subset G_M$ denote by $P(D_{s,t})$ the set of all allowed paths $\{\alpha_k\}_{k=0}^n$ such that $\alpha_0 = s$ and $\alpha_n = t$. Then $t \subset \alpha_k \subset s$ for any k, so that all α_k belong to $D_{s,t}$. Any path $\alpha \in P(D_{s,t})$ is called a *full chains* in $D_{s,t}$. With each n-cube $D = D_{s,t}$ let us associate a n-path $\omega = \omega(D)$ by

$$\omega(D) = \sum_{\alpha \in P(D)} (-1)^{\sigma(\alpha)} e_\alpha. \tag{4.5}$$

Since each n-path $e_\alpha = e_{\alpha_0 \ldots \alpha_n}$ is allowed in D, the n-path $\omega(D)$ is also allowed. We will show below that $\omega(D)$ is, in fact, is ∂-invariant in D.

Let $D = D_{s,t}$ be an n-cube in G_M. For any $(n-1)$-cube $D' \subset D$ define the number $\sigma(D, D')$ as follows. For D' there are two possibilities:

1. either $D' = D_{s',t}$ where $s \to s'$,
2. or $D' = D_{s,t'}$ where $t' \to t$.

In the first case consider any full chain $\alpha \in P(D)$ with $\alpha_1 = s'$ and set $\alpha' = \{\alpha_k\}_{k=1}^n$ so that $\alpha' \in P(D')$. Then define

$$\sigma\left(D, D'\right) = \sigma(\alpha) - \sigma\left(\alpha'\right). \tag{4.6}$$

In the second case consider a full chain $\alpha \in P(D)$ with $\alpha_{n-1} = t'$ and set $\alpha' = \{\alpha_k\}_{k=0}^{n-1}$ so that $\alpha' \in P(D')$. Then define

$$\sigma\left(D, D'\right) = (-1)^n \left(\sigma(\alpha) - \sigma\left(\alpha'\right)\right). \tag{4.7}$$

Note that by Lemma 4.3 the value of $\sigma(D, D')$ in the both cases does not depend on the choice of α: in the first case $\sigma(D, D')$ depends on s, s', t, in the second case – on s, t', t.

Lemma 4.4 *For any n-cube D in G_M we have*

$$\partial\omega(D) = \sum_{D' \subset D} (-1)^{\sigma(D,D')} \omega\left(D'\right) \tag{4.8}$$

where the sum is taken over all $(n-1)$-cubes $D' \subset D$. Consequently, $\omega(D)$ is a ∂-invariant path in the digraph D.

Proof. We have

$$\begin{aligned}
\partial\omega &= \sum_\alpha (-1)^{\sigma(\alpha)} \partial e_{\alpha_0\alpha_1\ldots\alpha_n} \\
&= \sum_\alpha (-1)^{\sigma(\alpha)} \sum_{k=0}^n (-1)^k e_{\alpha_0\ldots\widehat{\alpha_k}\ldots\alpha_n} \\
&= \sum_\alpha (-1)^{\sigma(\alpha)} e_{\alpha_1\ldots\alpha_n} + (-1)^n \sum_\alpha (-1)^{\sigma(\alpha)} e_{\alpha_0\ldots\alpha_{n-1}} \\
&\quad + \sum_{k=1}^{n-1} (-1)^k \sum_\alpha (-1)^{\sigma(\alpha)} e_{\alpha_0\ldots\widehat{\alpha_k}\ldots\alpha_n}.
\end{aligned}$$

Observe that for any $k = 1, \ldots, n-1$

$$\sum_\alpha (-1)^{\sigma(\alpha)} e_{\alpha_0\ldots\widehat{\alpha_k}\ldots\alpha_n} = 0.$$

Indeed, it suffices to show that

$$\sum_{\alpha_k} (-1)^{\sigma(\alpha)} e_{\alpha_0\ldots\widehat{\alpha_k}\ldots\alpha_n} = 0.$$

Since α_{k-1} and α_{k+1} are fixed, for α_k there are only two possibilities, and $\sigma(\alpha)$ for these two possibilities have different parity, so that the term $e_{\alpha_0...\widehat{\alpha_k}...\alpha_n}$ cancel out.

