Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations

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Abstract

The Euler-Maruyama scheme is known to diverge strongly and numerically weakly when applied to nonlinear stochastic differential equations (SDEs) with superlinearly growing and globally one-sided Lipschitz continuous drift coefficients. Classical Monte Carlo simulations do, however, not suffer from this divergence behavior of Euler's method because this divergence behavior happens on *rare events*. Indeed, for such nonlinear SDEs the classical Monte Carlo Euler method has been shown to converge by exploiting that the Euler approximations diverge only on events whose probabilities decay to zero very rapidly. Significantly more efficient than the classical Monte Carlo Euler method is the recently introduced multilevel Monte Carlo Euler method. The main observation of this article is that this multilevel Monte Carlo Euler method does – in contrast to classical Monte Carlo methods – not converge in general in the case of such nonlinear SDEs. More precisely, we establish divergence of the multilevel Monte Carlo Euler method for a family of SDEs with superlinearly growing and globally one-sided Lipschitz continuous drift coefficients. In particular, the multilevel Monte Carlo Euler method diverges for these nonlinear SDEs on an event that is not at all rare but has *probability one*. As a consequence for applications, we recommend not to use the multilevel Monte Carlo Euler method for SDEs with superlinearly growing nonlinearities. Instead we propose to combine the multilevel Monte Carlo method with a slightly modified Euler method. More precisely, we show that the multilevel Monte Carlo method combined with a tamed Euler method converges for nonlinear SDEs with globally one-sided Lipschitz continuous drift coefficients and preserves its strikingly higher order convergence rate from the Lipschitz case.

1 Introduction

We consider the following setting in this introductory section. Let $T \in (0,\infty)$, $d, m \in \mathbb{N} := \{1, 2, \dots\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $\xi \colon \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}\big[\|\xi\|_{\mathbb{R}^d}^p\big] < \infty$ for all $p \in [1,\infty)$. Moreover, let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ be a smooth globally one-sided Lipschitz continuous function with at most polynomially growing derivatives and let $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be a smooth globally Lipschitz continuous function with at most polynomially growing derivatives. In particular, we assume that there exists a real number $c \in (0,\infty)$ such that $\langle x - y, \mu(x) - \mu(y)\rangle_{\mathbb{R}^d} \leq c\|x-y\|_{\mathbb{R}^d}^2$ and $\|\sigma(x) - \sigma(y)\|_{\mathbb{R}^{d\times m}} \leq c\|x - y\|_{\mathbb{R}^d}$ for all $x, y \in \mathbb{R}^d$. These assumptions ensure the existence of an up to indistinguishability unique adapted stochastic process $X \colon [0,T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths solving the stochastic differential equation (SDE)

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad X_0 = \xi, \tag{1}$$

for $t \in [0,T]$ (see, e.g., Alyushina [1], Theorem 1 in Krylov [28] or Theorem 2.4.1 in Mao [30]). The function μ is the drift coefficient and the function σ is the diffusion coefficient of the SDE (1). Our goal in this introductory section is then to efficiently compute the deterministic real number

$$\mathbb{E}\Big[f(X_T)\Big] \tag{2}$$

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where $f\colon \mathbb{R}^d \to \mathbb{R}$ is a smooth function with at most polynomially growing derivatives. Note that this question is not treated in the standard literature in computational stochastics (see, for instance, Kloeden and Platen [27] and Milstein [33]) which concentrates on SDEs with globally Lipschitz continuous coefficients rather than the SDE (1). The computation of statistical quantities of the form (2) for SDEs with non-globally Lipschitz continuous coefficients is a major issue in financial engineering, in particular, in option pricing. For details the reader is refereed to the monographs Lewis [29], Glasserman [10], Higham [16] and Szpruch [40].

In order to simulate the quantity (2) on a computer, one has to discretize both the solution process $X\colon [0,T]\times\Omega \to \mathbb{R}^d$ of the SDE (1) as well as the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The simplest method for discretizing the SDE (1) is the Euler method (a.k.a. Euler-Maruyama method). More formally, the Euler approximations $Y_n^N\colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for the SDE (1) are defined recursively through $Y_0^N := \xi$ and

$$Y_{n+1}^N := Y_n^N + \mu(Y_n^N) \cdot \frac{T}{N} + \sigma(Y_n^N) \left(W_{\frac{(n+1)t}{n}} - w_{\frac{nt}{n}} \right) \tag{3}$$

for all $n \in \{0, 1, \ldots, N\}$ and all $N \in \mathbb{N}$. Convergence of Euler's method both in the strong as well as in the numerically weak sense is well-known in case of globally Lipschitz continuous coefficients μ and σ of the SDE (see, e.g., Section 14.1 in Kloeden and Platen [27] and Section 12 in Milstein [33]). The case of superlinearly growing and hence non-globally Lipschitz continuous coefficients of the SDE is more subtle. Indeed, Theorem 2.1 in the recent article [23] shows in the presence of noise that Euler's method diverges to infinity both in the strong and numerically weak sense if the coefficients of the SDE grow superlinearly (see Theorem 2.1 below for a generalization hereof). In this situation, Theorem 2.1 in [23] also proves the existence of events $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, and of real numbers $\theta, c \in (1, \infty)$ such that $\mathbb{P}[\Omega_N] \geq \theta^{(-N^\theta)}$ and $|Y_N^N(\omega)| \geq c^{(c^N)}$ for all $\omega \in \Omega_N$, $N \in \mathbb{N}$. Clearly, this implies the divergence of absolute moments of the Euler approximation, i.e., $\lim_{N\to\infty} \mathbb{E}\big[|Y_N^N|^p\big] = \infty$ for all $p \in (0, \infty)$.

The classical method for discretizing expectations is the Monte Carlo Euler method. Let $Y_n^{N,k}\colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for $k \in \mathbb{N}$ be independent copies of the Euler approximations (3) (see Section 3 for the precise definition). The Monte Carlo Euler approximation of (2) with $N \in \mathbb{N}$ time steps and N^2 Monte Carlo runs (see Duffie and Glynn [6] for more details on this choice) is then the random real number

$$\frac{1}{N^2} \left(\sum_{k=1}^{N^2} f\Big(Y_N^{N,k}\Big) \right). \tag{4}$$

Convergence of the Monte Carlo Euler approximations (4) is well-known in case of globally Lipschitz continuous coefficients μ and σ (see, e.g., Section 14.1 in Kloeden and Platen [27] and Section 12 in Milstein [33]). Recently, convergence of the Monte Carlo Euler approximations (4) has also been established for the SDE (1). More formally, Corollary 3.23 in [21] (which generalizes Theorem 2.1 in [20]) implies

$$\lim_{N\to\infty} \left| \mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2} \left(\sum_{k=1}^{N^2} f\Big(Y_N^{N,k}\Big) \right) \right| = 0 \tag{5}$$

$\mathbb{P}$-almost surely (see also Theorem 3.1 below). The Monte Carlo Euler method is thus *strongly consistent* (see, e.g., Nikulin [37], Cramér [2] or Appendix A.1 in Glasserman [10]) for the SDE (1). The reason why convergence (5) of the Monte Carlo Euler method does hold although the Euler approximations diverge is as follows. The events Ω_N, $N \in \mathbb{N}$, on which Euler's method diverges (see Theorem 2.1 below) are *rare events* and their probabilities decay to zero faster than any polynomial in N as $N \to \infty$, see Lemma 2.6 in [22] for details. Therefore, for large $N \in \mathbb{N}$ the event Ω_N is too unlikely to occur in any of N^2 Monte Carlo simulations in (4).

Considerably more efficient than the Monte Carlo Euler method is the so-called multilevel Monte Carlo Euler method in Giles [8] (see also Creutzig, Dereich, Müller-Gronbach and Ritter [3], Dereich [4], Giles [7], Giles, Higham and Mao [9], Heinrich [13, 14], Heinrich and Sindambiwe [15] and Kebaier [24] for related results). In this method, time is discretized through the Euler method and expectations are approximated by the multilevel Monte Carlo method. More formally, let $Y_n^{N,l,k}\colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for $l \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $k \in \mathbb{N}$ be independent copies of the Euler approximations (3) (see Section 6 for the precise definition). Then the multilevel Monte Carlo Euler approximations for the SDE (1) are defined as

$$\frac{1}{N} \sum_{k=1}^{N} f\Big(Y_1^{1,0,k}\Big) + \sum_{l=1}^{\log_2(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} f\Big(Y_{2^l}^{2^l,l,k}\Big) - f\Big(Y_{2^{(l-1)}}^{2^{(l-1)},l,k}\Big) \right) \tag{6}$$

for $N \in \{2^1, 2^2, 2^3, \ldots\}$. In the case of globally Lipschitz continuous coefficients of the SDE (1), this method has been shown to converge significantly faster to the target quantity (2) than the Monte Carlo Euler method (4). More precisely, in the case of globally Lipschitz continuous coefficients μ and σ, the multilevel Monte Carlo Euler method (6) converges with order $\frac{1}{2}-$ while the Monte Carlo Euler method converges with order $\frac{1}{3}-$ with respect to the computational effort (see Section 1 in Giles [8] or Creutzig, Dereich, Müller-Gronbach and Ritter [3] for details).

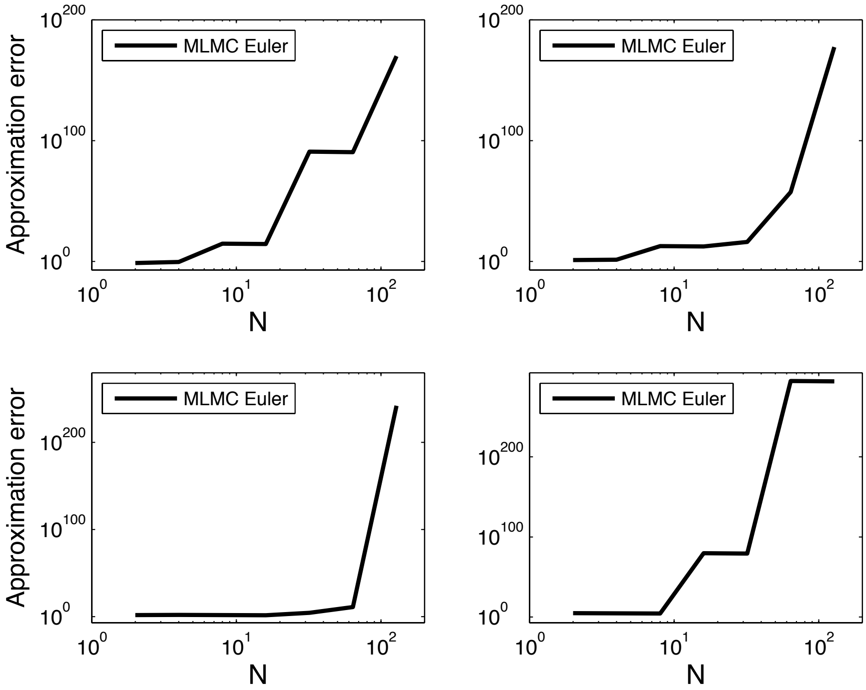

Figure 1: Four sample paths of the approximation error of the multilevel Monte Carlo Euler approximation (6) for the SDE (7) for $N \in \{2^1, 2^2, \ldots, 2^7\}$ with $T = 1$.

In the general setting of the SDE (1) where μ does not need to be globally Lipschitz continuous, convergence of the multilevel Monte Carlo Euler method (6) remained an open question.

The convergence (5) of the Monte Carlo Euler method and the fact that Euler's method diverges on very rare events only shaped our first guess that the multilevel Monte Carlo Euler method should converge too. However, convergence of the multilevel Monte Carlo Euler method fails to hold in the general setting of the SDE (1). To prove this it suffices to establish non-convergence for one counterexample which we choose to be as follows. Let $d = m = 1$, let $\mu(x) = -x^5$, $\sigma(x) = 0$, $f(x) = x^2$ for all $x \in \mathbb{R}$ and let $\xi : \Omega \to \mathbb{R}$ be standard normally distributed. Clearly, this choice satisfies the assumptions of the SDE (1) and the SDE (1) thus reduces to the random ordinary differential equation

$$dX_t = -X_t^5 \, dt, \qquad X_0 = \xi \tag{7}$$

for $t \in [0, T]$. The main observation of this article is that the approximation error of the multilevel Monte Carlo Euler method for the SDE (7) diverges to infinity. More formally, Theorem 4.1 below implies

$$\lim_{\substack{N \to \infty \\ \log_2(N) \in \mathbb{N}}} \left| \mathbb{E}\Big[(X_T)^2\Big] - \frac{1}{N} \sum_{k=1}^{N} \Big(Y_1^{1,0,k}\Big)^2 - \sum_{l=1}^{\log_2(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} \Big(Y_{2^l}^{2^l,l,k}\Big)^2 - \Big(Y_{2^{(l-1)}}^{2^{(l-1)},l,k}\Big)^2 \right) \right| = \infty \tag{8}$$

$\mathbb{P}$-almost surely. Note that the multilevel Monte Carlo Euler method diverges on an event that is not rare but has *probability one*. Thus – in contrast to classical Monte Carlo simulations – the multilevel Monte Carlo Euler method is very sensitive to the rare events on which Euler's method diverges in the sense of Theorem 2.1 below. To visualize the divergence (8), Figure 1 depicts four random sample paths of the approximation error of the multilevel Monte Carlo Euler method (6) for the SDE (7) with $T = 1$ and shows explosion even for small values of $N \in \{2^1, 2^2, 2^3, \ldots\}$. We emphasize that we are only able to establish the divergence (8) for the simple SDE (7). Even in this simple case, the proof of the divergence (8) is rather involved and requires precise estimates on the speed of divergence of Euler's method for the random ordinary differential equation (7) on an appropriate event of instability; see below for an outline.

Comparing the convergence result (5) for the Monte Carlo Euler method and the divergence result (8) for the multilevel Monte Carlo Euler method reveals a remarkable difference between the classical Monte Carlo Euler method and the new multilevel Monte Carlo Euler method. The classical Monte Carlo Euler method applies both to SDEs with globally Lipschitz continuous coefficients and to SDEs with possibly superlinearly growing coefficients such as our SDE (1). The multilevel Monte Carlo Euler method, however, produces often completely wrong values in the case of SDEs with superlinearly growing nonlinearities. This is particularly unfortunate as SDEs with superlinearly growing nonlinearities are very important in applications (see, e.g., [29, 41, 40] for applications in financial engineering). We recommend not to use the multilevel Monte Carlo Euler method for applications with such nonlinear SDEs.

Nonetheless, the multilevel Monte Carlo method can be used for SDEs with non-globally Lipschitz continuous coefficients when being combined with a strongly convergent numerical approximation method. For example, in [22] the following slight modification of the Euler method (3) is proposed. Let $Z_n^N\colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, be defined recursively through $Z_0^N := \xi$ and

$$Z_{n+1}^N := Z_n^N + \frac{\mu(Z_n^N) \cdot \frac{T}{N}}{1 + \frac{T}{N} \cdot \|\mu(Z_n^N)\|_{\mathbb{R}^d}} + \sigma(Z_n^N) \left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}} \right) \tag{9}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Following [22] we refer to this numerical approximation as a tamed Euler method. Additionally, let $Z_n^{N,l,k}\colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for $l \in \mathbb{N}_0$ and $k \in \mathbb{N}$ be independent copies of the tamed Euler approximations (9). In Theorem 6.2 below we then prove convergence of the multilevel Monte Carlo tamed Euler method for all locally Lipschitz continuous test functions on the path space whose local Lipschitz constants grow at most polynomially. In particular, Theorem 6.2 below implies the existence of finite random variables $C_\varepsilon\colon \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, such that

$$\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N} \sum_{k=1}^{N} f\Big(Z_1^{1,0,k} \Big) - \sum_{l=1}^{\log_2(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} f\Big(Z_{2^l}^{2^l,l,k} \Big) - f\Big(Z_{2^{(l-1)}}^{2^{(l-1)},l,k} \Big) \right) \right| \leq \frac{C_\varepsilon}{N^{\left(\frac{1}{2} - \varepsilon\right)}} \tag{10}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and all $\varepsilon \in (0, \frac{1}{2})$ $\mathbb{P}$-almost surely. To sum it up, the classical Monte Carlo Euler method converges (see (5)), the new multilevel Monte Carlo Euler method, in general, fails to converge (see (8)) and the new multilevel Monte Carlo tamed Euler method converges and preserves its striking higher convergence order from the Lipschitz case (see (10)). Thus, concerning applications, the message of this article is that the multilevel Monte Carlo Euler method (6) *needs to be modified* appropriately when being applied to SDEs with superlinearly growing nonlinearities. This is a crucial difference to the classical Monte Euler method which has been shown to converge for such SDEs and which does not need to be modified. However, when modified appropriately (see, e.g., (9)), the multilevel Monte Carlo method preserves its strikingly higher convergence order from the global Lipschitz case and is significantly more efficient than the classical Monte Carlo Euler method even for such nonlinear SDEs. Thereby, this article motivates future research in the construction and the analysis of "appropriately modified" numerical approximation methods.

For the interested reader, we now outline the central ideas in the proof of (8). For this we use the random variables $\xi^{l,k}\colon \Omega \to \mathbb{R}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, defined by $\xi^{l,k} := Y_0^{M,l,k}$ for all $M \in \mathbb{N}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$. Then we note for every $M, k \in \mathbb{N}$, $l \in \mathbb{N}_0$ and every $\omega \in \Omega$ that $|Y_n^{M,l,k}(\omega)|$ is strictly increasing in $n \in \{0, 1, \ldots, M\}$ if and only if $|\xi^{l,k}(\omega)| = |Y_0^{M,l,k}(\omega)| > (2M)^{\frac{1}{4}} T^{-\frac{1}{4}}$. It turns out that $|Y_n^{M,l,k}(\omega)|$ increases in $n \in \{0, 1, \ldots, M\}$ double exponentially fast for all $\omega \in \{|\xi^{l,k}| > (2M)^{\frac{1}{4}} T^{-\frac{1}{4}}\}$, $l \in \mathbb{N}_0$ and all $k, M \in \mathbb{N}$ (see Lemma 4.4 and Corollary 4.7 below for details). A central observation in our proof of the divergence (8) is then that the behavior of the multilevel Monte Carlo Euler method is dominated by the highest level that produces such double exponentially fast increasing trajectories. More precisely, a key step in our proof of (8) is to introduce the random variables $L_N\colon \Omega \to \{1, 2, \ldots, \mathrm{ld}(N)\}$, $N \in \{2^1, 2^2, 2^3, \ldots\}$, by

$$L_N := \max\Big(\{1\} \cup \Big\{ l \in \{1, 2, \ldots, \mathrm{ld}(N)\} \colon \exists\, k \in \{1, 2, \ldots, \tfrac{N}{2^l}\} \colon |\xi^{l,k}| > 2^{\frac{l}{4}} T^{-\frac{1}{4}} \Big\} \Big) \tag{11}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Using the random variables L_N, $N \in \{2^1, 2^2, 2^3, \ldots\}$, we now rewrite the multilevel Monte Carlo Euler method in (8) as

$$\frac{1}{N} \sum_{k=1}^{N} \Big(Y_1^{1,0,k} \Big)^2 + \sum_{l=1}^{\log_2(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} \Big(Y_{2^l}^{2^l,l,k} \Big)^2 - \Big(Y_{2^{(l-1)}}^{2^{(l-1)},l,k} \Big)^2 \right) \tag{12}$$

$$= \sum_{\substack{l \in \{0,1,\ldots,\log_2(N)\} \\ l \neq L_N - 1, L_N}} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big(Y_{2^l}^{2^l,l,k} \Big)^2 - \sum_{\substack{l \in \{1,2,\ldots,\log_2(N)\} \\ l \neq L_N}} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big(Y_{2^{(l-1)}}^{2^{(l-1)},l,k} \Big)^2 \tag{13}$$

$$+ \tfrac{2^{(L_N-1)}}{N} \sum_{k=1}^{\frac{N}{2^{(L_N-1)}}} \Big(Y_{2^{(L_N-1)}}^{2^{(L_N-1)},L_N-1,k} \Big)^2 + \tfrac{2^{L_N}}{N} \sum_{k=1}^{\frac{N}{2^{L_N}}} \Big(Y_{2^{L_N}}^{2^{L_N},L_N,k} \Big)^2 - \tfrac{2^{L_N}}{N} \sum_{k=1}^{\frac{N}{2^{L_N}}} \Big(Y_{2^{(L_N-1)}}^{2^{(L_N-1)},L_N,k} \Big)^2 \tag{14}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Due to the definition of L_N, $N \in \{2^1, 2^2, 2^3, \ldots\}$, it turns out that the asymptotic behavior of the multilevel Monte Carlo Euler method (12) is essentially determined by the three summands in (14) (see inequality (61), estimate (68) and inequalities (73), (74) in the proof of Theorem 4.1 for details). In order to investigate these three summands, we - roughly speaking - quantify the value of the largest summand in each of the three sums in (14). For this we introduce the random variables $\eta_N\colon \Omega \to [0, \infty)$ and $\theta_N\colon \Omega \to [0, \infty)$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ by

$$\eta_N := \max\Big\{ |\xi^{L_N,k}| \in \mathbb{R} \colon k \in \big\{1, 2, \ldots, \tfrac{N}{2^{L_N}}\big\} \Big\} \tag{15}$$

and

$$\theta_N := \max\Big\{|\xi^{L_N-1,k}| \in \mathbb{R} \colon k \in \big\{1, 2, \ldots, \tfrac{N}{2^{(L_N-1)}}\big\}\Big\} \tag{16}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Using the random variables $\eta_N : \Omega \to [0, \infty)$ and $\theta_N : \Omega \to [0, \infty)$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ we then distinguish between three different cases (see inequality (61), inequality (68) and inequalities (73), (74) below). First, on the events $\{\eta_N > 2^{\frac{(L_N+1)}{4}} T^{-\frac{1}{4}}\} \in \mathcal{F}$, $N \in \{2^1, 2^2, 2^3, \ldots\}$, the middle summand in (14) will be positive with large absolute value and will essentially determine the behavior of the multilevel Monte Carlo Euler approximations (12) (see estimate (61) for details). Second, on the events $\{\eta_N \leq 2^{\frac{(L_N+1)}{4}} T^{-\frac{1}{4}}\} \cap \{\eta_N < \theta_N\} \in \mathcal{F}$, $N \in \{2^1, 2^2, 2^3, \ldots\}$, the left summand in (14) will be positive with large absolute value and will essentially determine the behavior of the multilevel Monte Carlo Euler approximations (12) (see inequality (68) for details). Finally, on the events $\{\eta_N \leq 2^{\frac{(L_N+1)}{4}} T^{-\frac{1}{4}}\} \cap \{\eta_N > \theta_N\} \in \mathcal{F}$, $N \in \{2^1, 2^2, 2^3, \ldots\}$, the right summand in (14) will be negative with large absolute value and will essentially determine the behavior of the multilevel Monte Carlo Euler approximations (12) (see inequalities (73) and (74) for details). This very rough outline of the case-by-case analysis in our proof of (8) also illustrates that the multilevel Monte Carlo Euler approximations (12) assume both positive (first and second case) as well as negative values (third case) with large absolute values. We add that this case-by-case analysis argument in our proof of (8) requires that the probability that the random variables η_N and θ_N are close to each other in some sense must decay rapidly to zero as $N \in \{2^1, 2^2, 2^3, \ldots\}$ goes to infinity (see inequality (115) below). We verify the above decaying of the probabilities in Lemma 7.5 below which is a crucial step in our proof of (8). Additionally, we add that the level L_N is approximately of order $\log(\log(N))$ as N goes to infinity (see Lemma 7.1 for the precise assertion). In view of the above case-by-case analysis of the multilevel Monte Carlo Euler method, we find it quite remarkable to observe that the essential behaviour of the multilevel Monte Carlo Euler method in (8) is determined by the levels around the order $\log(\log(N))$ as N goes to infinity.

