Convergence of the stochastic Euler scheme for locally Lipschitz coefficients

Martin Hutzenthaler[1] and Arnulf Jentzen[2]

[1]LMU Biozentrum, Department Biologie II, University of Munich (LMU), D-82152 Planegg-Martinsried, Germany, e-mail: hutzenthaler (at) bio.lmu.de
Phone: +49-89-2180-74179, Fax: +49-89-2180-74104

[2]Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544-1000, USA, e-mail: ajentzen (at) math.princeton.edu
Phone: +1-609-258-2654, Fax: +1-609-258-1735

November 16, 2011

Abstract

Stochastic differential equations are often simulated with the Monte Carlo Euler method. Convergence of this method is well understood in the case of globally Lipschitz continuous coefficients of the stochastic differential equation. The important case of superlinearly growing coefficients, however, has remained an open question. The main difficulty is that numerically weak convergence fails to hold in many cases of superlinearly growing coefficients. In this paper we overcome this difficulty and establish convergence of the Monte Carlo Euler method for a large class of one-dimensional stochastic differential equations whose drift functions have at most polynomial growth.

1 Introduction

Many applications require the numerical approximation of moments or expectations of other functionals of the solution of a stochastic differential equation (SDE) whose coefficients are superlinearly growing. Moments are often approximated by discretizing time using the stochastic Euler scheme (see e.g. [**11**], [**15**], [**21**]) (a.k.a. Euler-Maruyama scheme) and by approximating expectations with the Monte Carlo method. This Monte Carlo Euler method has been shown to converge in the case of globally Lipschitz continuous coefficients of the SDE (see e.g. Section 14.1 in [**11**] and Section 12 in [**15**]). The important case of superlinearly growing coefficients, however, has remained an open problem. The main difficulty is that numerically weak convergence fails to hold in many cases of superlinearly growing coefficients; see [**7**]. In this paper we overcome this difficulty and establish convergence of the Monte Carlo Euler method for a large class of one-dimensional SDEs with at most polynomial growing drift functions and with globally Lipschitz continuous diffusion functions; see Section 2 for the exact statement.

For clarity of exposition, we concentrate in this introductory section on the following prominent example. Let $T \in (0,\infty)$ be fixed and let $(X_t)_{t\in[0,T]}$ be the unique strong solution of the one-dimensional SDE

$$dX_t = -X_t^3\,dt + \bar{\sigma}\,dW_t, \qquad X_0 = x_0 \tag{1}$$

for all $t \in [0,T]$, where $(W_t)_{t\in[0,T]}$ is a one-dimensional standard Brownian motion with continuous sample paths and where $\bar{\sigma} \in [0,\infty)$ and $x_0 \in \mathbb{R}$ are given constants. Our goal is then to solve the cubature approximation problem of the SDE (1). More formally, we want to compute moments and, more generally, the deterministic real number

$$\mathbb{E}\Big[f(X_T)\Big] \tag{2}$$

Communicated by Peter Kloeden. AMS subject classifications Primary 65C05; Secondary 60H35, 65C30.
Keywords: Euler scheme, stochastic differential equations, Monte Carlo Euler method, local Lipschitz condition.
Corresponding author: Arnulf Jentzen.

for a given smooth function $f\colon \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth.

A frequently used scheme for solving this problem is the Monte Carlo Euler method. In this method, time is discretized through the stochastic Euler scheme and expectations are approximated by the Monte Carlo method. More formally, the Euler approximation $(Y_n^N)_{n\in\{0,1,\ldots,N\}}$ of the solution $(X_t)_{t\in[0,T]}$ of the SDE (1) is defined recursively through $Y_0^N = x_0$ and

$$Y_{n+1}^N = Y_n^N - \frac{T}{N}\left(Y_n^N\right)^3 + \bar{\sigma}\cdot\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right) \tag{3}$$

for every $n \in \{0,1,\ldots,N-1\}$ and every $N \in \mathbb{N} := \{1,2,\ldots\}$. Moreover, let $Y_n^{N,m}$, $n \in \{0,1,\ldots,N\}$, $N \in \mathbb{N}$, for $m \in \mathbb{N}$ be independent copies of the Euler approximation defined in (3). The Monte Carlo Euler approximation of (2) with $N \in \mathbb{N}$ time steps and $M \in \mathbb{N}$ Monte Carlo runs is then the random real number

$$\frac{1}{M}\left(\sum_{m=1}^{M} f\big(Y_N^{N,m}\big)\right). \tag{4}$$

In order to balance the error due to the Euler method and the error due to the Monte Carlo method, it turns out to be optimal to have M increasing at the order of N^2; see [**2**]. We say that the Monte Carlo Euler method converges if

$$\lim_{N\to\infty}\left|\mathbb{E}\Big[f\big(X_T\big)\Big] - \frac{1}{N^2}\sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big)\right| = 0 \tag{5}$$

holds almost surely for every smooth function $f\colon \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth (see also Appendix A.1 in [**4**]).

In the literature, convergence of the Monte Carlo Euler method is usually established by estimating the bias and by estimating the statistical error (see e.g. Section 3.2 in [**20**]). More formally, the triangle inequality yields

$$\underbrace{\left|\mathbb{E}\Big[f\big(X_T\big)\Big] - \frac{1}{N^2}\sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big)\right|}_{\substack{\text{approximation error of the}\\ \text{Monte Carlo Euler method}}} \leq \underbrace{\left|\mathbb{E}\Big[f\big(X_T\big)\Big] - \mathbb{E}\Big[f\big(Y_N^N\big)\Big]\right|}_{\substack{\text{absolute value}\\ \text{of the bias}}} + \underbrace{\left|\mathbb{E}\Big[f\big(Y_N^N\big)\Big] - \frac{1}{N^2}\sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big)\right|}_{\text{statistical error}} \tag{6}$$

for every $N \in \mathbb{N}$ and every smooth function $f\colon \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth. The first summand on the right-hand side of (6) is the absolute value of the bias due to approximating the exact solution with Euler's method. The second summand on the right-hand side of (6) is the statistical error which is due to approximating an expectation with the arithmetic average over independent copies. The bias is usually the more difficult part to estimate. This is why the concept of numerically weak convergence, which concentrates on that error part, has been studied intensively in the literature (see for instance [**14**], [**12**], [**15**], [**20**], [**1**], [**5**], [**9**], [**17**] or Part VI in [**11**]). To give a definition, we say that the stochastic Euler scheme converges in the numerically weak sense (not to be confused with stochastic weak convergence) if the bias of the Monte Carlo Euler method converges to zero, i.e., if

$$\lim_{N\to\infty}\left|\mathbb{E}\Big[f\big(X_T\big)\Big] - \mathbb{E}\Big[f\big(Y_N^N\big)\Big]\right| = 0 \tag{7}$$

holds for every smooth function $f\colon \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth. If the coefficients of the SDE are globally Lipschitz continuous, then numerically weak convergence of Euler's method and convergence of the Monte Carlo Euler method is well-established; see e.g. Theorem 14.1.5 in [**11**] and Section 12 in [**15**].

The case of superlinearly growing coefficients is more subtle. The main difficulty in that case is that numerically weak convergence usually fails to hold; see [**7**] for a large class of examples. In particular, the sequence $\mathbb{E}\big[(Y_N^N)^2\big]$, $N \in \mathbb{N}$, of second moments of the Euler approximations (3) diverges to infinity if $\bar{\sigma} > 0$ although the second moment $\mathbb{E}\big[(X_T)^2\big]$ of the exact solution of the SDE (1) is finite and, hence, we have

$$\lim_{N\to\infty}\left|\mathbb{E}\Big[(X_T)^2\Big] - \mathbb{E}\Big[(Y_N^N)^2\Big]\right| = \infty \tag{8}$$

instead of (7). The absolute value of the bias thus diverges to infinity in case of SDEs with superlinearly growing coefficients. This in turn implies divergence of the Monte Carlo Euler method in the mean square sense, i.e.,

$$\underbrace{\mathbb{E}\left[\left|\mathbb{E}\Big[(X_T)^2\Big] - \frac{1}{N^2}\sum_{m=1}^{N^2}\big(Y_N^{N,m}\big)^2\right|^2\right]}_{\substack{\text{mean square error of the}\\ \text{Monte Carlo Euler method}}} = \underbrace{\left|\mathbb{E}\Big[(X_T)^2\Big] - \mathbb{E}\Big[(Y_N^N)^2\Big]\right|^2}_{\text{squared bias }\to\infty} + \underbrace{\operatorname{Var}\left(\frac{1}{N^2}\sum_{m=1}^{N^2}\big(Y_N^{N,m}\big)^2\right)}_{\substack{\text{variance of the Monte}\\ \text{Carlo Euler method}}} \to \infty \quad (9)$$

as $N \to \infty$. Clearly, the mean square divergence (9) does not exclude the almost sure convergence (5). Indeed, the main result of this article proves the convergence (5) of the Monte Carlo Euler method. For proving this result, we first need to understand why Euler's method does not converge in the sense of numerically weak convergence. In the deterministic case, that is, (1) and (3) with $\bar{\sigma} = 0$, the Euler approximation diverges if the starting point is sufficiently large. This divergence has been estimated in [**7**] and turns out to be at least double-exponentially fast. Now in the presence of noise ($\bar{\sigma} > 0$), the Brownian motion has an exponentially small chance to push the Euler approximation outside of $[-N, N]$. On this event, the Euler approximation grows at least double-exponentially fast due to the deterministic dynamics. Consequently, as being double-exponentially large over-compensates that the event has an exponentially small probability, the L^2-norm of the Euler approximation diverges to infinity and, hence, numerically weak convergence fails to hold.

Now we indicate for example (1) with $x_0 = 0$ and $\bar{\sigma} = 1$ why the Monte Carlo Euler method converges although the stochastic Euler scheme fails to converge in the sense of numerically weak convergence. Consider the event $\Omega_N := \{\sup_{0\le t\le T}|W_t| \le \sqrt{N/(2T)}\}$ and note that the probability of $(\Omega_N)^c$ is exponentially small in $N \in \mathbb{N}$. The key step in our proof is to show that the Euler approximation does not diverge on Ω_N as $N \in \mathbb{N}$ goes to infinity. More precisely, one can show that the Euler approximations (3) are uniformly dominated on Ω_N by twice the supremum of the Brownian motion, i.e.,

$$\sup_{N\in\mathbb{N}}\left(\mathbb{1}_{\Omega_N}\left|Y_N^N\right|\right) \le 2\left(\sup_{0\le t\le T}|W_t|\right) \tag{10}$$

holds. Consequently, the restricted absolute moments are uniformly bounded

$$\sup_{N\in\mathbb{N}}\mathbb{E}\Big[\mathbb{1}_{\Omega_N}\left|Y_N^N\right|^p\Big] \le 2^p\cdot\mathbb{E}\left[\Big(\sup_{0\le t\le T}|W_t|\Big)^p\right] < \infty \tag{11}$$

for all $p \in [1, \infty)$. This estimate complements the divergence

$$\lim_{N\to\infty}\mathbb{E}\Big[\mathbb{1}_{(\Omega_N)^c}\left|Y_N^N\right|^p\Big] = \infty \tag{12}$$

for all $p \in [1, \infty)$, which has been established in [**7**]. Now once the restricted absolute moments are uniformly bounded, an adaptation of the arguments of the globally Lipschitz case leads to the modified numerically weak convergence

$$\lim_{N\to\infty}\mathbb{E}\Big[\mathbb{1}_{\Omega_N}f\big(Y_N^N\big)\Big] = \mathbb{E}\Big[f(X_T)\Big] \tag{13}$$

for every smooth function $f\colon \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth, see Lemma 4.6. By substituting this into an inequality analogous to (6) and by using the exponential decay of the probability of $(\Omega_N)^c$ in $N \in \mathbb{N}$, one can establish convergence of the Monte Carlo Euler method. Note that a domination as strong as (10) holds for more general non-increasing drift functions if the diffusion function is identically equal to 1. For more general drift and diffusion functions, however, both Ω_N and the dominating process are more complicated in that they depend on the Euler approximation. Nevertheless, the dominating process can be shown to have uniformly bounded absolute moments; see Section 4 for the details.

Our main result, Theorem 2.1 below, establishes convergence of the Monte Carlo Euler method for SDEs with globally one-sided Lipschitz continuous drift functions and with globally Lipschitz continuous diffusion functions. Moreover, the coefficients of the SDE are assumed to have continuous fourth derivatives with at most polynomial growth, see Section 2 for the exact statement. The order of convergence turns out to be as in the globally Lipschitz case. In that case, the stochastic Euler scheme converges in the sense of numerically weak convergence with order 1. The Monte Carlo simulation of $\mathbb{E}\big[f\big(Y_N^N\big)\big]$ with M independent Euler approximations has convergence order $\frac{1}{2}-$. For a real number $r > 0$, we write $r-$ for the convergence order if the convergence order is better than $r - \varepsilon$ for every arbitrarily small $\varepsilon \in (0, r)$. We therefore choose $M = N^2$ in order to balance the error arising from Euler's approximation and the error arising from the Monte Carlo approximation. Both error terms are then bounded by a random multiple of $N^{(\varepsilon-1)}$ with $\varepsilon \in (0, 1)$. Since $O(M \cdot N) = O\big(N^3\big)$ function evaluations, arithmetical operations and random variables are needed to compute the Monte Carlo Euler approximation (4), the Monte Carlo Euler method converges with order $\frac{1}{3}-$ with respect to the computational effort in the case of global Lipschitz coefficients of the SDE (see [**2**]). Theorem 2.1 shows that $\frac{1}{3}-$ is also the convergence order in the case of superlinearly growing coefficients of the SDE. Simulations support this result, see Section 3.

Let us reconsider the standard splitting (6) of the approximation error into bias and statistical error. Theorem 2.1 of [**7**] implies that the absolute value of the bias diverges to infinity as $N \to \infty$. This together with our Theorem 2.1 below yields that also the statistical error diverges to infinity. More formally, we see that

$$\underbrace{\left|\mathbb{E}\Big[f\big(X_T\big)\Big] - \frac{1}{N^2}\sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big)\right|}_{\substack{\text{approximation error of the}\\ \text{Monte Carlo Euler method}}\ \to 0} \leq \underbrace{\left|\mathbb{E}\Big[f\big(X_T\big)\Big] - \mathbb{E}\Big[f\big(Y_N^N\big)\Big]\right|}_{\substack{\text{absolute value}\\ \text{of the bias}}\ \to\infty} + \underbrace{\left|\mathbb{E}\Big[f\big(Y_N^N\big)\Big] - \frac{1}{N^2}\sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big)\right|}_{\text{statistical error }\to\infty} \tag{14}$$

$\mathbb{P}$-a.s. as $N \to \infty$ for every smooth function $f \colon \mathbb{R} \to \mathbb{R}$ with at most polynomially growing derivatives and with $f(x) \geq c\,|x|^c - 1/c$ for all $x \in \mathbb{R}$ and some $c \in (0, \infty)$. This emphasizes that the standard splitting of the approximation error of the Monte Carlo Euler method into bias and statistical error is not appropriate in case of SDEs with superlinearly growing coefficients.

2 Main result

We establish convergence of the Monte Carlo Euler method for more general one-dimensional diffusions than our introductory example (1). More precisely, we pose the following assumptions on the coefficients. The drift function is assumed to be globally one-sided Lipschitz continuous and the diffusion function is assumed to be globally Lipschitz continuous. Additionally, both the drift function and the diffusion function are assumed to have a continuous fourth derivative with at most polynomial growth.

We introduce further notation for the formulation of our main result. Fix $T \in (0, \infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let $\xi^{(m)} \colon \Omega \to \mathbb{R}$, $m \in \mathbb{N}$, be a sequence of independent, identically distributed $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mappings with $\mathbb{E}\big[|\xi^{(1)}|^p\big] < \infty$ for every $p \in [1, \infty)$ and let $W^{(m)} \colon [0, T] \times \Omega \to \mathbb{R}$, $m \in \mathbb{N}$, be a sequence of independent scalar standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motions with continuous sample paths. Furthermore, let $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable functions. Generalizing (1), let $(X_t)_{t\in[0,T]}$ be a one-dimensional diffusion with drift function μ and diffusion function σ. More precisely, let $X \colon [0, T] \times \Omega \to \mathbb{R}$ be an (up to indistinguishability unique) adapted stochastic process with continuous sample paths which satisfies

$$\mathbb{P}\bigg[X_t = \xi^{(1)} + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s^{(1)}\bigg] = 1 \tag{15}$$

for every $t \in [0, T]$. The functions μ and σ^2 are the infinitesimal mean and the infinitesimal variance respectively.

Next we introduce independent versions of the Euler approximation. Define $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings $Y_n^{N,m} \colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, $m \in \mathbb{N}$, by $Y_0^{N,m}(\omega) := \xi^{(m)}(\omega)$ and by

$$Y_{n+1}^{N,m}(\omega) := Y_n^{N,m}(\omega) + \frac{T}{N} \cdot \mu\big(Y_n^{N,m}(\omega)\big) + \sigma\big(Y_n^{N,m}(\omega)\big) \cdot \bigg(W^{(m)}_{\frac{(n+1)T}{N}}(\omega) - W^{(m)}_{\frac{nT}{N}}(\omega)\bigg) \tag{16}$$

for every $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ and every $m \in \mathbb{N}$. Now we formulate the main result of this article.

Theorem 2.1. *Suppose that* $f, \mu, \sigma\colon \mathbb{R} \to \mathbb{R}$ *are four times continuously differentiable functions with*

$$\left|f^{(n)}(x)\right| + \left|\mu^{(n)}(x)\right| + \left|\sigma^{(n)}(x)\right| \leq L\left(1 + |x|^{\delta}\right) \quad \forall\, x \in \mathbb{R} \tag{17}$$

for every $n \in \{0, 1, \ldots, 4\}$*, where* $L \in (0, \infty)$ *and* $\delta \in (1, \infty)$ *are fixed constants. Moreover, assume that the drift coefficient is globally one-sided Lipschitz continuous*

$$(x - y) \cdot (\mu(x) - \mu(y)) \leq L\,(x - y)^2 \quad \forall\, x, y \in \mathbb{R} \tag{18}$$

and that the diffusion coefficient is globally Lipschitz continuous

$$|\sigma(x) - \sigma(y)| \leq L\,|x - y| \quad \forall\, x, y \in \mathbb{R}. \tag{19}$$

Then there are $\mathcal{F}/\mathcal{B}([0,\infty))$*-measurable mappings* $C_\varepsilon\colon \Omega \to [0, \infty)$*,* $\varepsilon \in (0,1)$*, and a set* $\tilde{\Omega} \in \mathcal{F}$ *with* $\mathbb{P}[\tilde{\Omega}] = 1$ *such that*

$$\left|\mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2}\left(\sum_{m=1}^{N^2} f(Y_N^{N,m}(\omega))\right)\right| \leq C_\varepsilon(\omega) \cdot \frac{1}{N^{(1-\varepsilon)}} \tag{20}$$

holds for every $\omega \in \tilde{\Omega}$*,* $N \in \mathbb{N}$ *and every* $\varepsilon \in (0,1)$*.*

The proof is deferred to Section 4. For further numerical approximation results for SDEs with superlinearly growing coefficients, see e.g. [**6**], [**13**], [**16**], [**19**] and the references in the introductory section of [**7**].

Note that the assumptions of Theorem 2.1 ensure the existence of an adapted stochastic process $X\colon [0,T] \times \Omega \to \mathbb{R}$ with continuous sample paths which satisfies (15) and

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^p\right] < \infty \tag{21}$$

for all $p \in [1, \infty)$ (see Theorem 2.6.4 in [**3**]). Therefore, the expression $\mathbb{E}\big[f(X_T)\big]$ in (20) in Theorem 2.1 is well-defined.

Since $O\big(N^3\big)$ function evaluations, arithmetical operations and random variables are needed to compute the expression $\frac{1}{N^2}\Big(\sum_{m=1}^{N^2} f(Y_N^{N,m}(\omega))\Big)$ in (20) for $\omega \in \Omega$, Theorem 2.1 shows that the Monte Carlo Euler method converges under the above assumptions with order $\frac{1}{3}-$ with respect to the computational effort. This is the standard convergence order as in the global Lipschitz case (see e.g. [**2**]).

