Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients

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The stochastic Euler scheme is known to converge to the exact solution of a stochastic differential equation with globally Lipschitz continuous drift and diffusion coefficient. Recent results extend this convergence to coefficients which grow at most linearly. For superlinearly growing coefficients finite-time convergence in the strong mean square sense remained an open question according to [Higham, Mao & Stuart (2002); Strong convergence of Euler-type methods for nonlinear stochastic differential equations, *SIAM J. Numer. Anal.* **40**, no. 3, 1041-1063]. In this article we answer this question to the negative and prove for a large class of stochastic differential equations with non-globally Lipschitz continuous coefficients that Euler's approximation converges neither in the strong mean square sense nor in the numerically weak sense to the exact solution at a finite time point. Even worse, the difference of the exact solution and of the numerical approximation at a finite time point diverges to infinity in the strong mean square sense and in the numerically weak sense.

Keywords: Euler scheme, Euler-Maruyama, stochastic differential equations, weak approximation, weak divergence, strong approximation, strong divergence, non-globally Lipschitz continuous

1. Introduction

An important numerical scheme for simulating stochastic differential equations (SDEs) is Euler's method (see, e.g., Kloeden & Platen 1992, Milstein 1995 and Higham 2001). If the coefficients of an SDE are globally Lipschitz continuous, then standard results (see, for instance, Chapter 10 and Chapter 14 in Kloeden & Platen 1992) show convergence of the Euler approximation in the strong and numerically weak sense to the exact solution of the SDE. It remained an open question whether the Euler approximation also converges in the strong or numerically weak sense at a finite time point if the coefficients of the SDE are not globally Lipschitz continuous, see, e.g., Section 1 in Higham, Mao & Stuart (2002) for a detailed description of this open problem. In this paper we answer this question to

the negative. More precisely, we prove for a large class of SDEs with superlinearly growing coefficient functions that both the distance in the strong L^p-sense and the distance between the p-th absolute moments of the Euler approximation and of the exact solution of the SDE diverge to infinity for all $p \in [1, \infty)$. Thus the Euler scheme does not produce an approximation in the strong or numerically weak sense of the exact solution of such an SDE.

For clarity of exposition we concentrate in this section on the following prominent example Let $(X_t)_{t \geq 0}$ be the unique solution process of the one-dimensional SDE

$$dX_t = -X_t^3 \, dt + dW_t, \qquad X_0 = x_0 \tag{1.1}$$

for $t \geq 0$ where $(W_t)_{t \geq 0}$ is a one-dimensional standard Brownian motion and where $x_0 \in \mathbb{R}$ is a constant. The Euler approximation $(Y_n^N)_{n \in \{0, 1, \dots, N\}}$ of $(X_t)_{t \in [0, T]}$, $T \in (0, \infty)$ fixed, is defined recursively through $Y_0^N := x_0$ and

$$Y_{n+1}^N := Y_n^N - \frac{T}{N} \left(Y_n^N \right)^3 + \left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}} \right) \tag{1.2}$$

for all $n \in \{0, 1, 2, \dots\}$ and all $N \in \mathbb{N} := \{1, 2, \dots\}$.

Two results motivated us to try to prove convergence of the stochastic Euler approximation Y_N^N to the exact solution X_T of the SDE (1.1) in the strong mean square sense as the number of time steps N goes to infinity. Gyöngy (1998) established pathwise convergence for SDEs with locally Lipschitz continuous coefficients. More precisely, Theorem 1 in Gyöngy (1998) implies

$$\lim_{N \to \infty} \left| X_T - Y_N^N \right| = 0 \tag{1.3}$$

$\mathbb{P}$-a.s. pathwise convergence of the stochastic Euler approximation (1.2) to the exact solution of the SDE (1.1). This implies convergence of expectations of continuous and bounded functionals of the difference between the Euler approximation and the exact solution. Of course, the squared difference needed for mean square convergence is not a bounded but an unbounded functional. Another motivation was an instructive conditional result of Higham, Mao & Stuart (2002). They assume in their Theorem 2.2 local Lipschitz continuity of the coefficients of the SDE and boundedness of the p-th moment of the Euler approximation and of the exact solution in the sense that

$$\sup_{N \in \mathbb{N}} \mathbb{E} \left[\sup_{n \in \{0, 1, \dots, N\}} \left| Y_n^N \right|^p \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^p \right] < \infty \tag{1.4}$$

for one arbitrary $p \in (2, \infty)$. Under these assumptions they establish in Theorem 2.2 in Higham, Mao & Stuart (2002) strong mean square convergence of Euler's method to the exact solution of the SDE. Under assumption (1.4) they, in particular, establish strong mean square convergence

$$\lim_{N \to \infty} \mathbb{E} \left[\left| X_T - Y_N^N \right|^2 \right] = 0 \tag{1.5}$$

of the Euler approximation (1.2) to the exact solution of the SDE (1.1). Boundedness of moments (1.4) of the Euler approximation, however, remained an open

question. Higham, Mao & Stuart (2002) say on page 1060 that in "general, it is not clear when such moment bounds can be expected to hold for explicit methods with $f, g \in C^1$" (f and g denote in Higham, Mao & Stuart (2002) the drift function and the diffusion function respectively).