The remainder of this article is organized as follows. Theorem 2.1 in Section 2 slightly generalizes the result on strong and weak divergence of the Euler method of Hutzenthaler, Jentzen and Kloeden [23]. Convergence of the Monte Carlo Euler method is reviewed in Section 3. The main result of this article, i.e., divergence of the multilevel Monte Carlo Euler method for the SDE (7), is presented and proved in Section 4. We believe that the multilevel Monte Carlo Euler method diverges more generally and formulate this as Conjecture 5.1 in Section 5. Section 6 contains our proof of almost sure and strong convergence of the multilevel Monte Carlo tamed Euler method for all locally Lipschitz continuous test functions on the path space whose local Lipschitz constants grow at most polynomially.

2 Divergence of the Euler method

Throughout this section assume that the following setting is fulfilled. Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $W \colon [0, T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion. Additionally, let $\xi \colon \Omega \to \mathbb{R}$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mapping and let $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$ be two $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable mappings. We then define the Euler approximations $Y_n^N \colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, recursively by $Y_0^N := \xi$ and

$$Y_{n+1}^N := Y_n^N + \mu(Y_n^N) \cdot \frac{T}{N} + \sigma(Y_n^N) \cdot \Big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\Big) \tag{17}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. The following theorem generalizes Theorem 2.1 in Hutzenthaler, Jentzen and Kloeden [23].

Theorem 2.1 (Strong and weak divergence of the Euler method)**.** *Assume that the above setting is fulfilled and let $\alpha, c \in (1, \infty)$ be real numbers such that $|\mu(x)| + |\sigma(x)| \geq \frac{|x|^\alpha}{c}$ for all $x \in \mathbb{R}$ with $|x| \geq c$. Moreover, assume that $\mathbb{P}\big[\sigma(\xi) \neq 0\big] > 0$ or that there exists a real number $\beta \in (1, \infty)$ such that $\mathbb{P}\big[|\xi| \geq x\big] \geq \beta^{(-x^\beta)}$ for all $x \in [1, \infty)$. Then there exists a real number $\theta \in (1, \infty)$ and a sequence of nonempty events $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, such that $\mathbb{P}\big[\Omega_N\big] \geq \theta^{(-N^\theta)}$ and $\big|Y_N^N(\omega)\big| \geq c^{\left((\frac{\alpha+1}{2})^N\right)}$ for all $\omega \in \Omega_N$ and all $N \in \mathbb{N}$. In particular, the Euler approximations (17) satisfy $\lim_{N\to\infty} \mathbb{E}\big[|Y_N^N|^p\big] = \infty$ for all $p \in (0, \infty)$.*

Theorem 2.1 immediately follows from Lemma 2.2 and Lemma 2.3 below. More results on Euler's method for SDEs with possibly superlinearly growing nonlinearities can, e.g., be found in [12, 11, 34, 35] and in the references therein.

Lemma 2.2 (Tails of Y_1^N, $N \in \mathbb{N}$)**.** *Assume that the above setting is fulfilled and let $\mathbb{P}\big[\sigma(\xi) \neq 0\big] > 0$. Then there exists a real number $\beta \in (1, \infty)$ such that $\mathbb{P}\big[|Y_1^N| \geq x\big] \geq \beta^{\left(-(Nx)^\beta\right)}$ for all $x \in [1, \infty)$ and all $N \in \mathbb{N}$.*

Proof of Lemma 2.2. By assumption we have $\mathbb{P}\big[|\sigma(\xi)| > 0\big] > 0$. Therefore, there exists a real number $K \in (1, \infty)$ such that

$$\vartheta := \mathbb{P}\bigg[|\sigma(\xi)| \geq \frac{1}{K},\ |\xi| + T|\mu(\xi)| \leq K\bigg] \in (0, \infty). \tag{18}$$

Moreover, we have

$$\begin{aligned}\mathbb{P}\Big[|Y_1^N|\ge x\Big] &= \mathbb{P}\Big[\big|\xi+\mu(\xi)\tfrac{T}{N}+\sigma(\xi)W_{\frac{T}{N}}\big|\ge x\Big] \ge \mathbb{P}\Big[\big|\sigma(\xi)W_{\frac{T}{N}}\big|-|\xi|-T|\mu(\xi)|\ge x\Big]\\ &\ge \mathbb{P}\Big[|\sigma(\xi)|\ge \tfrac{1}{K},\ |\xi|+T|\mu(\xi)|\le K,\ \big|\sigma(\xi)W_{\frac{T}{N}}\big|-|\xi|-T|\mu(\xi)|\ge x\Big]\\ &\ge \mathbb{P}\Big[|\sigma(\xi)|\ge \tfrac{1}{K},\ |\xi|+T|\mu(\xi)|\le K,\ \tfrac{1}{K}\big|W_{\frac{T}{N}}\big|-K\ge x\Big]\end{aligned}$$

for all $x\in[1,\infty)$ and all $N\in\mathbb{N}$. Definition (18) and Lemma 4.1 in [23] therefore show

$$\begin{aligned}\mathbb{P}\Big[|Y_1^N|\ge x\Big] &\ge \mathbb{P}\Big[|\sigma(\xi)|\ge \tfrac{1}{K},\ |\xi|+T|\mu(\xi)|\le K\Big]\cdot\mathbb{P}\Big[\tfrac{1}{K}\big|W_{\frac{T}{N}}\big|-K\ge x\Big] = \vartheta\cdot\mathbb{P}\Big[T^{-\frac12}\big|W_T\big|\ge T^{-\frac12}N^{\frac12}K(x+K)\Big]\\ &\ge \frac{\vartheta}{4\sqrt{T}}\cdot\exp\Big(-T^{-1}NK^2(x+K)^2\Big)\ge \frac{\vartheta}{4\sqrt{T}}\cdot\exp\Big(-4T^{-1}K^4(Nx)^2\Big)\\ &= \frac{\vartheta}{4\sqrt{T}}\cdot\Big(e^{4T^{-1}K^4}\Big)^{(-(Nx)^2)}\ge\left(e^{4T^{-1}K^4}+\frac{4\sqrt{T}}{\vartheta}\right)^{(-2(Nx)^2)}\end{aligned}$$

for all $x\in[1,\infty)$ and all $N\in\mathbb{N}$. This completes the proof of Lemma 2.2. □

Lemma 2.3. *Assume that the above setting is fulfilled and let $\alpha,c\in(1,\infty)$ be real numbers such that $|\mu(x)|+|\sigma(x)|\ge\frac{|x|^\alpha}{c}$ for all $x\in\mathbb{R}$ with $|x|\ge c$. Moreover, assume that there exist real numbers $N_0\in\{0,1,2,\dots\}$, $\beta\in(1,\infty)$ such that $\mathbb{P}\big[|Y_{N_0}^N|\ge x\big]\ge\beta^{(-(Nx)^\beta)}$ for all $x\in[1,\infty)$ and all $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$. Then there exists a real number $\theta\in(1,\infty)$ and a sequence of nonempty events $\Omega_N\in\mathcal{F}$, $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$, such that $\mathbb{P}\big[\Omega_N\big]\ge\theta^{(-N^\theta)}$ and $\big|Y_N^N(\omega)\big|\ge c^{((\frac{\alpha+1}{2})^N)}$ for all $\omega\in\Omega_N$ and all $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$. In particular, the Euler approximations* (17) *satisfy $\lim_{N\to\infty}\mathbb{E}\big[|Y_N^N|^p\big]=\infty$ for all $p\in(0,\infty)$.*

Proof of Lemma 2.3. Define real numbers $r_N\in[0,\infty)$, $N\in\mathbb{N}$, by

$$r_N:=\max\left(c,\left(\frac{2Nc}{T}\right)^{\frac{2}{(\alpha-1)}}\right)\tag{19}$$

for all $N\in\mathbb{N}$. We also use the function $\operatorname{sgn}\colon\mathbb{R}\to\mathbb{R}$ defined by $\operatorname{sgn}(x):=1$ for all $x\in[0,\infty)$ and by $\operatorname{sgn}(x):=-1$ for all $x\in(-\infty,0)$. Furthermore, we define events $\Omega_N\in\mathcal{F}$, $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$, by

$$\begin{aligned}\Omega_N:=\left(\cap_{n=N_0}^{N-1}\left\{\omega\in\Omega:\operatorname{sgn}\Big(\mu(Y_n^N(\omega))\cdot\sigma(Y_n^N(\omega))\Big)\cdot\Big(W_{\frac{(n+1)T}{N}}(\omega)-W_{\frac{nT}{N}}(\omega)\Big)\ge\frac{T}{N}\right\}\right)\\ \bigcap\left\{\omega\in\Omega:|Y_{N_0}^N(\omega)|\ge(r_N)^{((\frac{\alpha+1}{2})^{N_0})}\right\}\end{aligned}\tag{20}$$

for all $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$. In particular, the definition of $(\Omega_N)_{N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}}$ implies

$$\left|\mu(Y_n^N(\omega))\cdot\frac{T}{N}+\sigma(Y_n^N(\omega))\cdot\Big(W_{\frac{(n+1)T}{N}}(\omega)-W_{\frac{nT}{N}}(\omega)\Big)\right|=\frac{T}{N}\left|\mu(Y_n^N(\omega))\right|+\left|\sigma(Y_n^N(\omega))\right|\cdot\left|W_{\frac{(n+1)T}{N}}(\omega)-W_{\frac{nT}{N}}(\omega)\right|\tag{21}$$

for all $n\in\{N_0,N_0+1,\dots,N-1\}$, $\omega\in\Omega_N$ and all $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$.

In the next step let $N\in\mathbb{N}\cap\{N_0,N_0+1,\dots\}$ and $\omega\in\Omega_N$ be arbitrary. We then claim

$$\big|Y_n^N(\omega)\big|\ge(r_N)^{((\frac{\alpha+1}{2})^n)}\tag{22}$$

for all $n\in\{N_0,N_0+1,\dots,N\}$. We now show (22) by induction on $n\in\{N_0,N_0+1,\dots,N\}$. The base case $n=N_0$ follows from definition (20) of Ω_N. For the induction step assume that (22) holds for one $n\in\{N_0,N_0+1,\dots,N-1\}$. In particular, this implies

$$\big|Y_n^N(\omega)\big|\ge(r_N)^{((\frac{\alpha+1}{2})^n)}\ge r_N\ge c>1.\tag{23}$$

Moreover, definition (17), the triangle inequality and equation (21) yield

$$\begin{aligned}\big|Y_{n+1}^N(\omega)\big|&\ge\left|\mu(Y_n^N(\omega))\cdot\frac{T}{N}+\sigma(Y_n^N(\omega))\cdot\Big(W_{\frac{(n+1)T}{N}}(\omega)-W_{\frac{nT}{N}}(\omega)\Big)\right|-\big|Y_n^N(\omega)\big|\\ &=\frac{T}{N}\left|\mu(Y_n^N(\omega))\right|+\left|\sigma(Y_n^N(\omega))\right|\cdot\left|W_{\frac{(n+1)T}{N}}(\omega)-W_{\frac{nT}{N}}(\omega)\right|-\big|Y_n^N(\omega)\big|\ge\frac{T}{N}\Big(\left|\mu(Y_n^N(\omega))\right|+\left|\sigma(Y_n^N(\omega))\right|\Big)-\big|Y_n^N(\omega)\big|\end{aligned}$$

and the estimate $|\mu(x)| + |\sigma(x)| \geq \frac{|x|^\alpha}{c}$ for all $x \in \mathbb{R}$ with $|x| \geq c$, inequality (23) and definition (19) therefore show

$$\begin{aligned}
\left|Y_{n+1}^N(\omega)\right| &\geq \frac{T}{Nc}\left|Y_n^N(\omega)\right|^\alpha - \left|Y_n^N(\omega)\right| \geq \frac{T}{Nc}\left|Y_n^N(\omega)\right|^\alpha - \left|Y_n^N(\omega)\right|^{\frac{(\alpha+1)}{2}} \\
&= \left|Y_n^N(\omega)\right|^{\frac{(\alpha+1)}{2}}\left(\frac{T}{Nc}\left|Y_n^N(\omega)\right|^{\frac{(\alpha-1)}{2}} - 1\right) \geq \left|Y_n^N(\omega)\right|^{\frac{(\alpha+1)}{2}}\left(\frac{T}{Nc}\left(r_N\right)^{\frac{(\alpha-1)}{2}} - 1\right) \geq \left|Y_n^N(\omega)\right|^{\frac{(\alpha+1)}{2}}.
\end{aligned}$$

The induction hypothesis hence yields

$$\left|Y_{n+1}^N(\omega)\right| \geq \left|Y_n^N(\omega)\right|^{\frac{(\alpha+1)}{2}} \geq \left((r_N)^{\left(\left(\frac{\alpha+1}{2}\right)^n\right)}\right)^{\frac{(\alpha+1)}{2}} = (r_N)^{\left(\left(\frac{\alpha+1}{2}\right)^{(n+1)}\right)}.$$

Inequality (22) thus holds for all $n \in \{N_0, N_0+1, \ldots, N\}$, $\omega \in \Omega_N$ and all $N \in \mathbb{N} \cap \{N_0, N_0+1, \ldots\}$. In particular, we obtain

$$\left|Y_N^N(\omega)\right| \geq (r_N)^{\left(\left(\frac{\alpha+1}{2}\right)^N\right)} \geq c^{\left(\left(\frac{\alpha+1}{2}\right)^N\right)} \tag{24}$$

for all $\omega \in \Omega_N$ and all $N \in \mathbb{N} \cap \{N_0, N_0+1, \ldots\}$. Additionally, Lemma 4.1 in [23] yields

$$\begin{aligned}
&\mathbb{E}\left[1_{\left\{\operatorname{sgn}(\mu(Y_n^N)\cdot\sigma(Y_n^N))\cdot\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right) \geq \frac{T}{N}\right\}} \,\middle|\, \mathcal{F}_{\frac{nT}{N}}\right] \qquad (25)\\
&= \mathbb{P}\left[\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right) \geq \frac{T}{N}\right] = \mathbb{P}\left[W_{\frac{T}{N}} \geq \frac{T}{N}\right] = \mathbb{P}\left[T^{-\frac{1}{2}}W_T \geq \sqrt{\frac{T}{N}}\right] \geq \frac{e^{-\frac{T}{N}}\sqrt{T}}{8\sqrt{N}}
\end{aligned}$$

$\mathbb{P}$-almost surely for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
\mathbb{P}\big[\Omega_N\big] &= \mathbb{P}\left[\left|Y_{N_0}^N\right| \geq (r_N)^{\left(\left(\frac{\alpha+1}{2}\right)^{N_0}\right)}\right] \cdot \left(\mathbb{P}\left[T^{-\frac{1}{2}}W_T \geq \sqrt{\tfrac{T}{N}}\right]\right)^{(N-N_0)} \\
&\geq \beta^{\left(-(Nr_N)^{\left(\left(\frac{\alpha+1}{2}\right)^{N_0}\beta\right)}\right)} \cdot \left(\mathbb{P}\left[T^{-\frac{1}{2}}W_T \geq \sqrt{\tfrac{T}{N}}\right]\right)^N \geq \beta^{\left(-(Nr_N)^{\left(\left(\frac{\alpha+1}{2}\right)^{N_0}\beta\right)}\right)} \cdot \left(\frac{e^{-\frac{T}{N}}\sqrt{T}}{8\sqrt{N}}\right)^N \\
&\geq e^{-T} \cdot \beta^{\left(-(Nr_N)^{\left(\left(\frac{\alpha+1}{2}\right)^{N_0}\beta\right)}\right)} \cdot \left(\frac{\sqrt{T}}{8\sqrt{N}}\right)^N
\end{aligned}$$

for all $N \in \mathbb{N} \cap \{N_0, N_0+1, \ldots\}$. This shows the existence of a real number $\theta \in (1, \infty)$ such that

$$\mathbb{P}\big[\Omega_N\big] \geq \theta^{\left(-N^\theta\right)} \tag{26}$$

for all $N \in \mathbb{N} \cap \{N_0, N_0+1, \ldots\}$. Combining (24) and (26) finally gives

$$\lim_{N\to\infty} \mathbb{E}\Big[\left|Y_N^N\right|^p\Big] \geq \lim_{N\to\infty} \mathbb{E}\Big[1_{\Omega_N}\left|Y_N^N\right|^p\Big] \geq \lim_{N\to\infty}\left(\mathbb{P}\big[\Omega_N\big] \cdot c^{\left(p\cdot\left(\frac{\alpha+1}{2}\right)^N\right)}\right) \geq \lim_{N\to\infty}\left(\theta^{\left(-N^\theta\right)} \cdot c^{\left(p\cdot\left(\frac{\alpha+1}{2}\right)^N\right)}\right) = \infty$$

for all $p \in (0, \infty)$. This, (24) and (26) then complete the proof of Theorem 2.1. □

3 Convergence of the Monte Carlo Euler method

The Monte Carlo Euler method has been shown to converge with probability one for one-dimensional SDEs with superlinearly growing and globally one-sided Lipschitz continuous drift coefficients and with globally Lipschitz continuous diffusion coefficients according to [20]. The Monte Carlo Euler method is thus *strongly consistent* (see, e.g., Nikulin [37], Cramér [2] or Appendix A.1 in Glasserman [10]) for such SDEs. After having reviewed this convergence result of the Monte Carlo Euler method, we complement in this section this convergence result with the behavior of moments of the Monte Carlo Euler approximations for such SDEs. More precisely, an immediate consequence of Theorem 2.1 is Corollary 3.2 below which shows for such SDEs that the Monte Carlo Euler approximations diverge in the strong L^p-sense for every $p \in [1, \infty)$. We emphasize that this strong divergence result does not reflect the behavior of the Monte Carlo Euler method in a simulation and it is presented for completeness only. Indeed, the events on which the Euler approximations diverge (see Theorem 2.1) are *rare events* and their probabilities decay to zero very rapidly (see, e.g., Lemma 4.5 in [20] for details). This is the reason why the Monte Carlo Euler method is strongly consistent and thus does converge according to [20] (see also Theorem 3.1 below and Corollary 3.23 in [21]).

Throughout this section assume that the following setting is fulfilled. Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W^k\colon [0,T]\times\Omega \to \mathbb{R}$, $k \in \mathbb{N}$, be a family of independent one-dimensional standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motions and let $\xi^k\colon \Omega \to \mathbb{R}$, $k \in \mathbb{N}$, be a family of independent identically distributed

$\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mappings with $\mathbb{E}\big[|\xi^1|^p\big] < \infty$ for all $p \in [1,\infty)$. Moreover, let $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$ be two $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable mappings such that there exists a predictable stochastic process $X \colon [0,T] \times \Omega \to \mathbb{R}$ which satisfies $\int_0^T |\mu(X_s)| + |\sigma(X_s)|^2 \, ds < \infty$ $\mathbb{P}$-almost surely and

$$X_t = \xi^1 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s^1 \tag{27}$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$. The drift coefficient μ is the infinitesimal mean of the process X and the diffusion coefficient σ is the infinitesimal standard deviation of the process X. We then define a family $Y_n^{N,k} \colon \Omega \to \mathbb{R}$, $n \in \{0,1,\ldots,N\}$, $N,k \in \mathbb{N}$, of Euler approximations by $Y_0^{N,k} := \xi^k$ and

$$Y_{n+1}^{N,k} := Y_n^{N,k} + \mu(Y_n^{N,k}) \cdot \frac{T}{N} + \sigma(Y_n^{N,k}) \cdot \left(W^k_{\frac{(n+1)T}{N}} - W^k_{\frac{nT}{N}} \right) \tag{28}$$

for all $n \in \{0,1,\ldots,N-1\}$ and all $N,k \in \mathbb{N}$. For clarity of exposition we recall the following convergence theorem from [20]. Its proof can be found in [20].