Denoting by s' any successor of s and by t' any predecessor of t, we obtain

$$\begin{aligned}\partial\omega &= \sum_{\alpha}(-1)^{\sigma(\alpha)}e_{\alpha_1...\alpha_n}+(-1)^n\sum_{\alpha}(-1)^{\sigma(\alpha)}e_{\alpha_0...\alpha_{n-1}}\\ &= \sum_{s'}\sum_{\alpha:\alpha_1=s'}(-1)^{\sigma(\alpha)}e_{\alpha_1...\alpha_n}+(-1)^n\sum_{t'}\sum_{\alpha_{n-1}=t'}(-1)^{\sigma(\alpha)}e_{\alpha_0...\alpha_{n-1}}.\end{aligned}$$

The sequence $\alpha_1...\alpha_n$ with $\alpha_1=s'$ and $\alpha_n=t$ determines a $(n-1)$-subcube $D'=D_{s',t}$ of $D_{s,t}$. Denoting $\alpha'=\alpha_1...\alpha_n$ that is a full chain of $D_{s',t}$, we obtain

$$\begin{aligned}\sum_{\alpha:\alpha_1=s'}(-1)^{\sigma(\alpha)}e_{\alpha_1...\alpha_n} &= \sum_{\alpha'\in P(D')}(-1)^{\sigma(\alpha)}e_{\alpha'_1...\alpha'_n}\\ &= \sum_{\alpha'\in P(D')}(-1)^{\sigma(\alpha)-\sigma(\alpha')}(-1)^{\sigma(\alpha')}e_{\alpha'_1...\alpha'_n}\\ &= (-1)^{\sigma(\sigma,\sigma)}\omega\left(D'\right)\end{aligned}$$

where we have used (4.6). Hence,

$$\sum_{\alpha}(-1)^{\sigma(\alpha)}e_{\alpha_1...\alpha_n}=\sum_{D'\subset D}(-1)^{\sigma(D,D')}\omega\left(D'\right) \tag{4.9}$$

where the summation extends to all $(n-1)$-cubes $D'\subset D$ with the same target t.

Similarly, a sequence $\alpha_0...\alpha_{n-1}$ with $\alpha_{n-1}=t'$ determines a $(n-1)$-subcube $D'=D_{s,t'}$ of $D_{s,t}$. Denoting $\alpha'=\alpha_0...\alpha_{n-1}$ we obtain

$$(-1)^n\sum_{\alpha'\in P(D')}(-1)^{\sigma(\alpha)}e_{\alpha'_0...\alpha'_{n-1}}=(-1)^{\sigma(D,D')}\omega\left(D'\right)$$

where we have used (4.7). Therefore,

$$(-1)^n\sum_{\alpha}(-1)^{\sigma(\alpha)}e_{\alpha_0...\alpha_{n-1}}=\sum_{D'\subset D}(-1)^{\sigma(D,D')}\omega\left(D'\right) \tag{4.10}$$

where the summation extends to all $(n-1)$-cubes $D'\subset D$ with the same source s. Combining together (4.9) and (4.10) we obtain (4.8).

Finally, since all $\omega(D')$ are allowed paths in D, we obtain that $\partial\omega(D)$ is allowed and, hence, ω is ∂-invariant. ■

4.3 Spaces of n-forms and n-paths on cubical graphs

The main result of this section is the following lemma.

Lemma 4.5 *Let G_S be a cubical graph based in a set M. Denote by K_n the number of n-cubes that are contained in the graph G_S. Then*

$$\dim\Omega^n(G_S)=\dim\Omega_n(G_S)=K_n.$$

Remark 4.6 This statement is not true for a general digraph. Although any n-cube D in an arbitrary digraph always gives rise to the ∂-invariant n-path $\omega(D)$ as in Lemma 4.4, the paths $\omega(D)$ associated with different cubes D can be linearly dependent as it was shown in Example 3.4.