In this article we answer Higham, Mao & Stuart's question in the case of the explicit Euler method and superlinearly growing coefficients of the SDE to the negative (see Theorem 1 below for details) and establish strong L^p-divergence

$$\lim_{N \to \infty} \mathbb{E}\left[\left| X_T - Y_N^N \right|^p \right] = \infty \tag{1.6}$$

of the stochastic Euler appoximation (1.2) in finite time $T \in (0, \infty)$ for all $p \in [1, \infty)$ (see (2.6) for details). In addition numerically weak convergence (see, e.g., Section 9.4 in Kloeden & Platen 1992) (not to be confused with stochastic weak convergence) fails to hold, even more,

$$\lim_{N \to \infty} \left| \mathbb{E}\left[|X_T|^p \right] - \mathbb{E}\left[\left| Y_N^N \right|^p \right] \right| = \infty \tag{1.7}$$

for all $p \in [1, \infty)$, see (2.6). In particular, Theorem 1 implies that the absolute moments of the Euler approximation (1.2) at a finite time point diverge to infinity, that is,

$$\lim_{N \to \infty} \mathbb{E}\left[\left| Y_N^N \right|^p \right] = \infty \tag{1.8}$$

for all $p \in [1, \infty)$. Thus the moment bound assumption in Theorem 2.2 in Higham, Mao & Stuart (2002) (see also inequality (1.4) here) is not satisfied for the SDE (1.1).

Note that the strong divergence (1.6) and the weak divergence (1.7) of the Euler approximation is not a special property of equation (1.1). We establish this divergence for a large class of SDEs with superlinearly growing coefficients in Section 2. Moreover, our estimates are easily adapted to prove divergence of other numerical schemes such as the Milstein scheme. For simplicity we restrict ourselves to the Euler scheme. Presence of noise, however, is essential. In the deterministic case, the Euler scheme does converge and both (1.6) and (1.7) fail to hold.

Next we relate the divergence result (1.8) regarding finite time intervals to a divergence result of Mattingly, Stuart & Higham (2002) regarding infinite time intervals (see also Theorem 3.2 in Roberts & Tweedie 1996 and Lemma 4.1 in Higham, Mao & Stuart 2003). Their Lemma 6.3 shows for the SDE (1.1) that the second moment of the Euler approximation diverges in infinite time for any fixed discretization step size, that is,

$$\lim_{n \to \infty} \mathbb{E}\left[\left| Y_n^N \right|^2 \right] = \infty \tag{1.9}$$

for every fixed $N \in \mathbb{N}$. Note that this divergence result is rather different to our divergence result (1.8). First the time discretisation step size does not converge to zero in (1.9). Secondly the Euler approximation diverges even pathwise with positive probability on an infinite time interval. More precisely, Lemma 6.3 in Mattingly, Stuart & Higham (2002) proves

$$\mathbb{P}\left[\left| Y_n^N \right| \geq 2^n \sqrt{N} \ \ \forall n \in \mathbb{N} \right] > 0 \tag{1.10}$$

for every fixed $N \in \mathbb{N}$. The divergence (1.9) is then an immediate consequence of this pathwise divergence result. On a finite time interval the Euler approximation does converge pathwise to the exact solution according to Gyöngy (1998) (see (1.3) here). So here the divergence (1.8) of the moments is not a consequence of the pathwise behavior.

In the next step we compare the divergence results (1.6)-(1.8) in this article with a few further related results in the literature. It is a classical result (see, e.g., Kloeden & Platen 1992 and Milstein 1995) that the Euler approximation converges in case of globally Lipschitz continuous drift and diffusion coefficient functions. In contrast to our results on superlinearly growing coefficients, Yan (2002) proves numerically weak convergence of the Euler scheme if both drift and diffusion function have at most linear growth. Zhang (2006) and Berkaoui, Bossy and Diop (2008) prove strong convergence for a class of drift and diffusion functions with a singularity. Yuan & Mao (2008) obtain the rate of strong L^2-convergence of the stochastic Euler scheme for locally Lipschitz continuous coefficients if these grow at most linearly. An explicit bound for the strong L^1-error is established in Higham & Mao (2005) for the mean-reverting square-root process (a.k.a. CIR process) which has linearly bounded coefficients. Bernard & Fleury (2001) establish convergence in probability under weak hypotheses like local Lipschitz continuity. Pathwise convergence results can be found in Gyöngy (1998), Fleury (2005), Jentzen & Kloeden (2009) or in Jentzen, Kloeden & Neuenkirch (2009). A number of authors obtain convergence for modified Euler schemes. Milstein & Tretyakov (2005) consider a modified Euler scheme for non-globally Lipschitz continuous coefficients. They obtain numerically weak convergence by discarding trajectories of the approximation which leave a sufficiently large sphere. Lamba, Mattingly & Stuart (2007) prove strong convergence of an Euler scheme with an adaptive time-stepping algorithm for locally Lipschitz continuous coefficients. A completely different adaptive time-stepping algorithm with focus on long time approximation is proposed by Lemaire (2007). Finally, note that in contrast to the explicit Euler method (1.2), the implicit Euler method converges in the root mean square sense according to Higham, Mao & Stuart (2002) (see also Hu 1996, Talay 2002, Szpruch & Mao 2010 and the references therein for more convergence results on implicit numerical methods for SDEs).