Theorem 3.1 (Strong consistency and convergence with probability one of the Monte Carlo Euler method)**.** *Assume that the above setting is fulfilled, let $\mu, \sigma, f \colon \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable and let $c \in [0,\infty)$ be a real number such that $(x-y)\cdot(\mu(x)-\mu(y)) \leq c\,|x-y|^2$, $|\sigma(x)-\sigma(y)| \leq c\,|x-y|$ and $\big|\mu^{(4)}(x)\big| + \big|\sigma^{(4)}(x)\big| + \big|f^{(4)}(x)\big| \leq c(1+|x|^c)$ for all $x \in \mathbb{R}$. Then there exist finite $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $C_\varepsilon \colon \Omega \to [0,\infty)$, $\varepsilon \in (0,1)$, such that*

$$\left| \mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2} \left(\sum_{k=1}^{N^2} f(Y_N^{N,k}) \right) \right| \leq \frac{C_\varepsilon}{N^{(1-\varepsilon)}} \tag{29}$$

for all $N \in \mathbb{N}$ and all $\varepsilon \in (0,1)$ $\mathbb{P}$-almost surely.

In contrast to pathwise convergence of the Monte Carlo Euler method for SDEs with globally one-sided Lipschitz continuous drift and globally Lipschitz continuous diffusion coefficients (see Theorem 3.1 above for details), strong convergence of the Monte Carlo Euler method, in general, fails to hold for such SDEs which is established in the following corollary of Theorem 2.1, i.e., in Corollary 3.2. As mentioned above we emphasize that Corollary 3.2 does not reflect the behavior of the Monte Carlo Euler method in a practical simulation because the events on which the Euler approximations diverge (see Theorem 2.1) are rare events and their probabilities decay to zero very rapidly (see Lemma 4.5 in [20] for details).

Corollary 3.2 (Strong divergence of the Monte Carlo Euler method)**.** *Assume that the above setting is fulfilled and let $\alpha, c \in (1,\infty)$ be real numbers such that $|\mu(x)| + |\sigma(x)| \geq \frac{|x|^\alpha}{c}$ for all $x \in \mathbb{R}$ with $|x| \geq c$. Moreover, assume that $\mathbb{P}\big[\sigma(\xi^1) \neq 0\big] > 0$ or that there exists a real number $\beta \in (1,\infty)$ such that $\mathbb{P}\big[|\xi^1| \geq x\big] \geq \beta^{(-x^\beta)}$ for all $x \in [1,\infty)$. Moreover, let $f \colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable with $f(x) \geq \frac{1}{c}|x|^{\frac{1}{c}} - c$ for all $x \in \mathbb{R}$. Then*

$$\lim_{N\to\infty} \mathbb{E}\left[\left| \mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2} \left(\sum_{k=1}^{N^2} f(Y_N^{N,k}) \right) \right|^p \right] = \infty \tag{30}$$

for all $p \in [1,\infty)$.

Proof of Corollary 3.2. The triangle inequality, Jensen's inequality and the estimate $f(x) \geq \frac{1}{c}|x|^{\frac{1}{c}} - c$ for all $x \in \mathbb{R}$ give

$$\begin{aligned} &\left\| \mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2} \left(\sum_{k=1}^{N^2} f(Y_N^{N,k}) \right) \right\|_{L^p(\Omega;\mathbb{R})} \geq \frac{1}{N^2} \left\| \sum_{k=1}^{N^2} f(Y_N^{N,k}) \right\|_{L^p(\Omega;\mathbb{R})} - \mathbb{E}\Big[\big|f(X_T)\big|\Big] \\ &\geq \frac{1}{N^2}\, \mathbb{E}\left[\sum_{k=1}^{N^2} f(Y_N^{N,k}) \right] - \mathbb{E}\Big[\big|f(X_T)\big|\Big] = \mathbb{E}\Big[f(Y_N^{N,1})\Big] - \mathbb{E}\Big[\big|f(X_T)\big|\Big] \geq \frac{1}{c} \cdot \mathbb{E}\left[\Big|Y_N^{N,1}\Big|^{\frac{1}{c}} \right] - c - \mathbb{E}\Big[\big|f(X_T)\big|\Big] \end{aligned} \tag{31}$$

for all $N \in \mathbb{N}$ and all $p \in [1,\infty)$. Combining (31) and Theorem 2.1 then shows (30) in the case $\mathbb{E}\big[|f(X_T)|\big] < \infty$. In the case $\mathbb{E}\big[|f(X_T)|\big] = \infty$, the estimate $f(x) \geq -c$ for all $x \in \mathbb{R}$ shows $\mathbb{E}\big[f(X_T)\big] = \infty$ and this implies (30) in the case $\mathbb{E}\big[|f(X_T)|\big] = \infty$. The proof of Corollary 3.2 is thus completed. □

4 Counterexamples to convergence of the multilevel Monte Carlo Euler method

Theorem 4.1 below establishes divergence with probability one of the multilevel Monte Carlo Euler method (6) for the SDE (7). This, in particular, proves that the multilevel Monte Carlo Euler method is in contrast to the classical Monte Carlo Euler method not *consistent* (see, e.g., Nikulin [37], Cramér [2] or Appendix A.1 in Glasserman [10]) for the SDE (7).

Throughout this section assume that the following setting is fulfilled. Let $T, \bar{\sigma} \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\xi^{l,k} \colon \Omega \to \mathbb{R}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, be a family of independent normally distributed $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings with mean zero and standard deviation $\bar{\sigma}$. Moreover, let $X \colon [0, T] \times \Omega \to \mathbb{R}$ be the unique stochastic process with continuous sample paths which fulfills the SDE

$$dX_t = -X_t^5 \, dt, \qquad X_0 = \xi \tag{32}$$

for $t \in [0, T]$. We then define a family of Euler approximations $Y_n^{N,l,k} \colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, by $Y_0^{N,l,k} := \xi^{l,k}$ and

$$Y_{n+1}^{N,l,k} := Y_n^{N,l,k} - \big(Y_n^{N,l,k}\big)^5 \cdot \frac{T}{N} \tag{33}$$

for all $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$ and all $k \in \mathbb{N}$.

Theorem 4.1 (Main result of this article: Divergence with probability one of the multilevel Monte Carlo Euler method for the SDE (32))**.** *Assume that the above setting is fulfilled. Then*

$$\lim_{\substack{N \to \infty \\ \mathrm{ld}(N) \in \mathbb{N}}} \left| \frac{1}{N} \sum_{k=1}^{N} \big|Y_1^{1,0,k}\big|^p + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big(\big|Y_{2^l}^{2^l,l,k}\big|^p - \big|Y_{2^{(l-1)}}^{2^{(l-1)},l,k}\big|^p \Big) \right| = \infty \tag{34}$$

$\mathbb{P}$-*almost surely for all* $p \in (0, \infty)$.

The proof of Theorem 4.1 is postponed to Subsection 4.2 below.

4.1 Simulations

We illustrate Theorem 4.1 with numerical simulations. To this end we observe that the exact solution of the random ordinary differential equation (32) satisfies

$$X_t = \frac{\xi}{(1 + 4t\xi^4)^{\frac{1}{4}}} \tag{35}$$

for all $t \in [0, 1]$. The real number $\mathbb{E}\big[(X_1)^2\big]$ can then be computed approximatively by numerical integration or by the Monte Carlo method. Figure 1 depicts four random sample paths of the approximation error of the multilevel Monte Carlo Euler approximations in the case $T = 1$ and $\bar{\sigma} = 1$ in (32) where $\mathbb{E}\big[(X_1)^2\big] \approx 0.28801$ (calculated with the `integrate`-function of `R`). The sample paths clearly diverge even for small $N \in \{2^1, 2^2, 2^3, \ldots\}$. For some other SDEs, however, pathwise divergence does not emerge for small $N \in \{2^1, 2^2, 2^3, \ldots\}$. For example, let us choose a standard deviation as small as $\bar{\sigma} = 0.1$ in (32) where $T = 1$. Here the exact value satisfies $\mathbb{E}\big[(X_1)^2\big] \approx 0.009971$ (calculated with the `integrate`-function of `R`). Then sample paths of the multilevel Monte Carlo Euler approximation seem to converge even for reasonably large $N \in \{2^1, 2^2, 2^3, \ldots\}$ (see Figure 2 for four sample paths). So the sample paths of the multilevel Monte Carlo Euler method for some SDEs first seem to converge but diverge as $N \in \{2^1, 2^2, 2^3, \ldots\}$ becomes sufficiently large. To see this in a plot, we tried different values of $\bar{\sigma}$ and found sample paths in case of $\bar{\sigma} = \frac{1}{3}$ and $T = 1$ which first seem to convergence to the exact value $\mathbb{E}\big[(X_1)^2\big] \approx 0.09248$ (calculated with the `integrate`-function of `R`) but diverge for larger values of $N \in \{2^1, 2^2, 2^3, \ldots\}$ (see Figure 3 for four sample paths).

4.2 Proof of Theorem 4.1

First of all, we introduce more notation in order to prove Theorem 4.1. Let $y_n^{N,x} \in \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, $x \in \mathbb{R}$, be defined recursively through $y_0^{N,x} := x$ and

$$y_{n+1}^{N,x} := y_n^{N,x} - \big(y_n^{N,x}\big)^5 \cdot \frac{T}{N} = y_n^{N,x} \left(1 - \big(y_n^{N,x}\big)^4 \cdot \frac{T}{N}\right) \tag{36}$$

for all $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ and all $x \in \mathbb{R}$ and let $p \in (0, \infty)$ be fixed for the rest of this section. This notation enables us to rewrite the multilevel Monte Carlo Euler approximation in (34) as

$$\begin{aligned} &\frac{1}{N} \sum_{k=1}^{N} \big|Y_1^{1,0,k}\big|^p + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big(\big|Y_{2^l}^{2^l,l,k}\big|^p - \big|Y_{2^{(l-1)}}^{2^{(l-1)},l,k}\big|^p \Big) \\ &= \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big|Y_{2^l}^{2^l,l,k}\Big|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big|Y_{2^{(l-1)}}^{2^{(l-1)},l,k}\Big|^p = \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big|y_{2^l}^{2^l,\xi^{l,k}}\Big|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big|y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}}\Big|^p \end{aligned} \tag{37}$$

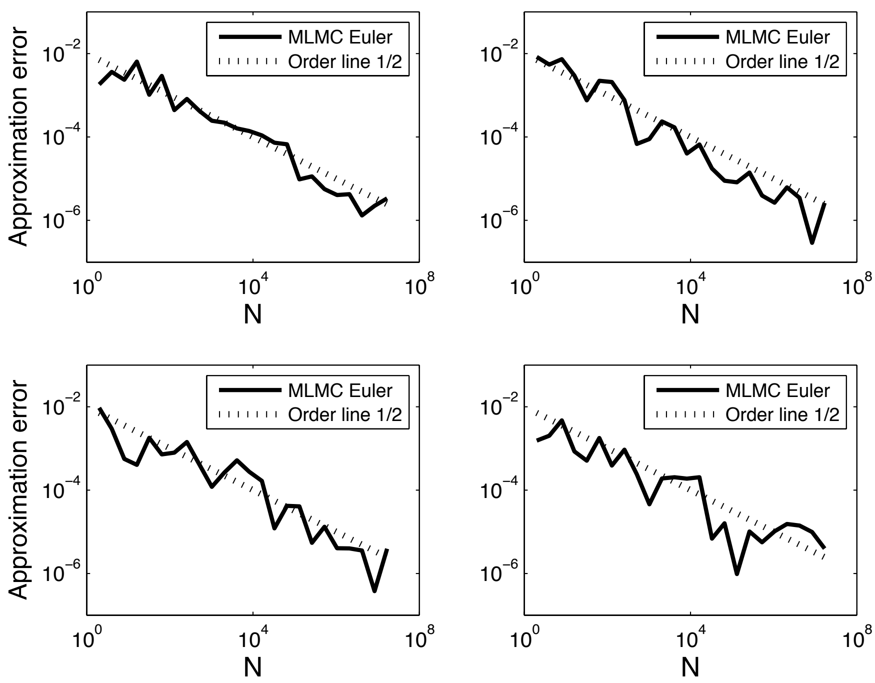

Figure 2: Four sample paths of the approximation error of the multilevel Monte Carlo Euler approximation in (34) where $T = 1$, $\bar{\sigma} = 0.1$, $p = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{22}\}$.

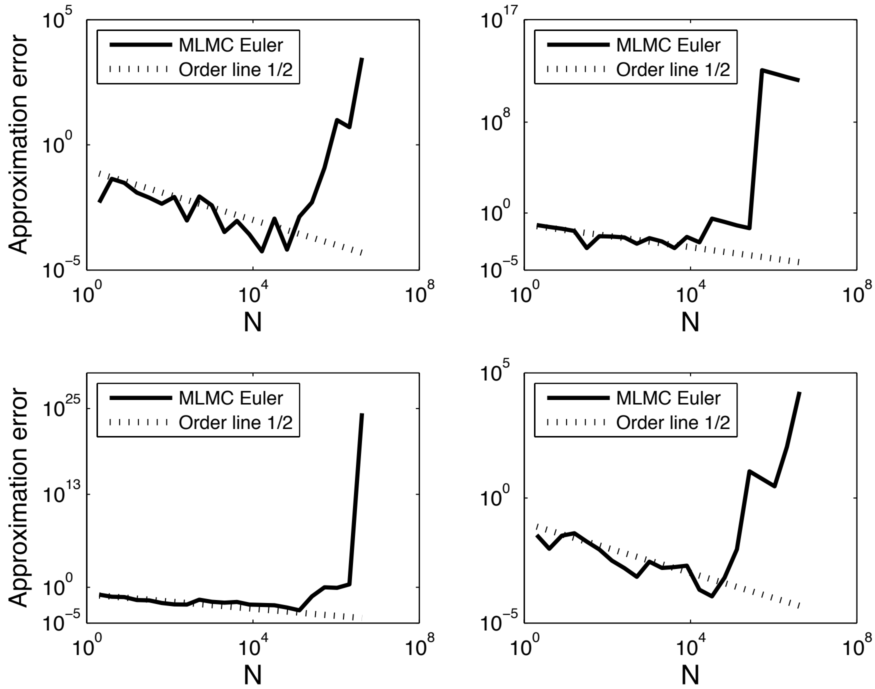

Figure 3: Four sample paths of the approximation error of the multilevel Monte Carlo Euler approximation in (34) where $T = 1$, $\bar{\sigma} = \frac{1}{3}$, $p = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{22}\}$.

for all $N \in \{2^1, 2^2, 2^3, \dots\}$. Additionally, let $L_N \colon \Omega \to \{1, 2, \dots, \mathrm{ld}(N)\}$ be defined as

$$L_N := \max\Big(\{1\} \cup \Big\{l \in \{1, 2, \dots, \mathrm{ld}(N)\} \colon \exists\, k \in \{1, 2, \dots, \tfrac{N}{2^l}\} \colon |\xi^{l,k}| > 2^{\frac{l}{4}} T^{-\frac{1}{4}}\Big\}\Big) \tag{38}$$

for every $N \in \{2^1, 2^2, 2^3, \dots\}$. Furthermore, define $\eta_N \colon \Omega \to [0, \infty)$ and $\theta_N \colon \Omega \to [0, \infty)$ by

$$\eta_N := \max\Big\{|\xi^{L_N,k}| \in \mathbb{R} \colon k \in \big\{1, 2, \dots, \tfrac{N}{2^{L_N}}\big\}\Big\} \tag{39}$$

and

$$\theta_N := \max\Big\{|\xi^{(L_N-1),k}| \in \mathbb{R} \colon k \in \big\{1, 2, \dots, \tfrac{N}{2^{(L_N-1)}}\big\}\Big\} \tag{40}$$

for every $N \in \{2^1, 2^2, 2^3, \dots\}$. Moreover, we define the mappings $\lceil\cdot\rceil, \lfloor\cdot\rfloor \colon \mathbb{R} \to \mathbb{Z}$ by $\lceil x \rceil := \min\{z \in \mathbb{Z} \colon z \geq x\}$ and by $\lfloor x \rfloor := \max\{z \in \mathbb{Z} \colon z \leq x\}$ for all $x \in \mathbb{R}$. Additionally, we fix a real number $\delta \in (0, \frac{1}{2})$ for the rest of this section. In the next step the following events are used in our analysis of the multilevel Monte Carlo Euler method. Let $A_N^{(1)}$, $A_N^{(2)}$, $A_N^{(3)}$, $A_N^{(4)} \in \mathcal{F}$, $N \in \{2^1, 2^2, 2^3, \dots\}$, be defined by

$$A_N^{(1)} := \Big\{L_N < \big\lfloor 2\,\mathrm{ld}\big(\bar{\sigma}^2 T^{\frac{1}{2}} \ln(N)\big)\big\rfloor\Big\} \tag{41}$$

$$A_N^{(2)} := \Big\{\exists\, l \in \{0, 1, 2, \dots, \mathrm{ld}(N)\} \colon \big(\exists\, k \in \{1, 2, \dots, \tfrac{N}{2^l}\} \colon |\xi^{l,k}| \geq 2^{\frac{(l-1)}{4}} T^{-\frac{1}{4}} N\big)\Big\} \tag{42}$$

$$A_N^{(3)} := \Big\{\exists\, l \in \mathbb{N}, \big\lfloor 2\,\mathrm{ld}\big(\bar{\sigma}^2 T^{\frac{1}{2}} \ln(N)\big)\big\rfloor \leq l \leq \mathrm{ld}(N) + 1 \colon 2^{\frac{l}{4}} T^{-\frac{1}{4}} \leq \eta_N < 2^{\frac{l}{4}} T^{-\frac{1}{4}} \big(1 + 5^{(-\delta \cdot 2^{(l-1)})}\big)\Big\} \tag{43}$$

$$A_N^{(4)} := \Big\{|\eta_N - \theta_N| \leq 4^{\left(-2^{(L_N-1)}\right)} \eta_N\Big\} \tag{44}$$

for all $N \in \{2^1, 2^2, 2^3, \dots\}$. Additionally, define $N_0 \in \{2^1, 2^2, 2^3, \dots\}$ and $N_1 \colon \Omega \to \{2^1, 2^2, 2^3, \dots\} \cup \{\infty\}$ by $N_0 := 2^{\left\lceil \exp\left(4\bar{\sigma}^{-2} T^{-\frac{1}{2}}\right) + \bar{\sigma}^8 T^2 \right\rceil}$ and by

$$N_1(\omega) := \min\Big(\{\infty\} \cup \Big\{n \in \{N_0, 2^1 N_0, 2^2 N_0, \dots\} \colon \forall\, m \in \{n, 2^1 n, 2^2 n, \dots\} \colon \omega \notin A_m^{(1)} \cup A_m^{(2)} \cup A_m^{(3)} \cup A_m^{(4)}\Big\}\Big) \tag{45}$$

for all $\omega \in \Omega$. Next we prove a few lemmas that we use in our proof of Theorem 4.1.

Lemma 4.2 (Dynamics for small initial values)**.** *Assume that the above setting is fulfilled. Then we have* $\big|y_n^{N,x}\big| \leq |x| \leq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$ *for all* $n \in \{0, 1, \dots, N\}$, $|x| \geq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$ *and all* $N \in \mathbb{N}$.

Proof of Lemma 4.2. Fix $N \in \mathbb{N}$ and $|x| \leq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$. We prove $\big|y_n^{N,x}\big| \leq |x|$ by induction on $n \in \{0, 1, \dots, N\}$. The base case $n = 0$ is trivial. For the induction step $n \to n+1$, note that the induction hypothesis implies

$$\big|y_{n+1}^{N,x}\big| = \big|y_n^{N,x}\big| \cdot \left|1 - \frac{T}{N}\big(y_n^{N,x}\big)^4\right| \leq \big|y_n^{N,x}\big| \leq |x| \tag{46}$$

for all $n \in \{0, 1, \dots, N-1\}$. This completes the proof of Lemma 4.2. □

Lemma 4.3 (Dynamics for large initial values)**.** *Assume that the above setting is fulfilled. Then we have* $\big|y_n^{N,x}\big| \geq |x| \geq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$ *for all* $n \in \{0, 1, \dots, N\}$, $|x| \geq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$ *and all* $N \in \mathbb{N}$. *In particular, we have*

$$\big|y_{n+1}^{N,x}\big| = \big|y_n^{N,x}\big| \left(\frac{T}{N}\big(y_n^{N,x}\big)^4 - 1\right) \tag{47}$$

for all $n \in \{0, 1, \dots, N-1\}$, $|x| \geq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$ *and all* $N \in \mathbb{N}$.

Proof of Lemma 4.3. Fix $N \in \mathbb{N}$ and $|x| \geq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$. We prove $\big|y_n^{N,x}\big| \geq |x|$ by induction on $n \in \{0, 1, \dots, N\}$. The base case $n = 0$ is trivial. For the induction step $n \to n+1$, note that the induction hypothesis implies

$$\big|y_{n+1}^{N,x}\big| = \big|y_n^{N,x}\big| \cdot \left|\frac{T}{N}\big(y_n^{N,x}\big)^4 - 1\right| = \big|y_n^{N,x}\big| \left(\frac{T}{N}\big(y_n^{N,x}\big)^4 - 1\right) \geq \big|y_n^{N,x}\big| \geq |x| \tag{48}$$

for all $n \in \{0, 1, \dots, N-1\}$. This completes the induction. The assertion (47) then immediately follows by taking absolute values in (36). □

Lemma 4.4 (Growth bound for large initial values)**.** *Assume that the above setting is fulfilled. Then we have*

$$\big(\tfrac{T}{N}\big)^{\frac{1}{4}} \big|y_n^{N,x}\big| \leq \Big(\big(\tfrac{T}{N}\big)^{\frac{1}{4}} |x|\Big)^{(5^n)} \tag{49}$$

for all $n \in \{0, 1, \dots, N\}$, $|x| \geq \big(\frac{2N}{T}\big)^{\frac{1}{4}}$ *and all* $N \in \mathbb{N}$.