3 Simulations

In this section, we simulate the second moment of two stochastic differential equations. First we simulate the stochastic Ginzburg-Landau equation with multiplicative noise, which we choose as there exists an explicit solution for this SDE. Let $(X_t)_{t\in[0,1]}$ be the solution of

$$dX_t = \left(\frac{1}{2}X_t - X_t^3\right) dt + X_t\, dW_t, \qquad X_0 = 1 \tag{22}$$

for all $t \in [0,1]$. The exact solution at time 1 is known explicitly (see e.g. Section 4.4 in [**11**]) and is given by

$$X_1 = \frac{\exp\left(W_1\right)}{\sqrt{1 + 2\int_0^1 \exp\left(2W_s\right) ds}}. \tag{23}$$

The exact value of the second moment $\mathbb{E}\big[(X_T)^2\big] = \mathbb{E}\big[(X_1)^2\big]$ is not known. Instead we use the exact solution (23) at time 1 to approximate the second moment. For this, we approximate the Lebesgue integral in the denominator of (23) with a Riemann sum with $3 \cdot 10^3$ summands. Moreover, we approximate the second moment at time 1 by a Monte Carlo simulation with 10^7 independent approximations of X_1. This results in the approximate value $\mathbb{E}\big[(X_1)^2\big] \approx 0.4945$.

$N = 2^0$	$N = 2^1$	$N = 2^2$	$N = 2^3$	$N = 2^4$
1.1379	0.9118	0.4258	0.2942	0.4386
$N = 2^5$	$N = 2^6$	$N = 2^7$	$N = 2^8$	$N = 2^9$
0.4641	0.4663	0.4859	0.4904	0.4935

Table 1: Monte Carlo Euler approximations (24) of $\mathbb{E}\left[(X_1)^2\right]$ of the SDE (22).

Next we approximate the second moment at time 1 with the Monte Carlo Euler method. We will sample one random $\omega \in \Omega$ and calculate the Monte Carlo Euler approximations for this $\omega \in \Omega$ for different discretization step sizes. More precisely, Table 1 shows the Monte Carlo Euler approximation

$$\frac{1}{N^2} \sum_{m=1}^{N^2} \left(Y_N^{N,m}(\omega) \right)^2 \tag{24}$$

of the second moment at time 1 of the SDE (22) for every $N \in \left\{2^0, 2^1, 2^2, \ldots, 2^9\right\}$ and one random $\omega \in \Omega$. In Figure 1, the approximation error of these Monte Carlo Euler approximations, i.e., the quantity

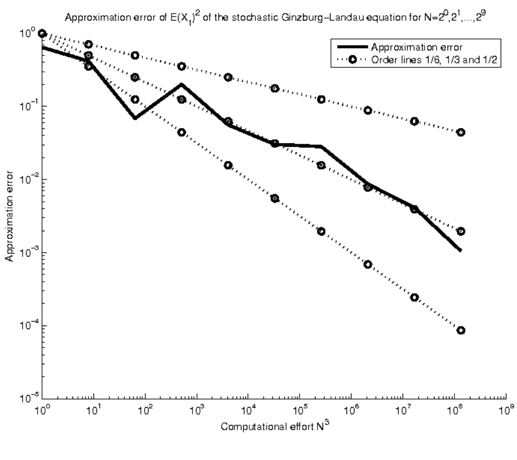

Figure 1: Approximation error (25) of the Monte Carlo Euler approximations (24) of $\mathbb{E}\left[(X_1)^2\right]$ of the SDE (22).

$$\left| 0.4945 - \frac{1}{N^2} \sum_{m=1}^{N^2} \left(Y_N^{N,m}(\omega) \right)^2 \right|, \tag{25}$$

is plotted against N^3 for every $N \in \left\{2^0, 2^1, 2^2, \ldots, 2^9\right\}$. Note that N^3 is the computational effort up to a constant. The three order lines in Figure 1 correspond to the convergence orders $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$. Hence, Figure 1 indicates that the Monte Carlo Euler method converges in the case of the stochastic Ginzburg-Landau equation (22) with its theoretically predicted order $\frac{1}{3}-$.

$N=2^0$	$N=2^1$	$N=2^2$	$N=2^3$	$N=2^4$
1.4516	0.5166	0.4329	0.5308	0.4285
$N=2^5$	$N=2^6$	$N=2^7$	$N=2^8$	$N=2^9$
0.4452	0.4602	0.4517	0.4548	0.4537

Table 2: Monte Carlo Euler approximations (27) of $\mathbb{E}\big[(X_1)^2\big]$ of the SDE (26).

Next we simulate our introductory example. Let $(X_t)_{t\in[0,T]}$ be the solution of the SDE (1) with $T=1$, $\bar{\sigma}=1$ and $x_0=0$. The SDE (1) thus reads as

$$dX_t = -X_t^3\, dt + dW_t, \qquad X_0 = 0 \tag{26}$$

for all $t\in[0,1]$. Here there exists no explicit expression for the solution or its second moments. As an approximation of the exact value $\mathbb{E}\big[(X_1)^2\big]$, we now take a Monte Carlo Euler approximation with a larger N. We choose $N=2^{12}$ and obtain the value $0.4529\approx\mathbb{E}\big[(X_1)^2\big]$ as an approximation of $\mathbb{E}\big[(X_1)^2\big]$. Table 2 shows the value of the Monte Carlo Euler approximation

$$\frac{1}{N^2}\sum_{m=1}^{N^2}\Big(Y_N^{N,m}(\omega)\Big)^2 \tag{27}$$

of the second moment at time 1 of the SDE (26) for every $N\in\big\{2^0,2^1,2^2,\ldots,2^9\big\}$ and one random $\omega\in\Omega$. In Figure 2, the approximation error of these Monte Carlo Euler approximations, i.e., the quantity

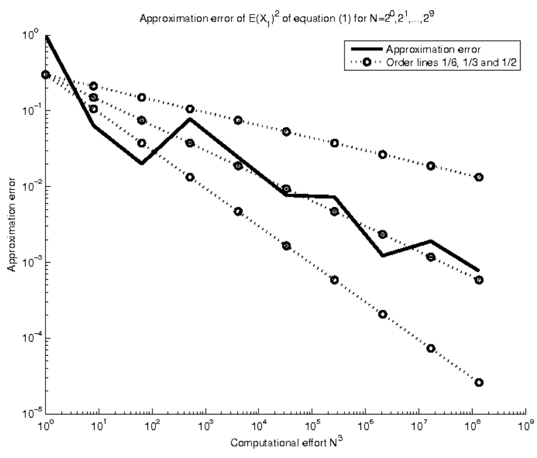

Figure 2: Approximation error (28) of the Monte Carlo Euler approximations (27) of $\mathbb{E}\big[(X_1)^2\big]$ of the SDE (26).

$$\left|0.4529 - \frac{1}{N^2} \sum_{m=1}^{N^2} (Y_N^{N,m}(\omega))^2\right|, \tag{28}$$

is plotted against N^3 for every $N \in \{2^0, 2^1, 2^2, \ldots, 2^9\}$. Note that N^3 is the computational effort up to a constant. The three order lines in Figure 2 correspond to the convergence orders $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$. Therefore, Figure 2 suggests that the Monte Carlo Euler method converges in the case of the SDE (26) with its theoretically predicted order $\frac{1}{3}-$.

4 Proof of Theorem 2.1

First we introduce more notation. Recall the standard Brownian motion $W^{(1)} \colon [0,T] \times \Omega \to \mathbb{R}$ and the Euler approximations $Y_n^{N,1} \colon \Omega \to \mathbb{R}$, $n \in \{0,1,\ldots,N\}$, $N \in \mathbb{N}$, from Section 2. Throughout this section, we use the stochastic process $W \colon [0,T] \times \Omega \to \mathbb{R}$ and the $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings $Y_n^N \colon \Omega \to \mathbb{R}$, $n \in \{0,1,\ldots,N\}$, $N \in \mathbb{N}$, given by

$$W_t(\omega) := W_t^{(1)}(\omega) \qquad \text{and} \qquad Y_n^N(\omega) := Y_n^{N,1}(\omega) \tag{29}$$

for every $t \in [0,T]$, $n \in \{0,1,\ldots,N\}$, $\omega \in \Omega$ and every $N \in \mathbb{N}$.

4.1 Outline

For general drift and diffusion functions, our proof of Theorem 2.1 somewhat buries the main new ideas. To explain these ideas, we give now a very rough outline (the precise estimates and assertions can be found in Subsections 4.2-4.7 below). The main step will be to establish uniform boundedness of the restricted absolute moments of the Euler approximations

$$\sup_{N \in \mathbb{N}} \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \big|Y_N^N\big|\Big] < \infty \tag{30}$$

where $(\Omega_N)_{N \in \mathbb{N}}$ is a sequence of events whose probabilities converge to 1 sufficiently fast. From here, one can then adapt the arguments of the global Lipschitz case to derive the modified numerically weak convergence (13) and to obtain Theorem 2.1.

The idea behind (30) is now explained on the example of negative cubic drift and multiplicative noise. Formally, we consider $\mu(x) = -x^3$, $\sigma(x) = x$ for all $x \in \mathbb{R}$ and $\xi^{(1)}(\omega) = 1$ for all $\omega \in \Omega$. The Euler approximation (29) is then given by $Y_0^N = 1$ and

$$Y_{n+1}^N = Y_n^N - \frac{T}{N}\big(Y_n^N\big)^3 + Y_n^N \cdot \Big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\Big) \tag{31}$$

for all $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$ and assume that the Euler approximation $(Y_k^N)_{k \in \{0,1,\ldots,N\}}$ does not change sign until and including the n-th approximation step for some $n \in \{0,1,\ldots,N\}$ which we fix for now, that is, $Y_k^N \geq 0$ for all $k \in \{0,1,\ldots,n\}$. Then, using $1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} Y_k^N &= Y_{k-1}^N - \frac{T}{N}\big(Y_{k-1}^N\big)^3 + Y_{k-1}^N \cdot \Big(W_{\frac{kT}{N}} - W_{\frac{(k-1)T}{N}}\Big) \\ &\leq Y_{k-1}^N \Big(1 + W_{\frac{kT}{N}} - W_{\frac{(k-1)T}{N}}\Big) \\ &\leq Y_{k-1}^N \exp\Big(W_{\frac{kT}{N}} - W_{\frac{(k-1)T}{N}}\Big) \end{aligned} \tag{32}$$

and iterating this inequality shows

$$\begin{aligned} Y_k^N &\leq Y_{k-2}^N \exp\Big(W_{\frac{(k-1)T}{N}} - W_{\frac{(k-2)T}{N}}\Big) \exp\Big(W_{\frac{kT}{N}} - W_{\frac{(k-1)T}{N}}\Big) \\ &= Y_{k-2}^N \exp\Big(W_{\frac{kT}{N}} - W_{\frac{(k-2)T}{N}}\Big) \\ &\leq \ldots \leq Y_0^N \exp\Big(W_{\frac{kT}{N}}\Big) =: \tilde{D}_k^N \end{aligned} \tag{33}$$

for all $k \in \{0, 1, \ldots, n\}$. Thus the Euler approximation $(Y_k^N)_{k\in\{0,1,\ldots,n\}}$ is bounded above by the dominating process $(\tilde{D}_k^N)_{k\in\{0,1,\ldots,n\}}$. This dominating process has uniformly bounded absolute moments. So the absolute moments of the Euler approximation can only be unbounded if the Euler approximation changes its sign. Now if Y_k happens to be very large for one $k \in \{0, 1, \ldots, N-1\}$, then Y_{k+1} is negative with absolute value being very large because of the negative cubic drift. Through a sequence of changes in sign, it could happen that the absolute value of the Euler approximation increases more and more. To avoid this, we restrict the Euler approximation to an event Ω_N on which the drift alone cannot change the sign of the Euler approximation. On Ω_N, the Euler approximation changes sign only due to the diffusive part. As the diffusion function is at most linearly growing, these changes of sign can be controlled. In between consecutive changes of sign, the Euler approximation is again bounded by a dominating process as above. Through this pathwise comparison with a dominating process, we will establish the uniform boundedness (30) of the restricted absolute moments. For the details, we refer to Lemma 4.2, which is the key result in our proof of Theorem 2.1.

4.2 Notation and auxiliary lemmas

In order to show Theorem 2.1, the following objects are needed. First of all, define $t_n^N := \frac{nT}{N}$ for every $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$ and let the $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable mapping $\tilde{\sigma} \colon \mathbb{R} \to \mathbb{R}$ be given by

$$\tilde{\sigma}(x) := \begin{cases} \frac{(\sigma(x)-\sigma(0))}{x} & : x \neq 0 \\ 0 & : x = 0 \end{cases} \tag{34}$$

for all $x \in \mathbb{R}$. Moreover, let the $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings $\alpha_n^{N,m} \colon \Omega \to \mathbb{R}$ and $\beta_n^{N,m} \colon \Omega \to \mathbb{R}$ be given by

$$\alpha_n^{N,m}(\omega) := \frac{TL}{N} + \tilde{\sigma}(Y_n^{N,m}(\omega)) \cdot \left(W_{t_{n+1}^N}^{(m)}(\omega) - W_{t_n^N}^{(m)}(\omega) \right) \tag{35}$$

and

$$\beta_n^{N,m}(\omega) := \frac{T\mu(0)}{N} + \sigma(0) \cdot \left(W_{t_{n+1}^N}^{(m)}(\omega) - W_{t_n^N}^{(m)}(\omega) \right) \tag{36}$$

for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N-1\}$ and every $N, m \in \mathbb{N}$. For simplicity we also use $\alpha_n^N, \beta_n^N \colon \Omega \to \mathbb{R}$ given by $\alpha_n^N(\omega) := \alpha_n^{N,1}(\omega)$ and $\beta_n^N(\omega) := \beta_n^{N,1}(\omega)$ for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Using these ingredients, we now define the dominating process. Let the $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping $D_{v,w}^{N,m} \colon \Omega \to \mathbb{R}$ be given by

$$\begin{aligned} D_{v,w}^{N,m}(\omega) &:= e^{\left(\sum_{l=v}^{w-1} \alpha_l^{N,m}(\omega)\right)} \left(T\,|\mu(0)| + |\sigma(0)| + \left|\xi^{(m)}(\omega)\right| + 1 \right) \\ &\quad + \sum_{k=v}^{w-1} \operatorname{sgn}(Y_k^{N,m}(\omega))\, e^{\left(\sum_{l=k+1}^{w-1} \alpha_l^{N,m}(\omega)\right)} \beta_k^{N,m}(\omega) \end{aligned} \tag{37}$$

for every $\omega \in \Omega$, $v, w \in \{0, 1, \ldots, N\}$ and every $N, m \in \mathbb{N}$, where $\operatorname{sgn} \colon \mathbb{R} \to \{-1, 1\}$ is given by $\operatorname{sgn}(x) := 1$ for every $x \in [0, \infty)$ and $\operatorname{sgn}(x) := -1$ for every $x \in (-\infty, 0)$. As usual, $\sum_{l=v}^{w-1} \alpha_l^{N,m}(\omega) = 0$ for every $v \in \{w, w+1, \ldots, N\}$, $w \in \{0, 1, \ldots, N\}$, $\omega \in \Omega$ and every $N, m \in \mathbb{N}$. Note that $D_{v,w}^{N,m} \colon \Omega \to \mathbb{R}$ only depends on the Brownian motion $W^{(m)} \colon [0, T] \times \Omega \to \mathbb{R}$ and the initial random variable $\xi^{(m)} \colon \Omega \to \mathbb{R}$ for every $v, w \in \{0, 1, \ldots, N\}$ and every $N, m \in \mathbb{N}$. Therefore, $D_{v,w}^{N,m}$, $m \in \mathbb{N}$, is a sequence of independent random variables for every $v, w \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$. We will show that the Euler approximation is dominated by the dominating process since the last change of sign. More formally, let the $\mathcal{F}/\mathcal{P}(\{0, 1, \ldots, n\})$-measurable mapping $\tau_n^N \colon \Omega \to \{0, 1, \ldots, n\}$ be given by

$$\tau_n^N(\omega) := \max\left(\{0\} \cup \left\{ k \in \{1, 2, \ldots, n\} \,\middle|\, \operatorname{sgn}(Y_{k-1}^N(\omega)) \neq \operatorname{sgn}(Y_k^N(\omega)) \right\} \right) \tag{38}$$

for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$. The random time $\tau_n^N \colon \Omega \to \{0, 1, \ldots, n\}$ is the last time of a change of sign of Y_k^N, $k \in \{0, 1, \ldots, n\}$, for every $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$. Lemma 4.2 below shows that $|Y_n^N|$ is bounded by $D_{\tau_n^N,n}^{N,1}$ on a certain event $\Omega_{N,n}$ for every $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$. Next we define these events $\Omega_{N,n}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, such that the drift alone cannot cause a change of sign and such that the increment of the Brownian motion is not extraordinary large. Let the real number $r_N \in [0,\infty)$ be given by

$$r_N := \min\left(\frac{N^{\frac{1}{4}}}{L}, \left[\max\left(0, \frac{N}{T\left(\sup_{s\in[-1,1]}|\mu'(s)|+3L\right)}-1\right)\right]^{\frac{1}{(\delta-1)}}\right) \tag{39}$$

for every $N \in \mathbb{N}$. Now define sets $\Omega_{N,n}, \Omega_N^m, \Omega_N \in \mathcal{F}$ by

$$\Omega_{N,n} := \left\{\omega\in\Omega \;\middle|\; \sup_{v,w\in\{0,1,\dots,n\}} \left|D_{v,w}^{N,1}(\omega)\right| \le r_N, \sup_{k\in\{0,1,\dots,n-1\}} \left|W_{t_{k+1}^N}(\omega) - W_{t_k^N}(\omega)\right| \le N^{-\frac{1}{4}}\right\} \tag{40}$$

by

$$\Omega_N^m := \left\{\omega\in\Omega \;\middle|\; \sup_{v,w\in\{0,1,\dots,N\}} \left|D_{v,w}^{N,m}(\omega)\right| \le r_N, \sup_{n\in\{0,1,\dots,N-1\}} \left|W^{(m)}_{t_{n+1}^N}(\omega) - W^{(m)}_{t_n^N}(\omega)\right| \le N^{-\frac{1}{4}}\right\} \tag{41}$$

and by

$$\Omega_N := \Omega_{N,N} = \Omega_N^1 \tag{42}$$

for every $n \in \{0,1,\dots,N\}$ and every $N,m \in \mathbb{N}$. Finally, define $\tilde{\Omega} \in \mathcal{F}$ by

$$\begin{aligned}\tilde{\Omega} := &\left(\bigcup_{N\in\mathbb{N}}\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega_M^m\right)\\ &\bigcap\left(\bigcap_{\varepsilon>0}\left\{\omega\in\Omega \;\middle|\; \sup_{N\in\mathbb{N}}\frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega_N^m}(\omega)\cdot f(Y_N^{N,m}(\omega)) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right)\right|}{N^{(1+\varepsilon)}} < \infty\right\}\right).\end{aligned} \tag{43}$$

Note that $\tilde{\Omega}$ is indeed in $\mathcal{F}$. Moreover, we write $\|Z\|_{L^p} := (\mathbb{E}[|Z|^p])^{\frac{1}{p}} \in [0,\infty]$ for all $p \in [1,\infty)$ and all $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings $Z\colon \Omega \to \mathbb{R}$. Our proof of Theorem 2.1 uses the following lemmas.

Lemma 4.1 (Burkholder-Davis-Gundy inequality)**.** *Let $N \in \mathbb{N}$ and let $Z_1,\dots,Z_N : \Omega \to \mathbb{R}$ be $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings with $\mathbb{E}|Z_n|^2 < \infty$ for all $n \in \{1,\dots,N\}$ and with $\mathbb{E}[Z_{n+1}|Z_1,\dots,Z_n] = 0$ for all $n \in \{1,\dots,N-1\}$. Then we obtain*

$$\|Z_1+\ldots+Z_N\|_{L^p} \le K_p\cdot\left(\|Z_1\|_{L^p}^2+\ldots+\|Z_N\|_{L^p}^2\right)^{\frac{1}{2}} \tag{44}$$

for every $p \in [2,\infty)$, where K_p, $p\in[2,\infty)$, are universal constants.