The rest of this article is organized as follows. Our main result (Theorem 1) and several examples for the divergence of the Euler scheme are provided in Section 2. Simulations in Section 3 illustrate this divergence. The proof of Theorem 1 is postponed to the final section.

2. Main result and examples

Throughout this section assume that the following setting is fulfilled. Fix $T \in (0, \infty)$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with normal filtration $(\mathcal{F}_t)_{t \in [0,T]}$. Additionally, let $W \colon [0,T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion and let $\xi \colon \Omega \to \mathbb{R}$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mapping. Moreover, let $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$ be two $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable functions such that the SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad X_0 = \xi \qquad (2.1)$$

for $t \in [0, T]$ has a solution. More precisely, we assume the existence of a predictable stochastic process $X \colon [0, T] \times \Omega \to \mathbb{R}$ satisfying

$$\mathbb{P}\left[\int_0^T |\mu(X_s)| + |\sigma(X_s)|^2 \, ds < \infty \right] = 1 \tag{2.2}$$

and

$$\mathbb{P}\left[X_t = \xi + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s \right] = 1 \tag{2.3}$$

for all $t \in [0, T]$. The drift function $\mu(\cdot)$ is the infinitesimal mean of the process X and the diffusion function $\sigma(\cdot)$ is the infinitesimal standard deviation of the process X. The Euler approximation for the SDE (2.1) – denoted by $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings $Y_n^N \colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, – is given by $Y_0^N(\omega) := \xi(\omega)$ and

$$Y_{n+1}^N(\omega) := Y_n^N(\omega) + \frac{T}{N} \cdot \mu\left(Y_n^N(\omega)\right) + \sigma\left(Y_n^N(\omega)\right) \cdot \left(W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega)\right) \tag{2.4}$$

for all $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ and all $\omega \in \Omega$. Now we formalize the main result of this article which asserts the divergence of Euler's method for the SDE (2.1) if at least one coefficient grows superlinearly.

Theorem 1. *Assume that the setting above is fulfilled with $\mathbb{P}\left[\sigma(\xi) \neq 0\right] > 0$ and let $C \geq 1$, $\beta > \alpha > 1$ be constants such that*

$$\max\left(|\mu(x)|, |\sigma(x)|\right) \geq \frac{|x|^\beta}{C} \quad and \quad \min\left(|\mu(x)|, |\sigma(x)|\right) \leq C|x|^\alpha \tag{2.5}$$

for all $|x| \geq C$. Then there exists a constant $c \in (1, \infty)$ and a sequence of nonempty events $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, with $\mathbb{P}[\Omega_N] \geq c^{(-N^c)}$ and $\left|Y_N^N(\omega)\right| \geq 2^{\left(\alpha^{(N-1)}\right)}$ for all $\omega \in \Omega_N$ and all $N \in \mathbb{N}$. Moreover, if the exact solution $X \colon [0, T] \times \Omega \to \mathbb{R}$ of the SDE (2.1) satisfies $\mathbb{E}\left[|X_T|^p\right] < \infty$ for one $p \in [1, \infty)$, then

$$\lim_{N \to \infty} \mathbb{E}\left[\left|X_T - Y_N^N\right|^p \right] = \infty \quad and \quad \lim_{N \to \infty} \left| \mathbb{E}\left[\left|X_T\right|^p \right] - \mathbb{E}\left[\left|Y_N^N\right|^p \right] \right| = \infty. \tag{2.6}$$

Roughly speaking, the theorem asserts that in the presence of noise there exists a sequence of events of at least exponentially small probability on which the Euler approximations grow at least double-exponentially fast. Consequently as being **double-exponentially large** over-compensates that the events have at least **exponentially small** probability, the L^1-norm of the Euler approximations Y_N^N are unbounded in $N \in \mathbb{N}$. The proof of the double exponential growth of the Euler approximations is based on elementary calculations. For details the reader is referred to Section 4 where the proof of Theorem 1 can be found.

Condition (2.5) should be read as follows. Either the drift function grows in a higher polynomial order than linearly and the diffusion function function grows slower than that or the diffusion function grows in a higher polynomial order than linearly and the drift function grows slower than that. More formally it suffices to show that either

$$|\mu(x)| \geq \frac{|x|^\beta}{C}, \qquad |\sigma(x)| \leq C \, |x|^\alpha \tag{2.7}$$

for all $|x| \geq C$ or

$$|\sigma(x)| \geq \frac{|x|^\beta}{C}, \qquad |\mu(x)| \leq C\,|x|^\alpha \qquad (2.8)$$

for all $|x| \geq C$ and some constants $\beta > 1$, $\beta > \alpha \geq 0$, $C > 0$. Note that our estimates need $\beta > 1$. A drift function of the form $\mu(x) = x \log|x|$ for all $x \in \mathbb{R}$ is too small for our estimates. The assumption that the diffusion function does not vanish on the starting point ensures the presence of noise in the first time step.

In the remainder of this section we apply Theorem 1 to a selection of examples. Note that the coefficients in the following examples satisfy an appropriate one-sided linear growth condition. Therefore, the p-th absolute moment of the exact solution is finite in each example for every $p \in [1, \infty)$ according to Theorem 2.4.1 in Mao (1997). For all examples we check that assumption (2.7) is satisfied. Thus both the distance in the strong L^p-sense and the distance between the p-th absolute moments of the Euler approximation and of the exact solution diverges to infinity for every $p \in [1, \infty)$ in each of the following examples.