Proof of Lemma 4.4. Fix $N \in \mathbb{N}$ and $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}}$. We prove (49) by induction on $n \in \{0, 1, \dots, N\}$. The base case $n = 0$ is trivial. For the induction step $n \to n + 1$, note that Lemma 4.3 and the induction hypothesis imply

$$\left(\tfrac{T}{N}\right)^{\frac{1}{4}} \left|y_{n+1}^{N,x}\right| = \left(\tfrac{T}{N}\right)^{\frac{1}{4}} \left|y_n^{N,x}\right| \left(\tfrac{T}{N}\left(y_n^{N,x}\right)^4 - 1\right) \leq \left(\left(\tfrac{T}{N}\right)^{\frac{1}{4}} \left|y_n^{N,x}\right|\right)^5 \leq \left(\left(\left(\tfrac{T}{N}\right)^{\frac{1}{4}} |x|\right)^{(5^n)}\right)^5 = \left(\left(\tfrac{T}{N}\right)^{\frac{1}{4}} |x|\right)^{\left(5^{(n+1)}\right)} \tag{50}$$

for all $n \in \{0, 1, \dots, N-1\}$. This completes the proof of Lemma 4.4. ☐

Lemma 4.5 (Monotonicity)**.** *Assume that the above setting is fulfilled. Then we have*

$$\left|y_n^{N,x}\right| \geq \left|y_n^{N,y}\right| \tag{51}$$

for all $n \in \{0, 1, \dots, N\}$*, all* $x, y \in \mathbb{R}$ *satisfying* $|x| \geq |y|$*,* $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}}$ *and all* $N \in \mathbb{N}$*.*

Proof of Lemma 4.5. Fix $N \in \mathbb{N}$ and $x, y \in \mathbb{R}$ with $|x| \geq |y|$, $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}}$. We prove (51) by induction on $n \in \{0, 1, \dots, N\}$. The base case $n = 0$ is trivial. For the induction step $n \to n + 1$, note that Lemma 4.3 and the induction hypothesis imply

$$\left|y_{n+1}^{N,x}\right| = \left|y_n^{N,x}\right| \left(\frac{T}{N}\left|y_n^{N,x}\right|^4 - 1\right) \geq \left|y_n^{N,y}\right| \left(\frac{T}{N}\left|y_n^{N,x}\right|^4 - 1\right) \geq \left|y_n^{N,y}\right| \left|\frac{T}{N}\left|y_n^{N,y}\right|^4 - 1\right| = \left|y_{n+1}^{N,y}\right| \tag{52}$$

for all $n \in \{0, 1, \dots, N-1\}$. This completes the proof of Lemma 4.5. ☐

Lemma 4.6 (Dynamics of multiples of the initial value)**.** *Assume that the above setting is fulfilled. Then we have*

$$\left|y_n^{N,Mx}\right| \geq M^{(5^n)} \left|y_n^{N,x}\right| \tag{53}$$

for all $n \in \{0, 1, \dots, N\}$*,* $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}}$*,* $M \in [1, \infty)$ *and all* $N \in \mathbb{N}$*.*

Proof of Lemma 4.6. Fix $N \in \mathbb{N}$. We prove (53) by induction on $n \in \{0, 1, \dots, N\}$. The base case $n = 0$ is trivial. For the induction step $n \to n + 1$, note that Lemma 4.3 and the induction hypothesis imply

$$\begin{aligned} \left|y_{n+1}^{N,Mx}\right| &= \left|y_n^{N,Mx}\right| \left(\frac{T}{N}\left|y_n^{N,Mx}\right|^4 - 1\right) \geq M^{(5^n)} \left|y_n^{N,x}\right| \left(\frac{T}{N}\left(M^{(5^n)} \left|y_n^{N,x}\right|\right)^4 - 1\right) \\ &\geq M^{(5^n)} \left|y_n^{N,x}\right| \left(\frac{T}{N}\left(M^{(5^n)} \left|y_n^{N,x}\right|\right)^4 - M^{(4 \cdot 5^n)}\right) = M^{\left(5^{(n+1)}\right)} \left|y_{n+1}^{N,x}\right| \end{aligned} \tag{54}$$

for all $n \in \{0, 1, \dots, N-1\}$, $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}}$ and all $M \in [1, \infty)$. This completes the proof of Lemma 4.6. ☐

Corollary 4.7. *Assume that the above setting is fulfilled. Then we have* $\left|y_n^{N,x}\right| \geq M^{(5^n)} \left(\frac{2N}{T}\right)^{\frac{1}{4}}$ *for all* $n \in \{0, 1, \dots, N\}$*,* $|x| \geq M\left(\frac{2N}{T}\right)^{\frac{1}{4}}$*,* $M \in [1, \infty)$ *and all* $N \in \mathbb{N}$*.*

Proof of Corollary 4.7. Lemma 4.5, Lemma 4.6 and $|y_n^{N,(\frac{2N}{T})^{\frac{1}{4}}}| = (\frac{2N}{T})^{\frac{1}{4}}$ imply

$$\left|y_n^{N,x}\right| \geq \left|y_n^{N,M\left(\frac{2N}{T}\right)^{\frac{1}{4}}}\right| \geq M^{(5^n)} \left|y_n^{N,(\frac{2N}{T})^{\frac{1}{4}}}\right| = M^{(5^n)} \left(\tfrac{2N}{T}\right)^{\frac{1}{4}} \tag{55}$$

for all $n \in \{0, 1, \dots, N\}$, $|x| \geq M\left(\frac{2N}{T}\right)^{\frac{1}{4}}$, $M \in [1, \infty)$ and all $N \in \mathbb{N}$. This completes the proof of Corollary 4.7. ☐

Lemma 4.8. *Assume that the above setting is fulfilled. Then we have*

$$\left|y_N^{N,x}\right| \geq \left(\tfrac{2N}{T}\right)^{\frac{1}{4}} \sqrt{e}^{\left(5^{(1-r)N}\right)} \tag{56}$$

for all $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}} \left(1 + 5^{(-rN)}\right)$*,* $N \in \mathbb{N}$ *and all* $r \in (0, \infty)$*.*

Proof of Lemma 4.8. We apply the inequality $1 + z \geq \exp(\frac{z}{2})$ for all $z \in [0, 2]$. Noting that $5^{(-rN)} \leq 1 \leq 2$ for all $N \in \mathbb{N}$ and all $r \in (0, \infty)$, we infer from Corollary 4.7

$$\left|y_N^{N,x}\right| \geq \left(\tfrac{2N}{T}\right)^{\frac{1}{4}} \left(1 + 5^{(-rN)}\right)^{(5^N)} \geq \left(\tfrac{2N}{T}\right)^{\frac{1}{4}} \left[\exp\left(\tfrac{1}{2}5^{(-rN)}\right)\right]^{(5^N)} = \left(\tfrac{2N}{T}\right)^{\frac{1}{4}} \sqrt{e}^{\left(5^{(1-r)N}\right)} \tag{57}$$

for all $|x| \geq \left(\frac{2N}{T}\right)^{\frac{1}{4}} \left(1 + 5^{(-rN)}\right)$, $N \in \mathbb{N}$ and all $r \in (0, \infty)$. This completes the proof of Lemma 4.8. ☐

Lemma 4.9 (Almost sure finiteness of N_1)**.** *Assume that the above setting is fulfilled. Then* $\mathbb{P}\big[N_1 < \infty\big] = 1$*.*

The proof of Lemma 4.9 is postponed to the appendix in Section 7. We now present the proof of Theorem 4.1. It makes of use of Lemma 4.9.

Proof of Theorem 4.1. Fix $p \in (0,\infty)$ throughout this proof. Our proof of Theorem 4.1 is then divided into four parts. In the first part we analyze the behavior of the multilevel Monte Carlo Euler approximations on the events $\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\} = \{\omega \in \Omega \colon \eta_N(\omega) > 2^{(L_N(\omega)+1)/4}T^{-1/4}, N_1(\omega) \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ (see inequality (61)). In the second part of this proof we concentrate on the events $\{\theta_N \geq \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ (see inequality (68)). In the third part of this proof we investigate the events $\{2^{L_N/4}T^{-1/4} < \theta_N\} \cap \{\theta_N < \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ (see inequality (73)) and in the fourth part we analyze the behavior of the multilevel Monte Carlo Euler approximations on the events $\{2^{L_N/4}T^{-1/4} \geq \theta_N\} \cap \{\theta_N < \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ (see inequality (74)). Combining all four parts (inequalities (61), (68), (73) and (74)) and Lemma 4.9 will then complete the proof of Theorem 4.1 as we will show below. In these four parts we will frequently use

$$\{N_1 \leq N\} \subseteq \big(A_N^{(1)}\big)^c \cap \big(A_N^{(2)}\big)^c \cap \big(A_N^{(3)}\big)^c \cap \big(A_N^{(4)}\big)^c \tag{58}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$.

We begin with the first part and consider the events $\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$. Note that Lemma 4.5, the inequalities $\eta_N \geq 2^{(L_N+1)/4}T^{-1/4}\big(1 + 5^{(-\delta \cdot 2^{L_N})}\big)$ on $\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ (see (43)) and $|\xi^{l,k}| < 2^{(l-1)/4}T^{-1/4}N$ on $\{N_1 \leq N\}$ for all $k \in \{1, 2, \ldots, \frac{N}{2^l}\}$, $l \in \{1, 2, \ldots, \mathrm{ld}(N)\}$ (see (42)) and the definition (38) of L_N imply

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^l}^{2^l, \xi^{l,k}} \Big|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \Big|^p \\
&\geq \frac{2^{L_N}}{N} \Big| y_{2^{L_N}}^{2^{L_N}, \eta_N} \Big|^p - \sum_{l=1}^{L_N} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \Big|^p - \sum_{l=L_N+1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \Big|^p \\
&\geq \frac{2^{L_N}}{N} \left| y_{2^{L_N}}^{2^{L_N}, \left(\frac{2^{(L_N+1)}}{T}\right)^{\frac14}\left(1+5^{(-\delta 2^{L_N})}\right)} \right|^p - \sum_{l=1}^{L_N} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \left(\frac{2^{(l-1)}}{T}\right)^{\frac14} N} \right|^p - \sum_{l=L_N+1}^{\mathrm{ld}(N)} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \left(\frac{2^l}{T}\right)^{\frac14}} \right|^p
\end{aligned} \tag{59}$$

on $\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ and Lemma 4.8, Lemma 4.4 and Lemma 4.2 hence yield

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^l}^{2^l, \xi^{l,k}} \Big|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \Big|^p \\
&\geq \frac{1}{N} \left| \left(\frac{2 \cdot 2^{L_N}}{T}\right)^{\frac14} \sqrt{e}^{\left(5^{(1-\delta)2^{L_N}}\right)} \right|^p - \sum_{l=1}^{L_N} \left(\frac{2^{(l-1)}}{T}\right)^{\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^{(l-1)}\right)}\right)} - \sum_{l=L_N+1}^{\mathrm{ld}(N)} \left(\frac{2^l}{T}\right)^{\frac{p}{4}} \\
&\geq N^{-1}\, T^{-\frac{p}{4}} \cdot \exp\!\Big(\tfrac{p}{2} \cdot 5^{(1-\delta)2^{L_N}}\Big) - L_N \left(\frac{2^{(L_N-1)}}{T}\right)^{\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)} - \mathrm{ld}(N) \left(\frac{2^{\mathrm{ld}(N)}}{T}\right)^{\frac{p}{4}}
\end{aligned}$$

on $\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Therefore, we obtain

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^l}^{2^l, \xi^{l,k}} \Big|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \Big|^p \\
&\geq N^{-1} T^{-\frac{p}{4}} \cdot \exp\!\Big(\tfrac{p}{2} \cdot 5^{(1-\delta)2^{L_N}}\Big) - \mathrm{ld}(N) N^{\frac{p}{4}} T^{-\frac{p}{4}} \cdot N^{\left(p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)} - \mathrm{ld}(N) N^{\frac{p}{4}} T^{-\frac{p}{4}} \\
&\geq T^{-\frac{p}{4}} \cdot \exp\!\Big(\tfrac{p}{2} \cdot 5^{(1-\delta)2^{L_N}} - \ln(N)\Big) - T^{-\frac{p}{4}} \cdot N^{\left(1 + \frac{p}{4} + p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)}
\end{aligned} \tag{60}$$

on $\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\} \cap \{N_1 \leq N\}$ and the estimate $2^{L_N} \geq \bar{\sigma}^2 \sqrt{T} \ln(N)$ on $\{N_1 \leq N\}$ (see (41)) hence shows

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^l}^{2^l, \xi^{l,k}} \Big|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \Big| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \Big|^p \\
&\geq \inf_{x \in [\bar{\sigma}^2 \sqrt{T} \ln(N), \infty)} \Big[\exp\!\Big(\tfrac{p}{2} \cdot 5^{((1-\delta)x)} - \ln(N)\Big) - \exp\!\Big(\ln(N)\big(1 + \tfrac{p}{4} + p \cdot 5^{\frac{x}{2}}\big)\Big) \Big] \cdot T^{-\frac{p}{4}} \geq r(N) \cdot T^{-\frac{p}{4}}
\end{aligned} \tag{61}$$

on $\left\{\eta_N > 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ where $r\colon \mathbb{N} \to \mathbb{R}$ is a function defined by

$$r(N) := \inf_{x \in \left[\bar{\sigma}^2 \sqrt{T} \ln(N), \infty\right)} \left[\exp\Big(\tfrac{p}{2} \cdot 5^{((1-\delta)x)} - \ln(2N)\Big) - \exp\Big(\ln(N)\left(1 + \tfrac{p}{4} + p \cdot 5^{\frac{x}{2}}\right)\Big)\right]$$

for all $N \in \mathbb{N}$.

In the next step we analyze the behavior of the multilevel Monte Carlo Euler approximations on the events $\{\theta_N \geq \eta_N\} \cap \left\{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$. To this end note that Lemma 4.5, the inequalities $\theta_N \geq \left(1+4^{\left(-2^{(L_N-1)}\right)}\right)\eta_N$ on $\{\theta_N \geq \eta_N\} \cap \{N_1 \leq N\}$ (see (44)) and $|\xi^{l,k}| < 2^{(l-1)/4}T^{-1/4}N$ on $\{N_1 \leq N\}$ for all $k \in \{1, 2, \ldots, \frac{N}{2^l}\}$, $l \in \{1, 2, \ldots, \mathrm{ld}(N)\}$ (see (42)) and the definition (38) of L_N imply

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \frac{1}{N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \theta_N} \right|^p - \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p - \sum_{l=1}^{L_N-1} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p - \sum_{l=L_N+1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \frac{1}{N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \left(1+4^{\left(-2^{(L_N-1)}\right)}\right)\eta_N} \right|^p - \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p - \sum_{l=1}^{L_N-1} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \left(\frac{2^{(l-1)}}{T}\right)^{\frac{1}{4}} N} \right|^p - \sum_{l=L_N+1}^{\mathrm{ld}(N)} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \left(\frac{2^l}{T}\right)^{\frac{1}{4}}} \right|^p
\end{aligned}$$

on $\{\theta_N \geq \eta_N\} \cap \left\{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Lemma 4.5, Lemma 4.4 and Lemma 4.2 therefore show

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \left(1+4^{\left(-2^{(L_N-1)}\right)}\right)\eta_N} \right|^p - \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p + \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad - \sum_{l=1}^{L_N-1} \left(\frac{2^{(l-1)}}{T}\right)^{\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^{(l-1)}\right)}\right)} - \sum_{l=L_N+1}^{\mathrm{ld}(N)} \left(\frac{2^l}{T}\right)^{\frac{p}{4}}
\end{aligned} \tag{62}$$

on $\{\theta_N \geq \eta_N\} \cap \left\{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. By definition of η_N and of L_N we have $\eta_N \geq 2^{L_N/4}T^{-1/4}$ on $\{N_1 \leq N\}$ (see (41)) for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Consequently we get the inequality $\eta_N \geq 2^{L_N/4}T^{-1/4}\left(1 + 5^{\left(-\delta \cdot 2^{(L_N-1)}\right)}\right)$ on $\{N_1 \leq N\}$ (see (43)) for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Lemma 4.6 and Lemma 4.5 hence yield

$$\begin{aligned}
\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p &\geq \left(\frac{1}{2N} \left(1 + 4^{\left(-2^{(L_N-1)}\right)}\right)^{\left(p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)} - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad + \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \left(\frac{2^{L_N}}{T}\right)^{\frac{1}{4}} \left(1+5^{\left(-\delta 2^{(L_N-1)}\right)}\right)} \right|^p \\
&\quad - L_N \left(\frac{2^{(L_N-2)}}{T}\right)^{\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^{(L_N-2)}\right)}\right)} - \mathrm{ld}(N) \left(\frac{2^{\mathrm{ld}(N)}}{T}\right)^{\frac{p}{4}}
\end{aligned} \tag{63}$$

on $\{\theta_N \geq \eta_N\} \cap \left\{\eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Lemma 4.8 and $L_N \leq \mathrm{ld}(N)$ therefore imply

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \left(\frac{1}{2N} \left(1 + 2^{\left(-2^{L_N}\right)}\right)^{\left(p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)} - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad + \frac{1}{2N} \left(\frac{2^{L_N}}{T}\right)^{\frac{p}{4}} \exp\!\left(\frac{p}{2} \cdot 5^{\left((1-\delta)2^{(L_N-1)}\right)}\right) - \mathrm{ld}(N) \left(\frac{2^{L_N}}{T}\right)^{\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^{(L_N-2)}\right)}\right)} - \mathrm{ld}(N) N^{\frac{p}{4}} T^{-\frac{p}{4}}
\end{aligned}$$

on $\{\theta_N \geq \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. The inequalities $1 \leq L_N \leq \mathrm{ld}(N)$ and $1 + 2^{(-x)} \geq \exp(2^{(-x-1)})$ for all $x \in [0, \infty)$ hence give

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \left(\frac{1}{2N} \cdot \exp\Big(2^{\left(-2^{L_N}-1\right)} \cdot p \cdot 5^{\left(2^{(L_N-1)}\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad + \frac{1}{2NT^{\frac{p}{4}}} \cdot \exp\Big(\tfrac{p}{2} \cdot 5^{\left((1-\delta)2^{(L_N-1)}\right)} \Big) - 2\,\mathrm{ld}(N) N^{\frac{p}{4}} T^{-\frac{p}{4}} \cdot N^{\left(p \cdot 5^{\left(2^{(L_N-2)}\right)}\right)}
\end{aligned} \tag{64}$$

on $\{\theta_N \geq \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. This shows

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \left(\tfrac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot \left(\tfrac{5}{4}\right)^{\left(2^{(L_N-1)}\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad + \frac{1}{2NT^{\frac{p}{4}}} \cdot \exp\Big(\tfrac{p}{2} \cdot 5^{\left((1-\delta)2^{(L_N-1)}\right)} \Big) - T^{-\frac{p}{4}} \cdot N^{\left(1 + \frac{p}{4} + p \cdot 5^{\left(2^{(L_N-2)}\right)}\right)} \\
&\geq \left(\tfrac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot \left(\tfrac{5}{4}\right)^{\left(2^{(L_N-1)}\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad + \inf_{x \in \left[2^{(L_N-1)}, \infty\right)} \left[\exp\Big(\tfrac{p}{2} \cdot 5^{((1-\delta)x)} - \ln(2N) \Big) - \exp\Big(\ln(N) \left(1 + \tfrac{p}{4} + p \cdot 5^{\frac{x}{2}}\right) \Big) \right] \cdot T^{-\frac{p}{4}}
\end{aligned}$$

on $\{\theta_N \geq \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and, using the estimate $2^{(L_N-1)} \geq \bar{\sigma}^2 \sqrt{T} \ln(N)$ on $\{N_1 \leq N\}$ (see (41)),

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \left(\tfrac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot \left(\tfrac{5}{4}\right)^{\left(\bar{\sigma}^2 \sqrt{T} \ln(N)\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad + \inf_{x \in \left[\bar{\sigma}^2 \sqrt{T} \ln(N), \infty\right)} \left[\exp\Big(\tfrac{p}{2} \cdot 5^{((1-\delta)x)} - \ln(2N) \Big) - \exp\Big(\ln(N) \left(1 + \tfrac{p}{4} + p \cdot 5^{\frac{x}{2}}\right) \Big) \right] \cdot T^{-\frac{p}{4}}
\end{aligned} \tag{65}$$

on $\{\theta_N \geq \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. It follows from

$$\lim_{N \to \infty} \tfrac{1}{2N} \exp\Big(\tfrac{p}{2} \cdot N^{\left(\bar{\sigma}^2 \sqrt{T} \ln(5/4)\right)} \Big) = \infty \tag{66}$$

that there exists an $N_2 \in \{2^1, 2^2, 2^3, \ldots\}$ such that

$$\frac{1}{2N} \exp\Big(\tfrac{p}{2} \cdot N^{\left(\bar{\sigma}^2 \sqrt{T} \ln(5/4)\right)} \Big) - 1 \geq 0 \tag{67}$$

for all $N \in [N_2, \infty)$. Using this, we deduce from (65)

$$\begin{aligned}
&\sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \\
&\geq \left(\tfrac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot N^{\left(\bar{\sigma}^2 \sqrt{T} \ln(5/4)\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p + r(N) \cdot T^{-\frac{p}{4}} \geq r(N) \cdot T^{-\frac{p}{4}}
\end{aligned} \tag{68}$$

on $\{\theta_N \geq \eta_N\} \cap \{\eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\} \cap \{N_1 \leq N\}$ for all $N \in \{N_2, 2^1 N_2, 2^2 N_2, \ldots\}$.