Proof. The identity of $\dim \Omega^n$ and $\dim \Omega_n$ is a consequence of the duality of these spaces. As follows from Lemma 4.4, for any n-cube D from G_S, the n-path $\omega(D)$ is ∂-invariant in D and, hence, in G_S. If $D_1, D_2, ..., D_{K_n}$ are all different n-cubes in G_S then the corresponding n-paths $\omega(D_j)$ are linearly independent because the sets of the basis elements of Ω_n that are used in each $\omega(D_j)$ are disjoint, which follows from the obvious fact that the families $P(D_i)$ and $P(D_j)$ of the full chains are disjoint provided $i \neq j$. Hence, we obtain

$$\dim \Omega_n \geq K_n.$$

Let us prove that

$$\dim \Omega^n \leq K_n.$$

Any allowed n-path α in G_S is a full chain in a n-cube $D_{s,t}$ with $s = \alpha_0$ and $t = \alpha_n$. Consider the associated allowed n-form $e^\alpha = e^{\alpha_0 \dots \alpha_n}$ and show that if α and β are full chains in the same cube $D_{s,t}$ then

$$e^\alpha \simeq \pm e^\beta \tag{4.11}$$

(see (3.7) for the definition of the equivalence relation $\simeq$).

Given a full chain α in $D_{s,t}$ and some index $k = 1, ..., n-1$, define another full chain α' as follows. Observe that the cube $D_{\alpha_{k-1}\alpha_{k+1}}$ is a square that has among the vertices $\alpha_{k-1}, \alpha_k, \alpha_{k+1}$. Denote by α'_k the forth vertex of this square (see Fig. 9) and define α'_j for $j \neq k$ simply by setting $\alpha'_j = \alpha_j$. Hence, we obtain a full chain α' in $D_{s,t}$ that will be called the *transposition* of α at position k.

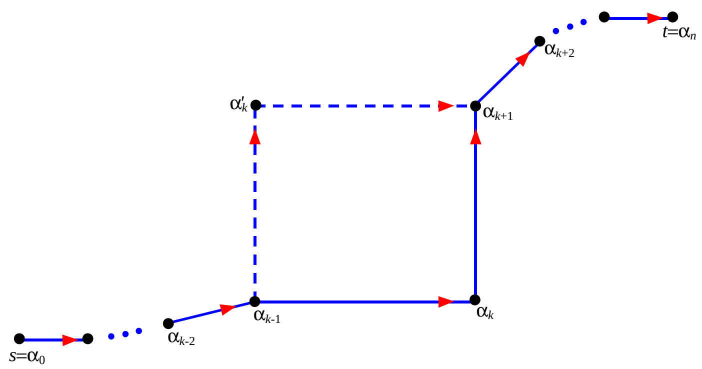

Figure 9: A full chain α and its transposition α' (dashed)

Let us show that

$$e^\alpha \simeq -e^{\alpha'}. \tag{4.12}$$

For that consider a regular form $\psi = e^{\alpha_0 \dots \alpha_{k-1}\alpha_{k+1} \dots \alpha_n}$ where the index α_k is dropped out, and observe that ψ is non-allowed because $\alpha_{k-1}\alpha_{k+1}$ is not an edge (this is a consequence of the fact that G_S contains no triangles). Next, we have

$$d\psi = \sum_{\tau \in S} e^{\tau \alpha_0 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_n} - \sum_{\tau \in S} e^{\alpha_0 \tau \alpha_1 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_n} + \ldots \tag{4.13}$$

$$+ (-1)^{k-1} \sum_{\tau \in S} e^{\alpha_0 \ldots \alpha_{k-1} \tau \alpha_{k+1} \ldots \alpha_n} \tag{4.14}$$

$$+ \ldots + (-1)^n \sum_{\tau \in S} e^{\alpha_0 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_n \tau}. \tag{4.15}$$