The following lemma is the key result in our proof of Theorem 2.1.

Lemma 4.2 (Dominator Lemma)**.** *Let $Y_n^N\colon \Omega\to\mathbb{R}$, $D_{n,m}^{N,1} : \Omega \to \mathbb{R}$, $\tau_n^N\colon \Omega\to\mathbb{R}$ and $\Omega_{N,n} \in \mathcal{F}$ for $n,m \in \{0,1,\dots,N\}$ and $N\in\mathbb{N}$ be given by (29), (37), (38) and (40). Then we have*

$$\left|Y_n^N(\omega)\right| \le D^{N,1}_{\tau_n^N(\omega),n}(\omega) \tag{45}$$

for every $\omega \in \Omega_{N,n}$, $n \in \{0,1,\dots,N\}$ and every $N \in \mathbb{N}$.

The domination (45) might not look helpful at first view since $D^{N,1}_{\tau_n^N,n}$ depends on the Euler approximation and since τ_n^N is in general not a stopping time for $n \in \{1,\dots,N\}$ and $N\in\mathbb{N}$. However, the dependence of the dominating process on the Euler approximation can be controlled as $\tilde{\sigma}$ is bounded and the dependence of $D^{N,1}_{\tau_n^N,n}$ on τ_n^N for all $n \in \{0,1,\dots,N\}$ and all $N\in\mathbb{N}$ is no problem as $D^{N,1}_{v,w}$ can be controlled uniformly in $v,w \in \{0,1,\dots,N\}$ and $N \in \mathbb{N}$. This is subject of the following lemma.

Lemma 4.3 (Uniformly bounded absolute moments of the dominator)**.** *Let $D_{n,m}^N\colon \Omega \to [0,\infty)$ for $n,m \in \{0,1,\dots,N\}$ and $N\in\mathbb{N}$ be given by (37). Then we have*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\sup_{v,w\in\{0,1,\dots,N\}} \left|D_{v,w}^{N,1}\right|^p\right] < \infty \tag{46}$$

for all $p \in [1,\infty)$.

From Lemma 4.2 and from Lemma 4.3, we immediately conclude that the restricted absolute moments of the Euler approximation are uniformly bounded.

Corollary 4.4 (Bounded moments of the Euler approximation)**.** *Let the Euler approximation $Y_n^N\colon \Omega \to \mathbb{R}$ for $n \in \{0, 1, \ldots, N\}$ and $N \in \mathbb{N}$ be given by* (29)*. Then we have*

$$\sup_{N\in\mathbb{N}} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n}} \left|Y_n^N\right|^p\Big] < \infty \tag{47}$$

for every $p \in [1, \infty)$.

Next we estimate the probability of Ω_N for every large $N \in \mathbb{N}$.

Lemma 4.5 (Full probability)**.** *Let $\Omega_N \in \mathcal{F}$ for $N \in \mathbb{N}$ and $\tilde{\Omega} \in \mathcal{F}$ be given by* (42) *and* (43)*. Then we have that*

$$\sup_{N\in\mathbb{N}} \Big(N^4 \cdot \mathbb{P}\Big[(\Omega_N)^c\Big]\Big) < \infty \qquad \textit{and} \qquad \mathbb{P}\Big[\tilde{\Omega}\Big] = 1 \tag{48}$$

holds.

Using the above lemmas, we establish the following modification of numerical weak convergence.

Lemma 4.6 (Modified weak convergence)**.** *Let $X\colon [0,T] \times \Omega \to \mathbb{R}$, $\Omega_N \in \mathcal{F}$ and $Y_n^N\colon \Omega \to \mathbb{R}$ for $n \in \{0, 1, \ldots, N\}$ and $N \in \mathbb{N}$ be given by* (15), (42) *and* (29)*. Then we obtain that*

$$\sup_{N\in\mathbb{N}} \left(N \cdot \left|\mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(Y_N^N)\Big]\right|\right) < \infty \tag{49}$$

holds.

While the proof of Lemma 4.1 can be found in Theorem 6.3.10 in [**18**], the proofs of Lemma 4.2, Lemma 4.3, Lemma 4.5 and Lemma 4.6 are given in Sections 4.3-4.7 below.

4.3 Proof of Theorem 2.1

Proof of Theorem 2.1. Consider the $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $Z\colon \Omega \to [0,\infty)$ given by

$$Z(\omega) := \sup_{N\in\mathbb{N}} \left(\frac{\mathbb{1}_{\tilde{\Omega}}(\omega)}{N} \left(\sum_{m=1}^{N^2} \left|f(Y_N^{N,m}(\omega))\right|\right) \left(1 - \mathbb{1}_{\left(\cap_{M=N}^{\infty} \cap_{m=1}^{M^2} \Omega_M^m\right)}(\omega)\right)\right) \tag{50}$$

for every $\omega \in \Omega$. Finiteness in (50) follows from

$$\lim_{N\to\infty} \left(\mathbb{1}_{\left(\cap_{M=N}^{\infty} \cap_{m=1}^{M^2} \Omega_M^m\right)}(\omega)\right) = \mathbb{1}_{\left(\cup_{N\in\mathbb{N}} \cap_{M=N}^{\infty} \cap_{m=1}^{M^2} \Omega_M^m\right)}(\omega) \geq \mathbb{1}_{\tilde{\Omega}}(\omega) = 1 \tag{51}$$

for every $\omega \in \tilde{\Omega}$. Moreover, define $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $R_\varepsilon\colon \Omega \to [0,\infty)$, $\varepsilon \in (0,1)$, by

$$R_\varepsilon(\omega) := \sup_{N\in\mathbb{N}} \left(\mathbb{1}_{\tilde{\Omega}}(\omega) \frac{\left|\sum_{m=1}^{N^2} \left(\mathbb{1}_{\Omega_N^m}(\omega) \cdot f(Y_N^{N,m}(\omega)) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f(Y_N^N)\right]\right)\right|}{N^{(1+\varepsilon)}}\right) \tag{52}$$

for every $\omega \in \Omega$ and every $\varepsilon \in (0,1)$. By definition of $\tilde{\Omega}$, the mappings $R_\varepsilon\colon \Omega \to [0,\infty)$, $\varepsilon \in (0,1)$, are also finite. Additionally, let the real number $C \in [0,\infty)$ be given by

$$C := \sup_{N\in\mathbb{N}} \left(N \cdot \left|\mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(Y_N^N)\Big]\right|\right) + \sup_{N\in\mathbb{N}} \left(N^2 \cdot \mathbb{P}\Big[(\Omega_N)^c\Big]\right), \tag{53}$$

which is finite due to Lemma 4.5 and Lemma 4.6. Moreover, we have

$$\begin{aligned}
&\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \right) \right| \\
&\leq \left| \mathbb{E}\Big[f(X_T) \Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T) \Big] \right| + \left| \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T) \Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(Y_N^N) \Big] \right| \\
&\quad + \left| \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(Y_N^N) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \cdot \mathbb{1}_{\Omega_N^m}(\omega) \right) \right| \\
&\quad + \left| \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \cdot \mathbb{1}_{\Omega_N^m}(\omega) \right) - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \right) \right|
\end{aligned}$$

for every $\omega \in \tilde{\Omega}$ and every $N \in \mathbb{N}$. Using (53), we then obtain

$$\begin{aligned}
&\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \right) \right| \\
&\leq \left| \mathbb{E}\Big[\Big(1 - \mathbb{1}_{\Omega_N} \Big) \cdot f(X_T) \Big] \right| + C \cdot \frac{1}{N} \\
&\quad + \frac{1}{N^2} \left| \sum_{m=1}^{N^2} \left(f\left(Y_N^{N,m}(\omega) \right) \cdot \mathbb{1}_{\Omega_N^m}(\omega) - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f\left(Y_N^N \right) \Big] \right) \right| \\
&\quad + \frac{1}{N^2} \left(\sum_{m=1}^{N^2} \left| f\left(Y_N^{N,m}(\omega) \right) \right| \cdot \left(1 - \mathbb{1}_{\Omega_N^m}(\omega) \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \right) \right| \\
&\leq \left| \mathbb{E}\Big[\mathbb{1}_{(\Omega_N)^c} \cdot f(X_T) \Big] \right| + C \cdot \frac{1}{N} \\
&\quad + \frac{N^{(\varepsilon-1)}}{N^{(1+\varepsilon)}} \left| \sum_{m=1}^{N^2} \left(\mathbb{1}_{\Omega_N^m}(\omega) \cdot f\left(Y_N^{N,m}(\omega) \right) - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f\left(Y_N^N \right) \Big] \right) \right| \\
&\quad + \frac{1}{N^2} \left(\sum_{m=1}^{N^2} \left| f\left(Y_N^{N,m}(\omega) \right) \right| \right) \left(\sup_{m \in \{1,2,\ldots,N^2\}} \left(1 - \mathbb{1}_{\Omega_N^m}(\omega) \right) \right)
\end{aligned} \tag{54}$$

for every $\omega \in \tilde{\Omega}$, $N \in \mathbb{N}$ and every $\varepsilon \in (0, 1)$. Hence, we get

$$\begin{aligned}
&\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega) \right) \right) \right| \\
&\leq \mathbb{E}\Big[\mathbb{1}_{(\Omega_N)^c} \cdot |f(X_T)| \Big] + C \cdot \frac{1}{N} \\
&\quad + \left(\frac{\left| \sum_{m=1}^{N^2} \left(\mathbb{1}_{\Omega_N^m}(\omega) \cdot f\left(Y_N^{N,m}(\omega) \right) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(Y_N^N \right) \right] \right) \right|}{N^{(1+\varepsilon)}} \right) \cdot N^{(\varepsilon-1)} \\
&\quad + \frac{1}{N^2} \left(\sum_{m=1}^{N^2} \left| f\left(Y_N^{N,m}(\omega) \right) \right| \right) \left(\sup_{m \in \{1,2,\ldots,N^2\}} \left(1 - \mathbb{1}_{\Omega_N^m}(\omega) \right) \right)
\end{aligned}$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2}\left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega)\right)\right)\right| \\
&\leq \left(\mathbb{E}\Big[\mathbb{1}_{(\Omega_N)^c}\Big]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\Big[\big|f(X_T)\big|^2\Big]\right)^{\frac{1}{2}} + C \cdot \frac{1}{N} + R_\varepsilon(\omega) \cdot N^{(\varepsilon-1)} \\
&\quad + \frac{1}{N^2}\left(\sum_{m=1}^{N^2} \left|f\left(Y_N^{N,m}(\omega)\right)\right|\right)\left(1 - \inf_{m\in\{1,2,\ldots,N^2\}} \mathbb{1}_{\Omega_N^m}(\omega)\right)
\end{aligned} \tag{55}
$$

for every $\omega \in \tilde{\Omega}$, $N \in \mathbb{N}$ and every $\varepsilon \in (0,1)$ by using Hölder's inequality and the definition (52) of $R_\varepsilon \colon \Omega \to [0,\infty)$, $\varepsilon \in (0,1)$. Therefore, the polynomial growth assumption (17) implies

$$
\begin{aligned}
&\left|\mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2}\left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega)\right)\right)\right| \\
&\leq \left(\mathbb{P}\Big[(\Omega_N)^c\Big]\right)^{\frac{1}{2}} \cdot \left(\mathbb{E}\Big[L^2\left(1 + |X_T|^\delta\right)^2\Big]\right)^{\frac{1}{2}} + C \cdot \frac{1}{N} + R_\varepsilon(\omega) \cdot N^{(\varepsilon-1)} \\
&\quad + \frac{1}{N^2}\left(\sum_{m=1}^{N^2} \left|f\left(Y_N^{N,m}(\omega)\right)\right|\right)\left(1 - \mathbb{1}_{\left(\cap_{m=1}^{N^2}\Omega_N^m\right)}(\omega)\right) \\
&\leq \left(\mathbb{P}\Big[(\Omega_N)^c\Big]\right)^{\frac{1}{2}} \cdot L \cdot \left(\mathbb{E}\Big[2\left(1 + |X_T|^{2\delta}\right)\Big]\right)^{\frac{1}{2}} + C \cdot \frac{1}{N} + R_\varepsilon(\omega) \cdot N^{(\varepsilon-1)} \\
&\quad + \left(\frac{1}{N}\left(\sum_{m=1}^{N^2} \left|f\left(Y_N^{N,m}(\omega)\right)\right|\right)\left(1 - \mathbb{1}_{\left(\cap_{m=1}^{N^2}\Omega_N^m\right)}(\omega)\right)\right) \cdot N^{-1}
\end{aligned}
$$

for every $\omega \in \tilde{\Omega}$, $N \in \mathbb{N}$ and every $\varepsilon \in (0,1)$. Furthermore, we have

$$
\begin{aligned}
&\left|\mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2}\left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega)\right)\right)\right| \\
&\leq \left(\mathbb{P}\Big[(\Omega_N)^c\Big]\right)^{\frac{1}{2}} \cdot 2L \cdot \left(1 + \mathbb{E}\Big[|X_T|^{2\delta}\Big]\right)^{\frac{1}{2}} + C \cdot \frac{1}{N} + R_\varepsilon(\omega) \cdot N^{(\varepsilon-1)} \\
&\quad + \left(\frac{1}{N}\left(\sum_{m=1}^{N^2} \left|f\left(Y_N^{N,m}(\omega)\right)\right|\right)\left(1 - \mathbb{1}_{\left(\cap_{M=N}^{\infty}\cap_{m=1}^{M^2}\Omega_M^m\right)}(\omega)\right)\right) \cdot N^{-1}
\end{aligned}
$$

for every $\omega \in \tilde{\Omega}$, $N \in \mathbb{N}$ and every $\varepsilon \in (0,1)$. Using the definition (50) of $Z \colon \Omega \to [0,\infty)$ then yields

$$
\begin{aligned}
&\left|\mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2}\left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega)\right)\right)\right| \\
&\leq 2L \cdot \left(\mathbb{P}\Big[(\Omega_N)^c\Big]\right)^{\frac{1}{2}} \cdot \left(1 + \mathbb{E}\Big[|X_T|^{2\delta}\Big]\right)^{\frac{1}{2}} + C \cdot \frac{1}{N} + R_\varepsilon(\omega) \cdot N^{(\varepsilon-1)} + Z(\omega) \cdot N^{-1} \\
&\leq 2L \cdot \left(\mathbb{P}\Big[(\Omega_N)^c\Big]\right)^{\frac{1}{2}} \cdot \left(1 + \mathbb{E}\Big[|X_T|^{2\delta}\Big]\right)^{\frac{1}{2}} + C \cdot N^{(\varepsilon-1)} + R_\varepsilon(\omega) \cdot N^{(\varepsilon-1)} + Z(\omega) \cdot N^{(\varepsilon-1)}
\end{aligned}
$$

and finally

$$\begin{aligned}
&\left|\mathbb{E}\Big[f(X_T)\Big] - \frac{1}{N^2}\left(\sum_{m=1}^{N^2} f\left(Y_N^{N,m}(\omega)\right)\right)\right| \\
&\leq 2L\cdot\left(\mathbb{P}\Big[(\Omega_N)^c\Big]\right)^{\frac{1}{2}}\cdot\left(1+\mathbb{E}\Big[|X_T|^{2\delta}\Big]\right)+\Big(C+R_\varepsilon(\omega)+Z(\omega)\Big)\cdot N^{(\varepsilon-1)} \\
&\leq 2L\sqrt{C}N^{-1}\left(1+\mathbb{E}\left[|X_T|^{2\delta}\right]\right)+\left(C+R_\varepsilon(\omega)+Z(\omega)\right)\cdot N^{(\varepsilon-1)} \\
&\leq \left(2L\sqrt{C}\left(1+\mathbb{E}\left[|X_T|^{2\delta}\right]\right)+C+R_\varepsilon(\omega)+Z(\omega)\right)\cdot N^{(\varepsilon-1)}
\end{aligned}$$

for every $\omega\in\tilde{\Omega}$, $N\in\mathbb{N}$ and every $\varepsilon\in(0,1)$ due to the definition (53) of C. The right-hand side is finite according to Theorem 2.6.4 in [**3**]. This completes the proof of Theorem 2.1. □

4.4 Proof of Lemma 4.2

Proof of Lemma 4.2. Roughly speaking, r_N is chosen such that the drift term alone cannot change the sign of the N-th Euler approximation as long as the N-th Euler approximation is bounded by r_N where $N\in\mathbb{N}$. Now we formalize this and observe that

$$\begin{aligned}
&x\cdot\left(x+\frac{T}{N}\cdot\mu(x)-\frac{T\mu(0)}{N}\right)=x^2+\frac{T}{N}\cdot x\cdot\mu(x)-x\cdot\frac{T\mu(0)}{N} \\
&\geq x^2-\left|\frac{T}{N}\cdot x\cdot\mu(x)-x\cdot\frac{T\mu(0)}{N}\right|=x^2-\frac{T}{N}\cdot|x|\cdot|\mu(x)-\mu(0)| \\
&=x^2-\frac{T\,|x|}{N}\left(\mathbb{1}_{[-1,1]}(x)\cdot|\mu(x)-\mu(0)|+\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)\cdot|\mu(x)-\mu(0)|\right) \\
&\geq x^2-\frac{T\,|x|}{N}\left(\mathbb{1}_{[-1,1]}(x)\cdot|x|\cdot\left(\sup_{s\in[-1,1]}|\mu'(s)|\right)+\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)\cdot|\mu(x)-\mu(0)|\right)
\end{aligned}$$

holds for every $x\in\mathbb{R}$. Moreover, using that $\mu\colon\mathbb{R}\to\mathbb{R}$ has at most polynomial growth (see (17)) shows

$$\begin{aligned}
&x\cdot\left(x+\frac{T}{N}\cdot\mu(x)-\frac{T\mu(0)}{N}\right) \\
&\geq x^2-\frac{T\,|x|}{N}\left(\mathbb{1}_{[-1,1]}(x)\cdot|x|\cdot\left(\sup_{s\in[-1,1]}|\mu'(s)|\right)+\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)\cdot\left(L\left(1+|x|^\delta\right)+L\right)\right) \\
&\geq x^2-\frac{T\,|x|}{N}\left(\mathbb{1}_{[-1,1]}(x)\cdot|x|\cdot\left(\sup_{s\in[-1,1]}|\mu'(s)|\right)+\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)\cdot L\cdot\left(2+|x|^\delta\right)\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&x\cdot\left(x+\frac{T}{N}\cdot\mu(x)-\frac{T\mu(0)}{N}\right) \\
&\geq x^2-\frac{T\,|x|}{N}\left(\mathbb{1}_{[-1,1]}(x)\cdot|x|\cdot\left(\sup_{s\in[-1,1]}|\mu'(s)|\right)+\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)\cdot 3L\cdot|x|^\delta\right) \\
&\geq x^2-\frac{T\,|x|}{N}\left(\sup_{s\in[-1,1]}|\mu'(s)|+3L\right)\left(\mathbb{1}_{[-1,1]}(x)\cdot|x|+\mathbb{1}_{\mathbb{R}\setminus[-1,1]}(x)\cdot|x|^\delta\right) \\
&\geq x^2-\frac{T\,|x|}{N}\left(\sup_{s\in[-1,1]}|\mu'(s)|+3L\right)\left(|x|+|x|^\delta\right)
\end{aligned}$$

for every $x \in \mathbb{R}$. This implies that

$$
\begin{aligned}
x \cdot \left(x + \frac{T}{N} \cdot \mu(x) - \frac{T\mu(0)}{N} \right) &\geq x^2 - \frac{T}{N} \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right) \left(x^2 + |x|^{(1+\delta)} \right) \\
&= x^2 \left(1 - \frac{T}{N} \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right) \left(1 + |x|^{(\delta-1)} \right) \right) \\
&\geq x^2 \left(1 - \frac{T}{N} \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right) \left\{ 1 + (r_N)^{(\delta-1)} \right\} \right)
\end{aligned}
\tag{56}
$$