Next, we analyze the behavior of the multilevel Monte Carlo Euler approximations on the events $\{2^{L_N/4} T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$. Note that Lemma 4.5 and the inequality

$|\xi^{l,k}| < 2^{(l-1)/4} T^{-1/4} N$ on $\{N_1 \leq N\}$ for all $k \in \{1, 2, \ldots, \frac{N}{2^l}\}$, $l \in \{0, 1, 2, \ldots, \mathrm{ld}(N)\}$ (see (42)) imply

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \right| \\
&\geq \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p - \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p \\
&\geq \frac{1}{N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p - \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \theta_N} \right|^p - \sum_{l=0}^{L_N-2} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=L_N}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p \\
&\geq \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p - \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \theta_N} \right|^p + \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p - \sum_{l=0}^{L_N-2} \left| y_{2^l}^{2^l, \left(\frac{2^{(l-1)}}{T}\right)^{\frac{1}{4}} N} \right|^p - \sum_{l=L_N}^{\mathrm{ld}(N)} \left| y_{2^l}^{2^l, \left(\frac{2^{(l+1)}}{T}\right)^{\frac{1}{4}}} \right|^p
\end{aligned} \tag{69}$$

on $\left\{2^{L_N/4} T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Therefore Lemma 4.5, the inequality $\eta_N \geq \left(1 + 4^{\left(-2^{(L_N-1)}\right)}\right) \theta_N$ on $\{\theta_N < \eta_N\} \cap \{N_1 \leq N\}$ (see (44)) and Lemma 4.2 result in

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \right| \\
&\geq \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \left(1+4^{\left(-2^{(L_N-1)}\right)}\right)\theta_N} \right|^p - \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \theta_N} \right|^p + \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \eta_N} \right|^p \\
&\quad - \sum_{l=0}^{L_N-2} \left| y_{2^l}^{2^l, \left(\frac{2^l}{T}\right)^{\frac{1}{4}} N} \right|^p - \sum_{l=L_N}^{\mathrm{ld}(N)} \left(\frac{2^{(l+1)}}{T} \right)^{\frac{p}{4}}
\end{aligned} \tag{70}$$

on $\left\{2^{L_N/4} T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Lemma 4.6, Lemma 4.5 and the estimate $\eta_N \geq 2^{L_N/4} T^{-1/4} \left(1 + 5^{\left(-\delta \cdot 2^{(L_N-1)}\right)}\right)$ on $\{N_1 \leq N\}$ (see (41) and (43)) and Lemma 4.4 hence yield

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \right| \\
&\geq \left(\frac{1}{2N} \left(1 + 4^{\left(-2^{(L_N-1)}\right)}\right)^{\left(p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)} - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \theta_N} \right|^p \\
&\quad + \frac{1}{2N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \left(\frac{2^{L_N}}{T}\right)^{\frac{1}{4}} \left(1+5^{-\delta \cdot 2^{(L_N-1)}}\right)} \right|^p - \sum_{l=0}^{L_N-2} \left(\frac{2^l}{T} \right)^{\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^l\right)}\right)} - 2^{\frac{p}{4}} \, \mathrm{ld}(N) N^{\frac{p}{4}} T^{-\frac{p}{4}}
\end{aligned}$$

on $\left\{2^{L_N/4} T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Therefore Lemma 4.8 implies

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l, \xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)}, \xi^{l,k}} \right|^p \right| \\
&\geq \left(\frac{1}{2N} \left(1 + 2^{\left(-2^{L_N}\right)}\right)^{\left(p \cdot 5^{\left(2^{(L_N-1)}\right)}\right)} - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)}, \theta_N} \right|^p \\
&\quad + \frac{1}{2N} \left(\frac{2^{L_N}}{T} \right)^{\frac{p}{4}} \exp\!\left(\frac{p}{2} \cdot 5^{\left((1-\delta) 2^{(L_N-1)}\right)} \right) - 2 \, \mathrm{ld}(N) N^{\frac{p}{4}} T^{-\frac{p}{4}} N^{\left(p \cdot 5^{\left(2^{(L_N-2)}\right)}\right)}
\end{aligned} \tag{71}$$

on $\left\{2^{L_N/4} T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4} T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. The inequality $1 + 2^{(-x)} \geq$

$\exp(2^{(-x-1)})$ for all $x \in [0,\infty)$ hence shows

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}} \right|^p \right| \\
&\geq \left(\frac{1}{2N} \cdot \exp\Big(2^{\left(-2^{L_N}-1\right)} \cdot p \cdot 5^{\left(2^{(L_N-1)}\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)},\theta_N} \right|^p \\
&\quad + \frac{1}{2NT^{\frac{p}{4}}} \cdot \exp\Big(\tfrac{p}{2} \cdot 5^{\left((1-\delta)2^{(L_N-1)}\right)} \Big) - T^{-\frac{p}{4}} \cdot N^{\left(1+\frac{p}{4}+p\cdot 5^{\left(2^{(L_N-2)}\right)}\right)}
\end{aligned} \tag{72}$$

on $\left\{2^{L_N/4}T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Consequently

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}} \right|^p \right| \\
&\geq \left(\frac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot \left(\tfrac{5}{4}\right)^{\left(2^{(L_N-1)}\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)},\theta_N} \right|^p \\
&\quad + \inf_{x \in \left[2^{(L_N-1)},\infty\right)} \left[\exp\Big(\tfrac{p}{2} \cdot 5^{((1-\delta)x)} - \ln(2N) \Big) - \exp\Big(\ln(N) \left(1 + \tfrac{p}{4} + p \cdot 5^{\frac{x}{2}}\right) \Big) \right] \cdot T^{-\frac{p}{4}}
\end{aligned}$$

on $\left\{2^{L_N/4}T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. The estimate $2^{(L_N-1)} \geq \bar{\sigma}^2\sqrt{T}\ln(N)$ on $\{N_1 \leq N\}$ (see (41)) therefore implies

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}} \right|^p \right| \\
&\geq \left(\frac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot \left(\tfrac{5}{4}\right)^{\left(\bar{\sigma}^2\sqrt{T}\ln(N)\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)},\theta_N} \right|^p \\
&\quad + \inf_{x \in \left[\bar{\sigma}^2\sqrt{T}\ln(N),\infty\right)} \left[\exp\Big(\tfrac{p}{2} \cdot 5^{((1-\delta)x)} - \ln(2N) \Big) - \exp\Big(\ln(N) \left(1 + \tfrac{p}{4} + p \cdot 5^{\frac{x}{2}}\right) \Big) \right] \cdot T^{-\frac{p}{4}}
\end{aligned}$$

on $\left\{2^{L_N/4}T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3 \ldots\}$. Finally, we obtain

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}} \right|^p \right| \\
&\geq \left(\tfrac{1}{2N} \cdot \exp\Big(\tfrac{p}{2} \cdot N^{\left(\bar{\sigma}^2\sqrt{T}\ln(5/4)\right)} \Big) - 1 \right) \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)},\theta_N} \right|^p + r(N) \cdot T^{-\frac{p}{4}} \geq r(N) \cdot T^{-\frac{p}{4}}
\end{aligned} \tag{73}$$

on $\left\{2^{L_N/4}T^{-1/4} < \theta_N < \eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{N_2, 2^1N_2, 2^2N_2, \ldots\}$.

Finally, we analyze the behavior of the multilevel Monte Carlo Euler approximations on the events $\left\{\theta_N \leq 2^{L_N/4}T^{-1/4}\right\} \cap \left\{\theta_N < \eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$. Note that Lemma 4.5 and the inequality $\left|\xi^{l,k}\right| < 2^{(l-1)/4}T^{-1/4}N$ on $\{N_1 \leq N\}$ for all $k \in \{1, 2, \ldots, \frac{N}{2^l}\}$, $l \in \{0, 1, \ldots, \mathrm{ld}(N)\}$ (see (42)) imply

$$\begin{aligned}
&\left| \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}} \right|^p \right| \\
&\geq \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}} \right|^p - \sum_{l=0}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p \\
&\geq \frac{1}{N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)},\eta_N} \right|^p - \sum_{l=0}^{L_N-2} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p - \sum_{l=L_N-1}^{\mathrm{ld}(N)} \frac{2^l}{N} \sum_{k=1}^{\frac{N}{2^l}} \left| y_{2^l}^{2^l,\xi^{l,k}} \right|^p \\
&\geq \frac{1}{N} \left| y_{2^{(L_N-1)}}^{2^{(L_N-1)},\left(\frac{2^{L_N}}{T}\right)^{\frac{1}{4}}\left(1+5^{\left(-\delta\cdot 2^{(L_N-1)}\right)}\right)} \right|^p - \sum_{l=0}^{L_N-2} \left| y_{2^l}^{2^l,\left(\frac{2^{(l-1)}}{T}\right)^{\frac{1}{4}}N} \right|^p - \sum_{l=L_N-1}^{\mathrm{ld}(N)} \left| y_{2^l}^{2^l,\left(\frac{2\cdot 2^l}{T}\right)^{\frac{1}{4}}} \right|^p
\end{aligned}$$

on $\left\{\theta_N \leq 2^{L_N/4}T^{-1/4}\right\} \cap \left\{\theta_N < \eta_N \leq 2^{(L_N+1)/4}T^{-1/4}\right\} \cap \{N_1 \leq N\}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and, applying

Lemma 4.8, Lemma 4.5, Lemma 4.4 and Lemma 4.2,

$$\begin{aligned}&\left|\sum_{l=0}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^l}^{2^l,\xi^{l,k}}\right|^p-\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}}\right|^p\right|\\&\geq\frac{1}{N}\left|\left(\frac{2^{L_N}}{T}\right)^{\frac{1}{4}}\sqrt{e}^{\left(5^{(1-\delta)2^{(L_N-1)}}\right)}\right|^p-\sum_{l=0}^{L_N-2}\left(\frac{2^l}{T}\right)^{\frac{p}{4}}N^{\left(p\cdot 5^{\left(2^l\right)}\right)}-\sum_{l=L_N-1}^{\mathrm{ld}(N)}\left(\frac{2\cdot 2^l}{T}\right)^{\frac{p}{4}}\\&\geq N^{-1}\,T^{-\frac{p}{4}}\cdot\exp\!\Big(\tfrac{p}{2}\cdot 5^{(1-\delta)2^{(L_N-1)}}\Big)-L_N\left(\frac{2^{(L_N-2)}}{T}\right)^{\frac{p}{4}}N^{\left(p\cdot 5^{\left(2^{(L_N-2)}\right)}\right)}-\mathrm{ld}(N)\left(\frac{2\cdot 2^{\mathrm{ld}(N)}}{T}\right)^{\frac{p}{4}}\end{aligned}$$

on $\left\{\theta_N\leq 2^{L_N/4}T^{-1/4}\right\}\cap\left\{\theta_N<\eta_N\leq 2^{(L_N+1)/4}T^{-1/4}\right\}\cap\{N_1\leq N\}$ for all $N\in\{2^1,2^2,2^3,\ldots\}$. Therefore, we obtain

$$\begin{aligned}&\left|\sum_{l=0}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^l}^{2^l,\xi^{l,k}}\right|^p-\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}}\right|^p\right|\\&\geq N^{-1}T^{-\frac{p}{4}}\cdot\exp\!\Big(\tfrac{p}{2}\cdot 5^{(1-\delta)2^{(L_N-1)}}\Big)-\mathrm{ld}(N)N^{\frac{p}{4}}T^{-\frac{p}{4}}\cdot N^{\left(p\cdot 5^{\left(2^{(L_N-2)}\right)}\right)}-\mathrm{ld}(N)\left(\frac{2N}{T}\right)^{\frac{p}{4}}\\&\geq T^{-\frac{p}{4}}\cdot\exp\!\Big(\tfrac{p}{2}\cdot 5^{(1-\delta)2^{(L_N-1)}}-\ln(N)\Big)-T^{-\frac{p}{4}}\cdot N^{\left(1+\frac{p}{4}+p\cdot 5^{\left(2^{(L_N-2)}\right)}\right)}\end{aligned}$$

on $\left\{\theta_N\leq 2^{L_N/4}T^{-1/4}\right\}\cap\left\{\theta_N<\eta_N\leq 2^{(L_N+1)/4}T^{-1/4}\right\}\cap\{N_1\leq N\}$ and hence, using $2^{(L_N-1)}\geq\bar{\sigma}^2\sqrt{T}\ln(N)$ on $\{N_1\leq N\}$,

$$\begin{aligned}&\left|\sum_{l=0}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^l}^{2^l,\xi^{l,k}}\right|^p-\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}}\right|^p\right|\\&\geq\inf_{x\in[\bar{\sigma}^2\sqrt{T}\ln(N),\infty)}\left[\exp\!\Big(\tfrac{p}{2}\cdot 5^{((1-\delta)x)}-\ln(2N)\Big)-\exp\!\Big(\ln(N)\left(1+\tfrac{p}{4}+p\cdot 5^{\frac{x}{2}}\right)\Big)\right]\cdot T^{-\frac{p}{4}}=r(N)\cdot T^{-\frac{p}{4}}\end{aligned}\tag{74}$$

on $\left\{\theta_N\leq 2^{L_N/4}T^{-1/4}\right\}\cap\left\{\theta_N<\eta_N\leq 2^{(L_N+1)/4}T^{-1/4}\right\}\cap\{N_1\leq N\}$ for all $N\in\{2^1,2^2,2^3,\ldots\}$.

Combining (61), (68), (73) and (74) then shows

$$\left|\sum_{l=0}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^l}^{2^l,\xi^{l,k}}\right|^p-\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}}\right|^p\right|\geq r(N)\cdot T^{-\frac{p}{4}}\tag{75}$$

on $\{N_1\leq N\}$ for all $N\in\{N_2,2^1N_2,2^2N_2,\ldots\}$. Equation (37) and inequality (75) imply

$$\begin{aligned}&\left|\frac{1}{N}\sum_{k=1}^{N}\left|Y_1^{1,0,k}(\omega)\right|^p+\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\Big(\left|Y_{2^l}^{2^l,l,k}(\omega)\right|^p-\left|Y_{2^{(l-1)}}^{2^{(l-1)},l,k}(\omega)\right|^p\Big)\right|\\&=\left|\sum_{l=0}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^l}^{2^l,\xi^{l,k}(\omega)}\right|^p-\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\left|y_{2^{(l-1)}}^{2^{(l-1)},\xi^{l,k}(\omega)}\right|^p\right|\geq r(N)\cdot T^{-\frac{p}{4}}\end{aligned}\tag{76}$$

for all $N\in\{N_1(\omega),2^1\cdot N_1(\omega),2^2\cdot N_1(\omega),\ldots\}\cap\big[N_2,\infty\big)$ and all $\omega\in\{N_1<\infty\}$. The fact $\lim_{N\to\infty}r(N)=\infty$ therefore shows

$$\lim_{\substack{N\to\infty\\ \mathrm{ld}(N)\in\mathbb{N}}}\left|\frac{1}{N}\sum_{k=1}^{N}\left|Y_1^{1,0,k}(\omega)\right|^p+\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\Big(\left|Y_{2^l}^{2^l,l,k}(\omega)\right|^p-\left|Y_{2^{(l-1)}}^{2^{(l-1)},l,k}(\omega)\right|^p\Big)\right|=\infty\tag{77}$$

for all $\omega\in\{N_1<\infty\}$. Hence, Lemma 4.9 finally yields

$$\lim_{\substack{N\to\infty\\ \mathrm{ld}(N)\in\mathbb{N}}}\left|\frac{1}{N}\sum_{k=1}^{N}\left|Y_1^{1,0,k}\right|^p+\sum_{l=1}^{\mathrm{ld}(N)}\frac{2^l}{N}\sum_{k=1}^{\frac{N}{2^l}}\Big(\left|Y_{2^l}^{2^l,l,k}\right|^p-\left|Y_{2^{(l-1)}}^{2^{(l-1)},l,k}\right|^p\Big)\right|=\infty\tag{78}$$

$\mathbb{P}$-almost surely. This completes the proof of Theorem 4.1. □

5 Divergence of the multilevel Monte Carlo Euler method

Motivated by Figure 4 below and by the divergence result of the multilevel Monte Carlo Euler method in Section 4, we conjecture in this section that the multilevel Monte Carlo Euler method diverges with probability one whenever one of the coefficients of the SDE grows superlinearly (see Conjecture 5.1). Whereas divergence with probability one seems to be quite difficult to establish, strong divergence is a rather immediate consequence of the divergence of the Euler method in Theorem 2.1 above. We derive this strong divergence in Corollary 5.2 below. For practical simulations the much more important question is, however, consistency and inconsistency respectively; see, e.g., Nikulin [37], Cramér [2], Appendix A.1 in Glasserman [10] and also Theorem 4.1 above and Conjecture 5.1 below.

Throughout this section assume that the following setting is fulfilled. Let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W^{l,k}\colon [0,T]\times\Omega \to \mathbb{R}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, be a family of independent one-dimensional standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motions and let $\xi^{l,k}\colon \Omega \to \mathbb{R}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, be a family of independent identically distributed $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mappings with $\mathbb{E}\big[|\xi^{0,1}|^p\big] < \infty$ for all $p \in [1,\infty)$. Moreover, let $\mu, \sigma\colon \mathbb{R} \to \mathbb{R}$ be two continuous mappings such that there exists a predictable stochastic process $X\colon [0,T]\times\Omega \to \mathbb{R}$ which satisfies $\int_0^T |\mu(X_s)| + |\sigma(X_s)|^2\, ds < \infty$ $\mathbb{P}$-almost surely and

$$X_t = \xi^{0,1} + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s^{0,1} \tag{79}$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$. The drift coefficient μ is the infinitesimal mean of the process X and the diffusion coefficient σ is the infinitesimal standard deviation of the process X. We then define a family of Euler approximations $Y_n^{N,l,k}\colon \Omega \to \mathbb{R}$, $n \in \{0,1,\dots,N\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, by $Y_0^{N,l,k} := \xi^{l,k}$ and

$$Y_{n+1}^{N,l,k} := Y_n^{N,l,k} + \mu(Y_n^{N,l,k}) \cdot \frac{T}{N} + \sigma(Y_n^{N,l,k}) \cdot \left(W^{l,k}_{\frac{(n+1)T}{N}} - W^{l,k}_{\frac{nT}{N}} \right)$$

for all $n \in \{0,1,\dots,N-1\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$ and all $k \in \mathbb{N}$.

Conjecture 5.1 (Divergence with probability one of the multilevel Monte Carlo Euler method)**.** *Assume that the above setting is fulfilled and let $\alpha, c \in (1,\infty)$ be real numbers such that $\frac{|x|^\alpha}{c} \leq |\mu(x)| + |\sigma(x)| \leq c\,|x|^c$ for all $x \in \mathbb{R}$ with $|x| \geq c$. Moreover, assume that $\mathbb{P}\big[\sigma(\xi^{0,1}) \neq 0\big] > 0$ or that there exists a real number $\beta \in (1,\infty)$ such that $\mathbb{P}\big[\,|\xi^{0,1}| \geq x\big] \geq \beta^{(-x^\beta)}$ for all $x \in [1,\infty)$. Moreover, let $f\colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable with $\frac{1}{c}|x|^{\frac{1}{c}} - c \leq f(x) \leq c(1+|x|^c)$ for all $x \in \mathbb{R}$. Then we conjecture*

$$\lim_{\substack{N\to\infty \\ \mathrm{ld}(N)\in\mathbb{N}}} \left| \mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(Y_1^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} f(Y_{2^l}^{2^l,l,k}) - f(Y_{2^{(l-1)}}^{2^{(l-1)},l,k}) \right) \right| = \infty \tag{80}$$

$\mathbb{P}$-almost surely.

To support this conjecture, we ran simulations for the stochastic Ginzburg-Landau equation given by the solution $(X_t)_{t\in[0,1]}$ of

$$dX_t = \left(2X_t - X_t^3\right) dt + 2X_t\, dW_t, \qquad X_0 = 1 \tag{81}$$

for all $t \in [0,1]$. Its solution is known explicitly (e.g. Section 4.4 in [27]) and is given by

$$X_t = \frac{\exp\left(2W_t\right)}{\sqrt{1 + 2\int_0^t \exp\left(4W_s\right) ds}} \tag{82}$$

for $t \in [0,1]$. We used this explicit solution to compute $\mathbb{E}[(X_1)^2] \approx 0.8114$. Figure 4 shows four sample paths of the approximation error of the multilevel Monte Carlo Euler method for the Ginzburg-Landau equation (81). Only finite values of the sample paths are plotted. The next corollary is an immediate consequence of Theorem 2.1 above.

Corollary 5.2 (Strong divergence of the multilevel Monte Carlo Euler method)**.** *Assume that the above setting is fulfilled and let $\alpha, c \in (1,\infty)$ be real numbers such that $\frac{|x|^\alpha}{c} \leq |\mu(x)| + |\sigma(x)| \leq c\,|x|^c$ for all $x \in \mathbb{R}$ with $|x| \geq c$. Moreover, assume that $\mathbb{P}\big[\sigma(\xi^{0,1}) \neq 0\big] > 0$ or that there exists a real number $\beta \in (1,\infty)$ such that $\mathbb{P}\big[|\xi^{0,1}| \geq x\big] \geq \beta^{(-x^\beta)}$ for all $x \in [1,\infty)$. Additionally, let $f\colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable with $\frac{1}{c}|x|^{\frac{1}{c}} - c \leq f(x) \leq c(1+|x|^c)$ for all $x \in \mathbb{R}$. Then we obtain*

$$\lim_{\substack{N\to\infty \\ \mathrm{ld}(N)\in\mathbb{N}}} \mathbb{E}\left[\left| \mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(Y_1^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} f(Y_{2^l}^{2^l,l,k}) - f(Y_{2^{(l-1)}}^{2^{(l-1)},l,k}) \right) \right|^p \right] = \infty \tag{83}$$

for all $p \in [1,\infty)$.