All the terms in the right hand side of (4.13) and (4.15) are non-allowed because $\alpha_{k-1}\alpha_{k+1}$ is not an edge. The term in (4.14) is equal to

$$(-1)^{k-1} \left(e^{\alpha_0 \ldots \alpha_{k-1} \alpha_k \alpha_{k+1} \ldots \alpha_n} + e^{\alpha_0 \ldots \alpha_{k-1} \alpha'_k \alpha_{k+1} \ldots \alpha_n} \right) + \text{non-allowed terms},$$

where we have used the fact that the only values of τ for which $\alpha_{k-1} \to \tau \to \alpha_{k+1}$ are $\tau = \alpha_k$ and $\tau = \alpha'_k$. It follows that

$$d\psi = (-1)^{k-1} \left(e^{\alpha} + e^{\alpha'} \right) + \varphi$$

where both φ and ψ are non-allowed. By (3.8) this means that $e^{\alpha} + e^{\alpha'} \simeq 0$, which proves (4.12).

Since any full chain β in $D_{s,t}$ can be obtained from α by a sequence of transpositions, we see that (4.11) follows from (4.12). Hence, all the full chains of the same cube determine the same (up to a multiple) element of the space Ω^n, which implies $\dim \Omega^n \leq K_n$. ■

5 Identity of homologies of S and G_S

Now we can prove the main result of this paper stated in Introduction. All homologies are considered over a fixed field $\mathbb{K}$.

Theorem 5.1 *For any finite simplicial complex S and for any $n \geq 0$, we have isomorphism*

$$H_n(C_*(S)) \cong H_n(G_S).$$

Proof. Let Q_S be the cubical complex associated with S, and $C_*(Q_S)$ be the corresponding chain complex as described in Section 2. Then by (2.3) we have

$$H_n(C_*(S)) \cong H_n(C_*(Q)). \tag{5.1}$$

As it follows from the construction of Q_S in Section 2 and G_S in Section 4.1, the graph G_S can be embedded into Q_S so that the vertices of G_S become the vertices of Q_S, and the edges of G_S become the 1-dimensional cubes in Q_S. Moreover, this embedding provides a bijection between the set of (geometric) n-cubes in Q_S and the set of discrete n-cubes in G_S.

For simplicity of notation, let us identify the cubes from Q_S and G_S. For example, one can always assume that the vertices of G_S are the barycenters of the simplexes from S (cf. Fig. 1). As before, denote by $M = \{1, ..., m\}$ the set of vertices of S, so that any simplex of S is determined by a subset of M.

Let us establish an one-to-one correspondence between the space $C_n \equiv C_n(Q_S)$ of n-chains on Q_S and the space Ω_n of ∂-invariant n-paths on G_S. Indeed, for any cube $D \in Q_S$ we have defined in (4.5) $\omega(D) \in \Omega_n$. Extending the mapping ω by $\mathbb{K}$-linearity, we obtain a linear mapping $\omega : C_n \to \Omega_n$. As it follows from Lemma 4.5, this mapping is bijective, so that the spaces C_n and Ω_n are $\mathbb{K}$-linearly isomorphic.

Let us show that the boundary operators ∂ on C_n and on Ω_n commute with this isomorphism, that is, the following diagram is commutative:

$$\begin{array}{ccc} C_{n-1} & \stackrel{\partial}{\longleftarrow} & C_n \\ \downarrow^{\omega} & & \downarrow^{\omega} \\ \Omega_{n-1} & \stackrel{\partial}{\longleftarrow} & \Omega_n \end{array} \tag{5.2}$$

By (4.5) we have, for any n-cube D from Q_S,

$$\partial\omega(D) = \omega\left(\sum_{D' \subset D} (-1)^{\sigma(D,D')} D'\right),$$

where D' runs over all $(n-1)$-subcubes of . Hence, it remains to show that

$$\sum_{D' \subset D} (-1)^{\sigma(D,D')} D' = \partial D,$$

which by (2.4) amounts to verifying that $(-1)^{\sigma(D,D')}$ coincides with the relative orientation $\mathcal{O}(D, D')$.