(a) *The introductory example*

The example in Section 1 is

$$dX_t = -X_t^3\,dt + dW_t, \qquad X_0 = x_0 \in \mathbb{R} \qquad (2.9)$$

for $t \in [0, T]$. The dominating coefficient is the drift function with dominating exponent $\beta = 3$. The diffusion function has exponent zero and we may choose $\alpha = 0$. The constant C in condition (2.7) can be chosen as $C = 1$.

(b) *Stochastic Ginzburg-Landau equation*

The Ginzburg-Landau equation is from the theory of superconductivity. It has been introduced by Ginzburg & Landau (1950) to describe a phase transition. Its stochastic version with multiplicative noise can be written as

$$dX_t = \left(\left(\eta + \frac{1}{2}\bar{\sigma}^2 \right) X_t - \lambda X_t^3 \right) dt + \bar{\sigma} X_t\,dW_t, \qquad X_0 = x_0 \in (0, \infty) \qquad (2.10)$$

for $t \in [0, T]$ where $\eta \geq 0$, $\lambda, \bar{\sigma} > 0$. Its solution is known explicitly (see, e.g., Section 4.4 in Kloeden & Platen 1992)

$$X_t = \frac{x_0 \exp\left(\eta t + \bar{\sigma} W_t \right)}{\sqrt{1 + 2x_0^2 \lambda \int_0^t \exp\left(2\eta s + 2\bar{\sigma} W_s \right) ds}} \qquad (2.11)$$

for $t \in [0, T]$. Since

$$
\begin{aligned}
\left| \left(\eta + \frac{\bar{\sigma}^2}{2} \right) x - \lambda x^3 \right| &\geq \lambda\,|x|^3 - \left(\eta + \frac{\bar{\sigma}^2}{2} \right) |x| \\
&\geq \frac{\lambda}{2}\,|x|^3 + \frac{\lambda}{2}\,|x|^3 - \left(\eta + \frac{\bar{\sigma}^2}{2} \right) |x| \\
&= \frac{\lambda}{2}\,|x|^3 + \frac{\lambda}{2}\,|x| \left(|x|^2 - \frac{2\eta + \bar{\sigma}^2}{\lambda} \right) \\
&\geq \frac{\lambda}{2}\,|x|^3 + \frac{\lambda}{2}\,|x| \left(|x| - \frac{2\eta + \bar{\sigma}^2}{\lambda} \right) \geq \frac{\lambda}{2}\,|x|^3
\end{aligned}
\qquad (2.12)
$$

for all $|x| \geq \max\left(1, \frac{2\eta + \bar{\sigma}^2}{\lambda}\right)$, the constants in condition (2.7) can be chosen as $\beta = 3$, $\alpha = 1$, $C = \max\left(1, \bar{\sigma}, \frac{2}{\lambda}, \frac{2\eta + \bar{\sigma}^2}{\lambda}\right)$.

(c) Stochastic Verhulst equation

The Verhulst equation is an ordinary differential equation and is a simple model for a population with competition between individuals. Its stochastic version with multiplicative noise can be written as

$$dX_t = \left(\left(\eta + \frac{1}{2}\bar{\sigma}^2\right) X_t - \lambda X_t^2\right) dt + \bar{\sigma} X_t \, dW_t, \qquad X_0 = x_0 \in (0, \infty) \qquad (2.13)$$

for $t \in [0, T]$ where $\eta, \lambda, \bar{\sigma} > 0$. Its solution is known explicitly (see, e.g., Section 4.4 in Kloeden & Platen 1992) and is given by

$$X_t = \frac{x_0 \exp\left(\eta t + \bar{\sigma} W_t\right)}{1 + x_0 \lambda \int_0^t \exp\left(\eta s + \bar{\sigma} W_s\right) ds} \qquad (2.14)$$

for $t \in [0, T]$. The dominant exponent of the drift function is two and of the diffusion function is one. Thus we may choose $\beta = 2$ and $\alpha = 1$. The constant C in condition (2.7) can be chosen as $C = \max\left(\bar{\sigma}, \frac{2}{\lambda}, \frac{2\eta + \bar{\sigma}^2}{\lambda}\right)$.

(d) Feller diffusion with logistic growth

The branching process with logistic growth (see, e.g., Lambert 2005) is a stochastic Verhulst equation with Feller noise. It solves

$$dX_t = \lambda X_t \left(K - X_t\right) dt + \bar{\sigma} \sqrt{X_t} \, dW_t, \qquad X_0 = x_0 \in (0, \infty) \qquad (2.15)$$

for $t \in [0, T]$ where $\lambda, K, \bar{\sigma} > 0$. There is no explicit solution for this equation. However, it features the following self-duality

$$\mathbb{E}^x \exp\left(-\frac{2\lambda}{\bar{\sigma}^2} X_t y\right) = \mathbb{E}^y \exp\left(-\frac{2\lambda}{\bar{\sigma}^2} x X_t\right) \qquad (2.16)$$

for all $t \in [0, T]$ and all $x, y \in [0, \infty)$ where $\mathbb{E}^x$ refers to the starting point $X_0 = x$, see Hutzenthaler & Wakolbinger (2007). As constants for condition (2.7) serve $\beta = 2$, $\alpha = 1$, $C = \max(1, \frac{2}{\lambda}, \bar{\sigma}, 2K)$.