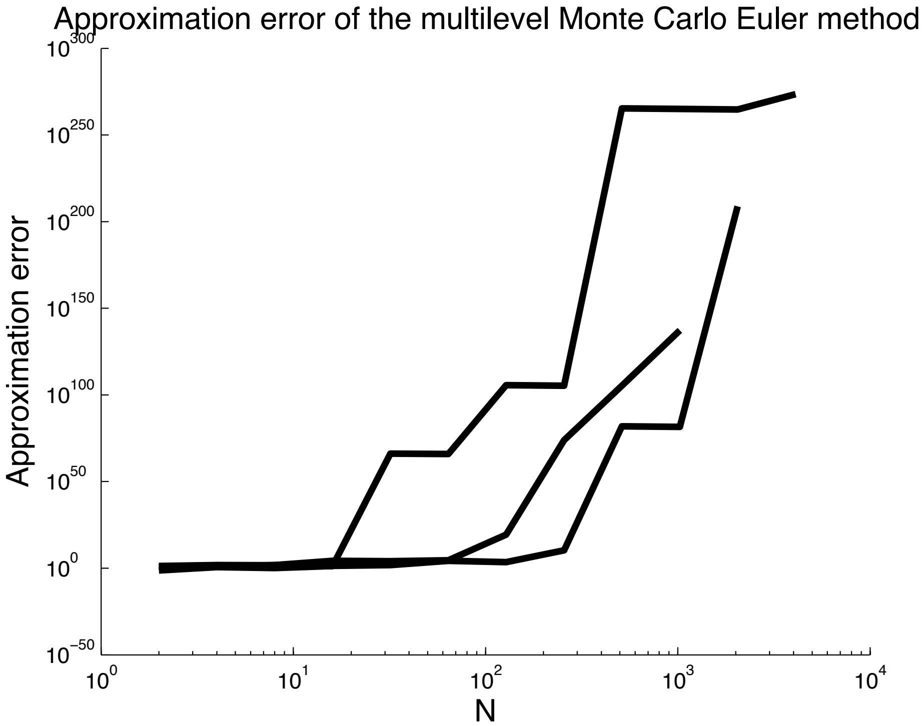

Figure 4: Four sample paths of the approximation error of the multilevel Monte Carlo Euler approximation for the Ginzburg-Landau equation (81).

Proof of Corollary 5.2. First of all, note that the assumption $\mathbb{E}\big[|\xi^{0,1}|^p\big] < \infty$ for all $p \in [1,\infty)$, the continuity of $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$, the inequality $|\mu(x)| + |\sigma(x)| \leq c\,|x|^c$ for all $x \in \mathbb{R}$ with $|x| \geq c$ and the estimate $|f(x)| \leq c(1+|x|^c)$ for all $x \in \mathbb{R}$ imply $\mathbb{E}\big[|f(Y_N^{N,0,1})|\big] < \infty$ for all $N \in \mathbb{N}$. Therefore, we obtain

$$\mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N} f(Y_1^{1,0,k}) + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(Y_{2^l}^{2^l,l,k}) - f(Y_{2^{(l-1)}}^{2^{(l-1)},l,k})\right)\right] = \mathbb{E}\left[f(Y_N^{N,0,1})\right]$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. The estimate $f(x) \geq \frac{1}{c}|x|^{\frac{1}{c}} - c$ for all $x \in \mathbb{R}$ and Theorem 2.1 hence give

$$\lim_{\substack{N\to\infty\\ \mathrm{ld}(N)\in\mathbb{N}}} \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N} f(Y_1^{1,0,k}) + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(Y_{2^l}^{2^l,l,k}) - f(Y_{2^{(l-1)}}^{2^{(l-1)},l,k})\right)\right] \geq \frac{1}{c}\left(\lim_{N\to\infty}\mathbb{E}\left[\left|Y_N^{N,0,1}\right|^{\frac{1}{c}}\right]\right) - c = \infty.$$

In the case $\mathbb{E}\big[|f(X_T)|\big] < \infty$, the triangle inequality and Jensen's inequality then yield

$$\begin{aligned}
&\lim_{\substack{N\to\infty\\ \mathrm{ld}(N)\in\mathbb{N}}} \left\|\mathbb{E}\big[f(X_T)\big] - \frac{1}{N}\sum_{k=1}^{N} f(Y_1^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(Y_{2^l}^{2^l,l,k}) - f(Y_{2^{(l-1)}}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\geq \lim_{\substack{N\to\infty\\ \mathrm{ld}(N)\in\mathbb{N}}} \mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N} f(Y_1^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(Y_{2^l}^{2^l,l,k}) - f(Y_{2^{(l-1)}}^{2^{(l-1)},l,k})\right)\right] - \mathbb{E}\big[|f(X_T)|\big] = \infty \qquad (84)
\end{aligned}$$

for all $p \in [1,\infty)$. This shows (83) in the case $\mathbb{E}\big[|f(X_T)|\big] < \infty$. In the case $\mathbb{E}\big[|f(X_T)|\big] = \infty$, the estimate $f(x) \geq -c$ for all $x \in \mathbb{R}$ shows $\mathbb{E}\big[f(X_T)\big] = \infty$ and this implies (83) in the case $\mathbb{E}\big[|f(X_T)|\big] = \infty$. The proof of Corollary 5.2 is thus completed. ☐

6 Convergence of the multilevel Monte Carlo tamed Euler method

In this section we combine the multilevel Monte Carlo method with a tamed Euler method. We aim at path-dependent payoff functions. Therefore, we consider piecewise linear time interpolations of the numerical approximations, which have continuous sample paths and which are implementable. Thereorem 6.1 shows that these piecewise linear interpolations of the tamed Euler approximations converge in the strong sense with the optimal convergence order according to Müller-Gronbach's lower bound in the Lipschitz case in [36]. Theorem 6.2 then establishes almost sure and strong convergence of the multilevel Monte Carlo method combined with the tamed Euler method. The payoff function is allowed to depend on the whole path. We assume the payoff function only to be locally Lipschitz continuous and the local Lipschitz constant to grow at most polynomially.

Throughout this section assume that the following setting is fulfilled. Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $d, m \in \mathbb{N}$, let $W^{l,k}\colon [0,T] \times \Omega \to \mathbb{R}^m$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, be a family of independent standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motions and let $\xi^{l,k}\colon \Omega \to \mathbb{R}^d$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, be a family of independent identically distributed $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mappings with $\mathbb{E}\big[\|\xi^{0,1}\|_{\mathbb{R}^d}^p\big] < \infty$ for all $p \in [1, \infty)$. Here and below we use the Euclidean norm $\|x\|_{\mathbb{R}^n} := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and all $n \in \mathbb{N}$. Moreover, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable and globally one-sided Lipschitz continuous function whose derivative grows at most polynomially and let $\sigma = (\sigma_{i,j})_{i\in\{1,2,\ldots,d\},j\in\{1,2,\ldots,m\}}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be a globally Lipschitz continuous function. More formally, suppose that there exists a real number $c \in [0, \infty)$ such that $\langle x - y, \mu(x) - \mu(y)\rangle_{\mathbb{R}^d} \leq c\|x - y\|_{\mathbb{R}^d}^2$, $\|\mu'(x)\|_{L(\mathbb{R}^d)} \leq c(1 + \|x\|_{\mathbb{R}^d}^c)$ and $\|\sigma(x) - \sigma(y)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\|x - y\|_{\mathbb{R}^d}$ for all $x, y \in \mathbb{R}^d$. Here and below we use $\|x\| := (\sum_{i=1}^d |x_i|^2)^{\frac{1}{2}}$ and $\langle x, y\rangle_{\mathbb{R}^d} := \sum_{i=1}^d x_i \cdot y_i$ for all $x = (x_1, x_2, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. Then consider the SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t^{0,1}, \qquad X_0 = \xi^{0,1} \tag{85}$$

for $t \in [0, T]$. Under the assumptions above, the SDE (85) is known to have a unique solution. More formally, there exists an up to indistinguishability unique adapted stochastic process $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ with continuous sample paths fulfilling

$$X_t = \xi^{0,1} + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s^{0,1} \tag{86}$$

$\mathbb{P}$-almost surely for all $t \in [0, T]$ (see, e.g., Theorem 2.4.1 in Mao [30]). The drift coefficient μ is the infinitesimal mean of the process X and the diffusion coefficient σ is the infinitesimal standard deviation of the process X. In the next step we define a family of tamed Euler approximations $Y_n^{N,l,k}\colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, by $Y_0^{N,l,k} := \xi^{l,k}$ and

$$Y_{n+1}^{N,l,k} := Y_n^{N,l,k} + \frac{\mu(Y_n^{N,l,k}) \cdot \frac{T}{N}}{1 + \left\|\mu(Y_n^{N,l,k}) \cdot \frac{T}{N}\right\|_{\mathbb{R}^d}} + \sigma(Y_n^{N,l,k})\left(W_{\frac{(n+1)T}{N}}^{l,k} - W_{\frac{nT}{N}}^{l,k}\right) \tag{87}$$

for all $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$ and all $k \in \mathbb{N}$. In order to formulate our convergence theorem for the multilevel Monte Carlo tamed Euler approximations, we now introduce piecewise continuous time interpolations of the time discrete numerical approximations (87). More formally, let $\bar{Y}^{N,l,k}\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$, $k \in \mathbb{N}$, be a family of stochastic processes with continuous sample paths defined by

$$\bar{Y}_t^{N,l,k} := Y_n^{N,l,k} + \frac{\left(t - \frac{nT}{N}\right)}{\frac{T}{N}}\left(Y_{n+1}^{N,l,k} - Y_n^{N,l,k}\right) = \left(\frac{tN}{T} - n\right)Y_{n+1}^{N,l,k} + \left(n + 1 - \frac{tN}{T}\right)Y_n^{N,l,k} \tag{88}$$

for all $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$, $l \in \mathbb{N}_0$ and all $k \in \mathbb{N}$.

The following corollary is a direct consequence of Hutzenthaler, Jentzen and Kloeden [22] and Müller-Gronbach [36] (see also Ritter [38]). It asserts that the piecewise linear approximations $\bar{Y}^N$, $N \in \mathbb{N}$, converge in the strong sense to the exact solution. The convergence order is $\frac{1}{2}$ except for a logarithmic term.

Corollary 6.1 (Strong convergence of the tamed Euler method)**.** *Assume that the above setting is fulfilled. Then there exists a family $R_p \in [0, \infty)$, $p \in [1, \infty)$, of real numbers such that*

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left\|X_t - \bar{Y}_t^{N,0,1}\right\|_{\mathbb{R}^d}^p\right]\right)^{\frac{1}{p}} \leq R_p \cdot \frac{\sqrt{1 + \mathrm{ld}(N)}}{\sqrt{N}} \tag{89}$$

for all $N \in \mathbb{N}$ and all $p \in [1, \infty)$.

The convergence rate $N^{-\frac{1}{2}}(1+\mathrm{ld}(N))^{\frac{1}{2}}$ for $N \in \mathbb{N}$ obtained in (89) is sharp according to Müller-Gronbach's lower bound established in Theorem 3 in [36] in the case of globally Lipschitz continuous coefficients (see also Hofmann, Müller-Gronbach and Ritter [18]).

Proof of Corollary 6.1. Let $\tilde{Y}^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be stochastic processes defined by

$$\tilde{Y}_t^N := Y_n^{N,0,1} + \frac{\mu(Y_n^{N,0,1}) \cdot \left(t - \frac{nT}{N}\right)}{1 + \|\mu(Y_n^{N,0,1}) \cdot \frac{T}{N}\|_{\mathbb{R}^d}} + \sigma(Y_n^{N,0,1})\left(W_t^{0,1} - W_{\frac{nT}{N}}^{0,1}\right)$$

for all $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. Theorem 1.1 in [22] then shows the existence of a family $\tilde{R}_p \in [0, \infty)$, $p \in [1, \infty)$, of real numbers such that $\left\| \sup_{t\in[0,T]} \|X_t - \tilde{Y}_t^N\|_{\mathbb{R}^d} \right\|_{L^p(\Omega;\mathbb{R})} \leq \frac{\tilde{R}_p}{\sqrt{N}}$ for all $N \in \mathbb{N}$ and all $p \in [1, \infty)$. The triangle inequality hence yields

$$\left\| \sup_{t\in[0,T]} \left\| X_t - \bar{Y}_t^{N,0,1} \right\|_{\mathbb{R}^d} \right\|_{L^p(\Omega;\mathbb{R})} \leq \frac{\tilde{R}_p}{\sqrt{N}} + \left\| \sup_{t\in[0,T]} \left\| \tilde{Y}_t^N - \bar{Y}_t^{N,0,1} \right\|_{\mathbb{R}^d} \right\|_{L^p(\Omega;\mathbb{R})} \tag{90}$$

for all $N \in \mathbb{N}$ and all $p \in [1, \infty)$. Moreover, we have

$$\begin{aligned} \left\| \tilde{Y}_t^N - \bar{Y}_t^{N,0,1} \right\|_{\mathbb{R}^d} &= \left\| \sigma(Y_n^{N,0,1}) \left(W_t^{0,1} - W_{\frac{nT}{N}}^{0,1} \right) - \left(\frac{tN}{T} - n \right) \sigma(Y_n^{N,0,1}) \left(W_{\frac{(n+1)T}{N}}^{0,1} - W_{\frac{nT}{N}}^{0,1} \right) \right\|_{\mathbb{R}^d} \\ &\leq \left\| \sigma(Y_n^{N,0,1}) \right\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \left\| W_t^{0,1} - W_{\frac{nT}{N}}^{0,1} - \left(\frac{tN}{T} - n \right) \left(W_{\frac{(n+1)T}{N}}^{0,1} - W_{\frac{nT}{N}}^{0,1} \right) \right\|_{\mathbb{R}^m} \end{aligned} \tag{91}$$

for all $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. Combining (90), (91) and Hölder's inequality then gives

$$\begin{aligned} &\left\| \sup_{t\in[0,T]} \left\| X_t - \bar{Y}_t^{N,0,1} \right\|_{\mathbb{R}^d} \right\|_{L^p(\Omega;\mathbb{R})} \leq \frac{\tilde{R}_p}{\sqrt{N}} + \left\| \max_{n\in\{0,1,\dots,N-1\}} \left\| \sigma(Y_n^{N,0,1}) \right\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \right\|_{L^{2p}(\Omega;\mathbb{R})} \\ &\cdot \left\| \max_{n\in\{0,1,\dots,N-1\}} \sup_{t\in[\frac{nT}{N},\frac{(n+1)T}{N}]} \left\| W_t^{0,1} - W_{\frac{nT}{N}}^{0,1} - \left(\frac{tN}{T} - n \right) \left(W_{\frac{(n+1)T}{N}}^{0,1} - W_{\frac{nT}{N}}^{0,1} \right) \right\|_{\mathbb{R}^m} \right\|_{L^{2p}(\Omega;\mathbb{R})} \\ &\leq \frac{\tilde{R}_p}{\sqrt{N}} + \sqrt{\frac{T}{N}} \left(c \cdot \sup_{M\in\mathbb{N}} \left\| \max_{n\in\{0,1,\dots,M\}} \left\| Y_n^{M,0,1} \right\|_{\mathbb{R}^d} \right\|_{L^{2p}(\Omega;\mathbb{R})} + \|\sigma(0)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \right) \left\| \max_{n\in\{1,2,\dots,N\}} \sup_{t\in[0,1]} |\beta_t^n - t \cdot \beta_1^n| \right\|_{L^{2p}(\Omega;\mathbb{R})} \end{aligned} \tag{92}$$

for all $N \in \mathbb{N}$ and all $p \in [1, \infty)$ where $\beta^n \colon [0, 1] \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of independent one-dimensional standard Brownian motions. Moreover, Theorem 1.1 in [22], in particular, implies

$$\sup_{M\in\mathbb{N}} \left\| \max_{n\in\{0,1,\dots,M\}} \left\| Y_n^{M,0,1} \right\|_{\mathbb{R}^d} \right\|_{L^p(\Omega;\mathbb{R})} < \infty \tag{93}$$

for all $p \in [1, \infty)$. Additionally, Corollary 2 in Müller-Gronbach [36] (see also Ritter [38]) shows

$$\sup_{N\in\mathbb{N}} \left((1 + \mathrm{ld}(N))^{-\frac{1}{2}} \left\| \max_{n\in\{0,1,\dots,N\}} \sup_{t\in[0,1]} |\beta_t^n - t \cdot \beta_1^n| \right\|_{L^p(\Omega;\mathbb{R})} \right) < \infty \tag{94}$$

for all $p \in [1, \infty)$. Combining (92), (93) and (94) finally completes the proof of Corollary 6.1. □

Proposition 6.2 (Strong consistency, converence with probability one and strong convergence of the multilevel Monte Carlo tamed Euler method)**.** *Assume that the above setting is fulfilled, let $c \in [0, \infty)$ and let $f \colon C([0, T], \mathbb{R}^d) \to \mathbb{R}$ be a function from the space of continuous functions $C([0, T], \mathbb{R}^d)$ into the real numbers $\mathbb{R}$ satisfying*

$$\|f(v) - f(w)\|_{C([0,T],\mathbb{R}^d)} \leq c \left(1 + \|v\|_{C([0,T],\mathbb{R}^d)}^c + \|w\|_{C([0,T],\mathbb{R}^d)}^c \right) \|v - w\|_{C([0,T],\mathbb{R}^d)} \tag{95}$$

for all $v, w \in C([0, T], \mathbb{R}^d)$. Then there exists a family $C_p \in [0, \infty)$, $p \in [1, \infty)$, of real numbers such that

$$\left(\mathbb{E}\left[\left| \mathbb{E}\Big[f(X) \Big] - \frac{1}{N} \sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k}) \right) \right|^p \right] \right)^{\frac{1}{p}} \leq C_p \cdot \frac{(1 + \mathrm{ld}(N))^{\frac{3}{2}}}{\sqrt{N}} \tag{96}$$

for all $N \in \{2^1, 2^2, 2^3, \dots\}$ and all $p \in [1, \infty)$. In particular, there are finite $\mathcal{F}/\mathcal{B}([0, \infty))$-measurable mappings $\tilde{C}_\varepsilon \colon \Omega \to [0, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, such that

$$\left| \mathbb{E}\Big[f(X) \Big] - \frac{1}{N} \sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N} \left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k}) \right) \right| \leq \frac{\tilde{C}_\varepsilon}{N^{(\frac{1}{2}-\varepsilon)}} \tag{97}$$

for all $N \in \mathbb{N}$ and all $\varepsilon \in (0, \frac{1}{2})$ $\mathbb{P}$-almost surely.

The convergence rate $N^{-\frac{1}{2}}(1+\mathrm{ld}(N))^{\frac{3}{2}}$ for $N \in \mathbb{N}$ obtained in (96) is the same as in Remark 8 in Creutzig, Dereich, Müller-Gronbach and Ritter [3]. For numerical approximation results for SDEs with globally Lipschitz continuous coefficients but under less restrictive smoothness assumption on the payoff function, the reader is referred to Giles, Higham and Mao [9] and Dörsek and Teichmann [5]. Moreover, numerical approximation results for SDEs with non-globally Lipschitz continuous and at most linearly growing coefficients can be found in Yan [43], for instance.