holds for every $x \in [-r_N, r_N]$. Therefore, we finally obtain

$$
\begin{aligned}
& x \cdot \left(x + \frac{T}{N} \cdot \mu(x) - \frac{T\mu(0)}{N} \right) \\
& \geq x^2 \left(1 - \frac{T}{N} \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right) \left\{ 1 + \max \left(0, \frac{N}{T \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right)} - 1 \right) \right\} \right) \\
& = x^2 \left(1 - \frac{T}{N} \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right) \frac{N}{T \left(\sup_{s \in [-1,1]} |\mu'(s)| + 3L \right)} \right) \\
& = x^2 \left(1 - 1 \right) = 0.
\end{aligned}
\tag{57}
$$

for every $x \in [-r_N, r_N]$. Hence, we have shown that

$$
x + \frac{T}{N} \cdot \mu(x) - \frac{T\mu(0)}{N} \geq 0 \tag{58}
$$

holds for all $x \in [0, r_N]$ and that

$$
x + \frac{T}{N} \cdot \mu(x) - \frac{T\mu(0)}{N} \leq 0 \tag{59}
$$

holds for all $x \in [-r_N, 0)$. The estimates (58) and (59) give us control on the effect of the drift function in one direction. In addition, we need to bound the growth in the other direction. The one-sided Lipschitz continuity of μ (see (18)) implies

$$
x \cdot (\mu(x) - \mu(0)) \leq L \cdot x^2
$$

and hence

$$
x \cdot (\mu(x) - L \cdot x - \mu(0)) \leq 0
$$

for every $x \in \mathbb{R}$. Therefore, we obtain

$$
\mu(x) - L \cdot x - \mu(0) \leq 0 \tag{60}
$$

for every $x \in [0, \infty)$ and

$$
\mu(x) - L \cdot x - \mu(0) \geq 0 \tag{61}
$$

for every $x \in (-\infty, 0)$. With these inequalities at hand, we now establish (45) by induction on $n \in \{0, 1, \ldots, N\}$ where $N \in \mathbb{N}$ is fixed. First of all, we have $\tau_0^N(\omega) = 0$ and therefore

$$
\begin{aligned}
\left| Y_0^N(\omega) \right| &= |\xi(\omega)| \leq T\, |\mu(0)| + |\sigma(0)| + |\xi(\omega)| + 1 \\
&= e^{\left(\sum_{l=0}^{-1} \alpha_l^N(\omega) \right)} \left(T\, |\mu(0)| + |\sigma(0)| + |\xi(\omega)| + 1 \right) \\
&= e^{\left(\sum_{l=\tau_0^N(\omega)}^{-1} \alpha_l^N(\omega) \right)} \left(T\, |\mu(0)| + |\sigma(0)| + |\xi(\omega)| + 1 \right) \\
&\quad + \sum_{k=\tau_0^N(\omega)}^{-1} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{-1} \alpha_l^N(\omega) \right)} \beta_k^N(\omega) = D_{\tau_0^N(\omega),0}^{N,1}(\omega)
\end{aligned}
$$

for every $\omega \in \Omega_{N,0}$, which shows (45) in the base case $n = 0$. Suppose now that (45) holds for one fixed $n \in \{0, 1, 2, \ldots, N-1\}$. Moreover, we fix an arbitrary $\omega \in \Omega_{N,n+1} \subset \Omega_{N,n}$ and we now show (45) for ω and $n+1$. For this we distinguish between the following four different cases.
1.) First of all suppose that $Y_n^N(\omega) \geq 0$ and that $Y_{n+1}^N(\omega) \geq 0$ holds. Then $\tau_n^N(\omega) = \tau_{n+1}^N(\omega)$ and

$$\begin{aligned}
0 &\leq Y_{n+1}^N(\omega) \\
&= Y_n^N(\omega) + \frac{T}{N} \cdot \mu\left(Y_n^N(\omega)\right) + \sigma\left(Y_n^N(\omega)\right) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right) \\
&= Y_n^N(\omega) + \frac{T}{N} \cdot \left(\mu\left(Y_n^N(\omega)\right) - L \cdot Y_n^N(\omega) - \mu(0)\right) + \frac{TL}{N} \cdot Y_n^N(\omega) \\
&\quad + \frac{T\mu(0)}{N} + \left(\sigma\left(Y_n^N(\omega)\right) - \sigma(0)\right) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right) + \sigma(0) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)
\end{aligned}$$

and hence

$$\begin{aligned}
0 \leq Y_{n+1}^N(\omega) \leq Y_n^N(\omega) &+ \frac{TL}{N} \cdot Y_n^N(\omega) + \frac{T\mu(0)}{N} \\
&+ \left(\sigma\big(Y_n^N(\omega)\big) - \sigma(0)\right) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right) + \sigma(0) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)
\end{aligned}$$

due to (60). Therefore, using $1 + x \leq e^x$ for all $x \in \mathbb{R}$ yields

$$\begin{aligned}
0 \leq Y_{n+1}^N(\omega) &\leq Y_n^N(\omega) \cdot \left(1 + \frac{TL}{N} + \tilde{\sigma}\left(Y_n^N(\omega)\right) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\right) \\
&\quad + \left(\frac{T\mu(0)}{N} + \sigma(0) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\right) \\
&\leq Y_n^N(\omega) \cdot e^{\left(\frac{TL}{N} + \tilde{\sigma}\left(Y_n^N(\omega)\right) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\right)} \\
&\quad + \left(\frac{T\mu(0)}{N} + \sigma(0) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\right) \\
&= Y_n^N(\omega) \cdot e^{\alpha_n^N(\omega)} + \beta_n^N(\omega) \\
&= \left|Y_n^N(\omega)\right| \cdot e^{\alpha_n^N(\omega)} + \operatorname{sgn}(Y_n^N(\omega)) \cdot \beta_n^N(\omega).
\end{aligned}$$

The induction hypothesis then implies

$$\begin{aligned}
0 \leq Y_{n+1}^N(\omega) &\leq D_{\tau_n^N(\omega),n}^{N,1}(\omega) \cdot e^{\alpha_n^N(\omega)} + \operatorname{sgn}(Y_n^N(\omega)) \cdot \beta_n^N(\omega) \\
&= e^{\left(\sum_{l=\tau_n^N(\omega)}^{n} \alpha_l^N(\omega)\right)} \left(T\left|\mu(0)\right| + \left|\sigma(0)\right| + \left|\xi(\omega)\right| + 1\right) \\
&\quad + \sum_{k=\tau_n^N(\omega)}^{n-1} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n} \alpha_l^N(\omega)\right)} \beta_k^N(\omega) + \operatorname{sgn}(Y_n^N(\omega)) \cdot \beta_n^N(\omega)
\end{aligned}$$

and hence

$$\begin{aligned}
\left|Y_{n+1}^N(\omega)\right| &\leq e^{\left(\sum_{l=\tau_n^N(\omega)}^{n} \alpha_l^N(\omega)\right)} \left(T\left|\mu(0)\right| + \left|\sigma(0)\right| + \left|\xi(\omega)\right| + 1\right) \\
&\quad + \sum_{k=\tau_n^N(\omega)}^{n} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n} \alpha_l^N(\omega)\right)} \beta_k^N(\omega) \\
&= D_{\tau_n^N(\omega),n+1}^{N,1}(\omega) = D_{\tau_{n+1}^N(\omega),n+1}^{N,1}(\omega),
\end{aligned} \tag{62}$$

which shows that (45) holds in this case for ω and $n+1$.
2.) Suppose now that $Y_n^N(\omega) < 0$ and that $Y_{n+1}^N(\omega) < 0$ holds. Then we also have $\tau_n^N(\omega) = \tau_{n+1}^N(\omega)$ and

$$\begin{aligned}
0 &> Y_{n+1}^N(\omega) \\
&= Y_n^N(\omega) + \frac{T}{N} \cdot \mu\left(Y_n^N(\omega)\right) + \sigma\left(Y_n^N(\omega)\right) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big) \\
&= Y_n^N(\omega) + \frac{T}{N} \cdot \left(\mu\left(Y_n^N(\omega)\right) - L \cdot Y_n^N(\omega) - \mu(0)\right) + \frac{TL}{N} \cdot Y_n^N(\omega) \\
&\quad + \frac{T\mu(0)}{N} + \left(\sigma\left(Y_n^N(\omega)\right) - \sigma(0)\right) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big) + \sigma(0) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big)
\end{aligned}$$

and hence

$$\begin{aligned}
0 > Y_{n+1}^N(\omega) &\geq Y_n^N(\omega) + \frac{TL}{N} \cdot Y_n^N(\omega) + \frac{T\mu(0)}{N} \\
&\quad + \left(\sigma(Y_n^N(\omega)) - \sigma(0)\right) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big) + \sigma(0) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big)
\end{aligned}$$

due to (61). Therefore, we obtain

$$\begin{aligned}
0 > Y_{n+1}^N(\omega) &\geq Y_n^N(\omega) \cdot \left(1 + \frac{TL}{N} + \tilde{\sigma}\left(Y_n^N(\omega)\right) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big)\right) \\
&\quad + \left(\frac{T\mu(0)}{N} + \sigma(0) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big)\right) \\
&\geq Y_n^N(\omega) \cdot e^{\left(\frac{TL}{N} + \tilde{\sigma}\left(Y_n^N(\omega)\right) \cdot \left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\right)} \\
&\quad + \left(\frac{T\mu(0)}{N} + \sigma(0) \cdot \Big(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\Big)\right) \\
&= Y_n^N(\omega) \cdot e^{\alpha_n^N(\omega)} + \beta_n^N(\omega) \\
&= -\left(\left|Y_n^N(\omega)\right| \cdot e^{\alpha_n^N(\omega)} + \operatorname{sgn}(Y_n^N(\omega)) \cdot \beta_n^N(\omega)\right).
\end{aligned}$$

Hence, the induction hypothesis yields

$$\begin{aligned}
\left|Y_{n+1}^N(\omega)\right| &\leq D_{\tau_n^N(\omega),n}^{N,1}(\omega) \cdot e^{\alpha_n^N(\omega)} + \operatorname{sgn}(Y_n^N(\omega)) \cdot \beta_n^N(\omega) \\
&= e^{\left(\sum_{l=\tau_n^N(\omega)}^{n} \alpha_l^N(\omega)\right)} \left(T\left|\mu(0)\right| + \left|\sigma(0)\right| + \left|\xi(\omega)\right| + 1\right) \\
&\quad + \sum_{k=\tau_n^N(\omega)}^{n-1} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n} \alpha_l^N(\omega)\right)} \beta_k^N(\omega) + \operatorname{sgn}(Y_n^N(\omega)) \cdot \beta_n^N(\omega) \\
&= e^{\left(\sum_{l=\tau_n^N(\omega)}^{n} \alpha_l^N(\omega)\right)} \left(T\left|\mu(0)\right| + \left|\sigma(0)\right| + \left|\xi(\omega)\right| + 1\right) \\
&\quad + \sum_{k=\tau_n^N(\omega)}^{n} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n} \alpha_l^N(\omega)\right)} \beta_k^N(\omega) \\
&= D_{\tau_n^N(\omega),n+1}^{N,1}(\omega) = D_{\tau_{n+1}^N(\omega),n+1}^{N,1}(\omega),
\end{aligned}$$

which shows that (45) also holds in this case for ω and $n+1$.
3.) In the third case assume that $Y_n^N(\omega) \geq 0$ and that $Y_{n+1}^N(\omega) < 0$ holds. Then we obtain $\tau_{n+1}^N(\omega) = n+1$. Additionally, note that

$$\left|Y_n^N(\omega)\right| \leq \sup_{k,l \in \{0,1,\ldots,n\}} \left|D_{k,l}^{N,1}(\omega)\right| \leq r_N \tag{63}$$

holds due to the induction hypothesis and since $\omega \in \Omega_{N,n+1} \subset \Omega_{N,n}$. Hence, (58) yields

$$Y_n^N(\omega) + \frac{T}{N}\mu\left(Y_n^N(\omega)\right) - \frac{T\mu(0)}{N} \geq 0 \tag{64}$$

and therefore

$$\begin{aligned}
0 &\geq Y_{n+1}^N(\omega)\\
&= Y_n^N(\omega) + \frac{T}{N}\mu\left(Y_n^N(\omega)\right) + \sigma\left(Y_n^N(\omega)\right)\cdot\left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\\
&= \left(Y_n^N(\omega) + \frac{T}{N}\mu\left(Y_n^N(\omega)\right) - \frac{T\mu(0)}{N}\right) + \frac{T\mu(0)}{N} + \sigma\left(Y_n^N(\omega)\right)\cdot\left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\\
&\geq \frac{T\mu(0)}{N} + \sigma\left(Y_n^N(\omega)\right)\cdot\left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
0 \geq Y_{n+1}^N(\omega) &\geq -\left|\frac{T\mu(0)}{N}\right| - \left|\sigma\left(Y_n^N(\omega)\right)\right|\cdot\left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right|\\
&\geq -T\left|\mu(0)\right| - \left(L\cdot\left|Y_n^N(\omega)\right| + |\sigma(0)|\right)\cdot\left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right|.
\end{aligned}$$

This implies

$$\begin{aligned}
0 \geq Y_{n+1}^N(\omega) &\geq -T\left|\mu(0)\right| - \left(L\cdot D^{N,1}_{\tau_n^N(\omega),n}(\omega) + |\sigma(0)|\right)\cdot\left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right|\\
&\geq -T\left|\mu(0)\right| - \left(L\cdot r_N + |\sigma(0)|\right)\cdot\left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right|\\
&\geq -T\left|\mu(0)\right| - \left(L\cdot\frac{N^{\frac{1}{4}}}{L} + |\sigma(0)|\right)\cdot\left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right|
\end{aligned}$$

and

$$0 \geq Y_{n+1}^N(\omega) \geq -T\left|\mu(0)\right| - \left(|\sigma(0)| + N^{\frac{1}{4}}\right)\cdot\frac{1}{N^{\frac{1}{4}}} = -T\left|\mu(0)\right| - \frac{|\sigma(0)|}{N^{\frac{1}{4}}} - 1 \geq -T\left|\mu(0)\right| - |\sigma(0)| - 1 \tag{66}$$

and finally

$$\begin{aligned}
&\left|Y_{n+1}^N(\omega)\right| \leq T\left|\mu(0)\right| + |\sigma(0)| + 1 + |\xi(\omega)|\\
&= e^{\left(\sum_{l=n+1}^{n}\alpha_l^N(\omega)\right)}\left(T\left|\mu(0)\right| + |\sigma(0)| + 1 + |\xi(\omega)|\right) + \sum_{k=n+1}^{n}\operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n}\alpha_l^N(\omega)\right)}\beta_k^N(\omega)\\
&= e^{\left(\sum_{l=\tau_{n+1}^N(\omega)}^{n}\alpha_l^N(\omega)\right)}\left(T\left|\mu(0)\right| + |\sigma(0)| + 1 + |\xi(\omega)|\right) + \sum_{k=\tau_{n+1}^N(\omega)}^{n}\operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n}\alpha_l^N(\omega)\right)}\beta_k^N(\omega)\\
&= D^{N,1}_{\tau_{n+1}^N(\omega),n+1}(\omega),
\end{aligned}$$

which shows that (45) also holds in this case for ω and $n+1$.
4.) In the last case assume that $Y_n^N(\omega) < 0$ and that $Y_{n+1}^N(\omega) \geq 0$ holds. Then we also obtain $\tau_{n+1}^N(\omega) = n+1$. Note that

$$\left|Y_n^N(\omega)\right| \leq \sup_{k,l\in\{0,1,\ldots,n\}}\left|D^{N,1}_{k,l}(\omega)\right| \leq r_N \tag{67}$$

holds due to the induction hypothesis and since $\omega \in \Omega_{N,n+1} \subset \Omega_{N,n}$. Therefore, (59) implies

$$Y_n^N(\omega) + \frac{T}{N}\mu\big(Y_n^N(\omega)\big) - \frac{T\mu(0)}{N} \leq 0 \tag{68}$$

and hence

$$\begin{aligned}
0 &\leq Y_{n+1}^N(\omega)\\
&= Y_n^N(\omega) + \frac{T}{N}\mu\left(Y_n^N(\omega)\right) + \sigma\left(Y_n^N(\omega)\right)\cdot\left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\\
&= \left(Y_n^N(\omega) + \frac{T}{N}\mu\left(Y_n^N(\omega)\right) - \frac{T\mu(0)}{N}\right) + \frac{T\mu(0)}{N} + \sigma\left(Y_n^N(\omega)\right)\cdot\left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)\\
&\leq \frac{T\mu(0)}{N} + \sigma\left(Y_n^N(\omega)\right)\cdot\left(W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right)
\end{aligned} \tag{69}$$

and

$$
\begin{aligned}
0 \le Y_{n+1}^N(\omega) &\le \left|\frac{T\mu(0)}{N}\right| + \left|\sigma\left(Y_n^N(\omega)\right)\right| \cdot \left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right| \\
&\le T\,|\mu(0)| + \left(L \cdot \left|Y_n^N(\omega)\right| + |\sigma(0)|\right) \cdot \left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right| \\
&\le T\,|\mu(0)| + \left(L \cdot D_{\tau_n^N(\omega),n}^{N,1}(\omega) + |\sigma(0)|\right) \cdot \left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right|.
\end{aligned}
\tag{70}
$$

This shows

$$
\begin{aligned}
\left|Y_{n+1}^N(\omega)\right| &\le T\,|\mu(0)| + \left(L \cdot r_N + |\sigma(0)|\right) \cdot \left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right| \\
&\le T\,|\mu(0)| + \left(L \cdot \frac{N^{\frac{1}{4}}}{L} + |\sigma(0)|\right) \cdot \left|W_{t_{n+1}^N}(\omega) - W_{t_n^N}(\omega)\right| \\
&\le T\,|\mu(0)| + \left(|\sigma(0)| + N^{\frac{1}{4}}\right) \cdot \frac{1}{N^{\frac{1}{4}}} = T\,|\mu(0)| + \frac{|\sigma(0)|}{N^{\frac{1}{4}}} + 1
\end{aligned}
\tag{71}
$$

and finally

$$
\begin{aligned}
&\left|Y_{n+1}^N(\omega)\right| \le T\,|\mu(0)| + |\sigma(0)| + 1 + |\xi(\omega)| \\
&= e^{\left(\sum_{l=n+1}^{n} \alpha_l^N(\omega)\right)} \left(T\,|\mu(0)| + |\sigma(0)| + 1 + |\xi(\omega)|\right) + \sum_{k=n+1}^{n} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n} \alpha_l^N(\omega)\right)} \beta_k^N(\omega) \\
&= e^{\left(\sum_{l=\tau_{n+1}^N(\omega)}^{n} \alpha_l^N(\omega)\right)} \left(T\,|\mu(0)| + |\sigma(0)| + 1 + |\xi(\omega)|\right) \\
&\quad + \sum_{k=\tau_{n+1}^N(\omega)}^{n} \operatorname{sgn}(Y_k^N(\omega))\, e^{\left(\sum_{l=k+1}^{n} \alpha_l^N(\omega)\right)} \beta_k^N(\omega) = D_{\tau_{n+1}^N(\omega),n+1}^{N,1}(\omega),
\end{aligned}
$$

which shows that (45) holds in this case for ω and $n+1$ and which finally yields (45) for every $\omega \in \Omega_{N,n}$, $n \in \{0, 1, 2, \ldots, N\}$ and every $N \in \mathbb{N}$ by induction. $\square$

4.5 Proof of Lemma 4.3

In order to bound the moments of the dominating process, we need to estimate the absolute moments of a normally distributed random variable.

Lemma 4.7. *Let $Y\colon \Omega \to \mathbb{R}$ be a normally distributed $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping. Then we obtain that*

$$
\|Y\|_{L^p} \le p\,\|Y\|_{L^2} \tag{72}
$$

holds for every $p \in [1, \infty)$.