(e) Protein Kinetics

The proportion x of one form of a certain protein can be modelled by an ordinary differential equation whose appropriate stochastic version is given by

$$dX_t = \left(\eta - X_t + \lambda X_t \left(1 - X_t\right) + \frac{1}{2}\bar{\sigma}^2 X_t \left(1 - X_t\right) \left(1 - 2X_t\right)\right) dt$$
$$+ \bar{\sigma} X_t \left(1 - X_t\right) dW_t$$

for $t \in [0, T]$ where $X_0 = x_0 \in (0, 1)$, $\eta \in [0, 1]$ and $\lambda, \bar{\sigma} > 0$, see Section 7.1 in Kloeden & Platen (1992). Here Theorem 1 applies with $\beta = 3$, $\alpha = 2$ and C sufficiently large.

	$\bar{\sigma} = 2$	$\bar{\sigma} = 4$	$\bar{\sigma} = 5$	$\bar{\sigma} = 6$	$\bar{\sigma} = 7$
Exact solution $\mathbb{E}\left[(X_3)^2\right]$	0.4739	0.9598	1.2357	1.5423	1.8861
Simulation 1	0.4577	0.7448	0.7243	NaN	NaN
Simulation 2	0.4549	0.7518	0.7094	0.5097	NaN
Simulation 3	0.4632	0.7330	0.6912	NaN	NaN
Simulation 4	0.4580	0.7544	0.7150	0.5378	NaN
Simulation 5	0.4619	0.7458	0.6998	0.5197	NaN
Simulation 6	0.4610	0.7664	0.7327	0.5243	NaN
Simulation 7	0.4606	0.7542	0.7189	NaN	NaN
Simulation 8	0.4690	0.7451	0.7122	NaN	NaN
Simulation 9	0.4667	0.7336	0.7146	0.5475	NaN
Simulation 10	0.4591	0.7605	0.6659	NaN	NaN

Table 1. *Simulation of $\mathbb{E}\left[(X_3)^2\right]$ and $\mathbb{E}\left[(Y_N^N)^2\right]$ for the SDE (3.1) where the number of time steps is $N = 10^3$. The number of Monte Carlo runs for the Euler approximation is 10^5 and for the exact solution is 10^7.*

3. Simulations

In this section we present two numerical simulations which illustrate the divergence of Euler's method as formulated in Theorem 1. For this purpose we choose an equation with an explicit solution to compare with. Consider the stochastic differential equation

$$dX_t = -\left(\frac{1}{2}\bar{\sigma}^2 X_t - X_t^3\right) dt + \bar{\sigma} X_t \, dW_t, \qquad X_0 = 1 \tag{3.1}$$

for $t \in [0, 3]$ where $\bar{\sigma} > 0$ is a constant. This equation agrees with the stochastic Ginzburg-Landau equation (2.10) in the case $\eta = 0, \lambda = 1, x_0 = 1$. Its explicit solution is given by equation (2.11) with $\eta = 0, \lambda = 1, x_0 = 1$.

Table 1 shows Monte Carlo simulations of the second moment both of the exact solution $\mathbb{E}\left[(X_3)^2\right]$ and of the Euler approximation $\mathbb{E}\left[(Y_N^N)^2\right]$ for five values of $\bar{\sigma}$. For each value of $\bar{\sigma}$ the table contains ten simulation runs for the Euler approximation. The number of time steps is $N = 10^3$. In the case $\bar{\sigma} = 7$ the parameters of the simulation are such that the Euler approximation produces the value "NaN" (NaN is the IEEE arithmetic representation for "not-a-number", here this is due to an operation "Inf − Inf" where Inf is the IEEE arithmetic representation for positive infinity). In contrast to this our Monte Carlo simulations in the cases $\bar{\sigma} \in \{2, 4, 5\}$ are all finite. In these cases the probability of the event on which the Euler approximation diverges seems to be rather small. The last but one column in Table 1 exemplifies a parameter setting in which the event of large growth has a probability such that in 10^5 runs in some Monte Carlo simulations no explosion occurs and in some Monte Carlo simulations at least one explosion occurs.