Proof of Proposition 6.2. The triangle inequality gives

$$\begin{aligned}
&\left\| \mathbb{E}\Big[f(X)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \left|\mathbb{E}\Big[f(X)\Big] - \mathbb{E}\Big[f(\bar{Y}^{N,0,1})\Big]\right| + \frac{1}{N}\left\|\sum_{k=1}^{N}\left(\mathbb{E}\Big[f(\bar{Y}^{1,0,1})\Big] - f(\bar{Y}^{1,0,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left\|\sum_{k=1}^{\frac{N}{2^l}}\left(\mathbb{E}\Big[f(\bar{Y}^{2^l,0,1})\Big] - \mathbb{E}\Big[f(\bar{Y}^{2^{(l-1)},0,1})\Big] - f(\bar{Y}^{2^l,l,k}) + f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})}
\end{aligned}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and all $p \in [1,\infty)$ and the Burkholder-Davis-Gundy inequality in Theorem 6.3.10 in Stroock [39] shows the existence of real numbers $K_p \in [0,\infty)$, $p \in [1,\infty)$, such that

$$\begin{aligned}
&\left\| \mathbb{E}\Big[f(X)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \mathbb{E}\Big[\big|f(X) - f(\bar{Y}^{N,0,1})\big|\Big] + \frac{K_p}{\sqrt{N}}\left\|\mathbb{E}\Big[f(\bar{Y}^{1,0,1})\Big] - f(\bar{Y}^{1,0,1})\right\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^{\frac{l}{2}}K_p}{\sqrt{N}}\left\|\mathbb{E}\Big[f(\bar{Y}^{2^l,0,1})\Big] - \mathbb{E}\Big[f(\bar{Y}^{2^{(l-1)},0,1})\Big] - f(\bar{Y}^{2^l,0,1}) + f(\bar{Y}^{2^{(l-1)},0,1})\right\|_{L^p(\Omega;\mathbb{R})}
\end{aligned}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and all $p \in [1,\infty)$. In the next step estimate (95), Hölder's inequality and the triangle inequality show

$$\begin{aligned}
&\left\| \mathbb{E}\Big[f(X)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq c\left(1 + \|X\|^c_{L^{2c}(\Omega;C([0,T],\mathbb{R}^d))} + \big\|\bar{Y}^{N,0,1}\big\|^c_{L^{2c}(\Omega;C([0,T],\mathbb{R}^d))}\right)\big\|X - \bar{Y}^{N,0,1}\big\|_{L^2(\Omega;C([0,T],\mathbb{R}^d))} \\
&\quad + \frac{2K_p}{\sqrt{N}}\big\|f(\bar{Y}^{1,0,1})\big\|_{L^p(\Omega;\mathbb{R})} + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^{(\frac{l}{2}+1)}K_p}{\sqrt{N}}\left\|f(\bar{Y}^{2^l,0,1}) - f(\bar{Y}^{2^{(l-1)},0,1})\right\|_{L^p(\Omega;\mathbb{R})}
\end{aligned}$$

and Corollary 6.1 and again estimate (95) hence give

$$\begin{aligned}
&\left\| \mathbb{E}\Big[f(X)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq 2cR_2\left(1 + \sup_{M\in\mathbb{N}}\big\|\bar{Y}^{M,0,1}\big\|^c_{L^{2c}(\Omega;C([0,T],\mathbb{R}^d))}\right)\frac{\sqrt{1+\mathrm{ld}(N)}}{\sqrt{N}} + \frac{2K_p}{\sqrt{N}}\big\|f(\bar{Y}^{1,0,1})\big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^{(\frac{l}{2}+2)}cK_p}{\sqrt{N}}\left(1 + \sup_{M\in\mathbb{N}}\big\|\bar{Y}^{M,0,1}\big\|^c_{L^{2pc}(\Omega;C([0,T],\mathbb{R}^d))}\right)\left\|\bar{Y}^{2^l,0,1} - \bar{Y}^{2^{(l-1)},0,1}\right\|_{L^{2p}(\Omega;C([0,T],\mathbb{R}^d))}
\end{aligned}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and all $p \in [1,\infty)$. The triangle inequality, again Corollary 6.1 and the estimate

$\|f(v)\|_{C([0,T],\mathbb{R}^d)} \leq (2c + \|f(0)\|_{C([0,T],\mathbb{R}^d)})(1 + \|v\|^{(c+1)}_{C([0,T],\mathbb{R}^d)})$ for all $v \in C([0,T],\mathbb{R}^d)$ then yield

$$\begin{aligned}
&\left\| \mathbb{E}\Big[f(X)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq 2cR_2\left(1 + \sup_{M\in\mathbb{N}} \left\|\bar{Y}^{M,0,1}\right\|^c_{L^{2c}(\Omega;C([0,T],\mathbb{R}^d))}\right)\frac{\sqrt{1+\mathrm{ld}(N)}}{\sqrt{N}} \\
&\quad + 2K_p\left(2c + \|f(0)\|_{C([0,T],\mathbb{R}^d)}\right)\left(1 + \left\|\bar{Y}^{1,0,1}\right\|^{(c+1)}_{L^{p(c+1)}(\Omega;C([0,T],\mathbb{R}^d))}\right)\frac{1}{\sqrt{N}} \\
&\quad + cK_pR_{2p}\left(1 + \sup_{M\in\mathbb{N}} \left\|\bar{Y}^{M,0,1}\right\|^c_{L^{2pc}(\Omega;C([0,T],\mathbb{R}^d))}\right)\sum_{l=1}^{\mathrm{ld}(N)} \frac{2^{(\frac{l}{2}+3)}\sqrt{1+\mathrm{ld}(2^l)}}{2^{\frac{(l-1)}{2}}\sqrt{N}}
\end{aligned}$$

and finally

$$\begin{aligned}
&\left\| \mathbb{E}\Big[f(X)\Big] - \frac{1}{N}\sum_{k=1}^{N} f(\bar{Y}^{1,0,k}) - \sum_{l=1}^{\mathrm{ld}(N)} \frac{2^l}{N}\left(\sum_{k=1}^{\frac{N}{2^l}} f(\bar{Y}^{2^l,l,k}) - f(\bar{Y}^{2^{(l-1)},l,k})\right)\right\|_{L^p(\Omega;\mathbb{R})} \\
&\leq 2cR_2\left(1 + \sup_{M\in\mathbb{N}} \left\|\bar{Y}^{M,0,1}\right\|^c_{L^{2c}(\Omega;C([0,T],\mathbb{R}^d))}\right)\frac{(1+\mathrm{ld}(N))^{\frac{3}{2}}}{\sqrt{N}} \\
&\quad + 2K_p\left(2c + \|f(0)\|_{C([0,T],\mathbb{R}^d)}\right)\left(1 + \left\|\bar{Y}^{1,0,1}\right\|^{(c+1)}_{L^{p(c+1)}(\Omega;C([0,T],\mathbb{R}^d))}\right)\frac{(1+\mathrm{ld}(N))^{\frac{3}{2}}}{\sqrt{N}} \\
&\quad + 12cK_pR_{2p}\left(1 + \sup_{M\in\mathbb{N}} \left\|\bar{Y}^{M,0,1}\right\|^c_{L^{2pc}(\Omega;C([0,T],\mathbb{R}^d))}\right)\frac{(1+\mathrm{ld}(N))^{\frac{3}{2}}}{\sqrt{N}}
\end{aligned}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and all $p \in [1,\infty)$. This shows (96). Inequality (97) then immediately follows from Lemma 2.1 in Kloeden und Neuenkirch [26]. This completes the proof of Proposition 6.2. □

It is well-known that the multilevel Monte Carlo method combined with the (fully) implicit Euler method converges too. The following simulation indicates that this multilevel Monte Carlo implicit Euler method is considerably slower than the multilevel Monte Carlo tamed Euler method. We choose a multi-dimensional Langevin equation as example. More precisely, we consider the motion of a Brownian particle of unit mass in the d-dimensional potential $\frac{1}{4}\|x\|^4 - \frac{1}{2}\|x\|^2$, $x \in \mathbb{R}^d$, with $d = 10$. The corresponding force on the particle is then $x - \|x\|^2 \cdot x$ for $x \in \mathbb{R}^d$. More formally, let $T = 1$, $m = d = 10$, $\xi = (0, 0, \ldots, 0)$, $\mu(x) = x - \|x\|^2 \cdot x$ for all $x \in \mathbb{R}^d$ and let $\sigma(x) = \mathbf{I}$ be the identity matrix for all $x \in \mathbb{R}^d$. Thus the SDE (85) reduces to the Langevin equation

$$dX_t = \left(X_t - \|X_t\|^2 \cdot X_t\right)dt + dW^{0,1}_t, \qquad X_0 = \xi \tag{98}$$

for $t \in [0,1]$. Then the implicit Euler scheme for the SDE (98) is given by mappings $\tilde{\tilde{Y}}^N_n \colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, satisfying $\tilde{\tilde{Y}}^N_0 = \xi$ and

$$\tilde{\tilde{Y}}^N_{n+1} = \tilde{\tilde{Y}}^N_n + \frac{T}{N}\cdot\left(\tilde{\tilde{Y}}^N_{n+1} - \left\|\tilde{\tilde{Y}}^N_{n+1}\right\|^2 \cdot \tilde{\tilde{Y}}^N_{n+1}\right) + \left(W^{0,1}_{\frac{(n+1)T}{N}} - W^{0,1}_{\frac{nT}{N}}\right) \tag{99}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Note that we used the Matlab function fsolve(...) in our implementation of the implicit Euler scheme (99). Figure 5 displays the root mean square approximation error of the multilevel Monte Carlo implicit Euler method for the uniform second moment $\mathbb{E}\Big[\sup_{t\in[0,1]} \|X_t\|^2\Big]$ of the exact solution of (98) as function of the runtime when $N \in \{2^5, 2^6, \ldots, 2^{18}\}$. In addition Figure 5 shows the root mean square approximation error of the multilevel Monte Carlo tamed Euler method for the uniform second moment $\mathbb{E}\Big[\sup_{t\in[0,1]} \|X_t\|^2\Big]$ of the exact solution of (98) as function of the runtime when $N \in \{2^5, 2^6, \ldots, 2^{25}\}$. We see that both numerical approximations of the SDE (98) apparantly converge with rate close to $\frac{1}{2}$. Moreover the multilevel Monte Carlo implicit Euler method was considerably slower than the multilevel Monte Carlo tamed Euler method. This is presumably due to the additional computational effort which is required to determine the zero of a nonlinear equation in each time step of the implicit Euler method (99). More results on implicit numerical methods for SDEs can be found in [19, 17, 42, 41, 40, 31, 32], for instance.

7 Appendix: Proof of Lemma 4.9

Before the proof of Lemma 4.9 is presented, a few auxiliary results (Lemmas 7.1–7.5) are established.

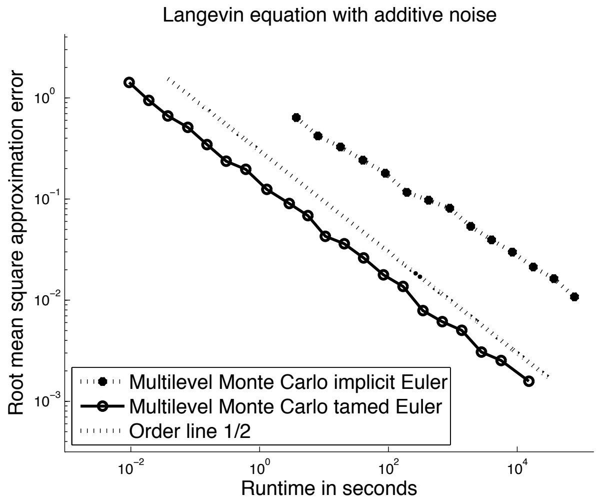

Figure 5: Root mean square approximation error for the uniform second moment $\mathbb{E}\Big[\sup_{t\in[0,1]}\|X_t\|^2\Big]$ of the exact solution of (98) as function of the runtime both for the multilevel Monte Carlo implicit Euler method and for the multilevel Monte Carlo tamed Euler method.

Lemma 7.1. *Assume that the setting described in Section 4 and Subsection 4.2 is fulfilled. Then we have*

$$\sum_{n=1}^{\infty}\mathbb{P}\Big[A_{2^n}^{(1)}\Big]<\infty. \tag{100}$$

Proof of Lemma 7.1. The definition (38) of L_N, $N\in\{2^1,2^2,2^3,\dots\}$, and independence of $\xi^{l,k}$, $l\in\mathbb{N}_0$, $k\in\mathbb{N}$, imply

$$\begin{aligned}\mathbb{P}\Big[A_N^{(1)}\Big]&=\mathbb{P}\Big[\forall\, l\in\mathbb{N},\, \big\lfloor 2\,\mathrm{ld}\big(\bar{\sigma}^2T^{\frac12}\ln(N)\big)\big\rfloor\le l\le \mathrm{ld}(N)\,\forall\, k\in\{1,2,\dots,\tfrac{N}{2^l}\}\colon |\xi^{l,k}|\le 2^{\frac{l}{4}}T^{-\frac14}\Big]\\
&=\prod_{l=\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)}\prod_{k=1}^{\frac{N}{2^l}}\mathbb{P}\Big[|\xi^{l,k}|\le 2^{\frac{l}{4}}T^{-\frac14}\Big]=\prod_{l=\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)}\left(\mathbb{P}\Big[|\xi^{0,1}|\le 2^{\frac{l}{4}}T^{-\frac14}\Big]\right)^{\frac{N}{2^l}}\\
&=\prod_{l=\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)}\left(1-\mathbb{P}\Big[\bar{\sigma}^{-1}|\xi^{0,1}|>2^{\frac{l}{4}}\bar{\sigma}^{-1}T^{-\frac14}\Big]\right)^{\frac{N}{2^l}}\end{aligned}$$

for all $N\in\{N_0,2^1N_0,2^2N_0,\dots\}$. The inequality

$$\begin{aligned}\mathbb{P}\big[\bar{\sigma}^{-1}|\xi^{0,1}|>x\big]&=2\cdot\mathbb{P}\big[\bar{\sigma}^{-1}\xi^{0,1}>x\big]=2\int_x^\infty\frac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}}\,dy\ge 2\int_x^{x\sqrt{\frac32}}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{y^2}{2}}\,dy\\
&\ge\frac{2}{\sqrt{2\pi}}\left(x\sqrt{\frac32}-x\right)e^{-\frac{3x^2}{2\cdot 2}}=\frac{x}{\sqrt{\pi}}\left(\sqrt3-\sqrt2\right)e^{-\frac34x^2}=\frac{xe^{-\frac34x^2}}{\sqrt{\pi}\left(\sqrt3+\sqrt2\right)}\ge\frac16xe^{-\frac34x^2}\end{aligned} \tag{101}$$

for all $x \in [0, \infty)$ therefore yields

$$\begin{aligned}
\mathbb{P}\Big[A_N^{(1)}\Big] &\leq \prod_{l=\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)} \left(1 - \frac{1}{6}\frac{2^{\frac{l}{4}}}{\bar{\sigma}T^{\frac{1}{4}}} \cdot \exp\!\left(-\frac{3}{4}\cdot\frac{2^{\frac{l}{2}}}{\bar{\sigma}^2\sqrt{T}}\right)\right)^{\frac{N}{2^l}} \\
&\leq \left(1 - \frac{2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor/4}}{6\bar{\sigma}T^{\frac{1}{4}}} \cdot \exp\!\left(-\frac{3\cdot 2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor/2}}{4\bar{\sigma}^2\sqrt{T}}\right)\right)^{\frac{N}{2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}}} \\
&\leq \left(1 - \frac{1}{6\bar{\sigma}T^{\frac{1}{4}}} \cdot \exp\!\left(-\frac{3\cdot 2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor/2}}{4\bar{\sigma}^2\sqrt{T}}\right)\right)^{\frac{N}{2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}}} \\
&\leq \left(1 - \frac{1}{6\bar{\sigma}T^{\frac{1}{4}}} \cdot \exp\!\left(-\frac{3\cdot 2^{\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))}}{4\bar{\sigma}^2\sqrt{T}}\right)\right)^{\frac{N}{2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))\rfloor}}} \\
&\leq \left(1 - \frac{1}{6\bar{\sigma}T^{\frac{1}{4}}} \cdot \exp\!\left(-\frac{3\cdot \bar{\sigma}^2\sqrt{T}\ln(N)}{4\bar{\sigma}^2\sqrt{T}}\right)\right)^{\frac{N}{2^{2\,\mathrm{ld}(\bar{\sigma}^2\sqrt{T}\ln(N))}}} \\
&= \left(1 - \frac{1}{6\bar{\sigma}T^{\frac{1}{4}}} \cdot \exp\!\left(-\frac{3}{4}\ln(N)\right)\right)^{\frac{N}{2^{\mathrm{ld}((\bar{\sigma}^2\sqrt{T}\ln(N))^2)}}} = \left(1 - \frac{1}{6\bar{\sigma}T^{\frac{1}{4}}} \cdot N^{-\frac{3}{4}}\right)^{\frac{N}{(\bar{\sigma}^2\sqrt{T}\ln(N))^2}}
\end{aligned} \tag{102}$$

for all $N \in \{N_0, 2^1N_0, 2^2N_0, \ldots\}$. Next we estimate $1 - x \leq \exp(-x)$ for all $x \in \mathbb{R}$ to get

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(1)}\Big] &= \sum_{\substack{N\in\{2^1,2^2,2^3,\ldots\}\\ N<N_0}} \mathbb{P}\Big[A_N^{(1)}\Big] + \sum_{\substack{N\in\{2^1,2^2,2^3,\ldots\}\\ N\geq N_0}} \mathbb{P}\Big[A_N^{(1)}\Big] \leq N_0 + \sum_{\substack{N\in\{2^1,2^2,2^3,\ldots\}\\ N\geq N_0}} \left(1 - \frac{1}{6\bar{\sigma}T^{\frac{1}{4}}}\cdot N^{-\frac{3}{4}}\right)^{\frac{N}{(\bar{\sigma}^2\sqrt{T}\ln(N))^2}} \\
&\leq N_0 + \sum_{N=N_0}^{\infty} \left[\exp\left(-\frac{1}{6\bar{\sigma}T^{\frac{1}{4}}}\cdot N^{-\frac{3}{4}}\right)\right]^{\frac{N}{(\bar{\sigma}^2\sqrt{T}\ln(N))^2}} = N_0 + \sum_{N=N_0}^{\infty} \exp\left(-\frac{N^{\frac{1}{4}}}{6\bar{\sigma}^5T^{\frac{5}{4}}(\ln(N))^2}\right) < \infty.
\end{aligned}$$

This completes the proof of Lemma 7.1. ☐

Lemma 7.2. *Assume that the setting described in Section 4 and Subsection 4.2 is fulfilled. Then we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(2)}\Big] < \infty. \tag{103}$$

Proof of Lemma 7.2. Subadditivity of the probability measure $\mathbb{P}$ and the inequality $\mathbb{P}[\bar{\sigma}^{-1}|\xi^{0,1}| \geq x] \leq \frac{1}{x}\exp\left(-\frac{x^2}{2}\right)$ for all $x \in (0, \infty)$ (e.g., Lemma 22.2 in [25]) imply

$$\begin{aligned}
\mathbb{P}\Big[A_N^{(2)}\Big] &= \mathbb{P}\Big[\exists\, l \in \{0,1,2,\ldots,\mathrm{ld}(N)\}\,\exists\, k \in \{1,2,\ldots,\tfrac{N}{2^l}\}\colon |\xi^{l,k}| \geq 2^{\frac{(l-1)}{4}}T^{-\frac{1}{4}}N\Big] \leq \sum_{l=0}^{\mathrm{ld}(N)}\sum_{k=1}^{\frac{N}{2^l}} \mathbb{P}\Big[|\xi^{l,k}| \geq 2^{\frac{(l-1)}{4}}T^{-\frac{1}{4}}N\Big] \\
&= \sum_{l=0}^{\mathrm{ld}(N)} \frac{N}{2^l}\cdot\mathbb{P}\Big[|\xi^{0,1}| \geq 2^{\frac{(l-1)}{4}}T^{-\frac{1}{4}}N\Big] = \sum_{l=0}^{\mathrm{ld}(N)} \frac{N}{2^l}\cdot\mathbb{P}\left[\bar{\sigma}^{-1}|\xi^{0,1}| \geq \frac{2^{\frac{(l-1)}{4}}N}{\bar{\sigma}T^{\frac{1}{4}}}\right] \leq \sum_{l=0}^{\mathrm{ld}(N)} \frac{N}{2^l}\cdot\frac{\bar{\sigma}T^{\frac{1}{4}}}{2^{\frac{(l-1)}{4}}N}\exp\left(-\frac{2^{\frac{(l-1)}{2}}N^2}{2\bar{\sigma}^2T^{\frac{1}{2}}}\right) \\
&\leq \sum_{l=0}^{\mathrm{ld}(N)} \frac{\bar{\sigma}T^{\frac{1}{4}}}{2^{\frac{-1}{4}}}\exp\left(-\frac{2^{\frac{-1}{2}}N^2}{2\bar{\sigma}^2T^{\frac{1}{2}}}\right) = (\mathrm{ld}(N)+1)\,\bar{\sigma}2^{\frac{1}{4}}T^{\frac{1}{4}}\exp\left(-\frac{N^2}{2^{\frac{3}{2}}\bar{\sigma}^2T^{\frac{1}{2}}}\right)
\end{aligned} \tag{104}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Summing over $N \in \{2^1, 2^2, 2^3, \ldots\}$ results in

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(2)}\Big] \leq \sum_{N\in\{2^1,2^2,2^3,\ldots\}} (\mathrm{ld}(N)+1)\,\bar{\sigma}2^{\frac{1}{4}}T^{\frac{1}{4}}\exp\left(-\frac{N^2}{2^{\frac{3}{2}}\bar{\sigma}^2T^{\frac{1}{2}}}\right) < \infty \tag{105}$$

and this completes the proof of Lemma 7.2. ☐

Lemma 7.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $Z\colon \Omega \to \mathbb{R}$ be a standard normally distributed $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping. Then*

$$\mathbb{P}\Big[|Z| < x + y\,\Big|\,|Z| \geq x\Big] \leq 5xy \tag{106}$$

for all $x \in [\frac{1}{2}, \infty)$ and all $y \in [0, \infty)$.