Proof of Lemma 4.7. First of all, Hölder's inequality implies

$$
\|Y\|_{L^p} \le \|Y\|_{L^2} \le p\,\|Y\|_{L^2} \tag{73}
$$

for every $p \in [1, 2)$, which shows (72) in the case $p \in [1, 2)$. Denote now the mean of $Y\colon \Omega \to \mathbb{R}$ by $c := \mathbb{E}[Y] \in \mathbb{R}$ and the standard deviation by $\sigma := \sqrt{\mathbb{E}\left[(Y-c)^2\right]} \in [0, \infty)$. If $\hat{Y}\colon \Omega \to \mathbb{R}$ is a standard normally distributed $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping, then

$$
\begin{aligned}
\left\|Y\right\|_{L^p} = \left\|\sigma\hat{Y} + c\right\|_{L^p} \le \left\|\sigma\hat{Y} + c\right\|_{L^{\lceil p \rceil}} &= \left(\mathbb{E}\left[\left|\sigma\hat{Y} + c\right|^{\lceil p \rceil}\right]\right)^{\frac{1}{\lceil p \rceil}} \\
&\le \left(\mathbb{E}\left[\sum_{k=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{k} |\sigma|^k \left|\hat{Y}\right|^k |c|^{(\lceil p \rceil - k)}\right]\right)^{\frac{1}{\lceil p \rceil}} \\
&= \left(\sum_{k=0}^{\lceil p \rceil} \binom{\lceil p \rceil}{k} |\sigma|^k\, \mathbb{E}\left|\hat{Y}\right|^k |c|^{(\lceil p \rceil - k)}\right)^{\frac{1}{\lceil p \rceil}}
\end{aligned}
\tag{74}
$$

holds for every $p \in [2,\infty)$. Using $\mathbb{E}|\hat{Y}|^k \leq (k-1)^{k/2}$ for all $k \in \{2,3,\dots\}$, $\mathbb{E}|\hat{Y}| \leq \sqrt{\mathbb{E}(\hat{Y})^2} = 1$ and $(|\sigma|+|c|)^2 \leq 2(\sigma^2+c^2)$ yields

$$\begin{aligned}\|Y\|_{L^p} &\leq \left(\sum_{k=0}^{\lceil p\rceil} \binom{\lceil p\rceil}{k} |\sigma|^k \left(\lceil p\rceil - 1\right)^{\frac{\lceil p\rceil}{2}} |c|^{(\lceil p\rceil - k)}\right)^{\frac{1}{\lceil p\rceil}} \\ &= \sqrt{(\lceil p\rceil - 1)}\,(|\sigma|+|c|) \leq \sqrt{p}\,(|\sigma|+|c|) \leq p\sqrt{(\sigma^2+c^2)} = p\,\|Y\|_{L^2}\end{aligned} \tag{75}$$

for every $p \in [2,\infty)$, which finally shows (72). $\square$

Lemma 4.8. *Let $Y\colon \Omega \to \mathbb{R}$ be a standard normally distributed $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping. Then we obtain that*

$$\left\|e^{cY}-1\right\|_{L^p} \leq |c|\, e^{(c^2+1)p} \tag{76}$$

holds for every $c \in \mathbb{R}$ and every $p \in [1,\infty)$.

Proof of Lemma 4.8. We establish (76) in the case $c \in (0,\infty)$ since the case $c=0$ is trivial and since the case $c \in (-\infty,0)$ immediately follows from the case $c \in (0,\infty)$. In order to show (76) in the case $c \in (0,\infty)$, note that

$$\begin{aligned}\mathbb{E}\left[\left|e^{cY}-1\right|^p\right] &= \mathbb{E}\left[\mathbb{1}_{\{Y\geq 0\}}\, e^{cpY}\left(1-e^{-cY}\right)^p\right] + \mathbb{E}\left[\mathbb{1}_{\{Y<0\}}\left(1-e^{cY}\right)^p\right] \\ &\leq c^p \cdot \mathbb{E}\left[\mathbb{1}_{\{Y\geq 0\}}\, e^{cpY}\, |Y|^p\right] + c^p \cdot \mathbb{E}\left[\mathbb{1}_{\{Y<0\}}\, |Y|^p\right] \\ &= c^p \cdot \mathbb{E}\left[\left(\mathbb{1}_{\{Y\geq 0\}}\, e^{cpY} + \mathbb{1}_{\{Y<0\}}\right)|Y|^p\right] \\ &\leq c^p\left(\mathbb{E}\big[e^{2cpY}\big] + \mathbb{P}[Y<0]\right)^{\frac{1}{2}} \sqrt{\mathbb{E}\,|Y|^{2p}} = c^p\left(e^{2c^2p^2}+\frac{1}{2}\right)^{\frac{1}{2}} \sqrt{\mathbb{E}\,|Y|^{2p}}\end{aligned}$$

and the estimate $\|Y\|_{L^{2p}} \leq \sqrt{2p-1}$ therefore shows that

$$\left\|e^{cY}-1\right\|_{L^p} \leq c\left(e^{2c^2p^2}+\frac{1}{2}\right)^{\frac{1}{2p}} \|Y\|_{L^{2p}} \leq c\, 2^{\frac{1}{2p}} e^{c^2 p}\, \|Y\|_{L^{2p}} \leq c\cdot e^{(c^2+1)p}$$

for all $c \in (0,\infty)$ and all $p \in [1,\infty)$. This completes the proof of Lemma 4.8. $\square$

Lemma 4.9. *Let $\alpha_n^N : \Omega \to \mathbb{R}$ for $n \in \{0,1,\dots,N-1\}$ and $N \in \mathbb{N}$ be given by (35). Then we obtain*

$$\left\|e^{-\alpha_n^N}-1\right\|_{L^p} \leq L N^{-\frac{1}{2}}\left(\sqrt{T} e^{\left(L^2 T+1\right)p} + T\right) \tag{77}$$

for all $n \in \{0,1,\dots,N\}$, $N \in \mathbb{N}$ and all $p \in [1,\infty)$.

Proof of Lemma 4.9. The triangle inequality and Lemma 4.8 impliy

$$\begin{aligned}\left\|e^{-\alpha_n^N}-1\right\|_{L^p} &\leq \left\|e^{-\alpha_n^N}-e^{-\frac{TL}{N}}\right\|_{L^p} + \left\|e^{-\frac{TL}{N}}-1\right\|_{L^p} \\ &= e^{-\frac{TL}{N}}\left\|e^{-\tilde{\sigma}(Y_n^N)\left(W_{t_{n+1}^N}-W_{t_n^N}\right)}-1\right\|_{L^p} + \left(1-e^{-\frac{TL}{N}}\right) \\ &\leq L\sqrt{\frac{T}{N}}\, e^{\left(L^2\frac{T}{N}+1\right)p} + \frac{TL}{N} \\ &\leq L N^{-\frac{1}{2}}\left(\sqrt{T} e^{\left(L^2 T+1\right)p} + T\right)\end{aligned}$$

for all $n \in \{0,1,\dots,N-1\}$, $N \in \mathbb{N}$ and all $p \in [1,\infty)$. This completes the proof of Lemma 4.9. $\square$

Proof of Lemma 4.3. Let $C \in (0,\infty)$ be a real number satisfying

$$5+L+\delta+T+|\mu(0)|+|\sigma(0)| \leq C, \qquad |\mu(x)|+|\mu'(x)| \leq C\left(1+|x|^C\right) \tag{78}$$

for all $x \in \mathbb{R}$. Such a real number $C < \infty$ indeed exists since the derivative of μ is assumed to grow at most polynomially according to (17). Since the exponential function is convex, we obtain that

$$\exp\left(\sum_{l=0}^{n-1} z \cdot \tilde{\sigma}(Y_l^N) \cdot \left(W_{t_{l+1}^N} - W_{t_l^N}\right)\right) \tag{79}$$

is a positive submartingale in $n \in \{0, 1, \ldots, N\}$ for every $z \in \{-1, 1\}$ and every $N \in \mathbb{N}$. Therefore, Doob's inequality (see e.g. Theorem 11.2 (ii) in [**8**]) shows

$$\begin{aligned}
&\mathbb{E}\left[\left|\sup_{n\in\{0,1,\ldots,N\}} e^{\left(\sum_{l=0}^{n-1} z\cdot\tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)}\right|^p\right] \\
&\leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[\left|e^{\left(\sum_{l=0}^{N-1} z\cdot\tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)}\right|^p\right] \\
&= \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[e^{pz\left(\sum_{l=0}^{N-2} \tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)} \mathbb{E}\left[e^{pz\left(\tilde{\sigma}(Y_{N-1}^N)\cdot\left(W_{t_N^N}-W_{t_{N-1}^N}\right)\right)}\middle|\mathcal{F}_{t_{N-1}^N}\right]\right] \\
&= \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[e^{pz\left(\sum_{l=0}^{N-2} \tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)} e^{\frac{1}{2}\left(pz\cdot\tilde{\sigma}(Y_{N-1}^N)\right)^2 \frac{T}{N}}\right]
\end{aligned} \tag{80}$$

due to the moment generating function $\mathbb{E}[\exp(cY)] = \exp(c^2/2)$, $c \in \mathbb{R}$, of the standard normally distributed random variable $Y := \sqrt{\frac{N}{T}}\left(W_{t_N^N} - W^N_{t_{N-1}}\right)$ and hence, using $|\tilde{\sigma}(x)| \leq L$ for every $x \in \mathbb{R}$,

$$\begin{aligned}
&\left\|\sup_{n\in\{0,1,\ldots,N\}} e^{\left(\sum_{l=0}^{n-1} z\cdot\tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)}\right\|_{L^p} \\
&\leq \left\{\left(\frac{p}{p-1}\right)^p \mathbb{E}\left[e^{pz\left(\sum_{l=0}^{N-2} \tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)}\right] e^{\frac{1}{2}p^2L^2\frac{T}{N}}\right\}^{\frac{1}{p}} \\
&\leq \ldots \leq \left\{\left(\frac{p}{p-1}\right)^p \prod_{l=0}^{N-1} e^{\frac{1}{2}p^2L^2\frac{T}{N}}\right\}^{\frac{1}{p}} = \left(\frac{p}{p-1}\right) e^{\frac{1}{2}pL^2T}
\end{aligned} \tag{81}$$

for every $p \in (1, \infty)$, $z \in \{-1, 1\}$ and every $N \in \mathbb{N}$. This implies

$$\begin{aligned}
&\left\|\sup_{n\in\{0,1,\ldots,N\}} e^{z\left(\sum_{l=0}^{n-1} \alpha_l^N\right)}\right\|_{L^p} \\
&\leq e^{TL}\left\|\sup_{n\in\{0,1,\ldots,N\}} e^{\left(\sum_{l=0}^{n-1} z\cdot\tilde{\sigma}(Y_l^N)\cdot\left(W_{t_{l+1}^N}-W_{t_l^N}\right)\right)}\right\|_{L^p} \\
&\leq e^{TL}\left(\frac{p}{p-1}\right) e^{\frac{1}{2}pL^2T} \leq 2e^{\frac{p}{2}(L^2+L)T} \leq e^{\frac{p}{2}(1+C^2T+C^2)} \leq e^{\frac{p}{2}C^3} \leq e^{\frac{p}{10}C^4}
\end{aligned} \tag{82}$$

for every $p \in [2, \infty)$, $z \in \{-1, 1\}$ and every $N \in \mathbb{N}$. Moreover, Lemma 4.9 shows

$$\begin{aligned}
\left\|e^{-\alpha_n^N} - 1\right\|_{L^p} &\leq LN^{-\frac{1}{2}}\left(\sqrt{T}e^{\left(L^2T+1\right)p} + T\right) \\
&\leq C^2N^{-\frac{1}{2}}\left(\frac{1}{2}e^{\left(C^3+1\right)p} + 1\right) \leq e^{\left(C+C^3+1\right)p}N^{-\frac{1}{2}} \leq \frac{e^{\frac{p}{4}C^4}}{\sqrt{N}}
\end{aligned} \tag{83}$$

for all $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ and all $p \in [2, \infty)$. With these estimates at hand, we now bound the p-th absolute moment

$$\mathbb{E}\left[\sup_{v,w\in\{0,1,\ldots,N\}} \left|D_{v,w}^{N,1}\right|^p\right] \tag{84}$$

of the dominating process for every $N \in \mathbb{N}$ and every $p \in [2, \infty)$. By definition (37) and by the triangle inequality we have

$$\begin{aligned}\left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq & \left\|\left(\sup_{v,w\in\{0,1,\ldots,N\}} e^{\left(\sum_{l=v}^{w-1}\alpha_l^N\right)}\right)\left(C^2+|\xi|\right)\right\|_{L^p} \\ & + \left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|\sum_{k=v}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{\left(\sum_{l=k+1}^{w-1}\alpha_l^N\right)} \beta_k^N\right|\right\|_{L^p}\end{aligned}$$

and, using Hölder's inequality,

$$\begin{aligned}\left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq & \left\|\sup_{0\leq v\leq w\leq N} e^{\left(\sum_{l=v}^{w-1}\alpha_l^N\right)}\right\|_{L^{2p}} \left(C^2+\|\xi\|_{L^{2p}}\right) \\ & + \left\|\sup_{0\leq v\leq w\leq N} \left|\sum_{k=v}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{\left(\sum_{l=k+1}^{w-1}\alpha_l^N\right)} \beta_k^N\right|\right\|_{L^p}\end{aligned}$$

for all $N \in \mathbb{N}$ and all $p \in [2, \infty)$. Therefore, we obtain

$$\begin{aligned}\left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq & \left\|\sup_{0\leq v\leq w\leq N} e^{\left(\sum_{l=v}^{w-1}\alpha_l^N\right)}\right\|_{L^{2p}} \left(C^2+\|\xi\|_{L^{2p}}\right) \\ & + \left\|\sup_{0\leq v\leq w\leq N} \left(e^{\left(\sum_{l=0}^{w-1}\alpha_l^N\right)} \left|\sum_{k=v}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)} \beta_k^N\right|\right)\right\|_{L^p}\end{aligned} \tag{85}$$

and

$$\begin{aligned}&\left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|D^{N,1}_{v,w}\right|\right\|_{L^p} \\ &\leq \left\|\sup_{w\in\{0,1,\ldots,N\}} e^{\left(\sum_{l=0}^{w-1}\alpha_l^N\right)}\right\|_{L^{4p}} \left\|\sup_{v\in\{0,1,\ldots,N\}} e^{-\left(\sum_{l=0}^{v-1}\alpha_l^N\right)}\right\|_{L^{4p}} \left(C^2+\|\xi\|_{L^{2p}}\right) \\ &\quad + \left\|\sup_{w\in\{0,1,\ldots,N\}} e^{\left(\sum_{l=0}^{w-1}\alpha_l^N\right)}\right\|_{L^{2p}} \left\|\sup_{0\leq v\leq w\leq N} \left|\sum_{k=v}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)} \beta_k^N\right|\right\|_{L^{2p}}\end{aligned} \tag{86}$$

for all $N \in \mathbb{N}$ and all $p \in [2, \infty)$. Inequality (82) therefore yields

$$\begin{aligned}&\left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|D^{N,1}_{v,w}\right|\right\|_{L^p} \\ &\leq e^{\frac{4p}{10}C^4} e^{\frac{4p}{10}C^4} \left(C^2+\|\xi\|_{L^{2p}}\right) + e^{\frac{2p}{10}C^4} \left\|\sup_{0\leq v\leq w\leq N} \left|\sum_{k=v}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)} \beta_k^N\right|\right\|_{L^{2p}} \\ &\leq e^{pC^4} \left(C^2+\|\xi\|_{L^{2p}}\right) + 2e^{\frac{p}{5}C^4} \left\|\sup_{w\in\{0,1,\ldots,N\}} \left|\sum_{k=0}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)} \beta_k^N\right|\right\|_{L^{2p}}\end{aligned} \tag{87}$$

for all $N \in \mathbb{N}$ and all $p \in [2, \infty)$. By definition of β_n^N, $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$, (see (36)) we then obtain

$$\begin{aligned}\left\|\sup_{v,w\in\{0,1,\ldots,N\}} \left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq & e^{pC^4}(C^2+\|\xi\|_{L^{2p}}) + 2e^{\frac{p}{5}C^4} \left(\sum_{k=0}^{N-1} \left\|\operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)} \frac{T}{N}\mu(0)\right\|_{L^{2p}}\right) \\ & + 2e^{\frac{p}{5}C^4} \left\|\sup_{w\in\{0,1,\ldots,N\}} \left|\sum_{k=0}^{w-1} \operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)} \sigma(0)\left(W_{t_{k+1}^N} - W_{t_k^N}\right)\right|\right\|_{L^{2p}}\end{aligned}$$

and therefore

$$\left\|\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq e^{pC^4}(C^2+\|\xi\|_{L^{2p}}) + 2C^2N^{-1}e^{\frac{p}{5}C^4}\left(\sum_{k=0}^{N-1}\left\|e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)}\right\|_{L^{2p}}\right)$$
$$+ 2Ce^{\frac{p}{5}C^4}\left\|\sup_{w\in\{0,1,\dots,N\}}\left|\sum_{k=0}^{w-1}\operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k}\alpha_l^N\right)}\left(W_{t^N_{k+1}}-W_{t^N_k}\right)\right|\right\|_{L^{2p}}$$

for all $N\in\mathbb{N}$ and all $p\in[2,\infty)$. The triangle inequality and again estimate (82) hence yield

$$\left\|\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq e^{pC^4}(C^2+\|\xi\|_{L^{2p}}) + 2C^2e^{\frac{2p}{5}C^4}$$
$$+ 2Ce^{\frac{p}{5}C^4}\left\|\sup_{w\in\{0,1,\dots,N\}}\left|\sum_{k=0}^{w-1}\operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k-1}\alpha_l^N\right)}\left(e^{-\alpha_k^N}-1\right)\left(W_{t^N_{k+1}}-W_{t^N_k}\right)\right|\right\|_{L^{2p}}$$
$$+ 2Ce^{\frac{p}{5}C^4}\left\|\sup_{w\in\{0,1,\dots,N\}}\left|\sum_{k=0}^{w-1}\operatorname{sgn}(Y_k^N)\, e^{-\left(\sum_{l=0}^{k-1}\alpha_l^N\right)}\left(W_{t^N_{k+1}}-W_{t^N_k}\right)\right|\right\|_{L^{2p}}$$

and Doob's inequality (see, e.g., Theorem 11.2 (ii) in [**8**]) and Davis-Burkholder-Gundy's inequality (44) then show

$$\left\|\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq e^{2pC^4}(C^2+\|\xi\|_{L^{2p}})$$
$$+ e^{\frac{p}{4}C^4}\left(\sum_{k=0}^{N-1}\left\|e^{-\left(\sum_{l=0}^{k-1}\alpha_l^N\right)}\left(e^{-\alpha_k^N}-1\right)\left(W_{t^N_{k+1}}-W_{t^N_k}\right)\right\|_{L^{2p}}\right)$$
$$+ e^{\frac{p}{4}C^4}K_{2p}\left(\sum_{k=0}^{N-1}\left\|e^{-\left(\sum_{l=0}^{k-1}\alpha_l^N\right)}\left(W_{t^N_{k+1}}-W_{t^N_k}\right)\right\|^2_{L^{2p}}\right)^{\frac{1}{2}}$$

for all $N\in\mathbb{N}$ and all $p\in[2,\infty)$. Hölder's inequality thus gives

$$\left\|\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right\|_{L^p} \leq e^{2pC^4}(C^2+\|\xi\|_{L^{2p}})$$
$$+ e^{\frac{p}{4}C^4}\left(\sum_{k=0}^{N-1}\left\|e^{-\left(\sum_{l=0}^{k-1}\alpha_l^N\right)}\right\|_{L^{6p}}\left\|e^{-\alpha_k^N}-1\right\|_{L^{6p}}\left\|W_{\frac{T}{N}}\right\|_{L^{6p}}\right)$$
$$+ e^{\frac{p}{4}C^4}K_{2p}\left\|W_{\frac{T}{N}}\right\|_{L^{4p}}\left(\sum_{k=0}^{N-1}\left\|e^{-\left(\sum_{l=0}^{k-1}\alpha_l^N\right)}\right\|^2_{L^{4p}}\right)^{\frac{1}{2}}$$

and inequality (82), inequality (83) and Lemma 4.7 finally yield

$$\begin{aligned}\left\|\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right\|_{L^p} &\leq e^{2pC^4}(C^2+\|\xi\|_{L^{2p}}) + e^{\frac{p}{4}C^4}\left(\sum_{k=0}^{N-1}e^{\frac{6p}{10}C^4}\cdot e^{\frac{6p}{4}C^4}\frac{1}{\sqrt{N}}\cdot 6p\sqrt{\frac{T}{N}}\right)\\ &\quad + e^{\frac{p}{4}C^4}K_{2p}\cdot 4p\sqrt{\frac{T}{N}}\left(\sum_{k=0}^{N-1}e^{\frac{8p}{10}C^4}\right)^{\frac{1}{2}}\\ &\leq e^{2pC^4}(C^2+\|\xi\|_{L^{2p}}) + e^{3pC^4}6p\sqrt{T} + e^{pC^4}K_{2p}4p\sqrt{T}\end{aligned}\tag{88}$$

for all $N\in\mathbb{N}$ and all $p\in[2,\infty)$. This shows the assertion in the case $p\in[2,\infty)$. The case $p\in[1,2)$ then follows from Jensen's inequality and this completes the proof of Lemma 4.3. □