So far we have not yet defined any orientation of the cubes in Q_S. Let us choose the orientation as follows. Each n-cube D has the form $D = D_{s,t}$ where s and t are two simplexes of S such that $s \supset t$ and $|s \setminus t| = n$. Let us define a mapping $\varphi : D_{s,t} \to \mathbb{R}^n$ such that the image $\varphi(D_{s,t})$ is the unit cube I^n in $\mathbb{R}^n$. It suffices to define φ on the vertices of $D_{s,t}$ and check that the images are all the vertices of I^n. Let us enumerate the elements of the set $s \setminus t$ (that are integers from 1 to m) in the increasing order as follows: $i_1, ..., i_n$. For any vertex $u \in D_{s,t}$,the set $s \setminus u$ has the form

$$s \setminus u = \{i_{k_1}, i_{k_2}, ..., i_{k_l}\}$$

where $l = |s \setminus u|$ and $k_1 < k_2 < ... < k_l$; in other words, the number $N(u)$ satisfies the identity

$$N(u) - N(s) = 2^{i_{k_1}} + ... + 2^{i_{k_l}}. \tag{5.3}$$

Denoting by $\mathbf{e}_1, ..., \mathbf{e}_n$ the standard basis in $\mathbb{R}^n$, define $\varphi(u)$ by

$$\varphi(u) = \mathbf{e}_{k_1} + \mathbf{e}_{k_2} + ... + \mathbf{e}_{k_l}. \tag{5.4}$$

For example, for the vertex s the sequence $\{i_{k_j}\}$ is empty, that is $l = 0$, and, hence, $\varphi(s) = 0$, while for the vertex t the sequence $\{i_{k_j}\}_{j=1}^{l}$ coincides with the full sequence $\{i_k\}_{k=1}^{n}$ so that $\varphi(t) = \mathbf{e}_1 + ... + \mathbf{e}_n$ (cf. Fig. 10). Clearly, φ maps $D_{s,t}$ onto I^n. Then define the orientation of $D_{s,t}$ by the sequence of vectors $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$.

Let D' be a face of D attached to t, that is, $D' = D_{s',t}$ with $s \to s'$. Then

$$N(s') = N(s) + 2^{i_\gamma}$$

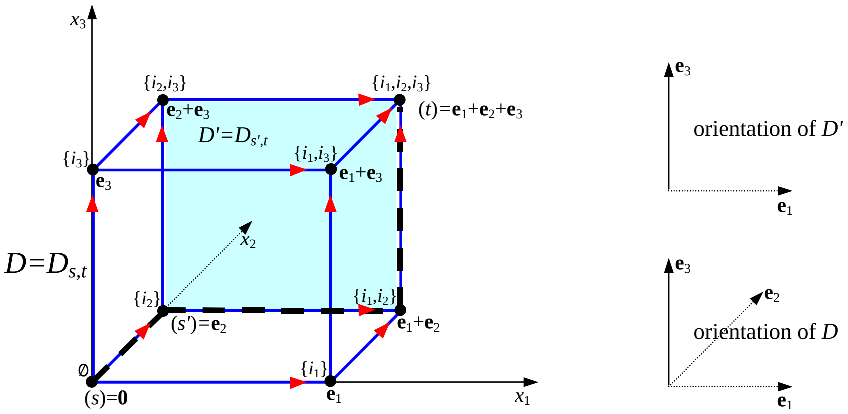

Figure 10: A 3-cube $D = D_{s,t}$ embedded in $\mathbb{R}^3$, its face $D' = D_{s',t}$, a full chain (dashed) from s to t, and the orientations of D, D'. For any vertex u of D, the set $s \setminus u$ is written near u.