Proof of Lemma 7.3. Monotonicity of the exponential function yields

$$\mathbb{P}[x \le |Z| < x+y] = 2 \cdot \mathbb{P}[x \le Z < x+y] = 2\int_x^{x+y} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \le \frac{2}{\sqrt{2\pi}} y e^{-\frac{x^2}{2}} \tag{107}$$

for all $x, y \in [0,\infty)$. Apply the standard estimate $\mathbb{P}[|Z| \ge x] \ge \frac{x}{1+x^2}\frac{2}{\sqrt{2\pi}}\exp\big(-\frac{x^2}{2}\big)$ for all $x \in (0,\infty)$ (e.g., Lemma 22.2 in [25]), inequality (107) and $\frac{x^2}{1+x^2} \ge \frac{1}{5}$ for all $x \in [\frac{1}{2},\infty)$ to get

$$\mathbb{P}\Big[|Z| < x+y \,\Big|\, |Z| \ge x\Big] = \frac{\mathbb{P}[x \le |Z| < x+y]}{\mathbb{P}[|Z| \ge x]} \le \frac{\frac{2}{\sqrt{2\pi}} y e^{-\frac{x^2}{2}}}{\frac{x}{1+x^2}\frac{2}{\sqrt{2\pi}}\exp\big(-\frac{x^2}{2}\big)} = \frac{xy}{\Big(\frac{x^2}{1+x^2}\Big)} \le 5xy$$

for all $x \in [\frac{1}{2},\infty)$ and all $y \in [0,\infty)$. This completes the proof of Lemma 7.3. □

Lemma 7.4. *Assume that the setting described in Section 4 and Subsection 4.2 is fulfilled. Then we have*

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(3)}_{2^n}\Big] < \infty. \tag{108}$$

Proof of Lemma 7.4. Let $K\colon \mathbb{N}_0 \times \mathbb{N}_0 \times \Omega \to \mathbb{N} \cup \{\infty\}$ be defined as

$$K(v,l) := \min\Big(\Big\{k \in \mathbb{N}\colon\ \big|\xi^{v,k}\big| \ge 2^{\frac{l}{4}} T^{-\frac{1}{4}}\Big\} \cup \{\infty\}\Big) \tag{109}$$

for all $v, l \in \mathbb{N}_0$. Inserting definition (43) we get

$$\begin{aligned}
\mathbb{P}\Big[A_N^{(3)}\Big] &= \mathbb{P}\Big[\exists\, l \in \mathbb{N},\, \big\lfloor 2\,\mathrm{ld}\big(\bar\sigma^2 T^{\frac{1}{2}} \ln(N)\big)\big\rfloor \le l \le \mathrm{ld}(N)+1\colon 2^{\frac{l}{4}} T^{-\frac{1}{4}} \le \eta_N < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big)\Big] \\
&\le \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \mathbb{P}\Big[2^{\frac{l}{4}} T^{-\frac{1}{4}} \le \eta_N < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big)\Big] \\
&\le \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \mathbb{P}\Big[\exists\, v \in \{1,2,\dots,\mathrm{ld}(N)\}\colon 2^{\frac{l}{4}} T^{-\frac{1}{4}} \le \max_{k \in \{1,2,\dots,\frac{N}{2^v}\}} \big|\xi^{v,k}\big| < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big)\Big] \\
&\le \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \sum_{v=1}^{\mathrm{ld}(N)} \mathbb{P}\Big[\Big\{\exists\, k \in \{1,\dots,\tfrac{N}{2^v}\}\colon\ \big|\xi^{v,k}\big| \ge 2^{\frac{l}{4}} T^{-\frac{1}{4}}\Big\} \cap \\
&\qquad\qquad \Big\{\forall\, k \in \{1,\dots,\tfrac{N}{2^v}\}\colon\ \big|\xi^{v,k}\big| < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big)\Big\}\Big] \\
&\le \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \sum_{v=1}^{\mathrm{ld}(N)} \mathbb{P}\Big[\big\{K(v,l) \le \tfrac{N}{2^v}\big\} \cap \Big\{\big|\xi^{v,K(v,l)}\big| < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big)\Big\}\Big]
\end{aligned} \tag{110}$$

for all $N \in \{N_0, 2^1 N_0, 2^2 N_0, \dots\}$. The method of rejection sampling hence results in

$$\begin{aligned}
&\mathbb{P}\Big[A_N^{(3)}\Big] \\
&\le \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \sum_{v=1}^{\mathrm{ld}(N)} \mathbb{P}\Big[|\xi^{v,K(v,l)}| < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big), K(v,l) < \infty\Big] \\
&= \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \sum_{v=1}^{\mathrm{ld}(N)} \mathbb{P}\Big[|\xi^{0,1}| < 2^{\frac{l}{4}} T^{-\frac{1}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big) \,\Big|\, |\xi^{0,1}| \ge 2^{\frac{l}{4}} T^{-\frac{1}{4}}\Big] \\
&= \sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} \mathrm{ld}(N) \cdot \mathbb{P}\Big[\big|\bar\sigma^{-1}\xi^{0,1}\big| < \frac{2^{\frac{l}{4}}\big(1+5^{(-\delta\cdot 2^{(l-1)})}\big)}{\bar\sigma T^{\frac{1}{4}}} \,\Big|\, \big|\bar\sigma^{-1}\xi^{0,1}\big| \ge \frac{2^{\frac{l}{4}}}{\bar\sigma T^{\frac{1}{4}}}\Big]
\end{aligned} \tag{111}$$

for all $N \in \{N_0, 2^1 N_0, 2^2 N_0, \dots\}$. In order to apply Lemma 7.3, we note that

$$\begin{aligned}
\frac{2^{\frac{l}{4}}}{\bar\sigma T^{\frac{1}{4}}} &\ge \frac{2^{\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor/4}}{\bar\sigma T^{\frac{1}{4}}} \ge \frac{2^{(2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))-1)/4}}{\bar\sigma T^{\frac{1}{4}}} = \frac{2^{\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))/2}}{\bar\sigma T^{\frac{1}{4}} 2^{\frac{1}{4}}} \\
&= \frac{\sqrt{\bar\sigma^2\sqrt{T}\ln(N)}}{\bar\sigma T^{\frac{1}{4}} 2^{\frac{1}{4}}} = \frac{\sqrt{\ln(N)}}{2^{\frac{1}{4}}} \ge \frac{\sqrt{\ln(2)}}{2^{\frac{1}{4}}} \ge \frac{1}{2}
\end{aligned} \tag{112}$$

for all $l \in \mathbb{N} \cap \big[\lfloor 2\,\mathrm{ld}(\bar\sigma^2 T^{\frac12}\ln(N))\rfloor, \infty\big)$ and all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Lemma 7.3 applied to the standard normally distributed variable $\bar\sigma^{-1}\xi^{0,1}$ thus leads to

$$\begin{aligned}
\mathbb{P}\Big[A_N^{(3)}\Big] &\le \mathrm{ld}(N)\left[\sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} 5\cdot\frac{2^{\frac{l}{4}}}{\bar\sigma T^{\frac14}}\cdot\frac{2^{\frac{l}{4}}\cdot 5^{(-\delta\cdot 2^{(l-1)})}}{\bar\sigma T^{\frac14}}\right]\\
&= \mathrm{ld}(N)\left[\sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} 5\cdot\frac{2^{\frac{l}{2}}}{\bar\sigma^2 T^{\frac12}}\cdot 5^{(-\frac{\delta}{2}\cdot 2^{l})}\right]\\
&\le \mathrm{ld}(N)\left[\sum_{l=\lfloor 2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))\rfloor}^{\mathrm{ld}(N)+1} 5\cdot\frac{2^{(\mathrm{ld}(N)+1)}}{\bar\sigma^2 T^{\frac12}}\cdot 5^{\left(-\frac{\delta}{2}\cdot 2^{\left(2\,\mathrm{ld}(\bar\sigma^2\sqrt{T}\ln(N))-1\right)}\right)}\right]\\
&\le (\mathrm{ld}(N))^2\cdot\frac{10N}{\bar\sigma^2 T^{\frac12}}\cdot 5^{\left(-\frac{\delta}{4}\cdot(\bar\sigma^2\sqrt{T}\ln(N))^2\right)} \le \frac{10N^3}{\bar\sigma^2 T^{\frac12}}\cdot\exp\!\left(-\frac{\delta\bar\sigma^4\,T\,(\ln(N))^2}{4}\right)
\end{aligned} \tag{113}$$

for all $N \in \{N_0, 2^1N_0, 2^2N_0, \ldots\}$. Summing over $N \in \{2^1, 2^2, 2^3, \ldots\}$ results in

$$\begin{aligned}
\sum_{n=1}^{\infty}\mathbb{P}\Big[A_{2^n}^{(3)}\Big] &= \sum_{\substack{N\in\{2^1,2^2,2^3,\ldots\}\\ N<N_0}}\mathbb{P}\Big[A_N^{(3)}\Big] + \sum_{\substack{N\in\{2^1,2^2,2^3,\ldots\}\\ N\ge N_0}}\mathbb{P}\Big[A_N^{(3)}\Big]\\
&\le N_0 + \sum_{N=N_0}^{\infty}\frac{10}{\bar\sigma^2 T^{\frac12}}\cdot N^{\left(3-\delta\bar\sigma^4 T\ln(N)/4\right)} < \infty.
\end{aligned} \tag{114}$$

This completes the proof of Lemma 7.4. □

Lemma 7.5. *Assume that the setting described in Section 4 and Subsection 4.2 is fulfilled. Then we have*

$$\sum_{n=1}^{\infty}\mathbb{P}\Big[A_{2^n}^{(4)}\Big] < \infty. \tag{115}$$

Proof of Lemma 7.5. First of all, define a filtration $\tilde{\mathcal{F}}_l^N$, $l \in \{0, 1, \ldots, \mathrm{ld}(N)-1\}$, through

$$\tilde{\mathcal{F}}_l^N := \sigma_\Omega\Big(\xi^{v,k}, k\in\mathbb{N}, v\in\big\{\mathrm{ld}(N)-l, \mathrm{ld}(N)-l+1, \ldots, \mathrm{ld}(N)\big\}\Big) \tag{116}$$

for all $l \in \{0, 1, \ldots, \mathrm{ld}(N)-1\}$ and every $N \in \{2^1, 2^2, 2^3, \ldots\}$ where $\sigma_\Omega(\cdot)$ denotes the smallest sigma-algebra generated by its argument. Moreover, define an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping $\tilde L_N\colon \Omega \to \{0, 1, \ldots, \mathrm{ld}(N)-1\}$ through $\tilde L_N := \mathrm{ld}(N) - L_N$ for every $N \in \{2^1, 2^2, 2^3, \ldots\}$. Next observe that the identity

$$\begin{aligned}
\tilde L_N &= \mathrm{ld}(N) - L_N = \mathrm{ld}(N) - \max\Big(\{1\}\cup\Big\{l\in\{1,2,\ldots,\mathrm{ld}(N)\}\colon \exists\, k\in\{1,2,\ldots,\tfrac{N}{2^l}\}\colon |\xi^{l,k}| > 2^{\frac{l}{4}}T^{-\frac14}\Big\}\Big)\\
&= \min\Big(\{\mathrm{ld}(N)-1\}\cup\Big\{l\in\{0,1,\ldots,\mathrm{ld}(N)-1\}\colon \exists\, k\in\{1,2,\ldots,\tfrac{N}{2^{(\mathrm{ld}(N)-l)}}\}\colon |\xi^{\mathrm{ld}(N)-l,k}| > 2^{\frac{(\mathrm{ld}(N)-l)}{4}}T^{-\frac14}\Big\}\Big)
\end{aligned}$$

for every $N \in \{2^1, 2^2, 2^3, \ldots\}$ shows that $\tilde L_N$ is a stopping time with respect to the filtration $\tilde{\mathcal{F}}_l^N$, $l \in \{0, 1, \ldots, \mathrm{ld}(N)-1\}$, for every $N \in \{2^1, 2^2, 2^3, \ldots\}$. Consequently, the sigma-algebras $\tilde{\mathcal{F}}_{\tilde L_N}^N := \big\{A \in \mathcal{F}\colon \big(\forall\, l \in \{0, 1, \ldots, \mathrm{ld}(N)\}\colon A \cap \{\tilde L_N = l\} \in \tilde{\mathcal{F}}_l^N\big)\big\}$ for $N \in \{2^1, 2^2, 2^3, \ldots\}$ are well-defined. By definition (116) the random variables $\xi^{L_N-1,k}$, $k \in \mathbb{N}$, are independent of $\tilde{\mathcal{F}}_{\tilde L_N}^N$ for every $N \in \{2^1, 2^2, 2^3, \ldots\}$. Indeed, observe that (116) shows that

$$\begin{aligned}
&\mathbb{P}\Big[\big\{\xi^{L_N-1,k}\in A\big\}\cap B\Big] = \sum_{l=0}^{\mathrm{ld}(N)-1}\mathbb{P}\Big[\big\{\xi^{L_N-1,k}\in A\big\}\cap B\cap\big\{\tilde L_N = l\big\}\Big]\\
&= \sum_{l=0}^{\mathrm{ld}(N)-1}\mathbb{P}\Big[\big\{\xi^{\mathrm{ld}(N)-l-1,k}\in A\big\}\cap\underbrace{\big(B\cap\{\tilde L_N = l\}\big)}_{\in\tilde{\mathcal{F}}_l^N}\Big] = \sum_{l=0}^{\mathrm{ld}(N)-1}\mathbb{P}\big[\xi^{\mathrm{ld}(N)-l-1,k}\in A\big]\cdot\mathbb{P}\Big[B\cap\big\{\tilde L_N = l\big\}\Big]\\
&= \mathbb{P}\big[\xi^{0,1}\in A\big]\left(\sum_{l=0}^{\mathrm{ld}(N)-1}\mathbb{P}\Big[B\cap\big\{\tilde L_N = l\big\}\Big]\right) = \mathbb{P}\big[\xi^{0,1}\in A\big]\cdot\mathbb{P}\big[B\big]
\end{aligned} \tag{117}$$

for all $A \in \mathcal{B}(\mathbb{R})$, $B \in \tilde{\mathcal{F}}^N_{\tilde{L}_N}$, $k \in \mathbb{N}$ and all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Next we note that $\eta_N \colon \Omega \to [0, \infty)$ is $\tilde{\mathcal{F}}^N_{\tilde{L}_N}/\mathcal{B}(\mathbb{R})$-measurable for every $N \in \{2^1, 2^2, 2^3, \ldots\}$. Indeed, observe that

$$\begin{aligned} &\{\eta_N < c\} \cap \{\tilde{L}_N = l\} = \Big\{ \max\{|\xi^{L_N,k}| \in \mathbb{R} \colon k \in \{1, 2, \ldots, \tfrac{N}{2^{L_N}}\}\} < c \Big\} \cap \{\tilde{L}_N = l\} \\ &= \underbrace{\Big\{ \max\{|\xi^{\mathrm{ld}(N)-l,k}| \in \mathbb{R} \colon k \in \{1, 2, \ldots, \tfrac{N}{2^{\mathrm{ld}(N)-l}}\}\} < c \Big\}}_{\in \tilde{\mathcal{F}}^N_l} \cap \underbrace{\{\tilde{L}_N = l\}}_{\in \tilde{\mathcal{F}}^N_l} \in \tilde{\mathcal{F}}^N_l \end{aligned} \tag{118}$$

for all $c \in \mathbb{R}$, $l \in \{0, 1, \ldots, \mathrm{ld}(N) - 1\}$ and all $N \in \{2^1, 2^2, 2^3, \ldots\}$. In the next step we observe that (117), (118), the fact that $L_N \colon \Omega \to \{1, 2, \ldots, \mathrm{ld}(N)\}$ is measurable with respect to $\tilde{\mathcal{F}}^N_{\tilde{L}_N}$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$ and the inequality $\mathbb{P}\big[||\xi^{0,1}| - x| \le \varepsilon\big] \le \mathbb{P}\big[|\xi^{0,1}| \le 2\varepsilon\big] \le 2\varepsilon\bar{\sigma}^{-1}$ for all $x \in \mathbb{R}$ and all $\varepsilon \in (0, \infty)$ show

$$\begin{aligned} &\mathbb{P}\Big[|\theta_N - \eta_N| \le 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N \,\Big|\, \tilde{\mathcal{F}}^N_{\tilde{L}_N}\Big] \\ &\le \mathbb{P}\Big[\exists\, k \in \{1, 2, \ldots, \tfrac{N}{2^{(L_N-1)}}\} \colon \big||\xi^{L_N-1,k}| - \eta_N\big| \le 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N \,\Big|\, \tilde{\mathcal{F}}^N_{\tilde{L}_N}\Big] \\ &\le \sum_{k=1}^{\frac{N}{2^{(L_N-1)}}} \mathbb{P}\Big[\big||\xi^{L_N-1,k}| - \eta_N\big| \le 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N \,\Big|\, \tilde{\mathcal{F}}^N_{\tilde{L}_N}\Big] \\ &= \frac{N}{2^{(L_N-1)}} \cdot \mathbb{P}\Big[\big||\xi^{0,1}| - \eta_N\big| \le 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N \,\Big|\, \tilde{\mathcal{F}}^N_{\tilde{L}_N}\Big] \\ &\le \frac{N}{2^{(L_N-1)}} \cdot 2 \cdot 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N \cdot \bar{\sigma}^{-1} \le \frac{2N^2}{\bar{\sigma} T^{\frac{1}{4}} 2^{(2^{L_N})}} \end{aligned} \tag{119}$$

$\mathbb{P}$-almost surely for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Now we apply inequality (119) to obtain

$$\begin{aligned} &\mathbb{P}\Big[A^{(4)}_N \cap \big(A^{(2)}_N\big)^c \cap \big(A^{(1)}_N\big)^c\Big] \\ &\le \mathbb{P}\Big[\Big\{|\eta_N - \theta_N| \le 4^{(-2^{(L_N-1)})} \eta_N\Big\} \cap \Big\{\eta_N < 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N\Big\} \cap \big(A^{(1)}_N\big)^c\Big] \\ &\le \mathbb{P}\Big[\Big\{|\eta_N - \theta_N| \le 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N\Big\} \cap \big(A^{(1)}_N\big)^c\Big] \\ &= \mathbb{E}\Big[\mathbb{1}_{(A^{(1)}_N)^c} \cdot \mathbb{P}\Big[|\eta_N - \theta_N| \le 4^{(-2^{(L_N-1)})} 2^{\frac{(L_N-1)}{4}} T^{-\frac{1}{4}} N \,\Big|\, \tilde{\mathcal{F}}^N_{\tilde{L}_N}\Big]\Big] \le \mathbb{E}\Big[\mathbb{1}_{(A^{(1)}_N)^c} \cdot \frac{2N^2}{\bar{\sigma} T^{\frac{1}{4}} 2^{(2^{L_N})}}\Big] \end{aligned} \tag{120}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Next we observe that

$$2^{L_N} \ge 2^{\lfloor 2\,\mathrm{ld}(\bar{\sigma}^2 T^{\frac{1}{2}} \ln(N)) \rfloor} \ge 2^{(2\,\mathrm{ld}(\bar{\sigma}^2 T^{\frac{1}{2}} \ln(N)) - 1)} = \frac{1}{2} \cdot 2^{\mathrm{ld}((\bar{\sigma}^2 T^{\frac{1}{2}} \ln(N))^2)} = \frac{1}{2} \bar{\sigma}^4 T \,(\ln(N))^2 \tag{121}$$

on $\big(A^{(1)}_N\big)^c$ for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Inserting (121) into (120) results in

$$\begin{aligned} &\mathbb{P}\Big[A^{(4)}_N \cap \big(A^{(2)}_N\big)^c \cap \big(A^{(1)}_N\big)^c\Big] \le \mathbb{E}\Big[\mathbb{1}_{(A^{(1)}_N)^c} \cdot \frac{4N^2}{\bar{\sigma} T^{\frac{1}{4}} 2^{(\frac{1}{2}\bar{\sigma}^4 T (\ln(N))^2)}}\Big] \\ &\le \frac{2N^2}{\bar{\sigma} T^{\frac{1}{4}}} \exp\Big(-\frac{\ln(2)}{2} \bar{\sigma}^4 T \,(\ln(N))^2\Big) = 2\,\bar{\sigma}^{-1}\, T^{-\frac{1}{4}}\, N^{(2 - \ln(2)\bar{\sigma}^4 T \ln(N)/2)} \end{aligned} \tag{122}$$

for all $N \in \{2^1, 2^2, 2^3, \ldots\}$. Combining (122), Lemma 7.1 and Lemma 7.2 then shows

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(4)}_{2^n}\Big] &= \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(4)}_{2^n} \cap \big(A^{(2)}_{2^n}\big)^c \cap \big(A^{(1)}_{2^n}\big)^c\Big] + \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(4)}_{2^n} \cap \Big(\big(A^{(2)}_{2^n}\big)^c \cap \big(A^{(1)}_{2^n}\big)^c\Big)^c\Big] \\ &\le \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(4)}_{2^n} \cap \big(A^{(2)}_{2^n}\big)^c \cap \big(A^{(1)}_{2^n}\big)^c\Big] + \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(2)}_{2^n} \cup A^{(1)}_{2^n}\Big] \\ &\le \sum_{N=1}^{\infty} 4\,\bar{\sigma}^{-1}\, T^{-\frac{1}{4}}\, N^{(2 - \ln(2)\bar{\sigma}^4 T \ln(N)/2)} + \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(2)}_{2^n}\Big] + \sum_{n=1}^{\infty} \mathbb{P}\Big[A^{(1)}_{2^n}\Big] < \infty. \end{aligned} \tag{123}$$

This completes the proof of Lemma 7.5. □

We now present the proof of Lemma 4.9. It makes use of Lemmas 7.1–7.5 above.

Proof of Lemma 4.9. Combining the subadditivity of the probability measure $\mathbb{P}$ and Lemmas 7.1, 7.2, 7.4 and 7.5 shows

$$\sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(1)} \cup A_{2^n}^{(2)} \cup A_{2^n}^{(3)} \cup A_{2^n}^{(4)}\Big] \leq \sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(1)}\Big] + \sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(2)}\Big] + \sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(3)}\Big] + \sum_{n=1}^{\infty} \mathbb{P}\Big[A_{2^n}^{(4)}\Big] < \infty. \tag{124}$$

The lemma of Borel-Cantelli (e.g., Theorem 2.7 in [25]) therefore implies

$$\mathbb{P}\bigg[\limsup_{n\to\infty} \Big(A_{2^n}^{(1)} \cup A_{2^n}^{(2)} \cup A_{2^n}^{(3)} \cup A_{2^n}^{(4)}\Big)\bigg] = 0. \tag{125}$$

Hence, we obtain

$$\begin{aligned} \mathbb{P}\Big[N_1 < \infty\Big] &= \mathbb{P}\bigg[\Big\{\omega \in \Omega \colon \exists\, n \in \{N_0, 2^1 N_0, 2^2 N_0, \ldots\} \colon \forall\, m \in \{n, 2^1 n, 2^2 n, \ldots\} \colon \omega \notin \cup_{i=1}^{4} A_m^{(i)}\Big\}\bigg] \\ &= \mathbb{P}\bigg[\Big\{\omega \in \Omega \colon \exists\, n \in \mathbb{N} \colon \forall\, m \in \{n, n+1, \ldots\} \colon \omega \notin A_{2^m}^{(1)} \cup A_{2^m}^{(2)} \cup A_{2^m}^{(3)} \cup A_{2^m}^{(4)}\Big\}\bigg] \\ &= \mathbb{P}\bigg[\liminf_{n\to\infty} \Big(A_{2^n}^{(1)} \cup A_{2^n}^{(2)} \cup A_{2^n}^{(3)} \cup A_{2^n}^{(4)}\Big)^c\bigg] = 1. \end{aligned} \tag{126}$$

This completes the proof of Lemma 4.9. □

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