The values in Table 1 are either within distance two of the true value or "NaN". This is due to the double-exponential growth of the deterministic system for some initial values. If the simulation starts to grow, then it reaches "NaN" very quickly. We encountered similar behaviour for other exponents greater than two. In order to see double-exponential growth of the Euler approximation in a plot we consider an exponent close to one. More precisely, we plot the Monte Carlo simulation of the first absolute moment of the Euler approximation of X_{10} where
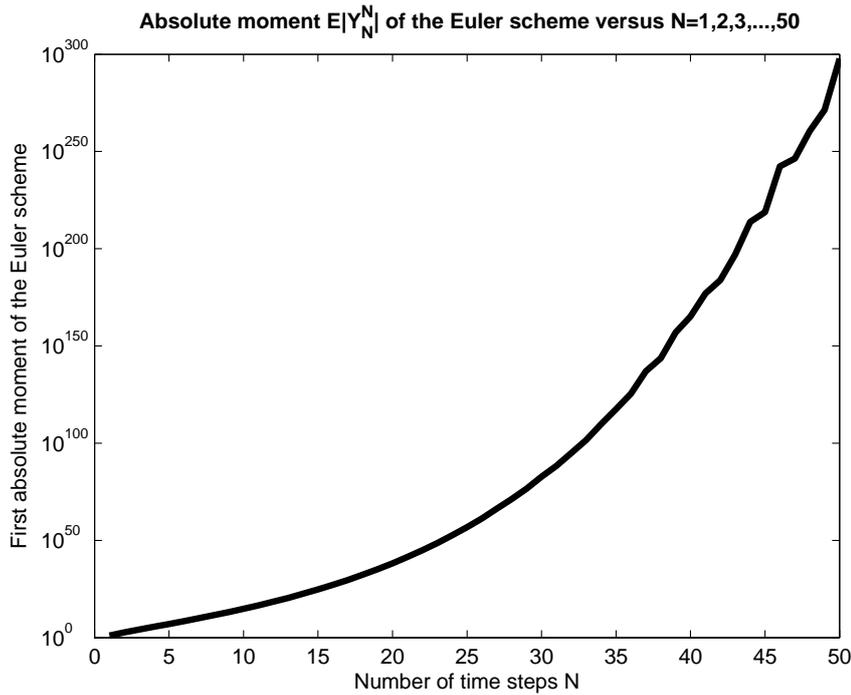
Figure 1. Simulation of the first absolute moment $\mathbb{E}\big[|Y_N^N|\big]$ of the Euler scheme for the SDE (3.2) and for number of time steps $N \in \{1, 2, 3, \ldots, 50\}$. The number of Monte Carlo runs is 10^4.

N	$\mathbb{E}\big[|Y_N^N|\big]$
$N = 1$	10.1160
$N = 2$	444.5797
$N = 3$	$1.3649 \cdot 10^4$
$N = 4$	$3.7566 \cdot 10^5$
$N = 5$	$1.0295 \cdot 10^7$
$\vdots$	$\vdots$

N	$\mathbb{E}\big[|Y_N^N|\big]$
$N = 48$	$3.4449 \cdot 10^{260}$
$N = 49$	$2.6631 \cdot 10^{271}$
$N = 50$	$6.1140 \cdot 10^{297}$
$N = 51$	$7.2342 \cdot 10^{301}$
$N = 52$	Inf
$N = 53$	Inf

Table 2. *Simulation values of the first absolute moment* $\mathbb{E}\big[|Y_N^N|\big]$ *for* $N \in \{1, 2, 3, \ldots, 53\}$ *and the SDE* (3.2).

$$dX_t = -10\,\mathrm{sgn}(X_t)\,|X_t|^{1.1}\,dt + 4\,dW_t, \qquad X_0 = 0 \qquad (3.2)$$

for $t \in [0, 10]$ see Figure 1 (see also the simulation values in Table 2). Note that the graph resembles an exponential function and that the y-axis is logarithmic. Thus the growth of the graph in Figure 1 is indeed close to a double-exponential function. In addition we should mention that some fine-tuning was necessary to obtain suitable parameters for which the simulated absolute moment grows but is not "NaN".

4. Proofs

Lemma 4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $Z \colon \Omega \to \mathbb{R}$ be an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mapping which is standard normal distributed. Then*

$$\mathbb{P}\Big[|Z| \geq x\Big] \geq \frac{x\,e^{-x^2}}{4}, \qquad \mathbb{P}\Big[|Z| \in [x, 2x]\Big] \geq \frac{x\,e^{-2x^2}}{2} \tag{4.1}$$

for all $x \in [0, \infty)$.

Proof of Lemma 4.1. We have

$$\begin{aligned}
\mathbb{P}\Big[|Z| \geq x\Big] = 2 \cdot \mathbb{P}\Big[Z \geq x\Big] &= 2 \left(\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds \right) \\
&\geq 2 \left(\int_x^{\sqrt{2}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds \right) \\
&\geq 2 \left(\sqrt{2} - 1 \right) x \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{2}x)^2}{2}} \right) \\
&= \left(\frac{2\left(\sqrt{2} - 1\right)}{\sqrt{2\pi}} \right) x e^{-x^2} \geq \frac{x e^{-x^2}}{4}
\end{aligned} \tag{4.2}$$

for all $x \in [0, \infty)$. Moreover, a similar estimate yields

$$\begin{aligned}
\mathbb{P}\Big[|Z| \in [x, 2x]\Big] = 2 \cdot \mathbb{P}\Big[Z \in [x, 2x]\Big] &= 2 \left(\int_x^{2x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds \right) \\
&\geq 2x \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(2x)^2}{2}} \right) = \left(\frac{2}{\sqrt{2\pi}} \right) x e^{-2x^2} \geq \frac{x\,e^{-2x^2}}{2}
\end{aligned} \tag{4.3}$$

for all $x \in [0, \infty)$. $\qquad \square$

Proof of Theorem 1. By assumption we have $\mathbb{P}[\,|\sigma(\xi)| > 0\,] > 0$. Therefore, there exists a real number $K \in (1, \infty)$, such that

$$\vartheta := \mathbb{P}\left[|\sigma(\xi)| \geq \frac{1}{K}, \ |\xi| + T\,|\mu(\xi)| \leq K \right] > 0. \tag{4.4}$$