4.6 Proof of Lemma 4.5

Proof of Lemma 4.5. First of all, we have

$$\mathbb{P}\left[\sup_{n\in\{0,1,\dots,N-1\}}\left|W_{t^N_{n+1}}-W_{t^N_n}\right|>N^{-\frac{1}{4}}\right]=\mathbb{P}\left[\cup_{n=0}^{N-1}\left\{\omega\in\Omega\,\middle|\,\left|W_{t^N_{n+1}}(\omega)-W_{t^N_n}(\omega)\right|>N^{-\frac{1}{4}}\right\}\right]$$
$$\leq\sum_{n=0}^{N-1}\mathbb{P}\left[\left|W_{t^N_{n+1}}-W_{t^N_n}\right|>N^{-\frac{1}{4}}\right]$$

and

$$\begin{aligned}&\mathbb{P}\left[\sup_{n\in\{0,1,\dots,N-1\}}\left|W_{t^N_{n+1}}-W_{t^N_n}\right|>N^{-\frac{1}{4}}\right]\\&\leq N\cdot\mathbb{P}\left[\left|W_{t^N_1}-W_{t^N_0}\right|>N^{-\frac{1}{4}}\right]\\&=2N\cdot\mathbb{P}\left[W_{\frac{T}{N}}>N^{-\frac{1}{4}}\right]=2N\cdot\mathbb{P}\left[\sqrt{\frac{N}{T}}W_{\frac{T}{N}}>N^{-\frac{1}{4}}\sqrt{\frac{N}{T}}\right]\end{aligned}\tag{89}$$

and

$$\begin{aligned}&\mathbb{P}\left[\sup_{n\in\{0,1,\dots,N-1\}}\left|W_{t^N_{n+1}}-W_{t^N_n}\right|>N^{-\frac{1}{4}}\right]\\&\leq 2N\int_{\frac{N^{\frac{1}{4}}}{\sqrt{T}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\leq 2N\int_{\frac{N^{\frac{1}{4}}}{\sqrt{T}}}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{x\sqrt{T}}{N^{\frac{1}{4}}}e^{-\frac{x^2}{2}}\,dx\\&=2N^{\frac{3}{4}}\sqrt{T}\int_{\frac{N^{\frac{1}{4}}}{\sqrt{T}}}^{\infty}\frac{1}{\sqrt{2\pi}}\,x\,e^{-\frac{x^2}{2}}\,dx=2N^{\frac{3}{4}}\sqrt{T}\frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{x^2}{2}}\right]_{x=\frac{N^{\frac{1}{4}}}{\sqrt{T}}}^{x=\infty}\\&=2N^{\frac{3}{4}}\sqrt{T}\frac{1}{\sqrt{2\pi}}e^{-\frac{\sqrt{N}}{2T}}=\sqrt{\frac{2}{\pi}}N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\leq N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\end{aligned}\tag{90}$$

for every $N\in\mathbb{N}$. Additionally, note that

$$(\Omega_N)^c=\left\{\omega\in\Omega\,\middle|\,\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}(\omega)\right|>r_N\right\}\cup\left\{\sup_{n\in\{0,1,\dots,N-1\}}\left|W_{t^N_{n+1}}-W_{t^N_n}\right|>N^{-\frac{1}{4}}\right\}\tag{91}$$

for every $N\in\mathbb{N}$. Therefore, inequality (90) implies

$$\begin{aligned}\mathbb{P}\Big[(\Omega_N)^c\Big]&\leq\mathbb{P}\left[\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|>r_N\right]+\mathbb{P}\left[\sup_{n\in\{0,1,\dots,N-1\}}\left|W_{t^N_{n+1}}-W_{t^N_n}\right|>N^{-\frac{1}{4}}\right]\\&\leq\mathbb{P}\left[\left(1+\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right)^p>(1+r_N)^p\right]+N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\end{aligned}\tag{92}$$

for every $N\in\mathbb{N}$ and every $p\in[1,\infty)$. Now we apply Markov's inequality to obtain

$$\begin{aligned}\mathbb{P}\Big[(\Omega_N)^c\Big]&\leq\frac{\mathbb{E}\left[\left(1+\sup_{v,w\in\{0,1,\dots,N\}}\left|D^{N,1}_{v,w}\right|\right)^p\right]}{(1+r_N)^p}+N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\\&\leq\left(\sup_{M\in\mathbb{N}}\left\|1+\sup_{v,w\in\{0,1,\dots,M\}}\left|D^{M,1}_{v,w}\right|\right\|_{L^p}^p\right)\frac{1}{(1+r_N)^p}+N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\end{aligned}\tag{93}$$

and

$$\begin{aligned}\mathbb{P}\Big[(\Omega_N)^c\Big]&\leq\left(\sup_{M\in\mathbb{N}}\left\|1+\sup_{v,w\in\{0,1,\dots,M\}}\left|D^{M,1}_{v,w}\right|\right\|_{L^p}^p\right)c^pN^{-p\cdot\min\left(\frac{1}{4},\frac{1}{(\delta-1)}\right)}+N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\\&\leq\left[c\left(1+\sup_{M\in\mathbb{N}}\left\|\sup_{v,w\in\{0,1,\dots,M\}}\left|D^{M,1}_{v,w}\right|\right\|_{L^p}\right)\right]^pN^{-p\cdot\min\left(\frac{1}{4},\frac{1}{(\delta-1)}\right)}+N^{\frac{3}{4}}\sqrt{T}e^{-\frac{\sqrt{N}}{2T}}\end{aligned}\tag{94}$$

for every $N \in \mathbb{N}$ and every $p \in [1, \infty)$, where we used

$$\frac{1}{(1 + r_N)} \leq c \cdot N^{-\min\left(\frac{1}{4}, \frac{1}{(\delta-1)}\right)}$$

for every $N \in \mathbb{N}$ with $c \in (0, \infty)$ given by $c := \sup_{N\in\mathbb{N}} \left(N^{\min\left(\frac{1}{4}, \frac{1}{(\delta-1)}\right)} / (1 + r_N)\right) < \infty$. Moreover, (94) with $p := 4\max(4, \delta - 1)$ yields

$$\begin{aligned}\mathbb{P}\Big[(\Omega_N)^c\Big] &\leq N^{-4}\left[c\left(1 + \sup_{M\in\mathbb{N}}\left\|\sup_{v,w\in\{0,1,\ldots,M\}}\left|D^{M,1}_{v,w}\right|\right\|_{L^{4\max(4,\delta-1)}}\right)\right]^{4\max(4,\delta-1)} + N\sqrt{T}e^{-\frac{\sqrt{N}}{2T}} \\ &\leq N^{-4}\left[c\left(1 + \sup_{M\in\mathbb{N}}\left\|\sup_{v,w\in\{0,1,\ldots,M\}}\left|D^{M,1}_{v,w}\right|\right\|_{L^{4\max(4,\delta-1)}}\right)\right]^{4\max(4,\delta-1)} \\ &\quad + N^{-4}\left[\sqrt{T}\left(\sup_{x\in[0,\infty)} x^5 e^{-\frac{\sqrt{x}}{2T}}\right)\right]\end{aligned} \tag{95}$$

for every $N \in \mathbb{N}$. The right-hand side is finite by Lemma 4.3. This proves

$$\tilde{c} := \sup_{N\in\mathbb{N}}\left(N^4 \cdot \mathbb{P}\Big[(\Omega_N)^c\Big]\right) \in [0, \infty). \tag{96}$$

Next we show that the event $\tilde{\Omega}$ has probability 1. We have

$$\mathbb{P}\left[\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega^m_M\right] = 1 - \mathbb{P}\left[\left(\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega^m_M\right)^c\right] = 1 - \mathbb{P}\left[\bigcup_{M=N}^{\infty}\bigcup_{m=1}^{M^2}(\Omega^m_M)^c\right] \tag{97}$$

and hence

$$\mathbb{P}\left[\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega^m_M\right] \geq 1 - \sum_{M=N}^{\infty}\sum_{m=1}^{M^2}\mathbb{P}\Big[(\Omega^m_M)^c\Big] = 1 - \sum_{M=N}^{\infty}\left(M^2 \cdot \mathbb{P}\Big[(\Omega_M)^c\Big]\right) \geq 1 - \left(\sum_{M=N}^{\infty}\tilde{c}M^{-2}\right)$$

for every $N \in \mathbb{N}$ due to (96). This implies

$$\mathbb{P}\left[\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega^m_M\right] \geq 1 - \tilde{c}\left(\int_{N-1}^{\infty}\frac{1}{s^2}\,ds\right) = 1 - \tilde{c}\left[-\frac{1}{s}\right]_{s=N-1}^{s=\infty} = 1 - \frac{\tilde{c}}{(N-1)}$$

for every $N \in \{2, 3, \ldots\}$, which shows

$$\mathbb{P}\left[\bigcup_{N\in\mathbb{N}}\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega^m_M\right] = \lim_{N\to\infty}\mathbb{P}\left[\bigcap_{M=N}^{\infty}\bigcap_{m=1}^{M^2}\Omega^m_M\right] = 1. \tag{98}$$

Moreover, we have

$$\begin{aligned}&\mathbb{E}\left[\left(\sup_{N\in\mathbb{N}}\frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega^m_N}\cdot f(Y^{N,m}_N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N}f(Y^N_N)\right]\right)\right|}{N^{(1+\varepsilon)}}\right)^p\right] \\ &\leq \mathbb{E}\left[\sum_{N=1}^{\infty}\frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega^m_N}\cdot f(Y^{N,m}_N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N}f(Y^N_N)\right]\right)\right|^p}{N^{(1+\varepsilon)p}}\right] \\ &= \sum_{N=1}^{\infty}\frac{\left\|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega^m_N}\cdot f(Y^{N,m}_N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N}f(Y^N_N)\right]\right)\right\|^p_{L^p}}{N^{(1+\varepsilon)p}} \\ &\leq \sum_{N=1}^{\infty}\frac{\left(K_p\left(\sum_{m=1}^{N^2}\left\|\mathbb{1}_{\Omega_N}f(Y^N_N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N}f(Y^N_N)\right]\right\|^2_{L^p}\right)^{\frac{1}{2}}\right)^p}{N^{(1+\varepsilon)p}}\end{aligned}$$

due to Lemma 4.1 and therefore

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sup_{N\in\mathbb{N}} \frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega_N^m}\cdot f(Y_N^{N,m}) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right)\right|}{N^{(1+\varepsilon)}}\right)^p\right] \\
&\leq \sum_{N=1}^{\infty} \frac{\left(K_p \cdot N \cdot \left\|\mathbb{1}_{\Omega_N} f(Y_N^N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right\|_{L^p}\right)^p}{N^{(1+\varepsilon)p}} \\
&= \sum_{N=1}^{\infty} \left(\frac{(K_p)^p \cdot \left\|\mathbb{1}_{\Omega_N} f(Y_N^N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right\|_{L^p}^p}{N^{\varepsilon p}}\right) \\
&\leq (K_p)^p \left(\sum_{N=1}^{\infty} N^{-\varepsilon p}\right)\left(\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left|\mathbb{1}_{\Omega_N} f(Y_N^N) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right|^p\right]\right)
\end{aligned}
\tag{99}
$$

for every $p \in [2, \infty)$ and every $\varepsilon \in (0, \infty)$. Hence, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sup_{N\in\mathbb{N}} \frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega_N^m}\cdot f(Y_N^{N,m}) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right)\right|}{N^{(1+\varepsilon)}}\right)^p\right] \\
&\leq (2^p (K_p)^p) \left(\sum_{N=1}^{\infty} N^{-\varepsilon p}\right)\left(\sup_{N\in\mathbb{N}} \mathbb{E}\left[\left|\mathbb{1}_{\Omega_N} f(Y_N^N)\right|^p\right]\right) \\
&\leq (2^p (K_p)^p) \left(\sum_{N=1}^{\infty} N^{-\varepsilon p}\right)\left(\sup_{N\in\mathbb{N}} \mathbb{E}\left[\mathbb{1}_{\Omega_N} L^p \left(1 + \left|Y_N^N\right|^\delta\right)^p\right]\right) \\
&\leq (2LK_p)^p \left(\sum_{N=1}^{\infty} N^{-\varepsilon p}\right)\left(\sup_{N\in\mathbb{N}} \mathbb{E}\left[\mathbb{1}_{\Omega_N} \left(1 + \left|Y_N^N\right|^\delta\right)^p\right]\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sup_{N\in\mathbb{N}} \frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega_N^m}\cdot f(Y_N^{N,m}) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right)\right|}{N^{(1+\varepsilon)}}\right)^p\right] \\
&\leq (4LK_p)^p \left(\sum_{N=1}^{\infty} N^{-\varepsilon p}\right)\left(1 + \sup_{N\in\mathbb{N}} \mathbb{E}\left[\mathbb{1}_{\Omega_N} \left|Y_N^N\right|^{p\delta}\right]\right) < \infty
\end{aligned}
$$

for every $p \in \left(\frac{1}{\varepsilon}, \infty\right)$ and every $\varepsilon \in (0, 1)$ due to Corollary 4.4. This implies

$$
\mathbb{P}\left[\left\{\omega \in \Omega \,\middle|\, \sup_{N\in\mathbb{N}} \frac{\left|\sum_{m=1}^{N^2}\left(\mathbb{1}_{\Omega_N^m}(\omega)\cdot f(Y_N^{N,m}(\omega)) - \mathbb{E}\left[\mathbb{1}_{\Omega_N} f(Y_N^N)\right]\right)\right|}{N^{(1+\varepsilon)}} < \infty\right\}\right] = 1 \tag{100}
$$

for every $\varepsilon \in (0, \infty)$ (see also Lemma 2.1 in Kloeden & Neuenkirch [**10**]). Putting together (98) and (100) shows $\mathbb{P}[\tilde{\Omega}] = 1$. This completes the proof of Lemma 4.5. □

4.7 Proof of Lemma 4.6

It is somewhat inconvenient to compare the exact solution, which is a continuous time process, with the Euler approximations, which are time-discrete stochastic processes. Therefore, we consider the following interpolation process of the Euler approximation. Let $\tilde{Y}^N \colon [0, T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, be given by

$$
\tilde{Y}_t^N(\omega) := Y_n^N(\omega) + \left(t - t_n^N\right) \cdot \mu(Y_n^N(\omega)) + \sigma(Y_n^N(\omega)) \cdot \left(W_t(\omega) - W_{t_n^N}(\omega)\right) \tag{101}
$$

for every $t \in [t_n^N, t_{n+1}^N]$, $n \in \{0, 1, \ldots, N-1\}$, $\omega \in \Omega$ and every $N \in \mathbb{N}$. Note that $\tilde{Y}_{t_n^N}^N(\omega) = Y_n^N(\omega)$ for every $\omega \in \Omega$, $n \in \{0, 1, \ldots, N\}$ and every $N \in \mathbb{N}$. Before we prove Lemma 4.6, we show that the restricted moments of the interpolation processes are uniformly bounded.

Lemma 4.10. *Let $\tilde{Y}^N\colon [0,T]\times\Omega\to\mathbb{R}$ for $N\in\mathbb{N}$ be given by* (101)*. Then we have*

$$\sup_{N\in\mathbb{N}}\sup_{0\leq t\leq T}\mathbb{E}\left[\mathbb{1}_{\Omega_{N,\lfloor\frac{tN}{T}\rfloor}}\big|\tilde{Y}_t^N\big|^p\right]<\infty \tag{102}$$

for every $p\in[1,\infty)$.

Proof of Lemma 4.10. Inserting the definition (101) of the interpolation process and the polynomial growth of μ and σ shows

$$\left\|\mathbb{1}_{\Omega_{N,n}}\left|\tilde{Y}_t^N\right|\right\|_{L^p}\leq\left\|\mathbb{1}_{\Omega_{N,n}}\left(1+\left|Y_n^N\right|^\delta\right)\left(1+LT+L\left|W_t-W_{t_n^N}\right|\right)\right\|_{L^p}$$

for every $t\in[t_n^N,t_{n+1}^N]$, $n\in\{0,1,\dots,N-1\}$ and every $N\in\mathbb{N}$. Now we apply Hölder's inequality, the triangle inequality and Lemma 4.7 to arrive at

$$\begin{aligned}\left\|\mathbb{1}_{\Omega_{N,n}}\left|\tilde{Y}_t^N\right|\right\|_{L^p}&\leq\left(1+\left\|\mathbb{1}_{\Omega_{N,n}}\left|Y_n^N\right|^\delta\right\|_{L^{2p}}\right)\cdot\left(1+LT+L\left\|W_t-W_{t_n^N}\right\|_{L^{2p}}\right)\\&\leq\left(1+\left(\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n}}\left|Y_n^N\right|^{2\delta p}\right]\right)^{\frac{1}{2p}}\right)\cdot\left(1+LT+L2p\sqrt{T}\right)\end{aligned} \tag{103}$$

for every $t\in[t_n^N,t_{n+1}^N]$, $n\in\{0,1,\dots,N-1\}$ and every $N\in\mathbb{N}$. The right-hand side of (103) is uniformly bounded in $n\in\{0,1,\dots,N\}$ and $N\in\mathbb{N}$ according to Corollary 4.4. This completes the proof. $\square$

Proof of Lemma 4.6. Let $X^{s,x}\colon[s,T]\times\Omega\to\mathbb{R}$, $s\in[0,T]$, $x\in\mathbb{R}$, be a family of adapted stochastic processes with continuous sample paths given by

$$X_t^{s,x}=x+\int_s^t\mu\left(X_u^{s,x}\right)du+\int_s^t\sigma\left(X_u^{s,x}\right)dW_u\qquad\mathbb{P}\text{-a.s.} \tag{104}$$

for every $t\in[s,T]$, $s\in[0,T]$ and every $x\in\mathbb{R}$. Moreover, assume that the mapping $X_t^{s,\cdot}(\cdot)\colon\mathbb{R}\times\Omega\to\mathbb{R}$ with $(x,\omega)\mapsto X_t^{s,x}(\omega)$ for all $x\in\mathbb{R}$, $\omega\in\Omega$ is $(\mathcal{B}(\mathbb{R})\otimes\mathcal{F}_t)/(\mathcal{B}(\mathbb{R}))$-measurable for every $s,t\in[0,T]$ with $s\leq t$ and assume that the mapping $X_t^{s,\cdot}(\omega)\colon\mathbb{R}\to\mathbb{R}$ with $x\mapsto X_t^{s,x}(\omega)$ for all $x\in\mathbb{R}$ is continuous for every $s,t\in[0,T]$ with $s\leq t$ and every $\omega\in\Omega$. We will show in (128) below that the difference between $X_{t_{n+1}}^{t_n^N,Y_n^N}$ and Y_{n+1}^N is of order $O\big(\frac{1}{N^2}\big)$ in a suitable weak sense for every $n\in\{0,1,\dots,N-1\}$. Summing over $n\in\{0,1,\dots,N-1\}$ for each $N\in\mathbb{N}$ will then prove the assertion.