for some γ, which implies that $\varphi(s') = \mathbf{e}_\gamma$. For any vertex $u \in D'$, the expansion (5.3) contains the term 2^{i_γ}, which implies that $\varphi(u)$ in (5.4) contains the term $\mathbf{e}_\gamma$. Hence, u lies on the face

$$I^n_\gamma = I^n \cap \{x_\gamma = 1\}$$

of I^n. Clearly, $\varphi(D')$ coincides with the set of all vertices of I^n_γ (cf. Fig. 10 where $\gamma = 2$). Identifying $\mathbb{R}^{n-1}$ with the hyperplane $\{x_\gamma = 1\}$ of $\mathbb{R}^n$, we see that the orientation of D' is determined by the sequence of vectors

$$\{\mathbf{e}_1, ...\mathbf{e}_{\gamma-1}, \mathbf{e}_{\gamma+1}, ..\mathbf{e}_n\}.$$

Since $\mathbf{e}_\gamma$ is the outer normal to D', the relative orientation of D' in D is given by the orientation of the sequence

$$\{\mathbf{e}_\gamma, \mathbf{e}_1, ..., \mathbf{e}_{\gamma-1}, \mathbf{e}_{\gamma+1}, ...\mathbf{e}_n\}$$

that is

$$\mathcal{O}\left(D, D'\right) = (-1)^{\gamma-1}.$$

Let us show that

$$\sigma\left(D, D'\right) = \gamma - 1, \tag{5.5}$$

that will settle the claim. Indeed, consider the sequence $\alpha = \{\alpha_k\}_{k=0}^n$ such that $\alpha_0 = s$ and each of α_k is obtained from the previous one by successive removal of the following elements of M, in the specified order:

$$i_\gamma, i_1, ..., i_{\gamma-1}, i_{\gamma+1}, ..., i_n. \tag{5.6}$$

In particular, $\alpha_n = t$ and, hence, α is a full chain in $D = D_{s,t}$. Since $\alpha_1 = s'$, we see that the sequence $\alpha' = \{\alpha_k\}_{k=1}^n$ is a full chain in $D' = D_{s',t}$. By definition, $\sigma(\alpha)$ is equal to the number of inversions in the sequence (5.6), whence

$$\sigma(\alpha) = \gamma - 1.$$

Similarly, $\sigma(\alpha')$ is equal to the number of inversions in the truncated sequence (5.6) without the first term i_γ, whence $\sigma(\alpha') = 0$. By (4.6) we obtain (5.5). The case when the face D' is attached to s is handled similarly.

The commutative diagram (5.2) implies that the chain complexes C_* and Ω_* are isomorphic. Hence, we have

$$H_n(C_*) \cong H_n(\Omega_*) \cong H_n(G_S),$$

which together with (5.1) finishes the proof. ■

Let us state one consequence of Theorem 5.1 for Hochschild cohomologies. For any finite simplicial complex S consider a digraph E_S where the vertices are all simplexes from S and a couple (s,t) is a (directed) edge if and only if $s \supset t$ (where the dimensions of s and t can be arbitrary, unlike (4.3)). Define an algebra A_S as a set of all finite $\mathbb{K}$-linear combinations of elements of E_S with a multiplication given by the rule:

$$(s,t)(s',t') = \begin{cases} (s,t'), & \text{if } t = s' \\ 0, & \text{otherwise.} \end{cases}$$

Define the space $C^n(A_S)$ of n-chains as the set of all $\mathbb{K}$-multilinear functions $f : A_S^n \to A_S$, and define the differential $D : C^n \to C^{n+1}$ by

$$Df(x_1, x_2, ..., x_{n+1}) = x_1 f(x_2, ..., x_n) + \sum_{i=1}^{n} (-1)^i f(x_1, ..., x_i x_{i+1}, ... x_n) + (-1)^{n+1} f(x_0, ..., x_{n-1}) x_n.$$

It is known that $D^2 = 0$ so that $\{C^*(A_S)\}$ is a cochain complex (see [8]). Its cohomologies are called the Hochschild cohomologies of the algebra A_S and are denoted by $HH^*(A_S)$.

It was proved in [5] that

$$HH^*(A_S) \cong H^*(C^*(S)).$$

Combining with Theorem 5.1, we obtain the following.

Corollary 5.2 *We have the following isomorphism:*

$$HH^*(A_S) \cong H^*(G_S).$$

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