Then we define

$$r_N := \max\left(2, C, \left(\frac{2CN}{T} + 2C^2 \right)^{\frac{1}{(\beta - \alpha)}} \right) \in [C, \infty) \tag{4.5}$$

and consider the sets $\Omega_N \in \mathcal{F}$ given by

$$\begin{aligned}
\Omega_N := \Bigg\{ \omega \in \Omega \ \Bigg| \ &\left| W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega) \right| \in \left[\frac{T}{N}, \frac{2T}{N} \right] \forall\, n \in \{1, \ldots, N-1\}, \\
&\left| W_{\frac{T}{N}}(\omega) - W_0(\omega) \right| \geq K\left(r_N + K \right), \\
&|\sigma(\xi(\omega))| \geq \frac{1}{K}, \ |\xi(\omega)| + T\,|\mu(\xi(\omega))| \leq K \Bigg\}
\end{aligned} \tag{4.6}$$

for all $N \in \mathbb{N}$. Note that

$$\frac{T}{NC} (r_N)^{(\beta-\alpha)} \geq \frac{T}{NC} \left(\frac{2CN}{T} + 2C^2 \right) \geq 2 + \frac{2TC}{N} \tag{4.7}$$

for all $N \in \mathbb{N}$ due to the definition of r_N, $N \in \mathbb{N}$.

Let now $N \in \mathbb{N}$ and $\omega \in \Omega_N$ be arbitrary. Then we claim

$$\left| Y_n^N(\omega) \right| \geq (r_N)^{\left(\alpha^{(n-1)} \right)} \tag{4.8}$$

for all $n \in \{1, 2, \ldots, N\}$. We prove equation (4.8) by induction on $n \in \{1, 2, \ldots, N\}$. In the base case $n = 1$ we have

$$
\begin{aligned}
\left| Y_1^N(\omega) \right| &= \left| \xi(\omega) + \frac{T}{N} \mu(\xi(\omega)) + \sigma(\xi(\omega)) \left(W_{\frac{T}{N}}(\omega) - W_0(\omega) \right) \right| \\
&\geq \left| \sigma(\xi(\omega)) \right| \cdot \left| W_{\frac{T}{N}}(\omega) - W_0(\omega) \right| - \left| \xi(\omega) \right| - \frac{T}{N} \left| \mu(\xi(\omega)) \right| \\
&\geq \left| \sigma(\xi(\omega)) \right| \cdot \left| W_{\frac{T}{N}}(\omega) - W_0(\omega) \right| - K \\
&\geq \frac{1}{K} \left| W_{\frac{T}{N}}(\omega) - W_0(\omega) \right| - K \geq \frac{K(r_N + K)}{K} - K = r_N
\end{aligned}
\tag{4.9}
$$

due to the definition (2.4) of Y_1^N and due to the definition (4.6) of Ω_N. For the induction step $n \to n + 1$ we assume that equation (4.8) holds for one $n \in \{1, 2, \ldots, N-1\}$. In particular, we then obtain

$$\left| Y_n^N(\omega) \right| \geq r_N \geq C \geq 1 \tag{4.10}$$

since $r_N \geq 2$. Additionally, we have

$$
\begin{aligned}
\left| Y_{n+1}^N(\omega) \right| &= \left| Y_n^N(\omega) + \frac{T}{N} \mu(Y_n^N(\omega)) + \sigma(Y_n^N(\omega)) \left(W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega) \right) \right| \\
&\geq \left| \frac{T}{N} \mu(Y_n^N(\omega)) + \sigma(Y_n^N(\omega)) \left(W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega) \right) \right| - \left| Y_n^N(\omega) \right| \\
&\geq \max \left(\frac{T}{N} \left| \mu(Y_n^N(\omega)) \right|, \left| \sigma(Y_n^N(\omega)) \right| \cdot \left| W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega) \right| \right) \\
&\quad - \min \left(\frac{T}{N} \left| \mu(Y_n^N(\omega)) \right|, \left| \sigma(Y_n^N(\omega)) \right| \cdot \left| W_{\frac{(n+1)T}{N}}(\omega) - W_{\frac{nT}{N}}(\omega) \right| \right) \\
&\quad - \left| Y_n^N(\omega) \right|
\end{aligned}
$$

and

$$
\begin{aligned}
\left| Y_{n+1}^N(\omega) \right| &\geq \max \left(\frac{T}{N} \left| \mu(Y_n^N(\omega)) \right|, \frac{T}{N} \left| \sigma(Y_n^N(\omega)) \right| \right) \\
&\quad - \min \left(\frac{T}{N} \left| \mu(Y_n^N(\omega)) \right|, \frac{2T}{N} \left| \sigma(Y_n^N(\omega)) \right| \right) - \left| Y_n^N(\omega) \right| \\
&\geq \frac{T}{N} \max \left(\left| \mu(Y_n^N(\omega)) \right|, \left| \sigma(Y_n^N(\omega)) \right| \right) \\
&\quad - \frac{2T}{N} \min \left(\left| \mu(Y_n^N(\omega)) \right|, \left| \sigma(Y_n^N(\omega)) \right| \right) - \left| Y_n^N(\omega) \right|
\end{aligned}
\tag{4.11}
$$

due to definition (2.4) and definition (4.6). Therefore, the growth condition (2.5) and inequality (4.10) imply