First we need several preparations. According to Theorem 2.6.4 in [**3**], there are real numbers $\kappa_p\in[0,\infty)$, $p\in\{2,4,6,\dots\}$, such that

$$\mathbb{E}\Big[\big|X_t^{s,x}\big|^p\Big]\leq\kappa_p\left(1+|x|^p\right) \tag{105}$$

holds for every $s,t\in[0,T]$ with $s\leq t$ and every $p\in\{2,4,6,\dots\}$, $x\in\mathbb{R}$. This implies

$$\begin{aligned}\sup_{t_n^N\leq t\leq T}\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n}}\left|X_t^{t_n^N,Y_n^N}\right|^p\right]&=\sup_{t_n^N\leq t\leq T}\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n}}\mathbb{E}\left[\left|X_t^{t_n^N,Y_n^N}\right|^p\,\middle|\,\mathcal{F}_{t_n^N}\right]\right]\\&\leq\kappa_p\cdot\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n}}\left(1+\left|Y_n^N\right|^p\right)\Big]\\&\leq\kappa_p\left(1+\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n}}\left|Y_n^N\right|^p\right]\right)\end{aligned} \tag{106}$$

for every $p\in\{2,4,6,\dots\}$, $n\in\{0,1,\dots,N\}$ and every $N\in\mathbb{N}$. Corollary 4.4 hence implies

$$\sup_{N\in\mathbb{N}}\sup_{n\in\{0,1,\dots,N\}}\sup_{t_n^N\leq t\leq T}\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n}}\left|X_t^{t_n^N,Y_n^N}\right|^p\right]<\infty \tag{107}$$

for every $p\in[1,\infty)$.

Now define $u:[0,T]\times\mathbb{R}\to\mathbb{R}$ by $u(t,x)=\mathbb{E}\big[f\big(X_T^{t,x}\big)\big]$ for every $t\in[0,T]$ and every $x\in\mathbb{R}$. Moreover, let the n-th partial derivative of u with respect to the second argument be the function $u_n\colon[0,T]\times\mathbb{R}\to\mathbb{R}$ defined through

$$u_n(t,x)=\left(\frac{\partial^n}{\partial x^n}u\right)(t,x) \tag{108}$$

for every $n \in \{0, 1, \ldots, 4\}$, $t \in [0, T]$ and every $x \in \mathbb{R}$. Additionally, we use the functions $u, \tilde{u} \colon [0, T] \times \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{u}(t, x) = u_1(t, x) \cdot \mu(x) + \frac{1}{2} u_2(t, x) \cdot (\sigma(x))^2 \tag{109}$$

and

$$\tilde{\tilde{u}}(t, x) = \left(\frac{\partial}{\partial x} \tilde{u} \right) (t, x) \cdot \mu(x) + \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} \tilde{u} \right) (t, x) \cdot (\sigma(x))^2 \tag{110}$$

for every $t \in [0, T]$ and every $x \in \mathbb{R}$. Moreover, let $R \in [5\delta, \infty)$ be a real number which satisfies

$$\begin{aligned}
&|u(t,x)| \leq R \left(1 + |x|^R\right), & |u_1(t,x)| \leq R \left(1 + |x|^R\right), \\
&|u_2(t,x)| \leq R \left(1 + |x|^R\right), & |u_3(t,x)| \leq R \left(1 + |x|^R\right), \\
&|u_4(t,x)| \leq R \left(1 + |x|^R\right), & |\tilde{u}(t,x)| \leq R \left(1 + |x|^R\right), \\
&\left|\tilde{\tilde{u}}(t,x)\right| \leq R \left(1 + |x|^R\right), & \left| \left(\frac{\partial}{\partial x} \tilde{u} \right) (t,x) \right| \leq R \left(1 + |x|^R\right)
\end{aligned} \tag{111}$$

for every $t \in [0, T]$ and every $x \in \mathbb{R}$ (see also Corollary 2.8.1 and Theorem 2.8.1 in [**3**]). The existence of such a real number can be shown by exploiting (17), (18) and (19). In our estimates, we will need the real number $C \in [0, \infty)$ defined by

$$\begin{aligned}
C := \sup_{N \in \mathbb{N}} \left(N^4 \cdot \mathbb{P}\left[(\Omega_N)^c \right] \right) + \sup_{N \in \mathbb{N}} \sup_{0 \leq u \leq T} \left\| \mathbb{1}_{\Omega_{N, \lfloor \frac{uN}{T} \rfloor}} \left(2 + \left| \tilde{Y}_u^N \right|^R \right) \right\|_{L^6} \\
+ L + R + T + \sup_{N \in \mathbb{N}} \sup_{n \in \{0,1,\ldots,N\}} \sup_{t_n^N \leq t \leq T} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left| X_t^{t_n^N, Y_n^N} \right|^R \right) \right\|_{L^4} < \infty.
\end{aligned} \tag{112}$$

Indeed, $C \in [0, \infty)$ is finite due to Lemma 4.5, Lemma 4.10 and due to (107). Moreover, since

$$\tilde{Y}_t^N = Y_n^N + \int_{t_n^N}^t \mu(Y_n^N)\, ds + \int_{t_n^N}^t \sigma(Y_n^N)\, dW_s \qquad \mathbb{P}\text{-a.s.} \tag{113}$$

holds for every $t \in [t_n^N, t_{n+1}^N]$, $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$, Itô's formula yields

$$\begin{aligned}
u\big(t_{n+1}^N, Y_{n+1}^N\big) &= u\Big(t_{n+1}^N, \tilde{Y}_{t_{n+1}^N}^N\Big) \\
&= u\big(t_{n+1}^N, Y_n^N\big) + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \mu(Y_n^N)\, ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \sigma(Y_n^N)\, dW_s \\
&\quad + \frac{1}{2} \int_{t_n^N}^{t_{n+1}^N} u_2\Big(t_{n+1}^N, \tilde{Y}_s^N\Big) \big(\sigma(Y_n^N)\big)^2\, ds \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{114}$$

for every $n \in \{0, 1, \dots, N-1\}$ and every $N \in \mathbb{N}$. Again Itô's formula yields

$$
\begin{aligned}
u\big(t_{n+1}^N, Y_{n+1}^N\big) &= u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big) \\
&\quad + \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\mu(Y_n^N)\big)^2\,dr\,ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\sigma(Y_n^N)\,\mu(Y_n^N)\,dW_r\,ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\sigma(Y_n^N)\big)^2\,\mu(Y_n^N)\,dr\,ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\,\sigma(Y_n^N)\,dW_s \\
&\quad + \frac{1}{2}\int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\sigma(Y_n^N)\big)^3\,dW_r\,ds \\
&\quad + \frac{1}{4}\int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_4\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\sigma(Y_n^N)\big)^4\,dr\,ds \qquad \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{115}
$$

for every $n \in \{0, 1, \dots, N-1\}$ and every $N \in \mathbb{N}$.

Now we estimate all non-stochastic integrals on the right-hand side of (115) restricted to the events $\Omega_{N,n+1}$ for $n \in \{0, 1, \dots, N-1\}$ and $N \in \mathbb{N}$. For the first integral on the right hand side of (115), we obtain

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\mu(Y_n^N)\big)^2\,dr\,ds \right\|_{L^1} \\
&\leq \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\mu(Y_n^N)\big)^2 \right\|_{L^1} dr\,ds \\
&\leq \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} 2L^2 \left\| \mathbb{1}_{\Omega_{N,n+1}} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\Big(1 + \big|Y_n^N\big|^{2\delta}\Big) \right\|_{L^1} dr\,ds
\end{aligned}
\tag{116}
$$

and, using the polynomial growth estimate (111) of u_2,

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\mu(Y_n^N)\big)^2\,dr\,ds \right\|_{L^1} \\
&\leq \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} 2L^2 R \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(1 + \left|\tilde{Y}_r^N\right|^R\right) \Big(1 + \big|Y_n^N\big|^{2\delta}\Big) \right\|_{L^1} dr\,ds \\
&\leq \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} 2L^2 R \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(1 + \left|\tilde{Y}_r^N\right|^R\right) \Big(2 + \big|Y_n^N\big|^{R}\Big) \right\|_{L^1} dr\,ds
\end{aligned}
\tag{117}
$$

and, applying Hölder's inequality and the definition (112) of $C \in [0, \infty)$,

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\,\big(\mu(Y_n^N)\big)^2\,dr\,ds \right\|_{L^1} \\
&\leq 2L^2 R \int_{t_n^N}^{t_{n+1}^N}\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(2 + \left|\tilde{Y}_r^N\right|^R\right) \right\|_{L^2} \left\| \mathbb{1}_{\Omega_{N,n+1}} \Big(2 + \big|Y_n^N\big|^{R}\Big) \right\|_{L^2} dr\,ds \\
&\leq 2L^2 R \left(\sup_{0 \leq u \leq T} \left\| \mathbb{1}_{\Omega_{N,\lfloor \frac{uN}{T} \rfloor}} \left(2 + \left|\tilde{Y}_u^N\right|^R\right) \right\|_{L^2}^2 \right) \frac{1}{2}\left(\frac{T}{N}\right)^2 \\
&\leq L^2 R T^2 C^2 N^{-2} \leq C^7 N^{-2}
\end{aligned}
\tag{118}
$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. In addition, we have

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^2 \mu(Y_n^N)\, dr\, ds \right\|_{L^1} \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^2 \mu(Y_n^N) \right\|_{L^1} dr\, ds \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} 2L^3 \left\| \mathbb{1}_{\Omega_{N,n+1}} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(1 + \left|Y_n^N\right|^2\right) \left(1 + \left|Y_n^N\right|^\delta\right) \right\|_{L^1} dr\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^2 \mu(Y_n^N)\, dr\, ds \right\|_{L^1} \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} 2L^3 R \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(1 + \left|\tilde{Y}_r^N\right|^R\right) \left(1 + \left|Y_n^N\right|^2\right) \left(1 + \left|Y_n^N\right|^\delta\right) \right\|_{L^1} dr\, ds \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} 2L^3 R \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(1 + \left|\tilde{Y}_r^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right) \right\|_{L^1} dr\, ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^2 \mu(Y_n^N)\, dr\, ds \right\|_{L^1} \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} 2L^3 R \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(2 + \left|\tilde{Y}_r^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right)^2 \right\|_{L^1} dr\, ds \\
&\leq 2L^3 R \left(\sup_{0 \leq u \leq T} \left\| \mathbb{1}_{\Omega_{N,\lfloor \frac{uN}{T} \rfloor}} \left(2 + \left|\tilde{Y}_u^N\right|^R\right) \right\|_{L^3}^3 \right) \frac{1}{2} \left(\frac{T}{N}\right)^2 \\
&\leq L^3 R T^2 C^3 N^{-2} \leq C^9 N^{-2}
\end{aligned} \tag{119}
$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Next we use the estimates $|\sigma(x)| \leq L(1 + |x|)$ and $(1 + |x|)^4 \leq 8(1 + x^4)$ for every $x \in \mathbb{R}$ to obtain

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \frac{1}{4} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_4\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^4 dr\, ds \right\|_{L^1} \\
&\leq \frac{1}{4} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} u_4\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^4 \right\|_{L^1} dr\, ds \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} 2L^4 \left\| \mathbb{1}_{\Omega_{N,n+1}} u_4\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(1 + \left|Y_n^N\right|^4\right) \right\|_{L^1} dr\, ds
\end{aligned} \tag{120}
$$

and

$$
\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \frac{1}{4} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_4\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^4 dr\, ds \right\|_{L^1} \\
&\leq 2L^4 R \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(1 + \left|\tilde{Y}_r^N\right|^R\right) \left(1 + \left|Y_n^N\right|^4\right) \right\|_{L^1} dr\, ds \\
&\leq 2L^4 R \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(1 + \left|\tilde{Y}_r^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right) \right\|_{L^1} dr\, ds
\end{aligned} \tag{121}
$$

and

$$\begin{aligned}
&\left\| \mathbb{1}_{\Omega_{N,n+1}} \frac{1}{4} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_4\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^4 dr\, ds \right\|_{L^1} \\
&\leq 2L^4 R \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(2 + \left|\tilde{Y}_r^N\right|^R\right) \right\|_{L^2} \left\| \mathbb{1}_{\Omega_{N,n+1}} \left(2 + \left|Y_n^N\right|^R\right) \right\|_{L^2} dr\, ds \\
&\leq 2L^4 R \left(\sup_{0 \leq u \leq T} \left\| \mathbb{1}_{\Omega_{N, \lfloor \frac{uN}{T} \rfloor}} \left(2 + \left|\tilde{Y}_u^N\right|^R\right) \right\|_{L^2}^2 \right) \frac{1}{2} \left(\frac{T}{N}\right)^2 \qquad (122) \\
&\leq L^4 R T^2 C^2 N^{-2} \leq C^9 N^{-2}
\end{aligned}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Combining (115), (118), (119) and (122) hence yields

$$\begin{aligned}
&\left| \mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \left\{ u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N} \tilde{u}\big(t_{n+1}^N, Y_n^N\big) \right\}\right] \right| \\
&\leq 3C^9 N^{-2} + \left| \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\, \sigma(Y_n^N)\, \mu(Y_n^N)\, dW_r\, ds \right] \right| \\
&\quad + \left| \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \sigma(Y_n^N)\, dW_s \right] \right| \\
&\quad + \left| \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \frac{1}{2} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^3 dW_r\, ds \right] \right|
\end{aligned}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Due to $\Omega_{N,n+1} \subset \Omega_{N,n}$ we have

$$\mathbb{1}_{\Omega_{N,n+1}} = \mathbb{1}_{(\Omega_{N,n+1} \cap \Omega_{N,n})} = \mathbb{1}_{\Omega_{N,n+1}} \cdot \mathbb{1}_{\Omega_{N,n}} \qquad (123)$$

and therefore

$$\begin{aligned}
&\left| \mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \left\{ u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N} \tilde{u}\big(t_{n+1}^N, Y_n^N\big) \right\}\right] \right| \\
&\leq 3C^9 N^{-2} + \left| \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\, \sigma(Y_n^N)\, \mu(Y_n^N)\, dW_r\, ds \right] \right| \\
&\quad + \left| \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \sigma(Y_n^N)\, dW_s \right] \right| \\
&\quad + \left| \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \frac{1}{2} \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^3 dW_r\, ds \right] \right|
\end{aligned}$$

and, using that the expectation of every involved stochastic integral is equal to zero,

$$\begin{aligned}
&\left| \mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \left\{ u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N} \tilde{u}\big(t_{n+1}^N, Y_n^N\big) \right\}\right] \right| \\
&\leq 3C^9 N^{-2} + \int_{t_n^N}^{t_{n+1}^N} \left| \mathbb{E}\left[\mathbb{1}_{(\Omega_{N,n+1})^c} \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\, \sigma(Y_n^N)\, \mu(Y_n^N)\, dW_r \right] \right| ds \\
&\quad + \left| \mathbb{E}\left[\mathbb{1}_{(\Omega_{N,n+1})^c} \int_{t_n^N}^{t_{n+1}^N} \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \sigma(Y_n^N)\, dW_s \right] \right| \\
&\quad + \frac{1}{2} \int_{t_n^N}^{t_{n+1}^N} \left| \mathbb{E}\left[\mathbb{1}_{(\Omega_{N,n+1})^c} \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^3 dW_r \right] \right| ds
\end{aligned}$$

for every $n \in \{0,1,\ldots,N-1\}$ and every $N \in \mathbb{N}$. This implies

$$
\begin{aligned}
&\left|\mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_k^N\big) + \frac{T}{N}\tilde{u}\left(t_{n+1}^N, Y_n^N\right)\Big\}\Big]\right| \\
&\leq 3C^9 N^{-2} + \left(\mathbb{P}\Big[\left(\Omega_{N,n+1}\right)^c\Big]\right)^{\frac{1}{2}} \int_{t_n^N}^{t_{n+1}^N} \left\| \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\, \sigma(Y_n^N)\, \mu(Y_n^N)\, dW_r \right\|_{L^2} ds \\
&+ \left(\mathbb{P}\Big[\left(\Omega_{N,n+1}\right)^c\Big]\right)^{\frac{1}{2}} \left\| \int_{t_n^N}^{t_{n+1}^N} \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \sigma(Y_n^N)\, dW_s \right\|_{L^2} \\
&+ \frac{1}{2}\left(\mathbb{P}\Big[\left(\Omega_{N,n+1}\right)^c\Big]\right)^{\frac{1}{2}} \int_{t_n^N}^{t_{n+1}^N} \left\| \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^3 dW_r \right\|_{L^2} ds
\end{aligned}
$$

and, using $(\Omega_{N,n+1})^c \subseteq (\Omega_N)^c$ and the Itô isometry,

$$
\begin{aligned}
&\left|\mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\left(t_{n+1}^N, Y_n^N\right)\Big\}\Big]\right| \\
&\leq 3C^9 N^{-2} + \left(\mathbb{P}\Big[\left(\Omega_N\right)^c\Big]\right)^{\frac{1}{2}} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} u_2\Big(t_{n+1}^N, \tilde{Y}_r^N\Big)\, \sigma(Y_n^N)\, \mu(Y_n^N) \right\|_{L^2}^2 dr \right)^{\frac{1}{2}} ds \\
&+ \left(\mathbb{P}\Big[\left(\Omega_N\right)^c\Big]\right)^{\frac{1}{2}} \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, \tilde{Y}_s^N\Big)\, \sigma(Y_n^N) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\
&+ \frac{1}{2}\left(\mathbb{P}\Big[\left(\Omega_N\right)^c\Big]\right)^{\frac{1}{2}} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} u_3\Big(t_{n+1}^N, \tilde{Y}_r^N\Big) \left(\sigma(Y_n^N)\right)^3 \right\|_{L^2}^2 dr \right)^{\frac{1}{2}} ds
\end{aligned}
$$

for every $n \in \{0,1,\ldots,N-1\}$ and every $N \in \mathbb{N}$. Hence, (112) shows

$$
\begin{aligned}
&\left|\mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\left(t_{n+1}^N, Y_n^N\right)\Big\}\Big]\right| \\
&\leq 3C^9 N^{-2} + CN^{-2} \\
&\cdot \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} R\left(1 + \left|\tilde{Y}_r^N\right|^R\right) L\left(1 + \left|Y_n^N\right|\right) L\left(1 + \left|Y_n^N\right|^\delta\right) \right\|_{L^2}^2 dr \right)^{\frac{1}{2}} ds \\
&+ CN^{-2} \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} R\left(1 + \left|\tilde{Y}_s^N\right|^R\right) L\left(1 + \left|Y_n^N\right|\right) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\
&+ \frac{1}{2} CN^{-2} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} R\left(1 + \left|\tilde{Y}_r^N\right|^R\right) 4L^3 \left(1 + \left|Y_n^N\right|^3\right) \right\|_{L^2}^2 dr \right)^{\frac{1}{2}} ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\left(t_{n+1}^N, Y_n^N\right)\Big\}\Big]\right| \\
&\leq 3C^9 N^{-2} \\
&+ RL^2 CN^{-2} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|\tilde{Y}_r^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right)^2 \right\|_{L^2}^2 dr \right)^{\frac{1}{2}} ds \\
&+ RLCN^{-2} \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|\tilde{Y}_s^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right) \right\|_{L^2}^2 ds \right)^{\frac{1}{2}} \\
&+ 2RL^3 CN^{-2} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|\tilde{Y}_r^N\right|^R\right) \left(2 + \left|Y_n^N\right|^R\right) \right\|_{L^2}^2 dr \right)^{\frac{1}{2}} ds
\end{aligned}
$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
&\left|\mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\left(t_{n+1}^N, Y_n^N\right)\Big\}\Big]\right| \\
&\leq 3C^9 N^{-2} + RL^2 C N^{-2} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left(\sup_{0 \leq u \leq T} \left\|\mathbb{1}_{\Omega_{N,\lfloor \frac{uN}{T} \rfloor}} \left(2 + \left|\tilde{Y}_u^N\right|^R\right)\right\|_{L^6}^2\right) dr\right)^{\frac{1}{2}} ds \\
&\qquad + RLCN^{-2} \left(\int_{t_n^N}^{t_{n+1}^N} \left(\sup_{0 \leq u \leq T} \left\|\mathbb{1}_{\Omega_{N,\lfloor \frac{uN}{T} \rfloor}} \left(2 + \left|\tilde{Y}_u^N\right|^R\right)\right\|_{L^4}^2\right) ds\right)^{\frac{1}{2}} \\
&\qquad + 2RL^3 C N^{-2} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left(\sup_{0 \leq u \leq T} \left\|\mathbb{1}_{\Omega_{N,\lfloor \frac{uN}{T} \rfloor}} \left(2 + \left|\tilde{Y}_u^N\right|^R\right)\right\|_{L^4}^2\right) dr\right)^{\frac{1}{2}} ds \\
&\leq 3C^9 N^{-2} + RL^2 C N^{-2} \int_{t_n^N}^{t_{n+1}^N} \sqrt{T} C \, ds + RLCN^{-2} \sqrt{T} C + 2RL^3 C N^{-2} \int_{t_n^N}^{t_{n+1}^N} \sqrt{T} C \, ds
\end{aligned}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. This finally shows that

$$\begin{aligned}
&\left|\mathbb{E}\big[\mathbb{1}_{\Omega_{N,n+1}} u\big(t_{n+1}^N, Y_{n+1}^N\big)\big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\left(t_{n+1}^N, Y_n^N\right)\Big\}\Big]\right| \\
&\leq 3C^9 N^{-2} + RL^2 C^2 \sqrt{T} T N^{-2} + RLC^2 \sqrt{T} N^{-2} + 2RL^3 C^2 \sqrt{T} T N^{-2} \\
&\leq 3C^9 N^{-2} + C^7 N^{-2} + C^5 N^{-2} + 2C^8 N^{-2} \leq 7C^9 N^{-2}
\end{aligned} \tag{124}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$.