$$
\begin{aligned}
\left|Y_{n+1}^N(\omega)\right| &\geq \frac{T}{NC}\left|Y_n^N(\omega)\right|^\beta - \frac{2TC}{N}\left|Y_n^N(\omega)\right|^\alpha - \left|Y_n^N(\omega)\right| \\
&\geq \frac{T}{NC}\left|Y_n^N(\omega)\right|^\beta - \frac{2TC}{N}\left|Y_n^N(\omega)\right|^\alpha - \left|Y_n^N(\omega)\right|^\alpha \\
&= \left|Y_n^N(\omega)\right|^\alpha\left(\frac{T}{NC}\left|Y_n^N(\omega)\right|^{(\beta-\alpha)} - \frac{2TC}{N} - 1\right) \\
&\geq \left|Y_n^N(\omega)\right|^\alpha\left(\frac{T}{NC}(r_N)^{(\beta-\alpha)} - \frac{2TC}{N} - 1\right) \geq \left|Y_n^N(\omega)\right|^\alpha
\end{aligned}
\tag{4.12}
$$

where the last step follows from inequality (4.7). The induction hypothesis hence yields

$$
\left|Y_{n+1}^N(\omega)\right| \geq \left|Y_n^N(\omega)\right|^\alpha \geq \left((r_N)^{\left(\alpha^{(n-1)}\right)}\right)^\alpha = (r_N)^{(\alpha^n)}. \tag{4.13}
$$

This proves inequality (4.8) for $n+1$. Therefore, equation (4.8) indeed holds for all $n \in \{1, 2, \ldots, N\}$. In particular – since $N \in \mathbb{N}$ and $\omega \in \Omega_N$ were arbitrary – we obtain

$$
\left|Y_N^N(\omega)\right| \geq (r_N)^{\left(\alpha^{(N-1)}\right)} \geq 2^{\left(\alpha^{(N-1)}\right)} \tag{4.14}
$$

for all $\omega \in \Omega_N$ and all $N \in \mathbb{N}$.

Furthermore, definition (4.4) gives

$$
\begin{aligned}
\mathbb{P}\big[\Omega_N\big] &= \vartheta \cdot \mathbb{P}\bigg[\left|W_{\frac{T}{N}}\right| \geq K(r_N + K)\bigg] \cdot \left(\mathbb{P}\bigg[\left|W_{\frac{T}{N}}\right| \in \left[\frac{T}{N}, \frac{2T}{N}\right]\bigg]\right)^{(N-1)} \\
&\geq \vartheta \cdot \mathbb{P}\bigg[|W_1| \geq N^{\frac{1}{2}}T^{-\frac{1}{2}}K(r_N + K)\bigg] \cdot \left(\mathbb{P}\bigg[|W_1| \in \left[\frac{\sqrt{T}}{\sqrt{N}}, \frac{2\sqrt{T}}{\sqrt{N}}\right]\bigg]\right)^N
\end{aligned}
$$

for all $N \in \mathbb{N}$. Lemma 4.1 therefore yields

$$
\begin{aligned}
\mathbb{P}\big[\Omega_N\big] &\geq \frac{\vartheta\sqrt{N}K(r_N + K)}{4\sqrt{T}} \cdot \exp\left(-\frac{NK^2(r_N + K)^2}{T} - 2T\right) \cdot \left(\frac{\sqrt{T}}{2\sqrt{N}}\right)^N \\
&\geq \frac{\vartheta e^{-2T}T^{\frac{(N-1)}{2}}}{2^{(N+2)}N^N} \cdot \exp\left(-NT^{-1}K^2(r_N + K)^2\right)
\end{aligned}
\tag{4.15}
$$

for all $N \in \mathbb{N}$. This shows the existence of a constant $c \in (1, \infty)$ such that

$$
\mathbb{P}\big[\Omega_N\big] \geq c^{(-N^c)} \tag{4.16}
$$

for all $N \in \mathbb{N}$.

Combining (4.14) and (4.16) then gives

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\Big[\left|Y_N^N\right|\Big] &\geq \lim_{N\to\infty} \mathbb{E}\Big[\mathbb{1}_{\Omega_N}\left|Y_N^N\right|\Big] \geq \lim_{N\to\infty}\left(\mathbb{P}\big[\Omega_N\big] \cdot (r_N)^{\left(\alpha^{(N-1)}\right)}\right) \\
&\geq \lim_{N\to\infty}\left(\mathbb{P}\big[\Omega_N\big] \cdot 2^{\left(\alpha^{(N-1)}\right)}\right) \geq \lim_{N\to\infty}\left(c^{(-N^c)} \cdot 2^{\left(\alpha^{(N-1)}\right)}\right) = \infty
\end{aligned}
\tag{4.17}
$$

and Jensen's inequality hence shows

$$\lim_{N \to \infty} \mathbb{E}\left[\left| Y_N^N \right|^p \right] = \infty. \qquad (4.18)$$

Since $\mathbb{E}\left[|X_T|^p\right] < \infty$ by assumption, we finally conclude

$$\lim_{N \to \infty} \mathbb{E}\left[\left| X_T - Y_N^N \right|^p \right] = \infty \quad \text{and} \quad \lim_{N \to \infty} \left| \mathbb{E}\left[\left| X_T \right|^p \right] - \mathbb{E}\left[\left| Y_N^N \right|^p \right] \right| = \infty. \quad (4.19)$$

This, (4.14) and (4.16) finally complete the proof of Theorem 1. $\qquad \square$

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