Next we aim at a similar estimate as (124) with Y_{n+1}^N replaced by $X_{t_{n+1}^N}^{t_n^N, Y_n^N}$ for $n \in \{0, 1, \ldots, N-1\}$ and $N \in \mathbb{N}$. Itô's formula implies

$$\begin{aligned}
&u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big) \\
&= u\big(t_{n+1}^N, Y_n^N\big) + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \mu\Big(X_s^{t_n^N, Y_n^N}\Big)\, ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s \\
&\quad + \frac{1}{2}\int_{t_n^N}^{t_{n+1}^N} u_2\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big) \Big(\sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\Big)^2 ds \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{125}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. This shows

$$\begin{aligned}
u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big) &= u\big(t_{n+1}^N, Y_n^N\big) + \int_{t_n^N}^{t_{n+1}^N} \tilde{u}\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

and

$$\begin{aligned}
&u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big) \\
&= u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big) + \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \tilde{\tilde{u}}\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, dr\, ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \left(\frac{\partial}{\partial x}\tilde{u}\right)\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_r^{t_n^N, Y_n^N}\Big)\, dW_r\, ds \\
&\quad + \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{126}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$ by Itô's formula. Hence, we obtain

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \left|\tilde{\tilde{u}}\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\right|\right] dr\, ds \\
&+ \int_{t_n^N}^{t_{n+1}^N} \left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{s} \left(\frac{\partial}{\partial x}\tilde{u}\right)\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_r^{t_n^N, Y_n^N}\Big)\, dW_r\right]\right| ds \\
&+ \left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s\right]\right|
\end{aligned}
$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. By (123) we obtain

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \int_{t_n^N}^{t_{n+1}^N} \int_{t_n^N}^{s} \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\, R\left(1 + \left|X_r^{t_n^N, Y_n^N}\right|^R\right)\right] dr\, ds \\
&+ \int_{t_n^N}^{t_{n+1}^N} \left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} \left(\frac{\partial}{\partial x}\tilde{u}\right)\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_r^{t_n^N, Y_n^N}\Big) dW_r\right]\right| ds \\
&+ \left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}} \int_{t_n^N}^{t_{n+1}^N} \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s\right]\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq RC\frac{1}{2}\left(\frac{T}{N}\right)^2 \\
&+ \int_{t_n^N}^{t_{n+1}^N} \left|\mathbb{E}\left[\mathbb{1}_{(\Omega_{N,n+1})^c} \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} \left(\frac{\partial}{\partial x}\tilde{u}\right)\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_r^{t_n^N, Y_n^N}\Big)\, dW_r\right]\right| ds \\
&+ \left|\mathbb{E}\left[\mathbb{1}_{(\Omega_{N,n+1})^c} \int_{t_n^N}^{t_{n+1}^N} \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s\right]\right|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} \Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \frac{1}{2}RCT^2N^{-2} \\
&+ \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} \int_{t_n^N}^{t_{n+1}^N} \left\| \int_{t_n^N}^{s} \mathbb{1}_{\Omega_{N,n}} \left(\frac{\partial}{\partial x}\tilde{u}\right)\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_r^{t_n^N, Y_n^N}\Big)\, dW_r\right\|_{L^2} ds \\
&+ \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} \left\| \int_{t_n^N}^{t_{n+1}^N} \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\, dW_s\right\|_{L^2}
\end{aligned}
$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Therefore, we have

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} \left(\frac{\partial}{\partial x}\tilde{u}\right)\Big(t_{n+1}^N, X_r^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_r^{t_n^N, Y_n^N}\Big)\right\|_{L^2}^2 dr\right)^{\frac{1}{2}} ds \\
&+ \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} u_1\Big(t_{n+1}^N, X_s^{t_n^N, Y_n^N}\Big)\, \sigma\Big(X_s^{t_n^N, Y_n^N}\Big)\right\|_{L^2}^2 ds\right)^{\frac{1}{2}} + \frac{C^4}{N^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \frac{C^4}{N^2} + \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} \\
&\cdot \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} R \left(1 + \left|X_r^{t_n^N, Y_n^N}\right|^R\right) L\left(1 + \left|X_r^{t_n^N, Y_n^N}\right|\right)\right\|_{L^2}^2 dr\right)^{\frac{1}{2}} ds \\
&+ \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} R \left(1 + \left|X_s^{t_n^N, Y_n^N}\right|^R\right) L\left(1 + \left|X_s^{t_n^N, Y_n^N}\right|\right)\right\|_{L^2}^2 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \frac{C^4}{N^2} + \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} LR \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|X_r^{t_n^N, Y_n^N}\right|^R\right)^2\right\|_{L^2}^2 dr\right)^{\frac{1}{2}} ds \\
&+ \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} LR \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|X_s^{t_n^N, Y_n^N}\right|^R\right)^2\right\|_{L^2}^2 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Hence, we finally obtain

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq \frac{C^4}{N^2} + \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} LR \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|X_r^{t_n^N, Y_n^N}\right|^R\right)\right\|_{L^4}^4 dr\right)^{\frac{1}{2}} ds \\
&+ \left(\mathbb{P}\left[(\Omega_N)^c\right]\right)^{\frac{1}{2}} LR \left(\int_{t_n^N}^{t_{n+1}^N} \left\| \mathbb{1}_{\Omega_{N,n}} \left(2 + \left|X_s^{t_n^N, Y_n^N}\right|^R\right)\right\|_{L^4}^4 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}} u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N, Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right| \\
&\leq C^4 N^{-2} + C N^{-2} LR \int_{t_n^N}^{t_{n+1}^N} \left(\int_{t_n^N}^{s} C^4 \, dr\right)^{\frac{1}{2}} ds + C N^{-2} LR \left(\int_{t_n^N}^{t_{n+1}^N} C^4 \, ds\right)^{\frac{1}{2}} \\
&\leq C^4 N^{-2} + C N^{-2} LR \int_{t_n^N}^{t_{n+1}^N} \sqrt{T} C^2 ds + LRC^3\sqrt{T} N^{-2}
\end{aligned}
$$

and hence

$$\begin{aligned}&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N,Y_n^N}\Big)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_{N,n+1}}\Big\{u\big(t_{n+1}^N, Y_n^N\big) + \frac{T}{N}\tilde{u}\big(t_{n+1}^N, Y_n^N\big)\Big\}\Big]\right|\\ &\leq C^4N^{-2} + LRC^3T\sqrt{T}N^{-2} + LRC^3\sqrt{T}N^{-2}\\ &\leq C^4N^{-2} + C^7N^{-2} + C^6N^{-2} \leq 3C^7N^{-2}\end{aligned} \tag{127}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Combining (124) and (127) yields

$$\left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}u\big(t_{n+1}^N, Y_{n+1}^N\big)\right] - \mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}u\Big(t_{n+1}^N, X_{t_{n+1}^N}^{t_n^N,Y_n^N}\Big)\right]\right| \leq 10C^9N^{-2} \tag{128}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$. Our interpretation of (128) is that the difference between the Euler approximation and the exact solution after a time of order $O(\frac{1}{N})$ is in a weak sense of order $O(\frac{1}{N^2})$. Now we split up the interval $[0, T]$ into $N \in \mathbb{N}$ subintervals and sum up all differences which arise in the subintervals. Rewriting the weak difference between the exact solution and the Euler approximation by a telescope sum yields

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f\big(Y_N^N\big)\Big]\\ &= \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{0,Y_0^N}\right)\right] - \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{T,Y_N^N}\right)\right]\\ &= \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{t_0^N,Y_0^N}\right)\right] - \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{t_1^N,Y_1^N}\right)\right]\\ &\quad + \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{t_1^N,Y_1^N}\right)\right] - \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{t_N^N,Y_N^N}\right)\right]\end{aligned} \tag{129}$$

and hence

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f\big(Y_N^N\big)\Big]\\ &= \sum_{n=0}^{N-1}\left(\mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{t_n^N,Y_n^N}\right)\right] - \mathbb{E}\left[\mathbb{1}_{\Omega_N} \cdot f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right]\right)\end{aligned} \tag{130}$$

for every $N \in \mathbb{N}$. Since $\Omega_N \subset \Omega_{N,n+1}$, we have

$$\mathbb{1}_{\Omega_N} = \mathbb{1}_{\Omega_{N,n+1}} - \mathbb{1}_{(\Omega_{N,n+1}\setminus\Omega_N)} \tag{131}$$

for every $n \in \{0, 1, \ldots, N-1\}$ and every $N \in \mathbb{N}$ and therefore

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f\big(Y_N^N\big)\Big]\\ &= \sum_{n=0}^{N-1}\mathbb{E}\left[\mathbb{1}_{\Omega_N}\left\{f\left(X_T^{t_n^N,Y_n^N}\right) - f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right\}\right]\\ &= \sum_{n=0}^{N-1}\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\left\{f\left(X_T^{t_n^N,Y_n^N}\right) - f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right\}\right]\\ &\quad - \sum_{n=0}^{N-1}\mathbb{E}\left[\mathbb{1}_{(\Omega_{N,n+1}\setminus\Omega_N)}\left\{f\left(X_T^{t_n^N,Y_n^N}\right) - f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right\}\right]\end{aligned} \tag{132}$$

and hence

$$\begin{aligned}&\left|\mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f(X_T)\Big] - \mathbb{E}\Big[\mathbb{1}_{\Omega_N} \cdot f\big(Y_N^N\big)\Big]\right|\\ &\leq \sum_{n=0}^{N-1}\left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\left\{f\left(X_T^{t_n^N,Y_n^N}\right) - f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right\}\right]\right|\\ &\quad + \sum_{n=0}^{N-1}\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\mathbb{1}_{(\Omega_N)^c}\left|f\left(X_T^{t_n^N,Y_n^N}\right) - f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right|\right]\end{aligned} \tag{133}$$

and

$$
\begin{aligned}
&\left|\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(X_T\right)\right]-\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(Y_N^N\right)\right]\right| \\
&\leq \sum_{n=0}^{N-1}\left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\mathbb{E}\left[f\left(X_T^{t_n^N,Y_n^N}\right)-f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\,\middle|\,\mathcal{F}_{t_{n+1}^N}\right]\right]\right| \\
&\quad+\sum_{n=0}^{N-1}\left(\mathbb{P}\left[\left(\Omega_N\right)^c\right]\right)^{\frac{1}{2}}\left\|\mathbb{1}_{\Omega_{N,n+1}}\left|f\left(X_T^{t_n^N,Y_n^N}\right)-f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right|\right\|_{L^2}
\end{aligned}
\tag{134}
$$

for every $N\in\mathbb{N}$. Therefore, we obtain

$$
\begin{aligned}
&\left|\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(X_T\right)\right]-\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(Y_N^N\right)\right]\right| \\
&\leq \sum_{n=0}^{N-1}\left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\mathbb{E}\left[f\left(X_T^{t_{n+1}^N,X_{t_{n+1}^N}^{t_n^N,Y_n^N}}\right)-f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\,\middle|\,\mathcal{F}_{t_{n+1}^N}\right]\right]\right| \\
&+\sum_{n=0}^{N-1}CN^{-2}\left(\left\|\mathbb{1}_{\Omega_{N,n+1}}\cdot f\left(X_T^{t_n^N,Y_n^N}\right)\right\|_{L^2}+\left\|\mathbb{1}_{\Omega_{N,n+1}}\cdot f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right\|_{L^2}\right)
\end{aligned}
$$

and, using the Markov property and inequality (128),

$$
\begin{aligned}
&\left|\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(X_T\right)\right]-\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(Y_N^N\right)\right]\right| \\
&\leq \sum_{n=0}^{N-1}\left|\mathbb{E}\left[\mathbb{1}_{\Omega_{N,n+1}}\left(u\left(t_{n+1}^N,X_{t_{n+1}^N}^{t_n^N,Y_n^N}\right)-u\left(t_{n+1}^N,Y_{n+1}^N\right)\right)\right]\right| \\
&+\sum_{n=0}^{N-1}CN^{-2}\left(\left\|\mathbb{1}_{\Omega_{N,n+1}}\cdot f\left(X_T^{t_n^N,Y_n^N}\right)\right\|_{L^2}+\left\|\mathbb{1}_{\Omega_{N,n+1}}\cdot f\left(X_T^{t_{n+1}^N,Y_{n+1}^N}\right)\right\|_{L^2}\right) \\
&\leq \sum_{n=0}^{N-1}10C^9N^{-2}+\sum_{n=0}^{N-1}CN^{-2}\left(\left\|\mathbb{1}_{\Omega_{N,n+1}}\,L\left(1+\left|X_T^{t_n^N,Y_n^N}\right|^{\delta}\right)\right\|_{L^2}\right. \\
&\left.+\left\|\mathbb{1}_{\Omega_{N,n+1}}\,L\left(1+\left|X_T^{t_{n+1}^N,Y_{n+1}^N}\right|^{\delta}\right)\right\|_{L^2}\right)
\end{aligned}
$$

for every $N\in\mathbb{N}$. Finally, using $\Omega_{N,n+1}\subseteq\Omega_{N,n}$ for all $n\in\{0,1,\ldots,N-1\}$, $N\in\mathbb{N}$ and $|x|^{\delta}\leq 1+|x|^R$ for all $x\in\mathbb{R}$, we arrive at

$$
\begin{aligned}
&\left|\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(X_T\right)\right]-\mathbb{E}\left[\mathbb{1}_{\Omega_N}\cdot f\left(Y_N^N\right)\right]\right|\leq 10C^9N^{-1}+LCN^{-2} \\
&\cdot\left\{\sum_{n=0}^{N-1}\left\|\mathbb{1}_{\Omega_{N,n}}\left(2+\left|X_T^{t_n^N,Y_n^N}\right|^R\right)\right\|_{L^2}+\left\|\mathbb{1}_{\Omega_{N,n+1}}\left(2+\left|X_T^{t_{n+1}^N,Y_{n+1}^N}\right|^R\right)\right\|_{L^2}\right\} \\
&\leq 10C^9N^{-1}+2LC^2N^{-1}\leq 12C^9N^{-1}
\end{aligned}
$$

for every $N\in\mathbb{N}$ due to (112). This proves the assertion of Lemma 4.6. □

Acknowledgement

This work has been partially supported by the Collaborative Research Centre 701 “Spectral Structures and Topological Methods in Mathematics” and by the research project “Numerical solutions of stochastic differential equations with non-globally Lipschitz continuous coefficients” both funded by the German Research Foundation. Additionally, the authors thank three anonymous referees for their very helpful comments.

References

[1] *V. Bally and D. Talay*, The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function, *Probab. Theory Related Fields 104* (1996), no. 1, 43–60.

[2] *D. Duffie and P. Glynn*, Efficient Monte Carlo simulation of security prices, *Ann. Appl. Probab. 5* (1995), no. 4, 897–905.

[3] *Ĭ. Ī. Gīhman and A. V. Skorohod*, Stochastic differential equations, *Springer-Verlag*, New York-Heidelberg, 1972. Translated from the Russian by Kenneth Wickwire, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 72.

[4] *P. Glasserman*, Monte Carlo methods in financial engineering, *Springer-Verlag*, New York, 2004.

[5] *D. J. Higham*, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev. 43* (2001), no. 3, 525–546 (electronic).

[6] *D. J. Higham, X. Mao and A. M. Stuart* Strong convergence of Euler-type methods for nonlinear stochastic differential equations, *SIAM J. Numer. Anal. 40* (2002), no. 3, 1041–1063 (electronic).

[7] *M. Hutzenthaler, A. Jentzen and P. E. Kloeden*, Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467* (2011), no. 2130, 1563–1576.

[8] *A. Klenke*, Probability theory. *Universitext, Springer-Verlag London Ltd.*, London, 2008. A comprehensive course. Translated from the 2006 German original.

[9] *P. E. Kloeden*, The systematic derivation of higher order numerical schemes for stochastic differential equations, *Milan J. Math. 70* (2002), 187–207.

[10] *P. E. Kloeden and A. Neuenkirch*, The pathwise convergence of approximation schemes for stochastic differential equations, *LMS J. Comput. Math. 10* (2007), 235–253.

[11] *P. E. Kloeden and E. Platen*, Numerical solution of stochastic differential equations, *Springer-Verlag*, Berlin, 1992.

[12] *P. E. Kloeden, E. Platen and H. Schurz*, Numerical solution of SDE through computer experiments, *Universitext, Springer-Verlag*, Berlin, 1994.

[13] *J. C. Mattingly, A. M. Stuart and D. J. Higham*, Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise, *Stochastic Process. Appl. 101* (2002), no. 2, 185–232.

[14] *G. N. Milstein*, A method with second order accuracy for the integration of stochastic differential equations, *Teor. Verojatnost. i Primenen. 23* (1978), no. 2, 414–419.

[15] *G. N. Milstein*, Numerical integration of stochastic differential equations, *Kluwer Academic Publishers Group*, Dordrecht, 1995.

[16] *G. N. Milstein and M. V. Tretyakov*, Numerical integration of stochastic differential equations with nonglobally Lipschitz coefficients, *SIAM J. Numer. Anal. 43* (2005), no. 3, 1139–1154 (electronic).

[17] *A. Rößler*, Runge-Kutta Methods for the Numerical Solution of Stochastic Differential Equations, *Shaker Verlag*, Aachen, 2003.

[18] *D. W. Stroock*, Probability theory, an analytic view, *Cambridge University Press*, Cambridge, 1993.

[19] *L. Szpruch*, Numerical Approximations of Nonlinear Stochastic Systems, *University of Strathclyde*, Glasgow, PhD thesis, 2010.

[20] *D. Talay*, Probabilistic numerical methods for partial differential equations: elements of analysis, in Probabilistic models for nonlinear partial differential equations, *Springer*, Berlin, 1996, pp. 148–196.

[21] *L. Yan*, The Euler scheme with irregular coefficients, *Ann. Probab. 30* (2002), no. 3, 1172–1194.